Edge Effects in the Parallel Plate Capacitor: The Maxwell Transformation and the Rogowski Profile

The parallel-plate capacitor is a common capacitor geometry, and if the plates are of infinite extent, it is one of the simplest problems in electrostatics. However the standard simple analysis neglects the fringing field at the edge of the plates. Now we will see an analysis first due to Maxwell, and developed further by Rogowski, which enables a quantitative calculation of the fringe field. This will also serve as an introduction to **conformal mapping**, a powerful technique for solving Laplace's equation.

Start by defining two functions u(x,y) and v(x,y) implicitly via the following equations:

$$x \equiv \left(\frac{a}{\pi}\right)\left(u+1+e^{u}\cos \mathrm{v}\right) \text{ and } y \equiv \left(\frac{a}{\pi}\right)\left(\mathrm{v}+e^{u}\sin \mathrm{v}\right)$$

Now it so happens that both u(x,y) and v(x,y) as defined above satisfy Laplace's equation in two dimensions, i.e. $\nabla^2 u = 0$ and $\nabla^2 \mathrm{v} = 0$. Let's verify this by taking partial derivatives of each equation with respect to x and y:

$$\frac{\partial}{\partial x}(x) = 1 = \left(\frac{a}{\pi}\right)\left[\frac{\partial u}{\partial x} + e^{u}\left(\frac{\partial u}{\partial x}\right)\cos \mathrm{v} - \left(\frac{\partial \mathrm{v}}{\partial x}\right)e^{u}\sin \mathrm{v}\right] \quad \text{Equation } \mathbf{[A]}$$

$$\frac{\partial}{\partial y}(x) = 0 = \left(\frac{a}{\pi}\right)\left[\frac{\partial u}{\partial y} + e^{u}\left(\frac{\partial u}{\partial y}\right)\cos \mathrm{v} - \left(\frac{\partial \mathrm{v}}{\partial y}\right)e^{u}\sin \mathrm{v}\right] \quad \text{Equation } \mathbf{[B]}$$

$$\frac{\partial}{\partial x}(y) = 0 = \left(\frac{a}{\pi}\right)\left[\frac{\partial \mathrm{v}}{\partial x} + e^{u}\left(\frac{\partial u}{\partial x}\right)\sin \mathrm{v} + \left(\frac{\partial \mathrm{v}}{\partial x}\right)e^{u}\cos \mathrm{v}\right] \quad \text{Equation } \mathbf{[C]}$$

$$\frac{\partial}{\partial y}(y) = 1 = \left(\frac{a}{\pi}\right)\left[\frac{\partial \mathrm{v}}{\partial y} + e^{u}\left(\frac{\partial u}{\partial y}\right)\sin \mathrm{v} + \left(\frac{\partial \mathrm{v}}{\partial y}\right)e^{u}\cos \mathrm{v}\right] \quad \text{Equation } \mathbf{[D]}$$

Now, if we add **[B]** to **[C]** we get

$$\left(\frac{\partial u}{\partial y} + \frac{\partial \mathrm{v}}{\partial x}\right)\left(1+e^{u}\cos \mathrm{v}\right) + \left(\frac{\partial u}{\partial x} - \frac{\partial \mathrm{v}}{\partial y}\right)e^{u}\sin \mathrm{v} = 0 \quad \text{Equation } \mathbf{[E]}$$

Similarly, if we subtract **[D]** from **[A]** we get:

$$\left(\frac{\partial u}{\partial x} - \frac{\partial \mathrm{v}}{\partial y}\right)\left(1+e^{u}\cos \mathrm{v}\right) - \left(\frac{\partial u}{\partial y} + \frac{\partial \mathrm{v}}{\partial x}\right)e^{u}\sin \mathrm{v} = \frac{\pi}{\mathrm{a}} - \frac{\pi}{a} = 0 \quad \text{Equation } \mathbf{[F]}$$

Now multiply [E] by $\left(\frac{\partial u}{\partial x} - \frac{\partial \mathrm{v}}{\partial y}\right)$ to give:

$$\left(\frac{\partial u}{\partial x} - \frac{\partial \mathrm{v}}{\partial y}\right)\left(\frac{\partial u}{\partial y} + \frac{\partial \mathrm{v}}{\partial x}\right)\left(1+e^{u}\cos \mathrm{v}\right) + \left(\frac{\partial u}{\partial x} - \frac{\partial \mathrm{v}}{\partial y}\right)^{2} e^{u}\sin \mathrm{v} = 0$$

and also multiply multiply [F] by $\left(\frac{\partial u}{\partial y}+\frac{\partial \text{v}}{\partial x}\right)$ to give:

$$\left(\frac{\partial u}{\partial x}-\frac{\partial \text{v}}{\partial y}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial \text{v}}{\partial x}\right)\left(1+e^{u}\cos\ \text{v}\right)-\left(\frac{\partial u}{\partial y}+\frac{\partial \text{v}}{\partial x}\right)^{2} e^{u}\sin\ \text{v}=0$$

Now subtract this second equation from the one which proceeds it and we get:

$$\left[\left(\frac{\partial u}{\partial x}-\frac{\partial \text{v}}{\partial y}\right)^{2}+\left(\frac{\partial u}{\partial y}+\frac{\partial \text{v}}{\partial x}\right)^{2}\right]e^{u}\sin\ \text{v}=0$$

Now for this equation to =0 for all u and v we must have the sum of squared terms equal to zero, and furthermore, since this is a sum of squares the individual terms must separately be equal to zero so we have finally

$\frac{\partial u}{\partial x}-\frac{\partial \text{v}}{\partial y}=0$ as well as $\frac{\partial u}{\partial y}+\frac{\partial \text{v}}{\partial x}=0$, which are the same as $\frac{\partial u}{\partial x}=\frac{\partial \text{v}}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial \text{v}}{\partial x}$

which are the **Cauchy-Riemann Equations** for u(x,y) and v(x,y), which as we have seen (and as is immediately apparent because of the equality of second partial derivatives with respect to x and y), imply that the functions u(x,y) and v(x,y) are (spatially) **harmonic,** that is they each satisfy the two dimensional form of Laplace's equation:

$$\nabla^{2}u(x,y)=\frac{\partial^{2}u}{\partial x^{2}}+\frac{\partial^{2}u}{\partial y^{2}}=0 \quad \text{and} \quad \nabla^{2}\text{v}(x,y)=\frac{\partial^{2}\text{v}}{\partial x^{2}}+\frac{\partial^{2}\text{v}}{\partial y^{2}}=0$$

Now the solution to our finite parallel-plate capacitor problem is almost trivial. Consider again our equation for y in terms of u and v, and now imagine that v(x,y) = V(x,y)/V_0 where V(x,y) is the electrical potential in all space, and V_0 is a normalizing quantity (units = Volts):

$$y\equiv\left(\frac{a}{\pi}\right)\left(\text{v}+e^{u}\sin\text{v}\right)$$

Clearly if we set v=0, then y = 0 for any value of u we choose. From the x equation $x\equiv\left(\frac{a}{\pi}\right)\left(u+1+e^{u}\cos\text{v}\right)=\left(\frac{a}{\pi}\right)\left(u+1+e^{u}\right)$, so x ranges from -∞ to +∞ as we vary u from -∞ to +∞.

On the other hand, if we set v=π, then y=a, again for any value of u we choose, but now we have $x\equiv\left(\frac{a}{\pi}\right)\left(u+1+e^{u}\cos\text{v}\right)=\left(\frac{a}{\pi}\right)\left(u+1-e^{u}\right)$, so now x ranges from -∞ to some finite (negative) value and then back to -∞ as we vary u from -∞ to +∞. Thus the plane y=0 is an infinite equipotential plane with V=0 Volts, and the plane y=a is an equipotential surface of finite extent extending from x=-∞ to a finite x value.

These two equipotential surfaces have the right geometries to model a capacitor plate of finite extent above an infinite ground plane -- which can also be the virtual ground midplane of a real 2-plate finite capacitor (infinite by symmetry). Now these equipotential surfaces emerged naturally as limiting cases for v(x,y), and we also know from above that that $\nabla^2 \mathrm{v}(x,y) = \frac{\partial^2 \mathrm{v}}{\partial x^2} + \frac{\partial^2 \mathrm{v}}{\partial y^2} = 0$. Because of this we are justified in concluding that the function v(x,y) in fact gives the potential (in terms of V_0) for ALL points in space (x,y). The uniqueness theorem then guarantees that this is in fact the one and only correct solution for the potential distribution around the edges of a finite-extent capacitor with plate spacing a. This is of course what we wanted to solve! We now can take our implicit expressions for v(x,y) and use them to arrive at actual field strengths in the fringe field region, and also understand the Rogowski electrode geometry—an important practical high-voltage application.

The figure below shows lines of constant potential v(x,y) and also u(x,y). It is taken from the classic reference: Gaseous Conductors: Theory and Engineering Applications by J.D. Cobine [Dover (1958)], page 178. Similar pictures can be found in many references. The original analysis is due to J.C. Maxwell.