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1) a) $T(n) = \begin{cases} \Theta(1) & \text{if } n=1 \\ 2\cdot T(n/2) + \Theta(n^2) & \text{, otherwise} \end{cases}$

b) $a=2$ $b=2$ $f(n)=n^2$

$\left(\begin{array}{l} \text{if } f(n) = \Omega(n^{\log_b a+\varepsilon}) \text{ , } \varepsilon>0 \\ \quad \text{and if } a\,f(n/b) \leq c.f(n) \text{ , } c<1 \end{array} \Rightarrow T(n) = \Theta(f(n)) \right)$

$\left. \begin{array}{l} \Rightarrow n^2 = \Omega(n^{1+1}) \checkmark \\ \quad \text{and } 2\cdot\left(\frac{n}{2}\right)^2 \leq c.n^2 \\ \quad \frac{n^2}{2} \leq n^2 \checkmark \end{array} \right\} T(n) = \Theta(n^2)$

2) a) $f(n) \stackrel{?}{=} \Theta(f(n/2))$, $\boxed{\text{False}}$

Counter Ex: $f(n)=2^n$ $f(n/2) = 2^{n/2}$

$\underbrace{c_1 \cdot 2^{n/2} \stackrel{?}{\leq} 2^n}_{\text{This holds}}$ but $\underbrace{2^n \stackrel{?}{\leq} c_2 \cdot 2^{n/2}}_{\text{This does not hold}}$ (e.g. $c_2=5$)

b) $f(n) + o(f(n)) \stackrel{?}{=} \Theta(f(n))$ $\boxed{\text{True}}$

$o(f(n)) = g(n) \Rightarrow c_1 \cdot f(n) \leq \underbrace{f(n)+g(n) \leq c_2 \cdot f(n)}_{\text{This holds}}$

$\hookrightarrow \forall c>0, \exists n_0>0, \forall n \geq n_0, 0 \leq g(n) \leq c.f(n)$

c) $f(n) \stackrel{?}{=} O((f(n))^2)$ $\boxed{\text{False}}$

Counter ex:

if $f(n) = \frac{1}{n}$, $f(n)^2 = \frac{1}{n^2}$ so $\frac{1}{n} > \frac{1}{n^2}$

d) $f(n) + g(n) \doteq \Theta(\min(f(n), g(n)))$ $\boxed{\text{False}}$

Counter example, $f(n) = n$ $g(n) = n^3$

$n + n^3 \neq \Theta(n)$

e) $\boxed{\text{True}}$

* Since log is a monotonic transformation way, equation does not chage.

if $f(n) = O(g(n)) \Rightarrow f(n) \leq c \cdot g(n)$

$\Rightarrow \underbrace{lg(f(n)) \leq lg\ c \cdot g(n)}$

$lg\ c + lg\ g(n)$

we can write $1 \cdot lg\ c$ since $lg\ g(n) \geq 1$

$lg(f(n)) \leq lg\ c \cdot lg\ g(n) + lg\ g(n)$

$lg(f(n)) \leq lg\ g(n) \underbrace{(lg\ c + 1)}_{\text{constant} = d}$

$lg(f(n)) \doteq O(lg\ g(n))$

3) $T(n) = T(n-5) + 2n^2$

Upper Bound

1) Assumption: $T(n) = O(n^3)$

$\exists c, n_0 \geq 0$ $T(n) \leq cn^3$

$\forall n \geq n_0$

2) Induction base: $T(1) \leq c \cdot 1^3 \leq 1$

3) Inductive step: $T(k) \leq c \cdot k^3$, $k < n$

$T(n) = c \cdot (n-5)^3 + 2n^2$

$= c[n^3 - 15n^2 + 75n - 125] + 2n^2$

$= cn^3 - \underbrace{(c15n^2 - c75n + c125 - 2n^2)}_{\geq 0,\ c \geq 1,\ n > 0}$

$\Rightarrow T(n) = O(n^3)$ ✓

Lower Bound

1) Assumption $T(n) = \Omega(n^3)$

$T(n) = c(n-5)^3 + 2n^2$

$T(n) = c[n^3 - 15n^2 + 75n - 125] + 2n^2$

$= cn^3 - \underbrace{15n^2 \cdot c + 75n \cdot c - 125 \cdot c + 2n^2}_{\geq 0,\ c \geq \frac{2}{15},\ n > 0}$

$\Rightarrow T(n) = \Omega(n^3)$

Since $\left.\begin{matrix} T(n) = \Omega(n^3) \\ T(n) = O(n^3) \end{matrix}\right\} T(n) = \Theta(n^3)$ tight bound

$T(n) = 2T(\sqrt{n}) + \log n$

$\log n = m \qquad n = 2^m \Rightarrow T(2^m) = 2T(2^{\frac{m}{2}}) + m$

$\Rightarrow T(2^m) = S(m)$

$\Rightarrow S(m) = \underbrace{2T(\frac{m}{2}) + m}$

this is in the same form with
$T(n) = 2T(\frac{n}{2}) + n$ which has time comp.
$T(n) = \Theta(n \log n)$

$\Rightarrow S(m) = \Theta(m \log m)$

$\Rightarrow T(2^m) = \Theta(m \log m)$

$\Rightarrow T(n) = \Theta(m \log m)$

$\boxed{T(n) = \Theta(\log n \cdot \log(\log n))}$

5) a) $T(n) = 4T(n/2) + n^2$

$a = 4. \quad b = 2 \quad \log_b a = 2 \Big\} \; f(n) = \Theta(n^{\log_b a}) \Rightarrow \boxed{T(n) = \Theta(n^2 \log n)}$
$n^2 = \Theta(n^2)$

b) $T(n) = 16T(n/4) + n$

$a = 16 \quad b = 4 \quad \log_b a = 2 \Big\} \; f(n) = O(n^{\log_b a - \varepsilon}), \varepsilon > 0 \Rightarrow \boxed{T(n) = \Theta(n^2)}$
$n = O(n^{2-\varepsilon})$

c) $T(n) = 0.75\,T(n/4) + 1/3n$

$a = 0{,}75 \quad b = 4 \Rightarrow$ Since a is smaller than 1, we can not use Master Theorem.

d) $T(n) = 3T(n/4) + n \log n$

$a = 3 \quad b = 4 \quad \log_4 3 = 0{,}792 \Big\} \; n \log n = \Omega(n^{0{,}792 + \varepsilon}), \; \varepsilon > 0$
$(\varepsilon = 0{,}2)$

and $3 \cdot \frac{n}{4} \log \frac{n}{4} \leq c \cdot n \log n$

$\Rightarrow \boxed{T(n) = \Theta(n \log n)}$