CECS 228 Name:

Lab 8.2 ID: Date:  
Objective:

* Be able to use mathematical induction
* Be able to solve recurrence relation

Exercise 1: Let P(n) be the statement that for the positive integer n.

a) What is the statement P(1)?

Plugging in n = 1 we have that P(1) is the statement 13 = [1 · (1 + 1)/2]2 .

b) Show that P(1) is true, completing the basis step of the proof.

Both sides of P(1) shown in part (a) equal 1.

c) What is the inductive hypothesis?  
The inductive hypothesis is the statement that

d) What do you need to prove in the inductive step?

For the inductive step, we want to show for each k ≥ 1 that P(k) implies P(k + 1). In other words, we

want to show that assuming the inductive hypothesis (see part (c)) we can prove

e) Complete the inductive step, identifying where you use the inductive hypothesis.

Replacing the quantity in brackets on the left-hand side of part (d) by what it equals by virtue of the

inductive hypothesis, we have

f ) Explain why these steps show that this formula is true whenever n is a positive integer.

We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n.

Exercise 2: Use mathematical induction to prove the inequality  
for all positive integers n ≥ 2.  
  
a) Plugging in n = 2, we see that P(2) is the statement 2! < 22 .  
b) Since 2! = 2, this is the true statement 2 < 4.  
c) The inductive hypothesis is the statement that k! < kk .  
d) For the inductive step, we want to show for each k ≥ 2 that P(k) implies P(k + 1). In other words, we want to show that assuming the inductive hypothesis (see part (c)) we can prove that (k + 1)! < (k + 1)k+1 .  
e) (k + 1)! = (k + 1)k! < (k + 1)kk < (k + 1)(k + 1)k = (k + 1)k+1  
f) We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n greater than 1.

Exercise 3: Prove that 1 · 1! + 2 · 2!+· · ·+n · n! = (n + 1)! – 1 whenever n is a positive integer.  
The basis step is clear, since 1 · 1! = 2! − 1. Assuming the inductive hypothesis, we then have   
1 · 1! + 2 · 2! + · · · + k · k! + (k + 1) · (k + 1)! = (k + 1)! − 1 + (k + 1) · (k + 1)!  
 = (k + 1)!(1 + k + 1) − 1 = (k + 2)! − 1, as desired.

Exercise 4: Prove that 6 divides n3 − n whenever n is a nonnegative integer.  
  
The statement is true for the base case, n = 0, since 6 | 0. Suppose that 6 | (n3 − n). We must show that 6 |[(n + 1)3 − (n + 1)]. If we expand the expression in question, we obtain n3 + 3n2 + 3n + 1 − n − 1 = (n3 − n) + 3n(n + 1). By the inductive hypothesis, 6 divides the first term, n3 − n. Furthermore clearly 3 divides the second term, and the second term is also even, since one of n and n + 1 is even; therefore 6 divides the second term as well. This tells us that 6 divides the given expression, as desired. (Note that here we have, as promised, used n as the dummy variable in the inductive step, rather than k .)  
  
Exercise 5:  
For each problem, we first write down the characteristic equation and find its roots. Using this we write down the general solution. We then plug in the initial conditions to obtain a system of linear equations. We solve these equations to determine the arbitrary constants in the general solution, and finally we write down the unique answer.  
  
a) an = an-1 + 6an-2 for n ≥ 2, a0 = 3, a1 = 6

r2 − r − 6 = 0 r = −2, 3

an = α1(−2)n + α23n

3 = α1 + α2

6 = −2 α1 + 3 α2

α1= 3/5 α2 = 12/5

an = (3/5)(−2)n + (12/5)3n

b) an = 7an-1 − 10an-2 for n ≥ 2, a0 = 2, a1 = 1

r2 − 7r + 10 = 0 r = 2, 5

an = α12n + α25n

2 = α1 + α2

1 = 2α1 + 5α2

α1= 3 α2 = -1

an = (3)2n - 5n

c) an= 2an−1 – an-2 for n ≥ 2, a0 = 4, a1 = 1

r2 − 2r + 1 = 0 r = 1, 1

an = α11n + α2 n1n = α1 + α2 n

4 = α1

1 = α1 + α2

α1= 4 α2 = -3

an = 4 – 3n

Exercise 5: Find the solution to an = 5an-2 − 4an-4 with a0 = 3, a1 = 2, a2 = 6, and a3 = 8.  
The characteristic equation is r4 − 5r2 + 4 = 0.   
This factors as (r2 − 1)(r2 − 4) = (r − 1)(r + 1)(r − 2)(r + 2) = 0, so the roots are 1, −1, 2, and −2. Therefore the general solution is an = α1 + α2 (−1)n + α32n + α4 (−2)n .  
Plugging in initial conditions gives   
3 = α1 + α2 + α3 + α4,   
2 = α1 − α2 + 2 α3 − 2 α4,   
6 = α1 + α2 + 4 α3 + 4 α4 ,  
8 = α1 − α2 + 8 α3 – 8 α4.   
The solution to this system of equations is α1 = α2 = α3= 1 and α4= 0.

Therefore the answer is an = 1 + (−1)n + 2n .