## Matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} A^{-1}+A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{bmatrix}$$

$\det A = |A| = \sum_{j=1}^{N}(-1)^{i+j}a_{ij}M_{ij} = \sum_{i=1}^{N}(-1)^{i+j}a_{ij}M_{ij}$
where $M$ is the minor: The *Minor* of entry $a_{ij}$, denoted by $M_{ij}$, is the determinant of the sub-matrix obtained by deleting ith row and jth col

**Cramer's Rule**: for equation $Ax = b$, Define $A_i$ to be equal to the the matrix but with col $i$ replaced by $b$, then we have

$$x_i = \frac{\det A_i}{\det A}$$

**Example**:

$$\begin{bmatrix} 3 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix}$$

thus $\det A = 2$ and

$$\det(A_1) = \begin{vmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & 4 \end{vmatrix} = 11$$

then $x = 11/2$, similarly, $y = -39/2, z = 7/2$

**Basis**: if $\mathbf{c}, \mathbf{b}$ are two basis, the coordinates under $\mathbf{c}$ is $x_c$, then the coordinates under $\mathbf{b}$ will satisfy

$$Bx_b = Cx_c \implies x_b = B^{-1}Cx_c$$

**Projection**: Suppose there a vector $\lambda w$ on $w$, and the new vector $v - \lambda w$ is perpendicular to $w$, then

$$(v-\lambda w)^{\mathrm{T}} w = 0 \implies \lambda = \frac{v^{\mathrm{T}}w}{w^{\mathrm{T}}w} \implies P_{wv} = \frac{v^{\mathrm{T}}w}{w^{\mathrm{T}}w}w$$

suppose a eigenvalue $\lambda_i$ has several eigenvectors $\xi_i, \xi_{i+1}, \ldots$, we could use Gram-Schmidt process to get the orthogonal vectors

$$\eta_i = \xi_i \tag{1}$$

$$\eta_{i+1} = \xi_{i+1} - \frac{\xi_{i+1}^{\mathrm{T}}\eta_i}{\eta_i^{\mathrm{T}}\eta_i}\eta_i \tag{2}$$

$$\eta_{i+2} = \xi_{i+2} - \frac{\xi_{i+2}^{\mathrm{T}}\eta_i}{\eta_i^{\mathrm{T}}\eta_i}\eta_i - \frac{\xi_{i+2}^{\mathrm{T}}\eta_{i+1}}{\eta_{i+1}^{\mathrm{T}}\eta_{i+1}}\eta_{i+1} \tag{3}$$

**Orthogonal**: just vertical **Orthonormal**: nomarlized and orthogonal
if $A$ is symmetric, for any $x$ we have $x^T Ax > 0$, we say $A$ is positive definite, if $x^T Ax >= 0$, we say $A$ is semi-positive definite

**Properties**:
1. $A$ is Positive-definite iff all Eigenvalues are positive
2. $A$ is Postive Semi-definite iff all eigenvalues are non-negative
3. If $A$ is Positive-definite, then so is $A^{-1}$
4. If $B_{n\times m}$ is any matrix, then $B^{\mathrm{T}}B, BB^{\mathrm{T}}$ are postive semi-definite

## Laplace Transform

$F(s) = \mathscr{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dx$
$\mathscr{L}[1] = \int_0^\infty e^{-st}\,dt = \frac{1}{s}$
$\mathscr{L}[t^n] = \int_0^\infty e^{-st}t^n\,dt = \frac{n!}{s^{n+1}}$
$\mathscr{L}[e^{-at}] = \int_0^\infty e^{-st}e^{-at}\,dt = \frac{1}{s+a}$
$\mathscr{L}[e^{at}f(t)] = F(s-a)$
Linearity: $\mathscr{L}[\alpha f(t) + \beta g(t)] = \alpha F(s) + \beta G(s)$
Scaling: $\mathscr{L}[f(at)] = \frac{1}{|a|}F\left(\frac{s}{a}\right)$
Multiplication: $\mathscr{L}[e^{ct}f(t)] = F(s-c)\mathscr{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$
Differentiation:
$\mathscr{L}[f'(t)] = sF(s) - f(0)$ $\mathscr{L}[f''(t)] = s^2F(s) - sf(0) - f'(0)$
$\mathcal{L}[f^{(n)}(t)] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$
Convolution: $\mathscr{L}\left[\int_0^t f(t-\tau)g(\tau)\,d\tau\right] = F(s)G(s)$
Example: $x' + x = e^{-2t}$

$$sX(s) - X(0) + X(s) = \frac{1}{s+2}$$

$$X(s) = \frac{x(0)}{s+1} + \frac{1}{s+1} - \frac{1}{s+2}$$

$$x(t) = x(0)e^{-t} + e^{-t} - e^{-2t}$$

## 1st Order Exact Equation

if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
then the solution is $u(x,y) = \int M\,dx + f(y) = \int N\,dy + g(x)$
given $P(x,t)dx + Q(x,t)dt = 0$
if $\frac{1}{P(x,t)}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial t}\right) = u(t)$, then $e^{\int u(t)dt}$ is an integrating factor
if $\frac{1}{Q(x,t)}\left(\frac{\partial P}{\partial t} - \frac{\partial Q}{\partial x}\right) = v(x)$, then $e^{\int v(x)dx}$ is an integrating factor
**Example**:solve $2xdx + (x^2 - t)dt = 0$
$\frac{1}{P(x,t)}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial t}\right) = \frac{1}{2x} \times 2x = 1$, then $e^{\int 1dt} = e^t$

$$\begin{aligned} u(x,t) &= \int e^t 2xdx + f(t) = e^t x^2 + f(t) \\ &= \int e^t(x^2 - t)dt + g(x) = e^t x^2 - \int e^t tdt + g(x) \\ &= e^t x^2 - e^t t + e^t + g(x) \end{aligned}$$

## 2nd Order ODE

for $y'' + ay' + by = 0$, we solve $\lambda^2 + a\lambda + b = 0$ and get the value $\lambda_1, \lambda_2$
- if $a^2 - 4b > 0$, then $y = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$
- if $a^2 - 4b = 0$, then $y = C_1e^{\lambda x} + C_2xe^{\lambda x}$
- if $a^2 - 4b < 0$, and $\lambda = \alpha \pm \beta i$, then $y = C_1e^{\alpha x}\cos\beta x + C_2e^{\alpha x}\sin\beta x$

## Linear System

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathrm{x} = \mathrm{P}\begin{bmatrix} C_1\mathrm{e}^{\lambda_1 t} \\ \vdots \\ C_n\mathrm{e}^{\lambda_n t} \end{bmatrix}$$

we do diagonalize and the eigenvalue is $\Lambda = diag(\lambda_1, \ldots\lambda_n)$, then
**example** $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$(A - 3I)u = 0 \Rightarrow \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$(A + I)u = 0 \Rightarrow \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

## Formula

$$\sin A\cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$$

$$\sin A\sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

$$\cos A\cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$$

$$\frac{d}{dx}a^x = a^x\ln a \qquad \int\sec^2 xdx = \tan x + C$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \qquad \int\csc^2 xdx = -\cot x + C$$

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \qquad \int\sec x\tan xdx = \sec x + C$$

$$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{x^2+1} \qquad \int\csc x\cot xdx = -\csc x + C$$

$$\frac{d}{dx}\cot^{-1}x = \frac{-1}{x^2+1} \qquad \int\frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}} \qquad \int\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\frac{d}{dx}\csc^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}} \qquad \int\frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{|x|}{a} + C$$

## Calculus

Suppose all partial derivatives of the function $G(x_1, x_2, \ldots, y)$ exists. and $G(x_1, x_2, \ldots, y) = C$, then $\frac{\partial y}{\partial x_i} = -\frac{\partial G}{\partial x_i} / \frac{\partial G}{\partial y}$
The 2nd-order is

$$\frac{\partial}{\partial x_i}\left(\frac{\partial G}{\partial x_i} + \frac{\partial G}{\partial y}\frac{\partial y}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial G}{\partial x_i}\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial G}{\partial y}\right)\frac{\partial y}{\partial x_i} + \frac{\partial G}{\partial y}\frac{\partial^2 y}{\partial x_i^2} = 0$$

and

$$\frac{\partial}{\partial x_i}\left(\frac{\partial G}{\partial x_i}\right) = \frac{\partial^2 G}{\partial x_i^2} + \frac{\partial^2 G}{\partial y x_i}\frac{\partial y}{\partial x_i} \qquad \frac{\partial}{\partial x_i}\left(\frac{\partial G}{\partial y}\right) = \frac{\partial^2 G}{\partial x_i y} + \frac{\partial^2 G}{\partial y^2}\frac{\partial y}{\partial x_i}$$

spherical polar coordinates:

$$r = (x^2+y^2+z^2)^{1/2}, \quad \theta = \cos^{-1}\frac{z}{(x^2+y^2+z^2)^{1/2}}, \quad \phi = \tan^{-1}\frac{y}{x}$$

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

**Hessian Matrix**: If all second partial derivatives are continuous, then Hessian Matrix is symmetric

$$[\nabla^2 f] = \frac{\partial^2 f}{\partial x\partial x^\top} = \begin{bmatrix} \frac{\partial^2 f}{\partial^2 x_1^2} & \frac{\partial^2 f}{\partial x_1\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1\partial x_n} \\ \frac{\partial^2 f}{\partial x_2\partial x_1} & \frac{\partial^2 f}{\partial^2 x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n\partial x_1} & \frac{\partial^2 f}{\partial x_n\partial x_2} & \cdots & \frac{\partial^2 f}{\partial^2 x_n} \end{bmatrix}$$

$$\frac{\partial \mathrm{x}^\top \mathrm{Ax}}{\partial \mathrm{x}^\top} = (\mathrm{A} + \mathrm{A}^\top)x = 2\mathrm{Ax}$$

**Taylor Expansion**: $f(x) = \sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_n(x)$

$$f(x) = f(x_0) + (x-x_0)^\top\nabla f(x_0) + \frac{1}{2!}(x-x_0)^\top[\nabla^2 f(x_0)](x-x_0) + \cdots$$

**Notice**: the quadratic term has coefficient 1/2!, but this not apply to $(x-x_0)(y-y_0)$
if $(x,y,z) \to (u,v,w)$, then Jacob matrix is

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

and

$$\begin{aligned} &\int_R f(x,y,z)dV(x,y,z) \\ &= \int_S f(x(u,v,w), y(u,v,w), z(u,v,w))|det(J)|dV(u,v,w) \end{aligned}$$

**Example**: $\iint_{x^2+y^2\le 4}(x^2+y^2+3)dA(x,y)$

$$\det(\mathbf{J}) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r\cos^2\theta - (-r\sin^2\theta) = r$$

$$\int_0^{2\pi}\int_0^2 (r^2+3)rdrd\theta = \int_0^{2\pi}\left[\frac{r^4}{4} + \frac{3r^2}{2}\right]_{r=0}^{r=2} d\theta = 20\pi$$

for spherical polar coordinates:

$$\iiint_R f(x,y,z)dV = \iiint_S f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)r^2\sin\phi d\rho\theta d\phi$$

## Optimization

For $f(x)$, $x = a$ is a critical point if $\nabla f(a) = 0$
To determine the nature of critical points at $x = a$, we need to analyze the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the Hessian Matrix of $f$:
- $\nabla f(a) = 0, \lambda_i(a) < 0$: maximum
- $\nabla f(a) = 0, \lambda_i(a) > 0$: minimum
- $\nabla f(a) = 0$, mixed signs: saddle point

if Hessian Matrix $A$ is positive semi-definite (i.e. $A >= 0$), then local minimum is global minimum

**Example**

$$\frac{1}{2}v\frac{\partial^2 f}{\partial x^2}+\rho\sigma v\frac{\partial^2 f}{\partial x\partial v}+\frac{1}{2}\sigma^2 v\frac{\partial^2 f}{\partial v^2}+(r+\beta v)\frac{\partial f}{\partial x}+(a-bv)\frac{\partial f}{\partial v}+\frac{\partial f}{\partial t}=0,$$

$$f(x,v,T)=e^{\phi x}$$

(a)with an ansatz $f(x,v,t)=\exp\{C(t)+D(t)v+\phi x\}$, Derive the equations driving $C(t)$ and $D(t)$
(b) Suppose that the function $D(t)$ for $t\in[0,T]$ is solvable. Solve $C(t)$
(c) suppose $\sigma=0$, solve fro $D(t)$

$$\frac{\phi^2}{2}vf+\rho\sigma\phi D(t)vf+\frac{1}{2}\sigma^2D^2(t)vf+(r+\beta v)\phi f+(a-bv)D(t)f$$
$$+\left(\frac{dC}{dt}+\frac{dD}{dt}v\right)f=0$$

with boundary condition $C(T)=0, D(T)=0$, since $f\neq 0$, we can organize the PDE in the order of $v$ as follows:

$$\left[\frac{dD}{dt}+\frac{\sigma^2}{2}D^2(t)+(\rho\sigma\phi-b)D(t)+\frac{\phi^2}{2}+\beta\phi\right]v+\left[\frac{dC}{dt}+aD(t)+r\right]=0$$

Note that the equation holds for all $v$. Hence, we have

$$\frac{dC}{dt}+aD(t)+r=0,\quad C(T)=0;\qquad (4)$$

$$\frac{dD}{dt}+\frac{\sigma^2}{2}D^2(t)+(\rho\sigma\phi-b)D(t)+\frac{\phi^2}{2}+\beta\phi=0,\quad D(T)=0.\qquad (5)$$

(b) Integrating both sides of the 1st equation from $t$ to $T$, we have

$$C(t)=r(T-t)+a\int_t^T D(s)ds$$

(c) In this case, we have

$$\frac{dD}{dt}-bD(t)+\frac{\phi^2}{2}+\beta\phi=0\Longleftrightarrow\frac{d(e^{-bt}D(t))}{dt}+e^{-bt}\left(\frac{\phi^2}{2}+\beta\phi\right)=0$$

Integrating from $t$ to $T$

$$e^{-bt}D(t)=\left(\frac{\phi^2}{2}+\beta\phi\right)\int_t^T e^{-bs}ds\Longrightarrow D(t)=\frac{\phi(\phi+2\beta)}{2b}(1-e^{-b(T-t)})$$

**Example**

$$\ln(x+z)+ze^y+x^2z=1\quad (x,y,z)=(0,0,1)$$

$$\frac{1}{x+z}+\frac{\partial z}{\partial x}\left(\frac{1}{x+z}+e^y+x^2\right)+2xz=0\Longrightarrow\left.\frac{\partial z}{\partial x}\right|_{(x,y,z)=(0,0,1)}=-\frac{1}{2}$$

$$\frac{\partial z}{\partial y}\left(\frac{1}{x+z}+e^y+x^2\right)+ze^y=0\Longrightarrow\left.\frac{\partial z}{\partial y}\right|_{(x,y,z)=(0,0,1)}=-\frac{1}{2}$$

$$\frac{\partial^2 z}{\partial x\partial y}\left(\frac{1}{x+z}+e^y+x^2\right)+\frac{\partial z}{\partial y}\left(-\frac{1}{(x+z)^2}\left(1+\frac{\partial z}{\partial x}\right)+2x\right)+\frac{\partial z}{\partial x}e^y=0$$

$$\Longrightarrow\left.\frac{\partial^2 z}{\partial x\partial y}\right|_{(x,y,z)=(0,0,1)}=\frac{1}{8}$$

$$\frac{\partial^2 z}{\partial y^2}\left(\frac{1}{x+z}+e^y+x^2\right)+\frac{\partial z}{\partial y}\left(-\frac{1}{(x+z)^2}\frac{\partial z}{\partial y}+e^y\right)+\frac{\partial z}{\partial y}e^y+ze^y=0$$

$$\Longrightarrow\left.\frac{\partial^2 z}{\partial y^2}\right|_{(x,y,z)=(0,0,1)}=\frac{1}{8}$$

$$-\frac{1}{(x+z)^2}\left(1+\frac{\partial z}{\partial x}\right)+\frac{\partial^2 z}{\partial x^2}\left(\frac{1}{x+z}+e^y+x^2\right)$$
$$+\frac{\partial z}{\partial x}\left(-\frac{1}{(x+z)^2}\left(1+\frac{\partial z}{\partial x}\right)+2x\right)+2z+2x\frac{\partial z}{\partial x}=0$$

$$\Longrightarrow\left.\frac{\partial^2 z}{\partial x^2}\right|_{(x,y,z)=(0,0,1)}=-\frac{7}{8}$$

Hence

$$z(x,y)\approx 1-\frac{1}{2}x-\frac{1}{2}y-\frac{7}{16}x^2+\frac{1}{8}xy+\frac{1}{16}y^2$$

**Example**

$\min_w\sigma(w):=\sqrt{w^\top\Sigma w}$ subject to $r_f+w^\top(\mu-r_f\mathbf{1}_p)=r^*$
The problem is equivalent to minimizing $\sigma^2(w)=w^T\Sigma w$ because square function is an increasing function.Define Lagrangian as follows

$$L(w;\lambda)=w^\top\Sigma w+\lambda(r_f+w^\top(\mu-r_f\mathbf{1}_p)-r^*)$$

Setting its derivative with respect to $w$ equal to 0 yields that

$$2\Sigma w+\lambda(\mu-r_f\mathbf{1}_p)=0\Longrightarrow w^*(\lambda)=-\frac{\lambda}{2}\Sigma^{-1}(\mu-r_f\mathbf{1}_p)$$

Plug it into constraint

$$r_f-\frac{\lambda}{2}(\mu-r_f\mathbf{1}_p)^\top\Sigma^{-1}(\mu-r_f\mathbf{1}_p)=r^*$$
$$\Longrightarrow\lambda^*=\frac{-2(r^*-r_f)}{(\mu-r_f\mathbf{1}_p)^\top\Sigma^{-1}(\mu-r_f\mathbf{1}_p)}$$

Hence

$$w^*=\frac{r^*-r_f}{(\mu-r_f1_p)^\top\Sigma^{-1}(\mu-r_f1_p)}\Sigma^{-1}(\mu-r_f1_p)$$
$$\sigma^*=\sigma(w^*)=\frac{r^*-r_f}{\sqrt{(\mu-r_f\mathbf{1}_p)^\top\Sigma^{-1}(\mu-r_f\mathbf{1}_p)}}$$

**Example**

$$V_t+rxV_x-\frac{\theta}{2}\frac{V_x^2}{V_{xx}}=0\quad\text{with}\quad V(T,x)=\frac{x^{1-\gamma}}{1-\gamma}$$

with an ansatz $V(t,x)=g(t)\frac{x^{1-\gamma}}{1-\gamma}$ derive the equation driving $g(t)$ and solve for it

$$V_t=\frac{dg}{dt}\frac{x^{1-\gamma}}{1-\gamma},V_x=g(t)x^{-\gamma},V_{xx}=-\gamma g(t)x^{-\gamma-1}$$

Hence, PDE becomes

$$\frac{dg}{dt}\frac{x^{1-\gamma}}{1-\gamma}+rg(t)x^{1-\gamma}+\frac{\theta}{2}\frac{g^2(t)x^{-2\gamma}}{\gamma g(t)x^{-\gamma-1}}=0,\quad g(T)=1$$
$$\Leftrightarrow\quad\frac{dg}{dt}+r(1-\gamma)g(t)+\frac{\theta(1-\gamma)}{2\gamma}g(t)=0,\quad g(T)=1$$
$$\frac{d(\ln g)}{dt}=-r(1-\gamma)-\frac{\theta(1-\gamma)}{2\gamma},\quad g(T)=1$$

Integrating from $t$ to $T$

$$-\ln g(t)=\ln g(T)-\ln g(t)=-\int_t^T\left[r(1-\gamma)+\frac{\theta(1-\gamma)}{2\gamma}\right]ds$$
$$\Leftrightarrow g(t)=\exp\left(\left[r(1-\gamma)+\frac{\theta(1-\gamma)}{2\gamma}\right](T-t)\right)$$

**Example**

let

$$\bar{x}_n=\frac{1}{n}\sum_{i=1}^n x_i,\quad e_n=x_n-\bar{x}_n,\quad Q_n=\sum_{i=1}^n(x_i-\bar{x}_n)(x_i-\bar{x}_n)^\intercal$$

Prove

$$\bar{x}_{n-1}=\bar{x}_n-\frac{e_n}{n-1},\ Q_{n-1}=Q_n-\left(1+\frac{1}{n-1}\right)e_ne_n^\intercal$$

$$Q_{n-1}=\sum_{i=1}^{n-1}(x_i-\bar{x}_{n-1})(x_i-\bar{x}_{n-1})^\top$$
$$=\sum_{i=1}^{n-1}\left(x_i-\bar{x}_n+\frac{e_n}{n-1}\right)\left(x_i-\bar{x}_n+\frac{e_n}{n-1}\right)^\top$$
$$=\sum_{i=1}^{n-1}[(x_i-\bar{x}_n)(x_i-\bar{x}_n)^\top+(x_i-\bar{x}_n)\frac{e_n^\top}{n-1}+$$
$$\frac{e_n}{n-1}(x_i-\bar{x}_n)^\top+\frac{e_ne_n^\top}{(n-1)^2}]$$
$$=Q_n-e_ne_n^\top-\frac{e_ne_n^\top}{n-1}-\frac{e_ne_n^\top}{n-1}+\frac{e_ne_n^\top}{n-1}$$
$$=Q_n-\left(1+\frac{1}{n-1}\right)e_ne_n^\top$$