Probability

Friday, March 20, 2020 12:39 PM

Axioms

① $P(A) \geq 0$ ← Probability of event A ≥ 0

② $P(\Omega) = 1$ ← Probability of sample space = 1

③ For disjoint sets:

$$P(A \cup B) = P(A) + P(B)$$

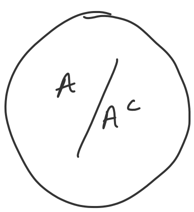
Consequences of the Axioms:

$$P(A) \leq 1$$

$$P(\phi) = 0$$

$$P(A) + P(A^c) = 1$$

$P(A \cup B \cup C) = P(A) + P(B) + P(C)$ & similarly for k disjoint sets.



$A \cup A^c = \Omega$ ← The set A & its compliment make up the whole sample space.

$A \cap A^c = \phi$ ← If an element is in A, it is not in A^c. So the intersection of A & A^c is the empty set.

$$1 \overset{\text{Axiom 2}}{=} P(\Omega) = P(A \cup A^c) \overset{\text{Axiom 3}}{=} P(A) + P(A^c)$$

$$1 = P(A) + P(A^c)$$

$$\boxed{P(A) = 1 - P(A^c) \leq 1}$$ ← Same thing as Axiom 1.

$$1 = \underbrace{P(\Omega)}_{1} + \underbrace{P(\Omega^c)}_{P(\emptyset)}$$ ← The probability of a set & it's complement is 1.

The probability of the whole set is 1

The compliment of the whole set is the empty set.

From this follows:

$$1 = 1 + P(\emptyset)$$

$$\boxed{P(\emptyset) = 0}$$ ← The probability of the empty set is 0.

$$P(A \cup B \cup C) = P(\underbrace{(A \cup B)}_{1} \cup \underbrace{C}_{2}) = P(A \cup B) + P(C)$$

Applying Axiom 3

Here we think of $(A \cup B)$ & C as 2 disjoint sets.

$$P(A \cup B) + P(C) = P(A) + P(B) + P(C)$$

↑ Applying axiom 3 a second time.

Hence $P(A \cup B \cup C) = P(A) + P(B) + P(C)$ for disjoint sets.

Axiom 3 can be applied as many times as needed. In the general case:

If $A_1, A_2, \ldots, A_k$ are disjoint then:

$$\boxed{P(A_1 \cup A_2 \cup \ldots \cup A_k) = \sum_{i=1}^{i=k} P(A_i)}$$

$$\underbrace{P(\{s_1, s_2, \ldots, s_k\})}_{\text{If we have a finite set of } k \text{ elements}} = P(\{s_1\} \cup \{s_2\} \cup \ldots \cup \{s_k\})$$

We can break it up as a probability of a union of single element sets.

$= P(\{s_1\}) + P(\{s_2\}) + \ldots + P(\{s_k\})$ axiom 3

$= \underbrace{P(s_1)} + \underbrace{P(s_2)} + \ldots + \underbrace{P(s_k)}$ By simplifying notation

Probability of events $s_1, s_2 \ldots$ since the sets are single element sets.

Problem 1 :

$P(A) + P(A^c) + P(B) = P(A \cup A^c \cup B)$ A, B, C are disjoint in the same sample space.

FALSE

Let $A = \emptyset$, then $A^c = \Omega$

Let $B = \Omega$

Then $LHS = P(\emptyset) + P(\Omega) + P(\Omega) = 0 + 1 + 1 = 2$

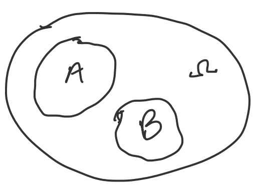
And $RHS = P(\emptyset \cup \Omega \cup \Omega) = P(\Omega) = 1$

$LHS \neq RHS$

Problem 2 :

$P(A) + P(B) \leq 1$ A, B, C are disjoint sets in same sample space.

sample space :



① $A \in \Omega$ & $B \in \Omega$

$P(A) \leq 1$ & $P(B) \leq 1$

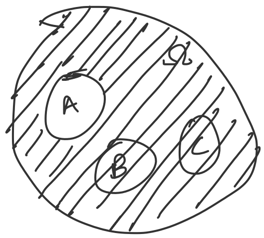
② $A \cap B = \emptyset$ since they are disjoint

① & ② taken together show that A & B are disjoint portions of Ω & hence cannot exceed it.

Problem 3 :

$P(A^c) + P(B) \leq 1$ A, B, C are disjoint sets in the same sample space.

Since A, B, C are disjoint in the sample space



Shaded region is A^c. This type of example can be described algebraically:

Let $A = \phi$, then $A^c = \Omega$

Let $B = \Omega$

Let $C = \phi$

$LHS = P(A^c) + P(B) = P(\Omega) + P(\Omega) = 2$

$RHS = 1$

$2 \neq 1$

$LHS \neq RHS$.

Problem 4:

$P(A \cup B \cup C) \geq P(A \cup B)$ $\quad$ A, B, C are disjoint in same sample space.

$LHS \geq RHS$

This is true simply because $RHS = P(A) + P(B)$

$LHS = P(A) + P(B) + P(C)$

& since A, B, C are disjoint:

$LHS \geq RHS$

$P(A) + P(B) + P(C) \geq P(A) + P(B)$

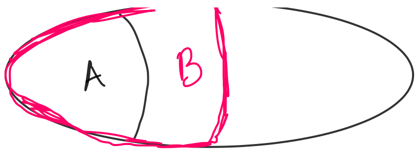
More consequences of the 3 Axioms:

If $A \subset B$, then $P(A) \leq P(B)$

Visually:

Mathematically:

$B = A \cup (B \text{ but not } A)$



$$B = \underbrace{A \cup (B \cap A^c)}$$

Disjoint, so we can apply the additivity axiom.

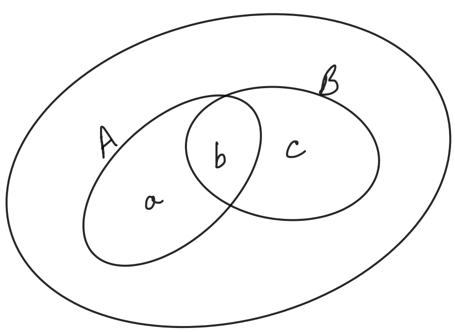
$$P(B) = P(A) + P(B \cap A^c)$$

The RHS $\longrightarrow$ $P(A) + P(B \cap A^c) \geq P(A)$ since probabilities are nonnegative.

Hence $P(B) \geq P(A)$

$P(A) \leq P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ where A, B are not necessarily disjoint.



$a = P(A) - P(A \cap B)$

$b = P(A \cap B)$

$c = P(B) - P(A \cap B)$

$$P(A \cup B) = a + b + c$$

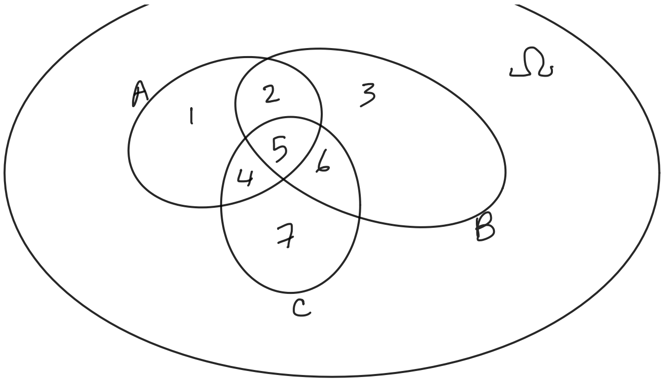
$$= P(A) - \cancel{P(A \cap B)} + \cancel{P(A \cap B)} + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A \cap B)$$

From this it also follows that

$P(A \cup B) \leq P(A) + P(B) \longleftarrow$ Union Bound.

$$P(A \cup B \cup C) = \underset{\substack{\uparrow \\ 1,2,4,5}}{P(A)} + \underset{\substack{\uparrow \\ 3,6}}{P(A^c \cap B)} + \underset{\substack{\uparrow \\ 7}}{P(A^c \cap B^c \cap C)}$$



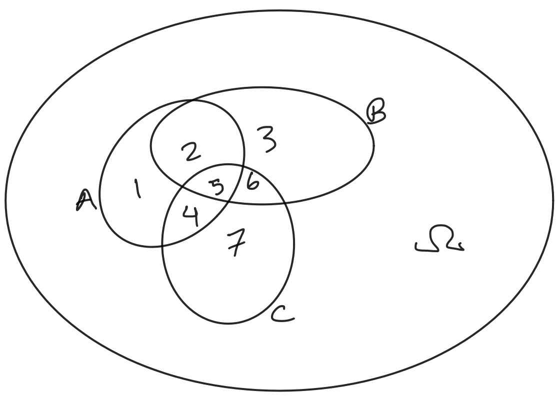
Together they cover

1, 2, 3, 4, 5, 6, 7.

Problem 1:

A, B, C are subsets of same sample space.
Not necessarily disjoint

$P((A \cap B) \cup (C \cap A^c)) \leq P(A \cup B \cup C)$



$A \cap B = 2, 5$

$C \cap A^c \to 6, 7$

TRUE

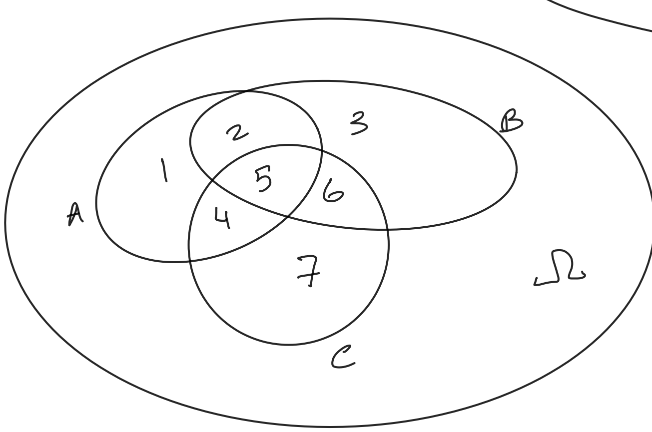
Problem 2:

$P(A \cup B \cup C) = P(A \cap C^c) + P(C) + P(B \cap A^c \cap C^c)$

1, 2,

4, 5, 6, 7

3



TRUE