Electricity and Magnetism I (P331)
M. R. Shepherd
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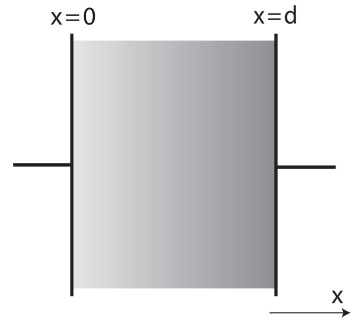
A capacitor with a non-uniform dielectric



Consider a parallel-plate capacitor filled with non-uniform dielectric. The dielectric is still linear, in other words, $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$, but this time χ_e varies as a function of position in the dielectric. Let's say that χ_e varies such that

$$\epsilon_r = \epsilon_{r0} + ax.$$

Here ϵ_{r0} is a dimensionless constant, x is the distance from one plate, and a is constant with units 1/length. Let's assume the area of the plates is A and their separation is d. What is the capacitance?

Remember the strategy for computing capacitance: put a charge Q on the capacitor and then compute the potential difference between the plates in terms of this charge. The capacitance can then be identified by comparing this expression to $V = Q/C$. In order to compute V we first need to find the electric field.

Let's charge up the plates so that we have a free surface charge density of $+\sigma_f$ on the left plate and $-\sigma_f$ on the right plate. We can then compute the electric displacement $\mathbf{D}$ based on the free charge, and obtain

$$\mathbf{D} = \sigma_f \ \hat{\mathbf{x}}.$$

This then gives

$$\mathbf{E} = \frac{1}{\epsilon}\mathbf{D} = \frac{\sigma_f}{\epsilon_0(\epsilon_{r0} + ax)}\hat{\mathbf{x}}$$

Now, to obtain the potential, we need to compute the line integral of this field from the negative plate to the positive plate. We have

$$V = -\int \mathbf{E} \cdot d\mathbf{l} = -\frac{\sigma_f}{\epsilon_0}\int_d^0 \frac{1}{\epsilon_{r0} + ax} dx = \frac{\sigma_f}{\epsilon_0 a}\left[\ln(\epsilon_{r0} + ad) - \ln(\epsilon_{r0})\right]$$

Let's rewrite the difference of logs as a log of a ratio and substitute in $\sigma_f = Q/A$. This allows us to identify the capacitance

$$V = \frac{Q \ln(1 + ad/\epsilon_{r0})}{A\epsilon_0 a} = \frac{Q}{C} \Longrightarrow C = \frac{A\epsilon_0 a}{\ln(1 + ad/\epsilon_{r0})}.$$

It is interesting to examine this expression in the limit that the dielectric is uniform. In this case we have $a \to 0$ and $\epsilon_r = \epsilon_{r0}$. This poses a little problem since, in the expression above, substituting in $a = 0$ we have $0/\ln(1) = 0/0$. It is helpful to remember that for small values of y we can write

$$\ln(1 + y) = y - \frac{y^2}{2} + \frac{y^3}{3} + ...$$

This means that for a very small we have

$$C = \frac{A\epsilon_0 a}{\ln(1 + ad/\epsilon_{r0})} \approx \frac{A\epsilon_0 a}{(ad/\epsilon_{r0}) - (ad/\epsilon_{r0})^2/2} = \frac{A\epsilon_0}{(d/\epsilon_{r0}) - a(d/\epsilon_{r0})^2/2}.$$

We can see that in the limit $a \to 0$ this expression reduces to $\epsilon_{r0}\epsilon_0 A/d$, which is exactly what we expect for a parallel plate capacitor with dielectric constant ϵ_{r0}. *Exercise:* Compute the bound charge densities σ_b on both sides of the capacitor. Which side should have the higher bound charge density? In this case, $\rho_b \neq 0$ – compute it also. Does its sign agree with your intuition?