Quadratic Roots Without Complex Numbers via UNNS Echo Cycles

Abstract

We show how quadratic equations with negative discriminant can be "solved" without explicitly using complex numbers by rephrasing the missing square roots as *UNNS* (Unbounded Nested Number Sequences) *echo cycles*—finite oscillatory recursions that live entirely in the real plane. Algebraically this matches the usual complex answers; dynamically it replaces i by a period-4 rotation orbit and $re^{i\theta}$ by a real 2×2 rotation–dilation. We illustrate with $x^2 + 4x + 5 = 0$, $x^2 + 2x + 2 = 0$, $x^2 + 6x + 13 = 0$ and give three more examples in the Appendix.

1 UNNS viewpoint on "imaginary" solutions

Completing the square

A quadratic

$$x^2 + bx + c = 0 \quad \Longleftrightarrow \quad (x + \tfrac{b}{2})^2 = \tfrac{b^2}{4} - c.$$

When the discriminant $\Delta = b^2 - 4c < 0$, the right-hand side is negative and the classical solution uses $\sqrt{-1}$. In UNNS, we interpret $\sqrt{-1}$ as the *generator of a period-4 echo cycle* in a real two-dimensional recursion (a quarter-turn rotation).

Definition 1 (UNNS echo cycle for $\sqrt{-1}$)**.** *Let R be the 90° rotation on $\mathbb{R}^2$:*

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad R^2 = -I, \quad R^4 = I.$$

For a real scalar $r > 0$ and any $v_0 \in \mathbb{R}^2$ with $\|v_0\| = 1$, define the orbit $v_k = r\, R^k v_0$. Then $v_{k+2} = -v_k$ and the map $v \mapsto Rv$ plays the role of multiplication by i without leaving $\mathbb{R}^2$.

Remark 1. *Thus "$x = -\frac{b}{2} \pm \sqrt{c - \frac{b^2}{4}}\, i$" may be recast as a* real *affine form $x = -\frac{b}{2} \pm \langle rR(\cdot) \rangle$, where the $\pm$ branch corresponds to advancing/retarding one step along the 90° rotation echo. Algebraically equivalent; ontologically real.*

2 Three core examples

2.1 Example A: $x^2 + 4x + 5 = 0$

Complete the square:

$$x^2 + 4x + 5 = (x+2)^2 + 1 = 0 \iff (x+2)^2 = -1.$$

Classical: $x = -2 \pm i$. **UNNS:** Let R be the 90° rotation. Seek $(x+2)\cdot(x+2) = -1$ in the sense that squaring corresponds to two steps of the echo: $R^2 = -I$. Thus take a unit vector u on the real plane, set

$$x + 2 \rightsquigarrow u \qquad \text{and} \qquad (x+2)^2 \rightsquigarrow R^2u = -u.$$

So the two solutions are the two first-step echoes along $\pm Ru$:

$$x = -2 \pm \langle Ru \rangle,$$

which numerically coincide with $-2 \pm i$ but are represented as real quarter-turn echoes rather than point values in $\mathbb{C}$.

2.2 Example B: $x^2 + 2x + 2 = 0$

$$(x+1)^2 + 1 = 0 \iff (x+1)^2 = -1.$$

Classical: $x = -1 \pm i$. **UNNS:** same echo as above, centered at -1:

$$x = -1 \pm \langle Ru \rangle.$$

2.3 Example C: $x^2 + 6x + 13 = 0$

$$(x+3)^2 + 4 = 0 \iff (x+3)^2 = -4.$$

Classical: $x = -3 \pm 2i$. **UNNS:** scale the echo by $r = 2$:

$$x = -3 \pm \langle 2\,Ru \rangle,$$

i.e., a radius-2 quarter-turn echo in $\mathbb{R}^2$ about -3.

3 A purely real 2×2 method (no complex, same content)

Negative discriminant systems admit a real 2×2 rotation–dilation model. Consider the companion form of $x_{k+1} = \alpha x_k - \beta x_{k-1}$ with $\alpha^2 < 4\beta$ (complex conjugate roots in the classical view). Then

$$\begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha & -\beta \\ 1 & 0 \end{bmatrix}}_{R(\rho,\theta)} \begin{bmatrix} x_k \\ x_{k-1} \end{bmatrix} \qquad \text{and} \qquad R \sim \rho \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Thus oscillatory behavior ("imaginary" parts) is just planar rotation. For a quadratic $(x + \frac{b}{2})^2 + (c - \frac{b^2}{4}) = 0$, the offset gives the center, and the (positive) quantity $r^2 = c - \frac{b^2}{4}$ gives the rotation radius.

4 Why this is not hand-waving

- The matrix R satisfies $R^2 = -I$ exactly, so it is an intrinsic realizer of "i" inside $\mathbb{R}^2$.
- All arithmetic remains real; the "imaginary" aspect is *geometric rotation* (UNNS echo) rather than a new number system.
- You can compute numerically by real vectors and rotations, obtaining the same numerical answers as the complex formula.

Conclusion

To "solve" a quadratic with negative discriminant without complex numbers, replace $\sqrt{-1}$ by a quarter-turn rotation echo in $\mathbb{R}^2$. Completing the square locates the center; the positive remainder gives the echo radius; the $\pm$ branch corresponds to advancing/retarding one 90° step along the echo. This is algebraically equivalent to the complex answer and fully real in computation.

Appendix: Three more worked examples

A1. $x^2 + 10x + 29 = 0$

Complete the square: $(x+5)^2 + 4 = 0 \Rightarrow (x+5)^2 = -4$.

- Classical: $x = -5 \pm 2i$.
- UNNS echo: $x = -5 \pm \langle 2\,Ru \rangle$ (radius 2 quarter-turn).

A2. $x^2 + 8x + 20 = 0$

$(x+4)^2 + 4 = 0 \Rightarrow (x+4)^2 = -4$.

- Classical: $x = -4 \pm 2i$.
- UNNS echo: $x = -4 \pm \langle 2\,Ru \rangle$.

A3. $x^2 + 2x + 5 = 0$

$(x+1)^2 + 4 = 0 \Rightarrow (x+1)^2 = -4$.

- Classical: $x = -1 \pm 2i$.
- UNNS echo: $x = -1 \pm \langle 2\,Ru \rangle$.

Optional numeric recipe (fully real):

1. Compute center $c = -\frac{b}{2}$ and radius $r = \sqrt{c - \frac{b^2}{4} + c}$ (equivalently $r = \sqrt{c - \frac{b^2}{4}}$ with c from $x^2 + bx + c$).
2. Pick $u = (1, 0)^\top$; form the two echoes $\pm r\, Ru$.
3. Report the two solutions as $c \pm \langle r\, Ru \rangle$ (identical numerically to $c \pm r\, i$, but computed entirely in $\mathbb{R}^2$).