Q1

Cox-Ingersoll-Ross (CIR) interest model. Let $W(t)$, $t \geq 0$, be a Brownian motion. The Cox-Ingersoll-Ross model for the interest rate process $R(t)$ is

$$dR(t) = (\alpha - \beta R(t))dt + \sigma\sqrt{R(t)}dW(t),$$

where α, β and σ are positive constants. The advantage of the CIR interest model over the Vasicek model is that the interest rate in the CIR model does not become negative but it does not have a closed-form solution. Like the Vasicek model, the CIR model is mean-reverting. The distribution of $R(t)$ for each positive t can be determined. Compute the expected value of $R(t)$. Compute the variance of $R(t)$.

We use the function $f(t,x) = e^{\beta t}x$ and the Itô-Doeblin formula to compute

$$d\left(e^{\beta t}R(t)\right) = \alpha e^{\beta t}dt + \sigma e^{\beta t}\sqrt{R(t)}dW(t).$$

Integration of both sides yields

$$e^{\beta t}R(t) = R(0) + \frac{\alpha}{\beta}\left(e^{\beta t} - 1\right) + \sigma\int_0^t e^{\beta u}\sqrt{R(u)}dW(u).$$

Recalling that the expectation of an Itô integral is zero, we obtain

$$\mathbb{E}[R(t)] = e^{-\beta t}R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right).$$

(ii) We set $X(t) = e^{\beta t}R(t)$. We use the function $f(x) = x^2$ and the Itô-Doeblin formula to compute

$$d\left(X^2(t)\right) = (2\alpha + \sigma^2)e^{\beta t}X(t)dt + 2\sigma e^{\frac{\beta t}{2}}X^{\frac{3}{2}}(t)dW(t).$$

Integration of both sides yields

$$X^2(t) = X^2(0) + (2\alpha + \sigma^2)\int_0^t e^{\beta u}X(u)du + 2\sigma\int_0^t e^{\frac{\beta u}{2}}X^{\frac{3}{2}}(u)dW(u).$$

Taking expectations, using the fact that the expectation of an Itô integral is zero and the formula already derived for $\mathbb{E}[X(t)]$, we obtain

$$\begin{aligned}
&\mathbb{E}[X^2(t)] \\
&= X^2(0) + (2\alpha + \sigma^2)\int_0^t e^{\beta u}\mathbb{E}[X(u)]du \\
&= R^2(0) + (2\alpha + \sigma^2)\int_0^t e^{\beta u}\left(R(0) + \frac{\alpha}{\beta}\left(e^{\beta u} - 1\right)\right)du \\
&= R^2(0) + \frac{2\alpha + \sigma^2}{\beta}\left(R(0) - \frac{\alpha}{\beta}\right)\left(e^{\beta t} - 1\right) + \frac{\alpha(2\alpha + \sigma^2)}{2\beta^2}\left(e^{2\beta t} - 1\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}[R^2(t)] &= e^{-2\beta t}\mathbb{E}[X^2(t)] \\
&= e^{-2\beta t}R^2(0) + \frac{\sigma^2}{\beta^2}R(0)(e^{-\beta t} - e^{-2\beta t}) + \frac{\alpha\sigma^2}{2\beta^2}(1 - e^{-2\beta t}).
\end{aligned}$$

Finally,

$$\mathrm{Var}[R(t)] = \mathbb{E}[R^2(t)] - (\mathbb{E}[R(t)])^2 = \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t}\right).$$

MGF

the condition pdf is

$$\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma_1}\frac{1}{\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\frac{1}{\sigma_1^2}\left(x - \frac{\rho\sigma_1}{\sigma_2}y\right)^2\right\}
\end{aligned}$$

More generally, if $X \sim N(\mu_X, \sigma_x^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$, then

$$X|Y \sim N\left(\mu_X + \frac{\rho\sigma_X}{\sigma_Y}(y - \sigma_Y), \sigma_X^2(1-\rho^2)\right)$$

if $Y = aX + b$, then $\Phi_y(\omega) = e^{b\omega}\Phi_x(a\omega)$

If $X \sim N(\eta, \sigma)$. Then its charateristic function is $\Phi_x(\omega) = \exp\left\{\eta\omega + \frac{1}{2}\sigma^2\omega^2\right\}$ Specially, If $X \sim N(0,1)$, $M_x(s) = \exp\{\frac{1}{2}s^2\}$

Q2

Suppose $B_1(t)$ and $B_2(t)$ are Brownian motions and

$$dB_1(t)dB_2(t) = \rho(t)dt,$$

where $\rho(t)$ is an adapted stochastic process taking values strictly between -1 and 1. Define processes $W_1(t)$ and $W_2(t)$ such that

$$B_1(t) = W_1(t), B_2(t) = \int_0^t \rho(s)dW_1(s) + \int_0^t \sqrt{1-\rho^2(s)}dW_2(s),$$

and show that $W_1(t)$ and $W_2(t)$ are independent Brownian motions.

Solution: Since $B_1(t)$ is a Brownian motion, so is $W_1(t)$. The differential $dB_2(t)$ can be expressed by

$$dB_2(t) = \rho(t)dW_1(t) + \sqrt{1-\rho^2(t)}dW_2(t).$$

Multiplying both sides by $dB_1(t)$, or equivalently $dW_1(t)$, we obtain

$$\begin{aligned}
dB_1(t)dB_2(t) &= \rho(t)dW_1(t)dW_1(t) + \sqrt{1-\rho^2(t)}dW_1(t)dW_2(t) \\
&= \rho(t)dt + \sqrt{1-\rho^2(t)}dW_1(t)dW_2(t).
\end{aligned}$$

But $dB_1(t)dB_2(t) = \rho(t)dt$. We conclude that the cross variation of $W_1(t)$ and $W_2(t)$ is zero, i.e.,

$$dW_1(t)dW_2(t) = 0.$$

Squaring both sides of $dB_2(t) = \rho(t)dW_1(t) + \sqrt{1-\rho^2(t)}dW_2(t)$ and using the fact that $B_2(t)$ is a Brownian motion together with the assumption that $\rho(t)$ is a stochastic process taking values strictly between -1 and 1, we obtain

$$\begin{aligned}
dt &= dB_2(t)dB_2(t) \\
&= \rho^2(t)dW_1(t)dW_1(t) + \left[1 - \rho^2(t)\right]dW_2(t)dW_2(t) \\
&\quad + 2\rho(t)\sqrt{1-\rho^2(t)}dW_1(t)dW_2(t) \\
&= \rho^2(t)dt + \left[1 - \rho^2(t)\right]dt.
\end{aligned}$$

As a result, $dW_2(t)dW_2(t) = dt$ or $[W_2(t), W_2(t)] = t$. From $dB_2(t) = \rho(t)dW_1(t) + \sqrt{1-\rho^2(t)}dW_2(t)$ we also have

$$W_2(t) = \int_0^t \frac{1}{\sqrt{1-\rho^2(u)}}dB_2(u) - \int_0^t \frac{\rho(u)}{\sqrt{1-\rho^2(u)}}dW_1(u),$$

which indicates that $W_2(t)$ has continuous paths and $W_2(0) = 0$. The final task is to show $W_2(t)$ is a martingale relative to a filtration $\mathcal{F}(t)$, $t \geq 0$. Let $0 \leq s \leq t$:

$$\begin{aligned}
&\mathbb{E}(W_2(t)|\mathcal{F}(s)) \\
&= \mathbb{E}\left[\int_0^t \frac{1}{\sqrt{1-\rho^2(u)}}dB_2(u) - \int_0^t \frac{\rho(u)}{\sqrt{1-\rho^2(u)}}dW_1(u)\Big|\mathcal{F}(s)\right] \\
&= \mathbb{E}\left[\int_0^s \frac{1}{\sqrt{1-\rho^2(u)}}dB_2(u) - \int_0^s \frac{\rho(u)}{\sqrt{1-\rho^2(u)}}dW_1(u)\Big|\mathcal{F}(s)\right] \\
&\quad + \mathbb{E}\left[\int_s^t \frac{1}{\sqrt{1-\rho^2(u)}}dB_2(u) - \int_s^t \frac{\rho(u)}{\sqrt{1-\rho^2(u)}}dW_1(u)\Big|\mathcal{F}(s)\right] \\
&= W_2(s) + \mathbb{E}\left[\int_s^t \frac{1}{\sqrt{1-\rho^2(u)}}dB_2(u) - \int_s^t \frac{\rho(u)}{\sqrt{1-\rho^2(u)}}dW_1(u)\Big|\mathcal{F}(s)\right].
\end{aligned}$$

The expected values of the two Itô integrals are zero. So, we have

$$\mathbb{E}(W_2(t)|\mathcal{F}(s)) = W_2(s).$$

According to Lévy's Theorem, Theorem 4.6.5, $W_2(t)$ is a Brownian motion, and it is independent of the Brownian motion $W_1(t)$.

Levy Theorem

One Dimension
Let $M(t), t \geq 0$ be a martingale relative to a filtration $\mathscr{F}(t), t \geq 0$. Assume $M(0) = 0$, $M(t)$ has continuous paths, and the Quadratic Variation $[M,M](t) = t$ for all $t \geq 0$, then $M(t)$ is a Brownian Motion

Two Dimension
Let $M_1(t)$ and $M_2(t)$, $t \geq 0$, be martingales relative to a filtration $\mathcal{F}(t)$, $t \geq 0$. Assume that for $i = 1, 2$, we have $M_i(0) = 0$, $M_i(t)$ has continuous paths, and $[M_i, M_i](t) = t$ for all $t \geq 0$. If, in addition, $[M_1, M_2] = 0$ for all $t \geq 0$, then $M_1(t)$ and $M_2(t)$ are independent Brownian motions

We need to prove 3 things: 1) Martingale; 2) Quadratic Variation; 3) Independent.

Q3

State price density process. Show that the risk-neutral pricing formula

$$D(t)V(t) = \mathbb{E}^{\mathbb{Q}}\left[D(T)V(T)|\mathcal{F}(t)\right], \quad 0 \leq t \leq T$$

may be rewritten as

$$D(t)Z(t)V(t) = \mathbb{E}\left[D(T)Z(T)V(T)|\mathcal{F}(t)\right],$$

where $Z(t)$ is the Radon-Nikodým derivative process

$$Z(t) = \exp\left\{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du\right\}$$

when the market price of risk process $\Theta(t)$ is given by

$$\Theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)},$$

and the conditional expectation on the right-hand side is taken under the actual probability measure $\mathbb{P}$, not the risk-neutral measure $\mathbb{Q}$.

In particular, if for some $A \in \mathcal{F}(T)$ a derivative security pays off $\mathbf{1}_A$ (i.e., pays 1 if A occurs and 0 if A does not occur), then the value of this derivative security at time zero is

$$\mathbb{E}\left[D(T)Z(T)\mathbf{1}_A\right].$$

The process $D(t)Z(t)$ is called the **state price density process**.

Solution: The Radon-Nikodým derivative process $Z(t)$ is $\mathcal{F}(t)$-measurable. Let $0 \leq t \leq T$. Based on Lemma 5.2.2,

$$\begin{aligned}
D(t)V(t) &= \mathbb{E}^{\mathbb{Q}}\left[D(T)V(T)|\mathcal{F}(t)\right] \\
&= \frac{1}{Z(t)}\mathbb{E}\left[D(T)V(T)Z(T)|\mathcal{F}(t)\right].
\end{aligned}$$

Hence,

$$D(t)Z(t)V(t) = \mathbb{E}\left[D(T)Z(T)V(T)|\mathcal{F}(t)\right].$$

The value of a derivative security at time zero is

$$\begin{aligned}
D(0)Z(0)V(0) &= \mathbb{E}\left[D(T)Z(T)V(T)|\mathcal{F}(0)\right] \\
&= \mathbb{E}\left[D(T)Z(T)V(T)\right],
\end{aligned}$$

or (given that $D(0)Z(0) = 1$)

$$V(0) = \mathbb{E}\left[D(T)Z(T)V(T)\right].$$

If for some $A \in \mathcal{F}(T)$ a derivative security pays off $\mathbf{1}_A$, then

$$V(0) = \mathbb{E}\left[D(T)Z(T)\mathbf{1}_A\right].$$

Radon Nikodym Derivative

Let P be the market prob measure and $\widetilde{P}$ be the risk-neutral prob measure. Assume

$$P(\omega) > 0, \widetilde{P}(\omega) > 0, \forall\omega \in \Omega$$

So that $P, \widetilde{P}$ are equivalent. The Radon-Nikodym Derivative of $\widetilde{P}$ w.r.t. P is

$$Z(\omega) = \frac{\widetilde{P}(\omega)}{P(\omega)}$$

Define the P-martingale

$$Z_k \triangleq \mathbb{E}[Z|\mathscr{F}_k]$$

If X is $\mathscr{F}_k$-measurable then $\widetilde{\mathbb{E}}[X] = \mathbb{E}[XZ_k]$ and for any $A \in \mathscr{F}_k$,

$$\widetilde{\mathbb{E}}[I_A X] = \mathbb{E}[Z_k I_A X]$$

or equivalently

$$\int_A X\, d\widetilde{P} = \int_A XZ_k\, dP$$

If X is $\mathscr{F}_k$-measurable and $0 \leq j \leq k$, then

$$\widetilde{\mathbb{E}}[X|\mathscr{F}_j] = \frac{1}{Z_j}\mathbb{E}[XZ_k|\mathscr{F}_j]$$

Q4

Let

$$c(t,x) = xN\left(d_+(T-t,x)\right) - Ke^{-r(T-t)}N\left(d_-(T-t,x)\right),$$

This option price assumes the underlying stock is a geometric Brownian motion with volatility $\sigma_1 > 0$. Suppose, however, that the underlying asset is really a geometric Brownian motion with volatility $\sigma_2 > \sigma_1 > 0$, i.e.,

$$dS(t) = \alpha S(t)dt + \sigma_2 S(t)dW(t).$$

Consequently, the market price of the call (as a function of σ_1) is incorrect.

We set up a portfolio whose value at each time t we denote by $X(t)$. We begin with $X(0) = 0$. At each time t, the portfolio is long one European call, is short $c_x(t, S(t))$ shares of stock, and thus has a cash position

$$X(t) - c(t, S(t)) + S(t)c_x(t, S(t)),$$

which is invested at the constant interest rate r. We also remove cash from this portfolio at a rate of $\frac{1}{2}(\sigma_2^2 - \sigma_1^2)S^2(t)c_{xx}(t, S(t))$. Therefore, the differential of the portfolio value is

$$dX(t) = dc - c_x dS + r\left[X - c + Sc_x\right]dt - \frac{1}{2}(\sigma_2^2 - \sigma_1^2)S^2 c_{xx}dt, \quad 0 \le t \le T.$$

Show that $X(t) = 0$ for all $t \in [0,T]$. In particular, because $c_{xx}(t, S(t)) > 0$ and $\sigma_2 > \sigma_1$, we have an arbitrage opportunity; we can start with zero initial capital, remove cash at a positive rate between 0 and T, and at time T have zero liability. (Hint: Compute $d\left(e^{-rt}X(t)\right)$.)

$$dc(t, S(t)) = c_t(t, S(t))dt + c_x(t, S(t))dS(t) + \frac{1}{2}c_{xx}(t, S(t))d\langle S(t)\rangle$$

$$= c_t(t, S(t))dt + c_x(t, S(t))dS(t) + \frac{1}{2}\sigma_1^2 c_{xx}(t, S(t))S^2(t)dt.$$

the following Black-Scholes-Merton partial differential equation holds:

$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma_1^2 x^2 c_{xx}(t,x) = rc(t,x),\ t \in [0,T],\ x > 0.$$

The differential $dc(t, S(t))$ can be further written as:

$$dc(t, S(t)) = \left[rc(t, S(t)) - rS(t)c_x(t, S(t)) - \frac{1}{2}\sigma_1^2 S^2(t)c_{xx}(t, S(t))\right]dt$$

$$+ c_x(t, S(t))dS(t) + \frac{1}{2}\sigma_2^2 S^2(t)c_{xx}(t, S(t))dt.$$

Using the Itô-Doob formula for an Itô process again and setting $f(t,x) = e^{-rt}x$, the differential $d(e^{-rt}X(t))$ is:

$$d\left(e^{-rt}X(t)\right)$$

$$= df(t, X(t)) = f_t dt + f_x dX + \frac{1}{2}f_{xx}dX(t)dX(t)$$

$$= -re^{-rt}X(t)dt + e^{-rt}dX(t)$$

$$= -re^{-rt}X(t)dt + e^{-rt}\left[dc - c_x dS\right] + e^{-rt}r\left[X - c + Sc_x\right]dt$$

$$- \frac{1}{2}e^{-rt}\left(\sigma_2^2 - \sigma_1^2\right)S^2 c_{xx}dt$$

$$= e^{-rt}\left[rc - rSc_x - \frac{1}{2}\sigma_1^2 S^2 c_{xx}\right]dt + e^{-rt}\left[c_x dS + \frac{1}{2}\sigma_2^2 S^2 c_{xx}dt\right]$$

$$- e^{-rt}c_x dS + e^{-rt}\left[-c + Sc_x\right]dt - \frac{1}{2}e^{-rt}\left(\sigma_2^2 - \sigma_1^2\right)S^2 c_{xx}dt$$

$$= 0.$$

We conclude that $e^{-rt}X(t)$ remains as a constant over time t. Since $X(0) = 0$, $X(t) = 0$.

Normal and log-Normal

A log-normal distribution is a continuous probability distribution of a random variable whose logarithm is Gaussian Distribution(Normal Distribution), denoted by $\ln X \sim N(\mu, \sigma^2)$

the pdf is

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}}\exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right)$$

The moments are

$$\mathbb{E}[X] = e^{\mu + \sigma^2/2} \qquad \mathrm{Var}[X] = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}$$

Q5

Hedging a cash flow. Let $W(t)$, $0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t)$, $0 \le t \le T$, be the filtration generated by this Brownian motion. Let the mean rate of return $\alpha(t)$, the interest rate $R(t)$, and the volatility $\sigma(t)$ be adapted processes, and assume that $\sigma(t)$ is never zero. Consider a stock price process whose differential is given by

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), \quad 0 \le t \le T.$$

Suppose an agent must pay a cash flow at rate $C(t)$ at each time t, where $C(t)$, $0 \le t \le T$, is an adapted process. If the agent holds $\Delta(t)$ shares of stock at each time t, then the differential of her portfolio value will be

$$dX(t) = \Delta(t)dS(t) + R(t)\left(X(t) - \Delta(t)S(t)\right)dt - C(t)dt.$$

Show that there is a nonrandom value of $X(0)$ and a portfolio process $\Delta(t)$, $0 \le t \le T$, such that $X(T) = 0$ almost surely.

Solution: The differential of the portfolio value is

$$dX(t) = \Delta(t)dS(t) + R(t)\left(X(t) - \Delta(t)S(t)\right)dt - C(t)dt$$

$$= R(t)X(t)dt + \left(\alpha(t) - R(t)\right)\Delta(t)S(t)dt + \sigma(t)\Delta(t)S(t)dW(t) - C(t)dt$$

$$= R(t)X(t)dt + \Delta(t)S(t)\sigma(t)\left[\Theta(t)dt + dW(t)\right] - C(t)dt,$$

where

$$\Theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}$$

is the usual market price of risk. Further, we define

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)du$$

and use Girsanov's Theorem to change to a measure $\widetilde{\mathbb{P}}$ under which $\widetilde{W}$ is a Brownian motion, so we may rewrite $dX(t)$ as

$$dX(t) = R(t)X(t)dt + \Delta(t)S(t)\sigma(t)d\widetilde{W}(t) - C(t)dt.$$

The discount process $D(t)$ is defined by

$$D(t) = e^{-\int_0^t R(s)ds}.$$

Its differential is

$$dD(t) = -R(t)D(t)dt.$$

Using Itô's product rule, we compute the differential of the discounted portfolio value

$$d[D(t)X(t)] = X(t)dD(t) + D(t)dX(t) + dD(t)dX(t)$$

$$= -R(t)D(t)X(t)dt + D(t)\left[R(t)X(t)dt + \Delta(t)S(t)\sigma(t)d\widetilde{W}(t) - C(t)dt\right]$$

$$= \Delta(t)D(t)S(t)\sigma(t)d\widetilde{W}(t) - D(t)C(t)dt.$$

Integrating both sides gives

$$D(t)X(t) = X(0) + \int_0^t \Delta(u)D(u)S(u)\sigma(u)d\widetilde{W}(u) - \int_0^t D(u)C(u)du,$$

or

$$D(t)X(t) + \int_0^t D(u)C(u)du = X(0) + \int_0^t \Delta(u)D(u)S(u)\sigma(u)d\widetilde{W}(u).$$

Under the risk-neutral measure $\widetilde{\mathbb{P}}$,

$$D(t)X(t) + \int_0^t D(u)C(u)du = \mathbb{E}^{\widetilde{\mathbb{P}}}\left[D(T)X(T) + \int_0^T D(u)C(u)du \middle| \mathcal{F}(t)\right]$$

is a martingale; and using the Martingale Representation Theorem, there is an adapted process $\widetilde{\Gamma}(u)$, $0 \le u \le T$, such that

$$D(t)X(t) + \int_0^t D(u)C(u)du = X(0) + \int_0^t \widetilde{\Gamma}(u)d\widetilde{W}(u), \quad 0 \le t \le T.$$

Particularly, to have $X(T) = 0$ almost surely, we have

$$X(0) + \int_0^T \widetilde{\Gamma}(u)d\widetilde{W}(u) = \mathbb{E}^{\widetilde{\mathbb{P}}}\left[D(T)X(T) + \int_0^T D(u)C(u)du \middle| \mathcal{F}(t)\right]$$

$$= \mathbb{E}^{\widetilde{\mathbb{P}}}\left[\int_0^T D(u)C(u)du \middle| \mathcal{F}(t)\right].$$

Hence, the initial value of the portfolio is

$$X(0) = \mathbb{E}^{\widetilde{\mathbb{P}}}\left[\int_0^T D(u)C(u)du\right],$$

and the portfolio process is

$$\Delta(t) = \frac{\widetilde{\Gamma}(t)}{D(t)S(t)\sigma(t)}.$$

Q6

Consider a stock, modeled as a generalized geometric Brownian motion, that pays dividends continuously over time at a rate $A(t)$ per unit time. Here $A(t)$, $0 \le t \le T$, is a nonnegative adapted process. Dividends paid by a stock reduce its value, and so we shall take as our model of the stock price:

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t) - A(t)S(t)dt, \quad 0 \le t \le T.$$

An agent who holds the stock receives both the capital gain or loss due to stock price movements and the continuously paying dividend. Thus, if $\Delta(t)$ is the number of shares held at time t, then the portfolio value $X(t)$ satisfies:

$$dX(t) = \Delta(t)dS(t) + A(t)\Delta(t)S(t)dt + R(t)\left[X(t) - \Delta(t)S(t)\right]dt.$$

(ii) Define

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)du$$

and use Girsanov's Theorem to change to a measure $\widetilde{\mathbb{P}}$ under which $\widetilde{W}$ is a Brownian motion. Show that the discounted portfolio value satisfies:

$$d[D(t)X(t)] = \Delta(t)D(t)S(t)\sigma(t)d\widetilde{W},$$

(iii) If we now wish to hedge a short position in a derivative security paying $V(T)$ at time T, where $V(T)$ is an $\mathscr{F}(T)$-measurable random variable, we will need to choose the initial capital $X(0)$ and the portfolio process $\Delta(t)$, $0 \le t \le T$, so that $X(T) = V(T)$. Because $D(t)X(t)$ is a martingale under $\widetilde{\mathbb{P}}$, we must have:

$$D(t)X(t) = \widetilde{\mathbb{E}}[D(T)V(T)|\mathscr{F}(t)],\ 0 \le t \le T.$$

The value $X(t)$ of this portfolio at each time t is the value (price) of the derivative security at that time, which we denote by $V(t)$. Making this replacement in the formula above, we obtain the risk-neutral pricing formula:

$$D(t)V(t) = \widetilde{\mathbb{E}}[D(T)V(T)|\mathscr{F}(t)],\ 0 \le t \le T.$$

Show that there is a nonrandom value of $X(0)$ and a portfolio process $\Delta(t)$, $0 \le t \le T$, such that $X(T) = V(T)$ almost surely.

(iv) Show that the discounted stock price $D(t)S(t)$ is not a martingale under the risk-neutral measure $\widetilde{\mathbb{P}}$.

(v) Show that the process

$$e^{\int_0^t A(u)du}D(t)S(t)$$

is a martingale under the risk-neutral measure $\widetilde{\mathbb{P}}$. This is the interest-rate-discounted value at time t of a bond that initially purchases one share of the stock and continuously reinvests the dividends into the bond.

(iii) We first show $D(t)V(t)$ is a martingale using iterated conditioning under $\widetilde{\mathbb{P}}$. For $0 \le s \le t \le T$,

$$\widetilde{\mathbb{E}}\left[D(t)V(t)|\mathscr{F}(s)\right] = \widetilde{\mathbb{E}}\left[\widetilde{\mathbb{E}}\left[D(T)V(T)|\mathscr{F}(t)\right]|\mathscr{F}(s)\right]$$

$$= \widetilde{\mathbb{E}}\left[D(T)V(T)|\mathscr{F}(s)\right] = D(s)V(s).$$

Since $D(t)V(t)$ is a $\widetilde{\mathbb{P}}$-martingale, $D(t)V(t)$ has a representation as (recall that $D(0)V(0) = V(0)$):

$$D(t)V(t) = V(0) + \int_0^t \widetilde{\Gamma}(u)d\widetilde{W}(u),\ 0 \le t \le T.$$

On the other hand, as derived in Part (ii), the differential of the discounted portfolio value is given by:

$$d\left[D(t)X(t)\right] = \Delta(t)\sigma(t)D(t)S(t)d\widetilde{W}(t),\ 0 \le t \le T.$$

Hence:

$$D(t)X(t) = X(0) + \int_0^t \Delta(u)\sigma(u)D(u)S(u)d\widetilde{W}(u),\ 0 \le t \le T.$$

In order to have $X(t) = V(t)$ for all t, we should choose $X(0) = V(0)$ and choose $\Delta(t)$ to satisfy:

$$\Delta(t)\sigma(t)D(t)S(t) = \widetilde{\Gamma}(t),\ 0 \le t \le T,$$

which is equivalent to:

$$\Delta(t) = \frac{\widetilde{\Gamma}(t)}{\sigma(t)D(t)S(t)},\ 0 \le t \le T.$$

Q7

Let $W(t)$, $0 \le t \le T$, be a Brownian motion defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Fix $m > 0$ and $\mu \in \mathbb{R}$. For $0 \le t < \infty$, define:

$$X(t) = \mu t + W(t), \quad \tau_m = \inf\{t \ge 0; X(t) = m\}.$$

As usual, we set $\tau_m = \infty$ if $X(t)$ never reaches the level m. Let σ be a positive number and set:

$$Z(t) = \exp\left\{\sigma X(t) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right) t\right\}.$$

(ii) Use (i) to conclude that:

$$\mathbb{E}\left[\exp\left\{\sigma X(t\wedge\tau_m) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t\wedge\tau_m)\right\}\right] = 1,\ t \ge 0.$$

(iii) Suppose $\mu \ge 0$. Show that, for $\sigma > 0$,

$$\mathbb{E}\left[\exp\left\{\sigma m - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\mathbf{1}_{\{\tau_m<\infty\}}\right] = 1,$$

where the notation $\mathbf{1}_{\{\tau_m<\infty\}}$ is used to indicate the random variable that takes the value 1 if $\tau_m < \infty$ and otherwise takes the value zero. Use this fact to show $\mathbb{P}(\tau_m < \infty) = 1$.

(iv) Suppose $\mu < 0$. Show that, for $\sigma > -2\mu$,

$$\mathbb{E}\left[\exp\left\{\sigma m - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\mathbf{1}_{\{\tau_m<\infty\}}\right] = 1.$$

Use this fact to show $\mathbb{P}(\tau_m < \infty) = e^{-2m|\mu|}$, which is strictly less than one.

(ii) Given that:

$$Z(t\wedge\tau_m) = \begin{cases} Z(t), & \text{if } t \le \tau_m,\\ Z(\tau_m), & \text{if } t > \tau_m.\end{cases}$$

We prove that $Z(t\wedge\tau_m)$ is a martingale.

(a) When $0 \le s \le t \le \tau_m$,

$$\mathbb{E}[Z(t\wedge\tau_m)|\mathscr{F}(s)] = \mathbb{E}[Z(t)|\mathscr{F}(s)] = Z(s) = Z(s\wedge\tau_m).$$

(b) When $0 \le s \le \tau_m \le t$,

$$\mathbb{E}[Z(t\wedge\tau_m)|\mathscr{F}(s)] = \mathbb{E}[Z(\tau_m)|\mathscr{F}(s)] = Z(s) = Z(s\wedge\tau_m).$$

(c) When $0 \le \tau_m \le s \le t$,

$$\mathbb{E}[Z(t\wedge\tau_m)|\mathscr{F}(s)] = \mathbb{E}[Z(\tau_m)|\mathscr{F}(s)] = Z(\tau_m) = Z(s\wedge\tau_m).$$

(iii) For a fixed $m > 0$ and $\sigma > 0$, $X(t)$ is always at or below level m for $t \le \tau_m$, and so:

$$0 \le \exp\{\sigma X(t\wedge\tau_m)\} \le e^{\sigma m}.$$

- If $\tau_m < \infty$: The term $\exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t\wedge\tau_m)\right\}$ is equal to $\exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}$ for large enough t.

- If $\tau_m = \infty$: The term $\exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t\wedge\tau_m)\right\}$ is equal to $\exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)t\right\}$, and as $t\to\infty$, this converges to zero, provided that $\mu \ge 0$ and $\sigma > 0$.

We capture these two cases by writing:

$$\lim_{t\to\infty}\exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t\wedge\tau_m)\right\} = \mathbf{1}_{\{\tau_m<\infty\}}\exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}.$$

- If $\tau_m < \infty$, then $\exp\{\sigma X(t\wedge\tau_m)\} = \exp\{\sigma X(\tau_m)\} = \exp\{\sigma m\}$ when t becomes large enough.

- If $\tau_m = \infty$, we do not know what happens to $\exp\{\sigma X(t\wedge\tau_m)\}$ as $t\to\infty$, but we at least know that this term is bounded because $0 \le \exp\{\sigma X(t\wedge\tau_m)\} \le e^{\sigma m}$ as derived earlier. That is enough to ensure that the product of $\exp\{\sigma X(t\wedge\tau_m)\}$ and $\exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)t\right\}$ has a limit zero in this case.

In conclusion, we have:

$$\lim_{t\to\infty}\mathbb{E}\left[\exp\left\{\sigma X(t\wedge\tau_m) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t\wedge\tau_m)\right\}\right]$$

$$= \mathbb{E}\left[\mathbf{1}_{\{\tau_m<\infty\}}\exp\left\{\sigma m - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\right].$$

We can now take the limit in:

$$\mathbb{E}\left[\exp\left\{\sigma X(t\wedge\tau_m) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t\wedge\tau_m)\right\}\right] = 1,\ t \ge 0$$

to obtain:

$$\mathbb{E}\left[\exp\left\{\sigma m - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\mathbf{1}_{\{\tau_m<\infty\}}\right] = 1,$$

or:

$$\mathbb{E}\left[\mathbf{1}_{\{\tau_m<\infty\}}\exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\right] = \exp\{-\sigma m\}.$$

The above equation holds when $m > 0$, $\mu \ge 0$, and $\sigma > 0$. We may not substitute $\sigma = 0$ into this equation. But since it holds for every positive σ, we may take the limit on both sides as $\sigma \to 0^+$. Using the Monotone Convergence Theorem, this yields:

$$\mathbb{P}(\tau_m < \infty) = 1.$$

Q7

(iv)

Now $m > 0$, $\mu < 0$, and $\sigma > -2\mu$. $X(t)$ is always at or below level m for $t \le \tau_m$, and so:

$$0 \le \exp\{\sigma X(t\wedge\tau_m)\} \le e^{\sigma m}.$$

- If $\tau_m < \infty$: The term:

$$\exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t\wedge\tau_m)\right\} = \exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}$$

for large enough t.

- On the other hand, if $\tau_m = \infty$: The term $\exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t\wedge\tau_m)\right\}$ is equal to $\exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)t\right\}$, and as $t\to\infty$, this converges to zero, provided that $\mu < 0$ and $\sigma > -2\mu$.

We capture these two cases by writing:

$$\lim_{t\to\infty}\exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t\wedge\tau_m)\right\} = \mathbf{1}_{\{\tau_m<\infty\}}\exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}.$$

- If $\tau_m < \infty$, then $\exp\{\sigma X(t\wedge\tau_m)\} = \exp\{\sigma X(\tau_m)\} = \exp\{\sigma m\}$ when t becomes large enough.

- If $\tau_m = \infty$, then we do not know what happens to $\exp\{\sigma X(t\wedge\tau_m)\}$ as $t\to\infty$, but we at least know that this term is bounded because of $0 \le \exp\{\sigma X(t\wedge\tau_m)\} \le e^{\sigma m}$, derived earlier. That is enough to ensure that the product of $\exp\{\sigma X(t\wedge\tau_m)\}$ and $\exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)t\right\}$ has limit zero in this case.

In conclusion, we have:

$$\lim_{t\to\infty}\exp\left\{\sigma X(t\wedge\tau_m) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t\wedge\tau_m)\right\}$$

$$= \mathbf{1}_{\{\tau_m<\infty\}}\exp\left\{\sigma m - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}.$$

We can now take the limit in:

$$\mathbb{E}\left[\exp\left\{\sigma X(t\wedge\tau_m) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t\wedge\tau_m)\right\}\right] = 1,\ t \ge 0,$$

to obtain:

$$\mathbb{E}\left[\exp\left\{\sigma m - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\mathbf{1}_{\{\tau_m<\infty\}}\right] = 1,$$

or:

$$\mathbb{E}\left[\mathbf{1}_{\{\tau_m<\infty\}}\exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\right] = \exp\{-\sigma m\}.$$

The above equation holds when $m > 0$, $\mu < 0$, and $\sigma > -2\mu$. We may not substitute $\sigma = -2\mu$ into this equation. But since it holds for every positive σ with $\sigma > -2\mu$, we may take the limit on both sides as $\sigma \to (-2\mu)^+$. Using the Monotone Convergence Theorem, this yields:

$$\mathbb{P}(\tau_m < \infty) = e^{-2m|\mu|}.$$

Q9

Show that the distribution of the scaled random walk $W^{(n)}(t)$ ($t \ge 0$ is fixed) evaluated at time t converges to the normal distribution with mean zero and variance t as $n\to\infty$.

If t is such that nt is an integer, then the moment-generating function for $W^{(n)}(t)$ is:

$$M_n(u) = \mathbb{E}e^{uW^{(n)}(t)} = \mathbb{E}\exp\left(\frac{u}{\sqrt{n}}M_{nt}\right) = \mathbb{E}\exp\left(\frac{u}{\sqrt{n}}\sum_{j=1}^{nt}X_j\right).$$

Simplifying further:

$$M_n(u) = \mathbb{E}\prod_{j=1}^{nt}\exp\left(\frac{u}{\sqrt{n}}X_j\right).$$

Because the random variables are independent, the right-hand side of the above equation may be written as:

$$\mathbb{E}\prod_{j=1}^{nt}\exp\left(\frac{u}{\sqrt{n}}X_j\right) = \prod_{j=1}^{nt}\mathbb{E}\exp\left(\frac{u}{\sqrt{n}}X_j\right).$$

Using the fact that X_j are i.i.d. with equal probabilities for ±1, we get:

$$\mathbb{E}\exp\left(\frac{u}{\sqrt{n}}X_j\right) = \frac{1}{2}e^{\frac{u}{\sqrt{n}}} + \frac{1}{2}e^{\frac{-u}{\sqrt{n}}} \implies M_n(u) = \left(\frac{1}{2}e^{\frac{u}{\sqrt{n}}} + \frac{1}{2}e^{\frac{-u}{\sqrt{n}}}\right)^{nt}$$

We need to show that, as $n\to\infty$,

$$M_n(u) \to \exp\left(\frac{u^2t}{2}\right).$$

To show that the moment-generating function $M_n(u)$ converges to $M(u) = e^{\frac{1}{2}u^2t}$, it suffices to consider the logarithm of $M_n(u)$ and show that:

$$\ln M_n(u) = nt\ln\left(\frac{1}{2}e^{\frac{u}{\sqrt{n}}} + \frac{1}{2}e^{-\frac{u}{\sqrt{n}}}\right)$$

converges to $\ln M(u) = \frac{1}{2}u^2t$.

For this final computation, we make the change of variable $x = \frac{1}{\sqrt{n}}$ so that:

$$\lim_{n\to\infty}\ln M_n(u) = t\lim_{x\to0+}\frac{\left(\frac{1}{2}e^{ux} + \frac{1}{2}e^{-ux}\right) - 1}{x^2}.$$

If we were to substitute $x = 0$ into the expression on the right-hand side, we would obtain $\frac{0}{0}$. In this situation, we may use L'Hôpital's rule. Therefore:

$$\lim_{n\to\infty}\ln M_n(u) = t\lim_{x\to0+}\frac{\frac{1}{2}ue^{ux} - \frac{1}{2}ue^{-ux}}{2x\left(\frac{1}{2}e^{ux} + \frac{1}{2}e^{-ux}\right)}$$

$$= t\lim_{x\to0+}\frac{\frac{1}{2}u^2e^{ux} + \frac{1}{2}u^2e^{-ux}}{2\left(\frac{1}{2}e^{ux} + \frac{1}{2}e^{-ux}\right) + 2x\left(\frac{1}{2}ue^{ux} - \frac{1}{2}ue^{-ux}\right)}$$

$$= t\lim_{x\to0+}\frac{\frac{1}{2}u^2e^{ux} + \frac{1}{2}u^2e^{-ux}}{1} = \frac{1}{2}u^2t.$$

QED

Q8

Two random variables X_i $(i = 1, 2)$ are said to be jointly normal if they have the joint density $f_{X_1,X_2}(x_1, x_2)$:

$$\frac{\exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]\right)}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

where $\sigma_1 > 0$, $\sigma_2 > 0$, $|\rho| < 1$, and μ_1, μ_2 are real numbers.

(a) Find the marginal density function of the random variable X_i, $i = 1, 2$.

(b) Show that the random variable X_1 and X_2 have a joint moment generating function:

$$\mathbb{E}\left[e^{u_1X_1+u_2X_2}\right] = e^{\mu_1u_1+\mu_2u_2+\frac{1}{2}\left(\sigma_1^2u_1^2+2\rho\sigma_1\sigma_2u_1u_2+\sigma_2^2u_2^2\right)}.$$

(c) Find the distribution of the random variable $Y = a_1X_1+a_2X_2$ where a_1 and a_2 are real-valued constants.

(a) Marginal density function of X_1

Let $f_{X_1}(x_1)$ be the marginal density function of the random variable X_1. Then:

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, x_2)\,dx_2$$

Compute we get

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]} dx_2 \\
&= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\frac{(x_1-\mu_1)^2}{\sigma_1^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[-\frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]} dx_2 \\
&= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\frac{(x_1-\mu_1)^2}{\sigma_1^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_2-\mu_2}{\sigma_2} - \frac{\rho(x_1-\mu_1)}{\sigma_1}\right)^2 - \frac{\rho^2(x_1-\mu_1)^2}{\sigma_1^2}\right]} dx_2 \\
&= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\frac{x_2-\mu_2}{\sigma_2\sqrt{1-\rho^2}} - \frac{\rho(x_1-\mu_1)}{\sigma_1\sqrt{1-\rho^2}}\right]^2} dx_2 = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}}.
\end{aligned}$$

Passage Time

From:

$$1 = Z(0) = \mathbb{E}Z(t\wedge\tau_m) = \mathbb{E}\left[\exp\left\{\sigma W(t\wedge\tau_m) - \frac{1}{2}\sigma^2(t\wedge\tau_m)\right\}\right],$$

by taking the limit on both sides as $t\to\infty$, we can derive:

$$\mathbb{E}\left[\mathbf{1}_{\{\tau_m<\infty\}}\exp\left\{-\frac{1}{2}\sigma^2\tau_m\right\}\right] = e^{-\sigma m},$$

where the notation $\mathbf{1}_{\{\tau_m<\infty\}}$ is used to indicate the random variable that takes the value 1 if $\tau_m < \infty$ and otherwise takes the value zero.

We may not substitute $\sigma = 0$ into equation (27), but since it holds for every positive σ, we may take the limit on both sides as $\sigma \to 0^+$. By using the Monotone Convergence Theorem, we obtain:

$$RHS = \lim_{\sigma\to0^+} e^{-\sigma m} = 1$$

and LHS is

$$\lim_{\sigma\to0^+}\mathbb{E}\left[\mathbf{1}_{\{\tau_m<\infty\}}\exp\left\{-\frac{1}{2}\sigma^2\tau_m\right\}\right] = \mathbb{E}\lim_{\sigma\to0^+}\left[\mathbf{1}_{\{\tau_m<\infty\}}\exp\left\{-\frac{1}{2}\sigma^2\tau_m\right\}\right].$$

This simplifies to:

$$\mathbb{E}\left[\mathbf{1}_{\{\tau_m<\infty\}}\right] = \mathbb{P}(\tau_m < \infty).$$

Thus:

$$\mathbb{P}(\tau_m < \infty) = 1.$$

From $\mathbb{P}\{\tau_m < \infty\} = 1$, we know that τ_m is finite with probability one (we say τ_m is finite almost surely), so we may drop the indicator of this event in (27) to obtain:

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}\sigma^2\tau_m\right\}\right] = e^{-\sigma m}.$$

Theorem 3.6.2 For $m \in \mathbb{R}$, the first passage time of Brownian motion to level m is finite almost surely, and the Laplace transform of its distribution is given by:

$$\mathbb{E}\left[\exp\{-\alpha\tau_m\}\right] = e^{-|m|\sqrt{2\alpha}} \quad \text{for all } \alpha > 0.$$

Remark: Differentiation of it with respect to α results in:

$$\mathbb{E}\left[\tau_m\exp\{-\alpha\tau_m\}\right] = \frac{|m|}{\sqrt{2\alpha}}e^{-|m|\sqrt{2\alpha}} \quad \text{for all } \alpha > 0.$$

Letting $\alpha \to 0^+$, we obtain $\mathbb{E}\tau_m = \infty$ so long as $m \neq 0$.

Reflection Principle

We consider the paths that reach level m prior to time t and are at or below a level w $(w \le m)$ at time t, and we consider their reflections, which are at or above $2m - w$ at time t. This leads to the key reflection equality:

$$\mathbb{P}\{\tau_m \le t, W(t) \le w\} = \mathbb{P}\{W(t) \ge 2m - w\}, \quad w \le m,\ m > 0.$$

Theorem 3.7.1 For all $m \neq 0$, the random variable τ_m has cumulative distribution function and density:

CDF:

$$\mathbb{P}\{\tau_m \le t\} = \frac{2}{\sqrt{2\pi}}\int_{|m|/\sqrt{t}}^{\infty} e^{-y^2/2}\,dy, \quad t \ge 0. \quad (32)$$

Density:

$$f_{\tau_m}(t) = \frac{d}{dt}\mathbb{P}\{\tau_m \le t\} = \frac{|m|}{t\sqrt{2\pi t}}e^{-m^2/2t}, \quad t \ge 0. \quad (33)$$

PROOF: We first consider the case $m > 0$. We substitute $w = m$ into the reflection formula (31) to obtain:

$$\mathbb{P}\{\tau_m \le t, W(t) \le m\} = \mathbb{P}\{W(t) \ge m\}.$$

On the other hand, if $W(t) \ge m$, then we are guaranteed that $\tau_m \le t$. In other words:

$$\mathbb{P}\{\tau_m \le t, W(t) \ge m\} = \mathbb{P}\{W(t) \ge m\}.$$

Adding these two equations, we obtain the cumulative distribution function for τ_m:

$$\mathbb{P}\{\tau_m \le t\} = \mathbb{P}\{\tau_m \le t, W(t) \le m\} + \mathbb{P}\{\tau_m \le t, W(t) \ge m\}.$$

Thus:

$$\mathbb{P}\{\tau_m \le t\} = 2\mathbb{P}\{W(t) \ge m\} = \frac{2}{\sqrt{2\pi}}\int_m^{\infty} e^{-x^2/2t}\,dx.$$

We make the change of variable $y = \frac{x}{\sqrt{t}}$ in the integral, and this leads to (32) when m is positive. If m is negative, then τ_m and $\tau_{|m|}$ have the same distribution, and (32) provides the cumulative distribution function of the latter.

Finally, (33) is obtained by differentiating (32).

We define the **maximum to date** for Brownian motion to be

$$M(t) = \max_{0\le s\le t} W(s)$$

For positive m, we have $M(t) \ge m$ if and only if $\tau_m \le t$. The reflection equality:

$$\mathbb{P}\{\tau_m \le t, W(t) \le w\} = \mathbb{P}\{W(t) \ge 2m - w\}, \quad w \le m,\ m > 0$$

can be rewritten as:

$$\mathbb{P}\{M(t) \ge m, W(t) \le w\} = \mathbb{P}\{W(t) \ge 2m - w\}, \quad w \le m,\ m > 0.$$

From this, we can obtain the joint distribution of $W(t)$ and $M(t)$.

Theorem 3.7.3

For $t > 0$, the joint distribution of $(W(t), M(t))$ is:

$$f_{M(t),W(t)}(m, w) = \frac{2(2m-w)}{t\sqrt{2\pi t}}e^{-\frac{(2m-w)^2}{2t}}, \quad w \le m,\ m > 0.$$

Corollary 3.7.4

The conditional distribution of $M(t)$ given $W(t) = w$ is:

$$f_{M(t)|W(t)}(m \mid w) = \frac{2(2m-w)}{t}e^{-\frac{2m(m-w)}{t}}, \quad w \le m,\ m > 0.$$

BSM

$$\begin{aligned}
&\mathbb{E}\left[e^{-rT}(S(T)-K)^+\right] \\
&= \int_{-\infty}^{\infty} e^{-rT}\left(S(0)e^{\left(r-\frac{1}{2}\sigma^2\right)T+x} - K\right)^+ \frac{1}{\sigma\sqrt{2\pi T}}e^{-\frac{x^2}{2\sigma^2T}}\,dx \\
&= \int_{-\infty}^{\infty} \left(S(0)e^{-\frac{1}{2}\sigma^2T+\sigma x} - e^{-rT}K\right)^+ \frac{1}{\sigma\sqrt{2\pi T}}e^{-\frac{x^2}{2\sigma^2T}}\,dx \\
&= \int_{\ln\frac{K}{S(0)}-\left(r-\frac{1}{2}\sigma^2\right)T}^{\infty} \left(S(0)e^{-\frac{1}{2}\sigma^2T+x} - e^{-rT}K\right) \frac{1}{\sigma\sqrt{2\pi T}}e^{-\frac{x^2}{2\sigma^2T}}\,dx \\
&= S(0)\int_{\ln\frac{K}{S(0)}-\left(r-\frac{1}{2}\sigma^2\right)T}^{\infty} e^{-\frac{1}{2}\sigma^2T+x}\frac{1}{\sigma\sqrt{2\pi T}}e^{-\frac{x^2}{2\sigma^2T}}\,dx \\
&\quad - Ke^{-rT}\int_{\ln\frac{K}{S(0)}-\left(r-\frac{1}{2}\sigma^2\right)T}^{\infty} \frac{1}{\sigma\sqrt{2\pi T}}e^{-\frac{x^2}{2\sigma^2T}}\,dx.
\end{aligned}$$

We use $-x$ to replace x and obtain

$$\begin{aligned}
&\mathbb{E}\left[e^{-rT}(S(T)-K)^+\right] \\
&= S(0)\int_{-\infty}^{-\ln\frac{K}{S(0)}+(r-\frac{1}{2}\sigma^2)T} e^{-\frac{1}{2}\sigma^2T-x}\frac{1}{\sigma\sqrt{2\pi T}}e^{-\frac{x^2}{2\sigma^2T}}\,dx - Ke^{-rT}\int_{-\infty}^{-\ln\frac{K}{S(0)}+(r-\frac{1}{2}\sigma^2)T}\frac{1}{\sigma\sqrt{2\pi T}}e^{-\frac{x^2}{2\sigma^2T}}\,dx \\
&= S(0)\int_{-\infty}^{\ln\frac{S(0)}{K}+(r-\frac{1}{2}\sigma^2)T}\frac{1}{\sqrt{2\pi T}}e^{-\frac{1}{2}\sigma^2T-x-\frac{x^2}{2\sigma^2T}}\,dx - Ke^{-rT}\int_{-\infty}^{\ln\frac{S(0)}{K}+(r-\frac{1}{2}\sigma^2)T}\frac{1}{\sigma\sqrt{2\pi T}}e^{-\frac{x^2}{2\sigma^2T}}\,dx \\
&= S(0)\int_{-\infty}^{\ln\frac{S(0)}{K}+(r-\frac{1}{2}\sigma^2)T}\frac{1}{\sigma\sqrt{2\pi T}}e^{-\frac{1}{2}\left(\sigma\sqrt{T}+\frac{x}{\sigma\sqrt{T}}\right)^2}dx - Ke^{-rT}\int_{-\infty}^{\ln\frac{S(0)}{K}+(r-\frac{1}{2}\sigma^2)T}\frac{1}{\sigma\sqrt{2\pi T}}e^{-\frac{x^2}{2\sigma^2T}}\,dx.
\end{aligned}$$

In the first integral, we make the change of variables $z = \sigma\sqrt{T} + \frac{x}{\sigma\sqrt{T}}$,

$$\begin{aligned}
&S(0)\int_{-\infty}^{\ln\frac{S(0)}{K}+(r-\frac{1}{2}\sigma^2)T}\frac{1}{\sigma\sqrt{2\pi T}}e^{-\frac{1}{2}\left(\sigma\sqrt{T}+\frac{x}{\sigma\sqrt{T}}\right)^2}dx \\
&= S(0)\int_{-\infty}^{\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{S(0)}{K}+(r+\frac{1}{2}\sigma^2)T\right]}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz \\
&= S(0)N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{S(0)}{K}+(r+\frac{1}{2}\sigma^2)T\right]\right) = S(0)N(d_+(T, S(0))).
\end{aligned}$$