Utility and Risk Averse

An inverstor is said to be **risk averse** if $u(w) \geq \mathbb{E}[u(\widetilde{w})]$ for any random $\widetilde{w}$ with mean w
A risk averse investor would prefer to avoid a fair bet, meaning that if $\widetilde{\varepsilon}$ is a zero-mean rv and w is a constant, then $u(w) \geq \mathbb{E}[u(w+\widetilde{\varepsilon})]$
A constant π is said to be the **Risk Premium** of $\widetilde{w}$ if $w-\pi$ is the **Certainty Equivalent**, that is $u(w-\pi) = \mathbb{E}[u(\widetilde{w})]$
For small gambles, $\pi \approx \frac{1}{2}\sigma^2\alpha(w)$ where σ^2 is the variance of $\widetilde{\varepsilon}$

$$u(w-\pi) = \mathbb{E}[u(w+\widetilde{\varepsilon})]$$

$$u(w) - u'(w)\pi + \underbrace{1/2u''(w)\pi^2}_{\text{too small}} \approx \mathbb{E}[u(w) + \underbrace{u'(w)\widetilde{\varepsilon}}_{\text{mean 0}} + 1/2u''(w)\widetilde{\varepsilon}^2]$$

$$-u'(w)\pi \approx 1/2u''(w)\sigma(\widetilde{\varepsilon})^2$$

$$\pi \approx [-u''(w)/2u'(w)]\sigma(\widetilde{\varepsilon})^2$$

The **coefficient of absolute risk aversion** at a wealth level w is defined as

$$\alpha(w) = -u''(w)/u'(w)$$

the 2nd derivative measures its concavity, the 1st derivative is used to eliminates the dependence on the arbitrary scaling of the utility – that is, the coefficient of absolute risk aversion is unaffected by a monotone affine transform of the utility function
$\alpha(w) \geq 0$ for any risk-averse investors, because concavity imply $u'' \leq 0$, and $u'0$ with basic logic. A high α indicates a high curvature of the utility function, which imply a high aversion of risk
The **coefficient of relative risk aversion** is defined as

$$\rho(w) = w\alpha(w) = -wu''(w)/u'(w)$$

This means how much fraction of wealth would an investor to pay to avoid a gamble
The **coefficient of risk tolerance** is defined as

$$\tau(w) = 1/\alpha(w) = -u''(w)/u'(w)$$

Stochastic Discount Factor

we define SDF as

$$\widetilde{m} = \frac{\delta u'(\widetilde{c}_1)}{u'(c_0)} \qquad p_i = \mathbb{E}[\widetilde{m}\widetilde{x}_i]$$

Property:
1. From the definition of SDF, we have $\mathbb{E}\left[\widetilde{m}\frac{\widetilde{x}_i}{p_i}\right] = \mathbb{E}[\widetilde{m}\widetilde{R}_i] = 1$
2. From the mean 1 property, for any two asset i, j, we have $\mathbb{E}[\widetilde{m}\widetilde{R}_i] - \mathbb{E}[\widetilde{m}\widetilde{R}_j] = \mathbb{E}[\widetilde{m}(\widetilde{R}_i - \widetilde{R}_j)] = 1-1 = 0$, and if $\widetilde{R}_j = R_f$ is a risk free asset, we have $\mathbb{E}[\widetilde{m}(\widetilde{R}_i - R_f)] = 0$
3. If a portfolio consists of θ_i shares of asset i. A portfolio $\widetilde{x} = \sum_{i=1}^n \theta_i\widetilde{x}_i$ and the price of portfolio is $p = \sum_{i=1}^n \theta_i p_i$, then we have $\mathbb{E}[\widetilde{m}\widetilde{x}] = p, \mathbb{E}[\widetilde{m}\widetilde{R}] = 1$, where $\widetilde{R} = \widetilde{x}/p$ is the return of the portfolio specially, if $\widetilde{R} = R_f$ is risk free asset, we have $\mathbb{E}[\widetilde{m}] = \frac{1}{R_f}$

$$\mathrm{Cov}[\widetilde{m}, \widetilde{R}] = \mathbb{E}[\widetilde{m}\widetilde{R}] - \mathbb{E}[\widetilde{m}]\mathbb{E}[\widetilde{R}] = 1 - \mathbb{E}[\widetilde{m}]\mathbb{E}[\widetilde{R}]$$

$$\Longrightarrow \ 1 = \mathrm{Cov}[\widetilde{m}, \widetilde{R}] - \mathbb{E}[\widetilde{m}]\mathbb{E}[\widetilde{R}]$$

since $\mathbb{E}[\widetilde{m}] = \frac{1}{R_f}$, then $\mathbb{E}[\widetilde{R}] - R_f = -R_f\mathrm{Cov}[\widetilde{m}, \widetilde{R}]$, this shows the Risk Premia is determined by the covariance with any SDF

$$p = \mathbb{E}[\widetilde{m}\widetilde{x}] = \mathbb{E}[\widetilde{m}]\mathbb{E}[\widetilde{x}] + \mathrm{Cov}[\widetilde{m}, \widetilde{x}] = \frac{\mathbb{E}[\widetilde{x}]}{R_f} + \mathrm{Cov}[\widetilde{m}, \widetilde{x}]$$

since $\widetilde{R} = \widetilde{x}/p$, then

$$\frac{\mathbb{E}[\widetilde{x}]}{p} - R_f = -R_f\mathrm{Cov}[\widetilde{m}, \widetilde{R}] \implies p = \frac{\mathbb{E}[\widetilde{x}]}{R_f - R_f\mathrm{Cov}[\widetilde{m}, \widetilde{R}]}$$

Equity Premium Puzzle

The Equity Premium Puzzle refers to the empirical observation that historically, stocks have offered much higher returns than government bonds.
Definition of Equity Premium: The equity premium refers to the difference between the average return on stocks and the return on risk-free assets. Historically, in the U.S. market, the average annual return on stocks is around 7%, while the risk-free rate is about 1-2%, resulting in an equity premium of approximately 5-6%.
Authors found that to explain the observed 5-6% equity premium, the model would require an implausibly high(30-40) risk-aversion coefficient. This implies that investors are extremely risk-averse, far beyond what is considered reasonable. With a more realistic risk-aversion coefficient (e.g., 2-4), the model would predict a much lower equity premium, significantly below the historical 5-6% observed.If investors were truly that risk-averse, they would avoid holding high-risk stocks, leading to low demand for equities. However, in reality, a significant number of investors do hold stocks, suggesting a more moderate level of risk aversion.

CARA and CRRA

Constant absolute risk aversion, CARA utility functions having constant absolute risk aversion; it means that absolute risk aversion is the same at every wealth level. For Linear Risk Tolerance $\tau(w) = A = \frac{1}{\alpha}$, every CARA utility function is a monotone affine transform of the utility function$u(w) = -e^{-\alpha w}$ where α is a constant and equal to the absolute risk aversion. This is called **negative exponential utility**
for **Risk Premium**, we have

$$u(w-\pi) = -e^{-\alpha w}e^{\alpha\pi}$$

$$u(w+\widetilde{\varepsilon}) = -e^{-\alpha w}e^{-\alpha\widetilde{\varepsilon}}$$

$$u(w-\pi) = \mathbb{E}[u(w+\widetilde{\varepsilon})] \Leftrightarrow e^{\alpha\pi} = \mathbb{E}[e^{-\alpha\widetilde{\varepsilon}}]$$

$$\pi = 1/\alpha \ln\mathbb{E}[e^{-\alpha\widetilde{\varepsilon}}]$$

so π is function independent of w. CARA utility is characterized by the absence of wealth. Thus, an individual with CARA utility will pay the same to avoid a fair gamble no matter what her initial wealth might be.
if $\widetilde{\varepsilon}$ is normal distri with mean 0 and variance σ, then

$$\mathbb{E}[e^{-\alpha\widetilde{\varepsilon}}] = e^{\frac{1}{2}\alpha^2\sigma^2}$$

and

$$\pi = 1/2\alpha\sigma^2$$

Constant Relative Risk Aversion, CRRA utility functions having constant relative risk aversion. It means the relative risk aversion is the same at all wealth levels. For Linear Risk Tolerance, $\tau(w) = Bw = \frac{w}{\rho}$
Any monotone CRRA utility function is a monotone affine transform of one of the following functions
1. $u(w) = \ln w$
2. $u(w) = \frac{1}{1-\rho}(w^{1-\rho} - 1), \rho > 0$, where ρ is relative risk aversion

$$\begin{aligned} \frac{d\ln(u'(w))}{dw} &= -\frac{\rho}{w}, \\ \ln(u'(w)) &= -\rho\ln(w) + C_1, \\ u'(w) &= \quad w^{-\rho}\exp(C_1), \\ u(w) &= \quad \frac{w^{1-\rho}}{1-\rho}\exp(C_1) + C_2; \end{aligned}$$

when $\rho = 1$, it's a special case

$$u(w) = \ln(w)$$

For Risk Premium, suppose investor would pay πw to avoid the gamle $\widetilde{\varepsilon}w$ if

$$u(w-\pi w) = \mathbb{E}[u(w+w\widetilde{\varepsilon})]$$

$$u(w-\pi w) = \ln(w-\pi w) = \ln w\ln(1-\pi)$$

$$\mathbb{E}[u(w+w\widetilde{\varepsilon})] = \mathbb{E}[\ln(w(1+\widetilde{\varepsilon}))] = \ln w\ \mathbb{E}[\ln(1+\widetilde{\varepsilon})]$$

$$\ln(1-\pi) = \mathbb{E}[\ln(1+\widetilde{\varepsilon})] \implies \pi = 1 - e^{\mathbb{E}[\ln(1+\widetilde{\varepsilon})]}$$

π is independent of w

$$u(w-\pi w) = \frac{1}{1-\rho}(w-\pi w)^{1-\rho} = \frac{1}{1-\rho}w^{1-\rho}(1-\pi)^{1-\rho}$$

$$\mathbb{E}[u(w+w\widetilde{\varepsilon})] = \mathbb{E}\left[\frac{1}{1-\rho}(w+w\widetilde{\varepsilon})^{1-\rho}\right] = \frac{1}{1-\rho}w^{1-\rho}\mathbb{E}[(1+\widetilde{\varepsilon})^{1-\rho}]$$

$$(1-\pi)^{1-\rho} = \mathbb{E}[(1+\widetilde{\varepsilon}^{1-\rho})]$$

then π is independent of w for CRRA utility
suppose $u(c) = \frac{c^{1-\rho}-1}{1-\rho}$, and the utility function is time additive, then
SDF

$$\widetilde{m} = \frac{\delta u'(\widetilde{c}_1)}{u'(c_0)} = \delta\left(\frac{\widetilde{c}_1}{c_0}\right)^{-\rho} = \delta\exp\{-\rho[\ln(\widetilde{c}_1) - \ln c_0]\}$$

denote $\widetilde{\Delta} = \ln\widetilde{c}_1 - \ln c_0$ as the consumption growth rate
and the **risk premia** $\mathbb{E}[\widetilde{R}_i] - R_f = -R_f\mathrm{Cov}[\widetilde{m}, \widetilde{R}_i]$
If $\widetilde{c}_1$ up, then $\widetilde{R}_i, \widetilde{\Delta}$ up, then $\widetilde{m}$ down, then risk premia down. That is to say, if one stock has same trend with the market, the its risk premia should be low
Shifted Logarithmic is a CRRA utility, it is: $u(w) = \ln(w-\zeta)$
Shifted Power is a CRRA utility which is $u(w) = \frac{\rho}{1-\rho}\left(\frac{w-\zeta}{\rho}\right)^{1-\rho}$, the risk tolerance is $\tau(w) = \frac{w-\zeta}{\rho}$
Quadratic Utility is a special case of Shifted Power Utility with $\rho = -1$, in this case

$$u(w) = -\frac{1}{2}(w-\zeta)^2 = -\frac{1}{2}\zeta^2 + \zeta w - \frac{1}{2}w^2$$

it implies mean-variance preferences
the expected utility(ignoring the constant term $-\zeta^2/2$) is

$$\mathbb{E}[u(\widetilde{w})] = \zeta\mathbb{E}[\widetilde{w}] - \frac{1}{2}\mathbb{E}[\widetilde{w}^2] = \zeta\mathbb{E}[\widetilde{w}] - \frac{1}{2}\mathbb{E}[\widetilde{w}]^2 - \frac{1}{2}\mathrm{Var}[\widetilde{w}]$$

Thus, preferences over gambles depend only on their means and variances when an investor has quadratic utility

Hansen and Jagannathan Bounds

since

$$\mathbb{E}[\widetilde{R}_i] - R_f = -R_f\mathrm{Cov}[\widetilde{m}, \widetilde{R}_i - R_f]$$

thus

$$\mathbb{E}[\widetilde{R}_i] - R_f = -\frac{1}{\mathbb{E}[\widetilde{m}]}\rho_{\widetilde{m}, \widetilde{R}_i - R_f} \cdot \sigma(\widetilde{m}) \cdot \sigma(\widetilde{R}_i - R_f)$$

$$\frac{\mathbb{E}[\widetilde{R}_i] - R_f}{\sigma(\widetilde{R}_i - R_f)} = -\frac{\sigma(\widetilde{m})}{\mathbb{E}[\widetilde{m}]}\rho_{\widetilde{m}, \widetilde{R}_i - R_f}$$

$$\left|\frac{\mathbb{E}[\widetilde{R}_i] - R_f}{\sigma(\widetilde{R}_i - R_f)}\right| = \left|\frac{\sigma(\widetilde{m})}{\mathbb{E}[\widetilde{m}]}\right| \left|\rho_{\widetilde{m}, \widetilde{R}_i - R_f}\right| \leq \left|\frac{\sigma(\widetilde{m})}{\mathbb{E}[\widetilde{m}]}\right|$$

which is

$$|\text{Sharpe Ratio}_i| \leq \left|\frac{\sigma(\widetilde{m})}{\mathbb{E}[\widetilde{m}]}\right|$$

Suppose $\widetilde{\Delta} \sim N(\mu, \nu)$. since $\widetilde{m} = \delta\exp\{-\rho\widetilde{\Delta}\}$, $\mathbb{E}[\widetilde{m}] = \delta\exp\{-\rho\mu + 0.5\rho^2\nu\}$, $\mathrm{Var}[\widetilde{m}] = \delta^2\exp\{-2\rho\mu + \rho^2\nu\}[\exp\{\rho^2\nu\} - 1]$, thus

$$\frac{\sigma(\widetilde{m})}{\mathbb{E}[\widetilde{m}]} = \sqrt{\exp(\rho^2\nu) - 1} \approx \rho\sqrt{\nu}$$

the Hansen and Jagannathan Bounds is

$$|\text{Sharpe Ratio}| \leq \rho\sqrt{\nu}$$

Kyle Model

Suppose a risky asset has a payoff $\widetilde{\theta} \sim N(0, \nu_\theta)$, and there are two types of investors: **A risk-neutral informed investor** observes $\widetilde{\theta}$ and admits a market order $\widetilde{x}$, **Uninformed liquidity traders** submit a market order $\widetilde{z} \sim N(0, \nu_z)$
A market maker observes the total order $\widetilde{y} = \widetilde{x} + \widetilde{z}$ and set the price

$$p = \mathbb{E}[\widetilde{\theta}|\widetilde{y}]$$

The uninformed will not think whether the price is too high or low, but for informed investor and market maker, they will play a game to achieve a equilibrium in price . While in Grossman-Stiglitz Model, all investors are price-takers
Here are still **assumptions/conjectures** about demand and price
1. The informed investor chooses to submit the market order (high payoff admits high demand)

$$\widetilde{x} = \beta\widetilde{\theta}$$

2. The market maker sets the price (high demand backs high price)

$$p = \lambda\widetilde{y}$$

3. β, λ are constant parameters to be determined
Given the price $p = \lambda\widetilde{y}$, the informed investor chooses the demand x to maximize his wealth:

$$\mathbb{E}[x(\widetilde{\theta} - p)|\widetilde{\theta}] = \mathbb{E}[x(\widetilde{\theta} - \lambda(x+\widetilde{z}))|\widetilde{\theta}] = x(\widetilde{\theta} - \lambda x)$$

The f.o.c. w.r.t. x imply $x = \frac{1}{2\lambda}\widetilde{\theta}$
This verify the demand, particularly $\beta = \frac{1}{2\lambda}$
Given the demand $x = \frac{1}{2\lambda}\widetilde{\theta}$, the market maker chooses the price as

$$p = \mathbb{E}\left[\widetilde{\theta}|\widetilde{y} = \frac{1}{2\lambda}\widetilde{\theta} + \widetilde{z}\right] = \frac{\mathrm{Cov}[\widetilde{\theta}, \widetilde{y}]}{\nu_y}\widetilde{y} \qquad = \frac{\frac{1}{2\lambda}\nu_\theta}{\frac{1}{(2\lambda)^2}\nu_\theta + \nu_z}\widetilde{y}$$

the 2nd equation is veried in Normal Distribution
since $p = \lambda\widetilde{y}$, if suffices that

$$\frac{\frac{1}{2\lambda}\nu_\theta}{\frac{1}{(2\lambda)^2}\nu_\theta + \nu_z} = \lambda$$

hence

$$\lambda = \frac{1}{2}\sqrt{\frac{\nu_\theta}{\nu_z}}$$

Low λ indicates high liquidity (i.e. a big buy order $\widetilde{y} > 0$ won't cause a big price increase), possible reasons:
ν_θ is low: informed investor has a smaller scale of information advantage
ν_z is high: there is a large scale of uninformed liquidity trades

Efficient Market Hypothesis

Strong form: Stock prices reflect all information, public and private.
Semi-strong form: Stock prices reflect all public, including past accounting information and stock (price and trading) information.
Weak form: Stock prices reflect past stock (price and trading) information.

m-v Analysis

Suppose there are n risky assets, $\widetilde{R}_i$ is the return of asset i, μ is the col vector of $\mathbb{E}[\widetilde{R}_i]$, Σ is the covariance matrix, ι is an n-dimensional col vector of 1's. The portfolio vector is π. If the portfolio is fully invested in risky assets, then components of π sum to 1, i.e. $\iota^{\mathrm{T}}\pi = 1$. The average return of the portfolio is $\pi^{\mathrm{T}}\mu$

We want to describe the mean-variance frontier of risky assets, so we ignore the risk-free asset and constrain portfolios that $\iota^{\mathrm{T}}\pi = 1$. We define the problem as

$$\min \frac{1}{2}\pi^{\mathrm{T}}\Sigma\pi \quad s.t. \mu^{\mathrm{T}}\pi = \mu_{tar}, \quad \iota^{\mathrm{T}}\pi = 1$$

where μ_{tar} is our target return. the Lagrangean is

$$\frac{1}{2}\pi^{\mathrm{T}}\Sigma\pi - \delta(\mu^{\mathrm{T}}\pi - \mu_{tar}) - \gamma(\iota^{\mathrm{T}}\pi - 1)$$

then

$$\Sigma\pi - \delta\mu - \gamma\iota = 0 \implies \pi = \delta\Sigma^{-1}\mu + \gamma\Sigma^{-1}\iota$$

which means π is a linear combination of two vectors $\Sigma^{-1}\mu$ and $\Sigma^{-1}\iota$, it's convenient to define the following constants:

$$A = \mu^{\mathrm{T}}\Sigma^{-1}\mu \qquad B = \mu^{\mathrm{T}}\Sigma^{-1}\iota \qquad C = \iota^{\mathrm{T}}\Sigma^{-1}\iota$$

from the constrains and applying the denotation we have

$$\mu_{tar} = \mu^{\mathrm{T}}\pi = \delta A + \gamma B$$

$$1 = \iota^{\mathrm{T}}\pi = \delta B + \gamma C$$

In general, we have

$$\delta = \frac{C\mu_{tar} - B}{AC - B^2} \qquad \gamma = \frac{A - B\mu_{tar}}{AC - B^2}$$

thus the return variance of the portfolio

$$\pi'\Sigma\pi = \quad \pi'\Sigma(\delta\Sigma^{-1}\mu + \gamma\Sigma^{-1}\iota) = \quad \delta\pi'\mu + \gamma\pi'\iota = \quad \delta\mu_{targ} + \gamma$$
$$= \frac{C\mu_{targ}^2 - 2B\mu_{targ} + A}{AC - B^2}$$

the variance-mean pairs for the frontier portfolio is

$$\left(\sqrt{\frac{C\mu_{targ}^2 - 2B\mu_{targ} + A}{AC - B^2}}, \mu_{targ}\right)$$

which is quadratic curve

if there is no target constrain, then we can find the minimum variance portfolio which is $\pi_{gmv} = \Sigma^{-1}\iota/\iota^{\mathrm{T}}\Sigma^{-1}\iota$

if there is a risk-free asset and a risky asset, then the expected return is $(1 - \iota^{\mathrm{T}}\pi)R_f + \mu^{\mathrm{T}}\pi = R_f + (\mu - R_f\iota)^{\mathrm{T}}\pi$, then the problem is

$$\min 1/2\pi^{\mathrm{T}}\Sigma\pi \qquad s.t. \quad R_f + (\mu - R_f\iota)^{\mathrm{T}}\pi = \mu_{tar}$$

from Lagrangian method we have $\pi = \delta\Sigma^{-1}(\mu - R_f\iota)$

the portfolios are

$$\pi = \frac{\mu_{\text{targ}} - R_f}{(\mu - R_f\iota)^{\mathrm{T}}\Sigma^{-1}(\mu - R_f\iota)}\Sigma^{-1}(\mu - R_f\iota)$$

set of (stdev, mean) paris is

$$\frac{|\mu_{\text{targ}} - R_f|}{\sqrt{(\mu - R_f\iota)^{\mathrm{T}}\Sigma^{-1}(\mu - R_f\iota)}}, \mu_{tar}$$

which is stright line of μ_{tar}

now return to the constraint $\pi = \delta\Sigma^{-1}(\mu - R_f\iota)$, we know that δ is a number, and $\Sigma^{-1}(\mu - R_f\iota)$ is a vector denoting the portfolio. And $\iota^T\pi$ is the fraction we investing in risky asset. If we normalize the vector, which means we do not invest any money in risk-free asset, then we get our **tangency portfolio**

$$\pi_{\tan} = \frac{1}{\iota^{\mathrm{T}}\Sigma^{-1}(\mu - R_f\iota)}\Sigma^{-1}(\mu - R_f\iota)$$

Denote ι_i as an $n \times 1$ vector with ith term 1 and other terms 0. Then the Tangency Portfolio return is

$$\widetilde{R}_i = \widetilde{R}^{\mathrm{T}}\iota_i = \iota_i\widetilde{R} \qquad \widetilde{R}_{\tan} = \pi_{\tan}^{\mathrm{T}}\widetilde{R}$$

then

$$\mathrm{Cov}(\tilde{R}_i, \tilde{R}_{tang}) = \quad \mathrm{Cov}(\iota_i^{\mathrm{T}}\widetilde{R}, \pi_{\tan}^{\mathrm{T}}\widetilde{R}) = \quad \iota_i'\Sigma\pi_{tang}$$
$$= \quad \iota_i'\Sigma\frac{\Sigma^{-1}(\mu - R_f\iota)}{\iota'\Sigma^{-1}(\mu - R_f\iota)} = \quad \frac{\iota_i'(\mu - R_f\iota)}{\iota'\Sigma^{-1}(\mu - R_f\iota)} = \quad \frac{E(\tilde{R}_i) - R_f}{\iota'\Sigma^{-1}(\mu - R_f\iota)}$$

This imply that

$$\mathrm{Var}[\widetilde{R}_{\tan}] = (E(\tilde{R}_{\tan}) - R_f)/(\iota'\Sigma^{-1}(\mu - R_f\iota))$$

therefore

$$\mathbb{E}[\widetilde{R}_i] - R_f = \frac{\mathrm{Cov}[\widetilde{R}_i, \widetilde{R}_{\tan}]}{\mathrm{Var}[\widetilde{R}_{\tan}]}(\mathbb{E}[\widetilde{R}_{\tan}] - R_f)$$

simplify it

$$\mathbb{E}[\widetilde{R}_i] - R_f = \beta_{i,\tan}(\mathbb{E}[\widetilde{R}_{\tan}] - R_f)$$

where $\beta_{i,\tan} = \mathrm{Cov}[\widetilde{R}_i, \widetilde{R}_{\tan}]/\mathrm{Var}[\widetilde{R}_{\tan}]$

CAMP

The market portfolio of risky assets provides the highest Sharpe Ratio. All investors allocates wealth between the risk-free asset and the market portfolio, thus the cross-section stock return:

$$\mathbb{E}[\widetilde{R}] - R_f = \frac{\mathrm{Cov}[\widetilde{R}, \widetilde{R}_m]}{\mathrm{Var}[\widetilde{R}_m]}(\mathbb{E}[\widetilde{R}_m] - R_f) = \beta_{i,m}(\mathbb{E}[\widetilde{R}_m] - R_f)$$

1. Corporate Finance: CAMP is used to compute cost of equity, the WACC and NPV
2. In active investment, it is used to identify overpriced (underpriced) stocks. Specifically, one runs a regression

$$\widetilde{R}_i - R_f = \alpha_i + \beta_{i,m}(\widetilde{R}_m - R_f) + \widetilde{\varepsilon}_i$$

overpriced stock have $\alpha_i < 0$

3. When evaluating fund managers' performance, look at the (Jensen's) α in the following regression of fund portfolio's performance

$$\widetilde{R}_p - R_f = \alpha_i + \beta_{p,m}(\widetilde{R}_m - R_f) + \widetilde{\varepsilon}_i$$

$\alpha_p > 0$ indicate good performance

Size Anomaly: The Size Anomaly refers to the observation that small-cap stocks (stocks with smaller market capitalizations) tend to outperform large-cap stocks on a risk-adjusted basis.

Reason: According to CAPM, expected returns should be a function of beta and not company size. However, small-cap stocks often deliver higher returns than predicted by their beta, possibly because they are perceived as riskier or less liquid, thus commanding a higher return as compensation.

Empirical Findings: Small-cap stocks' outperformance challenges market efficiency and suggests that CAPM fails to fully account for risks associated with firm size.

Value Anomaly: The Value Anomaly is the tendency for stocks with lower price-to-book ratios (P/B), lower price-to-earnings ratios (P/E), or other value metrics to outperform those with higher ratios, known as growth stocks. This anomaly was highlighted by Fama and French in their three-factor model introduced in 1992.

Reason: Value stocks are often perceived as undervalued by the market, while growth stocks may be overvalued. Value stocks usually represent companies that are financially distressed or in a down cycle, and the market may not fully price their potential for recovery.

Empirical Findings: The superior performance of value stocks contradicts CAPM, implying that additional factors, such as distress risk, are not adequately captured by traditional models

Momentum Anomaly: The Momentum Anomaly refers to the phenomenon where stocks that have performed well in the past (winners) continue to perform well in the near term, while stocks that have performed poorly (losers) continue to underperform.

Reason: Momentum effects may arise from behavioral biases such as investors' tendency to chase winners and avoid losers (herding behavior), or from delays in information dissemination and market underreaction to new data.

Empirical Findings: Momentum strategies contradict CAPM's assumptions, which suggest that asset returns should be solely related to systematic risk rather than historical price trends.

MGF

the condition pdf is

$$f_{X|Y}(x|y) = \quad \frac{f_{X,Y}(x, y)}{f_Y(y)}$$
$$= \quad \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma_1}\frac{1}{\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\frac{1}{\sigma_1^2}\left(x - \frac{\rho\sigma_1}{\sigma_2}y\right)^2\right\}$$

More generally, if $X \sim N(\mu_X, \sigma_x^2), Y \sim N(\mu_Y, \sigma_Y^2)$, then

$$X|Y \sim N\left(\mu_X + \frac{\rho\sigma_X}{\sigma_Y}(y - \sigma_Y), \sigma_X^2(1-\rho^2)\right)$$

if $Y = aX + b$, then $\Phi_y(\omega) = e^{b\omega}\Phi_x(a\omega)$

If $X \sim N(\eta, \sigma)$. Then its charateristic function is $\Phi_x(\omega) = \exp\left\{\eta\omega - \frac{1}{2}\sigma^2\omega^2\right\}$

Fama-French-4-Factor

The market factor

SMB (small stocks minus big stocks)

HML (high book-to-market stocks minus low book-to-market stocks)

Later, Carhart proposes a momentum factor (UMD, that is, winning stocks minus losing stocks)

$$\tilde{R}_i - R_f = \alpha_i + \beta_{i,M}(\tilde{R}_M - R_f) + \beta_{i,SMB}SMB$$
$$+ \beta_{i,HML}HML + \beta_{i,UMD}UMD + \tilde{\epsilon}_i.$$

Arbitrage Pricing Theory

Suppose that the stock returns are generated through a factor structure:

$$\widetilde{R}_i = \mu_i + \sum \beta_{ij}\widetilde{f}_j + \widetilde{\varepsilon}_i$$

where

- f_j is the systematic risk factor, $f_j \sim N(0,1), f_j \perp f_i, \forall i \neq j$, ε_i is the idiosyncratic risk, $\varepsilon_i \perp f_j, \forall i, j$ and $\varepsilon_i \perp \varepsilon_j, \forall i \neq j$, $\mu_i = \mathbb{E}[\widetilde{R}_i]$, β_{ij} often referred to as factor loadings

Recall SDF we have

$$p_i = \mathbb{E}[\widetilde{m}\widetilde{x}_i]$$

$\widetilde{m} > 0$ means no arbitrage, the return is

$$\mathbb{E}(\tilde{R}_i - R_f) = -R_f\,\mathrm{Cov}(\tilde{m}, \tilde{R}_i - R_f)$$
$$= -R_f\,\mathrm{Cov}(\tilde{m}, \mu_i + \sum_{j=1}^{k}(\beta_{ij}\tilde{f}_j) + \tilde{\epsilon}_i - R_f)$$
$$= \sum_{j=1}^{k}[\beta_{ij}\underbrace{(-R_f\,\mathrm{Cov}(\tilde{m}, \tilde{f}_j))}_{\text{same for all asset i}}] - R_f\,\mathrm{Cov}(\tilde{m}, \tilde{\epsilon}_i)$$
$$= \sum_{j=1}^{k}(\beta_{ij}\lambda_j) - R_f\mathrm{Cov}(\tilde{m}, \tilde{\epsilon}_i).$$

thus $-R_f\mathrm{Cov}[\widetilde{m}, \widetilde{\varepsilon}_i]$ represent the pricing error

compare this with CAMP, the difference is that CAMP only has one factor (with one $\beta_{i,m}$) , but in APT, there may be more that one factor

Well-diversified portfolios with $\widetilde{\varepsilon}_i \to 0$ have pricing error $\to 0$. then

$$\mathbb{E}(\tilde{R}_i - R_f) = \sum_{j=1}^{k}(\beta_{ij}\lambda_j)$$

where λ_j is referred to as the factor risk premia, often write as

$$\mathbb{E}(\tilde{R}_i - R_f) = \sum_{j=1}^{k}(\beta_{ij}\mathbb{E}[\widetilde{R}_j^F - R_f])$$

where $\widetilde{R}_j^F$ is a well-diversified port such that $\beta_{ij} = 1$ and other β are 0, thus

$$\mathbb{E}[\widetilde{R}_j^F - R_f] = \sum \beta_{ij}\lambda_j = \lambda_j$$

we call the port as the factor portfolio

Grossman-Stiglitz Model

Suppose a risky asset:

payoff $\widetilde{\theta} \sim N(0, \nu_\theta)$, stochastic supply $\widetilde{z} \sim N(0, \nu_z)$

there is a risk-free asset, the return is normalized to be 1

a mass 1 of investors with CARA α:

By spending $c > 0$, an investor can observe a private signal $\widetilde{s} = \widetilde{\theta} + \widetilde{\varepsilon}$, where $\widetilde{\varepsilon} \sim N(0, \nu_\varepsilon)$

in equilibrium, η fraction of the investors become informed; the remaining $1 - \eta$ stay uninformed

conjecture that the equilibrium price function $p = b\widetilde{\omega}$ where $\widetilde{\omega} = \widetilde{s} - \delta\widetilde{z}$, b, δ are constant parameters to be determined

An uninformed investor condition the trade on $\widetilde{\omega}$, based on the belief

$$\widetilde{\theta}|\widetilde{\omega} \sim N\left(\frac{\nu_\theta}{\nu_\omega}\widetilde{\omega}, \nu_{\theta|\omega}\right)$$

where $\nu_\omega = \nu_s + \delta^2\nu_z$ and $\nu_{\theta|\omega} = \nu_\theta\left(1 - \frac{\nu_\theta}{\nu_\omega}\right)$

the wealth at time 1 with demand x_U is

$$W_1 = W_0 + x_U(\widetilde{\theta} - p)$$

and the utility is

$$\mathbb{E}[U(W_1)|\widetilde{s}] = \mathbb{E}[-\exp\{-\alpha W_1\}|\widetilde{\omega}] = \mathbb{E}[-e^{-\alpha(W_0 + x_U(\widetilde{\theta} - p))}|\widetilde{\omega}]$$
$$= -\exp\left\{-\alpha(W_0 + x_U(\mathbb{E}[\widetilde{\theta}|\widetilde{\omega}] - p)) + \frac{1}{2}\alpha^2 x_U^2\nu_{\theta|\omega}\right\}$$

the f.o.c. w.r.t. x_U imply

$$x_U = (\mathbb{E}[\widetilde{\theta}|\widetilde{\omega}] - p)/\alpha\nu_{\theta|\omega} = (\frac{\nu_\theta}{\nu_\omega}\widetilde{\omega} - p)/\alpha\nu_{\theta|\omega}$$

the market clearing condition tells that $\eta x_I + (1-\eta)x_U = \widetilde{z}$ which is

$$\eta\frac{\frac{\nu_\theta}{\nu_s}\widetilde{s} - p}{\alpha\nu_{\theta|s}} + (1-\eta)\frac{\frac{\nu_\theta}{\nu_\omega}\widetilde{\omega} - p}{\alpha\nu_{\theta|\omega}} = \widetilde{z}$$

note that $p = b\widetilde{\omega}$, then

$$\delta = \frac{\alpha\nu_{\theta|s}}{\eta\frac{\nu_\theta}{\nu_s}}, \qquad b = \frac{\frac{\eta}{\nu_{\theta|s}}\frac{\nu_\theta}{\nu_s} + \frac{1-\eta}{\nu_{\theta|\omega}}\frac{\nu_\theta}{\nu_\omega}}{\frac{\eta}{\nu_{\theta|s}} + \frac{1-\eta}{\nu_{\theta|\omega}}}$$

In equilibrium, an investor must be indifferent between acquiring information (by spending c) and staying uninformed, thus $\exp\{\alpha c\}\sqrt{\frac{\nu_{\theta|s}}{\nu_{\theta|\omega}}} = 1$