UNNS Spectral Geometry: Recurrence Coefficients as Spectral Data for Lattices, Tilings and Waveguides

UNNS Research Notes

September 27, 2025

Abstract

We develop *UNNS Spectral Geometry*: a framework that interprets coefficients and spectral roots of Unbounded Nested Number Sequences (UNNS) as geometric and spectral parameters for lattices, tilings, and waveguides. We describe canonical mappings from recurrence data to graph Laplacians and Schrodinger operators, exhibit worked examples (Fibonacci and cyclotomic UNNS), discuss spectral gap formation, Cantor-type spectra, diffraction signatures, and relations to quasicrystals and Penrose tilings. We provide numerical experiment protocols and visualization guidelines for exploring these phenomena.

Contents

1 Introduction

Recurrence relations and nested sequences produce algebraic structures (characteristic polynomials, cyclotomic constants) that naturally suggest geometric realizations. UNNS Spectral Geometry formally ties the arithmetic data of UNNS to spectra of spatial operators on lattices and tilings. This provides a concrete geometric counterpart to UNNS sequences and allows comparison with known aperiodic solids (quasicrystals), substitution tilings (Fibonacci, Penrose), and engineered waveguides.

2 Definitions and canonical mappings

Definition 2.1 (UNNS recurrence)**.** *A UNNS recurrence of order r is*

$$x_{n+r} = c_1 x_{n+r-1} + \cdots + c_r x_n, \qquad c_j \in \mathbb{C},$$

possibly with coefficients arising from nested algebraic constructions (Gaussian/Eisenstein/cyclotomic integers).

Definition 2.2 (Characteristic data)**.** *Let $P(\lambda) = \lambda^r - c_1\lambda^{r-1} - \cdots - c_r$ be the characteristic polynomial with roots $\{\lambda_i\}_{i=1}^r$ (possibly algebraic integers). The unordered multiset $\{\lambda_i\}$ is the characteristic spectral data of the UNNS.*

We propose canonical maps from recurrence data to geometric operators. Three basic constructions follow.

2.1 Edge-weighted 1D lattice (chain) map

Given a UNNS sequence of coefficients (or derived weights) $\{w_n\}$, build a 1D chain graph with vertex set $\mathbb{Z}$ or finite approximant $\{1, \ldots, N\}$ and symmetric adjacency (or Laplacian) with weights $a_{n,n+1} = w_n$. The discrete Laplacian

$$(L\psi)_n = w_{n-1}\psi_{n-1} - (w_{n-1} + w_n)\psi_n + w_n\psi_{n+1}$$

defines a spectral problem $L\psi = E\psi$. When w_n derives from UNNS (e.g., values of a nested sequence or functions of characteristic roots), the operator's spectrum reflects UNNS arithmetic.

2.2 Schrödinger operator with UNNS potential

Define on $\ell^2(\mathbb{Z})$:

$$(H\psi)_n = \psi_{n+1} + \psi_{n-1} + V_n\psi_n,$$

with potential $V_n = f(\{x_n\})$ built from a UNNS sequence x_n (for instance $V_n = \phi(x_n)$, or V_n equals indicator of symbolic substitution). Classical results (e.g., for Sturmian potentials) show that such operators can display Cantor spectra and singular continuous spectral measures.

2.3 Tiling-lattice embedding

Given a UNNS-generated substitution or inflation rule (derived from coefficient algebra, e.g., Pisot substitutions), produce a tiling of $\mathbb{R}^d$. Edge vectors / tile shapes are chosen so that Fourier module (diffraction) is connected to cyclotomic fields determined by the UNNS coefficients.

3 Examples

3.1 Fibonacci UNNS (rank-2)

The Fibonacci recurrence $F_{n+1} = F_n + F_{n-1}$ produces the classic Sturmian sequence via substitutions $a \mapsto ab,\ b \mapsto a$. Take w_n equal to two-valued sequence (e.g., $w_n \in \{w_A, w_B\}$ according to the substitution), build a Schrödinger operator with that potential: this is the Fibonacci Hamiltonian, known to have a Cantor spectrum ("Cantor set of zero Lebesgue measure") and singular continuous spectral measures. This is a canonical UNNS spectral geometry example.

3.2 Cyclotomic UNNS and hexagonal lattices

When characteristic roots lie in cyclotomic fields (e.g., $\mathbb{Q}(\zeta_m)$), the lattice natural to use is hexagonal or triangular (Eisenstein lattice). Coefficients derived from Gauss/Jacobi sums can parameterize hopping phases on a 2D lattice producing Dirac-like points or engineered band structures. With suitable inhomogeneity the spectrum may split into bands and gaps determined by algebraic norms of UNNS constants.

4 Spectral phenomena: gaps, Cantor spectra, and localization

4.1 Spectral gaps and gap-labelling

For substitution-based UNNS, gap labels (integrated density of states at gap) often lie in a finitely generated group (the gap-labelling group) determined by cohomology of the tiling dynamical system. UNNS arithmetic influences this group via the module generated by recurrence frequencies and algebraic integers.

4.2 Cantor-spectrum and fractal IDS

Operators defined from substitution sequences (Fibonacci being the canonical case) exhibit fractal spectra: the spectrum is a Cantor set and the IDS is a devil's staircase. We conjecture similar phenomena for a broad class of UNNS derived substitution/transfer rules, especially when characteristic roots are Pisot numbers.

4.3 Localization vs extended states

Depending on disorder or deterministic aperiodicity, UNNS-lattices can show localized eigenstates (exponential decay), critical states (power-law decay), or extended states. The arithmetic of coefficients controls transfer-matrix growth and Lyapunov exponents.

5 Comparisons: Quasicrystals, Penrose tilings, and UNNS

- **Penrose / Ammann patterns:** Generated by projection methods or matching rules; diffraction shows pure point spectrum with icosahedral

/ decagonal symmetry. When UNNS coefficients produce substitution rules equivalent to these matching rules, identical diffraction features can be recovered.

- **Quasicrystals:** Can be generated by cut-and-project using modules over algebraic number fields. UNNS whose characteristic fields are cyclotomic or quadratic provide natural arithmetic modules for cut-and-project schemes, linking UNNS to quasicrystalline diffraction.
- **Waveguides and photonics:** Engineering hopping amplitudes and phases from UNNS constants yields waveguide arrays whose transmission spectra mirror UNNS spectral geometry. This enables prototyping via photonic lattices.

6 Numerical experiments and protocols

Below are reproducible experiments to demonstrate UNNS Spectral Geometry features.

6.1 Experiment A: Finite approximant spectra (1D)

1. Choose a UNNS sequence (Fibonacci or other substitution derived from UNNS coefficients).
2. Build finite chain of length N with potential V_n or coupling w_n.
3. Compute eigenvalues of the Hamiltonian H_N (sparse eigensolver). Plot eigenvalues vs approximant level to visualize gap opening.
4. Approximate IDS by cumulative histogram of eigenvalues scaled by $1/N$.

6.2 Experiment B: IDS and box-counting dimension

Compute IDS for increasing approximant sizes; perform box-counting on spectral measure to estimate fractal (Hausdorff) dimension.

6.3 Experiment C: Diffraction / structure factor (tilings)

Generate a finite patch of a UNNS-derived tiling; compute discrete Fourier transform of point set (intensity $|\widehat{\rho}(k)|^2$); visualize Bragg peaks or continuous scattering.

6.4 Experiment D: Wave propagation

Simulate time-dependent wave packet propagation on the UNNS lattice; measure transmission/reflection across finite segments; look for sharp resonances or fractal bandpass behavior.

7 Analytic questions and conjectures

[UNNS Cantor spectra] If a UNNS substitution is primitive and has characteristic root a Pisot number, then the associated 1D Schrodinger operator with suitable two-letter potential has a Cantor spectrum of zero Lebesgue measure.

[Algebraic gap labelling] Gap labels of UNNS substitution tilings lie in the group generated by traces of cyclotomic integers associated to the UNNS characteristic field.

8 Visualization and interactive tools

Key interactive modules:

- Tiling generator (slider: substitution depth; export patch).
- Spectrum explorer: live eigenvalue computation for approximants; slider controls N.
- Diffraction viewer: DFT intensity map; toggle logarithmic scale.
- Waveguide demo: animated wavepacket, toggle coupling values from UNNS coefficients.

9 Applications and outlook

UNNS Spectral Geometry connects arithmetic sequences to physically observable spectral phenomena. Possible applications:

- Design of photonic/phononic metamaterials with algebraically determined band structure.
- Novel quasicrystalline patterns derived from UNNS arithmetic for materials design.
- A new classification of substitution tilings via UNNS characteristic fields.

10 Conclusion

Treating UNNS recurrence coefficients as spectral/geometric data yields a fertile domain connecting number theory, spectral theory, and physical wave phenomena. The program above provides both rigorous directions and concrete computational experiments to explore.

Acknowledgements

Notes and prototypes developed in the UNNS project.

References

[1] M. Baake, U. Grimm, *Aperiodic Order.* (reference for diffraction aperiodic order)

[2] D. Damanik, *The Spectrum of the Fibonacci Hamiltonian.*

[3] M. Senechal, *Quasicrystals and Geometry.*