Foundations for Systematic Construction of Fault-Tolerant Quantum Gates

Uladzimir Kasacheuski* and Guy Wiss†
Department of Physics, Indiana University-Purdue University Indianapolis
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The parallel computation power inherent in quantum computers has resulted in extreme interest in their realization which has manifest in both academia and industry. This paper demonstrates the implications of entanglement, how such states including the Bell states can be created, and standard quantum teleportation to build a background of quantum computing. Upon this background we prove several claims made in a paper which demonstrates a method by which a fault-tolerant logic gates can be produced in a systematic way and later reflect on what these conclusions mean.

I. INTRODUCTION

Computers have revolutionized our world with digital electronics, the personal computers, the internet, and "smart technology". Quantum computing has become of more and more interest as both the power of quantum computing in certain applications[6] and the limits of classical computer architectures[3] become more apparent.

Much of the excitement about quantum computing comes from these phenomena and the potential to utilize them to exponentially speed up certain types of computation. Other interest stems from the realization that classical computation can only be scaled down to a limit until quantum mechanics begins to rule.

This paper defines a background for quantum computing in section III. This background will then be used to demonstrate the foundations of a technique to construct simple single qubit teleportation circuits. Single qubit teleportation circuits have been shown to be sufficient for systematically construct fault-tolerant quantum logic gates for arbitrary numbers of qubits [9]; this is a significant finding as fault tolerance is critical for information processing and ad-hoc approaches to creating quantum gates are costly endeavors.

II. CLASSICAL COMPUTING BACKGROUND

Theories of computational logic which were born in the domain of classical computation, reversibility and universality, are very relevant to quantum computation. Further, it is essential to understand what computation is in the classical, intuitive realm in order to reflect on the consequences of quantum computation. Logic theory in computation can be decomposed into the representation of a state (i.e., memory) as well as processing of the state (i.e. time evolution).

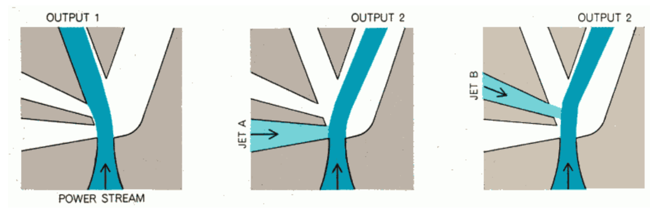

FIG. 1. Fluidic Logic OR Gate

A. Memory

The fundamental unit of information in classical computation is the binary unit, the bit. Each bit represents a Boolean value which is then utilized in operations and to represent more complex systems of information. The bit can be represented by anything that can be interpreted as having two distinct states: electronics typically use electric "pressure" (two distinct voltages), fluidics typically use water pressure [8], and more examples of physical representations of bits can be found[4].

B. Time Evolution

In classical computation architectures the time evolution of the state of the system, that is the state of the bits in the system, is simple to define and customize as needed.

As expected state of the system does not change unless it is acted on upon by some operator: these operators are called logic gates. Passing one or more bit through a logic gate results in a deterministic output of the gate.

The macroscopic nature of classical computation architectures results in a significant freedom in the ability to define theoretical logic gates and implement them in reality. In digital logic systems, electronic systems where bits are represented by a 5 volt difference between the Boolean values, logic gates are typically implemented in the form of transistors. Fluidic logic systems employ clever products of engineering to do the same, as demonstrated in figure 1.

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* Also at Department of Computer Science, Indiana University-Purdue University Indianapolis.
† Also at Department of Electrical Engineering, Indiana University-Purdue University Indianapolis.

Input	Output
0	1
1	0

TABLE I. Classical NOT Gate Truth Table

Input	Output
0 0	0
1 0	0
0 1	0
1 1	1

TABLE II. Classical AND Gate Truth Table

C. Logic Gates

There exists a ubiquitous set of elementary logic gates, with very clear parallels to formal logic theory, that can be seen as the theoretical foundation of all logical operators: the AND gate, the OR gate, and the NOT gate. All other classical logical gates can be generated as a combination of these basic logical operators, a property known as universality.

The NOT gate is a single input single output logic gate system that reverses the value of the bit; the truth table for this configuration is displayed in table I. The AND and OR gates both take two input bits and produce a single output bit. The truth tables for the AND and OR gates are displayed in tables II and III respectively.

More complex gates can be constructed through combinations of the elementary logic gates. Two more commonly found logic gates are the XOR and NAND gates. An XOR (exclusive-or) gate takes two inputs and produces a 1 if and only if the sum of the two inputs is 1, as shown in table IV. The NAND gate is simply an and gate followed by a not gate, as demonstrated in table V.

1. Reversibility

Logical reversibility requires that the input state of a logical operation can be deduced from the output of the logical operation. In some cases, like the NOT gate, this is trivial. In other cases this is impossible without the addition of more bits. A consequence of reversibility is that the operation of a logical gate can be expressed in matrix notation. Matrices that are reversible are unitary as defined by equation 1, where U^t is the transpose of the matrix and I is the identity matrix.

Input	Output
0 0	0
1 0	1
0 1	1
1 1	1

TABLE III. Classical OR Gate Truth Table

Input	Output
0 0	0
1 0	1
0 1	1
1 1	0

TABLE IV. Classical XOR Gate Truth Table

Input	Output
0 0	1
1 0	1
0 1	1
1 1	0

TABLE V. Classical NAND Gate Truth Table

$$U^t U = I \tag{1}$$

To make use of the matrix representation of logical gates we must first define the input state of a system in vector form as defined in equation 5, where $\delta_i j$ is the Kronecker delta function and the subscript b refers to the boolean value of the classical bit.

$$\psi = \begin{bmatrix} \delta_{0b} \\ \delta_{1b} \end{bmatrix} \tag{2}$$

With this representation it is easy to derive that the matrix representation of the NOT gate can be expressed by equation 3. We can see that the unitary requirement of reversible gates holds for this operation.

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{3}$$

To represent two bit input logical gates in matrix representation we need to define a two bit state vector representation. This representation is defined in equation 4 where b_1 is the boolean value of the first bit and b_2 is the boolean value of the second bit.

$$\psi = \begin{bmatrix} \delta_{0b_1} * \delta_{0b_2} \\ \delta_{0b_1} * \delta_{1b_2} \\ \delta_{1b_1} * \delta_{0b_2} \\ \delta_{1b_1} * \delta_{1b_2} \end{bmatrix} \tag{4}$$

With this state representation we can define a matrix which can be interpreted as a generalized version of XOR: the controlled not gate C.

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{5}$$

This logical gate operates in a way that applies the NOT function to the second bit if and only if the first bit, the control bit, is 1. We can note that this gate is reversible, in principle, because the first output bit remains unchanged and the output second bit records the result of the operation; because of this it is clear how the original input state may be logically found. Mathematically the reversibility of this gate can be proven by evaluating the unitary condition on this gate, which indeed resolves to the identity matrix.

2. *Universality*

Universality is a relevant topic of discussion in logical computation. A set of gates is considered universal if and only if every other possible logical gate can be created from that set of gates. This is seen with the set of the elementary logic gates, and, not, and or. A gate itself can be considered universal a set consisting only of itself is universal. Gate which match these conditions in the classical domain are the NAND and NOR gates. Universality provides a method of reducing engineering complexity in circuitry implementations: one only needs to be able to easily recreate one type of logical gate rather than many types physically.

III. QUANTUM COMPUTING BACKGROUND

Mechanics at the angstrom scale is subject to phenomena that are non-intuitive and non-observed at the macroscopic scale. The discrete nature of observable states, the superposition nature of states, and the interactions between multiple particles all breed consequences that make quantum computation distinctly different from classical computation.

A. Memory

The introduction of quantum mechanics in state representation brings properties to quantum mechanic states that have no clear parallels in the macroscopic domain. Representations of state in the quantum scale, then, are significantly different than representations in the macroscopic scale, as used by classical computing. The most prominent difference is superposition.

The discrete nature of quantum states is conducive to storing bits of information. A bit of information can be stored as a the spin up and down of a spin-1/2 particle, the polarization of a photon (clockwise -vs- counter clockwise), the first and second excited states of an electron in a hydrogen atom, and more [5, 2].

1. *Single Qubit*

The theoretical analog of a classical bit in a quantum system can be represented by a prepared quantum state of a two-level system, otherwise known as a qubit. It is important to note, however, that the qubit is a more general representation of information than a classical bit: in a quantum system with eigenstates 0 and $key1$ a qubit, a general state in the system, can be expressed as a linear combination of both of these eigenstates. A general state can formally be represented as demonstrated in equation 6. The parallelism between the qubit and the bit can be easily seen when comparing the vector representation of the bit: a bit can be expressed as a qubit with the probability of being in either state being equal to 1.

$$|\psi\rangle = c_0 |0\rangle + c_1 |1\rangle = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} \tag{6}$$

2. *Double Qubit*

States consisting of multiple qubits can also be analyzed. In a system with two qubits each represented in the basis of eigenstates 0 and $key1$ there exist four basis states. The general state in this system, expressed as a superposition of each possible eigenstate of the joint system, is defined in equation 7. Again, we find a similar parallelism between the quantum and classical states when both are represented in vector format.

$$\begin{aligned} |\psi\rangle = c_{00} |0\rangle |0\rangle + c_{01} |0\rangle |1\rangle \\ + c_{10} |1\rangle |0\rangle + c_{11} |1\rangle |1\rangle \\ = \begin{bmatrix} c_{00} \\ c_{01} \\ c_{10} \\ c_{11} \end{bmatrix} \end{aligned} \tag{7}$$

For scenarios when we are dealing with multiple states it is useful to adopt a notation which distinguishes between the states. For all intensive purposes of this paper, the relationship between the joint states demonstrated in equation 8 will always hold. Further to distinguish which qubit in particular we are talking in scenarios where the ϵ takes a particular value and it is important to keep explicit track of each qubit, we denote the state with a subscript as demonstrated in equation 9. It is important to keep in mind that each state of the joint state can in itself be expressed by 6.

$$|\epsilon_1 \epsilon_2\rangle = |\epsilon_1\rangle |\epsilon_2\rangle \tag{8}$$

$$|\epsilon_1\rangle = |\epsilon_1\rangle_1 \tag{9}$$

3. *Measurement*

It is relevant to note that if a state is normalized the modulus of each coefficient represents the probability of finding the particle represented by the state in eigenstate corresponding to the state. This arises due to the fact that in reality, in a two-level system, a qubit can only be in either of the two eigenstates discretely. In other words, if you attempt to measure the state of the particle you will find it to be in one of the two states. Further, the act of measurement itself will "collapse" the state of the qubit into the eigenstate corresponding to the value measured. This "collapse" of the state leads to consequences of irreversibility which is an important factor to consider in quantum computations.

A measurement in quantum circuit notation is demonstrated in figure 2, where M is the classical output.

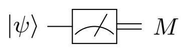

FIG. 2. Measurement Notation

4. *Entanglement*

A property of measurement in two qubit systems that is important to discuss is entanglement. Suppose a system is prepared in one of the Bell states (a.k.a. EPR states) such as defined in equation 10. In this senario if you measure qubit $|\epsilon_1\rangle$ you will know the value of qubit $|\epsilon_2\rangle$. As you measure $|\epsilon_1\rangle$ the state of the system collapses into the eigenfunction of the eigenvalue you measure; because the state is prepared in such a way that knowing the state one qubit is in tells you the state the other qubit is in the system is defined to be entangled.

$$|\psi\rangle = \frac{|0\rangle\,|0\rangle + |1\rangle\,|1\rangle}{\sqrt{2}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \tag{10}$$

To illustrate the point more clearly examine the state defined in equation 11. Measurement of qubit $|\epsilon_1\rangle$ will do nothing in providing information about which state qubit $|\epsilon_2\rangle$ could be in. This state is clearly not entangled as the measurement of $|\epsilon_2\rangle$ imposes no restriction upon which state qubit $|\epsilon_2\rangle$ could be in and will not result in a collapse of the state of qubit $|\epsilon_1\rangle$. In other words, the two states are not dependent on eachother.

$$|\psi\rangle = (a\,|0\rangle_1 + b\,|1\rangle_1)\,|0\rangle_2 = \begin{bmatrix} a \\ 0 \\ b \\ 0 \end{bmatrix} \tag{11}$$

This property of entanglement is especially important in quantum computing as it enables the measurement and classical readout of the state of the quantum computer in a way that does not change the uncertainty of the non-measured bit. This type of measurement, as long as it also is repeatable and does not alter the eigenstate of the measured particle after measurement, is defined as a Quantum Non-Demolition (QND) measurement [7].

B. Logical Gates

Information processing requires the ability to manipulate bits in order to calculate or transfer information. This is true in the classical and quantum computing domains. We will see that the superposition and entanglement properties of quantum states enables operations that have no parallels in classical computing.

Time evolution operator is unitary. Reversibility.

May notice that X matrix looks like pauli x. The notation is not a coincidence.

1. *Time Evolution*

The time evolution of quantum states depends on the, aptly named, time evolution operator. This time evolution operator is a unitary operator and thus dictates that all logical gates in quantum computation must be reversible. A simple mathematical representation of the unitary operator is defined in equation 12 where the operator, U, acts on a state, $|\Psi(0)\rangle$, and produces a new state $|\Psi(t)\rangle$.

$$|\Psi(t)\rangle = U(t,0)\,|\Psi(0)\rangle \tag{12}$$

2. *Single Qubit Logic Gates*

Contrary to the case in classical computing, there exists more than one nontrivial single qubit logic gate. As before, the reversible NOT gate is still applicable in quantum computing. We note that the NOT gate, equation 13, coincides exactly with the Pauli spin matrix in the x direction; it is not a coincidence that the matrix is also called the X gate. Similarly we can define the Z gate, demonstrated in equation 14. A unique quantum gate of particular importance is the Hadamard gate, represented in equation 15, which is able to take an eigenstate and produce an equal probability superposition of the basis states.

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{13}$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{14}$$

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{15}$$

For convenience and to further illustrate the effect of these single qubit gates the affect of applying them to a general qubit state, equation 6, is defined in equations 16, 17, and 18. Note that we replace nomenclature of the coefficients c_0 and c_1 with a and b respectively.

$$X|\psi\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix} = b|0\rangle + a|1\rangle \tag{16}$$

$$Z|\psi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -b \end{bmatrix} = a|0\rangle - b|1\rangle \tag{17}$$

$$\begin{aligned} H|\psi\rangle &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} a+b \\ a-b \end{bmatrix} \\ &= \frac{(a+b)}{\sqrt{2}}|0\rangle + \frac{(a-b)}{\sqrt{2}}|1\rangle \end{aligned} \tag{18}$$

Its relevant to point out, explicitly, the result of applying the H gate twice on the same state. Utilizing the result of equation 18 and evaluating further we find that this operation returns the original state $|\psi\rangle$. This is to be expected as we can see that $H^t = H$.

$$\begin{aligned} HH|\psi\rangle &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a+b \\ a-b \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} a+b+a-b \\ a+b-a+b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \\ &= |\psi\rangle \end{aligned} \tag{19}$$

3. *Double Qubit Logic Gates*

The CNOT gate from classical logic holds in the quantum logic domain due to the property of reversibility. A more precise nomenclature for the CNOT gate is required as the choice of the control qubit is arbitrary. The CNOT gate, defined in equation 20, operates on the two qubit state where i is the index of the control qubit and j is the index of the signal qubit.

$$C_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{20}$$

A useful and concise non-matrix notation of the CNOT operation is defined in equation 21, where the $\oplus$ operation, addition modulus 2, is represented by equation 22.

$$C_{ij}|\epsilon_i\rangle|\epsilon_j\rangle = |\epsilon_i\rangle|\epsilon_j \oplus \epsilon_i\rangle \tag{21}$$

$$x \oplus y = (x+y)\%2 \tag{22}$$

The CNOT gate is very important in quantum computing as it enables the entanglement of quantum states, which is critical for QND measurements, swaping states, and as we will see later for single qubit teleportation. This operation can also be expressed in quantum circuit notation as demonstrated in figure 3.

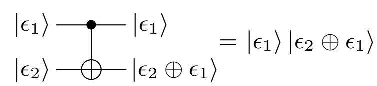

FIG. 3. CNOT Circuit Notation

An notable example of the entanglement capability of the CNOT gate can be demonstrated through creating Bell states. The quantum circuit in figure 4 defines the operations required for creating a Bell states. Equation 26 explicitly evaluates the operations defined in the circuit under the conditions that both $x, y = \{0, 1\}$ (i.e., both start in eigenstates), utilizing equation 23 in the process. We note that γ_+ and γ_- can be expressed in a more precise way under the bounds placed on x and y as defined by equations 24 and 25 respectively.

$$\begin{aligned} |\psi_1\rangle &= H_1|x\rangle_1|y\rangle_2 \\ &= \frac{a(|0\rangle_1 + |1\rangle_1)}{\sqrt{2}}|y\rangle_2 + \frac{b(|0\rangle_1 - |1\rangle_1)}{\sqrt{2}}|y\rangle_2 \\ &= \frac{a+b}{\sqrt{2}}|0\rangle_1|y\rangle_2 + \frac{a-b}{\sqrt{2}}|1\rangle_1|y\rangle_2 \\ &= \gamma_+|0\rangle_1|y\rangle_2 + \gamma_-|1\rangle_1|y\rangle_2 \end{aligned} \tag{23}$$

$$\gamma_+ = \frac{a+b}{\sqrt{2}} = \frac{1}{\sqrt{2}} \tag{24}$$

$$\gamma_- = \frac{a-b}{\sqrt{2}} = \frac{(-1)^x}{\sqrt{2}} \tag{25}$$

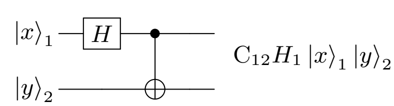

FIG. 4. Quantum Circuit for Creating Bell States

$$
\begin{aligned}
C_{12}H_1 |x\rangle_1 |y\rangle_2 &= C_{12} |\psi_1\rangle \\
&= C_{12}(\gamma_+ |0\rangle_1 |y\rangle_2 + \gamma_- |1\rangle_1 |y\rangle_2) \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \gamma_+\delta_{y0} \\ \gamma_+\delta_{y1} \\ \gamma_-\delta_{y0} \\ \gamma_-\delta_{y1} \end{bmatrix} \\
&= \begin{bmatrix} \gamma_+\delta_{y0} \\ \gamma_+\delta_{y1} \\ \gamma_-\delta_{y1} \\ \gamma_-\delta_{y0} \end{bmatrix} \\
&= \frac{|0\rangle_1 |y\rangle_2}{\sqrt{2}}(\delta_{y0} + \delta_{y1}) \\
&+ \frac{(-1)^x |1\rangle_1 |y\rangle_2}{\sqrt{2}}(\delta_{y0} + \delta_{y1}) \\
&= \frac{|0\rangle_1 |y\rangle_2 + (-1)^x |1\rangle_1 |y\rangle_2}{\sqrt{2}}
\end{aligned} \tag{26}
$$

4. *Universality*

It has been proven that, contrary to what was found in classical architectures, a two qubit universal gate is sufficient for universality [1]. This is relevant to note as the one qubit teleporter is a two qubit operation. Knowing that two qubit operators can be sufficient for universality enables the potential of the one qubit teleporter to construct any gate.

5. *Teleportation*

Teleportation is a phenomenon that is enabled by the entanglement property of quantum states. In this section it will be shown that armed with only two bits of classical information and a Bell state, two individuals can successfully transport (i.e., communicate) an arbitrary quantum state. This is particularly important as it is extremely difficult to classically communicate a quantum state due to the requirement of infinite measurements.

Consider the situation where Alice and Bob produced an entangled state, a Bell state in this case, and subsequently took a qubit from this state each. The bell state is defined in equation 27 where Alice's qubit is distinguished with the subscript a and Bob's qubit is distinguished with the subscript b.

$$
|\psi\rangle = \frac{|0\rangle_a |0\rangle_b + |1\rangle_a |1\rangle_b}{\sqrt{2}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \tag{27}
$$

Suppose that they have a general state $|\phi\rangle_s$, a signal state, that they wish to communicate between eachother.

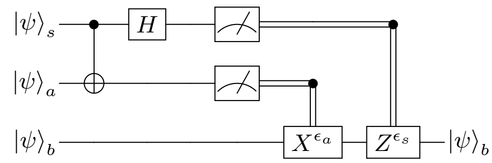

FIG. 5. Quantum circuit for standard quantum teleportation

With the circuitry presented in figure 5 we will prove that indeed Bob's half of the entangled bell state can be made to possess the information stored in the signal state. In the nomenclature, the gates X^{ϵ_a} and Z^{ϵ_s} represent classically controlled gates; i.e., if the eigenvalue of $|\psi\rangle_a$ is measured to be $\epsilon_a = 1$ then the gate operation X is conducted on state $|\psi\rangle_b$

The result of this circuit can be proven by carefully keeping track of the states and which operators act on which states. The goal is to somehow replace the original state of Bob's qubit, $|\psi\rangle_b \propto |0\rangle_b + |1\rangle_b$, with a general signal state, $|\psi\rangle_s = a|0\rangle + b|1\rangle$. Equations 28 through 30 complete the operations step by step. Note that in equation 30 we utilize equation 18.

$$
\begin{aligned}
|\psi_0\rangle &= |\psi\rangle_s |\psi\rangle_a |\psi\rangle_b \\
&= \frac{a}{\sqrt{2}} |0\rangle_s (|0\rangle_a |0\rangle_b + |1\rangle_a |1\rangle_b) \\
&+ \frac{b}{\sqrt{2}} |1\rangle_s (|0\rangle_a |0\rangle_b + |1\rangle_a |1\rangle_b)
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
|\psi_1\rangle &= C_{sa} |\psi_0\rangle \\
&= \frac{a}{\sqrt{2}} |0\rangle_s (|0\rangle_a |0\rangle_b + |1\rangle_a |1\rangle_b) \\
&+ \frac{b}{\sqrt{2}} |1\rangle_s (|1\rangle_a |0\rangle_b + |0\rangle_a |1\rangle_b)
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
|\psi_2\rangle &= H_g |\psi_1\rangle \\
&= \frac{a}{2}(|0\rangle_g + |1\rangle_g)(|0\rangle_a |0\rangle_b + |1\rangle_a |1\rangle_b) \\
&+ \frac{b}{2}(|0\rangle_g - |1\rangle_g)(|1\rangle_a |0\rangle_b + |0\rangle_a |1\rangle_b) \\
&= \frac{1}{2} |0\rangle_g |0\rangle_a (a|0\rangle_b + b|1\rangle_b) + \frac{1}{2} |0\rangle_g |1\rangle_a (b|0\rangle_b + a|1\rangle_b) \\
&+ \frac{1}{2} |1\rangle_g |0\rangle_a (a|0\rangle_b - b|1\rangle_b) + \frac{1}{2} |0\rangle_g |1\rangle_a (b|0\rangle_b - a|1\rangle_b) \\
&= \frac{1}{2} |0\rangle_g |0\rangle_a (|\psi\rangle_b) + \frac{1}{2} |0\rangle_g |1\rangle_a (X_b |\psi\rangle_b) \\
&+ \frac{1}{2} |1\rangle_g |0\rangle_a (Z_b |\psi\rangle_b) + \frac{1}{2} |0\rangle_g |1\rangle_a (X_b Z_b |\psi\rangle_b)
\end{aligned} \tag{30}
$$

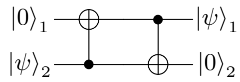

FIG. 6. Quantum circuit for swapping a general signal state, $|\psi\rangle_2$, and an ancilla of $|\psi\rangle_1 = |0\rangle_1$

The final relationship in equation 30 demonstrates that, due to entanglement, depending on the classically measured values of ϵ_s and ϵ_a, the eigenvalues corresponding to the collapsed eigenstate of states $|\psi\rangle_s$ and $|\psi\rangle_a$ respectively, after the H gate bob has in his possession a qubit which identically represents a transformed version of the original signal qubit. At this point, armed with the values of ϵ_s and ϵ_a, Bob can conditionally apply gates X^{ϵ_a} and Z^{ϵ_s} to reconstruct the original information of the signal state $|\psi\rangle_s$.

IV. SINGLE QUBIT TELEPORTATION

The paper "Methodology for quantum logic gate construction"[9] outlines the basis on which teleportation could be conducted in a more simple manner than demonstrated in section III B 5. They paper constructs the argument for the single qubit teleporter based on three facts. However, these Stanford slackers left gaps in their work, not proving the facts and not demonstrating explicitly how the facts resolve to their conclusion, which someone without a foundation of quantum mechanics knowledge carefully crafted by Dr. Ricardo Decca would not be able to fill in themselfs. Fortunately for all people without the gift of a quantum mechanics education, we will explicitly fill in the gaps and demonstrate why the conclusions of the original paper are valid.

A. Fact One

Claim: an ancilla state of $|0\rangle_1$ and a general signal state of $|\psi\rangle_2$ can be swaped with two consecutive CNOT operations.

The circuitry for this operation is demonstrated in figure 6. As usual, we prove this claim by applying the operations step by step and keeping track which states the operators act on and are dependent on.

$$\begin{aligned} |\psi_0\rangle &= |0\rangle_1 |\psi\rangle_2 \\ &= |0\rangle_1 (a|0\rangle_2 + b|1\rangle_2) \\ &= a|0\rangle_1 |0\rangle_2 + b|0\rangle_1 |1\rangle_2 \end{aligned} \tag{31}$$

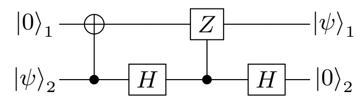

FIG. 7. An equivalent quantum circuit for swapping a general signal state, $|\psi\rangle_2$, and an ancilla of $|\psi\rangle_1 = |0\rangle_1$

$$\begin{aligned} |\psi_1\rangle &= C_{21} |\psi_0\rangle \\ &= a|0\rangle_1 |0\rangle_2 + b|1\rangle_1 |1\rangle_2 \end{aligned} \tag{32}$$

$$\begin{aligned} |\psi_2\rangle &= C_{12} |\psi_1\rangle \\ &= a|0\rangle_1 |0\rangle_2 + b|1\rangle_1 |0\rangle_2 \\ &= (a|0\rangle_1 + b|1\rangle_1) |0\rangle_2 \end{aligned} \tag{33}$$

As we see in the relationship defined at the end of equation 33, the information originally possessed in state $|\psi\rangle_2$ has been successfully swapped with the information possessed by $|\psi\rangle_1$.

B. Fact Two

Claim: Given that the gate $X = HZH$, the operation defined the circuit of figure 6 can be equally defined by the circuit of figure 7.

In proving this claim we evaluate step by step the conclusion of the circuitry in figure 7. We note that $|\psi_0\rangle$ and $|\psi_1\rangle$ defined in equations 31 and 32 respectively are equivalent for this case. We utilize equation 19 to evaluate equation 36.

$$\begin{aligned} |\psi_2\rangle &= H_2 |\psi_1\rangle \\ &= a|0\rangle_1 \frac{|0\rangle_2 + |1\rangle_2}{\sqrt{2}} + b|1\rangle_1 \frac{|0\rangle_2 - |1\rangle_2}{\sqrt{2}} \\ &= \big(\frac{a}{\sqrt{2}}|0\rangle_1 + \frac{b}{\sqrt{2}}|1\rangle_1\big)|0\rangle_2 \\ &+ \big(\frac{a}{\sqrt{2}}|0\rangle_1 - \frac{b}{\sqrt{2}}|1\rangle_1\big)|1\rangle_2 \end{aligned} \tag{34}$$

$$\begin{aligned} |\psi_3\rangle &= Z_{21} |\psi_2\rangle \\ &= \Big(\frac{a}{\sqrt{2}}|0\rangle_1 + \frac{b}{\sqrt{2}}|1\rangle_1\Big)|0\rangle_2 \\ &+ \Big(\frac{a}{\sqrt{2}}|0\rangle_1 + \frac{b}{\sqrt{2}}|1\rangle_1\Big)|1\rangle_2 \\ &= \frac{1}{\sqrt{2}}(a|0\rangle_1 + b|1\rangle_1)(|0\rangle_2 + |1\rangle_2) \end{aligned} \tag{35}$$

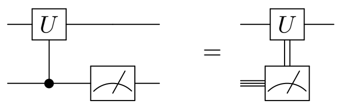

FIG. 8. A controlled gate followed by a measurement is equivalent to a classically controlled gate.

$$\begin{aligned} |\psi_4\rangle &= H_2 |\psi_3\rangle \\ &= (a|0\rangle_1 + b|1\rangle_1)|0\rangle_2 \end{aligned} \tag{36}$$

As promised, we find that again the information is swapped between $|\psi\rangle_2$ and $|\psi\rangle_1$ and that the two circuits are equivalent.

C. Fact Three

Claim: Conducting a measurement on the control bit after a quantum-controlled gate is processed is equivalent to measuring the control bit and enacting a classically controlled gate based on the result, such as represented in the circuit of figure 8.

This argument is based on a more conceptual basis than that used for prior facts. Consider the states before and after both measurement and the controlled-gate, $|\psi_0\rangle$ and $|\psi_1\rangle$ as defined by equations 37 and 38 respectively. Note, these states have nothing to do with the states defined for the previous facts.

$$|\psi_0\rangle = |\psi\rangle_1 |\psi\rangle_2 \tag{37}$$

$$\begin{aligned} |\psi_1\rangle &= U_{21} |\psi\rangle_1 |\psi\rangle_2 \\ &= a|0\rangle_2 |\psi\rangle_1 + b|1\rangle_2 U_1 |\psi\rangle_1 \end{aligned} \tag{38}$$

After measurement of $|\psi\rangle_2$ we find the total system to collapse into either state $|0\rangle_2 |\psi\rangle_1$ or state $|1\rangle_2 U_1 |\psi\rangle_1$, with a probability of $|a|^2$ and $|b|^2$ respectively. In other words, we are guaranteed that if we measure $|\psi\rangle_2$ to be in state $|0\rangle$ we are guaranteed that U_1 is not applied to $|\psi\rangle_1$. Conversely, we are guaranteed that it is applied if we find $|\psi\rangle_2$ to be in state $|1\rangle$.

It is due to this entangled nature of state $|\psi_1\rangle$, equation 38, that this guarantee exists. Thus, regardless of if we apply the conditional quantum gate and then collapse the system through measurement or if we measure and then apply a classically conditional gate we will find the same state, as this claim promises.

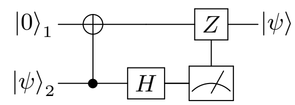

FIG. 9. Circuitry for a Z-Teleporter gate.

D. Z Teleporter

Claim: by applying fact 3 to the circuitry described in figure 7 we produce a cicruit which acts as a teleporter, as denoted by 9.

Fact 3 states that applying the circuitry in figure 9 results in a final system state, $|\psi\rangle_f$, which is equivalent to $|psi_3\rangle$ of equation 35. This state is not entangled, meaning that the value you measure for $|\psi\rangle_2$ does not affect $|\psi\rangle_1$ in any way. It also demonstrates that the information of the original signal qubit $|\psi\rangle_2$ is captured in the final qubit $|\psi\rangle_1$. Thus, we see that the circuit successfully teleports the information originally in qubit $|\psi\rangle_1$ to the qubit $|\psi\rangle_2$.

The reason this circuit is called a single qubit teleporter is because, contrary to the standard teleporter described in section III B 5, this teleporter does not need an entangled state as an ancilla state; it can operate with one single ancilla qubit, in this case $|0\rangle_1$.

V. DISCUSSION

This Z-teleporter defined in section IV D is particularly useful because it can be used to build fault-tolerant gates for any logical operation in a universal way[9]. Section III of that paper is devoted to demonstrating the systematic method by which any fault-tolerant quantum gate can be constructed based on these the Z-teleporter.

Fault tolerance is critical in information processing with quantum computation as circuitry which enables errors to propagate will corrupt the conclusions of the computation and make results not reliable. The requirement of fault tolerance makes certain combinations of logical operators not possible in quantum circuitry. This method, which uses the Z-teleporter to reduce quantum circuitry into fault-tolerant forms, provides a systematic method of creating a fault-tolerant gate for any logical circuit.

VI. CONCLUSION

Quantum computation is capable of generating unparalleled computational speed increases due to its ability to compute information in parallel with single operations, due to superposition. In order to trust the results of quantum computations it is critical to conduct the computation with fault tolerant circuitry. Single bit teleportation can be used to in a systematic way to produce fault tolerant gates for any logical circuit [9]. Armed with the basics of quantum mechanics and carefully evaluated mathematics it is possible to prove the foundations and conclusions that make the single bit teleporter possible.

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