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New tighter polynomial length formulations for the asymmetric traveling salesman problem with and without precedence constraints

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Abstract

We propose a new formulation for the asymmetric traveling salesman problem, with and without precedence relationships, which employs a polynomial number of subtour elimination constraints that imply an exponential subset of certain relaxed Dantzig–Fulkerson–Johnson subtour constraints. Promising computational results are presented, particularly in the presence of precedence constraints.

Keywords: Asymmetric traveling salesman problem; Precedence constraints; Subtour elimination constraints

1. Introduction

The traveling salesman problem (TSP) is perhaps the most widely researched combinatorial optimization problem. The TSP can be stated as follows: Given a finite set of cities $N = \{1, 2, \ldots, n\}$ and the cost of travel c_{ij} between each pair of cities $i, j \in N$, find a tour that visits each city exactly once, while minimizing the total cost of travel. In this paper, we address the *asymmetric traveling salesman problem* (ATSP) for which c_{ij} and c_{ji} might differ for any pair $i, j \in N$.

Mathematical programming formulations for the ATSP involve the assignment constraints along with subtour elimination constraints (SECs), besides the binary restrictions on the decision variables (see [5,10–12]). In this paper, we present a new formulation for ATSP based on modeling the subtour elimination constraints using a polynomial number of restrictions that imply an exponential subset of certain relaxed *Dantzig–Fulkerson–Johnson* (DFJ) [5] subtour elimination constraints. We establish the validity of this formulation and show that it is tighter than a similar ATSP formulation recently proposed by Gouveia and Pires [8,9]. (In the sequel, the formulation by Gouveia and Pires is referred to as RMTZ, as in their paper.) Furthermore, we provide an extension of the proposed formulation to model the *precedence constrained asymmetric traveling salesman problem* (PCATSP), also known as the *sequential ordering problem* (see [1,3,7]). As the name suggests, the PCATSP is similar to the ATSP except for some additional precedence constraints that require certain cities to be visited before others. In addition to comparisons

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with the RMTZ model, we present computational results to compare the linear programming relaxations and the branch-and-bound effort required for our formulation with that of the lifted *Miller–Tucker–Zemlin* (MTZ) [11] formulations of Desrochers and Laporte (DL) [6] and Sherali and Driscoll (SD) [13] (both of which imply the DFJ two-city SECs), as well as for these lifted MTZ formulations that additionally incorporate the DFJ three-city SECs.

The remainder of this paper is organized as follows. In Section 2, we present the proposed formulations for ATSP and PCATSP, and establish their validity and relationship with the formulation of Gouveia and Pires [8,9]. We also derive certain valid lifted inequalities that serve to tighten these proposed formulations. In Section 3, we present the results of our experimentation, in which we compare the LP relaxation bounds and branch-and-bound effort of the proposed formulations with that of other well-known formulations in the literature. We conclude in Section 4 with some recommendations for future research.

2. Proposed formulation: ATSPxy

To present the new formulation for the ATSP, let us designate city 1 as the base city, and define the familiar assignment binary variables x_{ij}, $\forall i,\ j \in N,\ i \neq j$, as follows:

$$x_{ij} = \begin{cases} 1 & \text{if city } i \text{ precedes city } j \text{ immediately} \\ & \text{in a tour,} \\ 0 & \text{otherwise.} \end{cases}$$

In addition, we define another set of *continuous* variables y_{ij}, $\forall i, j = 2, \ldots, n, i \neq j$, which nonetheless have a binary connotation, as follows:

$$y_{ij} = \begin{cases} 1 & \text{if city } i \text{ precedes city } j \\ & \text{(not necessarily immediately) in a tour} \\ 0 & \text{otherwise.} \end{cases}$$

Accordingly, consider the following model for the ATSP, which is justified in the sequel.

$$\text{ATSPxy:} \quad \text{minimize} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} c_{ij} x_{ij}$$

subject to

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} x_{ij} = 1, \quad \forall j = 1, \ldots, n, \tag{1}$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} x_{ij} = 1, \quad \forall i = 1, \ldots, n, \tag{2}$$

$$y_{ij} \geqslant x_{ij}, \quad \forall i, j = 2, \ldots, n, \ i \neq j, \tag{3}$$

$$y_{ij} + y_{ji} = 1, \quad \forall i, j = 2, \ldots, n, \ i \neq j \tag{4}$$

$$y_{ij} + y_{jk} + y_{ki} \leqslant 2, \quad \forall i, j, k = 2, \ldots, n, \quad i \neq j \neq k, \tag{5}$$

$$x_{ij} \text{ binary}, \quad \forall i, j = 1, \ldots, n, \ i \neq j, \tag{6}$$

$$y_{ij} = 0, \quad \forall i, j = 2, \ldots, n, \ i \neq j. \tag{7}$$

Constraints (1) and (2) are the usual assignment constraints. Constraints (3)–(5) prevent the occurrence of subtours and, together with the other restrictions, provide a valid formulation for the ATSP as shown below in Propositions 1 and 2. Furthermore, Proposition 3 later establishes that a valid formulation for the PCATSP (denoted PCATSPxy) can be obtained by adding the following set of constraints, which serve to impose the required precedence relationships:

$$y_{ij} = 1, \quad \forall j = 2, \ldots, n, \ \forall i \in SPC_j, \tag{8}$$

where SPC_j is the subset of cities in $\{2, \ldots, n\}$ that are required to precede city j in the Hamiltonian path that commences at the base city 1. Note that in the context of precedence constrained sequential ordering problems, the base city is typically a dummy entity that represents a starting point.

Proposition 1. *Consider any Hamiltonian tour on the cities in N that is represented by the corresponding assignment solution* $\boldsymbol{x}$ *to the constraints* (1), (2), *and* (6). *Then, there exists a* $\boldsymbol{y}$ *such that* $(\boldsymbol{x}, \boldsymbol{y})$ *is feasible to ATSPxy.*

Proof. Without loss of generality (noting the structure of the problem), suppose that the given tour is $1, 2, \ldots, n$, and back to 1, so that $x_{12} = x_{23} = \cdots = x_{n-1,n} = x_{n1} = 1$, and $x_{ij} = 0$ otherwise. Define $y_{ij} = 1$, if $i < j$, and $y_{ij} = 0$ otherwise, $\forall i, j = 2, \ldots, n,\ i \neq j$. Then (3) holds true since $y_{23} = \cdots = y_{n-1,n} = 1$, and (4) holds true because for each pair $i, j \in \{2, \ldots, n\}$, we have $\{y_{ij} = 1$ and $y_{ji} = 0\}$ if $i < j$, and $\{y_{ij} = 0$ and $y_{ji} = 1\}$ if $i > j$. Furthermore, (5) is satisfied because, if not, then there exists some triplet $i, j, k \in \{2, \ldots, n\}$ such that $y_{ij} = y_{jk} = y_{ki} = 1$, which implies that $i < j < k < i$, a contradiction. This completes the proof. □

Proposition 2. *ATSPxy is a valid formulation for the ATSP in that there exists a feasible solution* $(\boldsymbol{x}, \boldsymbol{y})$ *if and only if* $\boldsymbol{x}$ *represents a Hamiltonian tour on* N.

Proof. The if-part is established by Proposition 1. Conversely, suppose that $(\boldsymbol{x}, \boldsymbol{y})$ is feasible to ATSPxy. To show that the assignment solution $\boldsymbol{x}$ that is feasible to (1), (2) and (6) represents a Hamiltonian tour, we need to demonstrate that $\boldsymbol{x}$ does not admit any subtours on a set of cities contained in $\{2, \ldots, n\}$, i.e.,

$$\sum_{(i,j)\in C} x_{ij} \leqslant m - 1 \text{ for any circuit}$$

$$C \equiv \{(i_1, i_2),\ (i_2, i_3), \ldots, (i_m, i_1)\}, \tag{9}$$

where $\{i_1, \ldots, i_m\} \subseteq \{2, \ldots, n\}$ and where $m \in \{2, \ldots, n-1\}$.

From (3), it is sufficient to show that:

$$\sum_{(i,j)\in C} y_{ij} \leqslant m - 1 \text{ for any such circuit}$$

$$C \equiv \{(i_1, i_2),\ (i_2, i_3), \ldots, (i_m, i_1)\}. \tag{10}$$

For $m = 2$ or 3, this directly follows from constraints (4) and (5), respectively. Hence, by induction, suppose that the result holds true for any circuit involving m cities, and consider the case of $(m+1)$ cities, $\{i_1, i_2, \ldots, i_{m+1}\}$, where $3 \leqslant m \leqslant n-2$, and let us show that

$$y_{i_1 i_2} + y_{i_2 i_3} + \cdots + y_{i_m i_{m+1}} + y_{i_{m+1} i_1} \leqslant m. \tag{11}$$

Note that by the induction hypothesis, we have

$$y_{i_1 i_2} + y_{i_2 i_3} + \cdots + y_{i_m i_1} \leqslant m - 1. \tag{12}$$

Hence, adding (12) to $y_{i_1 i_m} + y_{i_m i_{m+1}} + y_{i_{m+1} i_1} \leqslant 2$ from (5), we get

$$y_{i_1 i_2} + y_{i_2 i_3} + \cdots + y_{i_m i_1} + y_{i_1 i_m} + y_{i_m i_{m+1}} + y_{i_{m+1} i_1} \leqslant m + 1. \tag{13}$$

Using the fact that $y_{i_m i_1} + y_{i_1 i_m} = 1$ from (4), the inequality (13) reduces to (11), and this completes the proof. □

Proposition 3. *Consider the model PCATSPxy as given by ATSPxy augmented with the constraints* (8). *Then PCATSPxy is a valid formulation of the ATSP that enforces the precedence relationships, namely, there exists a feasible solution* $(\boldsymbol{x}, \boldsymbol{y})$ *to PCATSPxy if and only if* $\boldsymbol{x}$ *represents a Hamiltonian tour on* N *for which city* i *precedes city* j *for each* $i \in SPC_j$, $\forall j \in \{2, \ldots, n\}$.

Proof. The if-part follows from the proof in Proposition 1, noting that (8) is satisfied for the stated $\boldsymbol{y}$ solution, given that $\boldsymbol{x}$ represents a Hamiltonian tour that satisfies the precedence relationships. Conversely, suppose that $(\boldsymbol{x}, \boldsymbol{y})$ is feasible to PCATSPxy. By Proposition 2, we have that $\boldsymbol{x}$ represents a Hamiltonian tour on N. Hence, all we need to show is that for any $i \in SPC_j,\ j \in \{2, \ldots, n\}$, we have that city i precedes city j in this tour. By contradiction, suppose not, i.e., suppose that the tour starting and ending at the base city 1 contains the path $\{j, k_1, \ldots, k_p, i\}$. Note that we must have $p \geqslant 1$, because otherwise, we would have $x_{ji} = 1$, which by (3) and (4) would imply that $y_{ji} = 1$ and $y_{ij} = 0$, contradicting (8). But $x_{jk_1} = \ldots = x_{k_p i} = 1$ on this path implies from (3), along with $y_{ij} = 1$, that

$$y_{jk_1} + \cdots + y_{k_p i} + y_{ij} = p + 2, \tag{14}$$

which contradicts (10) in Proposition 2 based on the circuit $C \equiv \{(j, k_1), \ldots, (k_p, i), (i, j)\}$ involving $(p+2)$ edges. This completes the proof. □

Remark 1. The Dantzig–Fulkerson–Johnson formulation (see [5]) of the ATSP models the SECs through the following set of constraints:

$$\sum_{i,j\in S} x_{ij} \leqslant |S| - 1, \quad \forall S \subseteq N \quad \text{with}$$

$$2 \leqslant |S| \leqslant n - 2. \tag{15}$$

We call this the ATSP-DFJ formulation. From the proof of Proposition 2, it is clear that our formulation, ATSPxy, implies a relaxed subset of the DFJ constraints of the type:

$$\sum_{(i,j)\in C} x_{ij} \leqslant |S| - 1, \quad \forall S \subseteq N \setminus \{1\}$$

$$\text{with} \quad 2 \leqslant |S| \leqslant n-2, \tag{16}$$

and for each circuit C involving all the cities in S.

Thus, a polynomial number of constraints in ATSPxy formulation capture an exponential number of, albeit, relaxed DFJ constraints. In particular, note that (16) includes the two-city DFJ constraints involving pairs of cities in $N \setminus \{1\}$. However, for $|S| = 2$, (15) also involves pairs of the type $\{1, j\}, j = 2, \ldots, n$. Hence, in order to tighten the LP relaxation of ATSPxy, we can include the constraints

$$x_{1j} + x_{j1} \leqslant 1, \quad \forall j = 2, \ldots, n \tag{17}$$

in the model formulation. Our computations in the following section include these constraints as well.

Remark 2. Note that the ATSPxy formulation uses the y-variables to convey a sequencing order much like the u-variables in the Miller–Tucker–Zemlin [11] formulation (and its lifted versions as discussed in Desrochers and Laporte [6] and Sherali and Driscoll [13]). Moreover, it permits a complete polynomial length formulation (unlike ATSP-DFJ) that, as demonstrated in the following section, affords a significantly tighter LP relaxation than the MTZ model. This is useful in the modeling of more complex routing and production problems that include ATSP as only a part of their structure. Furthermore, similar to the MTZ formulation, the ATSPxy model has an inherent structure that readily facilitates the formulation of precedence constraints via the restrictions (8).

Remark 3. Another formulation (RMTZ) that models the SECs through a reformulation of MTZ constraints has been proposed by Gouveia and Pires [8,9]. This formulation includes the assignment constraints (1) and (2) and the variable restrictions (6) and (7) along with the SECs in the following form

$$x_{ij} + y_{ki} \leqslant y_{kj} + 1, \quad \forall i, j, k = 2, \ldots, n, \quad i \neq j \neq k, \tag{18}$$

$$y_{ij} \geqslant x_{ij}, \quad \forall i, j = 2, \ldots, n, \quad i \neq j \tag{19}$$

$$x_{ij} + y_{ji} \leqslant 1, \quad \forall i, j = 2, \ldots, n, \quad i \neq j. \tag{20}$$

Examining these constraints in the light of our model ATSPxy, note that in view of (4), (20) is the same as (19), and is therefore implied by our model. Moreover, under (4), (18) is of the form

$$y_{ki} + x_{ij} + y_{jk} \leqslant 2, \quad \forall i, j, k = 2, \ldots, n, \quad i \neq j \neq k. \tag{21}$$

However, constraints (5) of ATSPxy together with constraints (3) imply (21). Hence, ATSPxy is potentially tighter than RMTZ.

Furthermore, Gouveia and Pires suggest the following two possible liftings of (18):

$$x_{ij} + x_{ji} + y_{ki} \leqslant y_{kj} + 1, \quad \forall i, j, k = 2, \ldots, n, \quad i \neq j \neq k \tag{22}$$

and

$$x_{ij} + x_{kj} + x_{ik} + y_{ki} \leqslant y_{kj} + 1, \quad \forall i, j, k = 2, \ldots, n, \quad i \neq j \neq k \tag{23}$$

In the sequel, we refer to the lifted formulation obtained by replacing (18) with (22) as L1RMTZ, and the one obtained by replacing (18) with (23) as L2RMTZ. Note that under (4) of ATSPxy, (22) is of the form

$$(x_{ij} + x_{ji}) + y_{jk} + y_{ki} \leqslant 2, \quad \forall i, j, k = 2, \ldots, n, \quad i \neq j \neq k. \tag{24}$$

However, constraints of the form

$$(y_{ij} + x_{ji}) + y_{jk} + y_{ki} \leqslant 2, \quad \forall i, j, k = 2, \ldots, n, \quad i \neq j \neq k \tag{25}$$

are valid for ATSPxy, and together with (3), they imply (24); hence, they are potentially tighter. The validity of (25) follows from the fact that when $x_{ji} = 0$, (25) is the same as (5), which is valid, and when $x_{ji} = 1$, (3) implies that $y_{ji} = 1$ and then, it follows from (4) that $y_{ij} = 0$ and hence, $y_{jk} + y_{ki} \leqslant 1$, since k either comes after or before the pair (j, i) in the tour. Symmetrically, the following two constraints are valid for ATSPxy:

$$y_{ij} + y_{jk} + y_{ki} + x_{kj} \leqslant 2, \quad \forall i, j, k = 2, \ldots, n, \quad i \neq j \neq k \tag{26}$$

and

$$y_{ij} + y_{jk} + y_{ki} + x_{ik} \leqslant 2, \quad \forall i,j,k = 2,\ldots,n, \ i \neq j \neq k. \tag{27}$$

Consequently, the following constraint obtained by surrogating (25), (26), and (27) is also valid for ATSPxy:

$$3(y_{ij} + y_{jk} + y_{ki}) + x_{ji} + x_{kj} + x_{ik} \leqslant 6, \quad \forall i,j,k = 2,\ldots,n, \ i \neq j \neq k \tag{28}$$

Also, note that under (4), (23) is of the form

$$x_{ij} + (y_{jk} + x_{kj}) + (y_{ki} + x_{ik}) \leqslant 2, \quad \forall i,j,k = 2,\ldots,n, \ i \neq j \neq k, \tag{29}$$

which is also valid for ATSPxy.

Notationally, we refer to the formulation obtained by replacing (5) with (25) as L1ATSPxy, the one obtained by replacing (5) with the surrogate (28) as SL1ATSPxy, and the one obtained by replacing (5) with (29) as L2ATSPxy. Additionally, we can replace (5) with one of the constraints (25), (26), or (27) depending on which of the variables x_{ji}, x_{kj}, or x_{ik} has the smallest cost associated with it and hence, has the greatest tendency to be positive in the continuous solution. We refer to the formulation obtained by this replacement strategy as ML1ATSPxy. Note that all these (lifted) ATSPxy and (lifted) RMTZ formulations have $O(n^2)$ variables and $O(n^3)$ constraints.

3. Computational results

In this section, we present computational results to compare the quality of the LP relaxation bound and the effort required by the branch-and-bound methodology using CPLEX 7.0 for the proposed formulations versus some alternative formulations presented in the literature. Standard test problems from the TSP library as well as some additional randomly generated problems for the precedence-constrained case have been used for this experimentation. All runs were made on a PC Model Optiplex GX1 550 Mbr+ under the Windows 2000 operating system, with a time limit of 100, 000 s.

We first compared the LP relaxation bounds of the lifted ATSPxy formulations with those of the corresponding lifted RMTZ formulations. While L1ATSPxy produced better bounds than SL1ATSPxy, ML1ATSPxy, as well as L1RMTZ (even with (17) incorporated in the formulation), by far, the bounds for the corresponding "L2" formulations were observed to be much tighter, and the corresponding LP relaxations were also solved considerably faster. Hence, we focused on the L2ATSPxy and the L2RMTZ formulations for further comparisons. In addition, we compared these formulations with the following lifted MTZ models, as well as a more "natural" polynomially-sized multicommodity formulation (which is a direct extension of the corresponding symmetric TSP formulation given in [4]).

a. MTZ lifted formulation of Desrochers and Laporte [6] (ATSP-DL) ($O(n^2)$ variables and constraints);
b. MTZ lifted formulation of Sherali and Driscoll [13] (ATSP-SD) ($O(n^2)$ variables and constraints);
c. The aforementioned MTZ lifted formulations augmented with the DFJ three-city SECs (referred to as (ATSP-DL + 3DFJ) and (ATSP-SD + 3DFJ), respectively) ($O(n^2)$ variables and $O(n^3)$ constraints); and
d. Multicommodity flow formulation (MCATSP—see [4]) ($O(n^3)$ variables and constraints).

Table 1 presents the bounds obtained for the initial LP relaxation and the results for solving the problems to optimality. Observe that, in general, both L2ATSPxy and MCATSP produced better LP relaxation bounds in comparison with the other formulations. (See [4] for the equivalence between the LP relaxations of such a multicommodity formulation and that of a model having DFJ-type subtour elimination constraints.) Also, note that although these models took longer to find optimal integer solutions than the fastest method (ATSP-DL), they enumerated significantly fewer branch-and-bound nodes to reach optimality. The longer computational times can be attributed to the greater effort required for solving the LP relaxation at each node. We expect that with ongoing improvements in solving linear programming problems, the relatively tighter bounds obtained via the proposed formulation will yield a more favorable compromise.

Table 1
Comparison of LP relaxation bounds and CPLEX 7.0 branch-and-bound effort for ATSP problems from TSPLIB

Problem	Formulation	Z_{LP}	CPU (LP)	Best integer solution	CPU (IP)	B and B nodes
br17	L2TSPxy	28.00	19.9	39	22424.3	66251
$Z_{IP}=39$	L2RMTZ	28.00	7.0	39	95670.3	623434
	ATSP-DL	22.00	0.6	39	4573.6	1122848
	ATSP-SD	27.68	0.4	39	624.1	6965
	MCATSP	39.00	60.0	39	301.0	0
	ATSP-DL+3DFJ	28.00	0.3	39	2246.8	327452
	ATSP-SD+3DFJ	30.57	0.7	39	55.2	682
ftv33	L2ATSPxy	1286.00	10.6	1286	18.8	0
$Z_{IP}=1286$	L2RMTZ	1286.00	8.3	1286	20.7	0
	ATSP-DL	1217.18	0.6	1286	16.1	235
	ATSP-SD	1224.50	5.6	1286	928.9	2704
	MCATSP	1286.00	3740.0	1286	361.0	0
	ATSP-DL+3DFJ	1270.72	2.0	1286	37.0	115
	ATSP-SD+3DFJ	1272.33	3.5	1286	126.6	37
ftv35	L2ATSPxy	1456.89	8.8	1473	4457.6	89
$Z_{IP}=1473$	L2RMTZ	1453.53	6.9	1473	1324.8	76
	ATSP-DL	1413.50	2.4	1473	43.0	1424
	ATSP-SD	1415.51	8.3	1473	2068.8	3594
	MCATSP	1457.33	5903.0	1473	1283.0	7
	ATSP-DL+3DFJ	1448.29	2.4	1473	54.3	312
	ATSP-SD+3DFJ	1448.67	4.2	1473	870.7	1205
ftv38	L2ATSPxy	1484.00	9.6	1503	8905.1	36
$Z_{IP}=1503$	L2RMTZ	1481.29	6.8	1503	746.2	49
	ATSP-DL	1456.34	2.3	1503	101.3	3081
	ATSP-SD	1458.22	10.2	1503	10841.8	7590
	MCATSP	1482.00	10046.0	1503	2499.0	15
	ATSP-DL+3DFJ	1478.00	2.8	1503	439.2	2980
	ATSP-SD+3DFJ	1478.00	4.2	1503	3770.6	3684
ftv44	L2ATSPxy	1584.88	21.3	1613	51962.1	82
$Z_{IP}=1613$	L2RMTZ	1583.38	11.4	1613	7801.6	159
	ATSP-DL	1573.75	2.5	1613	295.4	6554
	ATSP-SD	1573.75	12.2	1613	2381.2	2111
	MCATSP	1584.88	17333.0	1613	14458.0	43
	ATSP-DL+3DFJ	1584.63	4.3	1613	421.1	1393
	STSP-SD+3DFJ	1584.63	5.5	1613	3105.3	1194

However, the most promising results using the proposed model formulation were obtained for precedence constrained ATSP (PCATSP) problems. In this case, among the alternative proposed formulations (L1ATSPxy, SL1ATSPxy, ML1ATSPxy and L2ATSPxy) augmented with constraints (8), the best results were obtained using the first of these models. Hence, we focus on this formulation. Furthermore, note that we can derive a tighter representation by incorporating the following valid constraints in the precedence-constrained formulations:

$$x_{ik}=0, \quad \forall k=2,\dots,n, \ \ j\in SPC_k, \ \ i\in SPC_j. \quad (30)$$

Constraints (30) are valid because if city j must be visited before city k, and city i must be visited before city j, then city k cannot be visited immediately

Table 2
Comparison of LP relaxation bounds and CPLEX 7.0 branch-and-bound effort for PCATSP problems from TSPLIB

Problem	Formulation	Z_{LP}	CPU (LP)	Best integer solution	CPU (IP)	B and B nodes
esc07	L1PCATSPxy	2125.00	0.1	2125	0.1	0
$Z_{IP} = 2125$	PCATSP-SD	2087.5	0.1	2125	0.2	0
	PCATSP-DL	2005.83	0.1	2125	0.1	1
	PCMCATSP	312.5	5.0	2125	19.0	6
esc11	L1PCATSPxy	2058.83	0.3	2075	0.3	0
$Z_{IP} = 2075$	PCATSP-SD	2036.90	0.3	2075	0.3	0
	PCATSP-DL	2032.53	0.1	2075	0.1	0
	PCMCATSP	1258.67	20.0	2075	18.0	2
ecs12	L1PCATSPxy	1554.00	0.4	1675	6.5	83
$Z_{IP} = 1675$	PCATSP-SD	1520.35	0.4	1675	17.1	520
	PCATSP-DL	1508.52	0.2	1675	1.1	133
	PCMCATSP	889	13.0	1675	27.0	91
br17.10	L1PCATSPxy	27.29	2.3	55	580.4	1842
$Z_{IP} = 55$	PCATSP-SD	22.68	0.9	55	8536.9	175284
	PCATSP-DL	22.00	0.2	55	25354.1	5877686
	PCMCATSP	0	62.0	58	$1.00E+05$*	44432
esc25	L1PCATSPxy	1610.96	17.3	1681	487.4	69
$Z_{IP} = 1681$	PCATSP-SD	1540.32	2.9	1681	281.7	1107
	PCATSP-DL	1529.87	0.3	1681	72.1	5074
	PCMCATSP	1020.85	962.0	1681	6924.0	496

*Aborted after 10^5 CPU seconds.

after city i; city j must be visited in the sequence between i and k. Table 2 compares the LP relaxation bounds and the branch-and-bound effort required by L1PCATSPxy with the precedence-constrained formulations of Sherali and Driscoll [13] (PCATSP-SD) and Desrochers and Laporte [6] (PCATSP-DL) (both having (O(n^2) variables and constraints), as well as with an MTZ-augmented precedence constrained version of MCATSP (abbreviated PCMCATSP), which has O(n^3) variables and constraints. Note that constraints (30) have been incorporated in all these formulations. (The dual simplex method was used to obtain the LP relaxation bounds, since this option was found to be the fastest.) Observe that L1PCATSPxy produced tighter bounds and enumerated fewer branch-and-bound nodes in comparison with the other three formulations. Also, observe that although MCATSP was competitive for ATSP problems, its performance for the precedence constrained problems has deteriorated significantly in comparison with the other models. Furthermore, note that in the case of Problem br17.10, where the CPU time required to obtain an optimal solution is relatively large, L1ATSPxy converged to an optimum substantially faster (by more than an order of magnitude).

It so happens that for this test case, the precedence graph is less structured than for the "esc" labeled problems. Motivated by this observation, in order to further evaluate the potential usefulness of L1PCATSPxy for solving more complex precedence constrained ATSP problems, we conducted an experimentation using several precedence constrained problems that were randomly generated following the procedure outlined in Ascheuer et al. [2]. We constructed problems having 25 and 30 nodes, and for each problem

Table 3
Comparison of LP relaxation bounds and CPLEX 7.0 branch-and-bound effort for randomly generated PCATSP problems

Problem (Precedences)	Formulation	Z_{LP}	CPU (LP)	Best integer solution	CPU (IP)	B and B nodes
25 (7)	L1PCATSPxy	158.00	63.0	165	110	19
	PCATSP-SD	154.00	1.1	165	30	440
	PCATSP-DL	154.00	0.032	165	10	914
25 (13)	L1PCATSPxy	262.94	26.0	292	470	974
	PCATSP-SD	242.87	1.7	292	270	4511
	PCATSP-DL	236.18	0.015	292	590	28828
25 (19)	L1PCATSPxy	172.79	30.0	196	180	70
	PCATSP-SD	153.54	1.7	196	220	5412
	PCATSP-DL	152.48	0.016	196	230	10865
25 (25)	L1PCATSPxy	255.64	36.0	287	240	94
	PCATSP-SD	222.95	2.0	287	26000	240675
	PCATSP-DL	219.05	0.031	287	440	56168
25 (50)	L1PCATSPxy	652.00	0.19	652	$4.70E-02$	0
	PCATSP-SD	610.23	0.55	652	9.8	79
	PCATSP-DL	598.54	0.015	652	$6.30E-02$	0
30 (8)	L1PCATSPxy	182.79	200.0	200	2800	247
	PCATSP-SD	172.19	2.5	200	1100	12474
	PCATSP-DL	171.50	0.016	200	440	25511
30 (15)	L1PCATSPxy	164.19	320.0	194	8100	616
	PCATSP-SD	124.53	3.7	194	41000	308248
	PCATSP-DL	122.72	0.032	194	49000	2283573
30 (23)	L1PCATSPxy	256.48	130.0	281	890	54
	PCATSP-SD	184.50	3.2	281	29000	457954
	PCATSP-DL	178.79	0.015	281	32000	2014592
30 (30)	L1PCATSPxy	268.7	150.0	328	6500	2772
	PCATSP-SD	192.15	3.4	328	$1.00E+05^*$	464001
	PCATSP-DL	183.88	0.016	328	$1.00E+05^*$	3909502
30 (60)	L1PCATSPxy	589.23	7.4	637	48	90
	PCATSP-SD	376.37	2.3	644	$1.00E+05^*$	678833
	PCATSP-DL	358.44	0.015	637	29000	1815467

*Aborted after 10^5 s.

with p nodes, we generated $\lceil p/4 \rceil$, $\lceil p/2 \rceil$, $\lceil 3p/4 \rceil$, p, and $2p$ precedence relations. The results obtained are displayed in Table 3. (We suppress the results for PCMCATSP, which was unable to solve most of these problems in a time limit of 10^5 CPU seconds.) It is evident from these results that our formulation outperformed the other formulations on this set of test problems. Not only did it provide tighter LP relaxation bounds and explored fewer branch-and-bound nodes, but also, it usually required a significantly lesser (often by several orders of magnitude) computational effort to reach an optimal solution.

4. Conclusions

As demonstrated by our computational experiments on precedence-constrained ATSP problems, the new formulation proposed in this paper usually requires a significantly reduced effort over competing models for solving relatively more complex problems (often by more than an order of magnitude). While the relative tightness of this formulation does not generally translate to a competitive performance for ATSP problems without precedence relationships due to the size and structure of the LP relaxations, it is hoped that with advances in LP technology, it might prove to be more favorable in the future. Based on our preliminary results, it would be of interest to study the effectiveness of the formulations developed in this paper to model complex routing and production scheduling problems that have ATSP as an embedded substructure. We recommend such investigations for future research.

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