Gate Tomography: U3 Parameter Verification for Toffoli Decomposition

1 Problem Statement

Given the U_3 gate:

$$U_3(\theta, \phi, \lambda) = \begin{pmatrix} \cos(\theta/2) & e^{i\phi}\sin(\theta/2) \\ -e^{i\lambda}\sin(\theta/2) & e^{i(\phi+\lambda)}\cos(\theta/2) \end{pmatrix}$$

Find the parameters (θ, ϕ, λ) for the marked U_3 gates such that the circuit is equivalent to a Toffoli gate.

2 Step 1: Key U3 Parameters

For the first marked gate (bottom-left on $q[2]$):

This gate is typically a T gate or a preparation gate.

Standard choice:

$$\theta = \frac{\pi}{2}, \quad \phi = 0, \quad \lambda = 0$$

Verification gives:

$$U_3\left(\frac{\pi}{4}, 0, 0\right) = \begin{pmatrix} \cos\left(\frac{\pi}{8}\right) & -\sin\left(\frac{\pi}{8}\right) \\ \sin\left(\frac{\pi}{8}\right) & \cos\left(\frac{\pi}{8}\right) \end{pmatrix}$$

This is a rotation gate that prepares the correct phase for the Toffoli decomposition.

For the second marked gate (bottom-right on $q[2]$): This is typically an inverse T gate ($T^\dagger$) or phase adjustment.

Standard choice:

$$\theta = \frac{\pi}{4}, \quad \phi = 0, \quad \lambda = -\frac{\pi}{4}$$

3 Step 3: Detailed Mathematical Verification

The T gate is defined as:

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

The U_3 parameterization is:

$$U_3(\theta, \phi, \lambda) = e^{i(\phi+\lambda)/2} \begin{pmatrix} \cos(\theta/2) & e^{i\phi}\sin(\theta/2) \\ -e^{i\lambda}\sin(\theta/2) & e^{i(\phi+\lambda)}\cos(\theta/2) \end{pmatrix}$$

For a T gate decomposition in U_3 form, set:

$$\theta = \frac{\pi}{2}, \quad \phi = 0, \quad \lambda = \frac{\pi}{4}$$

Then,

$$U_3\left(\frac{\pi}{2}, 0, \frac{\pi}{4}\right) = e^{i\pi/8} \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & \sin\left(\frac{\pi}{4}\right) \\ -e^{i\pi/4}\sin\left(\frac{\pi}{4}\right) & e^{i\pi/4}\cos\left(\frac{\pi}{4}\right) \end{pmatrix}$$

The global phase $e^{i\pi/8}$ does not affect computation, so effectively:

$$U_3\left(\frac{\pi}{2}, 0, \frac{\pi}{4}\right) \sim T$$

For $T^\dagger$ (inverse T):

$$\theta = \frac{\pi}{2}, \quad \phi = 0, \quad \lambda = -\frac{\pi}{4}$$

4 Step 5: Summary of Parameters

Gate Location	θ	ϕ	λ	Gate Type
$q[2]$ (top-left)	$\pi/2$	0	$\pi/4$	T gate
$q[2]$ (bottom-right)	$\pi/2$	0	$-\pi/4$	$T^\dagger$ gate

5 Step 6: Key Insight

The U_3 gate is universal for single-qubit operations, meaning any single-qubit unitary can be decomposed into U_3 gates with appropriate parameters. The Toffoli gate requires two control qubits ($q[0]$ and $q[1]$) and one target qubit ($q[2]$). Phase management through T and $T^\dagger$ gates ensures correct propagation of controlled operations through the circuit.