Conditional probabilities obey the probability axioms

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① $P(A|B) \geq 0$ ← Ordinary probabilities are non-negative.

$\frac{P(A\cap B)}{P(B)}$ will be non-negative assuming $P(B) > 0$, where $P(B)$ is an ordinary probability

Hence conditional probabilities are also non-negative.

② $P(\Omega|B) \leftarrow \frac{P(\Omega \cap B)}{P(B)} \leftarrow \Omega \cap B = B$ since $B \subset \Omega$

So $\frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1.$

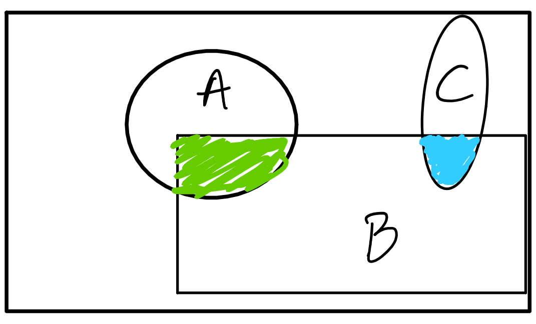
③ $P(B|B) \leftarrow \frac{P(B\cap B)}{P(B)} \leftarrow$ Again $B \cap B = B$

So $\frac{P(B\cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

④ if $A \cap C = \emptyset$ then $P(A\cup C \mid B) = P(A|B) + P(C|B)$

The above statement is asking if the additivity axiom holds if we are in a conditional sample space.

Visually:



Let's look at LHS and RHS individually:

$$\boxed{P(A\cup C \mid B) = \frac{P((A\cup C)\cap B)}{P(B)} = \frac{P((A\cap B)\cup(C\cap B))}{P(B)}}$$

Since A & C are disjoint, $(A\cap B)$ & $(C\cap B)$ are also disjoint. This can also be deduced by the above visual (green & blue shaded areas).

So because $(A\cap B)\cap(C\cap B) = \emptyset$

The above equation becomes:

$$\frac{P(A\cap B) + P(C\cap B)}{P(B)} = \frac{P(A\cap B)}{P(B)} + \frac{P(C\cap B)}{P(B)}$$

$$= P(A|B) + P(C|B)$$

The above can be proven similarly for the case of finitely many and countably many disjoint events.

Hence any formula that we derive for ordinary probabilities using the above axioms will also hold for conditional probabilities.