Recall that the advection equation which describes constant velocity and conserved mass is

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = 0$$

We can discretize this by saying

$$\frac{f_{t+\delta t} - f_t}{\delta t} = -v \frac{\partial f}{\partial x}$$

For the *upwind* scheme with positive velocity, we saw that the discretized derivative was just  $[f(x) - f(x - \delta x)]/\delta x$ .

In this problem, we replace that discretized derivative by one we calculate by looking at both our left and right neighbours:

$$\frac{\partial f}{\partial x} \sim \frac{f(x + \delta x) - f(x - \delta x)}{2\delta x}$$

In principle, this derivative should be accurate to second order, unlike the upwind derivative. But now we need to check stability. First, solve explicitly for  $f(\delta t, x)$  (assuming the initial time is t = 0).

$$f(x, \delta t) = f(x, 0) - \frac{v\delta t}{\delta x} \frac{f(x + \delta x) - f(x - \delta x)}{2\delta x}$$

The standard technique now is to let  $f(x,t=0)=\exp(ikx)$ , and define  $C\equiv\frac{v\delta t}{\delta x}$ . If we plug that in, we have

$$f(x, \delta t) = \exp(ikx) - \frac{C}{2} \left[ \exp(ik(x + \delta x)) - \exp(ik - \delta x) \right]$$

Each term has an  $\exp(ikx)$  in it, which we can factor out:

$$f(x,\delta t) = \exp(ikx) \left( 1 + \frac{C}{2} \left[ \exp(ik\delta x) - \exp(-ik\delta x) \right] \right) = \exp(ikx) \left[ 1 + iC\sin(k\delta x) \right]$$

Recall that we're unstable when the magnitude of a mode increases with time, which happens when the quantity in square brackets has an absolute value larger than 1. Note that we add a purely imaginary nuber to 1 in the brackets, which must always have an absolute value of at least 1, so this second-order spatial derivative method is *always* unstable.