Computational Physics
Lecture 5

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git clone https://github.com/ukzncompphys/lecture5.git
wget www.cita.utoronto.ca/~sievers/compphys/lecture5.tar.gz
(followed by tar -xzf lecture5.tar.gz)

FFT Review

- $F(k)=\int f(x)*\exp(-2\pi ikx)dx$ (where $k=1/\omega$)
- Integral gets rid of *x*, replaces with *k*. New function has amplitude and phase as a function of *k*.
- Quantum mechanics - de Broglie says $p = \hbar k$. So, Fourier transform position to get momentum.
- Fourier transform electric field $E(t)$ to get frequency spectrum.
- Fourier transform to get fast correlations, convolutions, many other things.

DFT (Discrete FT)

- Computers don't to continuous. Not enough RAM for starters…
- Function exists over finite range in x at finite number of points.
- If input function has *n* points, output can only have *n* k's.
- Gives rise to discrete Fourier Transform (DFT)
- $F(k)=\sum f(x)\exp(-2\pi ikx/N)$ for N points and $0\leq k<N$
- What would DFT of $f(0)=1$, otherwise $f(x)=0$ look like?
- What would DFT of $f(x)=1$ look like?
- DFTs have subtle behaviours not seen in continuous, infinite FTs.

Convolution Theorem

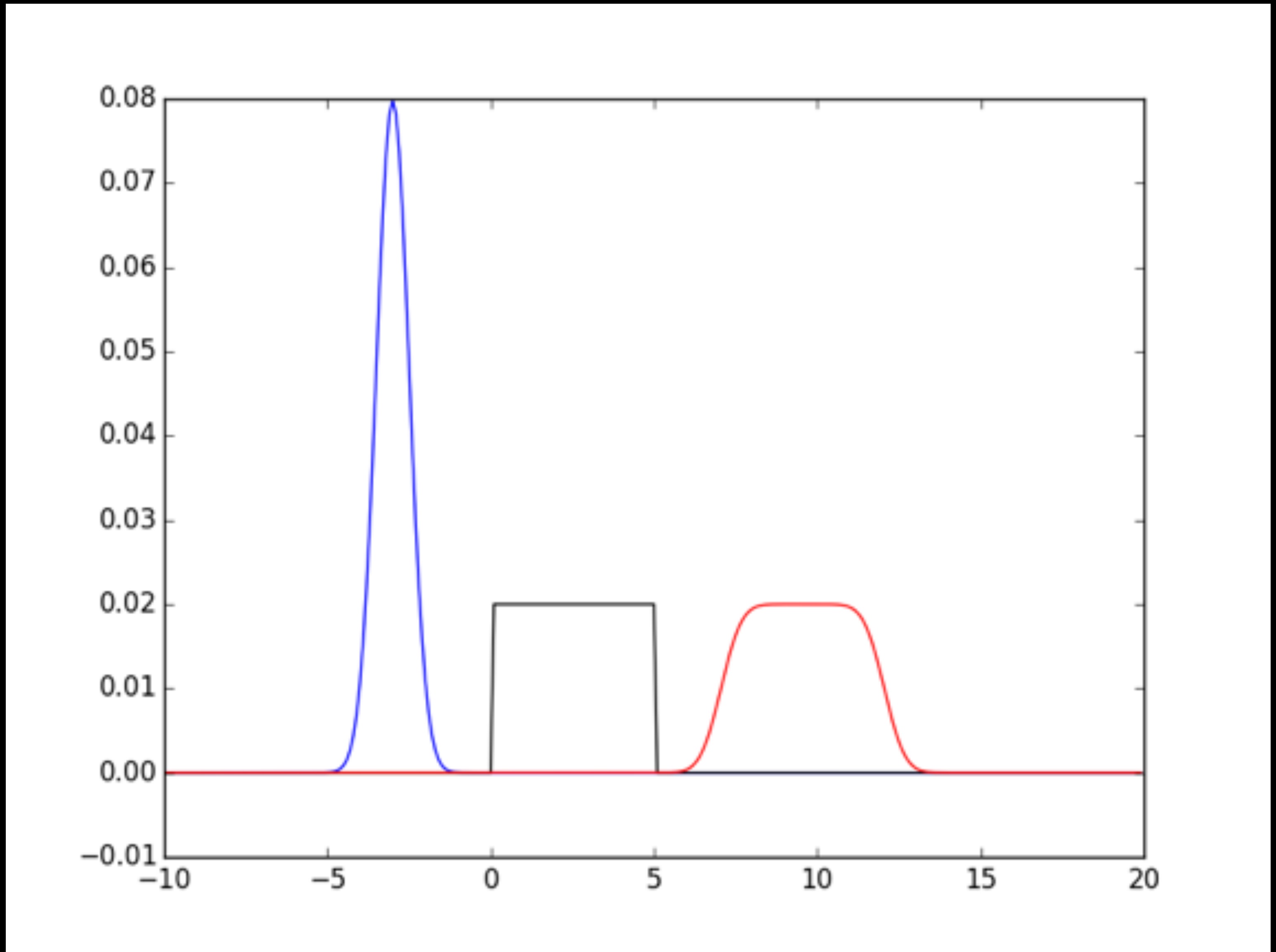
- Convolution defined to be $conv(y)=f\otimes g==\int f(x)g(y-x)dx$
- $\sum_x \sum F(k)exp(2\pi ikx) \sum conj(G(k'))exp(2\pi ik'x)exp(2\pi ik'y/N)$
- Reorder sum: $\sum\sum F(k)conj(G(k'))exp(-2\pi ik'y/N)\sum_x exp(2\pi i(k+k')x)$
- equals zero unless k'==-k. Cancels one sum, conjugates G
- $f\otimes g=\sum F(k)G(k)exp(2\pi iky/N)=ift(dft(f)*dft(g))$
- So, to convolve two functions, multiply their DFTs and take the IFT

Convolution Example

```
from numpy import arange,exp,real
from numpy.fft import fft,ifft
from matplotlib import pyplot as plt
def conv(f,g):
    ft1=fft(f)
    ft2=fft(g)
    return real(ifft(ft1*ft2))

x=arange(-10,20,0.1)
f=exp(-0.5*(x+3)**2/0.5**2)
g=0*x;g[ (x>0)&(x<5)]=1
g=g/g.sum()
f=f/f.sum()
h=conv(f,g)

plt.plot(x,f,'b')
plt.plot(x,g,'k')
plt.plot(x,h,'r')
plt.savefig('convolved')
plt.show()
```



Fast Fourier Transform

- How many operations does a DFT take?
- Have an N by N matrix operating on a vector of length N - clearly N^2 operations, right?
- Nope! Otherwise we'd never use them. What's actually going on?
- Note DFT=$\sum f(x)\exp(-2\pi ikx/N) = \sum f_{even}(x)\exp(-2\pi ik(2x)/N) + \sum f_{odd}(x)\exp(-2\pi ik(2x+1)/N)$
- $=F_{even}+\exp(-2\pi ik/N)F_{odd}$. Let $W_k=\exp(-2\pi ik/N)$
- if $k>N/2$, then $k^*=k-N/2$ and DFT=$F_{even}+\exp(-2\pi ik^*/N+i\pi)F_{odd}=F_{even}-W_kF_{odd}$.

FFT cont'd

- So $F(k)=F_{even}(k)+W_kF_{odd}(k)$ $(k<N/2)$ or $F_{even}(k)-W_kF_{odd}(k)$ $(k>=N/2)$
- So, can get *all* the frequencies if I have 2 half-length FFTs.
- Well, just do the same thing again. FFT of a single element is itself.
- This algorithm works for arrays whose length is a power of 2
- Popularized by Cooley/Tukey in early computer days. Later found to go back to Gauss in 1805. Changes computational work from n^2 to *nlogn*.

Sample FFT

- Routine uses *recursion* - function calls itself. Recursion can be very powerful, but also easy to goof.
- numpy.concatenate will combine arrays - note that they have to be passed in as a tuple, hence the extra set of parenthesis
- Modern FFT routines deal with arbitrary length arrays. Fastest Fourier Transform in the West (FFTW) standard packaged - usually used by numpy.

```python
from numpy import concatenate,exp,pi,arange,complex
def myfft(vec):
    n=vec.size
    #FFT of length 1 is itself, so quit
    if n==1:
        return vec
    #pull out even and odd parts of the data
    myeven=vec[0::2]
    myodd=vec[1::2]

    nn=n/2;
    j=complex(0,1)
    #get the phase factors
    twid=exp(-2*pi*j*arange(0,nn)/n)

    #get the dfts of the even and odd parts
    eft=myfft(myeven)
    oft=myfft(myodd)

    #Now that we have the partial dfts, combine them with
    #the phase factors to get the full DFT
    myans=concatenate((eft+twid*oft,eft-twid*oft))
    return myans
```

```
>>> import myft
>>> x=numpy.random.randn(32)
>>> xft1=numpy.fft.fft(x)
>>> xft2=myft.myfft(x)
>>> print numpy.sum(numpy.abs(xft1-xft2))
2.33937690259e-13
>>>
```

Dictionaries

- Dictionaries very useful built-in datatype in python
- Dictionaries have *keys*, each key stores a *value*.
- You can create a dictionary with {}
- Use square brackets to get the values

Dictionaries in Action

```
>>> x={} #initialize an empty dictionary
>>> x[0]=4
>>> x[-1]='sandwich'
>>> x['quest']='to find the holy grail'
>>> print x['quest']
to find the holy grail
>>> y=0
>>> print x[y]
4
>>> y=-1
>>> print x[y]
sandwich
>>> print x
{0: 4, 'quest': 'to find the holy grail', -1: 'sandwich'}
>>> for key in x.keys():
...     print 'key ' + repr(key) + ' has value ' + repr(x[key])
...
key 0 has value 4
key 'quest' has value 'to find the holy grail'
key -1 has value 'sandwich'
>>>
```

See how to assign and reference dictionary entries.

d.keys() gets all keys
you can use "del" to delete a dictionary entry (or any variable for that matter)

```
>>> sandwich={'bread' : 'rye','cheese' : 'swiss', 'meat' : 'pastrami','toppings': ('mustard','mayo')}
>>> for key in sandwich.keys():
...     print 'key ' + repr(key) + ' has value ' + repr(sandwich[key])
...
key 'cheese' has value 'swiss'
key 'toppings' has value ('mustard', 'mayo')
key 'meat' has value 'pastrami'
key 'bread' has value 'rye'
>>>
```

```
>>> del sandwich['bread']
>>> print sandwich.keys()
['cheese', 'toppings', 'meat']
>>>
```

Make a dictionary containing cubes from 1 to 10

```
Jonathans-MacBook-Pro:lecture5 sievers$ more cube_dict.py
#file cube_dict.py
cubes={}
for x in range(1,11):
    cubes[x]=x**3

for xx in cubes.keys():
    print repr(xx) + ' cubed is ' + repr(cubes[xx])
Jonathans-MacBook-Pro:lecture5 sievers$
```

```
>>> execfile('cube_dict.py')
1 cubed is 1
2 cubed is 8
3 cubed is 27
4 cubed is 64
5 cubed is 125
6 cubed is 216
7 cubed is 343
8 cubed is 512
9 cubed is 729
10 cubed is 1000
>>>
```

Classes

- Python is *object-oriented*. That means things called objects contain data and contain methods (i.e. functions) that do things with the data.
- You have seen this in action: e.g. vec=numpy.ones(10);print vec.sum()
- This is very different from e.g. C or (classic) Fortran. In C, you need to know how to sum an array. In Python, the array knows how to sum itself
- In python, objects are called *classes*. You can define them in files with the *class* keyword.
- data/methods of a class are accessed with a period '.'. The first argument to any method is the instance of the class itself. It is customary (and strongly encouraged) to name that variable "self".

Beginnings of a complex variable class

```python
#class_example1.py
import numpy
class Complex:
    #__init__ is a special function.  When you create a new
    #instance of a class, if it exists in the class definition,
    #__init__ will get called.  __init__ is assumed to return the first value

    def __init__(self,r=0,i=0):
        self.r=r
        self.i=i


if __name__=='__main__':
    num=Complex()
    print 'real part of num is ' + repr(num.r)
    print 'imaginary part of num is ' + repr(num.i)

    num2=Complex(2,5)
    print 'real part of num2 is ' + repr(num2.r)
    print 'imaginary part of num2 is ' + repr(num2.i)

    #we can assign new data to classes whenever we want.
    #you probably want to be really careful with this however
    num2.len=numpy.sqrt(num2.r**2+num2.i**2)
    print 'length of num2 is ' + repr(num2.len)
```

```
-uu-:---F1  class_example1.py    All L1     (Python)--
Loading python...done
```

Left: a bare-bones complex number class.

Below: output

```
Jonathans-MacBook-Pro:lecture5 sievers$ python class_example1.py
real part of num is 0
imaginary part of num is 0
real part of num2 is 2
imaginary part of num2 is 5
length of num2 is 5.385164807134504
Jonathans-MacBook-Pro:lecture5 sievers$
```

Class Methods

- We’ve made a class that can hold complex numbers.
- Right now the class just holds numbers, it doesn’t do anything.
- We did take an absolute value, but we had to know at the command line how to do that.
- Let’s add a method to the class to take its absolute value

Methods ctd.

```python
#class_example2.py
import numpy
class Complex:
    #__init__ is a special function.  When you create a new
    #instance of a class, if it exists in the class definition,
    #__init__ will get called.  __init__ is assumed to return the first value

    def __init__(self,r=0,i=0):
        self.r=r
        self.i=i
    def abs(self):
        return numpy.sqrt(self.r**2+self.i**2)

if __name__=='__main__':

    num=Complex(2,5)
    print 'real part of num is ' + repr(num.r)
    print 'imaginary part of num is ' + repr(num.i)
    myabs=num.abs()
    print 'absolute value is ' + repr(myabs)
```

We have added an abs() method to the complex class. Now you can get the absolute value without having to know anything about complex numbers.

```
Jonathans-MacBook-Pro:lecture5 sievers$ python class_example2.py
real part of num is 2
imaginary part of num is 5
absolute value is 5.385164807134504
Jonathans-MacBook-Pro:lecture5 sievers$
```

What's the difference?

```python
#class_example3.py

import numpy
class Complex:
    def __init__(self,r=0,i=0):
        self.r=r
        self.i=i
    def abs(self):
        return numpy.sqrt(self.r**2+self.i**2)
#####################################################
#
# What is the difference between these two classes?
#
#####################################################
class Complex2:
    def __init__(self,r=0,i=0):
        self.r=r
        self.i=i
def abs(self):
    return numpy.sqrt(self.r**2+self.i**2)


if __name__=='__main__':

    num=Complex(2,5)
    print num.abs()
    num2=Complex2(2,5)
    print num2.abs()
```

Classes Complex and Complex2 look similar, but they might have different behaviour. Why?

```
Jonathans-MacBook-Pro:lecture5 sievers$ python class_example3.py
5.38516480713
Traceback (most recent call last):
  File "class_example3.py", line 28, in <module>
    print num2.abs()
AttributeError: Complex2 instance has no attribute 'abs'
Jonathans-MacBook-Pro:lecture5 sievers$
```

What's the difference?

```
#class_example3.py

import numpy
class Complex:
    def __init__(self,r=0,i=0):
        self.r=r
        self.i=i
    def abs(self):
        return numpy.sqrt(self.r**2+self.i**2)
#####################################################
#
# What is the difference between these two classes?
#
#########################
class Complex2:
    def __init__(self,r=0
        self.r=r
        self.i=i
def abs(self):
    return numpy.sqrt(sel


if __name__=='__main__':

    num=Complex(2,5)
    print num.abs()
    num2=Complex2(2,5)
    print num2.abs()
```

```
>>> abs(-3)
3
>>> execfile('class_example3.py')
5.38516480713
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
  File "class_example3.py", line 28, in <module>
    print num2.abs()
AttributeError: Complex2 instance has no attribute 'abs'
>>> abs(num2)
5.385164807134537
>>> abs(num)
5.385164807134537
>>> abs(3)
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
  File "class_example3.py", line 20, in abs
    return numpy.sqrt(self.r**2+self.i**2)
AttributeError: 'int' object has no attribute 'r'
>>>
```

Always remember your indenting! By not indenting we closed the Complex2 definition *and* defined a global function abs that replaced the built-in function.

Python Uses References

- Python uses *references*. If *a* is an instance of a class, and you say *b*=*a*, then the contents of *b* will point to the same memory as the contents of *a*.
- This means that if I then change *b*, *a* will also change.
- General rule is if you change/assign a piece of *b*, same piece of *a* will change.
- Be very careful - don't change values inside of functions unless you meant to.

```
>>> a=Complex(3,5)
>>> b=a
>>> print a.r
3
>>> b.r=5
>>> print a.r
5
>>> 
```

Copy

- Because of this, it is often customary to have a copy() function.
- Copy should make a fully distinct version of the instance.

```python
#class_example4.py

import numpy
class Complex:
    def __init__(self,r=0,i=0):
        self.r=r
        self.i=i
    def copy(self):
        return Complex(self.r,self.i)
    def abs(self):
        return numpy.sqrt(self.r**2+self.i**2)


if __name__=='__main__':

    num=Complex(2,5)
    num2=num.copy()
    num2.r=10
    print 'real part of num is ' + repr(num.r)
    print 'real part of num2 is ' + repr(num2.r)
```

```
Jonathans-MacBook-Pro:lecture5 sievers$ python class_example4.py
real part of num is 2
real part of num2 is 10
Jonathans-MacBook-Pro:lecture5 sievers$
```

Overloading

- The operators in python (e.g. +,-,*…) just map to a set of special functions. You can use them on your classes if you include methods with those names.

- Entending the behaviour of the default operators is called overloading.

- __add__ is the keyword for '+'. __repr__ is the keyword for printing things.

- If you want to do this, you can google to get the rest of the special function names.

- Note that a+b is shorthand for a.__add__(b) so as written a+2 will work, but 2+a won't. Why?

```python
#overload.py

import numpy
class Complex:
    def __init__(self,r=0,i=0):
        self.r=r
        self.i=i
    def copy(self):
        return Complex(self.r,self.i)
    def __add__(self,val):
        ans=self.copy()
        if isinstance(val,Complex):
            ans.r=ans.r+val.r
            ans.i=ans.i+val.i
        else:
            ans.r=ans.r+val
        return ans
    def __repr__(self):
        if (self.i<0):
            return repr(self.r)+' - '+repr(-1*self.i) +'i'
        else:
            return repr(self.r)+' + '+repr(self.i) +'i'
```

```
>>> from overload import Complex
>>> a=Complex(2,5)
>>> b=Complex(4,-3)
>>> c=a+b
>>> print c
6 + 2i
>>> d=a+b+2
>>> print d
8 + 2i
>>>
```

Try/Except

- Sometimes things go wrong. Say a method is given invalid input

- Python has *try/except*. The code will execute the *try* block. As soon as that hits an error it jumps to the *except* block.

- If there is no error, *except* is skipped.

- Optionally, you can include a *finally* clause that always gets executed after the *try/except*. Useful for e.g. freeing memory/closing files etc.

```
def __add__(self,val):
    ans=self.copy()
    if isinstance(val,Complex):
        ans.r=ans.r+val.r
        ans.i=ans.i+val.i
    else:
        try:
            ans.r=ans.r+val
        except:
            print 'Invalid type in Complex.__add__'
            ans=None
    return ans
```

```
>>> a=Complex(2,5)
>>> b=3
>>> c=a+b
>>> print c
5 + 5i
>>> b='abc'
>>> print a+b
Invalid type in Complex.__add__
None
>>>
```

Tutorial Problems (due Tuesday)

- Write a function that will shift an array by an arbitrary amount using a convolution (yes, I know there are easier ways to do this). The function should take 2 arguments - an array, and an amount by which to shift the array. Plot a gaussian that started in the centre of the array shifted by half the array length. (10)

- The correlation function f★g is ∫f(x)g(x+y). Through a similar proof, one can show f★g=ift(dft(f)*conj(dft(g))). Write a routine to take the correlation function of two arrays. Plot the correlation function of a Gaussian with itself. (10)

- Using the results of part 1 and part 2, write a routine to take the correlation function of a Gausian (shifted by an arbitrary amount) with itself. How does the correlation function depend on the shift? Does this surprise you? (10)

- The circulant (wrap-around) nature of the dft can sometimes be problematic. Write a routine to take the convolution of two arrays *without* any danger of wrapping around. You may wish to add zeros to the end of the input arrays. (10)

Tutorial Problems 2

- Complete the complex definition to support -,*, and / (__sub__, __mul__, and __div__). Recall that a/b = a*conj(b)/(b*conj(b)). Show from a few sample cases that your functions work. (10)

- Next lecture we will look at n-body simulations. In preparation, write a class that contains masses and x and y positions for a collection of particles. The class should also contain a dictionary that can contain options. Two entries in the dictionary should be the # of particles and G (gravitational constant). The class should also contain a method that calculates the potential energy of every particle, sum($G\ m_1 m_2/r_{12}$) . (10)

Tutorial Bonus Problems

- Bonus: extend the complex class to also support arbitrary (i.e. non-integer) powers (keyword is __pow__). +3 if the routine works if a^b works for complex *a* and real *b*, +5 if it works for complex a and complex *b*. (you may ignore branch cuts).

- You have a sample code that calculates an FFT of an array whose length is a power of 2. Using that routine as a guideline, write an FFT routine that works on an array whose length is a power of 3 (e.g. 9, 27, 81). Verify that it gives the same answer as numpy.fft.fft (10)