Lenses and Lens Systems

Uladzimir Kasacheuski
Indiana University - Purdue University Indianapolis *

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Geometric optics can be employed to calculate the focal length of lenses and lens systems. Outlined in this report are the methods and analyses utilized to calculate the focal length of lens systems with the thin lens equation, Bessel's method, and Abbe's method. Lens systems consisting of a single convex lens, two convex lenses, and a Cooke triplet are assessed.

I. INTRODUCTION

The laws of geometrical optics can be derived in more than one way; for example, with Fermat's principle which is rooted in the idea that light propagates along the shortest path or with Huygens' principle which is rooted in the idea that each point on a wave front can be regarded as a source of spherical waves.

A. Thin Lenses

When analyzing a thin lens with a spherical surface we are able to utilize the small angle approximation (equation 1). This approximation combined with the analysis presented in figure 1 allows us to demonstrate equation 2. With algebraic manipulation, equation 2 can then be resolved into equation 3[1]. For a thin lens, which has two spherical surfaces, this equation equation can be utilized to generate the canonical thin lens equation, equation 4. This equation defines the relationship between object distance, s, image distance, s', and focal length f.

$$\sin(\theta) \approx \theta \tag{1}$$

$$n_2(\frac{h}{s} - \frac{h}{R}) = n_2(\frac{h}{s'} - \frac{h}{R}) \tag{2}$$

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R} \tag{3}$$

$$\frac{1}{s} + \frac{1}{s'} = \frac{n_2 - n_1}{n_1}(\frac{1}{R_1} - \frac{1}{R_2}) = \frac{1}{f} \tag{4}$$

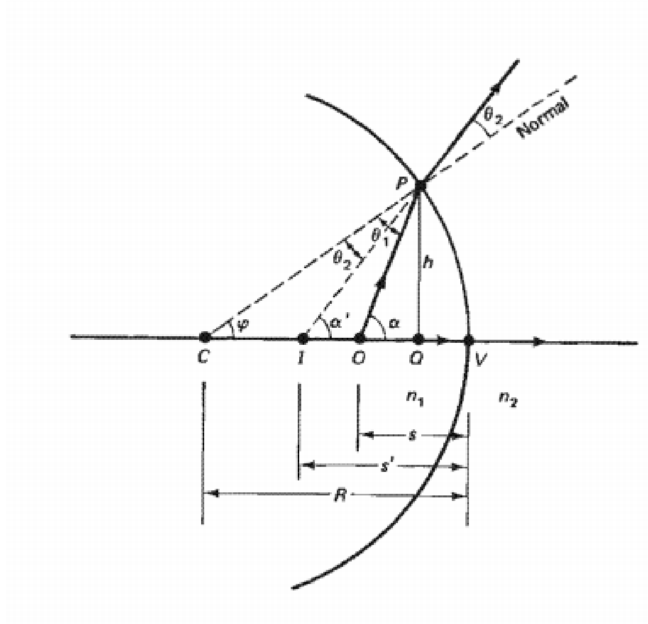

FIG. 1. Refraction at a Spherical Surface

B. Newtonian Form

Geometrical optics can be utilized to relate the focal length to image and object positions in an alternative way. Assessing the ray diagram presented in figure 2 presents the relationships of equation 5. This equation can be resolved into the Newtonian form of the thin lens equation, presented in equation 6.

$$\frac{h_i}{h_o} = \frac{f}{x} = \frac{x'}{f} \tag{5}$$

$$xx' = f^2 \tag{6}$$

II. METHODS

The focal length of lens system can be experimentally ascertained by multiple methods. This section outlines four methods which can be utilized to determine the focal length of a lens system based on knowing the image, object, and the position of the lenses of the lens system.

* Code base for this research is publicly available at https://github.com/UladKasach/Academic/tree/master/...
...Physics/Phys401/lens_systems

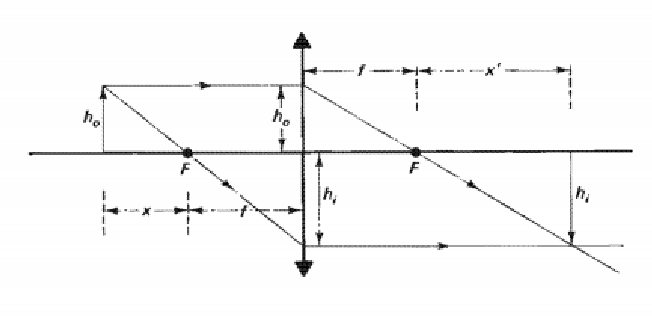

FIG. 2. Ray Diagram for a Convex Lens with Notation for Newtonian Equations

A. Thin Lens Equation

The thin lens relationship, presented in equation 4, can be used to determine the focal length given the object and image position. This can be done explicitly by solving equation 4 for f. It can also be accomplished by noting that equation 4 is of the form $y = mx + b$ where $b = 1/f$ and $m = -1$.

B. Bessel's Method

Bessel's method utilizes the measurement of two lens positions which produce the an image at the same image position for an object at a fixed position. In practice, this involves fixing the positions of the object and viewing screen and varying the position of the lens until two distinct lens positions are found which produce an image at the viewing screen. As long as $L > 4f$, we are guaranteed to find two positions which produce an image at the viewing screen.

The experimental setup defines the relationships presented in 7 and 8, where L is the distance between the object and the viewing screen and D is the distance between the two lens positions. The thin lens equation applied to the lens at position one and position two resolves into equations 9.

Analyzing the experimental setup along with equations 7 and 8 we can conclude that the object distance in lens position one is equal to the image distance in lens position two, $s_1 = s_2'$, and vice versa, $s_2 = s_1'$, therefore reducing equations 8 and 9 into equations 10 and 11 respectively.

$$s_1 + s_1' = s_2 + s_2' = L \tag{7}$$

$$s_2 - s_1 = s_1' - s_2' = D \tag{8}$$

$$f = \frac{s_1 s_1'}{s_1' + s_1} = \frac{s_1 s_1'}{L} \tag{9}$$

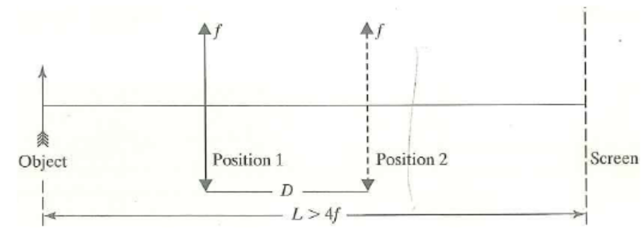

FIG. 3. Lens Positions and Notation for Bessel's Method

$$s_2 + s_1 = L \tag{10}$$

$$f = \frac{s_1 s_2}{s_2 + s_1} = \frac{s_1 s_2}{L} \tag{11}$$

Adding and subtracting equations 11 and 8 results in equations 12 and 13 respectively. Multiplying equations 12 and 13 then resolves into a relationship between the distance between the two lenses, L, the distance between the object and image D, and the focal length, f, as demonstrated in equation 15.

$$2s_2 = L + D \tag{12}$$

$$2s_1 = L - D \tag{13}$$

$$4s_1 s_2 = 4fL = (L - D)(L + D) = L^2 - D^2 \tag{14}$$

$$\therefore f = \frac{L^2 - D^2}{4L} \tag{15}$$

C. Abbe's Method

Abbe's method presents a method by which the focal length of a lens can be determined knowing only the magnification produced at two distinct object-lens distances. In practice, the focal length can be determined by varying the object distance and finding the respective image distance. From this one can utilize the definition of magnification, equation 16, and utilize the relationship defined for Abbe's method, 21.

$$M = \frac{-s'}{s} \tag{16}$$

The thin lens equation can be expressed in terms of s_1' as demonstrated in equation 17. Combined with equation

16 this results in equation 18. Similarly, for M_2 we find equation 25. Subtracting equations 18 and 25 we produce a relationship between M_1, M_2, s_1, s_2, and f, equation 21.

$$s_1' = \frac{s_1 f}{s_1 - f} \tag{17}$$

$$\frac{1}{M_1} = \frac{f - s_1}{f} \tag{18}$$

$$\frac{1}{M_2} = \frac{f - s_2}{f} \tag{19}$$

$$\frac{1}{M_1} - \frac{1}{M_2} = \frac{s_2 - s_1}{f} \tag{20}$$

$$\therefore f = \frac{s_2 - s_1}{\frac{1}{M_1} - \frac{1}{M_2}} \tag{21}$$

III. EXPERIMENTATION

The methods presented above were utilized to assess the focal length of several lens systems. Lens systems consisted of one convex lens, two convex lenses, and a Cooke triplet.

A lens system consisting of two convex lenses and a Cooke triplet were then analyzed utilizing the thin lens equation directly.

A. One Convex Lens

Focal length of a convex lens was assessed using the thin lens equation directly, using Bessel's method, and using Abbe's method. The convex lens was then used to verify the Newtonian form of the thin lens equation. The convex lens utilized was labeled with a focal length, $f_{labeled}$, of $138mm$.

1. Thin Lens Equation Analysis

In order to analyze the focal length of a lens using the thin lens equation directly multiple object distance, s, image distance, s', measurement pairs had to be captured. Due to the difficulty in assessing a precise image location, three independent image distance measurements were taken for each object distance; i.e., three people independently measured the image distance for each object distance. The results are recorded in table I.

As noted in section II A, the focal length of the system can be determined by modeling interpreting equation 4 in the form $y = mx + b$ and noting that the focal length is the inverse of the y-intercept of the line of best fit, $b = 1/f$. To this end, a line of best fit was found for the data in table I and the focal length found was $f = 126.8mm$. A plot of this analysis can be found in figure 4. Note, the expected slope is -1 while the slope found is -1.084; there is some error in the fitted data.

s	s'	s'	s'
330	217	212	218
230	305	319	324
188	461	490	477
363	202	202	203
441	185	186	183
527	173	172	173
573	167	166	169
601	166	160	163
618	166	164	162
395	196	190	200
mm	mm	mm	mm

TABLE I. Measurements of object distance, s, and image distance, s', for a single convex lens.

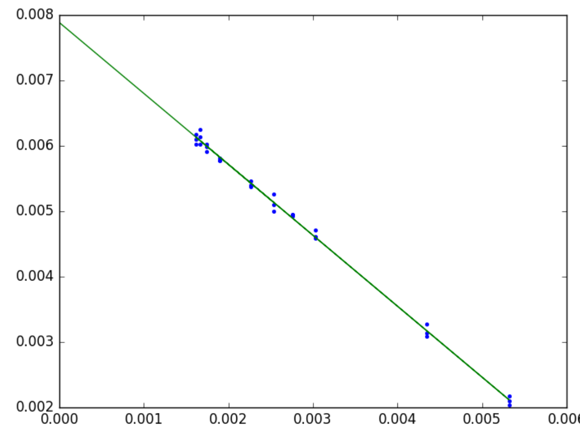

FIG. 4. Line of best fit for a single convex lens object distance, image distance measurements.

2. Bessel's Method

Bessel's method requires the measurement of both the distance between object and image, L, and the distance between two lens positions, D, which generate an image at the same image position for the given object position. As with the thin lens equation analysis, three people independently measured the two lens positions for each object to image distance combination. The results are shown in table II. The focal length given by 15 for these results has a mean of $f = 132.17mm$ and a standard deviation of $1.65mm$.

L	D	D	D
709	356	359	357
870	548	551	544
756	412	417	418
717	370	369	371
698	346	341	341
681	287	323	324
702	350	355	355
724	378	372	369
766	427	427	427
826	495	496	499
mm	mm	mm	mm

TABLE II. Measurements of object to image distance, L, and lens position one to lens position two distance, D, for a single convex lens.

3. Abbe's Method

Abbe's method requires the measurement of two distinct object distance, s, magnification, m measurement pairs in order to calculate the focal length. In practice, we are relegated to deriving the magnification by measuring the image position. The focal length values predicted by by Abbe's method is a combinatorial function of the number of measurement pairs that are available: i.e., T is equal to N choose 2, where N is the total number of measurement pairs available. The distinct measurement pairs are presented in table III.

When not restricting the combinations of measurement pairs, the total number of choices for two measurement pairs to calculate focal length with, given $N = 24$, is $T = 276$, where T is the total combinations possible.. For this data Abbe's method produces a focal length with a mean of $f = 110.82$ with a standard deviation of 35.606.

If restricting the measurement pair combinations to only allow combinations where the image distance is distinct between the two pairs, the total combinations of measurement pairs to calculate focal length with reduces to 252; i.e., $T-N$. In this case, the focal length found has a mean of 120.93 and a standard deviation of 12.66. This is understandably closer to the expected focal length and this makes sense, as it can be expected that $M_1^{-1} \approx M_2^{-1}$ and thus will give an asymptotic value; therefore, these combinations will produce outliers and should not be included.

4. Single Convex Lens Summary

A summary of the focal lengths evaluated with the thin lens equation directly, Bessel's method, and Abbe's method is presented in table IV.

s	s'	s'	s'
224.5	471.5	475.5	467.5
248	365.5	365	363
287	282	273	279
312	253.5	249	256
334	235.5	238	241
396	206	203	206
453	189.5	189	192
460	189.5	186	189
mm	mm	mm	mm

TABLE III. Measurements of object distance, s, and image distance, s' for a single convex lens again.

Method	f	σ
Direct	126.8	
Bessel	132.17	1.65
Abbe	120.93	12.66
	mm	mm

TABLE IV. Summary of the focal lengths found with each method for the single convex lens.

B. Two Convex Lenses

A lens system consisting of two convex lenses can be analyzed using the thin lens equation. By measuring several the object and image distances for the system the process demonstrated in section III A 1 can be utilized to derive the focal length. A set of data used for this analysis is presented in tables V, VI, and VII .

Data was collected for three lens systems characterized by the same focal lengths of constituent lenses, $f_{l1} = 136mm$ and $f_{l2} = 136mm$, and differing between-lens lengths, $L_1 = 60mm$, $L_2 = 100mm$, $L_3 = 150mm$. The focal lengths found were $f_1 = 53.5mm$ with $m_1 = -1.53$ for L_1, $f_2 = 57.61mm$ with $m_2 = -1.29$ for L_2, and $f_3 = 100.75mm$ with $m_3 = -0.404$ for L_3. The plots for this analysis are presented in figures 5, 6, and 7

We note that because the slopes m_1, m_2, and m_3 deviate from $m_{expected} = -1$ substantially, we further analyzed this data utilizing statistical analysis of the focal lengths given by equation 4 directly. The focal lengths found from this approach were $f_1 = 62.19mm$ with $\sigma_1 = 3.56mm$ for L_1, $f_2 = 70.15mm$ with $\sigma_2 = 3.62mm$, and $f_3 = 65.99mm$ with $\sigma_3 = -3.49mm$ for L_3.

C. Cooke Triplet

A Cooke Triplet lens system consists of a converging, diverging, and converging lens separated by distances L_1 and L_2. We repeat the process employed for the two convex lens system with this lens system. The focal lengths of the three constituent lenses are $f_1 = 48mm$, $f_2 = -22mm$, and $f_3 = 48mm$. The inter-lens distances were kept at $L_1 = L_2 = 38mm$. The object, image distances measured are displayed in table VIII. The focal length found with this method is $f = 32.16mm$, however

s	s'	s'	s'
315	69	71	70
290	73	75	76
260	76	78	78
240	81	79	81
220	87	87	88
200	95	93	97
170	110	108	105
120	158	157	160
mm	mm	mm	mm

TABLE V. Measurements of object distance, s, and image distance, s' for a two convex lens system with inter-lens distance $L = 60mm$.

s	s'	s'	s'
121	154	151	148
136	144	139	143
146	141	143	143
156	136	136	135
165	134	134	135
176	131	131	130
116	155	157	155
106	163	170	160
mm	mm	mm	mm

TABLE VI. Measurements of object distance, s, and image distance, s' for a two convex lens system with inter-lens distance $L = 100mm$.

the slope of the line was $m = -2.77$, which is exceptionally off. Due to this the thin lens equation was utilized directly to generate a statistical analysis of the focal length represented by the data. The average focal length found was $f = 51.12mm$ and the standard deviation was $\sigma = 6.39mm$.

s	s'	s'	s'
121	146	148	145
136	129	128	131
146	121	122	117
156	112	110	109
165	103	107	104
176	97	96	99
116	231	152	224
106	168	167	171
mm	mm	mm	mm

TABLE VII. Measurements of object distance, s, and image distance, s' for a two convex lens system with inter-lens distance $L = 150mm$.

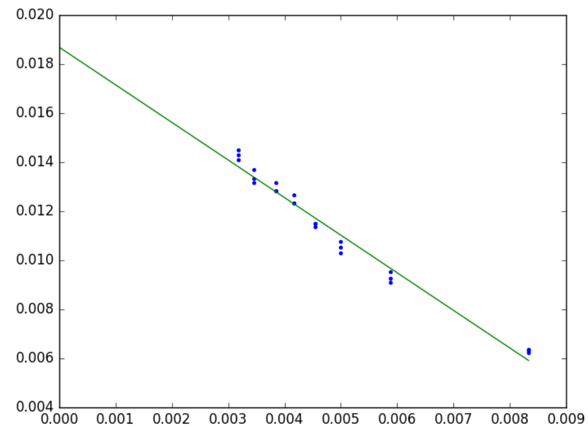

FIG. 5. Line of best fit for a two convex lens system object distance, image distance measurements, with an inter-lens length of $60mm$.

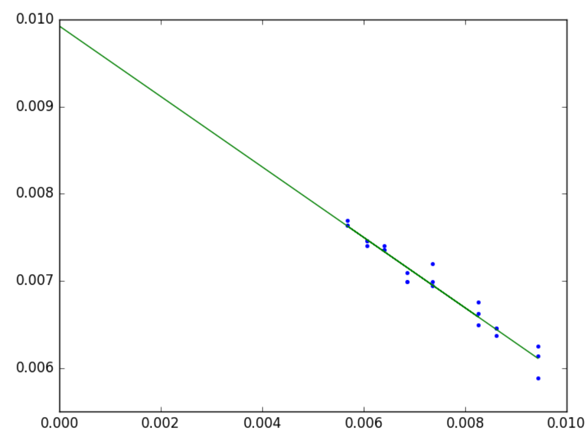

FIG. 6. Line of best fit for a two convex lens system object distance, image distance measurements, with an inter-lens length of $100mm$.

s	s'	s'
145	85	91
169	73	77
153	87	87
186	64	66
230	48	49
209	53	53
191	57	63
131	102	101
139	95	96
111	127	126
mm	mm	mm

TABLE VIII. Measurements of object distance, s, and image distance, s' for a two Cooke Triplet.

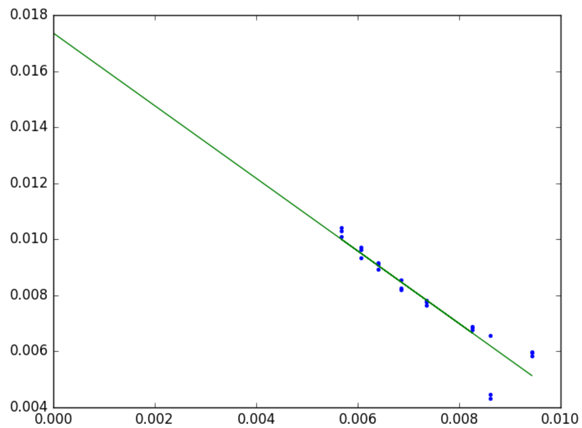

FIG. 7. Line of best fit for a two convex lens system object distance, image distance measurements, with an inter-lens length of $150mm$.

IV. ANALYSIS

A. One Convex Lens

1. Error Analysis

Analysis of the imprecision of the relationships we utilized combined with the scale of data we measured can be conducted by utilizing the log errors method for equations 4, 15, and 21. Equations 22, 23, and 24 represent the log error relationship between each before mentioned equation respectively. The average error, Δf, is presented for each method in table IX.

The data reveals that there should be very little inaccuracy contributed by the method of determining the focal lengths using methods demonstrated, with systematic imprecision actually decreasing as we use more sophisticated methods such as Bessel's and Abbe's.

$$\frac{\Delta f}{f} = \frac{\Delta s}{s} + \frac{\Delta s'}{s'} - \frac{\Delta s + \Delta s'}{s + s'} \tag{22}$$

$$\frac{\Delta f}{f} = \frac{2L\Delta L - 2D\Delta D}{L^2 - D^2} - \frac{\Delta L}{L} \tag{23}$$

$$\frac{\Delta f}{f} = \frac{\Delta s_1}{s_1} + \frac{\Delta s_2}{s_2} - \frac{s_1 \Delta s_2' + \Delta s_1 s_2' - \Delta s_1' s_2 - \Delta s_2 s_1'}{s_1 s_2' - s_2 s_1'} \tag{24}$$

2. Expected Values

We note that the expected values of the focal length, $f = 136mm$, is significantly off of the found values for the

Method	f	σ	Δf	$\sigma_{\Delta f}$
Direct	126.8		0.296	0.027
Bessel	132.17	1.65	0.0276	0.0058
Abbe	120.93	12.66	0.162	0.0743
	mm	mm	mm	mm

TABLE IX. Summary of the focal lengths found with each method for the single convex lens.

focal length as presented in table IX. This is likely due to measurement errors and difficulty in ascertaining the best "image" of the object, rather than the lens manufacturers producing flawed products.

Given the data found in figure for this lens system, presented in table I, we can verify the Newtonian form of the thin lens equation, equation 6. We compare the square root of the difference of both sides of the Newtonian form of the thin lens equation, utilizing the focal length that we found and the object, $f = 126.8mm$, as shown in equation **??**. We find that the average difference in focal length is $\Delta f = 33.56$ and that the standard deviation of these differences is $\sigma = 14.66$. The square root of the difference should be a reasonable measure of the difference between the focal length derived by the data and the true focal length of the system - as the Newtonian form of the thin lens equation should be upheld for this lens system. Therefore there is significant indication that our focal length of $126.8mm$ is inaccurate. This was predicted, as the slope of the vergence graph, figure 4, was not precisely $m = -1$.

$$\Delta f = \sqrt{xx' - f^2} \tag{25}$$

B. Two Convex Lenses

The matrix representation of this lens system, demonstrated in equation 26, can be utilized to predict the image position for any object distance utilizing L, the distance between the two lenses, f_1, focal length of the first lens, and f_2, the focal length of the second lens. By appending $_s$ and $T_{s'}$, the translation matrices for the image and output distances, as demonstrated in equation 27, we relate the image and output distance to the lens system. Setting matrix element $S_{12} = 0$, we define a relationship between s and s' which relates the input distance to the image distance, as $S_{12} = 0$ restricts the output plane at s'. The relationship S_{12}, simplified, is presented in equation 28. We also note that the equivalent focal length of a lens system can be calculated as $-1/M_{12}$.

From these relationships we can calculate the difference between the expected focal length, f_e, and the calculated focal length, f, as well as the expected image distance s'_e and the calculated image distance s' for each inter-lens distance. These calculations are presented in table X. From this it is clearly evident that as the distance between the lenses increases the expected focal length

L	Δf	$\sigma_{\Delta f}$	$\Delta s'$	$\sigma_{\Delta s'}$
60	25.05	3.55	5.65	2.22
100	37.38	3.62	12.995	7.49
150	85.61	3.49	11.79	17.19
mm	mm	mm	mm	mm

TABLE X. Statistical analysis of differences between expected focal length and calculated focal length as well as expected image position and calculated image position for a two convex lens system.

Δf	$\sigma_{\Delta f}$	$\Delta s'$	$\sigma_{\Delta s'}$
135.68	6.39	34.18	5.15
mm	mm	mm	mm

TABLE XI. Statistical analysis of differences between expected focal length and calculated focal length as well as expected image position and calculated image position for the Cooke Triplet .

and the found focal length diverge.

$$M = \begin{bmatrix} 1 & 0 \\ -f_2^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -f_1^{-1} & 1 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \quad (26)$$

$$\phi' = T_{s'} M T_s \phi = S\phi =$$

$$\begin{bmatrix} M_{11} + s'M_{21} & sM_{11} + s'(sM_{21} + M_{22}) \\ M_{21} & sM_{21} + M_{22} \end{bmatrix} \phi (27)$$

$$s' = \frac{(sM_{11} + M_{12})(-1)}{sSM_{21} + M_{22}} \quad (28)$$

C. Cooke Triplet

Similarly to the two lens system, the matrix representation of this lens system, demonstrated in equation 29, can be utilized to predict the image position and equivalent lens system focal length as demonstrated in equation 28 and by the relationship $f = -1/M_2 1$. These relationships were used to again calculate the difference between the expected focal length, f_e, and the calculated focal length, f, as well as the expected image distance s'_e and the calculated image distance s' for each inter-lens distance. These calculations are presented in table XI. It is clear that there is a high level of error between expected and found focal length.

$$M = \begin{bmatrix} 1 & 0 \\ -f_3^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & L_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -f_2^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & L_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -f_1^{-1} & 1 \end{bmatrix} \quad (29)$$

V. CONCLUSION

Geometric optics can be utilized to determine the focal length of arbitrary lens systems. It is clear, however, that as the number of lenses increases and the length between lenses increases the ability to accurately determine the focal length of a lens system decreases. It is possible to determine the focal length of a single convex lens with relative accuracy. However, with two or more lenses a higher degree of precision in measurements is required.

REFERENCES

[1] Leno M Pedrotti Frank L Pedrotti and Leno S Pedrotti. *Introduction to Optics*. Pearson Education, 2013.