Set operations & probability examples

Saturday, April 11, 2020 12:16 AM

Find the value of $P(A \cup (B^c \cup C^c)^c)$ for each of the following cases:

① The events A, B, C are disjoint and $P(A) = 2/5$

$$P(A \cup (B^c \cup C^c)^c) = P(A \cup (B \cap C))$$

$$= \underbrace{P(A)}_{2/5 \text{ (given)}} + \underbrace{P(B \cap C)}_{\emptyset \text{ since } B, C \text{ are disjoint.}}$$

Hence $P(A \cup (B^c \cup C^c)^c = 2/5$

② Events A, C are disjoint. $P(A) = 1/2$

$P(B \cap C) = 1/4$

from ① we know we are trying to find

$$P(A) + P(B \cap C)$$

$$= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

③ $P(A^c \cap (B^c \cup C^c)) = 0.7$

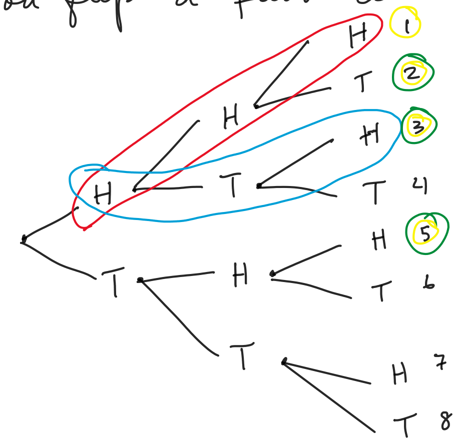
Notice that $P(A^c \cap (B^c \cup C^c))$ is the same as $P(A^c \cap (B^c \cup C^c))^c = P(A \cup (B^c \cup C^c)^c)$ which is what we are looking for.

Hence

$$P(A \cup (B^c \cup C^c)^c) = 0.7$$

3 Tosses of a fair coin:

- You flip a fair coin 3 times



$P(\{HHH\}) = \frac{1}{8}$

$P(\{HTH\}) = \frac{1}{8}$

$P(\{\text{Any sequence w/ 2 Heads, 1 Tail}\}) = 3/8$

$P(\{\text{Any sequence with more heads than tails}\}) = \frac{4}{8} = \frac{1}{2}$

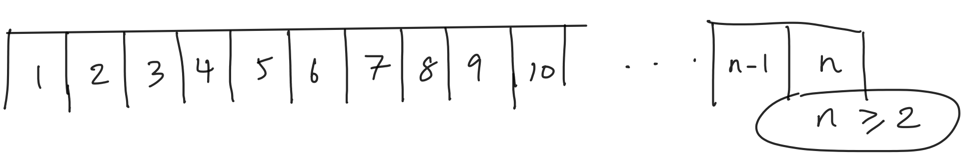
Parking lot problem:

- Mary & Tom park their cars in an empty parking lot. There are $n \geq 2$ consecutive parking spots.
- Only 1 car fits in each space.
- Mary and Tom pick spaces @ random. But they have to choose different spots.
- All pairs of distinct parking spaces are equally likely.

Event E = (At most 1 parking spot between Mary & Tom)

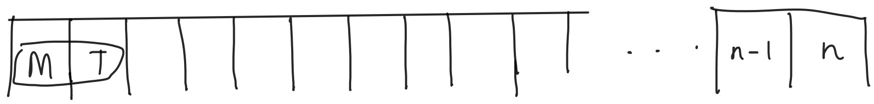
Find $P(E)$

Visual of parking lot:

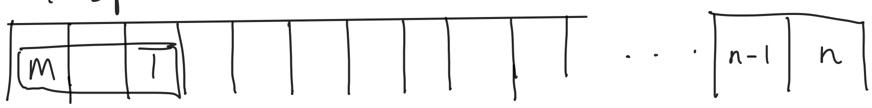


For our event E (At most 1 spot between M & T)
we have 2 scenarios.

① No spot between M & T $\longrightarrow$ $n-1$ different ways



② 1 spot between M & T. $\longrightarrow$ $n-2$ different ways.



Total ways to choose spots $= n(n-1)$.

$$P(E) = P(1) + P(2) = \frac{n-1}{n(n-1)} + \frac{n-2}{n(n-1)} = \frac{2n-3}{n(n-1)}$$

Probabilities on a continuous sample space.

- Alice & Bob choose a real # between 0, 1.
- The pair of #s is chosen according to the uniform probability law on the unit square. So $P(E)$ is equal to its area.
- We define the following events:

A $\longrightarrow$ E : { magnitude of the difference $|x-y| > \frac{1}{3}$ }

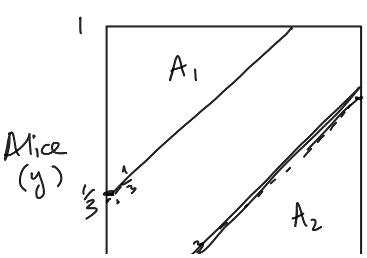
B : { At least one of the #s is $> \frac{1}{4}$ }

C : { $x+y=1$ }

D : { Alice's # $> 1/4$ }.

Find:
① $P(E)$
② $P(B)$
③ $P(A \cap B)$
④ $P(C)$
⑤ $P(D)$
⑥ $P(A \cap D)$

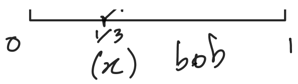
(E)



$|x-y| > \frac{1}{3} \longrightarrow x - \frac{1}{3} = y$.

We are looking for $A_1 + A_2$.

$A_1 = \frac{1}{2} bh \quad b = \frac{2}{3},\ h = \frac{2}{3}$

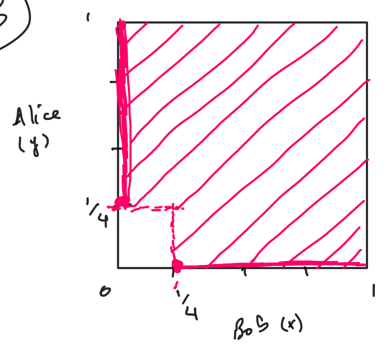


$$A_1 = \frac{1}{2}\,\frac{2}{3}\,\frac{2}{3} = \frac{4}{18} = \frac{2}{9}$$

$$A_2 = A_1 = 2/9$$

$$A = A_1 + A_2 = \frac{4}{9}$$

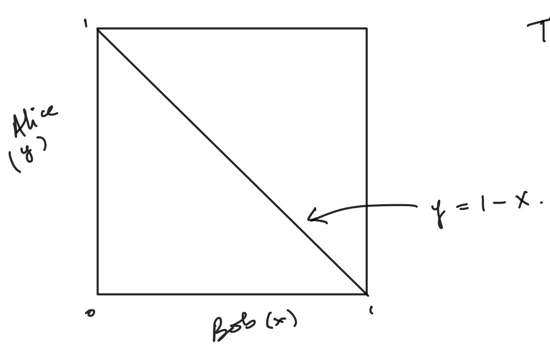
(B)



B = Area of unit square − Area of unselected $\left(\frac{1}{4}\right)^2$ space.

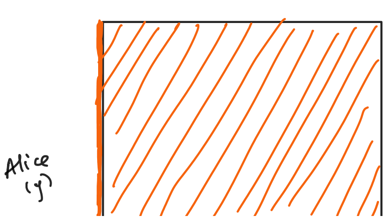
$$= \frac{16}{16} - \frac{1}{16} = \frac{15}{16}$$

(C) $x + y = 1 \longrightarrow y = 1 - x$



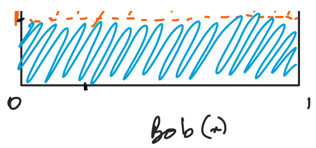
The line $y = 1 - x$ has no area, hence $P\{C\} = \phi$.

(D)



The restriction is only on Alice's #. being $> \frac{1}{4}$ (Orange area).

Blue area $= \frac{1}{4} \times 1 = \frac{1}{4}$.

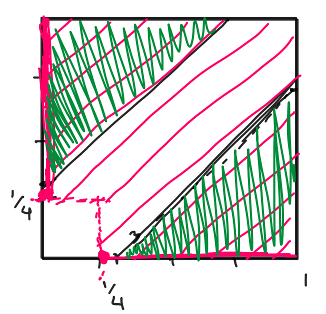


Orange area $= 1 - \frac{1}{4} = \frac{3}{4}$

① $P(A) = \frac{13}{18}$ from above. (same as $P(E)$)

② $P(B) = \frac{15}{16}$ from above.

③ $P(E \cap B) \leftarrow$ for this we can superimpose the graphs of E and B to get the intersection

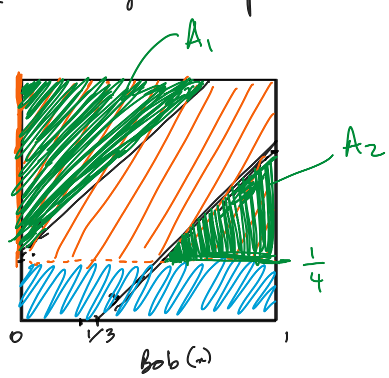


$P(E \cap B) = P(E) = \frac{4}{9}$

④ $P(C) = \emptyset$ from above.

⑤ $P(D) = 3/4$ from above.

⑥ $P(A \cap D) \leftarrow$ superimposing the above graphs:



$A_1 = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{2}{3}$

$= \frac{4}{18} = \frac{2}{9}$

$b_2 = \frac{2}{3} - \frac{1}{4} = \frac{8}{12} - \frac{3}{12}$

$A_2 = \frac{5}{12} \cdot \frac{5}{12} \cdot \frac{1}{2}$

$= \frac{25}{288}$

$A_1 + A_2 = \frac{2}{9} + \frac{25}{288}$

$= 0.3090$

Given 2 events A, B; $P(A) = 3/4$ $P(B) = 1/3$, what is the smallest possible value of $P(A \cap B)$? The largest?

Smallest: $P(A \cap B) = \underbrace{P(A)}_{3/4} + \underbrace{P(B)}_{1/3} - \underbrace{P(A \cup B)}_{\leq 1}$

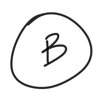
$$= \frac{3}{4} + \frac{1}{3} - 1$$

$$= \frac{9}{12} + \frac{4}{12} - 1$$

$$= \frac{13}{12} - 1 = \frac{1}{12}.$$

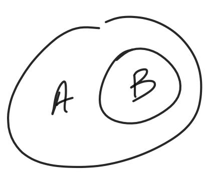
Greatest value can only be the smaller of the 2 probabilities:

$P(A) = \frac{3}{4}$



$P(B) = \frac{1}{3}$

If $B \subset A$, the intersection will have the max probability:



Hence $\max(P(A \cap B))$
$= P(B) = \frac{1}{3}$

and $P(A \cap B) \leq \frac{1}{3}$.