A UNNS Perspective on a Ball on a String
Discrete recurrences, operators and stability

UNNS Research Notes

September 26, 2025

Abstract

We reframe the classical pendulum ("ball on a string") within the Unbounded Nested Number Sequence (UNNS) substrate. The continuous pendulum equation is discretized into a second-order recurrence that naturally fits the UNNS language: nest states, operator actions (inletting, inlaying, collapse, repair), and spectral invariants. We present the discrete recurrence (implicit Verlet/centered scheme), a small-angle linearization producing a constant-coefficient UNNS recurrence, a stability lemma, and a discrete energy invariant. We then extend the discussion to the nonlinear (large-angle) regime via Poincaré maps and Floquet (monodromy) analysis, translated into UNNS terms: Floquet multipliers become UNNS spectral constants, and instability triggers repair/adoption operators. A TikZ illustration shows the physical pendulum above and the corresponding recursion spiral of discrete states below.

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1 Introduction

The simple pendulum (a ball on an ideal massless string) is a canonical oscillatory system. In classical mechanics it is modeled by the nonlinear ODE

$$mL^2\ddot{\theta}(t) + mgL\sin\theta(t) = 0,$$

or equivalently

$$\ddot{\theta}(t) + \omega_0^2 \sin\theta(t) = 0, \qquad \omega_0^2 = \frac{g}{L}. \tag{1}$$

In the UNNS viewpoint we regard time evolution as an iterative nest of states. We discretize (1) to obtain a second-order recurrence for the sequence $\{\theta_n\}$, interpret the recurrence coefficients as UNNS operators, and analyze stability and invariants within the UNNS substrate.

2 Discrete UNNS Recurrence

Let $\Delta t > 0$ be a fixed time step and define $\theta_n \approx \theta(n\Delta t)$. Using the centered finite-difference (Verlet / leapfrog) discretization we obtain:

$$\theta_{n+1} - 2\theta_n + \theta_{n-1} + \Delta t^2\, \omega_0^2 \sin\theta_n \;=\; 0. \tag{2}$$

Re-arranged,

$$\theta_{n+1} \;=\; 2\theta_n - \theta_{n-1} - \Delta t^2\, \omega_0^2 \sin\theta_n. \tag{3}$$

Equation (3) is a second-order nonlinear recurrence of the UNNS type: the next state is a fixed integer-linear combination (here coefficients are integers for the linear part) plus a nonlinear *echo* term $-\Delta t^2 \omega_0^2 \sin\theta_n$. UNNS operators interpret parts of this recurrence:

- **Inletting** ($\mathcal{I}$): propagation of inertial memory, realized by the term $2\theta_n - \theta_{n-1}$ (carry-forward of previous states).
- **Inlaying** ($\mathcal{J}$): the restoring action embedded into the state via the nonlinear term $-\Delta t^2\omega_0^2 \sin\theta_n$.
- **Collapse** ($\mathcal{C}$): turning points (velocity zero) where the sequence locally attains extrema; in UNNS language these are nests approaching absorbing states.
- **Repair/Normalization** ($\mathcal{R}$): any algorithmic step that enforces boundedness or numerical stabilization (e.g., limiting Δt or applying projection).

2.1 Small-angle linearization

For small oscillations $\theta \ll 1$ we use $\sin\theta \approx \theta$, leading to a linear recurrence:

$$\theta_{n+1} \;=\; 2\theta_n - \theta_{n-1} - \Delta t^2\,\omega_0^2\theta_n = (2-\kappa)\theta_n - \theta_{n-1}, \qquad \kappa := \Delta t^2\omega_0^2. \tag{4}$$

This is a constant-coefficient UNNS recurrence of order 2. Its characteristic polynomial is

$$\lambda^2 - (2-\kappa)\lambda + 1 = 0,$$

with roots

$$\lambda_{1,2} \;=\; \frac{(2-\kappa) \pm \sqrt{(2-\kappa)^2 - 4}}{2}.$$

Introduce the discrete angular frequency via

$$2\cos(\omega\Delta t) \;=\; 2-\kappa \quad\Longrightarrow\quad \omega\Delta t = \arccos\left(1-\tfrac{\kappa}{2}\right).$$

Then the general solution is

$$\theta_n = A\cos(n\omega\Delta t) + B\sin(n\omega\Delta t),$$

showing the expected oscillatory behavior when the roots are complex (i.e. stability).

3 A UNNS Stability Lemma

The linear recurrence (4) is stable (i.e. bounded oscillatory) precisely when the characteristic roots lie on the complex unit circle. This yields the familiar stability condition for Verlet time stepping.

Lemma 3.1 (Stability of the discrete UNNS pendulum)**.** *For the linearized recurrence* (4)*, the discrete solution* $\{\theta_n\}$ *is bounded (oscillatory) for all* n *iff*

$$0 < \kappa < 4 \quad\Longleftrightarrow\quad 0 < \Delta t < \frac{2}{\omega_0}.$$

When $\kappa = 4$ *the scheme is marginal (double root* $= -1$*), and if* $\kappa > 4$ *one gets exponentially growing (unstable) modes.*

Proof. The discriminant of the characteristic polynomial is $(2-\kappa)^2 - 4 = (\kappa^2 - 4\kappa)$. Complex conjugate roots (with unit modulus) occur when $(2-\kappa)^2 - 4 < 0$, i.e. $\kappa(4-\kappa) > 0$, which gives $0 < \kappa < 4$. Furthermore, the product of the roots is 1, so roots lie on unit circle when real part between -2 and 2, equivalent to above. □

Remark 3.2. *In UNNS language,* Δt *plays the role of a* nest resolution parameter*: too coarse a resolution (*Δt *large) breaks the recursion stability and triggers repair operators to restore bounded behavior.*

4 Discrete Energy (Approximate Invariant)

The Verlet discretization preserves a discrete analogue of mechanical energy up to $O(\Delta t^2)$ errors. One convenient discrete energy functional is

$$E_n \;=\; \tfrac{1}{2}mL^2\Big(\frac{\theta_{n+1}-\theta_n}{\Delta t}\Big)^2 + mgL\big(1-\cos\theta_n\big).$$

Using the recurrence (3) one checks that

$$E_{n+1}-E_n = O(\Delta t^3),$$

so energy is approximately conserved (symplecticity of the integrator). In UNNS terms, this near-invariance is a *recursive spectral invariant* — a constant of motion that survives under inlaying/inletting iterations if the repair operator acts rarely.

5 UNNS Operator Interpretation (Summary)

We summarize the mapping between classical pendulum elements and UNNS substrate concepts:

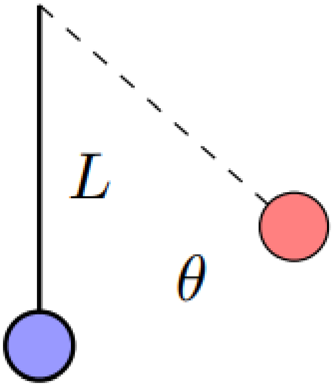
- **State** $\theta_n \longleftrightarrow$ UNNS nest element at depth n.
- **Time step** $\Delta t \longleftrightarrow$ nest resolution / sampling depth.
- **Verlet update** $\longleftrightarrow$ composition of inletting (inertia) and inlaying (restoration).
- **Turning point** $\longleftrightarrow$ collapse event (local nest extremum).
- **Energy invariant** $\longleftrightarrow$ UNNS constant (spectral invariant).
- **Stability threshold** $\kappa < 4 \longleftrightarrow$ admissibility condition for nests.

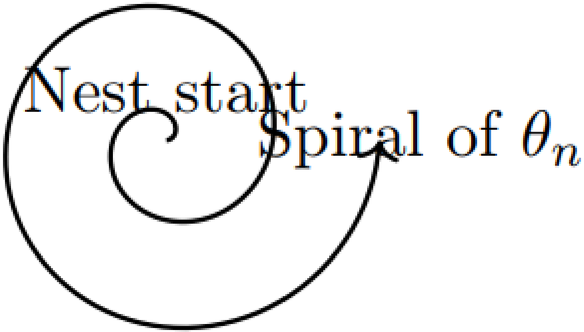
6 Worked example: numerical parameters

Choose $L = 1\,\mathrm{m}$, $g = 9.81\,\mathrm{m/s^2}$ so $\omega_0 \approx 3.1321\,\mathrm{s}^{-1}$. The stability bound gives $\Delta t < 2/\omega_0 \approx 0.6388\,\mathrm{s}$. For typical accuracy, choose $\Delta t = 0.01\,\mathrm{s}$ ($\kappa \approx 0.000981$), well within the stable UNNS regime. Under small initial angle $\theta_0 = 0.1$, the discrete solution closely matches the continuous solution; collapse events coincide with zeros of discrete velocity.

7 Visualization: Pendulum + UNNS Spiral

The figure below shows (top) the classical pendulum arc and (bottom) a schematic recursion spiral: each dot on the spiral corresponds to a discrete state θ_n; radial oscillations encode amplitude, angular position encodes phase (index n). Turning points appear as foldings on the spiral and map to collapse loci.


L
θ


Nest start
Spiral of θ_n

8 Nonlinear dynamics: Floquet and Poincaré analysis in UNNS language

Large-angle oscillations of the pendulum are fundamentally nonlinear: the period depends on amplitude, and the dynamics can exhibit richer behaviors (period doubling under perturbation, parametric resonances when driven, etc.). For a robust UNNS interpretation of nonlinear recurrence dynamics we use the classical tools of *Poincaré maps* and *Floquet (monodromy) analysis* and then translate their spectral outputs into UNNS spectral constants (multipliers) that inform operator actions (e.g., when to trigger repair/normalization). Below we outline the core definitions, state key results, provide UNNS translations, and explain numerics.

8.1 The continuous-time viewpoint and periodic orbits

Consider the continuous system in first-order form:

$$\dot{x} = f(x), \qquad x = (\theta, \dot{\theta}) \in \mathbb{R}^2,$$

with $f(\theta, \dot{\theta}) = (\dot{\theta},\ -\omega_0^2 \sin\theta)$. A nontrivial periodic orbit $\gamma(t)$ of period T (a large-angle oscillation) satisfies $\gamma(t + T) = \gamma(t)$.

Definition 8.1 (Poincaré section and map)**.** *Let Σ be a transverse section to the flow near the periodic orbit (e.g., $\Sigma = \{\theta = 0,\ \dot{\theta} > 0\}$). The* Poincaré map $P : \Sigma \to \Sigma$ *sends an initial point $x_0 \in \Sigma$ to the next intersection $x(T; x_0) \in \Sigma$ of its forward orbit with the section.*

Fixed points of P correspond to periodic orbits of the flow. The local stability of a periodic orbit is determined by the derivative (Jacobian) DP at the fixed point: eigenvalues of DP are the *Floquet multipliers* (equivalently eigenvalues of the monodromy matrix of the linearized flow).

8.2 Linearization and the monodromy matrix

Linearize the flow along a T-periodic solution $\gamma(t)$. The variational equation for perturbations $\xi(t)$ is

$$\dot{\xi}(t) = A(t)\,\xi(t), \qquad A(t) := Df(\gamma(t)),$$

a T-periodic linear system. Let $\Phi(t)$ be the fundamental matrix solution with $\Phi(0) = I$. The *monodromy matrix* is $\Phi(T)$. Its eigenvalues μ_i are the Floquet multipliers.

Theorem 8.2 (Floquet stability criterion)**.** *The periodic orbit γ is (linearly) stable iff all Floquet multipliers satisfy $|\mu_i| < 1$ (for the Poincaré map) or, equivalently, the continuous-time exponents have nonpositive real parts with simple exceptions for neutral directions due to time-shift.*

8.3 Discretization, stroboscopic map and UNNS recurrence

To integrate the continuous analysis with the UNNS discrete viewpoint, proceed as follows:

- Choose a sampling interval Δt (nest resolution) and integrate the ODE with a symplectic method (e.g., Verlet) to obtain the discrete sequence θ_n. For a periodic orbit of period T, choose sampling strobing at times $t_k = kT$ to build a *stroboscopic* or return map that coincides with the Poincaré map on a chosen section.

- The discrete update from x_k to x_{k+1} under one period is a nonlinear map $F : \mathbb{R}^2 \to \mathbb{R}^2$, the discrete analogue of P. Linearizing F at the fixed point gives the monodromy matrix $M = DF$ whose eigenvalues are the same Floquet multipliers μ_i up to numerical error.
- In UNNS language, the stroboscopic mapping over one period is a higher-level recurrence (a nest-of-nests): it maps a nest state to another nest state after one cycle. Its linearization yields a companion matrix whose spectrum forms UNNS spectral constants for that periodic behavior.

8.4 UNNS translation: multipliers as spectral constants

Floquet multipliers μ_i become UNNS spectral constants (multipliers of the nested map). Their magnitudes govern admissibility:

- If $\max_i |\mu_i| < 1$, the periodic nest is *attracting*: small perturbations decay and no repair is necessary.
- If some $|\mu_i| > 1$, the nest is *unstable*: perturbations grow, leading potentially to divergence or bifurcation. In the UNNS operator grammar this triggers the *repair/normalization* operator $\mathcal{R}$ (e.g., re-projection onto admissible coefficient rings) or an *adoption* operator that grafts new stabilizing sub-nests.
- If $|\mu_i| = 1$ with nontrivial Jordan blocks, the behavior is marginal (resonant); UNNS evaluation operator $\mathcal{E}$ must inspect higher-order terms (nonlinear normal forms) to decide action.

8.5 A UNNS Floquet lemma

Lemma 8.3 (UNNS Floquet lemma)**.** *Consider a periodic orbit γ of the pendulum and its associated Poincaré map P. Let $M = DP(x^*)$ be the monodromy (Jacobian) at the fixed point $x^* \in \Sigma$. Then the UNNS stroboscopic recurrence formed by sampling once per period has characteristic constants equal to the eigenvalues of M. The UNNS nest is stable iff $\rho(M) < 1$, where ρ is the spectral radius.*

Sketch. By construction the stroboscopic map is exactly the Poincaré map; its linearization is M. The UNNS stroboscopic recurrence is the discrete map $x_{k+1} = P(x_k)$. Near the fixed point one writes $x_{k+1} - x^* = M(x_k - x^*) + O(\|x_k - x^*\|^2)$. The characteristic behavior of the linearized UNNS recurrence is governed by M, hence the spectral radius criterion. □

8.6 Bifurcations and UNNS operator responses

When parameters vary (e.g., drive amplitude if the pendulum is periodically forced, or Δt in numerical realizations), Floquet multipliers cross the unit circle leading to bifurcations: period-doubling (a multiplier crosses -1), Neimark–Sacker (complex-conjugate pair crosses unit circle), saddle-node, etc. In UNNS vocabulary:

- **Detection:** The evaluation operator $\mathcal{E}$ monitors multipliers and computes $\rho(M)$.
- **Repair action:** If $\rho(M) > 1$, the repair operator $\mathcal{R}$ applies normalization (projection to stable coefficient set) or triggers decomposition $\mathcal{D}$ to split unstable nest into sub-nests that can be managed.
- **Adoption:** Adopting operator $\mathcal{A}$ may graft a stabilizing sub-nest (e.g., passive damping nest) onto the unstable nest.

8.7 Numerical computation of Floquet multipliers (practical note)

Practical steps to compute multipliers for the pendulum periodic orbit numerically:

1. Locate a periodic orbit (shooting method), i.e., find initial condition x_0 such that $\Phi(T; x_0) = x_0$.
2. Integrate variational equations $\dot{\Phi}(t) = A(t)\Phi(t)$ along the orbit to obtain $\Phi(T)$.
3. Compute eigenvalues of $\Phi(T)$ (numerical eigenvalue routine). These are the multipliers μ_i.

The same computation can be performed on the discrete UNNS stroboscopic map using the discrete Jacobian DF (finite differences or automatic differentiation).

8.8 Interpretation: Floquet spectrum as UNNS constants

The Floquet multipliers are natural candidates for UNNS spectral constants associated with a periodic nest: they quantify echo amplification/attenuation per period. In particular:

$$\text{UNNS spectral constants } \equiv \{\mu_1, \mu_2, \dots\}.$$

These constants feed directly into UNNS decision rules:

$$\begin{cases} \max_i |\mu_i| < 1 & \Rightarrow \text{No repair; nest admissible,} \\ \max_i |\mu_i| = 1 & \Rightarrow \text{Evaluate nonlinear terms } (\mathcal{E}), \\ \max_i |\mu_i| > 1 & \Rightarrow \text{Repair/Decompose/Adopt } (\mathcal{R}, \mathcal{D}, \mathcal{A}). \end{cases}$$

8.9 Example: large-angle periodic oscillation

For the undriven pendulum the only instabilities in the conservative setting are marginal ones associated with the separatrix (homoclinic orbit). For forced/damped variants the Floquet spectrum exhibits classical windows of period-doubling into chaos. UNNS analysis here recovers the same bifurcation diagram, but additionally labels epochs of instability with operator actions: when cascades toward chaos are detected (multipliers crossing unit circle), UNNS repair inserts damping nests or decomposes the nest into frequency bands (spectral decomposition) for control.

9 Discussion and extensions

- The UNNS perspective emphasizes that continuous oscillation can be seen as the limit of nested discrete recurrences. The choice of Δt is central: it is both a numerical sampling parameter and a UNNS nest-resolution constant.
- For larger amplitudes (nonlinear regime) the recurrence (3) is genuinely nonlinear; spectral analysis must be done via Floquet theory or numeric spectral decomposition of the monodromy map (this fits naturally into the UNNS spectral protocol).
- Repair operators in UNNS naturally model numerical stabilization or physical damping: they act when discrete amplitudes leave admissible bounds, projecting trajectories back into stable nest zones.

10 Conclusion

The ball-on-a-string example provides a clear and didactic laboratory for the UNNS substrate: discrete recurrences, operator interpretations, stability thresholds, and approximate invariants (all standard in numerical analysis) acquire an interpretive layer as UNNS nests, operators, and constants. Extending the analysis into the nonlinear regime via Floquet and Poincaré maps connects classical periodic-orbit stability theory with UNNS spectral constants and operator responses, providing a concrete mechanism for when and how repair/adoption/decomposition operators should be applied.

Acknowledgements. This note is part of the UNNS Research Notes series.