Recall that the advection equation which describes constant velocity and conserved mass is

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0$$

We can discretize this by saying

$$\frac{f_{t+\delta t} - f_t}{\delta t} = -v\frac{\partial f}{\partial x}$$

For the *upwind* scheme with positive velocity, we saw that the discretized derivative was just $\left[f(x) - f(x-\delta x)\right]/\delta x$.

In this problem, we replace that discretized derivative by one we calculate by looking at both our left and right neighbours:

$$\frac{\partial f}{\partial x} \sim \frac{f(x+\delta x) - f(x-\delta x)}{2\delta x}$$

In principle, this derivative should be accurate to second order, unlike the upwind derivative. But now we need to check stability. First, solve explicitly for $f(\delta t, x)$ (assuming the initial time is $t = 0$).

$$f(x,\delta t) = f(x,0) - \frac{v\delta t}{\delta x}\frac{f(x+\delta x) - f(x-\delta x)}{2\delta x}$$

The standard technique now is to let $f(x, t = 0) = \exp(ikx)$, and define $C \equiv \frac{v\delta t}{\delta x}$. If we plug that in, we have

$$f(x,\delta t) = \exp(ikx) - \frac{C}{2}\left[\exp(ik(x+\delta x)) - \exp(ik(x-\delta x))\right]$$

Each term has an $\exp(ikx)$ in it, which we can factor out:

$$f(x,\delta t) = \exp(ikx)\left(1 + \frac{C}{2}\left[\exp(ik\delta x) - \exp(-ik\delta x)\right]\right) = \exp(ikx)\left[1 + iC\sin(k\delta x)\right]$$

Recall that we're unstable when the magnitude of a mode increases with time, which happens when the quantity in square brackets has an absolute value larger than 1. Note that we add a purely imaginary nuber to 1 in the brackets, which must always have an absolute value of at least 1, so this second-order spatial derivative method is *always* unstable.