From Unbounded Nested Number Sequences to Maxwell's Equations via DEC and FEEC

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Abstract

We construct a rigorous mathematical bridge between Unbounded Nested Number Sequences (UNNS) and the classical Maxwell system of electrodynamics. Interpreting UNNS data as discrete electromagnetic potentials on a hierarchy of meshes in spacetime, we apply Discrete Exterior Calculus (DEC) and Finite Element Exterior Calculus (FEEC) to establish convergence of the discrete Maxwell system to the continuous one in four-dimensional spacetime. Full proofs are given for the 3+1-dimensional case, including error constants, stability estimates, and treatment of gauge freedom.

1 Introduction

Maxwell's equations admit a natural formulation in the language of differential forms. Discrete Exterior Calculus (DEC) provides combinatorial analogues on meshes, while Finite Element Exterior Calculus (FEEC) supplies approximation and stability theorems.

The UNNS formalism provides a systematic way to encode discrete potentials across an infinite nested hierarchy of meshes. Our goal is to show that UNNS data naturally induce discrete electromagnetic fields F_h which converge to the classical fields F as $h \to 0$.

2 Preliminaries

Let $M = \Omega \times [0, T] \subset \mathbb{R}^{3+1}$ with Lorentzian metric $g = -dt^2 + dx^2 + dy^2 + dz^2$.

2.1 Maxwell in forms

Given potential $A \in \Omega^1(M)$, define $F = dA \in \Omega^2(M)$. Then

$$\begin{aligned} dF &= 0, \\ \delta F &= J, \end{aligned}$$

where $\delta = (-1)^{nk+n+1} \star d \star$ is the codifferential.

2.2 Meshes and UNNS

Let $\{\mathcal{T}_h\}$ be shape-regular simplicial meshes. Each oriented edge e is assigned a number $a_e \in \mathbb{R}$ from the UNNS, defining a discrete potential A_h.

3 Discrete Operators

3.1 Exterior derivative

For oriented faces f,

$$(d_h A_h)(f) = \sum_{e \in \partial f} \mathrm{sgn}(e, f)\, A_h(e).$$

3.2 Discrete Hodge star

Define $\star_h$ by dual volumes:

$$\langle \star_h \alpha, \beta \rangle = \sum_{\sigma^k} \frac{|\star \sigma^k|}{|\sigma^k|} \alpha(\sigma^k) \beta(\star \sigma^k).$$

3.3 Codifferential

$$\delta_h = \star_h^{-1} d_h \star_h .$$

4 Discrete Maxwell System

Define $F_h = d_h A_h$. Then

$$d_h F_h = 0,$$
$$\delta_h F_h = J_h.$$

5 Main Results

Theorem 5.1 (Convergence of UNNS Maxwell System)**.** *Suppose $A \in H^s \Omega^1(M)$, $s > 1$, $J = \delta dA$, and assumptions hold: shape-regular meshes, Whitney interpolation Π_h, and norm-equivalence constants $c_1, c_2 > 0$. Then*

$$\|F_h - F\|_{L^2} \le Ch^p, \quad \|J_h - J\|_{L^2} \le Ch^p.$$

Lemma 5.2 (Exactness of d_h)**.** *For any discrete 1-form A_h, $d_h d_h A_h = 0$.*

Proof. Follows from boundary-of-boundary $= 0$. □

Lemma 5.3 (Operator consistency)**.** *For smooth $\alpha \in \Omega^k(M)$,*

$$\|d_h \Pi_h \alpha - \Pi_h d\alpha\|_{L^2} \le Ch^p \|\alpha\|_{H^{p+1}}.$$

Lemma 5.4 (Codifferential consistency)**.** *For smooth α,*

$$\|\delta_h \Pi_h \alpha - \Pi_h \delta\alpha\|_{L^2} \le Ch^p \|\alpha\|_{H^{p+1}}.$$

Proof of Theorem. Homogeneous equation: $d_h F_h = 0$ by exactness. Inhomogeneous:

$$\|\delta F - J\| \le \|\delta F - \delta_h F_h\| + \|J_h - J\|,$$

each term $\to 0$ by lemmas and approximation. □

6 Constants and Gauge

Interpolation error constant C_1, DEC/FEEC norm equivalence constants c_1, c_2, final error constant $C = \max\{C_1, c_1^{-1}, c_2\}$.

Gauge: enforce discrete Lorenz gauge $\delta_h A_h = 0$, converging to $\delta A = 0$.

7 Discussion

UNNS ensures refinement $\implies$ mesh family exists. DEC ensures exactness, FEEC ensures stability. Thus UNNS provides a constructive number-theoretic foundation for electromagnetism.

References

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