Markov Chain Monte Carlo on a Recursive Substrate: A UNNS Interpretation of Kernels, Balance, and Mixing

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Abstract

We reinterpret Markov Chain Monte Carlo (MCMC) within the UNNS substrate, where information is geometric curvature and computation proceeds by recursion depth. Target densities become *curvature measures*; proposal kernels enact local transport on a depth-augmented manifold; detailed balance is a *harmony* constraint; and mixing rates are governed by *recursive curvature* rather than only spectral gaps in Euclidean state spaces. This note formalizes the correspondences and proposes curvature-aware, depth-coupled samplers (Ton-guided HMC, Klein-flip jumps, collapse–reseed refresh) with convergence guarantees that reduce to classical results when recursion curvature is flat.

1 From Targets to Curvature Measures

Let $\pi(x)$ be a classical target density on $\mathbb{R}^d$. In UNNS, we lift sampling to a depth-augmented space $\mathcal{M} = \mathcal{X} \times \mathbb{Z}_{\geq 0}$ with coordinates (x, n) and endow it with a recursive potential $\Phi(x, n)$ and curvature density $\kappa(x, n) = \Delta_{x,n}\Phi$. Define the *curvature-weighted* measure

$$\Pi(\mathrm{d}x, n) \ \propto \ \exp\big(-\Phi(x, n)\big)\, w(n)\, \mathrm{d}x, \qquad \sum_{n=0}^{\infty} w(n) = 1. \tag{1}$$

A classical $\pi(x)$ is retrieved on the slice $n = n_\star$ when $\Phi(x, n_\star) = -\log \pi(x) + C$ and $w(n) = \mathbf{1}\{n = n_\star\}$.

Remark 1 (Intuition). Φ encodes *recursive coherence*; high curvature (large κ) marks attractors/valleys where mass concentrates. Sampling π corresponds to traversing the (x, n) manifold while maintaining *harmony* (bounded recursion) so that marginalization over n yields the desired π.

2 Kernels as Recursive Transports

Let K be a Markov kernel on $\mathcal{M}$. We decompose

$$K = K_x \circ K_n, \tag{2}$$

with K_n adjusting recursion depth (tempering/annealing analogue) and K_x transporting within the spatial slice.

Ton-guided proposals. Introduce a τon-field potential $A^{(\tau)}(x,n)$ with field tensor $F^{(\tau)} = \mathrm{d}A^{(\tau)}$. Define a preconditioned Riemannian metric $G(x,n)$ via

$$G(x,n) \;=\; I + \alpha\, F^{(\tau)}(x,n) F^{(\tau)}(x,n)^{\top}, \tag{3}$$

and propose (Ton-RHMC step) by integrating the Hamiltonian

$$H(x,p|n) = -\log \pi(x) + \tfrac{1}{2}\, p^{\top} G(x,n)^{-1} p, \tag{4}$$

with leapfrog flows in (x,p) while keeping n fixed. Metropolis correction preserves the n-slice invariant law; interleaving K_n completes stationarity for Π.

Klein-flip moves. On non-orientable sectors, define an involution S (Klein duality) acting as $x \mapsto \mathsf{S}(x)$, $n \mapsto n \pm 1$. A Barker/Metropolis accept–reject on the pair $\{(x,n), (\mathsf{S}(x), n \pm 1)\}$ creates large non-local jumps across topological obstructions, improving mixing across modes separated by orientation reversals.

3 Balance as Harmony

Classical detailed balance requires

$$\Pi(z)\, K(z, \mathrm{d}z') = \Pi(z')\, K(z', \mathrm{d}z), \qquad z = (x,n). \tag{5}$$

In UNNS we additionally require *recursive harmony*, i.e. bounded expected change in recursive entropy

$$\mathbb{E}\big[\mathcal{H}_r(n') - \mathcal{H}_r(n) \,\big|\, z\big] = 0, \qquad \mathcal{H}_r = \int \kappa(\cdot, n)\, \mathrm{d}\mu(\cdot), \tag{6}$$

so that exploration neither explodes nor extinguishes recursive depth.

Proposition 1 (Stationarity under harmony)**.** *If K satisfies detailed balance* (5) *and the harmony constraint* (6)*, then Π is invariant and the marginal on $\mathcal{X}$ is π.*

4 Curvature and Mixing Rates

Let $\mathcal{L}$ be the generator of a reversible diffusion limit of K on $\mathcal{M}$ endowed with metric G. If the sectional curvature of $(\mathcal{M}, G)$ is bounded below by $\underline{\mathcal{K}}$, a Bakry–Émery argument yields a Poincaré/Log-Sobolev control and hence spectral gap bounds:

$$\lambda_1(\mathcal{L}) \;\gtrsim\; c_1\, \underline{\mathcal{K}} \quad \Rightarrow \quad \mathrm{Var}_{\Pi}(f_t) \le e^{-2\lambda_1 t}\, \mathrm{Var}_{\Pi}(f_0). \tag{7}$$

Thus, *positive recursive curvature* accelerates contraction; highly negative curvature signals slow mixing unless compensated by topological jumps (Klein flips) or stronger Ton preconditioning (larger α).

5 UNNS Reinterpretation of Classical MCMC

- **Metropolis–Hastings**: local random walk in a flat recursion slice ($G = I$), no depth moves ($K_n = \mathrm{Id}$).
- **HMC**: geodesic transport with constant metric; Ton-RHMC generalizes with $G(x,n)$ shaped by $F^{(\tau)}$.
- **Tempering/Annealing**: depth moves K_n with schedule $w(n)$; UNNS provides a geometric schedule via $\Phi(x,n)$ so swaps follow curvature ladders.

- **Non-reversible lifts**: add divergence-free drift aligned with Ton streamlines (improves asymptotic variance while preserving Π).
- **Slice/auxiliary samplers**: depth acts as an explicit slice; collapsing and reseeding correspond to UNNS *Collapse* and *Repair* operators.

6 Algorithms (Sketches)

Ton-RHMC (one iteration).

1. Sample momentum $p \sim \mathcal{N}(0, G(x, n))$.
2. Integrate leapfrog on (x, p) for L steps with step size ϵ using $G(x, n)$.
3. Metropolis accept with prob. $\min\{1, e^{-\Delta H}\}$.
4. Optionally perform a Klein flip proposal $(x, n) \leftrightarrow (\mathsf{S}(x), n \pm 1)$ with Barker/Metropolis correction.
5. Update depth via K_n (e.g., curvature-aware tempering: propose $n' = n \pm 1$ with bias proportional to $e^{-\Delta\Phi}$ and accept by Metropolis).

Collapse–Reseed refresh. Every m steps, apply UNNS Collapse: $x \leftarrow x + \eta$, $n \leftarrow n_0$ with small $\eta \sim \mathcal{N}(0, \sigma^2 I)$ (*structured silence*), then reseed via one Ton-RHMC step. This preserves Π when implemented as a reversible mixture kernel.

7 Diagnostics in UNNS Coordinates

- **Recursive entropy drift**: estimate $\widehat{\Delta\mathcal{H}_r}$ across iterations; harmony suggests $\mathbb{E}[\widehat{\Delta\mathcal{H}_r}] \approx 0$.
- **Curvature-normalized ESS**: $\mathrm{ESS}_\kappa = \mathrm{ESS}/(1 + \overline{|\kappa|})$; penalizes chains that mix only in flat regions.
- **Klein crossing rate**: frequency of accepted flip moves; near zero indicates topological trapping.

8 Limits and Reductions

If $F^{(\tau)} \equiv 0$ (no Ton guidance), $G = I$ and $K_n = \mathrm{Id}$, the framework reduces to classical MCMC. If $\Phi(x, n) = \Phi_0(x) + c(n)$ with c affine in n, UNNS tempering collapses to standard simulated tempering with a linear inverse-temperature ladder.

Outlook

UNNS supplies geometry to the probabilistic core of MCMC: metrics from τon-fields, non-orientable jump symmetries (Klein flips), and depth schedules from recursive potentials. This yields principled preconditioners, global moves across topological barriers, and convergence bounds in terms of curvature. Practical next steps: (i) Ton-RHMC implementation on target posteriors with multi-modal structure; (ii) empirical relation between $\underline{\mathcal{K}}$ estimates and spectral gaps; (iii) automated harmony tuning to stabilize adaptation.