CS 301 HW5

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Cost of i^{th} operation $= \begin{cases} i & \text{if } i \text{ is an exact power of 2} \\ 1 & \text{otherwise} \end{cases}$

a) Aggregate Analysis Method

In this sequence (n) there are $\lfloor \log_2 n \rfloor + 1$ exatc powers of 2.
(Ex: 1, 2, 4, 8, ..., $\log_2 n$)

If we use geometric sum => $\sum_{i=0}^{\lfloor \log_2 n \rfloor} 2^i = \frac{2^{\log_2 n+1} - 1}{2-1} = 2^{\log_2 n} \cdot 2 - 1$

- Since $2^{\log_2 n} \cdot 2 - 1 \leq 2^{\log_2 n} \cdot 2$ => Cost is $2n$ for powers of 2.
- For the rest of the operations,
  They cost 1 and there are less than n operations.

Total cost is $T(n) \leq 2n + n = 3n = O(n)$

Since there are n Operations

Amortized cost per operation is $O(1)$

b) Accounting Method

- The actual cost of i^{th} operation:

$$c_i = \begin{cases} i & \text{if } i \text{ is an exact power of 2} \\ 1 & \text{otherwise} \end{cases}$$

- For the amortized cost we may assign the following:

$$\hat{c}_i = \begin{cases} 2 & \text{if } i \text{ is an exact power of 2} \\ 3 & \text{otherwise} \end{cases}$$

$\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i$ => must be satisfied

* In this conditions, if the operation is the exact power of 2, it uses all accumulated credit plus one unit of its amortized cost. If there is surplus, those remainings are assigned as Credit.

If operation is not exact power of 2, then it uses 1 unit and assigns remaining as credit.

To show there are always enough credit :

- if $i=1$ credit : 1
- after 2^{J-1} th operation where $J>0$ remaining credit is 1
- Between 2^{J-1} th and 2^J th operations there are $2^{J-1}-1$ operations
  (Ex: $J=3$ $8^{th}-4^{th}$ => 5, 6, 7 (3 operations))

Each non-power of 2 operation accumulate 2 credits.
So the total is $1+2\cdot(2^{J-1}-1)=2^J-1$ credits before 2^J th operation

So the amount of credit stays non-negative and the total amortized cost is an upper bond for the total cost.

$O(1)$

c) Potential Method

We may define the potential function like this:

$$\Phi(D_0)=0 \qquad \Phi(D_i)=2i-2^{\log_2 i+1}+1 \text{ for } i>0$$

Since potential is always ≥ 0 => $\Phi(D_i)\geq 0=\Phi(D_0)$ for all i.

=> For i, exact power of 2 potential difference:

$$\Phi(D_i)-\Phi(D_{i-1})=(2i-2i+1)-(2(i-1)-i+1)$$
$$=2-i$$

So amortized cost for i th operation

$$\hat{c_i}=c_i+\Phi(D_i)-\Phi(D_{i-1})=i+(2-i)=2$$

=> For i, not exact power of 2, potential difference is

$$(2i-i+1)-(2(i-1)-i+1)=2$$

Amortized cost for i^{th} operation:

$$\hat{c_i}=c_i+\Phi(D_i)-\Phi(D_{i-1})=1+2=3$$

So the amortized cost is $O(1)$