Chapter 3

Numerical Solution to Laplace Equation; Estimation of Capacitance

3.1 Introduction

Solving Laplace equation in practical applications often requires numerical methods. In this Chapter, a few typical methods are explained. The finite difference methods (FDM) exploits the fact that if the scalar potential Φ satisfies the Laplace equation, the potential at (x, y, z) can be approximated by the average of the 6 potentials at $(x+h, y, z)$, $(x-h, y, z)$, $(x, y+h, z)$, $(x, y-h, z)$, $(x, y, z+h),)\,(x, y, z-h)$ where h is a small distance compared with the system size. For two dimensional problems, the average involves four potentials at $(x \pm h, y)$, $(x, y \pm h)$. The error of FDM is of order h^4 and the accuracy of this method improves as a smaller size h is chosen at the cost of computation time. Another well known method is the Finite Element Method (FEM). This method is based on the Thomson's theorem that the electrostatic energy becomes minimum if the potential Φ satisfies Laplace equation. Laplace equation is in fact Euler's equation to minimize electrostatic energy in variational principle. FEM has been fully developed in the past 40 years together with the rapid increase in the speed of computation power.

3.2 Finite Difference Method

The basic element in numerically solving the Laplace equation is as follows. Consider a two-dimensional potential problem such as the one we analyzed in Chapter 2. If the potential satisfies the Laplace equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \Phi\,(x, y) = 0$$

the potential at the center of four surrounding potentials is equal to the algebraic average of the four potentials, namely,

$$\Phi(x, y) \simeq \frac{1}{4}\left[\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4\right] \tag{3.1}$$

where

$$\Phi_1,\ \Phi_3 = \Phi(x \pm h, y) \tag{3.2}$$

$$\Phi_2,\ \Phi_4 = \Phi(x, y \pm h) \tag{3.3}$$

The proof is straightforward. We may Taylor expand $\Phi(x \pm h, y)$ and $\Phi(x, y \pm h)$ as

$$\Phi(x \pm h, y) \simeq \Phi(x, y) \pm h\frac{\partial \Phi}{\partial x} + \frac{h^2}{2}\frac{\partial^2 \Phi}{\partial x^2} \pm \frac{h^3}{3!}\frac{\partial^3 \Phi}{\partial x^3} + \frac{h^4}{4!}\frac{\partial^4 \Phi}{\partial x^4} \cdots \tag{3.4}$$

$$\Phi(x, y \pm h) \simeq \Phi(x, y) \pm h\frac{\partial \Phi}{\partial x} + \frac{h^2}{2}\frac{\partial^2 \Phi}{\partial y^2} \pm \frac{h^3}{3!}\frac{\partial^3 \Phi}{\partial y^3} + \frac{h^4}{4!}\frac{\partial^4 \Phi}{\partial y^4} \cdots \tag{3.5}$$

assuming that h is a small distance. Adding these four equations, we obtain,

$$\begin{aligned} &\Phi(x+h, y) + \Phi(x-h, y) + \Phi(x, y+h) + \Phi(x, y-h) \\ &\simeq 4\Phi(x, y) + h^2\left[\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right] + \mathcal{O}(h^4) \end{aligned} \tag{3.6}$$

But by assumption, Φ satisfies Laplace equation,

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \tag{3.7}$$

Therefore, within accuracy to order h^4 , we obtain Eq. (3.1). Accuracy is obviously improved by letting h sufficiently small, but excessively small values of h will make computation time consuming.

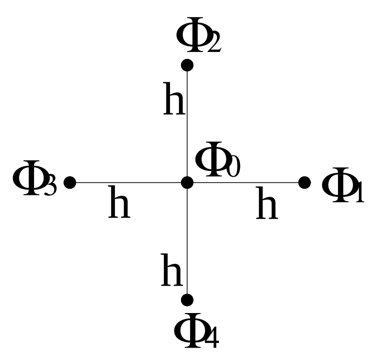


Figure 3-1: In charge free region where the potential Φ satisfies two dimensional Laplace equation $\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$, the potential Φ_0 at the center is given by the algebraic average of the four surrounding potentials $\Phi_0 = \frac{1}{4}\left(\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4\right)$.

Let us apply the theorem to the example shown in Fig. 2.2 and repeated here. Since we know analytic solution, we can check the accuracy of the numerical method. We assume a square cross-section with side a. For h, we choose $h = a/4$ and assign a total of 9 "mesh points" (actually they are "rods" extending in z direction) in the $x - y$ plane as shown in Fig. 3.3. The potentials at the mesh points on the boundary are known, and correspond to the boundary values. For the mesh

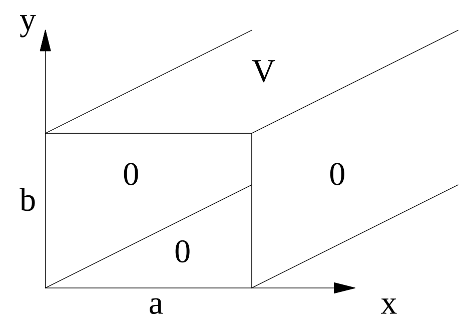


Figure 3-2: Rectangular conducting cylinder with the top plate at $\Phi = V$.

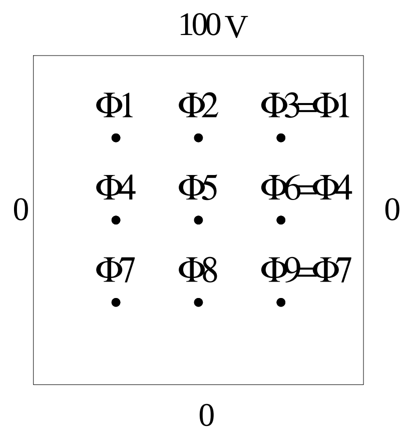


Figure 3-3: Unknown potentials at the mesh points. Note there are 6 unknowns because of the symmetry.

point 1, the potential Φ_1 is given in terms of the four surrounding potentials as

$$\Phi_1 = \frac{1}{4}(100 + \Phi_2 + \Phi_4 + 0) \tag{3.8}$$

Similarly,

$$\Phi_2 = \frac{1}{4}(100 + \Phi_1 + \Phi_3 + \Phi_5) = \frac{1}{4}(100 + 2\Phi_1 + \Phi_5) \tag{3.9}$$

$$\Phi_4 = \frac{1}{4}(0 + \Phi_1 + \Phi_5 + \Phi_7) \tag{3.10}$$

$$\Phi_5 = \frac{1}{4}(\Phi_2 + 2\Phi_4 + \Phi_8) \tag{3.11}$$

$$\Phi_7 = \frac{1}{4}(\Phi_4 + \Phi_8) \tag{3.12}$$

$$\Phi_8 = \frac{1}{4}(\Phi_5 + 2\Phi_7) \tag{3.13}$$

These are 6 simultaneous equations for 6 unknowns $\Phi_1 = \Phi_3$, $\Phi_2, \Phi_4 = \Phi_6, \Phi_5, \Phi_7 = \Phi_9$ and Φ_8. In matrix form,

$$
\begin{bmatrix} 4 & -1 & -1 & 0 & 0 & 0 \\ -2 & 4 & 0 & -1 & 0 & 0 \\ -1 & 0 & 4 & -1 & -1 & 0 \\ 0 & -1 & -2 & 4 & 0 & -1 \\ 0 & 0 & -1 & 0 & 4 & -1 \\ 0 & 0 & 0 & -1 & -2 & 4 \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_4 \\ \Phi_5 \\ \Phi_7 \\ \Phi_8 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.14}
$$

Solutions are

$$
\Phi_1 = 42.86, \Phi_2 = 52.68, \Phi_4 = 18.75, \Phi_5 = 25.0, \Phi_7 = 7.14, \Phi_8 = 9.82
$$

all in volts.

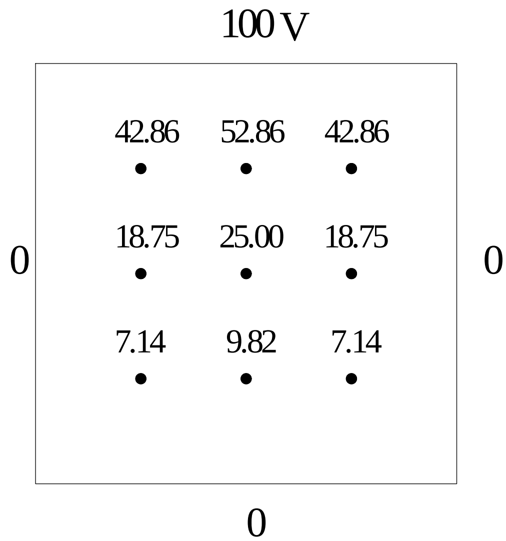


Figure 3-4: Solutions for the mesh potentials Φ_1˜Φ_9.

Let us compare the result with the analytic solution worked out in Chapter 2. When $a = b$, the solution is

$$
\Phi(x, y) = \frac{4V}{\pi} \sum_{n\ \mathrm{odd}} \frac{1}{n} \frac{1}{\sinh(n\pi)} \sin\left(\frac{n\pi}{a} x\right) \sinh\left(\frac{n\pi}{a} y\right) \tag{3.15}
$$

Equipotential surfaces are shown in Fig. 3.5 for the case $V = 100$ V. The potential Φ_1 at $x = a/4, y = 3a/4$ is

$$
\begin{aligned}
\Phi_1 &= \Phi(x = \frac{a}{4}, y = \frac{3}{4} a) \\
&= \frac{400}{\pi} \sum_{n\ \mathrm{odd}} \frac{1}{n} \frac{1}{\sinh(n\pi)} \sin\left(\frac{n\pi}{4}\right) \sinh\left(\frac{3n\pi}{4}\right) \\
&= 43.2 \text{ V}
\end{aligned}
$$

The error in the numerical method is about 0.7 % which would decrease if a smaller mesh size is chosen. Since $h = a/4$ in this example, the expected error is of order

$$\left(\frac{1}{4}\right)^4 = 0.4\%$$

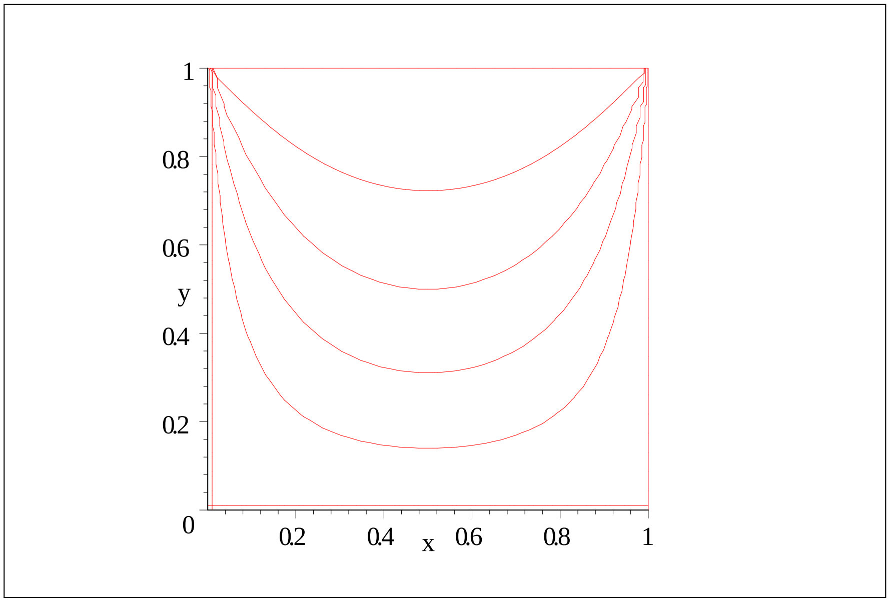


Figure 3-5: Potential inside a square ($a = b$) cylinder. The top plate is at $V = 100$ volt. Equipotential surafces are, from top, $\Phi = 50$, 25, 10, 5 V.

There is an alternative way to solve the system of equations in 3.14 which is much more efficient and less time consuming. In the so-called relaxation method, some guestimate values are initially assigned for the unknown potentials. Then, we iterate calculations of averages for each mesh point until Eq. (3.1) is satisfied for all mesh points. Initial guesses can be very rough, although the closer they are to the final values, the less number of iterations will be required. In the example, the number of unknown potentials is actually 6 because of the symmetry about the midplane. We assign 6 unknown potentials as shown in Fig 3.5. Such observation greatly reduces computation time. A MATLAB program for 30 iterations is as follows:

```
>> clear
V1(1)=0;
V2(1)=0;
V3(1)=0;
V4(1)=0;
V5(1)=0;
V6(1)=0;
for j=1:30
V1(j+1)=(100+V2(j)+V3(j))/4;
```

```
V2(j+1)=(100+V1(j)+V1(j)+V4(j))/4;
V3(j+1)=(V1(j)+V4(j)+V5(j))/4;
V4(j+1)=(V3(j)*2+V2(j)+V6(j))/4;
V5(j+1)=(V3(j)+V6(j))/4;
V6(j+1)=(V4(j)+V5(j)*2)/4;
end
>> [(1:31)', V1',V2',V3',V4',V5',V6']
```

The results are identical to those obtained by solving the simultaneous equations. In this example, a relatively large mesh size is chosen to illustrate the numerical procedure. A smaller mesh size can be chosen in practical applications in order to improve accuracy.

Assignment of mesh points does not have to be square-like. If a mesh size in the x direction is h_x, and that in the y direction is h_y, the center potential is to be modified as

$$\begin{aligned}\Phi(x,y) \quad = \quad & \frac{1}{2\left(h_x^2+h_y^2\right)}\left[h_y^2\left(\Phi(x+h_x)+\Phi(x-h_x)\right)\right. \\ & \left.+h_x^2\left(\Phi(y+h_y)+\Phi(y-h_y)\right)\right] \qquad (3.16)\end{aligned}$$

and the same numerical procedure can still be applied. Such nonuniform mesh assignment becomes necessary when the ratio a/b is an irrational number.

Example 1 ***Potential due to periodic anode***

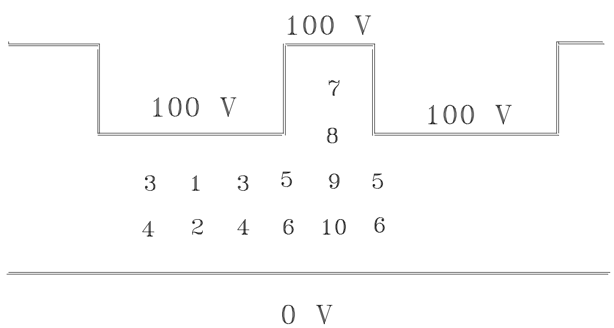


Figure 3-6: Cross-section of long cylinder with periodic anode structure.

Consider a conducting cylinder having a cross-section as shown in Fig. 3-6. The periodic upper electrode is at a potential V and the flat lower electrode is grounded. In order to apply the finite element method, we divide the cross section into sub-sections and allocate 10 nodes points as shown.

Applying Eq. (3.1) to the potentials $\Phi_i (i = 1 - 10)$, we obtain

$$
\begin{aligned}
4\Phi_1 &= 100 + \Phi_2 + 2\Phi_3, \\
4\Phi_2 &= \Phi_1 + 2\Phi_4, \\
4\Phi_3 &= 100 + \Phi_1 + \Phi_4 + \Phi_5, \\
4\Phi_4 &= \Phi_2 + \Phi_3 + \Phi_6, \\
4\Phi_5 &= 100 + \Phi_4 + \Phi_6 + \Phi_9, \\
4\Phi_6 &= \Phi_4 + \Phi_5 + \Phi_{10}, \\
4\Phi_7 &= 300 + \Phi_8, \\
4\Phi_8 &= 200 + \Phi_7 + \Phi_9, \\
4\Phi_9 &= 2\Phi_5 + \Phi_8 + \Phi_{10}, \\
4\Phi_{10} &= 2\Phi_6 + \Phi_9.
\end{aligned}
$$

Solutions are: $\Phi_1 = 63.7$, $\Phi_2 = 31.0$, $\Phi_3 = 61.8$, $\Phi_4 = 30.2$, $\Phi_5 = 53.5$, $\Phi_6 = 27.9$, $\Phi_7 = 97.1$, $\Phi_8 = 88.2$, $\Phi_9 = 55.7$, $\Phi_{10} = 27.9$ all in Volts. A larger number of node points will improve accuracy.

3.3 Graphical Method

Capacitance per unit length of cylindrical capacitors having odd cross sections can be estimated using graphical method. To become familiar with the method, let us consider the capacitance of a parallel plate capacitor,

$$C = \varepsilon_0 \frac{A}{d} = \varepsilon_0 \frac{ab}{d} \tag{3.17}$$

The capacitance per unit length along b direction is

$$\frac{C}{b} = \varepsilon_0 \frac{a}{d} \tag{3.18}$$

which can be interpreted as a capacitance consisting of a unit in parallel and d units in series as illustrated in Fig. 3.7. Each unit has unit area and its axial capacitance is $C/l = \varepsilon_0$. In charge free regions, the equi potential surfaces and electric field lines are normal to each other, the unit cross section can be approximated by a circle touching to neighboring circles.

Let us consider cylindrical capacitors as shown in Fig. 3.8. The first case has circular electrodes with outer radius $4a$ and inner radius $2a$ which are eccentric by a. We can draw 5.2 circles for the top half. Therefore, the capacitance is

$$\frac{C}{l} = \varepsilon_0 \frac{10.4}{1} = 10.4\varepsilon_0 \tag{3.19}$$

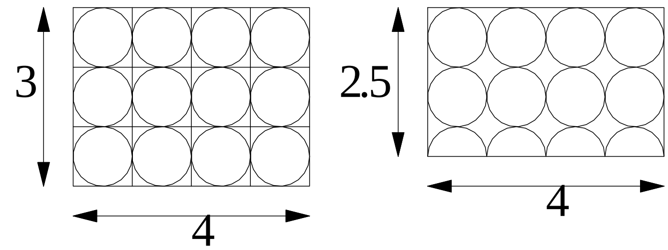


Figure 3-7: Capacitance per unit length is $\varepsilon_0 \times$ (number of parallel unit capacitors)/(number of series unit capacitors).

For this case, analytic formula is known

$$\frac{C}{l} = \frac{2\pi\varepsilon_0}{\cosh^{-1}\left(\frac{a^2+b^2-d^2}{2ab}\right)} = \frac{2\pi\varepsilon_0}{\cosh^{-1}\left(\frac{19}{16}\right)} = 10.5\varepsilon_0 \tag{3.20}$$

and the graphical method gives a reasonable estimate.

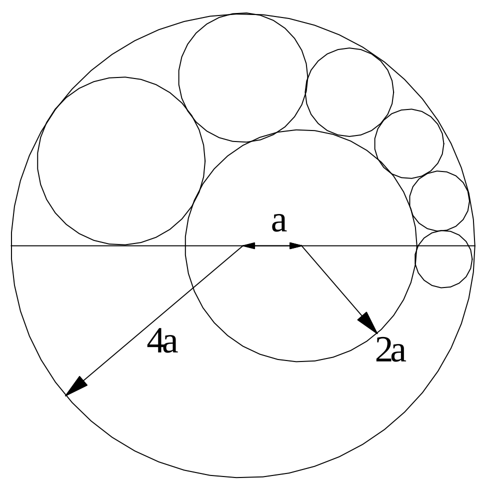


Figure 3-8: Eccentric cylindrical capacitor. 5.2 circles can be drawn in the upper half and the capacitance per unit length is about $10.4\varepsilon_0$.

The inductance per unit length of a transmission line having the same cross-section is given by

$$\frac{L}{l} = \frac{1}{c^2 C/l =} = \frac{1}{(3\times 10^8)^2 \times 10.5 \times 8.85 \times 10^{-12}} = 1.2 \times 10^{-7}\ \mathrm{H/m}$$

and the characteristic impedance is

$$Z = \sqrt{\frac{L/l}{C/l}} = \sqrt{\frac{1.2\times 10^{-7}\ \mathrm{H/m}}{10.5 \times 8.85 \times 10^{-12}\ \mathrm{F/m}}} = 35.9\ \Omega$$

The case shown in Fig. 3.9 has a square inner electrode with edge $4a$ and circular outer electrode with radius $4a$. In one quadrant, 6 parallel units and 2 series units can be drawn. Then

$$\frac{C}{l} = \varepsilon_0 \frac{24}{2} = 12\varepsilon_0 \tag{3.21}$$

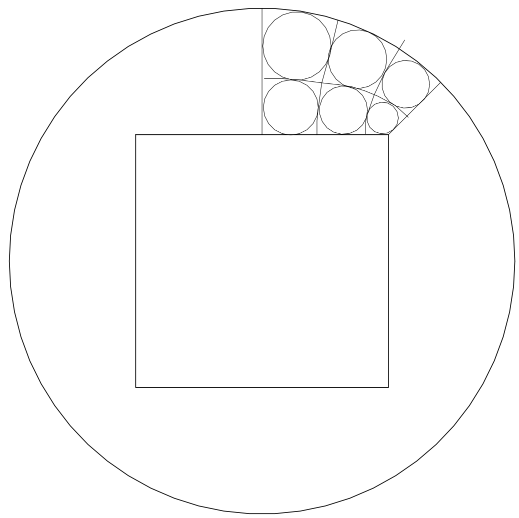


Figure 3-9: Cross-section of a cylindrical capacitor with square inner electrode (side $2a$) and circular outer electrode (radius $2a$).

3.4 Capacitance of Circular Disk

The problem of thin circular disk was solved by Cavendish in the 18-th century who found the following formula

$$C = 8\varepsilon_0 a \tag{3.22}$$

where a is the disk radius. This means that if a charge q is given to the disk, its potential relative to $\Phi = 0$ at ∞ is raised to

$$V = \frac{q}{C} = \frac{q}{8\varepsilon_0 a} \tag{3.23}$$

Comparing with the case of a sphere

$$V = \frac{q}{4\pi\varepsilon_0 a} \tag{3.24}$$

the difference in the two potentials by the factor $\pi/2$ can be attributed to the geometrical change and stems from

$$\cot^{-1}(0) = \frac{\pi}{2}$$

A disk is a limiting case of an oblate spheroid (a sphere compressed in the axial direction). The so-called oblate spheroidal coordinates (η, θ, ϕ) may be best suited for boundary value problems involving spheroidal electrodes. The oblate spheroidal coordinates (η, θ, ϕ) are related to the cartesian coordinates (x, y, z) through

$$\begin{cases} x &= a\cosh\eta\sin\theta\sin\phi \\ y &= a\cosh\eta\sin\theta\sin\phi \\ z &= a\sinh\eta\cos\theta \end{cases} \tag{3.25}$$

where a is a positive constant. Since

$$x^2 + y^2 = a^2\cosh^2\eta\sin^2\theta \tag{3.26}$$

$$z^2 = a^2\sinh^2\eta\cos^2\theta \tag{3.27}$$

we find (note $\cos^2\theta + \sin^2\theta = 1, \cosh^2\eta - \sinh^2\eta = 1$)

$$\frac{x^2+y^2}{a^2\cosh^2\eta} + \frac{z^2}{a^2\sinh^2\eta} = 1 \tag{3.28}$$

$$\frac{x^2+y^2}{a^2\sin^2\theta} - \frac{z^2}{a^2\cos^2\theta} = 1 \tag{3.29}$$

Eq. (3.28) indicates that a $\eta =$ const. surface is an oblate spheroid. In the limit $\eta \to 0$, the spheroid degenerates to a thin disk having a radius a. For large η, $\cosh\eta \simeq \sinh\eta$, and $\eta =$const. surface approaches a sphere with radius $r = a\cosh\eta$.

Let us calculate the metric coefficients for the oblate spheroidal coordinates, (η, θ, ϕ). Recalling the definition of metric coefficient in Chapter 1, we find

$$\begin{aligned} h_\eta &= \sqrt{\left(\frac{\partial x}{\partial\eta}\right)^2 + \left(\frac{\partial y}{\partial\eta}\right)^2 + \left(\frac{\partial z}{\partial\eta}\right)^2} \\ &= a\sqrt{\cosh^2\eta - \sin^2\theta} \end{aligned} \tag{3.30}$$

$$h_\theta = h_\eta$$

$$h_\phi = a\cosh\eta\sin\theta$$

(Calculations are left for an exercise.) Then, Laplace equation in the oblate spheroidal coordinates becomes

$$\begin{aligned} \nabla^2\Phi &= \frac{1}{h_\eta h_\theta h_\phi}\left[\frac{\partial}{\partial\eta}\left(\frac{h_\theta h_\phi}{h_\eta}\frac{\partial\Phi}{\partial\eta}\right) + \frac{\partial}{\partial\theta}\left(\frac{h_\eta h_\phi}{h_\theta}\frac{\partial\Phi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{h_\eta h_\phi}{h_\phi}\frac{\partial\Phi}{\partial\phi}\right)\right] \\ &= \frac{1}{a^2(\cosh^2\eta - \sin^2\theta)}\left[\frac{\partial^2}{\partial\eta^2} + \tanh\eta\frac{\partial}{\partial\eta} + \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta}\right]\Phi \\ &\quad + \frac{1}{a^2(\cosh^2\eta\sin^2\theta)}\frac{\partial^2\Phi}{\partial\phi^2} \\ &= 0 \end{aligned}$$

This looks complicated, and the reader may be wondering what merit we gain by introducing the exotic coordinates. In fact, the oblate spheroidal coordinates enormously simplify potential problems associated with an oblate spheroidal electrode (including conducting disk) because Laplace equation becomes **one dimensional** depending on the variable η only. Such simplification can never be achieved in the usual coordinates (cartesian, cylindrical and spherical). It should be recalled that the spherical coordinates are most convenient when boundary surfaces are spherical. For exotic boundary surfaces, often exotic coordinates are best suited.

Consider an electrode having a shape of oblate spheroid. Its surface is described by $\eta = \eta_0$ (const.), and we assume the electrode is at a potential $\Phi = V$. Thus, the boundary condition is

$$\Phi(\eta_0) = V.$$

The potential outside the electrode should be a function of η only,

$$\Phi = \Phi(\eta)$$

and Laplace equation in Eq. (**??**) becomes one dimensional,

$$\frac{d}{d\eta}\left(\cosh\eta \frac{d\Phi}{d\eta}\right)\Phi = 0 \tag{3.31}$$

Integrating once,

$$\frac{d\Phi}{d\eta} = \frac{\text{const}}{\cosh\eta} \tag{3.32}$$

Further integration yields

$$\Phi(\eta) = \text{const.} \cot^{-1}(\sinh\eta),$$

where use has been made of

$$\frac{d}{d\eta}\cot^{-1}(\sinh\eta) = -\frac{\cosh\eta}{1+\sinh^2\eta} = -\frac{1}{\cosh\eta}.$$

The boundary condition, $\Phi = V$ at $\eta = \eta_0$ determines the constant,

$$\text{const.} = \frac{V}{\cot^{-1}(\sinh\eta_0)},$$

and the final solution for the potential becomes

$$\Phi(\eta) = V\,\frac{\cot^{-1}(\sinh\eta)}{\cot^{-1}(\sinh\eta_0)} \tag{3.33}$$

For a thin conducting disk of radius a, $\eta_0 = 0$. Since

$$\cot^{-1}(0) = \frac{\pi}{2},$$

the potential due to a charged conducting disk is given by

$$\Phi(\eta) = \frac{2V}{\pi}\cot^{-1}(\sinh\eta) \tag{3.34}$$

$$C = \frac{q}{V} = 8\epsilon_0 a \quad \text{(F)} \tag{3.35}$$

3.5 Capacitance of Square Plate (Method of Integral Equation)

In this section, we will estimate the capacitance of a square conductor plate of side a. Mathematically speaking, this problem constitutes an integral equation for the potential Φ, which is constant at the conductor,

$$\Phi = \frac{1}{4\pi\varepsilon_0}\oint \frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dS' = -\frac{1}{4\pi}\oint \frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial\Phi}{\partial n'}dS' = V = \text{ constant}, \tag{3.36}$$

where

$$\sigma = \varepsilon_0 E_n = -\varepsilon_0\frac{\partial\Phi}{\partial n},$$

is the unknown surface charge density. The capacitance can be found from

$$C = \frac{1}{\Phi}\int \sigma dS. \tag{3.37}$$

As a very rough estimate, we recall that the capacitance of a circular disk of radius a is given by

$$C = 8\varepsilon_0 a, \tag{3.38}$$

and approximate the capacitance of the square plate by

$$C = 8\varepsilon_0 r_{\text{eff}} = 8\varepsilon_0 \times 0.564a, \tag{3.39}$$

where r_{eff} is the radius of a circular disk having the same area as the plate,

$$\pi r_{\text{eff}}^2 = a^2,\ r_{\text{eff}} = 0.564a.$$

The numerical method given below yields $C \simeq 8\varepsilon_0 \times 0.547a$.

The capacitance is the ratio between the total charge Q and the plate potential V, $C = Q/V$. We divide the plate into $n\times n$ sub-areas of equal size each with side a/n. Each sub-plate is at an equal potential V but charges on the sub-plates differ. To illustrate the procedure, we choose $n = 5$ (25 sub-areas) as shown in Fig. 3-10. Because of symmetry, there are 6 unknown charges to be found. The potential on each sub-plate can be calculated by summing contributions from charges on all sub-plates including the charge on itself. The self-potential of one unit can be estimated as follows. Consider a square plate of side δ carrying a uniform surface charge density σ (C/m^2). The

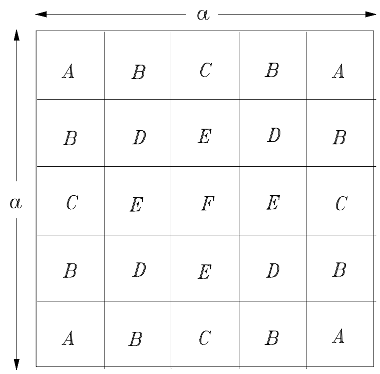


Figure 3-10: A square conducting plate of side a is divided into 25 sub-areas. Because of symmetry, the number of unknown potentials is reduced to 6.

potential at the center of the plate can be found from

$$
\begin{aligned}
\Phi &= \frac{\sigma}{4\pi\varepsilon_0}\int_{-\delta/2}^{\delta/2} dx \int_{-\delta/2}^{\delta/2} dy \frac{1}{\sqrt{x^2+y^2}} \\
&= \frac{\sigma}{4\pi\varepsilon_0} 4\int_0^{\delta/2}\left[\ln\left(\sqrt{x^2+\delta^2}+\delta\right)-\ln x\right] dx \\
&= \frac{\sigma}{4\pi\varepsilon_0} 4\delta \ln\left(1+\sqrt{2}\right) \\
&= \frac{q}{4\pi\varepsilon_0\delta}\times 4\ln\left(1+\sqrt{2}\right) = \frac{q}{4\pi\varepsilon_0\delta}\times 3.5255,
\end{aligned}
\tag{3.40}
$$

where $q=\sigma\delta^2$ is the charge carried by the sub-plate. With this preparation, we can write down the potential of sub-plate A as follows:

$$
\begin{aligned}
4\pi\varepsilon_0\Phi_A\delta &= \left(3.5255+\frac{2}{4}+\frac{1}{4\sqrt{2}}\right)q_A + \left(2+\frac{2}{3}+\frac{2}{\sqrt{17}}+\frac{2}{5}\right)q_B + \left(1+\frac{2}{\sqrt{20}}\right)q_C \\
&\quad + \left(\frac{1}{\sqrt{2}}+\frac{2}{\sqrt{10}}+\frac{1}{3\sqrt{2}}\right)q_D + \left(\frac{2}{\sqrt{5}}+\frac{2}{\sqrt{13}}\right)q_E + \frac{1}{2\sqrt{2}}q_F \\
&= 4.2023q_A + 3.5517q_B + 1.4472q_C + 1.5753q_D + 1.4491q_E + 0.3536q_F.
\end{aligned}
\tag{3.41}
$$

Similarly

$$
\begin{aligned}
4\pi\varepsilon_0\Phi_B\delta &= \left(1+\frac{1}{3}+\frac{1}{\sqrt{17}}+\frac{1}{5}\right)q_A+\left(3.5255+\frac{1}{2}+\frac{1}{\sqrt{2}}+\frac{1}{4}+\frac{2}{\sqrt{10}}+\frac{1}{3\sqrt{2}}+\frac{1}{\sqrt{13}}\right)q_B \\
&\quad+\left(1+\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{13}}+\frac{1}{\sqrt{17}}\right)q_C+\left(1+\frac{1}{\sqrt{5}}+\frac{1}{3}+\frac{1}{\sqrt{13}}\right)q_D \\
&\quad+\left(\frac{1}{\sqrt{2}}+\frac{1}{2}+\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{10}}\right)q_E+\frac{1}{\sqrt{5}}q_F \\
&= 1.7759q_A+6.1281q_B+1.9671q_C+2.0579q_D+1.9705q_E+0.44721q_F \qquad (3.42)
\end{aligned}
$$

$$
\begin{aligned}
4\pi\varepsilon_0\Phi_C\delta &= \left(\frac{2}{2}+\frac{2}{\sqrt{20}}\right)q_A+\left(2+\frac{2}{\sqrt{5}}+\frac{2}{\sqrt{13}}+\frac{2}{\sqrt{17}}\right)q_B+\left(3.5255+\frac{2}{2\sqrt{2}}+\frac{1}{4}\right)q_C \\
&\quad+\left(\frac{2}{\sqrt{2}}+\frac{2}{\sqrt{10}}\right)q_D+\left(1+\frac{2}{\sqrt{5}}+\frac{1}{3}\right)q_E+\frac{1}{2}q_F \\
&= 1.4472q_A+3.9342q_B+4.4826q_C+2.0467q_D+2.2278q_E+0.5q_F \qquad (3.43)
\end{aligned}
$$

$$
\begin{aligned}
4\pi\varepsilon_0\Phi_D\delta &= \left(\frac{1}{\sqrt{2}}+\frac{2}{\sqrt{10}}+\frac{1}{3\sqrt{2}}\right)q_A+\left(2+\frac{2}{\sqrt{5}}+\frac{2}{3}+\frac{2}{\sqrt{13}}\right)q_B+ \\
&\quad\left(\frac{2}{\sqrt{2}}+\frac{2}{\sqrt{10}}\right)q_C+\left(3.5255+\frac{2}{2}+\frac{1}{2\sqrt{2}}\right)q_D+\left(2+\frac{2}{\sqrt{5}}\right)q_E+\frac{1}{\sqrt{2}}q_F \\
&= 1.5753q_A+4.1158q_B+2.0467q_C+4.8791q_D+2.8944q_E+0.70711q_F \qquad (3.44)
\end{aligned}
$$

$$
\begin{aligned}
4\pi\varepsilon_0\Phi_E\delta &= \left(\frac{2}{\sqrt{5}}+\frac{2}{\sqrt{13}}\right)q_A+\left(1+\frac{2}{\sqrt{2}}+\frac{1}{\sqrt{2}}+\frac{2}{\sqrt{10}}\right)q_B+\left(1+\frac{2}{\sqrt{5}}+\frac{1}{3}\right)q_C \\
&\quad+\left(2+\frac{2}{\sqrt{5}}\right)q_D+\left(3.5255+\sqrt{2}+\frac{1}{2}\right)q_E+q_F \\
&= 1.4491q_A+3.7538q_B+2.2278q_C+2.8944q_D+5.4397q_E+q_F \qquad (3.45)
\end{aligned}
$$

$$
\begin{aligned}
4\pi\varepsilon_0\Phi_F\delta &= \frac{4}{2\sqrt{2}}q_A+\frac{8}{\sqrt{5}}q_B+2q_C+\frac{4}{\sqrt{2}}q_D+4q_E+3.5255q_F \\
&= 1.4142q_A+3.5777q_B+2.0q_C+2.8284q_D+4.0q_E+3.5255q_F \qquad (3.46)
\end{aligned}
$$

The 6 simultaneous equations can be put in a matrix form,

$$
\begin{bmatrix}
4.2023 & 3.5517 & 1.4472 & 1.5753 & 1.4491 & 0.3536 \\
1.7759 & 6.1284 & 1.9671 & 1.7246 & 1.9705 & 0.44721 \\
1.4472 & 3.9342 & 4.4826 & 2.0467 & 2.2278 & 0.5 \\
1.5753 & 4.1158 & 2.0467 & 4.8791 & 2.8944 & 0.70711 \\
1.4491 & 3.7538 & 2.2278 & 2.8944 & 5.4397 & 1 \\
1.4142 & 3.5777 & 2.0 & 2.8284 & 4 & 3.5255
\end{bmatrix}
\begin{bmatrix} q_A \\ q_B \\ q_C \\ q_D \\ q_E \\ q_F \end{bmatrix}
= 4\pi\varepsilon_0\Phi\delta \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
$$

Solutions for the charges q_i are

$$
\begin{aligned}
\begin{bmatrix} q_A \\ q_B \\ q_C \\ q_D \\ q_E \\ q_F \end{bmatrix}
&= 4\pi\varepsilon_0\Phi\delta
\begin{bmatrix}
0.322\,71 & -0.143\,06 & -0.02004 & -0.0431 & -0.00126 & -0.0024 \\
-0.07626 & 0.297\,57 & -0.07272 & -0.0225 & -0.04357 & -0.00290 \\
-0.02001 & -0.136\,55 & 0.342\,9 & -0.0554 & -0.05406 & -0.00285 \\
-0.02368 & -0.117\,96 & -0.03688 & 0.341\,93 & -0.106\,09 & -0.0159 \\
-0.01081 & -0.04450 & -0.06318 & -0.119\,99 & 0.332\,72 & -0.05461 \\
-0.00944 & -0.0220 & -0.01140 & -0.06662 & -0.216\,99 & 0.363\,92
\end{bmatrix}
\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
&= 4\pi\varepsilon_0\Phi\delta
\begin{bmatrix}
0.112\,87 \\
7.\,958\,2 \times 10^{-2} \\
7.\,402\,5 \times 10^{-2} \\
4.\,138\,8 \times 10^{-2} \\
3.\,961\,9 \times 10^{-2} \\
3.\,746\,2 \times 10^{-2}
\end{bmatrix}
\end{aligned}
$$

The total charge is

$$
\begin{aligned}
Q &= 4(q_A + q_C + q_D + q_E) + 8q_B + q_F \\
&= 1.7458 \times 4\pi\varepsilon_0\Phi\delta,
\end{aligned}
$$

and the capacitance is

$$
\begin{aligned}
C &\simeq \frac{Q}{\Phi} = 0.349 \times 4\pi\varepsilon_0 a \\
&= 0.548 \times 8\varepsilon_0 a. \qquad (3.47)
\end{aligned}
$$

Accuracy will improve if a larger number of sub-areas are used. The exact value is

$$C = 0.567 \times 8\varepsilon_0 a$$

The method can be applied to estimate the capacitance of a conducting cube as well. With 150 sub-areas (25 sub-areas on each side), the following capacitance emerges,

$$C \simeq 0.65 \times 4\pi\varepsilon_0 a, \qquad (3.48)$$

where a is the side of the cube. An estimate based on a sphere having the same surface area gives

$$C = 4\pi\varepsilon_0 r_{\mathrm{eff}} = 0.69 \times 4\pi\varepsilon_0 a, \qquad (3.49)$$

where

$$r_{\mathrm{eff}} = \sqrt{\frac{6}{4\pi}}a = 0.69a.$$

3.6 Finite Element Method (FEM)

Triangular Element

The finite element method has originally been developed for mechanical structural analysis. Its physical principle common to all disciplines is energy minimization. In electrostatics, if the potential Φ satisfies the Poisson's equation

$$\nabla^2\Phi = -\frac{\rho}{\varepsilon_0}$$

the total energy becomes minimum. That is, the Poisson's equation is the Euler's equation to minimize the energy functional

$$\int \left(\frac{1}{2}\varepsilon_0 \left(\nabla\Phi\right)^2 + \rho\Phi \right) dV$$

Here we consider two dimensional electrostatic boundary value problems with $\rho = 0$ and relevant PDE is the Laplace equation $\nabla^2\Phi(x, y) = 0$. The objective is to solve the equation for given potential specified on a boundary.

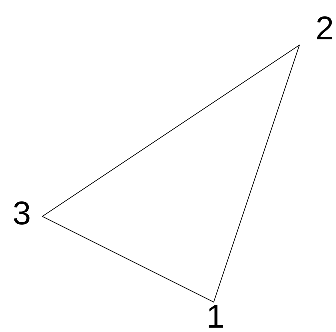


Figure 3-11: Triangle element. The nodes (vertexes) are numbered counterclockwise so that the area of triangle A $A = \frac{1}{2}\det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$ is positive.

In FEM, the region of interest is divided into small subareas. The shape of subarea is usually triangle because of greater flexibility. If the potentials at three vertexes, called nodes, of a triangle are known, the potential $\Phi\left(x, y\right)$ within the triangle can be estimated using linear interpolation as follows. Let us consider a triangular element whose vertexes (nodes) are defined by the coordinates, node 1 at $\left(x_1, y_1\right)$, node 2 at $\left(x_2, y_2\right)$ and node 3 at $\left(x_3, y_3\right)$ as shown. The numbering (1,2,3) of the nodes is counterclockwise so that the area of the triangle A is positive,

$$A = \frac{1}{2}\det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

The potential within the triangle is approximated by linear interpolation

$$V(x, y) = a + bx + cy \tag{3.50}$$

For small size of the triangle, this may be sufficiently accurate. If not, the size of the triangle can be reduced for higher accuracy. In matrix form, the potential $V(x, y)$ can be written as

$$V(x, y) = a + bx + cy = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{3.51}$$

The electric fields within the triangle is constant,

$$E_x = -\frac{\partial \Phi}{\partial x} = -b; \ E_y = -\frac{\partial \Phi}{\partial y} \tag{3.52}$$

The potential at the three nodes V_i are

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{3.53}$$

Solving this for $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, we obtain

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \tag{3.54}$$

where

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1}$$

is the inverse matrix of

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

and given by

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} = \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \tag{3.55}$$

Therefore, the potential within the triangle can be written in terms of the node potentials as

$$
\begin{aligned}
V(x,y) &= \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\
&= \begin{bmatrix} 1 & x & y \end{bmatrix} \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}
\end{aligned}
\tag{3.56}
$$

We define a row vector $\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}$ by

$$
\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 & x & y \end{bmatrix} \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}
$$

so that

$$
V(x,y) = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \tag{3.57}
$$

The components of α_i are

$$
\alpha_1 = \frac{1}{2A} \left[x_2y_3 - x_3y_2 + (y_2 - y_3)\, x + (x_3 - x_2)\, y \right] \tag{3.58}
$$

$$
\alpha_2 = \frac{1}{2A} \left[x_3y_1 - x_1y_3 + (y_3 - y_1)\, x + (x_1 - x_3)\, y \right] \tag{3.59}
$$

$$
\alpha_3 = \frac{1}{2A} \left[x_1y_2 - x_2y_1 + (y_1 - y_2)\, x + (x_2 - x_1)\, y \right] \tag{3.60}
$$

and they satisfy the following properties,

$$
\alpha_i (x_j, y_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} = \delta_{ij} \text{ (Kronecker's delta)} \tag{3.61}
$$

$$
\sum_{i=1}^{3} \alpha_i = 1 \tag{3.62}
$$

The electric energy can be calculated from

$$
U_e = \frac{1}{2} \varepsilon_0 \int E^2 dS = \frac{1}{2} \varepsilon_0 \int (\nabla V)^2 \, dS \tag{3.63}
$$

where $\nabla V(x,y)$ is

$$
\nabla V = \sum_{i=1}^{3} V_i \nabla \alpha_i \tag{3.64}
$$

Then

$$U_e = \frac{1}{2}\varepsilon_0 \sum_{i=1}^{3}\sum_{j=1}^{3} \int \nabla\alpha_i \cdot \nabla\alpha_j V_i V_j dS \tag{3.65}$$

We define the Dirichlet element coefficient $C_{ij}^{(e)}$ by

$$C_{ij}^{(e)} = \int \nabla\alpha_i \cdot \nabla\alpha_j dS \tag{3.66}$$

and write U_e as

$$\begin{aligned} U_e &= \frac{1}{2}\varepsilon_0 \sum_{i=1}^{3}\sum_{j=1}^{3} C_{ij}^{(e)} V_{ei} V_{ej} = \frac{1}{2}\varepsilon_0 \begin{bmatrix} V_{e1} & V_{e2} & V_{e3} \end{bmatrix} \begin{bmatrix} C_{11}^{(e)} & C_{12}^{(e)} & C_{13}^{(e)} \\ C_{21}^{(e)} & C_{22}^{(e)} & C_{23}^{(e)} \\ C_{31}^{(e)} & C_{32}^{(e)} & C_{33}^{(e)} \end{bmatrix} \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix} \\ &= \frac{1}{2}\varepsilon_0 \left[V_e\right]^T \left[C^{(e)}\right] \left[V_e\right] \end{aligned} \tag{3.67}$$

The components of $C_{ij}^{(e)}$ can be found as

$$\begin{aligned} C_{11}^{(e)} &= \frac{1}{4A}\left[\left(y_2 - y_3\right)^2 + \left(x_3 - x_2\right)^2\right] \\ C_{12}^{(e)} &= C_{21}^{(e)} = \frac{1}{4A}\left[\left(y_2 - y_3\right)\left(y_3 - y_1\right) + \left(x_3 - x_2\right)\left(x_1 - x_2\right)\right] \\ C_{13}^{(e)} &= C_{31}^{(e)} = \frac{1}{4A}\left[\left(y_2 - y_3\right)\left(y_1 - y_2\right) + \left(x_3 - x_2\right)\left(x_2 - x_1\right)\right] \\ C_{22}^{(e)} &= \frac{1}{4A}\left[\left(y_2 - y_1\right)^2 + \left(x_1 - x_2\right)^2\right] \\ C_{23}^{(e)} &= C_{32}^{(e)} = \frac{1}{4A}\left[\left(y_3 - y_1\right)\left(y_1 - y_2\right) + \left(x_1 - x_3\right)\left(x_2 - x_1\right)\right] \\ C_{33}^{(e)} &= \frac{1}{4A}\left[\left(y_1 - y_2\right)^2 + \left(x_2 - x_1\right)^2\right] \end{aligned}$$

Note that the $[C^{(e)}]$ matrix is symmetric $C_{ij}^{(e)} = C_{ji}^{(e)}$. It is convenient to define

$$P_1 = y_2 - y_3, P_2 = y_3 - y_1, P_3 = y_1 - y_2 \tag{3.68}$$

$$Q_1 = x_3 - x_2, Q_2 = x_1 - x_3, Q_3 = x_2 - x_1 \tag{3.69}$$

Note that

$$\sum P_i = 0, \; \sum Q_i = 0 \tag{3.70}$$

Then

$$C_{ij} = \frac{1}{4A}\left(P_i P_j + Q_i Q_j\right) \tag{3.71}$$

with

$$A = \frac{1}{2}\left(P_2 Q_3 - P_3 Q_2\right) \tag{3.72}$$

Assembling Elements

The next step is to learn how to assemble N elements. Suppose that after assembling, the system has N elements and n nodes. The total energy consists of the sum of the energy in each elements,

$$U = \sum_{e=1}^{N} U_e = \frac{1}{2}\varepsilon_0 [V]^T [C] [V] \tag{3.73}$$

where

$$[V] = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ . \\ . \\ V_n \end{bmatrix} \tag{3.74}$$

are the potentials at n nodes and $[C]$ is the $n \times n$ global coefficient matrix.

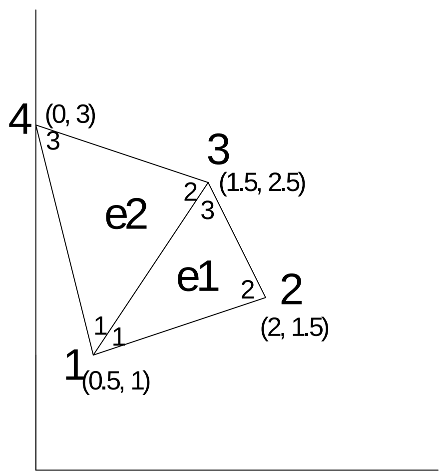


Figure 3-12: Assembling two triangle elements with a common edge. Note that numbering of element nodes is counterclockwise for both triangles.

Let us consider two triangular elements with a common edge. It is straightforward to find the following global coefficients,

$$[C] = \begin{bmatrix} C_{11}^{(1)} + C_{11}^{(2)} & C_{12}^{(2)} & C_{12}^{(1)} + C_{13}^{(2)} & C_{13}^{(1)} \\ C_{21}^{(2)} & C_{22}^{(2)} & C_{23}^{(2)} & 0 \\ C_{21}^{(1)} + C_{31}^{(2)} & C_{32}^{(2)} & C_{22}^{(1)} + C_{33}^{(2)} & C_{23}^{(1)} \\ C_{31}^{(1)} & 0 & C_{32}^{(1)} & C_{33}^{(1)} \end{bmatrix} \tag{3.75}$$

Note that the potentials at node 1 and node 3 are shared by both elements and thus $V_1^{(1)} = V_1^{(2)}, V_2^{(1)} = V_3^{(2)}$. The number of nodes before assembling is 6 but after assembling it is reduced to 4. There is no coupling between the node 2 and node 4 $C_{24} = C_{42} = 0$. For a system of three triangle elements, the number of node is $n = 5$ and we find

$$[C] = \begin{bmatrix} C_{11}^{(1)} + C_{11}^{(2)} & C_{13}^{(1)} & C_{12}^{(2)} & C_{12}^{(1)} + C_{13}^{(2)} & 0 \\ C_{31}^{(1)} & C_{33}^{(1)} & 0 & C_{32}^{(1)} & 0 \\ C_{21}^{(2)} & 0 & C_{22}^{(2)} + C_{33}^{(3)} & C_{23}^{(2)} + C_{13}^{(3)} & C_{12}^{(3)} \\ C_{21}^{(1)} + C_{31}^{(2)} & C_{23}^{(1)} & C_{32}^{(2)} + C_{31}^{(3)} & C_{22}^{(1)} + C_{33}^{(2)} + C_{33}^{(3)} & C_{32}^{(3)} \\ 0 & 0 & C_{21}^{(3)} & C_{23}^{(3)} & C_{22}^{(3)} \end{bmatrix}$$

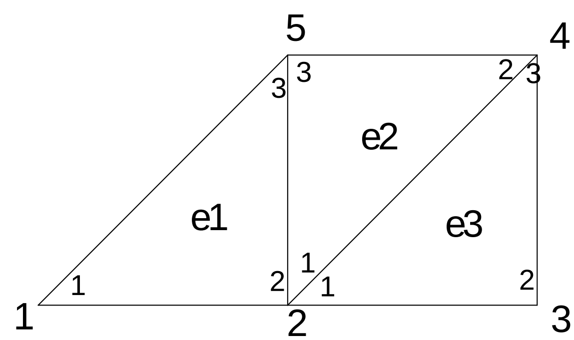


Figure 3-13: Assembling 3 elements. The number of global nodes is 4.

Energy Minimization and Finding the Unknown Node Potentials

To find the potential $[V]$ that minimizes the total electric energy, we impose

$$\frac{\partial U}{\partial V_i} = 0, i = 1, 2, \cdots n \tag{3.76}$$

for each node potential V_i. Since C_{ij} is symmetric $C_{ij} = C_{ji}$, we find, for example

$$\frac{\partial U}{\partial V_i} = 2(C_{11}V_1 + V_2 C_{12} + C_{13}V_3 + C_{14}V_4) = 0 \tag{3.77}$$

and we obtain the following simultaneous equations for V_i

$$\sum_{i=1}^{n} C_{ij} V_j = 0 \tag{3.78}$$

If the potentials at some nodes are known (or fixed), differentiation with respect to the known potentials should be excluded. In this case, instead of Eq. (3.78), we have

$$\begin{bmatrix} C_{ff} & C_{fp} \\ C_{pf} & C_{pp} \end{bmatrix} \begin{bmatrix} V_f \\ V_p \end{bmatrix} = 0$$

For example, if the global node number is $n=4$ and the potentials at $n=1$ and 3 are known, the potentials at $n=2$ and 4 can be found from

$$C_{22}V_2 + C_{24}V_4 + C_{21}V_1 + C_{23}V_3 = 0 \tag{3.79}$$

$$C_{42}V_2 + C_{44}V_4 + C_{41}V_1 + C_{43}V_3 = 0 \tag{3.80}$$

which can be written as

$$\begin{bmatrix} C_{22} & C_{24} \\ C_{42} & C_{44} \end{bmatrix} \begin{bmatrix} V_2 \\ V_4 \end{bmatrix} = - \begin{bmatrix} C_{21} & C_{23} \\ C_{41} & C_{43} \end{bmatrix} \begin{bmatrix} V_1 \\ V_3 \end{bmatrix} \tag{3.81}$$

Let us consider two-triangle system with 4 node coordinates $(x,y) = (0.5, 1.0), (2.0, 1.5), (1.5, 2.5), (0, 3.0)$ as shown. The potentials at nodes 1 and 3 are given, $V_1 = 5$ V and $V_3 = 10$ V and those at nodes 2 and 4 are unknown. The first element is defined by $(x,y)^{(1)} = (0.5, 1.0), (2.0, 1.5), (1.5, 2.5)$ and the second element by $(x,y)^{(2)} = (0.5, 1.0), (1.5, 2.5), (0, 3.0)$. The coefficient matrix of the first element can be calculated using P_i, Q_i and A,

$$P_i = [-1, 1.5, -0.5]$$

$$Q_i = [-0.5, -1, 1.5]$$

$$A = \frac{1}{2}(P_2 Q_3 - P_3 Q_2) = 0.875$$

$$C_{ij} = \frac{1}{4A}(P_i P_j + Q_i Q_j)$$

$$C_{ij}^{(1)} = \begin{bmatrix} 0.357\,142\,9 & -0.285\,714\,3 & 0.0714 \\ -0.285\,714\,3 & 0.928\,571\,4 & -0.642\,857\,1 \\ 0.0714 & -0.642\,857\,1 & 0.714\,285\,7 \end{bmatrix} \tag{3.82}$$

Similarly,

$$C_{ij}^{(2)} = \begin{bmatrix} 0.454\,545\,5 & -0.318\,181\,8 & -0.136\,363\,6 \\ -0.318\,181\,8 & 0.772\,727\,3 & -0.454\,545\,5 \\ -0.136\,363\,6 & -0.454\,545\,5 & 0.590\,909\,1 \end{bmatrix} \tag{3.83}$$

and the global coefficients are

$$
\begin{aligned}
C_{ij}^{G} &= \begin{bmatrix} C_{11}^{(1)}+C_{11}^{(2)} & C_{12}^{(2)} & C_{12}^{(1)}+C_{13}^{(2)} & C_{13}^{(1)} \\ C_{21}^{(2)} & C_{22}^{(2)} & C_{23}^{(2)} & 0 \\ C_{21}^{(1)}+C_{31}^{(2)} & C_{32}^{(2)} & C_{22}^{(1)}+C_{33}^{(2)} & C_{23}^{(1)} \\ C_{31}^{(1)} & 0 & C_{32}^{(1)} & C_{33}^{(1)} \end{bmatrix} \\
&= \begin{bmatrix} 0.811\,688\,4 & -0.318\,181\,8 & -0.422\,07 & -0.07142\,9 \\ -0.318\,181\,8 & 0.772\,727\,3 & -0.454\,545\,5 & 0 \\ -0.422\,07 & -0.454\,545\,5 & 1.\,519\,5 & -0.642\,86 \\ -0.07142\,9 & 0 & -0.642\,86 & 0.714\,29 \end{bmatrix}
\end{aligned}
\tag{3.84}
$$

Suppose that $V_2 = 5$ V and $V_4 = 10$ V are given and V_1 and V_3 are unknown. From

$$C_{11}V_1 + C_{12}V_2 + C_{13}V_3 + C_{14}V_4 = 0$$

$$C_{13}V_1 + C_{23}V_2 + C_{33}V_3 + C_{34}V_4 = 0$$

we get

$$\begin{bmatrix} C_{11} & C_{13} \\ C_{31} & C_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_3 \end{bmatrix} = -\begin{bmatrix} C_{12} & C_{14} \\ C_{23} & C_{34} \end{bmatrix} \begin{bmatrix} V_2 \\ V_4 \end{bmatrix}$$

$$\begin{bmatrix} 0.811\,688\,4 & -0.246\,781\,8 \\ -0.246\,781\,8 & 0.590\,909\,1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_3 \end{bmatrix} = -\begin{bmatrix} -0.318\,181\,8 & -0.07142\,9 \\ -0.454\,545\,5 & -0.642\,86 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

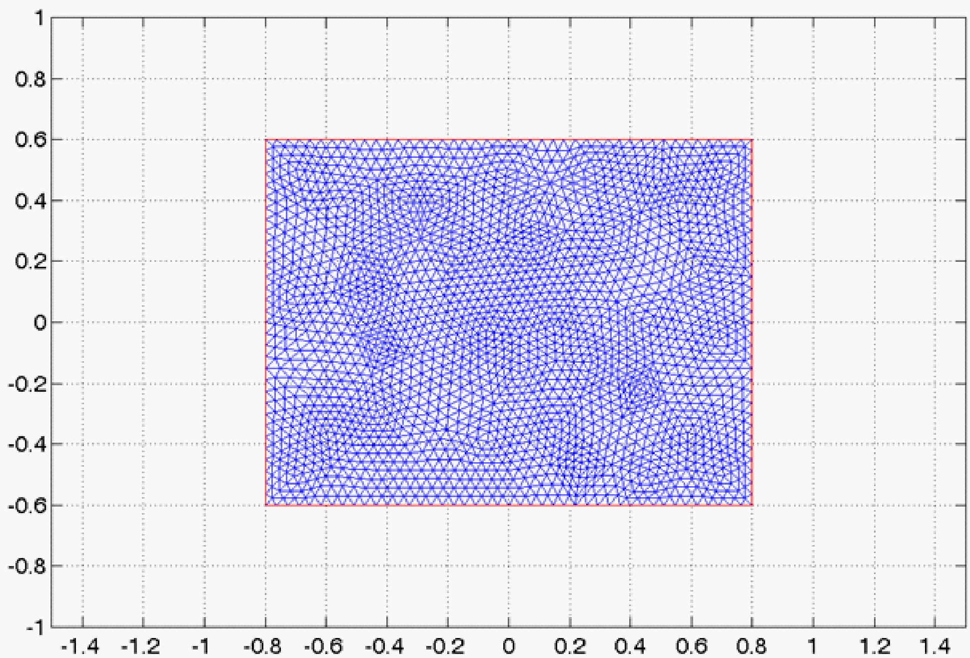
$$
\begin{aligned}
\begin{bmatrix} V_1 \\ V_3 \end{bmatrix} &= -\begin{bmatrix} 0.811\,688\,4 & -0.246\,781\,8 \\ -0.246\,781\,8 & 0.590\,909\,1 \end{bmatrix}^{-1} \begin{bmatrix} -0.285\,714\,3 & -0.136\,363\,6 \\ -0.642\,857\,1 & -0.454\,545\,5 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \end{bmatrix} \\
&= \begin{bmatrix} 8.\,513\,6 \\ 16.\,687 \end{bmatrix}
\end{aligned}
$$

Some examples obatined with MATLAB PDE Toolbox are shown below.

Example 1: 2D potential profile in rectangular box when the top plate is at 100 V and the rest are grounded.

Example 2: Transmission line with rectangular electrodes. The inner electrode is at 100 V and the outer is gorunded.

Example 3: Transmission line with eccentric circular electrodes. The inner electrode is at 100 V and the outer grounded.


1
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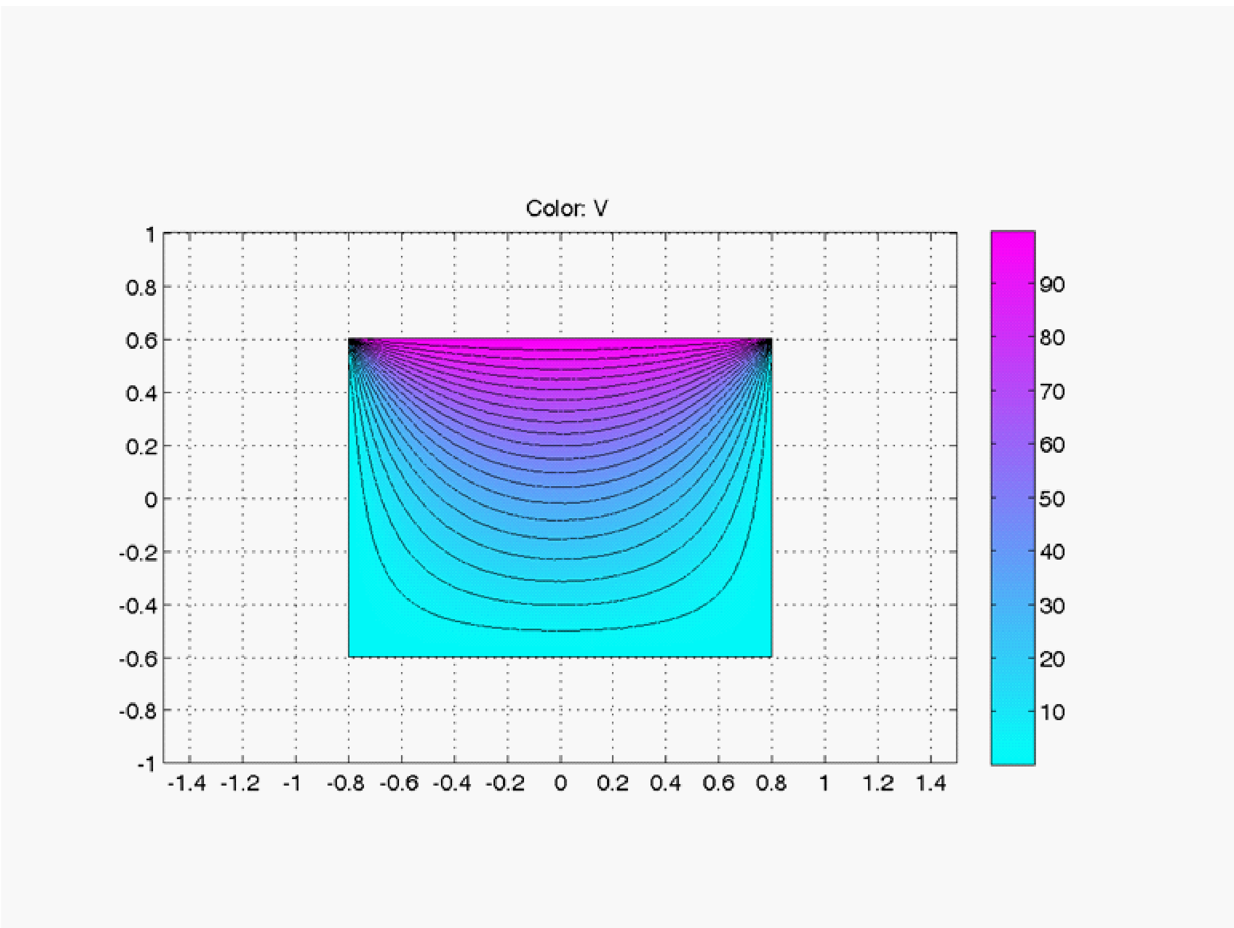
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.4
-1.2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4


Color: V
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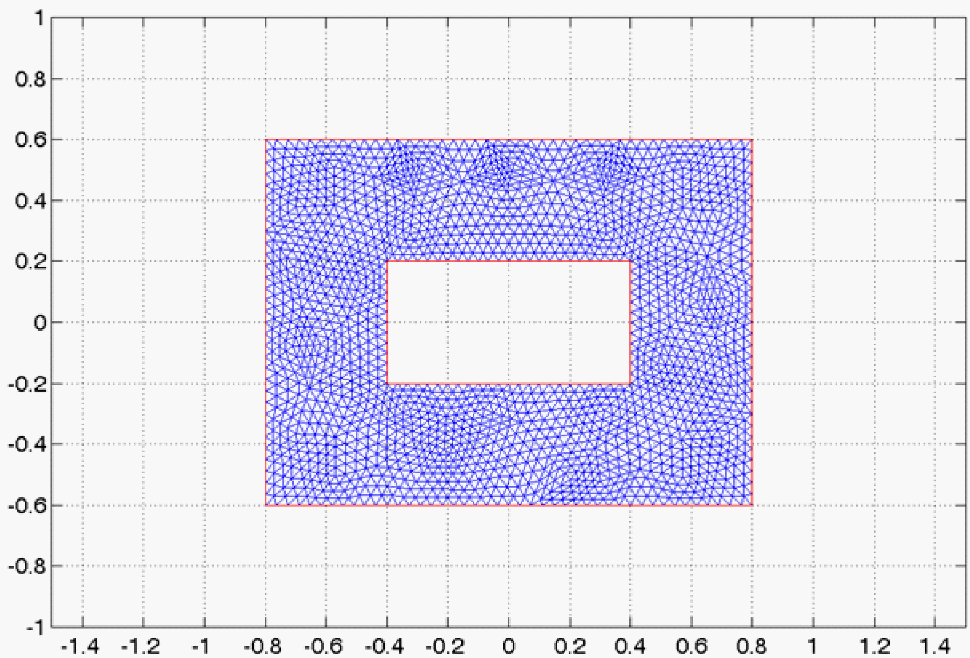
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.4
-1.2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
90
80
70
60
50
40
30
20
10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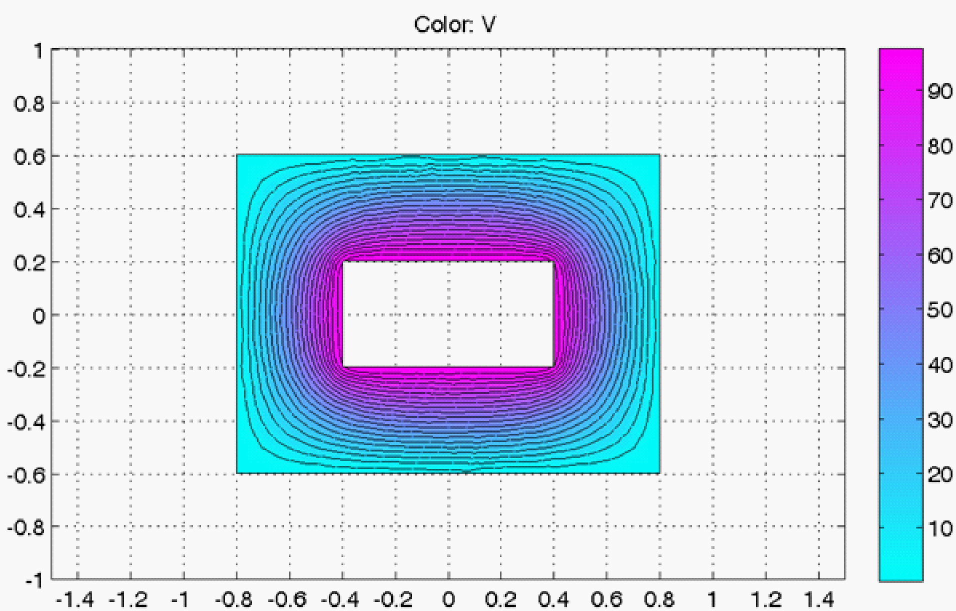
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.4 -1.2 -1 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1 1.2 1.4


Color: V
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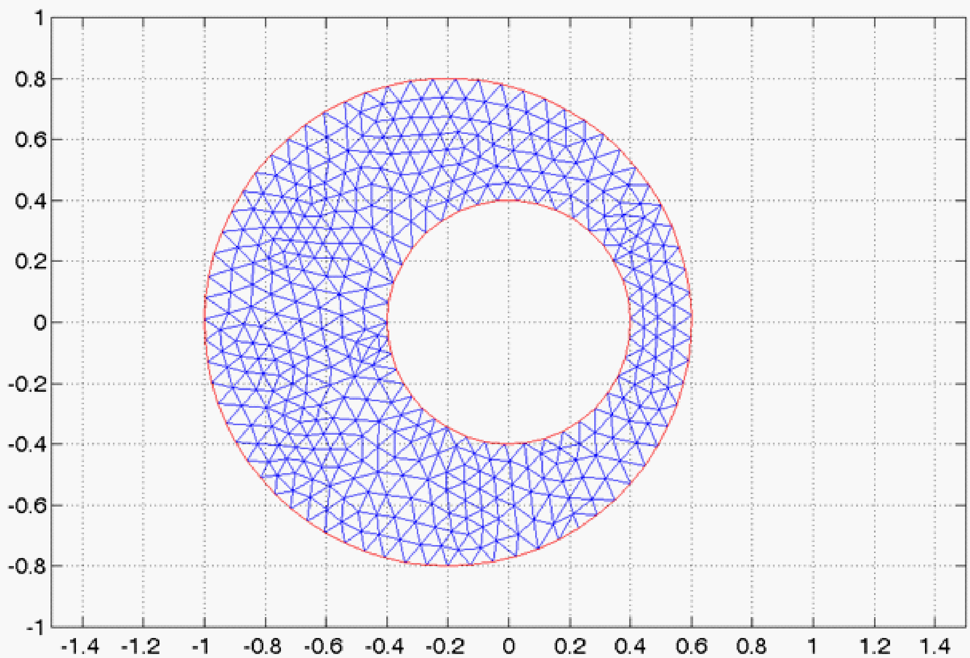
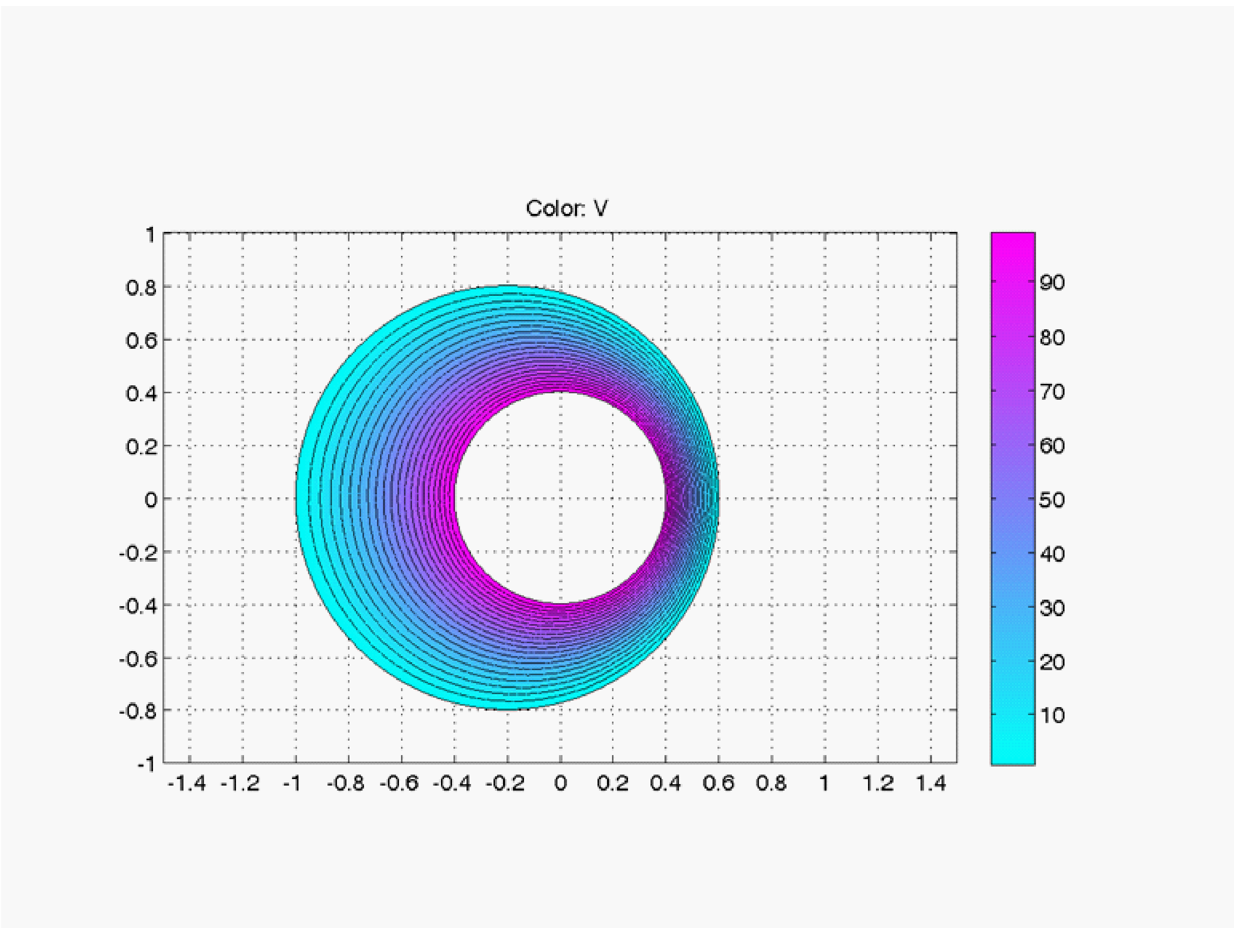
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.4 -1.2 -1 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1 1.2 1.4
90
80
70
60
50
40
30
20
10


1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.4
-1.2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4


Color: V
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.4 -1.2 -1 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1 1.2 1.4
90
80
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