Poisson's Ratio from Stiffness Matrix

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Stiffness Matrix Relations

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & \frac{-v_{12}}{E_1} & \frac{-v_{13}}{E_1} \\ \frac{-v_{12}}{E_1} & \frac{1}{E_2} & \frac{-v_{23}}{E_2} \\ \frac{-v_{13}}{E_1} & \frac{-v_{23}}{E_2} & \frac{1}{E_3} \end{bmatrix} * \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} \tag{1}$$

Simplify Notation

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{A} & \frac{-x}{A} & \frac{-y}{A} \\ \frac{-x}{A} & \frac{1}{B} & \frac{-z}{B} \\ \frac{-y}{A} & \frac{-z}{B} & \frac{1}{C} \end{bmatrix} * \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} \tag{2}$$

Reduce to Equations

$$\epsilon_1 = \frac{1 * \sigma_1}{A} + \frac{-x * \sigma_2}{A} + \frac{-y * \sigma_3}{A} \tag{3}$$

$$\epsilon_2 = \frac{-x * \sigma_1}{A} + \frac{1 * \sigma_2}{B} + \frac{-z * \sigma_3}{B} \tag{4}$$

$$\epsilon_3 = \frac{-y * \sigma_1}{A} + \frac{-z * \sigma_2}{B} + \frac{1 * \sigma_3}{C} \tag{5}$$

Isolate Unknowns and Simplify

$$\frac{x * \sigma_2}{A} + \frac{y * \sigma_3}{A} = \frac{\sigma_1}{A} - \epsilon_1 \tag{6}$$

$$\frac{x * \sigma_1}{A} + \frac{z * \sigma_3}{B} = \frac{\sigma_2}{B} - \epsilon_2 \tag{7}$$

$$\frac{y * \sigma_1}{A} + \frac{z * \sigma_2}{B} = \frac{\sigma_3}{C} - \epsilon_3 \tag{8}$$

Augmented Matrix Notation

$$\left[\begin{array}{ccc|c} \frac{\sigma_2}{A} & \frac{\sigma_3}{A} & 0 & \frac{\sigma_1}{A}-\epsilon_1 \\ \frac{\sigma_1}{A} & 0 & \frac{\sigma_3}{B} & \frac{\sigma_2}{B}-\epsilon_2 \\ 0 & \frac{\sigma_1}{A} & \frac{\sigma_2}{B} & \frac{\sigma_3}{C}-\epsilon_3 \end{array}\right] \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \tag{9}$$

Perform Gaussian Elimination

$$\left[\begin{array}{ccc|c} \frac{\sigma_2}{A} & \frac{\sigma_3}{A} & 0 & \frac{\sigma_1}{A}-\epsilon_1 \\ \frac{\sigma_1}{A} & 0 & \frac{\sigma_3}{B} & \frac{\sigma_2}{B}-\epsilon_2 \\ 0 & \frac{\sigma_1}{A} & \frac{\sigma_2}{B} & \frac{\sigma_3}{C}-\epsilon_3 \end{array}\right] \begin{matrix} *\frac{A}{\sigma_2} \\ *\frac{A}{\sigma_1} \\ \end{matrix} \tag{10}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{\sigma_3}{\sigma_2} & 0 & (\frac{\sigma_1}{A}-\epsilon_1)(\frac{A}{\sigma_2}) \\ 1 & 0 & \frac{\sigma_3 A}{\sigma_1 B} & (\frac{\sigma_2}{B}-\epsilon_2)(\frac{A}{\sigma_1}) \\ 0 & \frac{\sigma_1}{A} & \frac{\sigma_2}{B} & \frac{\sigma_3}{C}-\epsilon_3 \end{array}\right] -1*R_1 \tag{11}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{\sigma_3}{\sigma_2} & 0 & (\frac{\sigma_1}{A}-\epsilon_1)(\frac{A}{\sigma_2}) \\ 0 & \frac{(-1)\sigma_3}{\sigma_2} & \frac{\sigma_3 A}{\sigma_1 B} & (\frac{\sigma_2}{B}-\epsilon_2)(\frac{A}{\sigma_1})-(\frac{\sigma_1}{A}-\epsilon_1)(\frac{A}{\sigma_2}) \\ 0 & \frac{\sigma_1}{A} & \frac{\sigma_2}{B} & \frac{\sigma_3}{C}-\epsilon_3 \end{array}\right] \begin{matrix} \\ *\frac{(-1)\sigma_2}{\sigma_3} \\ *\frac{A}{\sigma_1} \end{matrix} \tag{12}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{\sigma_3}{\sigma_2} & 0 & (\frac{\sigma_1}{A}-\epsilon_1)(\frac{A}{\sigma_2}) \\ 0 & 1 & \frac{(-1)\sigma_2 A}{\sigma_1 B} & [(\frac{\sigma_1}{A}-\epsilon_1)(\frac{A}{\sigma_2})-(\frac{\sigma_2}{B}-\epsilon_2)(\frac{A}{\sigma_1})](\frac{\sigma_2}{\sigma_3}) \\ 0 & 1 & \frac{\sigma_2 A}{\sigma_1 B} & (\frac{\sigma_3}{C}-\epsilon_3)(\frac{A}{\sigma_1}) \end{array}\right] -1*R_2 \tag{13}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{\sigma_3}{\sigma_2} & 0 & (\frac{\sigma_1}{A}-\epsilon_1)(\frac{A}{\sigma_2}) \\ 0 & 1 & \frac{(-1)\sigma_2 A}{\sigma_1 B} & [(\frac{\sigma_1}{A}-\epsilon_1)(\frac{A}{\sigma_2})-(\frac{\sigma_2}{B}-\epsilon_2)(\frac{A}{\sigma_1})](\frac{\sigma_2}{\sigma_3}) \\ 0 & 0 & \frac{(2)\sigma_2 A}{\sigma_1 B} & (\frac{\sigma_3}{C}-\epsilon_3)(\frac{A}{\sigma_1})-[(\frac{\sigma_1}{A}-\epsilon_1)(\frac{A}{\sigma_2})-(\frac{\sigma_2}{B}-\epsilon_2)(\frac{A}{\sigma_1})](\frac{\sigma_2}{\sigma_3}) \end{array}\right] + \frac{1}{2} * R_3 \tag{14}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{\sigma_3}{\sigma_2} & 0 & (\frac{\sigma_1}{A}-\epsilon_1)(\frac{A}{\sigma_2}) \\ 0 & 1 & 0 & \frac{1}{2}[(\frac{\sigma_1}{A}-\epsilon_1)(\frac{A}{\sigma_2})-(\frac{\sigma_2}{B}-\epsilon_2)(\frac{A}{\sigma_1})](\frac{\sigma_2}{\sigma_3})+\frac{1}{2}(\frac{\sigma_3}{C}-\epsilon_3)(\frac{A}{\sigma_1}) \\ 0 & 0 & \frac{(2)\sigma_2 A}{\sigma_1 B} & (\frac{\sigma_3}{C}-\epsilon_3)(\frac{A}{\sigma_1})-[(\frac{\sigma_1}{A}-\epsilon_1)(\frac{A}{\sigma_2})-(\frac{\sigma_2}{B}-\epsilon_2)(\frac{A}{\sigma_1})](\frac{\sigma_2}{\sigma_3}) \end{array}\right] * \frac{\sigma_1 B}{(2)\sigma_2 A} \tag{15}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{\sigma_3}{\sigma_2} & 0 & (\frac{\sigma_1}{A}-\epsilon_1)(\frac{A}{\sigma_2}) \\ 0 & 1 & 0 & \frac{1}{2}[(\frac{\sigma_1}{A}-\epsilon_1)(\frac{A}{\sigma_2})-(\frac{\sigma_2}{B}-\epsilon_2)(\frac{A}{\sigma_1})](\frac{\sigma_2}{\sigma_3})+\frac{1}{2}(\frac{\sigma_3}{C}-\epsilon_3)(\frac{A}{\sigma_1}) \\ 0 & 0 & 1 & (\frac{\sigma_3}{C}-\epsilon_3)(\frac{A}{\sigma_1})-[(\frac{\sigma_1}{A}-\epsilon_1)(\frac{A}{\sigma_2})-(\frac{\sigma_2}{B}-\epsilon_2)(\frac{A}{\sigma_1})](\frac{\sigma_2}{\sigma_3}) \end{array}\right] - (\frac{\sigma_3}{\sigma_2}) * R_2 \tag{16}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & (\frac{\sigma_1}{A}-\epsilon_1)(\frac{A}{\sigma_2})-(\frac{\sigma_3}{\sigma_2})[\frac{1}{2}[(\frac{\sigma_1}{A}-\epsilon_1)(\frac{A}{\sigma_2})-(\frac{\sigma_2}{B}-\epsilon_2)(\frac{A}{\sigma_1})](\frac{\sigma_2}{\sigma_3})+\frac{1}{2}(\frac{\sigma_3}{C}-\epsilon_3)(\frac{A}{\sigma_1})] \\ 0 & 1 & 0 & \frac{1}{2}[(\frac{\sigma_1}{A}-\epsilon_1)(\frac{A}{\sigma_2})-(\frac{\sigma_2}{B}-\epsilon_2)(\frac{A}{\sigma_1})](\frac{\sigma_2}{\sigma_3})+\frac{1}{2}(\frac{\sigma_3}{C}-\epsilon_3)(\frac{A}{\sigma_1}) \\ 0 & 0 & 1 & (\frac{\sigma_3}{C}-\epsilon_3)(\frac{A}{\sigma_1})-[(\frac{\sigma_1}{A}-\epsilon_1)(\frac{A}{\sigma_2})-(\frac{\sigma_2}{B}-\epsilon_2)(\frac{A}{\sigma_1})](\frac{\sigma_2}{\sigma_3}) \end{array}\right] \tag{17}$$

Solution

$$v_{12} = (\frac{\sigma_1}{E_1} - \epsilon_1)(\frac{E_1}{\sigma_2}) - (\frac{\sigma_3}{\sigma_2})(v_{13}) \tag{18}$$

$$v_{13} = \frac{1}{2}[(\frac{\sigma_1}{E_1} - \epsilon_1)(\frac{E_1}{\sigma_2}) - (\frac{\sigma_2}{E_2} - \epsilon_2)(\frac{E_1}{\sigma_1})](\frac{\sigma_2}{\sigma_3}) + \frac{1}{2}(\frac{\sigma_3}{E_3} - \epsilon_3)(\frac{E_1}{\sigma_1}) \tag{19}$$

$$v_{23} = (\frac{\sigma_3}{E_3} - \epsilon_3)(\frac{E_1}{\sigma_1}) - [(\frac{\sigma_1}{E_1} - \epsilon_1)(\frac{E_1}{\sigma_2}) - (\frac{\sigma_2}{E_2} - \epsilon_2)(\frac{E_1}{\sigma_1})](\frac{\sigma_2}{\sigma_3}) \tag{20}$$