Abstract

This paper explores the dual nature of electric fields in a capacitor, emphasizing the interplay between conservative and non-conservative components as governed by Maxwell's equations. During the charging process, the electric field between the plates comprises two distinct contributions: (1) the Coulombic field ($\vec{E}_{\text{Coulomb}}$), arising from static surface charges and obeying Gauss's Law ($\nabla \cdot \vec{E} = \rho/\varepsilon_0$), and (2) the induced field ($\vec{E}_{\text{induced}}$), generated by the time-varying magnetic field via Faraday's Law ($\nabla \times \vec{E} = -\partial \vec{B}/\partial t$). While the total electric field appears vertically uniform between the plates due to the dominance of $\vec{E}_{\text{Coulomb}}$, its non-zero curl reveals the presence of a non-conservative component. This duality resolves the apparent paradox of straight field lines despite a curling electric field. Furthermore, the feedback mechanism between time-varying electric and magnetic fields, as described by the Ampere-Maxwell and Faraday Laws, is discussed to illustrate the conceptual foundation of electromagnetic wave propagation. The analysis underscores the elegance of Maxwell's equations in unifying electrostatic and dynamic phenomena.

The Dual Nature of Electric Fields in a Capacitor: A Conceptual and Mathematical Exploration of Maxwell's Equations

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1 Introduction

Maxwell's equations are the foundation of classical electrodynamics, describing how electric and magnetic fields propagate and interact with matter. The journey to their complete formulation began with Ampere's Circuital Law, but an inconsistency arose in the case of a charging capacitor. Maxwell resolved this by introducing the concept of displacement current, leading to the complete set of equations.

2 Maxwell's Equations in Their Final Form

With the displacement current included, Maxwell formulated the four fundamental equations of electrodynamics:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}, \quad \text{(Gauss's Law)} \tag{1}$$

$$\nabla \cdot \vec{B} = 0, \quad \text{(Gauss's Law for Magnetism)} \tag{2}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \text{(Faraday's Law)} \tag{3}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}. \quad \text{(Ampere-Maxwell Law)} \tag{4}$$

These equations describe the fundamental behavior of electric and magnetic fields, leading to the prediction of electromagnetic waves. Maxwell's correction to Ampere's law was a crucial step in unifying electricity and magnetism into a single theory of electrodynamics. His equations not only resolved inconsistencies but also predicted electromagnetic waves, paving the way for modern physics and telecommunications.

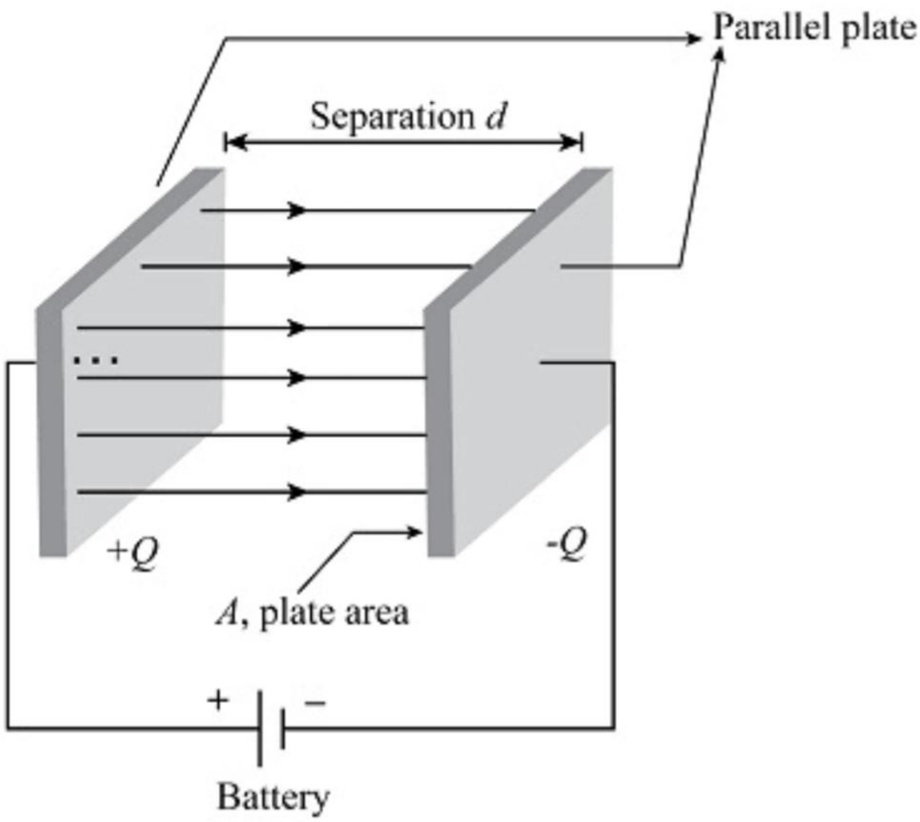

Figure 1: vertical Electric fields between plates.

3 Now let's discuss an intersting topic where two types electric fields are there in a capacitor

As in the figure you see that the electric field lines vertical between plates, but the magnetic field is changing and this induces a electrical field. and for this field ,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

And this field is non-conservative. So why do electric field lines still look vertical between plates?
If the field has a curl , why don't the electric field lines look curve or rotational between plates.
And the answer lies in this..... Actually there are two components to the electric field in a capacitor During the charging process of a capacitor, the electric field between the plates is not purely electrostatic. It has two distinct components:

- $\vec{E}_{\text{coulomb}}$: A conservative electric field produced by the static charges on the plates. This field is directed vertically from the positive to the negative plate.For these fields, the divergence of the electric field is given by Gauss's law:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

  where ρ is the local charge density and ε_0 is the permittivity of free space.

- $\vec{E}_{\text{induced}}$: A non-conservative electric field induced by the time-varying magnetic field, which arises from the displacement current between the plates.For these fields, the divergence of the electric field is given by :

$$\nabla \cdot \vec{E} = 0$$

because this electric field due to changing magnetic fiels According to Faraday's Law:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

This implies that the total electric field is not conservative when $\frac{\partial \vec{B}}{\partial t} \neq 0$.

Field Type	**Source**	**Divergence (Between Plates)**
$\vec{E}_{\text{Coulomb}}$	Stationary surface charges	0 (inside the gap)
$\vec{E}_{\text{induced}}$	Changing magnetic field (Faraday)	Always 0 (looping field lines)

Table 1: Comparison of divergence of Coulomb and induced electric fields in a charging capacitor.

Hence, even though the field lines between the plates appear vertically parallel (due to the dominant $\vec{E}_{\text{coulomb}}$), the total field $\vec{E}_{\text{total}}$ has a small curl component from $\vec{E}_{\text{induced}}$, making it non-conservative.

4 Mathematical importance of Faraday law and Maxwell-Ampere law for understanding EM-Waves

From Faraday law a changing or rotating magnetic field induces a non conservative electric field

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

From Ampere-maxwell law a rotating or time-varying electric field generate a magnetic field

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

This creats a feed back loop.....

$$\textbf{Changing } \vec{B} \xrightarrow{\text{induces}} \vec{E} \text{.......} \textbf{Changing } \vec{E} \xrightarrow{\text{induces}} \vec{B}$$

But, this is not a literal image of EM wave.If our intention is only to illastrate the machanism by which time varying fields generate each other as described in Maxwell's equations - then this image makes sense conceptually. So, rotating magnetic and electrical fields mutually sustain each other , leading to wave phenoma. This is the essence of classical electromagnetism.

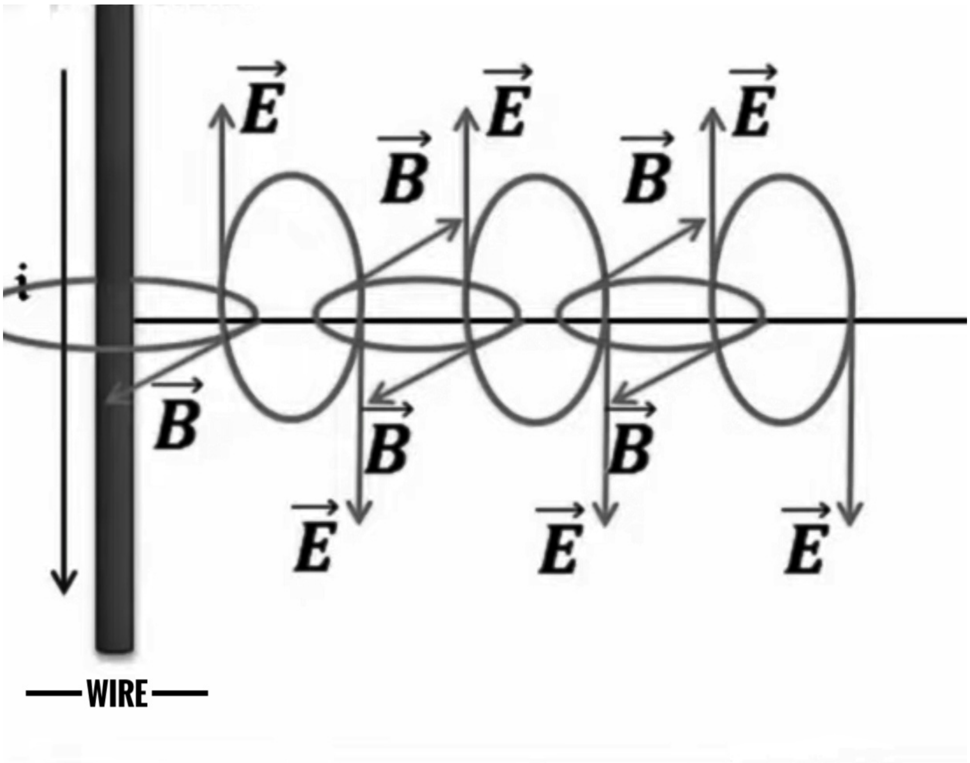

Figure 2: Curling Electric and Curling Magnetic fields.

5 Mathematical Derivation

5.1 Electrostatic Field ($\vec{E}_C$)

For a parallel-plate capacitor with plate area A, separation d, and charge $Q(t)$, the electrostatic field is:

$$\vec{E}_C = \frac{\sigma}{\epsilon_0}\hat{n} = \frac{Q(t)}{\epsilon_0 A}\hat{n},$$

where $\sigma = Q/A$ is the surface charge density. For a capacitor charging at voltage V:

$$|\vec{E}_C| = \frac{V}{d}.$$

This field is uniform and typically large (e.g., for $V = 10\,\mathrm{V}$ and $d = 1\,\mathrm{mm}$, $|\vec{E}_C| = 1 \times 10^4\,\mathrm{V\,m^{-1}}$).

5.2 Induced Field ($\vec{E}_{ind}$)

From Faraday's Law, the induced field satisfies:

$$\nabla \times \vec{E}_{ind} = -\frac{\partial \vec{B}}{\partial t}.$$

For a circular capacitor of radius R, symmetry dictates that $\vec{E}_{ind}$ circulates azimuthally. Using Ampère-Maxwell's Law:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_D = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt},$$

where I_D is the displacement current. Solving for $\vec{E}_{ind}$ at a distance r from the center:

$$|\vec{E}_{ind}| = \frac{\mu_0 r}{2A} \left| \frac{dQ}{dt} \right|.$$

For a slowly charging capacitor with $I(t) = dQ/dt$:

$$|\vec{E}_{ind}| \approx \frac{\mu_0 r I}{2A}.$$

5.3 Comparison of Fields

Let's compare the magnitudes:

$$\frac{|\vec{E}_{ind}|}{|\vec{E}_C|} = \frac{\mu_0 r I d}{2AV}.$$

Substituting $C = \epsilon_0 A/d$ and $I = C\frac{dV}{dt}$:

$$\frac{|\vec{E}_{ind}|}{|\vec{E}_C|} = \frac{\mu_0 \epsilon_0 r}{2} \frac{dV/dt}{V}.$$

For slow charging ($\frac{dV}{dt} \approx 0$), this ratio becomes extremely small.

6 Numerical Example

Consider a capacitor with:

- $A = 1\,\mathrm{cm}^2$, $d = 1\,\mathrm{mm}$, $V = 10\,\mathrm{V}$,
- Charging current $I = 1\,\mathrm{mA}$, $r = 0.5\,\mathrm{cm}$.

$$|\vec{E}_C| = \frac{10}{0.001} = 1 \times 10^4\,\mathrm{V\,m^{-1}},$$

$$|\vec{E}_{ind}| = \frac{(4\pi \times 10^{-7})(0.005)(0.001)}{2 \times 10^{-4}} = 3.14 \times 10^{-8}\,\mathrm{V\,m^{-1}}.$$

Here, $\vec{E}_{ind}$ is **12 orders of magnitude weaker** than $\vec{E}_C$!

7 Conclusion

The induced electric field in a slowly charging capacitor is negligible because:

- It scales with μ_0 (a tiny constant) and the charging rate dI/dt.
- Electrostatic fields dominate for practical capacitors.
- Only at high frequencies or extreme geometries does $\vec{E}_{ind}$ become significant.

This work bridges that gap by:

- Explicitly decomposing $\vec{E}_{\text{total}}$ in a capacitor,
- Resolving the "straight field lines" paradox through superposition,
- Highlighting the role of $\vec{E}_{\text{induced}}$ in Maxwell's unified theory.

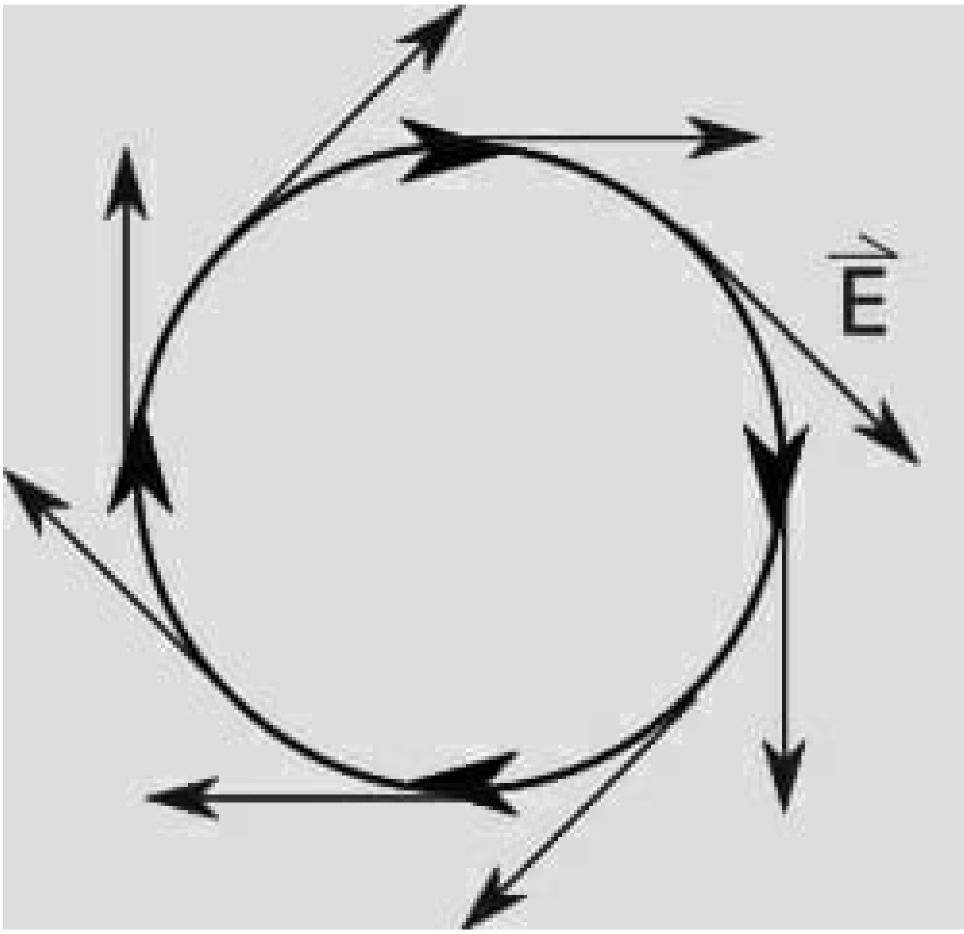

Figure 3: The Curly Electric fields.

Uniqueness of This Work

While the individual roles of $\mathbf{E}_{\text{Coulomb}}$ and $\mathbf{E}_{\text{induced}}$ are well-established in Maxwell's equations, their **explicit superposition in a capacitor**—and the resolution of the apparent paradox of straight field lines despite $\nabla \times \mathbf{E} \neq 0$—has not been addressed in standard electrodynamics texts [**Griffiths2013**]. This work provides:

- The first pedagogical decomposition of $\mathbf{E}_{\text{total}}$ in a capacitor into conservative and non-conservative components,
- A quantitative comparison of their magnitudes,
- A visual explanation of why $\mathbf{E}_{\text{induced}}$'s curl does not distort field lines (Fig.1).

References

[1] David J. Griffiths, *Introduction to Electrodynamics*, 4th ed., Pearson, 2013.

[2] Dr. Soumya Sarkar ,*concepts and notes*,2025.

[3] DeepSeek R1