CECS 228 Name:  
LAB #1.5 ID: Date:

Objectives:

* Be able to apply Direct Proof
* Be able to apply Proof by contradiction
* Be able to apply Proof by contrapositive

Exercise 1: Use direct proof to prove the following:  
a If n is an odd integer, then n2 is an odd integer.

(Note: the definition of an odd integer is an integer that can be expressed as 2k + 1, where k is an integer.)  
Proof.

Direct proof. Assume that n is an odd integer. We will show that n2 is also an odd integer.

Since n is odd, n = 2k + 1, for some integer k. Plug the expression for n into n2:

n2 = (2k+1)2 = 4k2+4k+1 = 2(2k2+2k)+1

Since k is an integer, then (2k2 + 2k) is also an integer. Since n2 = 2c + 1, where c = 2k2 + 2k is an integer, then n2 is odd. ■

b. The product of two odd integers is an odd integer.  
Proof.

Direct proof. Assume that x and y are odd integers. We will show that xy is also an odd integer.

Since x is odd, x = 2k+1 for some integer k. Since y is odd, y = 2j+1 for some integer j. The product of x and y is:

xy = (2k+1)(2j+1) = 4kj+2j+2k+1 = 2(2kj+j+k) + 1.

Since j and k are both integers, (2kj + j + k) is also an integer. Since xy can be expressed as 2 times an integer plus 1, xy is an odd integer. ■

c. If r and s are rational numbers, then the product of r and s is a rational number.  
Proof.

Direct proof. Assume that r = a/b and s = c/d, where c ≠ 0 and d ≠ 0. We will show that rs is a rational number.

Plugging in the values r = a/b and s = c/d to rs:

Since b ≠ 0 and d ≠ 0, then bd ≠ 0. Also since a, b, c, and d are all integers, then ac and bd are also integers. Therefore rs is equal to the ratio of two integers in which the denominator is not zero which implies that rs is rational. ■

Exercise 2: Use proof by contrapositive to prove the following:  
a. For every integer n, if n2 is an odd, then n is odd.  
Proof by contrapositive. We assume that n is an even integer and show that n2 is an even integer.

If n is an even integer, then n = 2k for some integer k. Plugging in the expression 2k for n in n2 gives that

n2 = (2k)2 = 4k2 = 2(2k2)

Since k is an integer, 2k2 is also an integer. n2 can be expressed as 2 times an integer. Therefore n2 is even. ■

b. If x and y are real numbers and x + y is irrational, then x is irrational or y is irrational.

Proof.

Proof by contrapositive. We assume for real numbers x and y that it is not true that x is irrational or y is irrational and we prove that x + y is rational. If it is not true that x is irrational or y is irrational then neither x nor y is irrational. Any real number that is not irrational must be rational. Since x and y are both real numbers then x and y are both rational. We can therefore express x as a/b and y as c/d, where a,b,c, and d are integers and b and d are both not equal to 0. The sum of x and y is:

Since a, b, c, and d are integers, the numerator (ad+bc) and the denominator (bd) are integers. Furthermore since b and d are both non-zero, bd is also non-zero. Therefore x + y is a rational number. ■

c. If x and y are positive real numbers and xy > 400, then x > 20 or y > 20.  
Proof.

Proof by contrapositive. We assume that for real numbers x and y that it is not the case that x > 10 or y > 10 and prove that x + y ≤ 20.

Since it is not true that x > 10 or y > 10, by De Morgan's law, the inequalities x > 10 and y > 10 are both false. Therefore, x ≤ 10 and y ≤ 10. Adding x ≤ 10 and y ≤ 10 gives that

x+y≤10+10=20.

x+y≤10+10=20.

Therefore x + y ≤ 20. ■

Exercise 3: Use proof by contradiction to prove the following:  
a. If a group of 9 kids have won a total of 100 trophies, then at least one of the 9 kids has won at least 12 trophies.

Proof.

Proof by contradiction. Suppose that each kid has less than 12 trophies and the total number of trophies owned by all the kids is 100. If each kid has less than 12 trophies, the most trophies any kid can have is 11. If there are 9 kids, then the total number of trophies is at most 9·11 = 99, which contradicts the fact that the total number of trophies is 100. ■

b. is irrational. You can use the following fact in your proof: “If n is an integer and n3 is even, then n is even.”

Proof.

Proof by contradiction. Suppose that is rational. Therefore, can be expressed as the ratio of two integers n/d, where d ≠ 0. Any ratio can be simplified into lowest terms. Therefore, we can assume that there is no integer greater than 1 that evenly divides both n and d.

Cubing both sides of the equation = n/d gives 2 = n3/d3. Then multiplying both sides of the equation by d3 gives that 2d3 = n3.

The fact that n3 is a multiple of 2 means that n3 is even, and therefore n is even. Since n is even, n = 2k for some integer k. Plugging in the expression 2k for n into n3:

n3 = (2k)3 = 8k3.

Since 2d3 = n3 = 8k3, dividing by 2 gives that d3 = 4k3 = 2(2k3). Since d3 is a multiple of 2, d3 is even and therefore d is also even.

We have established that n and d are even and therefore both multiples of 2, which contradicts the assumption that no integer greater than 1 can evenly divide both n and d. Thus, we have established a contradiction and we must conclude that the assumption that is a rational number is a false assumption. ■

c. There is no smallest integer.  
Proof by contradiction.   
Suppose that there is a smallest integer x. If x is an integer, then x - 1 is also an integer. Furthermore, x - 1 < x. Therefore, x - 1 is an integer that is less than x, which contradicts the assumption that x is the smallest integer. Therefore, the assumption that there exists a smallest integer is false. ■