Dynamic System Modeling and Control

by
Hugh Jack

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Course Number:EGR 345
Course Name:Dynamic Systems Modeling and Control
Academic Unit:Padnos School of Engineering
Semester:Fall 2002

Class Times:Lecture: 9-10am - Mon, Wed, Fri in EC312
Lab 1: 8-11am - Tues
Lab 2: 2-5pm - Tues
Lab 3: 8-11am - Thurs

Description: Mathematical modeling of mechanical, electrical, fluid, and thermal dynamic systems involving energy storage and transfer by lumped parameter linear elements. Topics include model building, Laplace transforms, transfer functions, system response and stability, Fourier methods, frequency response feedback control, control methods, and computer simulation. Emphasis on linear mechanical systems. Laboratory.

Prerequisites:EGR 214, MTH 302, ENG 150
Corequisites:EGR 314 - Dynamics

Instructor:Dr. Hugh Jack,
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Textbook:
Hugh Jack, Dynamic System Modelling and Control, Grand Valley State University, Version 2.2, 2002.
Frank Ayres, Philip Schmidt, College Mathematics, Second Edition, McGraw Hill, 1992

Software:Mathcad
Scilab (www.scilab.org)
Netscape Communicator
FTP/Telnet
Labview
Excel
A C/C++ compiler

Goals: The main objective of this course is to develop your knowledge and ability to mathematically model, simulate, and analyze dynamic systems. In the lab you will study the time and frequency response of dynamic systems and further develop your laboratory, data analysis, and report writing skills. During this course you will practice the application of differential equations to the solution of practical engineering problems and then verify some of these

solutions in the laboratory. The overall goal is to improve your engineering problem solving ability in the area of time-varying systems.
Another major objective is to improve your technical writing skills. To this end, this course has been designated a supplemental writing skills (SWS) course and significant time and effort will be spent on writing instruction and the creation of technical reports.

Instruction Methods: Lecture, discussion, laboratories, assignments and projects.

Prerequisite Topics:

1. Electric circuits
2. Statics
3. Trigonometry, algebra, matrices
4. Calculus and differential equations
5. Computer applications and programming
6. Physics

Topics:

Translation
Differential Equations
Numerical Analysis
Rotation
Input-Output Equations
Electrical systems
Feedback Control
Phasor Analysis
Bode Plots
Root Locus Analysis
Nonlinear Systems
Analog IO
Continuous Sensors/Actuators
Motion Control

Grading:	
Design project	10%
Labs and SWS writing skills	40%
Quizzes and assignments	20%
Final exam	30%

Tests and assignments will be given at natural points during the term as new material is covered. Laboratory work will be assigned to reinforce lecture material and expose the student to practical aspects of systems modeling and control. Special attention will be paid to writing skills in the laboratories.
A final examination will be given to conclude the work, and test the students global comprehension of the material. A design project will be done in class to emphasize lecture and lab topics. Details of this will be announced later.
Note: A PASSING GRADE MUST BE OBTAINED IN ALL GRADE COMPONENTS TO RECEIVE A PASSING GRADE IN THE CLASS.

SWS Required Statement:

This course is designated SWS (Supplemental Writing Skills). As a result you MUST have already taken and passed ENG150 with a grade of C or better, or have passed the advanced placement exam with a score of 3 or higher. If you have not already done this, please see the instructor.

The official university SWS statement is:

" This course is designated SWS (Supplemental Writing Skills). Completion of English 150 with a grade of C or better (not C-) is the prerequisite. SWS credit will not be given to a student who completes the course before the prerequisite. SWS courses adhere to certain guidelines. Students turn in a total of at least 3,000 words or writing during the term. Part of that total may be essay exams, but a substantial amount of it is made up of finished essays or reports or research papers. The instructor works with the students on revising drafts of their papers, rather than simply grading the finished pieces of writing. At least four hours of class time are devoted to writing instruction. At least one third of the final grade in the course is based on the writing assignments."

SWS Practical Implementation:

The main source of writing grades are the laboratories and they are worth 40% of the final grade. You may look at all of this grade as writing. If the level of writing is not acceptable it will be returned for rewriting and it will be awarded partial marks. It is expected that the level of writing improve based upon feedback given for previous laboratory reports. A lab that would have received a grade of 'A' at the beginning of the term may very well receive an 'F' at the end of the term. It is expected that a typical lab will include 500-1000 words, and there will be approximately 10 labs in the course. Writing instruction will be given in the labs at appropriate times and this will total four hours.

Grading Scale:	A	100 - 90
	A-	89-80
	B+	79-77
	B	76-73
	B-	72-70
	C+	69-67
	C	66-63
	C-	62-60
	D+	59-57
	D	56-55

Mathematics Review:

The list of topics from the College Mathematics book should help students experiencing difficulties with particular topics.

Chapter 1 Problems: 30, 31diln, 32begi, 33gjkm
Chapter 2 Problems: 11cd, 12cd, 13acegk, 14ace, 15bcg, 16ad
Chapter 5 Problems: 15bdf, 16, 18, 21
Chapter 6 Problems: 9 (solve with algebra), 10cd
Chapter 7 Problems: 23acf, 25acd, 27abe
Chapter 8 Problems: 18acde
Chapter 11 Problems:
Chapter 13 Problems:
Chapter 23 Problems: 7, 8, 9, 10a, 11d
Chapter 25 Problems: 8, 9
Chapter 26 Problems: 5, 6, 7, 8, 9, 13
Chapter 30 Problems: 12abc, 13cd, 14ab, 15ab, 17, 20, 21
Chapter 31 Problems: 11abcdef, 12abc, 13a
Chapter 32 Problems: 18a, 19ab, 20abcd
Chapter 34 Problems: 12, 13, 14, 15, 18, 19
Chapter 35 Problems: 12abcghi, 13abd, 15
Chapter 37 Problems: 20, 23, 24, 27, 31,42
Chapter 38 Problems: 27abh, 31ac
Chapter 39 Problems: 12, 14, 18
Chapter 40 Problems: 16, 17, 20, 24, 25, 26
Chapter 41 Problems: 19abcefgh, 20abcde, 23
Chapter 42 Problems: 31, 32, 33, 36, 58
Chapter 43 Problems: 20abefimpqr, 21, 23abd
Chapter 45 Problems: 10bce
Chapter 54 Problems: 20d, 21b, 22be, 23cd
Chapter 57 Problems: 17d, 21c, 22, 23, 24
Chapter 58 Problems: 9acdgjp, 10f
Chapter 59 Problems: 13df, 15d, 16
Chapter 60 Problems: 13, 20bd

PREFACE

How to use the book.

- read the chapters and do drill problems as you read
- examine the case studies - these pull together concepts from previous chapters
- problems at the ends of chapters are provided for further practice

Tools that should be used include,

- graphing calculator that can solve differential equations, such as a TI-85
- computer algebra software that can solve differential equations, such as Mathcad

Supplemental materials at the end of this book include,

- a lab guide for the course
- a writing guide
- a summary of math topics important for engineers
- a table of generally useful engineering units
- properties of common materials

Acknowledgement to,

Dr. Hal Larson for reviewing the calculus and numerical methods chapters
Dr. Wendy Reffeor for reviewing the translation chapter

Student background

a basic circuits course
a basic statics and mechanics of materials course
math up to differential equations
a general knowledge of physics
computer programming, preferably in ’C’

TO BE DONE

small
- italicize variables and important terms
- fix equation numbering (auto-numbering?)
- fix subscripts and superscripts
- fix problem forms to include therefores, mark FBDs, etc.

big
- proofread and complete writing chapters
- add more drill problems and solutions

chapter numerical
- add ti-89 integration methods

chapter rotation
- replace the rotational case with IC motor

chapter circuits

- complete the induction motor section
- complete the brushless motor section
- add a design case - implement a differential equation with op-amps

chapter frequency analysis
- add problems

chapter non-linear systems
- develop chapter

chapter motion control
- add acceleration profile for velocity control
- add a setpoint scheduler program
- add a multiaxis motion control program

chapter magnetic
- consider adding/writing this chapter

chapter fluids
- consider adding/writing this chapter

chapter thermal
- consider adding/writing this chapter

chapter laplace
- consider adding/writing these chapters

chapter lab guide
- update the labs and rewrite where necessary

chapter c programming
- review section
- add problems

1. INTRODUCTION

Topics:

Objectives:

1.1 BASIC TERMINOLOGY

- Lumped parameter (masses and springs)
- Distributed parameters (stress in a solid)
- Continuous vs. Discrete
- Linear vs. Non-linear
- linearity and superposition
- reversibility
- through and across variables
- Analog vs. Digital
- process vs. controllers

1.2 EXAMPLE SYSTEM

- Servo control systems
- Robot

1.3 SUMMARY

-

1.4 PRACTICE PROBLEMS

1.

2. TRANSLATION

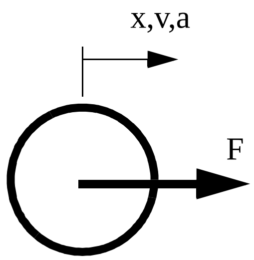
Topics:
- Basic laws of motion
- Gravity, inertia, springs, dampers, cables and pulleys, drag, friction, FBDs
- System analysis techniques
- Design case

Objectives:
- To be able to develop differential equations that describe translating systems.

2.1 INTRODUCTION

If the velocity and acceleration of a body are both zero then the body will be static. When forces act on the body they may cause motion. If the applied forces are balanced, and cancel each other out, the body will not accelerate. If the forces are unbalanced then the body will accelerate. If all of the forces act through the center of mass then the body will only translate. Forces that do not act through the center of mass will also cause rotation to occur. This chapter will focus only on translational systems.

The equations of motion for translating bodies are shown in Figure 1. These state simply that velocity is the first derivative of position, and velocity is the first derivative of acceleration. Conversely the acceleration can be integrated to find velocity, and the velocity can be integrated to find position. Therefore, if we know the acceleration of a body, we can determine the velocity and position. Finally, when a force is applied to a mass, an acceleration can be found by dividing the net force by the mass.



equations of motion

$$v(t) = \left(\frac{d}{dt}\right)x(t) \quad (1)$$

$$a(t) = \left(\frac{d}{dt}\right)^2 x(t) = \left(\frac{d}{dt}\right)v(t) \quad (2)$$

OR

$$x(t) = \int v(t)dt = \iint a(t)dt \quad (3)$$

$$v(t) = \int a(t)dt \quad (4)$$

$$a(t) = \frac{F(t)}{M} \quad (5)$$

where,

x, v, a = position, velocity and acceleration

M = mass of the body

F = an applied force

Figure 1 Velocity and acceleration of a translating mass

An example application of these fundamental laws is shown in Figure 2. The initial conditions of the system are supplied (and are normally required to solve this type of problem). These are then used to find the state of the system after a period of time. The solution begins by integrating the acceleration, and using the initial velocity value for the integration constant. So at t=0 the velocity will be equal to the initial velocity. This is then integrated once more to provide the position of the object. As before the initial position is used for the integration constant. This equation is then used to calculate the position after a period of time. Notice that the units are used throughout the calculations. This is good practice for any engineer.

Given an initial (t=0) state of x=5m, v=2m/s, a=3ms^{-2}, find the system state 5 seconds later assuming constant acceleration.

The initial conditions for the system at time t=0 are,

$$x_0 = 5m$$
$$v_0 = 2ms^{-1}$$
$$a_0 = 3ms^{-2}$$

Note: units are very important and should normally be carried through all calculations.

The constant acceleration can be integrated to find the velocity as a function of time.

$$v(t) = \int a_0 dt = a_0 t + C = a_0 t + v_0 \qquad (6)$$

Note:
$$v(t) = a_0 t + C$$
$$v_0 = a_0(0) + C$$
$$v_0 = C$$

Next, the velocity can be integrated to find the position as a function of time.

$$x(t) = \int v(t) dt = \int (a_0 t + v_0) dt = \frac{a_0}{2} t^2 + v_0 t + x_0 \qquad (7)$$

This can then be used to calculate the position of the mass after 5 seconds.

$$x(5) = \frac{a_0}{2} t^2 + v_0 t + x_0$$
$$\therefore \quad = \frac{3ms^{-2}}{2}(5s)^2 + 2ms^{-1}(5s) + 5m$$
$$\therefore \quad = 37.5m + 10m + 5m = 52.5m$$

Figure 2 Sample state calculation for a translating mass, with initial conditions

2.2 MODELING

When modeling translational systems it is common to break the system into parts. These parts are then described with Free Body Diagrams (FBDs). Common components that must be considered when constructing FBDs are listed below, and are discussed in following sections.

• gravity and other fields - apply non-contact forces
• inertia - opposes acceleration and deceleration
• springs - resist deflection
• dampers and drag - resist motion
• friction - opposes relative motion between bodies in contact

• cables and pulleys - redirect forces
• contact points/joints - transmit forces through up to 3 degrees of freedom

2.2.1 Free Body Diagrams

Free Body Diagrams (FBDs) allow us to reduce a complex mechanical system into smaller, more manageable pieces. The forces applied to the FBD can then be summed to provide an equation for the piece. These equations can then be used later to do an analysis of system behavior. These are required elements for any engineering problem involving rigid bodies.

An example of FBD construction is shown in Figure 3. In this case there is a mass sitting atop a spring. An FBD can be drawn for the mass. In total there are two obvious forces applied to the mass, gravity pulling the mass downward, and a spring pushing the mass upwards. The FBD for the spring has two forces applied at either end. Notice that the spring force acting on the mass, and on the spring have an equal magnitude, but opposite direction.

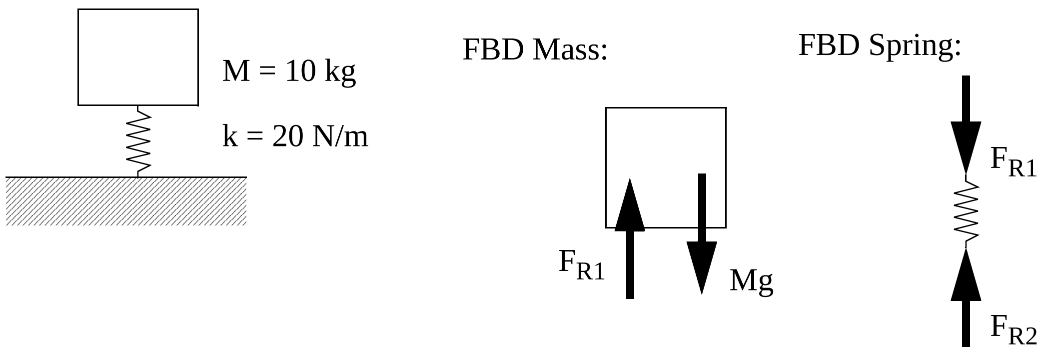


Figure 3 Free body diagram example

2.2.2 Mass and Inertia

In a static system the sum of forces is zero and nothing is in motion. In a dynamic system the sum of forces is not zero and the masses accelerate. The resulting imbalance in forces acts on the mass causing it to accelerate. For the purposes of calculation we create a virtual reaction force, called the inertial force. This force is also known as D'Alembert's (pronounced as daa-lamb-bear) force. It can be included in calculations in one of two ways. The first is to add the inertial force to the FBD and then add it into the sum of forces, which will equal zero. The second method is known as D'Alembert's equation where all of the forces are summed and set equal to the inertial force, as shown in Figure 4. The acceleration is proportional to the inertial force and inversely proportional to the mass.

$$\sum F = Ma \qquad \text{(Newton's)} \qquad (11)$$

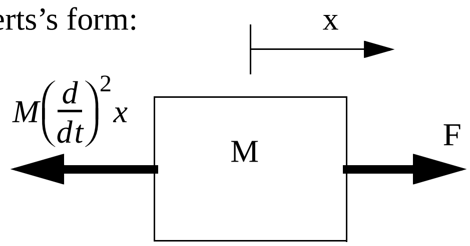
$$\sum F - Ma = 0 \qquad \text{(D'Alembert's)} \qquad (12)$$

Figure 4 D'Alembert's and Netwon's equation

An application of Newton's form to FBDs can be seen in Figure 5. In the first case an inertial force is added to the FBD. This force should be in an opposite direction (left here) to the positive direction of the mass (right). When the sum of forces equation is used then the force is added in as a normal force component. In the second case Newton's equation is used so the force is left off the FBD, but added to the final equation. In this case the sign of the inertial force is positive if the assumed positive direction of the mass matches the positive direction for the summation.

D'Alemberts's form:

$$Ma = M\left(\frac{d}{dt}\right)^2 x$$



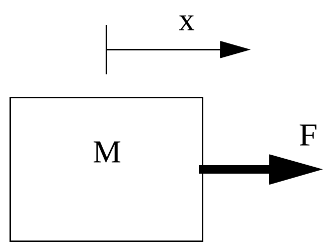
$$\xrightarrow{+} \sum F_x = F - M\left(\frac{d}{dt}\right)^2 x = 0$$

or

$$\xleftarrow{+} \sum F_x = -F + M\left(\frac{d}{dt}\right)^2 x = 0$$

Note: If using an inertial force then the direction of the force should be opposite to the positive motion direction for the mass.

Newton's form



$$\xrightarrow{+} \sum F_x = F = M\left(\frac{d}{dt}\right)^2 x$$

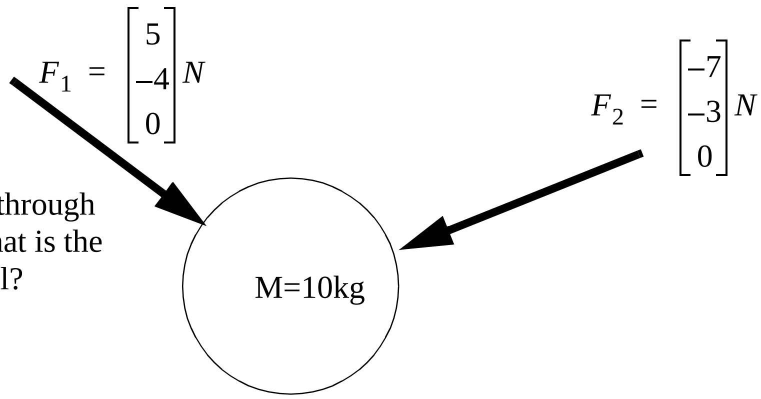
or

$$\xleftarrow{+} \sum F_x = -F = (-M)\left(\frac{d}{dt}\right)^2 x$$

Note: If using Newton's form the sign of the inertial force should be positive if the positive direction for the summation and the mass are the same, otherwise if they are opposite then the sign should be negative.

Figure 5 Free body diagram and inertial forces

An example of the application of Newton's equation is shown in Figure 6. In this example there are two unbalanced forces applied to a mass. These forces are summed and set equal to the inertial force. Solving the resulting equation results in acceleration values in the x and y directions. In this example the forces and calculations are done in vector format for convenience and brevity.



If both forces shown act through the center of mass, what is the acceleration of the ball?

$$\sum F = F_1 + F_2 = Ma$$

$$\therefore \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix} N + \begin{bmatrix} -7 \\ -3 \\ 0 \end{bmatrix} N = (10Kg)a$$

$$\therefore a = \left(\frac{1}{10Kg}\right) \begin{bmatrix} -2 \\ -7 \\ 0 \end{bmatrix} N = \begin{bmatrix} -0.2 \\ -0.7 \\ 0 \end{bmatrix} \left(\frac{Kgm}{s^2}\right) \frac{1}{Kg} = \begin{bmatrix} -0.2 \\ -0.7 \\ 0 \end{bmatrix} \frac{m}{s^2}$$

Figure 6 Sample acceleration calculation

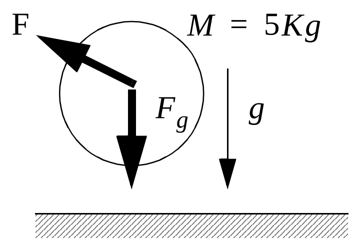
2.2.3 Gravity And Other Fields

Gravity is a weak force of attraction between masses. In our situation we are in the proximity of a large mass (the earth) which produces a large force of attraction. When analyzing forces acting on rigid bodies we add this force to our FBDs. The magnitude of the force is proportional to the mass of the object, with a direction toward the center of the earth (down).

The relationship between mass and force is clear in the metric system with mass having the units Kilograms (kg), and force the units Newtons (N). The relationship between these is the gravitational constant 9.81N/kg, which will vary slightly over the surface of the earth. The Imperial units for force and mass are both the pound (lb.) which often causes confusion. To reduce this confusion the units for force is normally modified to be, lbf.

An example calculation including gravitational acceleration is shown in Figure 7. The 5kg mass is pulled by two forces, gravity and the arbitrary force, F. These forces are described in vector form, with the positive z axis pointing upwards. To find the equations of motion the forces are summed. To eliminate the second derivative on the inertia term the equation is integrated twice. The result is a set of three vector equations that describe the x, y and z components of the motion. Notice that the units have been carried through these calculations.

Assume we have a mass that is acted upon by gravity and a second constant force vector. To find the position of the mass as a function of time we first define the gravity vector, and position components for the system.



$$g = \begin{bmatrix} 0 \\ 0 \\ -9.81 \end{bmatrix} \frac{N}{Kg} = \begin{bmatrix} 0 \\ 0 \\ -9.81 \end{bmatrix} \frac{m}{s^2} \qquad X(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \qquad F = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \qquad F_g = Mg$$

Next sum the forces and set them equal to inertial resistance.

$$\sum F = Mg + F = M\left(\frac{d}{dt}\right)^2 X(t)$$

$$5Kg \begin{bmatrix} 0 \\ 0 \\ -9.81 \end{bmatrix} \frac{m}{s^2} + \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = 5Kg\left(\frac{d}{dt}\right)^2 \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ -9.81 \end{bmatrix} \frac{m}{s^2} + 0.2Kg^{-1} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \left(\frac{d}{dt}\right)^2 \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

Integrate twice to find the position components.

$$\left[\begin{bmatrix} 0 \\ 0 \\ -9.81 \end{bmatrix} \frac{m}{s^2} + 0.2Kg^{-1} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}\right] t + \begin{bmatrix} v_{x_0} \\ v_{y_0} \\ v_{z_0} \end{bmatrix} = \left(\frac{d}{dt}\right) \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

$$\frac{1}{2}\left[\begin{bmatrix} 0 \\ 0 \\ -9.81 \end{bmatrix} \frac{m}{s^2} + 0.2Kg^{-1} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}\right] t^2 + \begin{bmatrix} v_{x_0} \\ v_{y_0} \\ v_{z_0} \end{bmatrix} t + \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 0.1f_x t^2 + v_{x_0} t + x_0 \\ 0.1f_y t^2 + v_{y_0} t + y_0 \\ \left(\frac{-9.81}{2} + 0.1f_z\right)t^2 + v_{z_0} t + z_0 \end{bmatrix} m$$

> Note: When an engineer solves a problem they will always be looking at the equations and unknowns. In this case there are three equations, and there are 9 constants/givens fx, fy, fz, vx0, vy0, xz0, x0, y0 and z0. There are 4 variables/unknowns x, y, z and t. Therefore with 3 equations and 4 unknowns only one value (4-3) is required to find all of the unknown values.

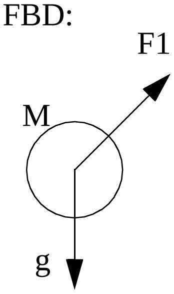
Figure 7 Gravity vector calculations

Like gravity, magnetic and electrostatic fields can also apply forces to objects. Magnetic forces are commonly found in motors and other electrical actuators. Electrostatic forces are less common, but may need to be considered for highly charged systems.

Given,

$$F_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} N \qquad g = \begin{bmatrix} 0 \\ -9.81 \\ 0 \end{bmatrix} \frac{N}{Kg} \qquad M = 2Kg$$

FBD:



Find the acceleration.

Figure 8 Drill problem: find the acceleration of the FBD

2.2.4 Springs

A spring is based on an elastic material that will provide an opposing force when deformed. The most common springs are made of metals or plastics. The properties of the spring are determined by the Young's modulus (E) of the material and the geometry of the spring. A primitive spring is shown in Figure 9. In this case the spring is a solid member. The relationship between force and displacement is determined by the basic mechanics of materials relationship. In practice springs are more complex, but the parameters (E, A and L) are combined into a more convenient form. This form is known as Hooke's Law.

$$\delta = \left(\frac{L}{EA}\right)F$$

$$F = \left(\frac{EA}{L}\right)\delta$$

$$F = K\delta \qquad \text{(Hooke's law)}$$

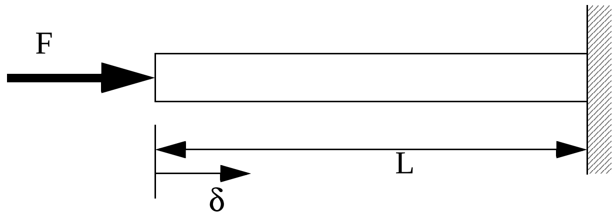
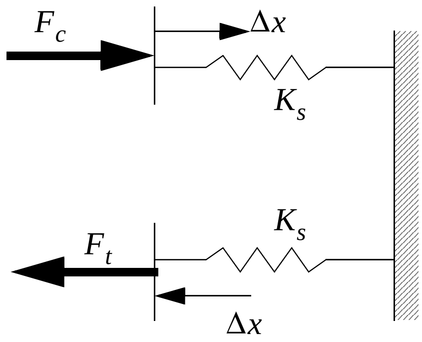


Figure 9 A solid member as a spring

Hooke's law does have some limitations that engineers must consider. The basic equation is linear, but as a spring is deformed the material approaches plastic deformation, and the modulus of elasticity will change. In addition the geometry of the object changes, also changing the effective stiffness. Springs are normally assumed to be massless. This allows the inertial effects to be ignored, such as a force propagation delay. In applications with fast rates of change the spring mass may become significant, and they will no longer act as an ideal device.

The cases for tension and compression are shown in Figure 10. In the case of compression the spring length has been made shorter than its' normal length. This requires that a compression force be applied. For tension both the displacement from neutral and the required force change direction. It is advisable when solving problems to assume a spring is either in tension or compression, and then select the displacement and force directions accordingly.



compression as positive

$$F_c = K_s \Delta x$$

tension as positive

$$F_t = K_s \Delta x$$

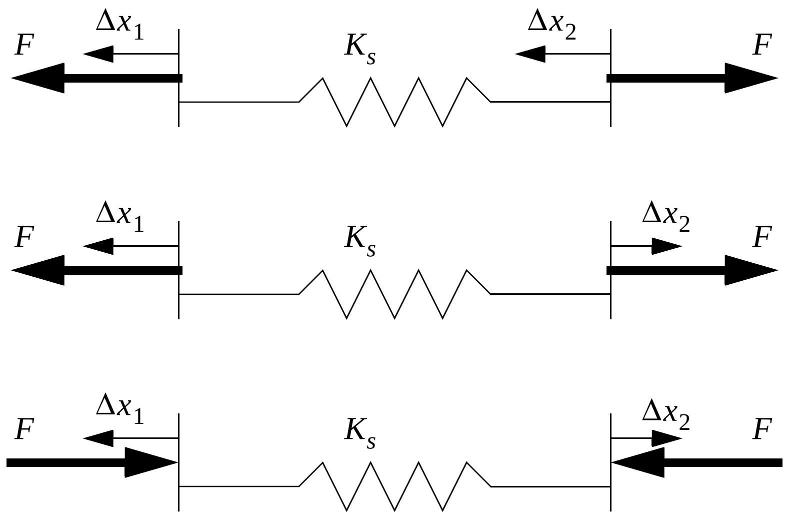
Δx = deformed length

ASIDE: a spring has a natural or undeformed length. When at this length it is neither in tension or compression

NOTE: the symbols for springs, resistors and inductors are quite often the same or similar. You will need to remember this when dealing with complex systems - and especially in this course where we deal with both types of systems.

Figure 10 Sign conventions for spring forces and displacements

Previous examples have shown springs with displacement at one end only. In Figure 11 springs are shown that have movement at both ends. In these cases the force applied to the spring is still related to the relative compression and tension. The primary difference is that care is required to correctly construct the expressions for the tension or compression forces. In all cases the forces on the springs must be assumed and drawn as either tensile or compressive. In the first example the displacement and forces are tensile. The displacement at the left is tensile, so it will be positive, but on the right hand side the displacement is compressive so it is negative. In the second example the force and both displacements are shown as tensile, so the terms are both positive. In the third example the force is shown as compressive, while the displacements are both shown as tensile, so both terms are negative.



$$F = K_s(\Delta x_1 - \Delta x_2)$$

$$F = K_s(\Delta x_1 + \Delta x_2)$$

$$F = K_s(-\Delta x_1 - \Delta x_2)$$

ETC....

Aside: it is useful to assume that the spring is either in tension or compression, and then make all decisions based on that assumption.

Figure 11 Examples of forces when both sides of a spring can move

Sometimes the true length of a spring is important, and the deformation alone is insufficient. In these cases the deformation can be defined as a deformed and undeformed length, as shown in Figure 12.

$$\Delta x = l_1 - l_0$$

where,

Δx = deformed length
l_0 = the length when undeformed
l_1 = the length when deformed

Figure 12 Using the actual spring length

In addition to providing forces, springs may be used as energy storage devices. Figure 13 shows the equation for energy stored in a spring.

$$E_P = \frac{K(\Delta x)^2}{2}$$

Figure 13 Energy stored in a spring

Given,

$$K_s = 10\frac{N}{m}$$

$$\Delta x_1 = 0.1m$$
$$\Delta x_2 = 0.1m$$

Find F1 and F2 separately (don’t try to solve at the same time)

Aside: it can help to draw a FBD of the pin.

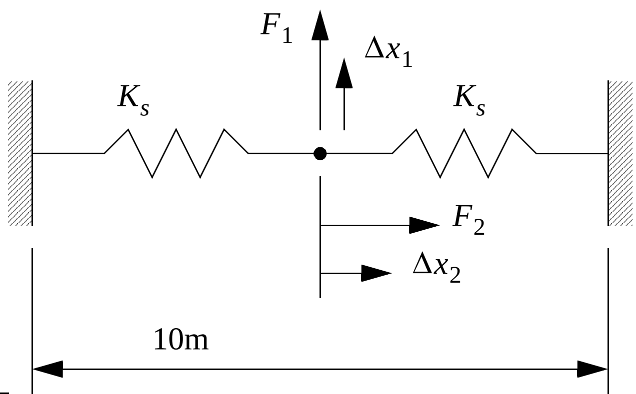


Figure 14 Drill problem: Deformation of a two spring system

Draw the FBDs and sum the forces for the masses

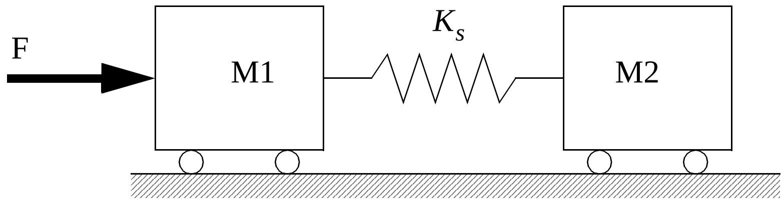


Figure 15 Drill problem: Draw the FBDs for the masses

2.2.5 Damping and Drag

A damper is a component that resists motion. The resistive force is relative to the rate of displacement. As mentioned before, springs store energy in a system but dampers dissipate energy. Dampers and springs are often used to compliment each other in designs.

Damping can occur naturally in a system, or can be added by design. A physical damper is pictured in Figure 16. This one uses a cylinder that contains a fluid. There is a moving rod and piston that can slide along the cylinder. As the piston moves fluid is forced through a small orifice in the cylinder. When moved slowly the fluid moves easily, but when moved quickly the pressure required to force the fluid through the orifice rises. This rise in pressure results in a higher force of resistance. In ideal terms any motion would result in an opposing force. In reality there is also a break-away force that needs to be applied before motion begins. Other manufacturing variations could also lead to other small differences between dampers. Normally these cause negligible effects.

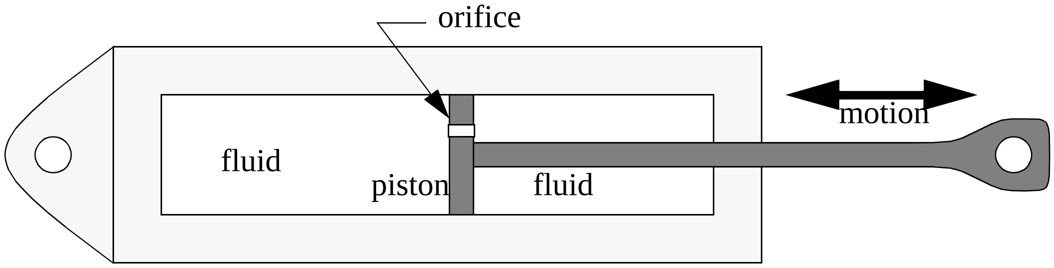
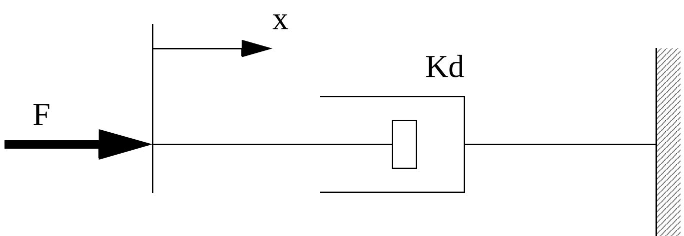


Figure 16 A physical damper

The basic equation for an ideal damper in compression is shown in Figure 17. In this case the force and displacement are both compressive. The force is calculated by multiplying the damping coefficient by the velocity, or first derivative of position. Aside from the use of the first derivative of position, the analysis of dampers in systems is similar to that of springs.

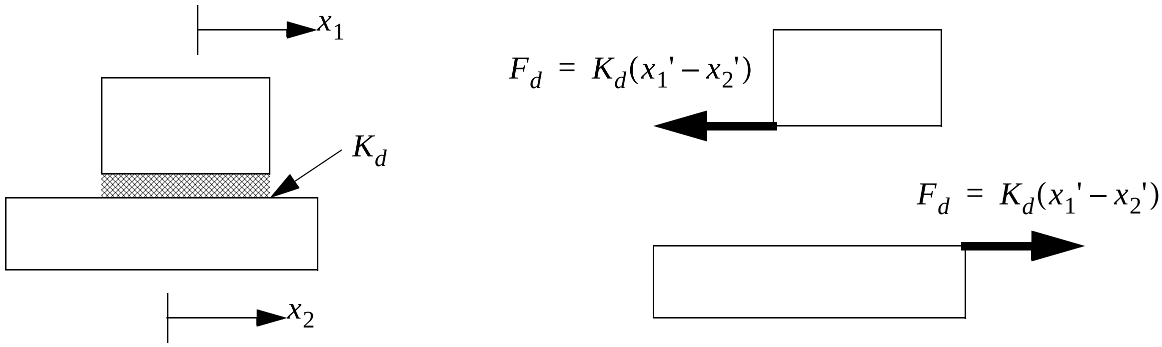


$$F = K_d\left(\frac{d}{dt}\right)x \tag{15}$$

Aside: The symbol shown is typically used for dampers. It is based on an old damper design called a dashpot. It was constructed using a small piston inside a larger pot filled with oil.

Figure 17 An ideal damper

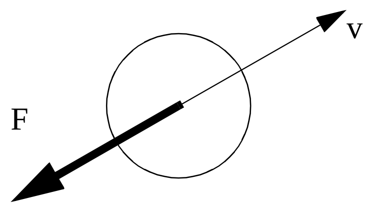
Damping can also occur when there is relative motion between two objects. If the objects are lubricated with a viscous fluid (e.g., oil) then there will be a damping effect. In the example in Figure 18 two objects are shown with viscous friction (damping) between them. When the system is broken into free body diagrams the forces are shown to be a function of the relative velocities between the blocks.



Aside: Fluids, such as oils, have a significant viscosity. When these materials are put in shear they resist the motion. The higher the shear rate, the greater the resistance to flow. Normally these forces are small, except at high velocities.

Figure 18 Viscous damping between two bodies with relative motion

A damping force is proportional to the first derivative of position (velocity). Aerodynamic drag is proportional to the velocity squared. The equation for drag is shown in Figure 19 in vector and scalar forms. The drag force increases as the square of velocity. Normally, the magnitude of the drag force coefficient 'D' is approximated theoretically and/or measured experimentally. The drag coefficient is a function of material type, surface properties, object size and object geometry.



$$F = -D|v|v$$

Figure 19 Aerodynamic drag

If we are pushing the cylinder at the given velocities below, what is the required force?

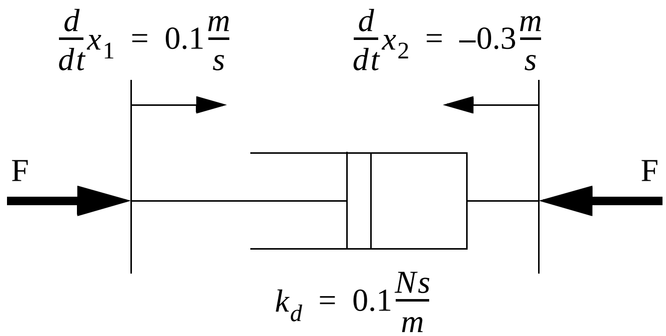
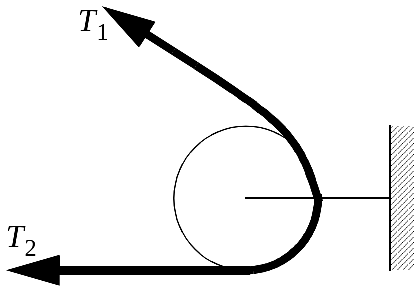


Figure 20 Drill problem: Find the required forces on the dampers

2.2.6 Cables And Pulleys

Cables are useful when transmitting tensile forces or displacements. The centerline of the cable becomes the centerline for the force. And, if the force becomes compressive, the cable goes limp, and will not transmit force. A cable by itself can be represented as a force vector. When used in combination with pulleys a cable can redirect a force vector, or multiply a force.

Typically we assume that a pulley is massless and frictionless (in the rotation chapter we will assume they are not). If this is the case then the tension in the cable on both sides of the pulley is equal as shown in Figure 21.

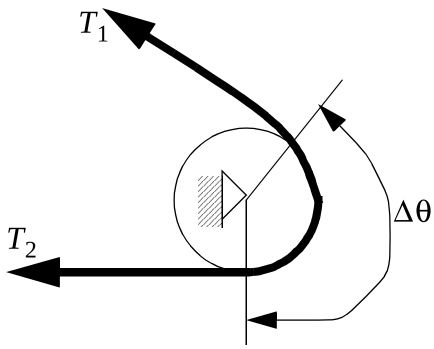


for a massless frictionless pulley

$$T_1 = T_2$$

Figure 21 Tension in a cable over a massless frictionless pulley

If we have a pulley that is fixed and cannot rotate the cable must slide over the surface of the pulley. In this case we can use the friction to determine the relative ratio of forces between the sides of the pulley, as shown in Figure 22.



$$\frac{T_2}{T_1} = e^{\mu_k(\Delta\theta)}$$

Figure 22 Friction of a belt over a fixed drum

Given,

$$\mu_k = 0.2$$

$$M = 1Kg$$

Find F to lift/drop the mass slowly.

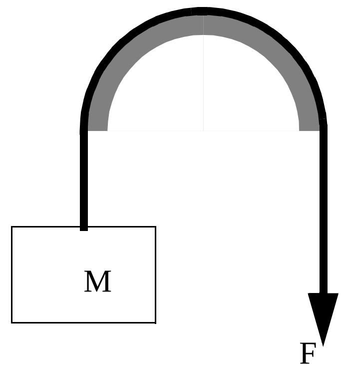


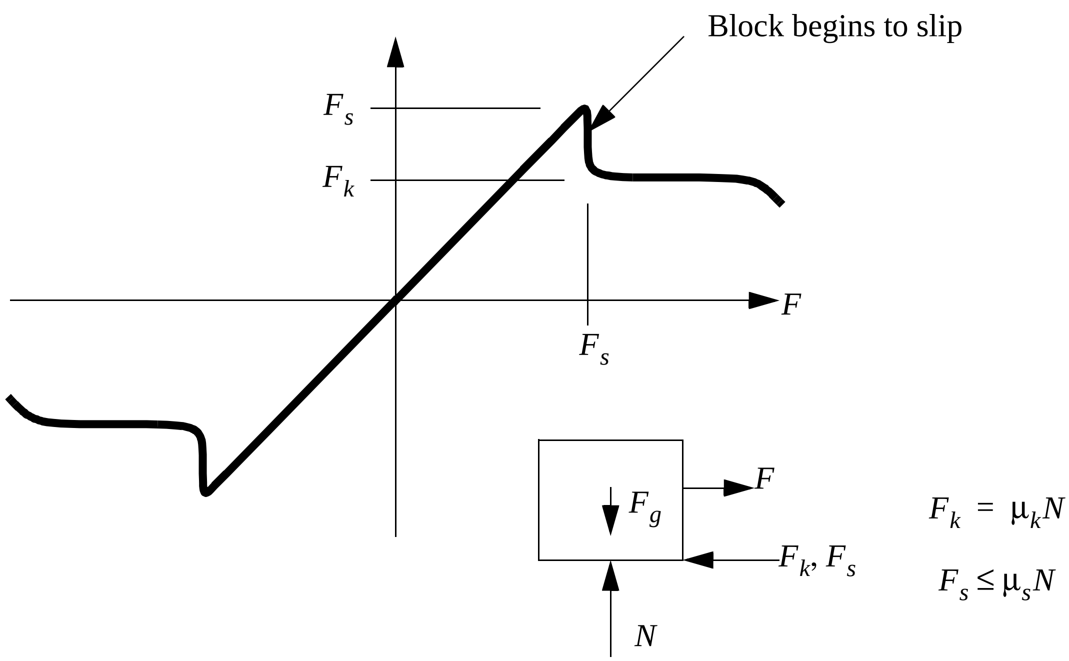
Figure 23 Drill problem: Find the force required to overcome friction

Although the discussion in this section has focused on cables and pulleys, the theory also applies to belts and drums.

2.2.7 Friction

Viscous friction was discussed before, where a lubricant would provide a damping effect between two moving objects. In cases where there is no lubricant, and the joint is dry, we may experience dry coulomb friction. In this case the object will stick in place until a maximum force is overcome. After that the object will begin to slide and a constant force will result.

Figure 24 shows the classic model for friction. The force on the horizontal axis is the force applied to the friction surfaces while the vertical axis is the resulting friction force. Beneath the slip force the object will stay in place. When the slip force is exceeded the object will begin to slip, and the resulting kinetic friction force is relatively constant. If the object begins to travel fast then the kinetic friction force will decrease. It is also common to forget that friction is bidirectional, but it always opposes the applied force. The friction force is a function of the normal force across the friction surface and the coefficient of friction. The coefficient of friction is a function of the materials, surface texture and surface shape.



Note: When solving problems with friction remember that the friction force will always equal the applied force until slip occurs. After that the friction is approximately constant. In addition, the friction forces will change direction to oppose an applied force, or motion.

Figure 24 Dry friction

Many systems use kinetic friction to dissipate energy from a system as heat, sound and vibration.

Find the acceleration of the block for both angles indicated.

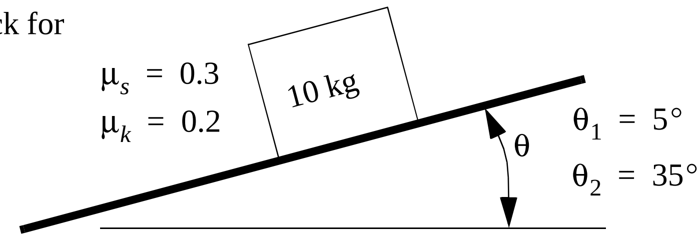


Figure 25 Drill problem: find the accelerations

2.2.8 Contact Points And Joints

A system is built by connecting components together. These connections can be rigid or moving. In solid connections all forces and moments are transmitted and the two pieces act as a single rigid body. In moving connections there is at least one degree of freedom. If we limit this thought to translation only, there are up to three degrees of freedom between parts, x, y and z. In any direction there is a degree of freedom, force cannot be transmitted.

When constructing FBDs for a system we must break all of the components into individual rigid bodies. Where the mechanism has been broken the contact forces must be added to both of the separated pieces. Consider the example in Figure 26. At joint A the forces are written as two components in the x and y directions. For joint B the force components with equal magnitudes but opposite directions are added to both FBDs.

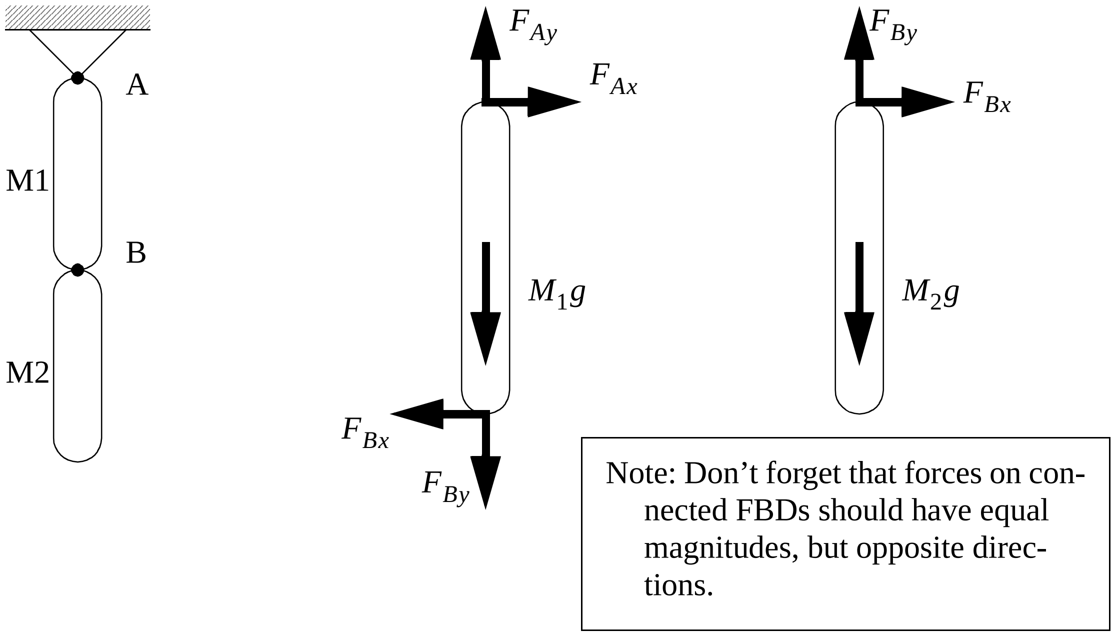


Figure 26 FBDs for systems with connected members

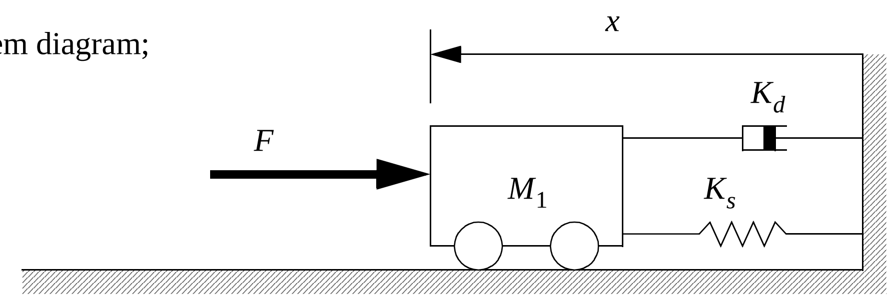
2.3 SYSTEM EXAMPLES

An orderly approach to system analysis can simplify the process of analyzing large systems. The list of steps below is based on general observations of good problem solving techniques.

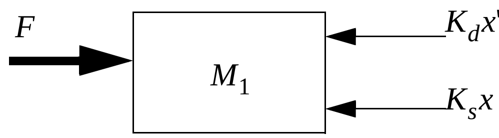
1. Assign letters/numbers to designate components (if not already done) - this will allow you to refer to components in your calculations.
2. Define positions and directions for any moving masses. This should include the selection of reference points.
3. Draw free body diagrams for each component, and add forces (inertia is optional).
4. Write equations for each component by summing forces.
5. (next chapter) Combine the equations by eliminating unwanted variables.
6. (next chapter) Develop a final equation that relates input (forcing functions) to outputs (results).

Consider the cart in Figure 27. On the left is a force, it is opposed by a spring and damper on the right. The basic problem definition already contains all of the needed definitions, so no others are required. The FBD for the mass shows the force and the reaction forces from the spring and damper. When the forces are summed the inertia is on the right in Newton’s form. This equation is then rearranged to a second-order non-homogeneous differential equation.

Given the system diagram;



The FBD for the cart is



The forces for the cart are in a single direction and can be summed as,

$$\overset{+}{\leftarrow} \quad \sum F_x = -F + K_d x' + K_s x = M_1 x''$$

This equation can be rearranged to a second-order non-homogeneous diff. eqn.

$$M_1 x'' + K_d x' + K_s x = -F$$

Aside: later on we will solve the differential equations, or use other methods to determine how the system will behave. It is useful to have all of the ’output’ variables for the system on the left hand side, and everything else on the other.

Figure 27 A simple translational system example

Develop the equation relating the input force to the motion (in terms of x) of the lefthand cart for the problem below.

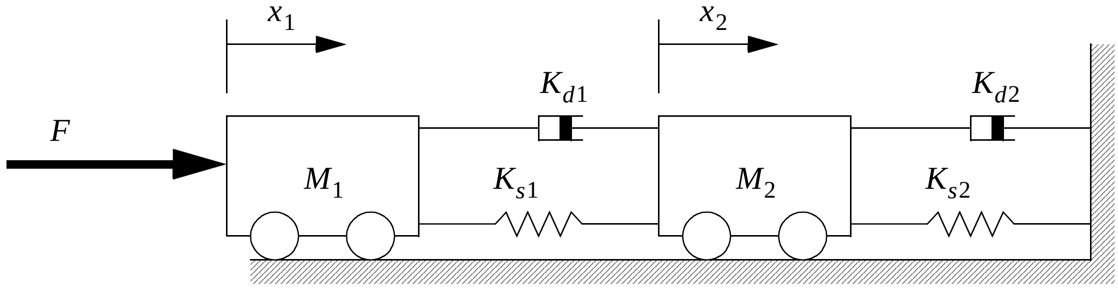
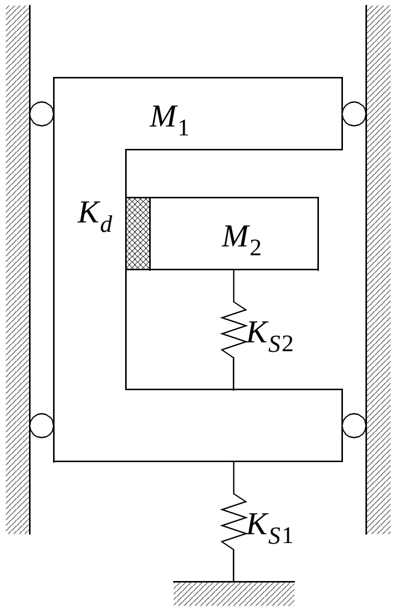


Figure 28 Drill problem: Find the differential equations

A simplified model of an elevator (M1) and a passenger (M2) are shown in Figure 29. In this example many of the required variables need to be defined. These are added to the FBDs. Care is also taken to ensure that all forces between bodies are equal in magnitude, but opposite in direction. The wall forces are ignored because they are statically indeterminate and being in the x-axis irrelevant to the problem in the y-axis.



Assign required quantities and draw the FBDs

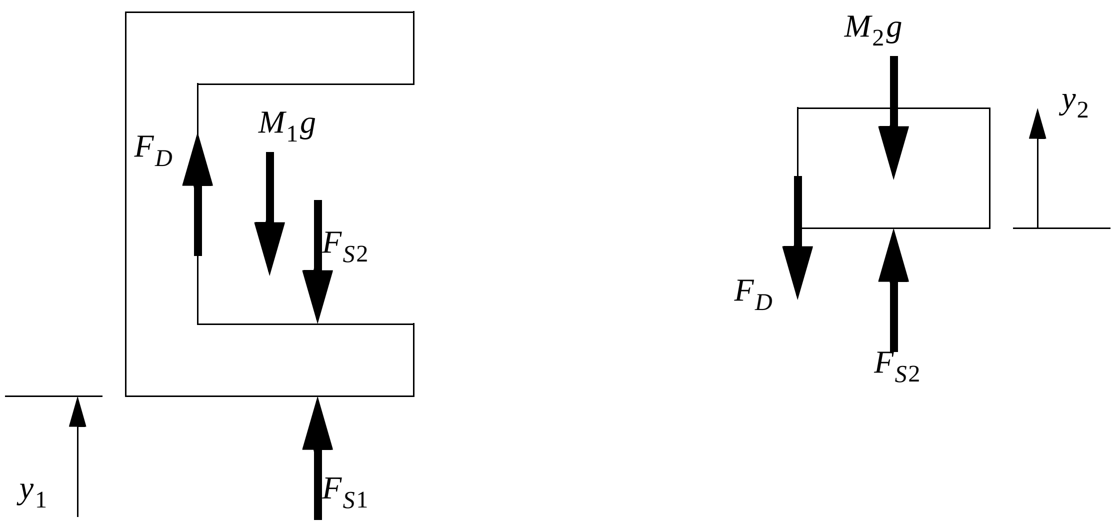


Figure 29 A multi-body translating system (an elevator with a passenger)

The forces on the FBDs are then summed, and the equations are expanded as shown in Figure 30.

Now, we write out the force balance equation (vector form), and substitute relationships

$$\sum F_{M_1} = F_{S1} + F_{S2} + M_1 g + F_D = M_1 a_1$$

$$\sum F_{M_2} = -F_{S2} + M_2 g - F_D = M_2 a_2$$

At this point we can expand the equations into vector components

$$K_{S1}(-\Delta y_1) + K_{S2}(\Delta y_2 - \Delta y_1) + M_1(-9.81) + K_d \frac{d}{dt}(y_2 - y_1) = M_1 a_y$$

$$-K_{S2}(\Delta y_2 - \Delta y_1) + M_2(-9.81) - K_d \frac{d}{dt}(y_2 - y_1) = M_2 a_y$$

Figure 30 Equations for the elevator

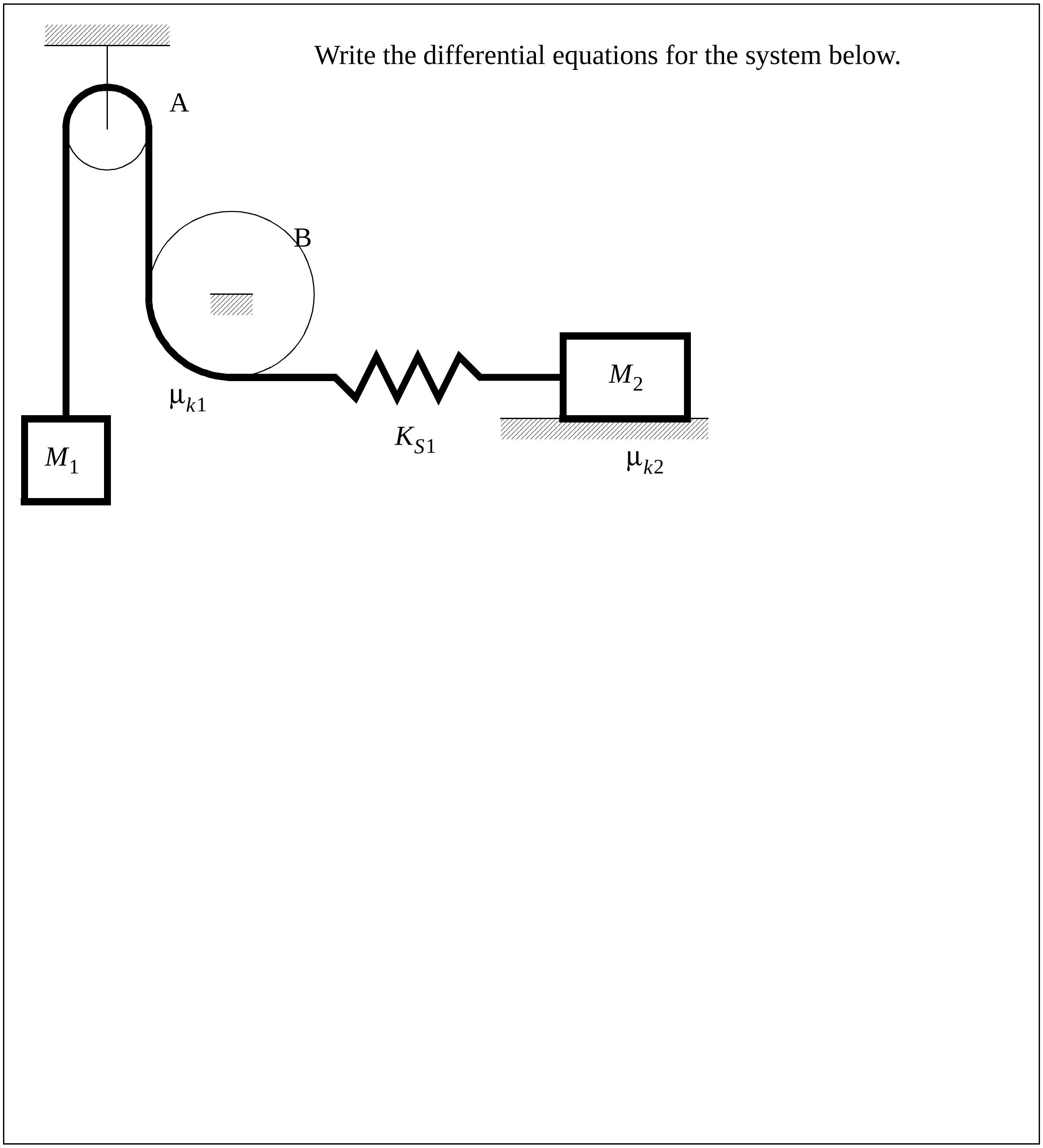


Figure 31 Drill problem: A more complex translational systems

Consider the springs shown in Figure 32. When two springs are combined in this manner they can be replaced with a single equivalent spring. In the parallel spring combination the overall stiffness of the spring would increase. In the series spring combination the overall stiffness would decrease.

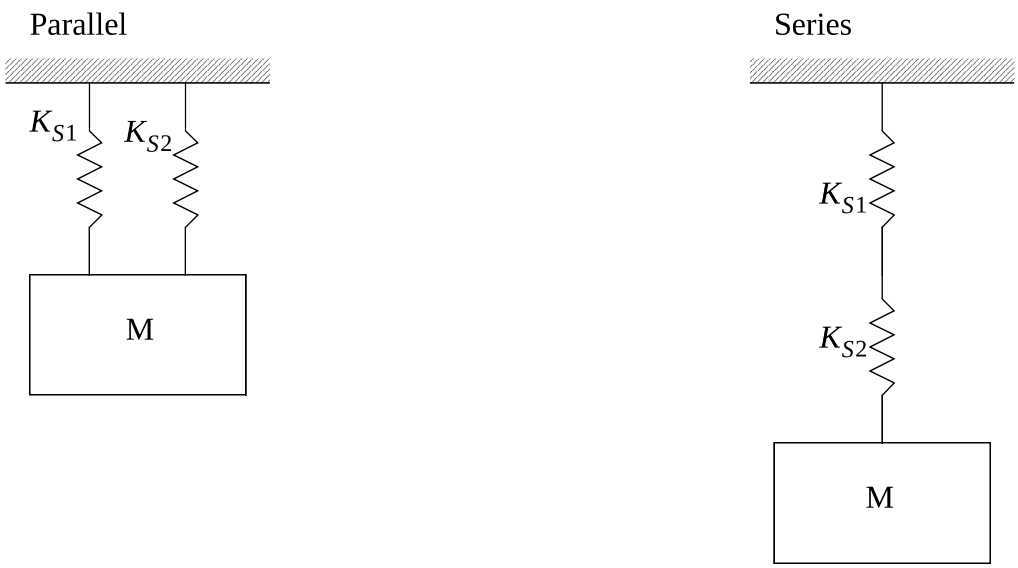
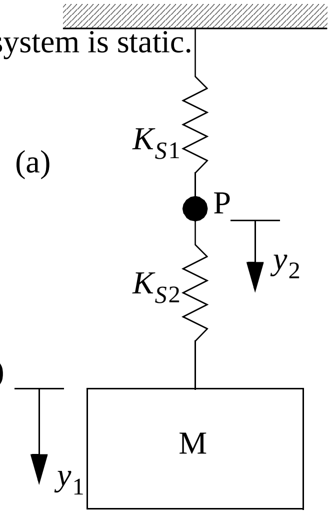
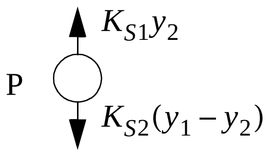


Figure 32 Springs in parallel and series (kinematically)

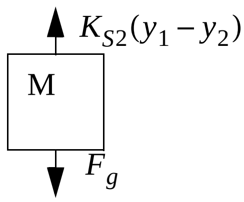
Figure 33 shows the calculations required to find a spring coefficient equivalent to the two springs in series. The first step is to draw a FBD for the mass at the bottom, and for a point between the two springs, P. The forces for both of these are then summed. The next process is to combine the two equations to eliminate the height variable created for P. After this the equation is rearranged into Hooke’s law, and the equivalent spring coefficient is found.

First, draw FBDs for P and M and sum the forces assuming the system is static.





$$+\uparrow \sum F_y = K_{S1}y_2 - K_{S2}(y_1 - y_2) = 0 \qquad \text{(a)}$$



$$+\uparrow \sum F_y = K_{S2}(y_1 - y_2) - F_g = 0 \qquad \text{(b)}$$

Next, rearrange the equations to eliminate y2 and simplify.

(b) becomes $$K_{S2}(y_1 - y_2) - F_g = 0$$

$$\therefore y_1 - y_2 = \frac{F_g}{K_{S2}}$$

$$\therefore y_2 = y_1 - \frac{F_g}{K_{S2}}$$

(a) becomes $$K_{S1}y_2 - K_{S2}(y_1 - y_2) = 0$$

$$\therefore y_2(K_{S1} + K_{S2}) = y_1 K_{S2}$$

sub (b) into (a) $$\left(y_1 - \frac{F_g}{K_{S2}}\right)(K_{S1} + K_{S2}) = y_1 K_{S2}$$

$$\therefore y_1 - \frac{F_g}{K_{S2}} = y_1 \frac{K_{S2}}{K_{S1} + K_{S2}}$$

$$\therefore F_g = y_1\left(1 - \frac{K_{S2}}{K_{S1} + K_{S2}}\right)K_{S2}$$

$$\therefore F_g = y_1\left(\frac{K_{S1} + K_{S2} - K_{S2}}{K_{S1} + K_{S2}}\right)K_{S2}$$

$$\therefore F_g = y_1\left(\frac{K_{S1}K_{S2}}{K_{S1} + K_{S2}}\right)$$

Finally, consider the basic spring equation to find the equivalent spring coefficient.

$$\therefore K_{equiv.} = \frac{K_{S1}K_{S2}}{K_{S1} + K_{S2}}$$

Figure 33 Calculation of an equivalent spring coefficient for springs in series

Figure 34 Drill problem: Find an equivalent spring for the springs in parallel

Consider the drill problem. When an object has no mass, the force applied to one side of the spring will be applied to the other. The only factor that changes is displacement.

Show that a force applied to one side of a massless spring is the reaction force at the other side.

Figure 35 Drill problem: Prove that the force on both sides is equal

2.4 OTHER TOPICS

Designing a system in terms of energy content can allow insights not easily obtained by the methods already discussed. Consider the equations in Figure 36. These equations show that the total energy in the system is the sum of kinetic and potential energy. Kinetic energy is half the product of mass times velocity squared. Potential energy in translating systems is a distance multiplied by a length that force was applied over. In addition the power, or energy transfer rate is the force applied multiplied by the velocity.

$$E = E_P + E_K \tag{7}$$

$$E_K = \frac{Mv^2}{2} \tag{8}$$

$$E_P = Fd = Mgd \tag{9}$$

$$P = Fv = \frac{d}{dt}E \tag{10}$$

Figure 36 Energy and power equations for translating masses

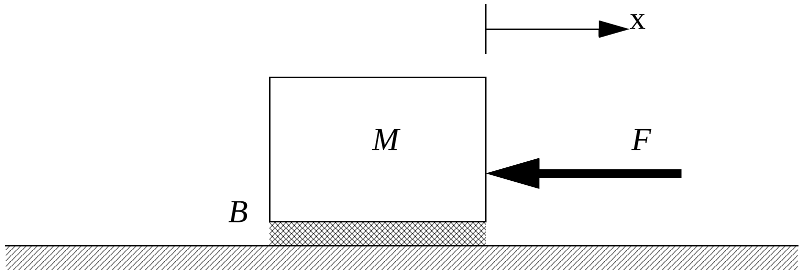
2.5 SUMMARY

- FBDs are useful for reducing complex systems to simpler parts.
- Equations for translation and rotation can be written for FBDs.
- The equations can be integrated for dynamic cases, or solved algebraically for static cases.

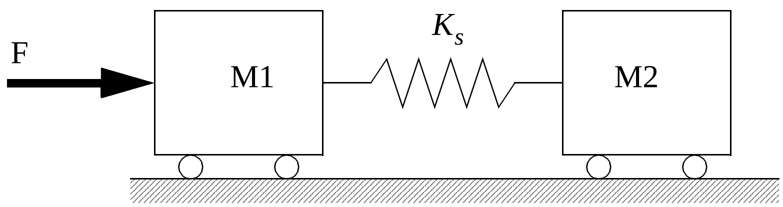
2.6 PRACTICE PROBLEMS

1. If a spring has a deflection of 6 cm when exposed to a static load of 200N, what is the spring constant? (ans. 33.3N/cm)

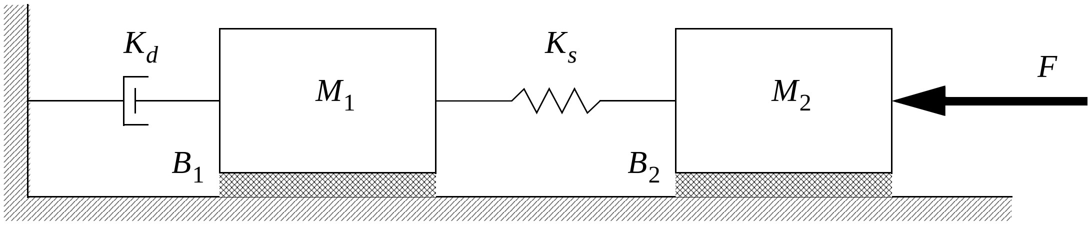
2. The mass, M, illustrated below starts at rest. It can slide across a surface, but the motion is opposed by viscous friction (damping) with the coefficient B. Initially the system starts at rest, when a constant force, F, is applied. Write the differential equation for the mass, and solve the differential equation. Leave the results in variable form.



3. Write the differential equations for the translating system below.

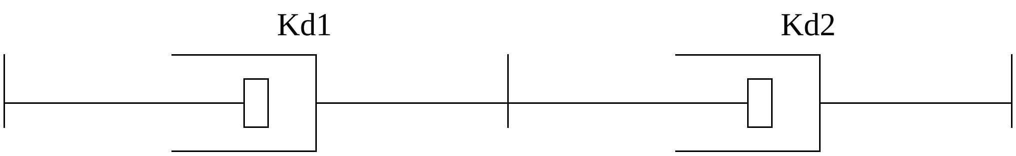


4. Write the differential equations for the system given below.

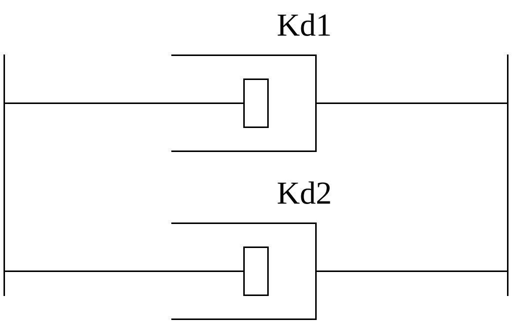


5. Find the effective damping coefficients for the pairs below,

a)

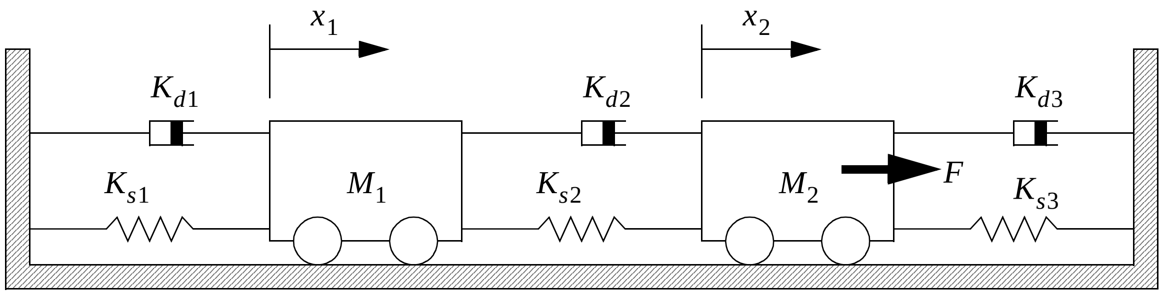


b)



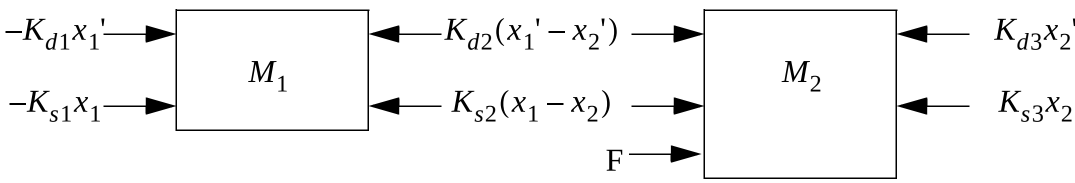
(ans. a) $K_{eq} = \dfrac{K_{d1}K_{d2}}{K_{d1} + K_{d2}}$ b) $K_{eq} = K_{d1} + K_{d2}$

6. Write the differential equations for the system below.



(ans.

FBDs:



For M1:

$$\xrightarrow{+} \sum F = -K_{d1}x_1' - K_{s1}x_1 - K_{d2}(x_1' - x_2') - K_{s2}(x_1 - x_2) = M_1 x_1''$$

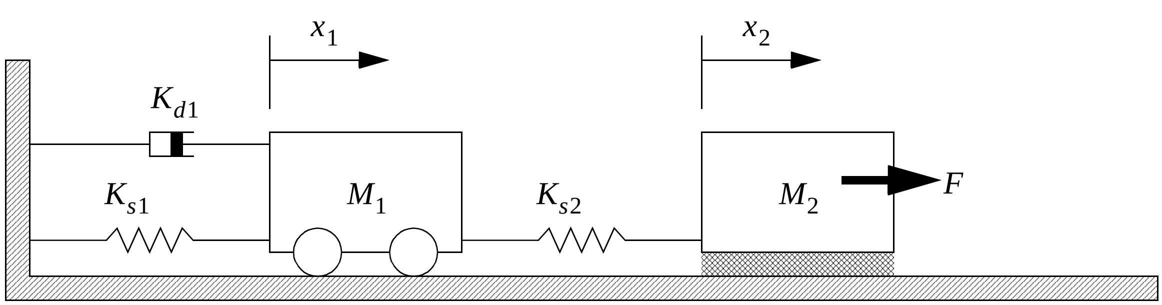
$$x_1''(M_1) + x_1'(K_{d1} + K_{d2}) + x_1(K_{s1} + K_{s2}) + x_2'(-K_{d2}) + x_2(-K_{s2}) = 0$$

For M2:

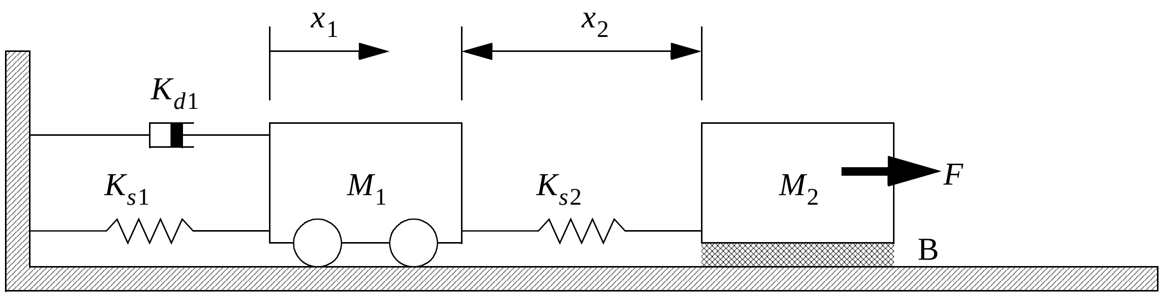
$$\xrightarrow{+} \sum F = K_{d2}(x_1' - x_2') + K_{s2}(x_1 - x_2) + F - K_{d3}x_2' - K_{s3}x_2 = M_2 x_2''$$

$$x_2''(M_2) + x_2'(K_{d2} + K_{d3}) + x_2(K_{s2} + K_{s3}) + x_1'(-K_{d2}) + x_1(-K_{s2}) = F$$

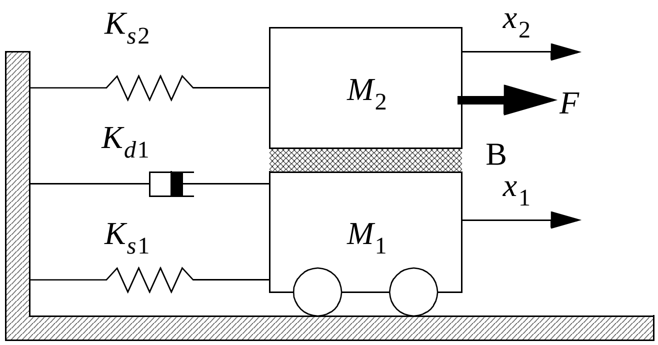
7. Write the differential equations for the system below.



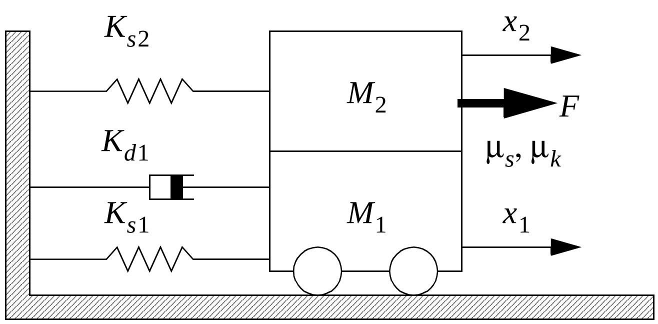
8. Write the differential equations for the system below.



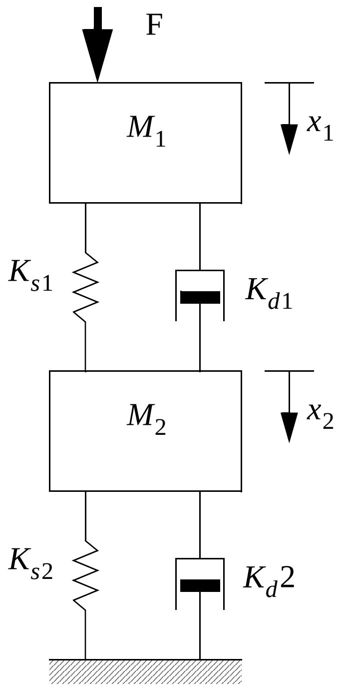
9. Write the differential equations for the system below.



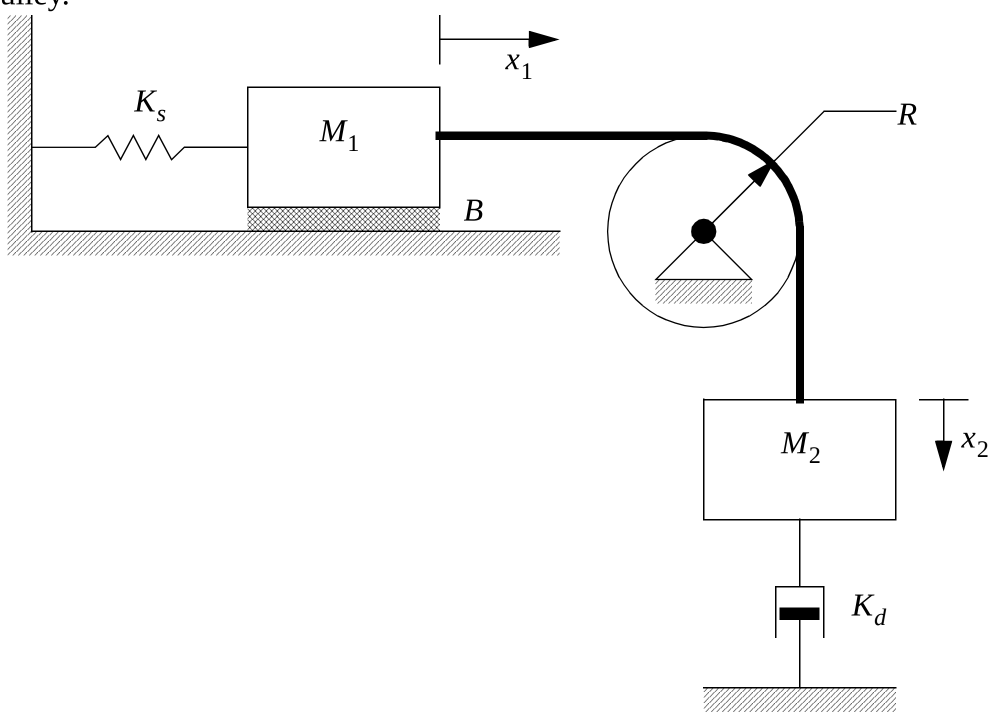
10. Write the differential equations for the system below.



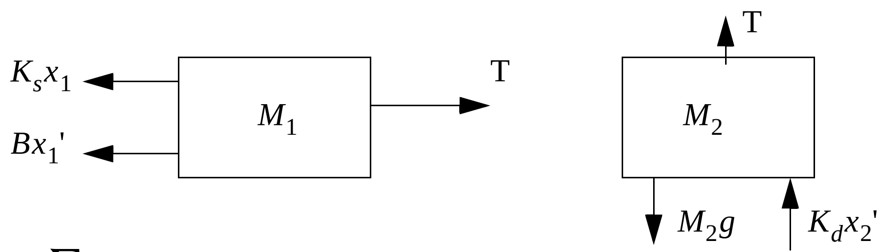
11. Write the differential equations for the system below.



12. Write the differential equations for the system below. In this system the upper mass, M1, is between a spring and a cable and there is viscous damping between the mass and the floor. The suspended mass, M2, is between the cable and a damper. The cable runs over a massless, frictionless pulley.



(ans. FBDs:



For M1:

$$\xrightarrow{+} \quad \sum F = -K_s x_1 - Bx_1' + T = M_1 x_1''$$

$$x_1''(M_1) + x_1'(B) + x_1(K_s) = T$$

For M2:

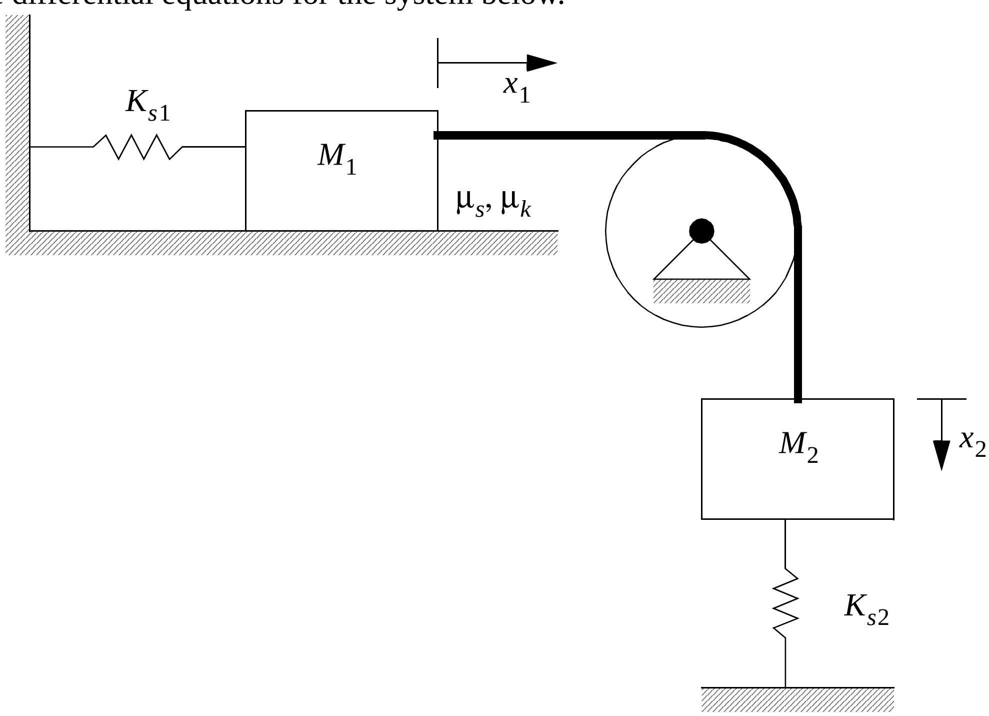
$$+\uparrow \quad \sum F = T + K_d x_2' - M_2 g = -M_2 x_2''$$

$$x_2''(-M_2) + x_2'(-K_d) = T - M_2 g$$

For T:

if T <= 0 then T = 0N
if T > 0 x1 = x2

13. Write the differential equations for the system below.



3. ANALYSIS OF DIFFERENTIAL EQUATIONS

Topics:
- First and second-order homogeneous differential equations
- Non-homogeneous differential equations
- First and second-order responses
- Non-linear system elements
- Design case

Objectives:
- To develop explicit equations that describe a system response.
- To recognize first and second-order equation forms.

3.1 INTRODUCTION

In the previous chapter we derived differential equations of motion. These equations can be used to analyze the behavior of the system and make design decisions. The most basic method is to select a standard input type, and initial conditions, and then solve the differential equation. It is also possible to estimate the system response without solving the differential equation.

Figure 37 shows an abstract description of a system used by engineers. The basic concept is that the system accepts inputs and the system changes the inputs to produce outputs. Say, for example, that the system to be analyzed is an elevator. Inputs to the system would be the mass of human riders and desired elevator height. The output response of the system would be the actual height of the elevator. For analysis, the system model could be developed using differential equations for the motor, elastic lift cable, mass of the car, etc. A basic test would involve assuming that the elevator starts at the ground floor and must travel to the top floor. Using assumed initial values and input functions the differential equation could be solved to get an explicit equation for elevator height. This output response can then be used as a guide to modify design choices (parameters). In practice, many of the assumptions and tests are mandated by law or groups such as Underwriters Laboratories (UL), Canadian Standards Association (CSA) and the European Commission (CE).

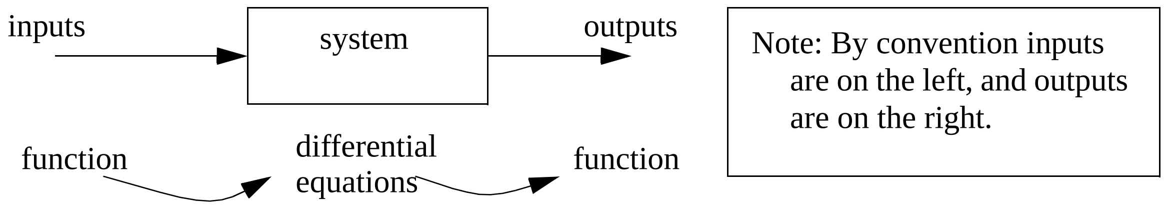


Figure 37 A system with and input and output response

There are several standard input types used to test a system. These are listed below in order of relative popularity with brief explanations.

- step - a sudden change of input, such as very rapidly changing a desired speed from 0Hz to 50Hz.
- ramp - a continuously increasing input, such as a motor speed that increases constantly at 10Hz per minute.
- sinusoidal - a cyclic input that varies continuously, such as a motor speed that is continually oscillating sinusoidally between 0Hz and 100Hz.
- parabolic - an exponentially increasing input, such as a motor speed that is 2Hz at 1 second, 4Hz at 2 seconds, 8Hz at 3 seconds, etc.

After the system has been modeled, an input type has been chosen, and the initial conditions have been selected, the system can be analyzed to determine its behavior. The most fundamental technique is to integrate the differential equation for the system.

3.2 EXPLICIT SOLUTIONS

Solving a differential equation results in an explicit solution. This explicit equation provides the general response, but it can also be used to find frequencies and other details of interest. This section will review techniques used to integrate first and second-order homogenous differential equations. These equations correspond to systems without inputs, also called unforced systems. Non-homogeneous differential equations will also be reviewed. Solving differential equations is necessary for analyzing systems with inputs.

The basic types of differential equations are shown in Figure 38. Each of these equations is linear. On the left hand side is the integration variable ’x’. If the right hand side is zero, then the equation is homogeneous. Each of these equations is linear because each of the terms on the left hand side is simply multiplied by a linear coefficient.

$Ax' + Bx = 0$	first-order homogeneous
$Ax' + Bx = Cf(t)$	first-order non-homogeneous
$Ax'' + Bx' + Cx = 0$	second-order homogeneous
$Ax'' + Bx' + Cx = Df(t)$	second-order non-homogeneous

Figure 38 Standard equation forms

A general solution for a first-order homogeneous differential equation is given in Figure 39. The solution begins with the solution of the homogeneous equation where a general form is ’guessed’. Substitution leads to finding the value of the coefficient ’Y’. Following this, the initial conditions for the equation are used to find the value of the coefficient ’X’. Notice that the final equation will begin at the initial displacement, but approach zero as time goes to infinity. The e-to-the-x behavior is characteristic for a first-order response.

Given,

$$Ax' + Bx = 0 \qquad \text{and} \qquad x(0) = x_0$$

initial condition

Guess a solution form and solve.

$$x = Xe^{-Yt} \qquad x' = -YXe^{-Yt}$$

$$\therefore A(-YXe^{-Yt}) + B(Xe^{-Yt}) = 0$$

$$\therefore A(-Y) + B = 0$$

$$\therefore Y = \frac{B}{A}$$

Therefore the general form is,

$$x_h = Xe^{-\frac{B}{A}t}$$

Note: The general form below is useful for finding almost all homogeneous equations

$$x_h(t) = Xe^{-Yt}$$

Next, use the initial conditions to find the remaining unknowns.

$$x_h = Xe^{-\frac{B}{A}t}$$

$$x_0 = Xe^{-\frac{B}{A}0}$$

$$x_0 = X$$

Therefore the final equation is,

$$x(t) = x_0 e^{-\frac{B}{A}t}$$

Figure 39 General solution of a first-order homogeneous equation

The general solution to a second-order homogeneous equation is shown in Figure 40. The solution begins with a guess of the homogeneous solution, but this time requires the solution of the quadratic equation. There are three possible cases that result from the solution of the quadratic equation: different but real roots; two identical roots; or two complex roots. The three cases result in three different forms of solutions, as shown. The complex result is the most notable because it results in sinusoidal oscillations. It is not shown, but after the homogeneous solution has been found, the initial conditions need to be used to find the remaining coefficient values.

Given,

$$Ax'' + Bx' + Cx = 0 \qquad x(0) = x_0 \qquad \text{and} \qquad x'(0) = v_0$$

Guess a general equation form and substitute it into the differential equation,

$$x_h = Xe^{-Yt} \qquad x'_h = -YXe^{-Yt} \qquad x''_h = Y^2Xe^{-Yt}$$

$$A(Y^2Xe^{-Yt}) + B(-YXe^{-Yt}) + C(Xe^{-Yt}) = 0$$

$$A(Y^2) + B(-Y) + C = 0$$

$$Y = \frac{-(-B) \pm \sqrt{(-B)^2 - 4(AC)}}{2A} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

Note: There are three possible outcomes of finding the roots of the equations: two different real roots, two identical real roots, or two complex roots. Therefore there are three fundamentally different results.

If the values for Y are both real, but different, the general form is,

$$Y = R_1, R_2 \qquad x_h = X_1e^{R_1t} + X_2e^{R_2t}$$

Note: The initial conditions are then used to find the values for X_1 and X_2.

If the values for Y are both real, and identical, the general form is,

$$Y = R_1, R_1 \qquad x_h = X_1e^{R_1t} + X_2te^{R_1t}$$

The initial conditions are then used to find the values for X_1 and X_2.

If the values for Y are complex, the general form is,

$$Y = \sigma \pm \omega j \qquad x_h = X_3e^{\sigma t}\cos(\omega t + X_4)$$

The initial conditions are then used to find the values of X_3 and X_4.

Figure 40 Solution of a second-order homogeneous equation

As mentioned above, a complex solution when solving the homogeneous equation results in a sinusoidal oscillation, as proven in Figure 41. The most notable part of the solution is that there is both a frequency of oscillation and a phase shift. This form is very useful for analyzing the frequency response of a system, as will be seen in a later chapter.

Consider the situation where the results of a homogeneous solution are the complex conjugate pair..

$$Y = R \pm Cj$$

This gives the general result, as shown below:

$$x = X_1 e^{(R + Cj)t} + X_2 e^{(R - Cj)t}$$

$$x = X_1 e^{Rt} e^{Cjt} + X_2 e^{Rt} e^{-Cjt}$$

$$x = e^{Rt}(X_1 e^{Cjt} + X_2 e^{-Cjt})$$

$$x = e^{Rt}(X_1(\cos(Ct) + j\sin(Ct)) + X_2(\cos(-Ct) + j\sin(-Ct)))$$

$$x = e^{Rt}(X_1(\cos(Ct) + j\sin(Ct)) + X_2(\cos((Ct) - j\sin(Ct))))$$

$$x = e^{Rt}((X_1 + X_2)\cos(Ct) + j(X_1 - X_2)\sin(Ct))$$

$$x = e^{Rt}((X_1 + X_2)\cos(Ct) + j(X_1 - X_2)\sin(Ct))$$

$$x = e^{Rt}\frac{\sqrt{(X_1 + X_2)^2 + j^2(X_1 - X_2)^2}}{\sqrt{(X_1 + X_2)^2 + j^2(X_1 - X_2)^2}}((X_1 + X_2)\cos(Ct) + j(X_1 - X_2)\sin(Ct))$$

$$x = e^{Rt}\frac{\sqrt{X_1^2 + 2X_1X_2 + X_2^2 - (X_1^2 + -2X_1X_2 + X_2^2)}}{\sqrt{X_1^2 + 2X_1X_2 + X_2^2 - (X_1^2 + -2X_1X_2 + X_2^2)}}((X_1 + X_2)\cos(Ct) + j(X_1 - X_2)\sin(Ct)$$

$$x = e^{Rt}\frac{\sqrt{4X_1X_2}}{\sqrt{4X_1X_2}}((X_1 + X_2)\cos(Ct) + j(X_1 - X_2)\sin(Ct))$$

$$x = e^{Rt}\sqrt{4X_1X_2}\left(\frac{(X_1 + X_2)}{\sqrt{4X_1X_2}}\cos(Ct) + j\frac{(X_1 - X_2)}{\sqrt{4X_1X_2}}\sin(Ct)\right)$$

$$x = e^{Rt}\sqrt{4X_1X_2}\cos\left(Ct + \operatorname{atan}\left(\frac{(X_1 - X_2)}{(X_1 + X_2)}\right)\right)$$

$$x = e^{Rt}X_3\cos(Ct + X_4)$$

frequency phase shift

where, $X_3 = \sqrt{4X_1X_2}$

$$X_4 = \operatorname{atan}\left(\frac{(X_1 - X_2)}{(X_1 + X_2)}\right)$$

Figure 41 The phase shift solution to a second-order homogeneous differential equation

The methods for solving non-homogeneous differential equations builds upon the methods used for the solution of homogeneous equations. This process adds a step to find the particular solution of the equation. An example of the solution of a first-order non-homogeneous equation is shown in Figure 42. To find the homogeneous solution the non-homogeneous part of the equation is set to zero. To find the particular solution the final form must be guessed. This is then substituted into the equation, and the values of the coefficients are found. Finally the homogeneous and particular are added to get the final equation form. The overall response of the system can be obtained by adding the homogeneous and particular parts because the equations are linear, and the principle of superposition applies. The homogeneous equation deals with the response to initial conditions, and the particular solution deals with the response to forced inputs.

Generally,

$$Ax' + Bx = Cf(t) \qquad\qquad x(0) = x_0$$

First, find the homogeneous solution as before in Figure 39.

$$x_h = x_0 e^{-\frac{B}{A}t}$$

Next, guess the particular solution by looking at the form of ’f(t)’. This step is highly subjective, and if an incorrect guess is made, it will be unsolvable. When this happens, just make another guess and repeat the process. An example is given below. In the case below the guess should be similar to the exponential forcing function.

For example, if we are given

$$6x' + 2x = 5e^{4t}$$

A reasonable guess for the particular solution is,

$$x_p = C_1 e^{4t} \qquad \therefore x'_p = 4C_1 e^{4t}$$

Substitute these into the differential equation and solve for A.

$$6(4C_1 e^{4t}) + 2(C_1 e^{4t}) = 5e^{4t}$$

$$24C_1 + 2C_1 = 5 \qquad\qquad \therefore C_1 = \frac{5}{26}$$

Combine the particular and homogeneous solutions.

$$x = x_p + x_h = \frac{5}{26}e^{4t} + x_0 e^{-\frac{6}{2}t}$$

Figure 42 Solution of a first-order non-homogeneous equation

The method for finding a particular solution for a second-order non-homogeneous differential equation is shown in Figure 43. In this example the forcing function is sinusoidal, so the particular result should also be sinusoidal. The final result is converted into a phase shift form.

Generally,

$$Ax'' + Bx' + Cx = Df(t) \qquad x(0) = x_0 \quad \text{and} \quad x'(0) = v_0$$

1. Find the homogeneous solution as before.

$$x_h = e^{\sigma t} X_3 \cos(\omega t + X_4)$$

2. Guess the particular solution by looking at the form of ’f(t)’. This step is highly subjective, and if an incorrect guess is made it will be unsolvable. When this happens, just make another guess and repeat the process. For the purpose of illustration an example is given below. In the case below it should be similar to the sine function.

For example, if we are given

$$2x'' + 6x' + 2x = 2\sin(3t + 4)$$

A reasonable guess is,

$$x_p = A\sin(3t) + B\cos(3t)$$
$$x'_p = A\cos(3t) - B\sin(3t)$$
$$x''_p = -A\sin(3t) - B\cos(3t)$$

Substitute these into the differential equation ans solve for A and B.

$$2(A\sin(3t) + B\cos(3t)) + 6(A\cos(3t) - B\sin(3t)) + 2(-A\sin(3t) - B\cos(3t)) = 2\sin(3t + 4$$

$$(2A - 6B - 2A)\sin(3t) + (2B + 6A - 2B)\cos(3t) = 2\sin(3t + 4)$$

$$(-6B)\sin(3t) + (6A)\cos(3t) = 2(\sin 3t \cos 4 + \cos 3t \sin 4)$$

$$-6B = 2\cos 4 \qquad B = 0.2179$$
$$6A = 2\sin 4 \qquad A = -0.2523$$

Next, rearrange the equation to phase shift form.

$$x_p = -0.2523\sin(3t) + 0.2179\cos(3t)$$
$$x_p = 0.3333(-0.7568\sin(3t) + 0.6536\cos(3t))$$
$$x_p = 0.3333(\sin(-0.8584)\sin(3t) + \cos(-0.8584)\cos(3t))$$
$$x_p = 0.3333\cos(3t + 0.8584)$$

3. Use the initial conditions to determine the coefficients in the homogeneous solution.

Figure 43 Solution of a second-order non-homogeneous equation

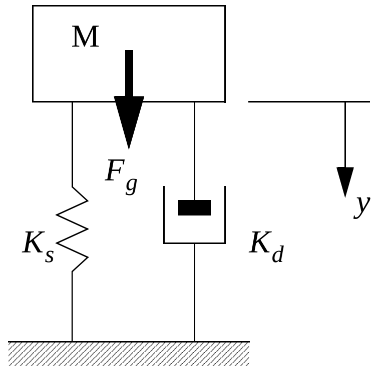
When guessing particular solutions, the forms in Figure 44 can be helpful.

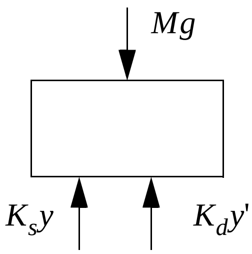
forcing function	Guess
A	C
$Ax + B$	$Cx + D$
e^{Ax}	Ce^{Ax} or Cxe^{Ax}
$B\sin(Ax)$ or $B\cos(Ax)$	$C\sin(Ax) + D\cos(Ax)$ or $Cx\sin(Ax) + xD\cos(Ax)$

Figure 44 General forms for particular solutions

An example of a second-order system is shown in Figure 45. As typical it begins with a FBD and summation of forces. This is followed with the general solution of the homogeneous equation. Real roots are assumed to allow the problem solution to continue in Figure 46.

Assume the system illustrated to the right starts from rest at a height h. At time t=0 the system is released and allowed to move.





$$+\uparrow \sum F_y = -Mg + K_s y + K_d y' = -My''$$

$$My'' + K_d y' + K_s y = Mg$$

Find the homogeneous solution.

$$y_h = e^{At} \qquad y'_h = Ae^{At} \qquad y''_h = A^2 e^{At}$$

$$My'' + K_d y' + K_s y = 0$$

$$M(A^2 e^{At}) + K_d(Ae^{At}) + K_s(e^{At}) = 0$$

$$MA^2 + K_d A + K_s = 0$$

$$A = \frac{-K_d \pm \sqrt{K_d^2 - 4MK_s}}{2M}$$

Let us assume that the values of M, K_d and K_s lead to the case of two different positive roots. This would occur if the damper value was much larger than the spring and mass values. Thus,

$$A = R_1, R_2$$

$$y_h = C_1 e^{R_1 t} + C_2 e^{R_2 t}$$

Figure 45 Second-order system example

The solution continues by finding the particular solution and then solving it using initial conditions in Figure 46. The final result is a second-order system that is overdamped, with no oscillation.

Next, find the particular solution.

$$y_p = C \qquad y'_h = 0 \qquad y''_h = 0$$

$$M(0) + K_d(0) + K_s(C) = Mg$$

$$C = \frac{Mg}{K_s}$$

Now, add the homogeneous and particular solutions and solve for the unknowns using the initial conditions.

$$y(t) = y_p + y_h = \frac{Mg}{K_s} + C_1 e^{R_1 t} + C_2 e^{R_2 t}$$

$$y(0) = h \qquad y'(0) = 0$$

$$h = \frac{Mg}{K_s} + C_1 e^0 + C_2 e^0$$

$$C_1 + C_2 = h - \frac{Mg}{K_s}$$

$$y'(t) = R_1 C_1 e^{R_1 t} + R_2 C_2 e^{R_2 t}$$

$$0 = R_1 C_1 e^0 + R_2 C_2 e^0$$

$$0 = R_1 C_1 + R_2 C_2 \qquad C_1 = \frac{-R_2}{R_1} C_2$$

$$-\frac{R_2}{R_1} C_2 + C_2 = h - \frac{Mg}{K_s}$$

$$C_2 = \left(\frac{K_s h - Mg}{K_s}\right)\left(\frac{-R_1}{R_1 - R_2}\right) \qquad C_1 = \frac{-R_2}{R_1}\left(\frac{K_s h - Mg}{K_s}\right)\left(\frac{-R_1}{R_1 - R_2}\right)$$

Now, combine the solutions and solve for the unknowns using the initial conditions.

$$y(t) = \frac{Mg}{K_s} + \left(\frac{K_s h - Mg}{K_s}\right)\left(\frac{-R_1}{R_1 - R_2}\right) e^{R_1 t} + \frac{-R_2}{R_1}\left(\frac{K_s h - Mg}{K_s}\right)\left(\frac{-R_1}{R_1 - R_2}\right) e^{R_2 t}$$

$$y(t) = \frac{Mg}{K_s} + \left(\frac{K_s h - Mg}{K_s}\right)\left(\frac{-R_1}{R_1 - R_2}\right) e^{R_1 t} + \left(\frac{K_s h - Mg}{K_s}\right)\left(\frac{R_2}{R_1 - R_2}\right) e^{R_2 t}$$

Figure 46 Second-order system example (continued)

3.3 RESPONSES

Solving differential equations tends to yield one of two basic equation forms. The e-to-the-negative-t forms are the first-order responses and slowly decay over time. They never naturally oscillate, they only oscillate if forced to do so. The second-order forms may include natural oscillation. In general the analysis of input responses focus on the homogeneous part of the solution.

3.3.1 First-order

A first-order system will result in a first-order differential equation. The solution for these systems is a natural decay or growth as shown in Figure 47. The time constant for the system can be found directly from the differential equation. It is a measure of how quickly the system responds to a change. When an input to a system has changed, the system output will be approximately 63% of the way to its final value when the elapsed time equals the time constant. The initial and final values of the function can be determined algebraically to find the first-order response with little effort.

If we have experimental results for a system, we can find the time constant, initial and final values graphically. The time constant can be found two ways, one by extending the slope of the first part of the curve until it intersects the final value line. That time is the time constant value. The other method is to look for the time when the output value has shifted 63.2% of the way from the initial to final values for the system. Assuming the change started at t=0, this time at this point corresponds to the time constant.

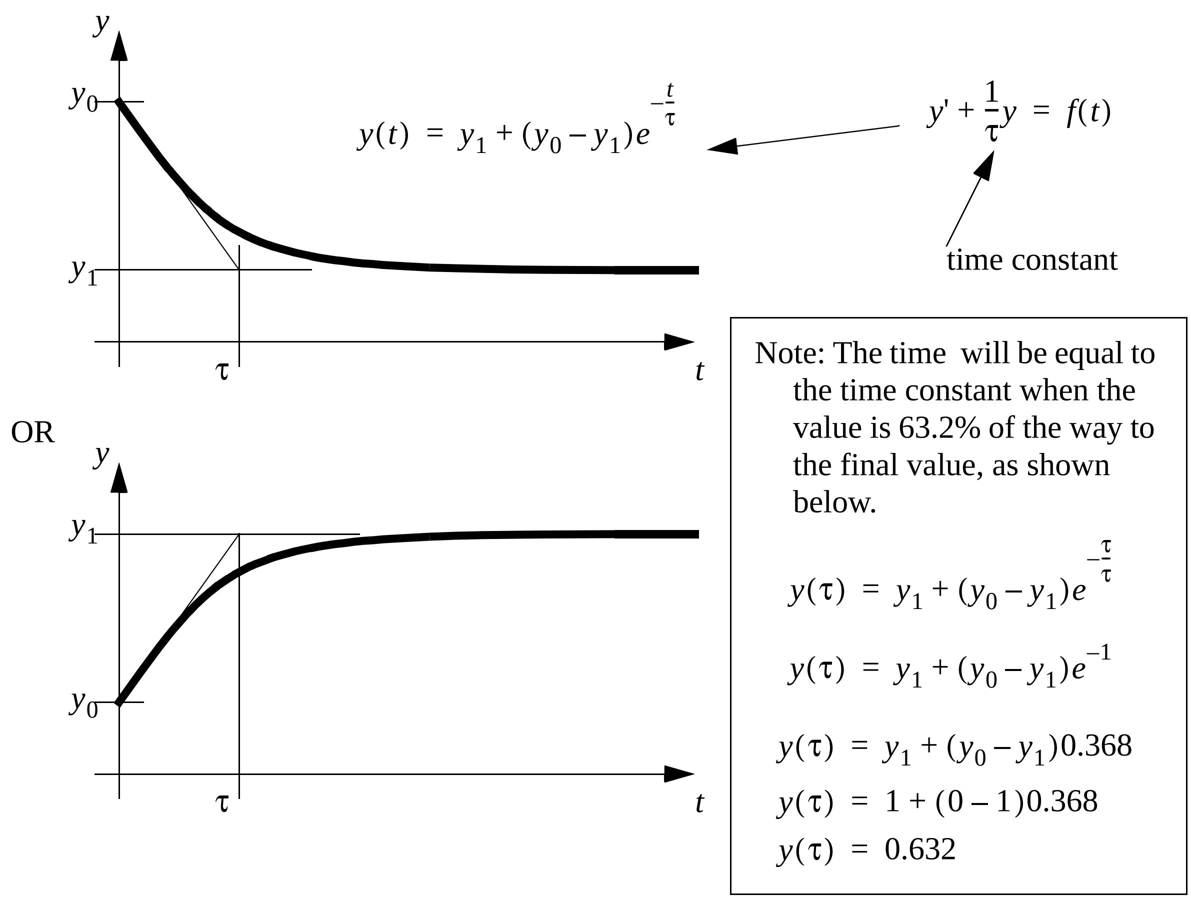
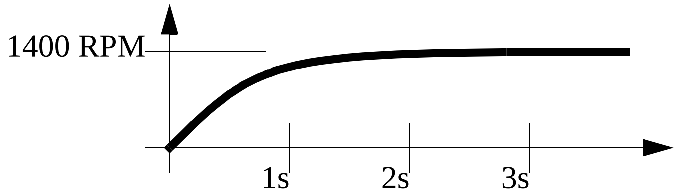


Figure 47 Typical first-order responses

The example in Figure 48 determines the coefficients for a first-order differential equation given a graphical output response to an input. The differential equation is for a permanent magnet DC motor, and will be developed in a later chapter. If we consider the steady state when the speed is steady at 1400RPM, the first derivative will be zero. This simplifies the equation and allows us to find a value for the parameter K in the differential equation. The time constant can be found by drawing a line asymptotic to the start of the motor curve, and finding the point where it intercepts the steady-state value. This gives an approximate time constant of 0.8 s. This can then be used to calculate the remaining coefficient. Some additional numerical calculation leads to the final differential equation as shown.

Figure 48 Finding an equation using experimental data

For the motor use the differential equation and the speed curve when Vs=10V is applied:



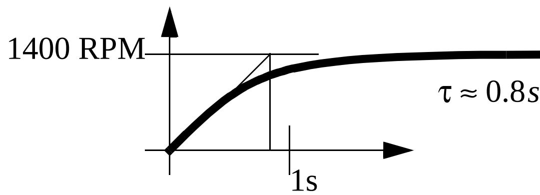
$$\left(\frac{d}{dt}\right)\omega + \left(\frac{K^2}{JR}\right)\omega = \left(\frac{K}{JR}\right)V_s$$

For steady-state

$$\left(\frac{d}{dt}\right)\omega = 0 \qquad \omega = 1400RPM = 146.6rads^{-1}$$

$$0 + \left(\frac{K^2}{JR}\right)146.6 = \left(\frac{K}{JR}\right)10$$

$$K = 0.0682$$



$$\left(\frac{K^2}{JR}\right) = \frac{1}{0.8s}$$

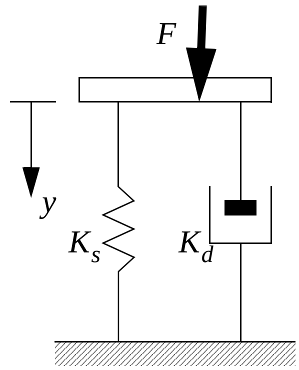
$$0.0682\left(\frac{K}{JR}\right) = \frac{1}{0.8s}$$

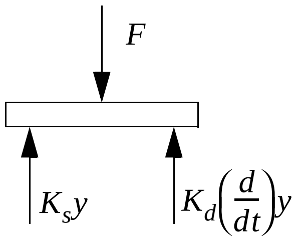
$$\frac{K}{JR} = 18.328$$

$$D\omega + \frac{1}{0.8}\omega = 18.328V_s$$

A simple mechanical example is given in Figure 49. As typical the modeling starts with a FBD and a sum of forces. After this, the homogenous solution is found by setting the non-homogeneous part to zero and solving. Next, the particular solution is found, and the two solutions are combined. The initial conditions are used to find the remaining unknown coefficients.

Find the response to the applied force if the force is applied at t=0s. Assume the system is initially deflected a height of *h*.





$$+\uparrow \sum F_y = -F + K_s y + K_d\left(\frac{d}{dt}\right)y = 0$$

$$K_d y' + K_s y = F$$

Find the homogeneous solution.

$$y_h = Ae^{Bt} \qquad y'_h = ABe^{Bt}$$

$$K_d(ABe^{Bt}) + K_s(Ae^{Bt}) = 0$$

$$K_d B + K_s = 0$$

$$B = \frac{-K_s}{K_d}$$

Next, find the particular solution.

$$y_p = C \qquad y'_p = 0$$

$$K_d(0) + K_s(C) = F \qquad \therefore C = \frac{F}{K_s}$$

Combine the solutions, and find the remaining unknown.

$$y(t) = y_p + y_h = Ae^{\frac{-K_s}{K_d}t} + \frac{F}{K_s}$$

$$y(0) = h$$

$$h = Ae^0 + \frac{F}{K_s} \qquad \therefore A = h - \frac{F}{K_s}$$

The final solution is,

$$y(t) = \left(h - \frac{F}{K_s}\right)e^{\frac{-K_s}{K_d}t} + \frac{F}{K_s}$$

Figure 49 First-order system analysis example

Use the general form given below to solve the problem in Figure 49 without solving the differential equation.

$$y' + \frac{1}{\tau}y = f(t) \longrightarrow y(t) = y_1 + (y_0 - y_1)e^{-\frac{t}{\tau}}$$

Figure 50 Drill problem: Developing the final equation using the first-order model form

A first-order system tends to be passive, meaning it doesn't deliver energy or power. A first-order system will not oscillate unless the input forcing function is also oscillating. Its output response lags its input and the delay is determined by the system's time constant.

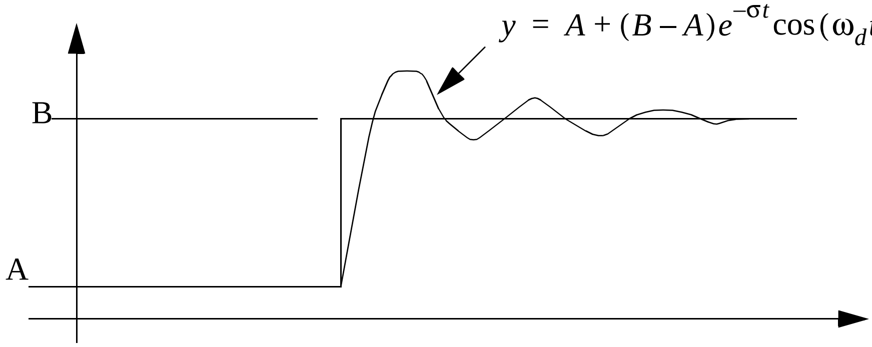
3.3.2 Second-order

A second-order system response typically contains two first-order responses, or a first-order response and a sinusoidal component. A typical sinusoidal second-order response is pictured in Figure 51. Notice that the coefficients of the differential equation include a damping coefficient and a natural frequency. These can be used to develop the final response, given the initial conditions and forcing function. Notice that the damped frequency of oscillation is the actual frequency of oscillation. The damped frequency will be lower than the natural frequency when the damping coefficient is between 0 and 1. If the damping coefficient is greater than one the damped frequency becomes negative, and the system will not oscillate - it is overdamped.

A second-order system, and a typical response to a stepped input.

$$y'' + 2\zeta\omega_n y' + \omega_n^2 y = f(t) \longrightarrow \sigma = \zeta\omega_n \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$y = A + (B-A)e^{-\sigma t}\cos(\omega_d t)$$



ω_n Natural frequency of system, approximate frequency of control system oscillations

ξ Damping factor of system. If < 1 underdamped, and system will oscillate. If =1 critically damped. If > 1 overdamped, and never any oscillation (more like a first-order system). As damping factor approaches 0, the first peak becomes infinite in height.

ω_d The actual frequency of the system. This is the resulting decrease in the natural frequency caused the damping

Figure 51 The general form for a second-order system

When only the damping coefficient is increased, the frequency of oscillation, and overall response time will slow, as seen in Figure 52. When the damping coefficient is 0 the system will oscillate indefinitely. Critical damping occurs when the damping coefficient is 1. At this point both roots of the differential equation are equal. The system will not oscillate is the damping coefficient is greater than or equal to 1.

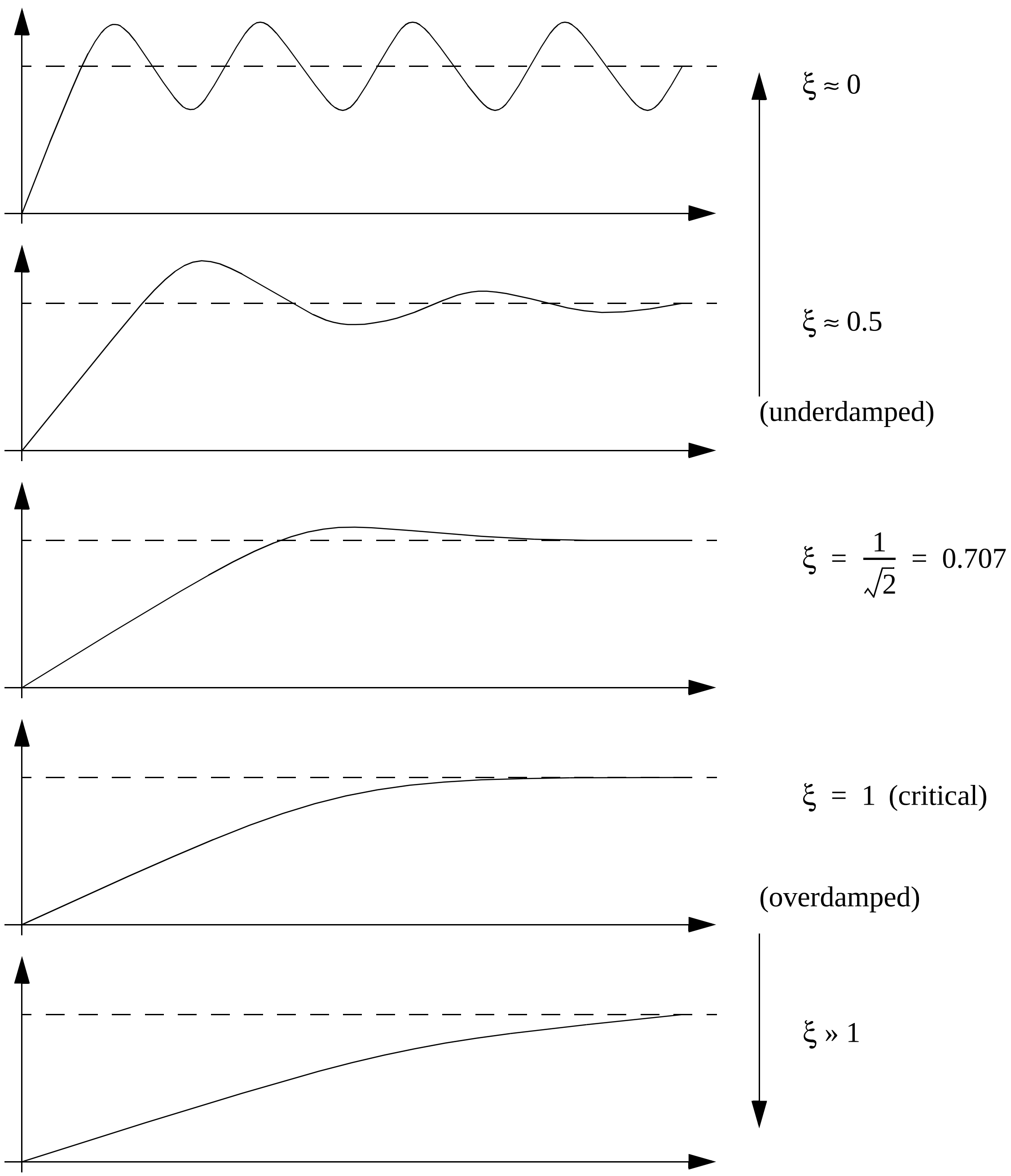
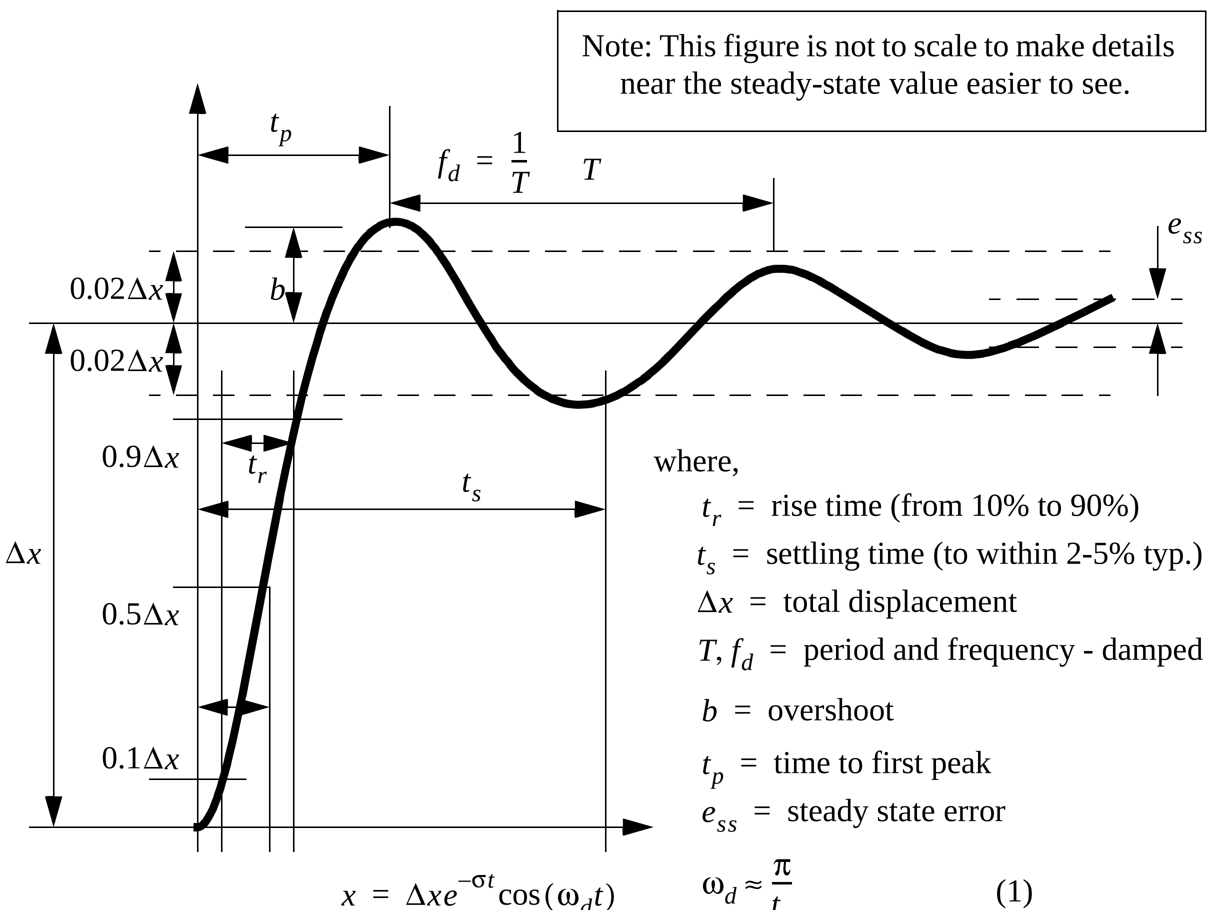


Figure 52 The effect of the damping coefficient

When observing second-order systems it is more common to use more direct measurements of the response. Some of these measures are shown in Figure 53. The rise time is the time it takes to go from 10% to 90% of the total displacement, and can be a good measure of general system responsiveness. The settling time indicates how long it takes for the system to pass within a tolerance band around the final value. Here the permissible zone is 2%, but if it were slightly larger the system would have a much smaller settling time. The period of oscillation can be measured directly as the time between peaks of the oscillation, the inverse is the damping frequency. (Note: don't forget to convert to radians.) The damped frequency can also be found using the time to the first peak, as half the period. The overshoot is the height of the first peak. Using the time to the first peak, and the overshoot the damping coefficient can be found.



where,

t_r = rise time (from 10% to 90%)

t_s = settling time (to within 2-5% typ.)

Δx = total displacement

T, f_d = period and frequency - damped

b = overshoot

t_p = time to first peak

e_{ss} = steady state error

$$\omega_d \approx \frac{\pi}{t_p} \qquad (1)$$

$$\sigma = \xi \omega_n \qquad (2)$$

$$\frac{b}{\Delta x} = e^{-\sigma t_p} \qquad (3)$$

$$\xi = \frac{1}{\sqrt{\left(\frac{\pi}{t_p \sigma}\right)^2 + 1}} \qquad (4)$$

Figure 53 Characterizing a second-order response (not to scale)

Note: We can calculate these relationships using the complex homogenous form, and the generic second order equation form.

$$A^2 + 2\xi\omega_n + \omega_n^2 = 0$$

$$A = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2} = \sigma \pm j\omega_d$$

$$\frac{-2\xi\omega_n}{2} = \sigma = -\xi\omega_n \qquad \omega_n = \frac{\sigma}{-\xi} \qquad (1)$$

$$\frac{\sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2} = j\omega_d$$

$$4\xi^2\omega_n^2 - 4\omega_n^2 = 4(-1)\omega_d^2$$

$$\omega_n^2 - \xi^2\omega_n^2 = \omega_d^2 \qquad \omega_n\sqrt{1-\xi^2} = \omega_d \qquad (2)$$

$$\frac{\sigma^2}{\xi^2} - \xi^2\frac{\sigma^2}{\xi^2} = \omega_d^2$$

$$\frac{1}{\xi^2} = \frac{\omega_d^2}{\sigma^2} + 1 \qquad \xi = \frac{1}{\sqrt{\frac{\omega_d^2}{\sigma^2} + 1}} \qquad (3)$$

The time to the first peak can be used to find the approximate decay constant

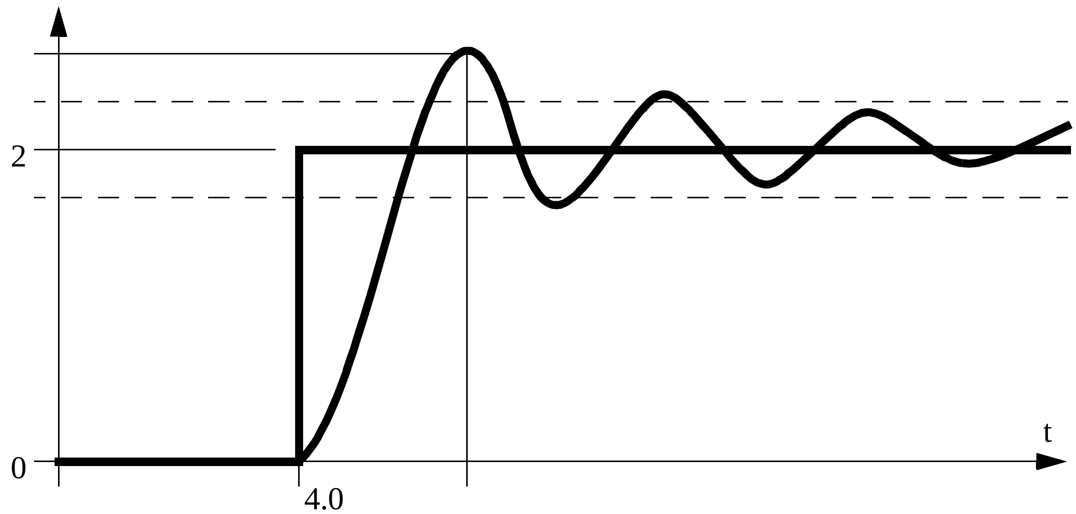
$$x(t) = C_1 e^{-\sigma t}\cos(\omega_d t + C_2)$$

$$\omega_d = \frac{\pi}{t_p} \qquad (4)$$

$$b \approx \Delta x e^{-\sigma t_p}(1)$$

$$\sigma = -\frac{\ln\left(\frac{b}{\Delta x}\right)}{t_p} \qquad (5)$$

Figure 54 Second order relationships between damped and natural frequency



Write a function of time for the graph given. (Note: measure using a ruler to get values.) Also find the natural frequency and damping coefficient to develop the differential equation. Using the dashed lines determine the settling time.

Figure 55 Drill problem: Find the equation given the response curve

3.3.3 Other Responses

First-order systems have e-to-the-t type responses. Second-order systems add another e-to-the-t response or a sinusoidal excitation. As we move to higher order linear systems we typically add more e-to-the-t terms, and more sinusoidal terms. A possible higher order system response is seen in Figure 56. The underlying function is a first-order response that drops at the beginning, but levels out. There are two sinusoidal functions superimposed, one with about one period showing, the other with a much higher frequency.

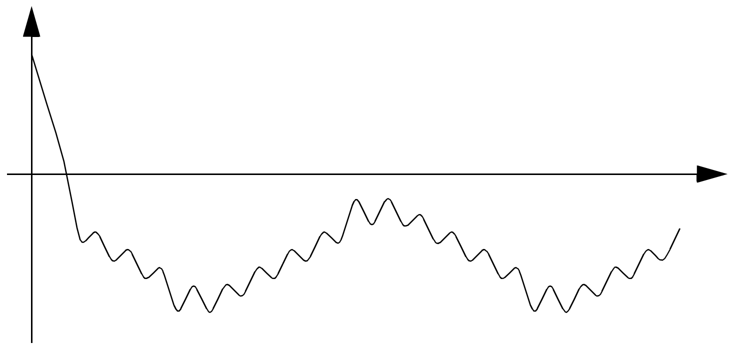


Figure 56 An example of a higher order system response

The basic techniques used for solving first and second-order differential equations can be applied to higher order differential equations, although the solutions will start to become complicated for systems with much higher orders.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

Given the homogeneous differential equation

$$\left(\frac{d}{dt}\right)^4 x + 13\left(\frac{d}{dt}\right)^3 x + 34\left(\frac{d}{dt}\right)^2 x + 42\left(\frac{d}{dt}\right) x + 20x = 5$$

Guess a solution for the homogeneous equation,

$$x_h = e^{At}$$

$$\frac{d}{dt}x_h = Ae^{At} \qquad \left(\frac{d}{dt}\right)^2 x_h = A^2 e^{At} \qquad \left(\frac{d}{dt}\right)^3 x_h = A^3 e^{At} \qquad \left(\frac{d}{dt}\right)^4 x_h = A^4 e^{At}$$

Substitute the values into the differential equation and find a value for the unknown.

$$A^4 e^{At} + 13A^3 e^{At} + 34A^2 e^{At} + 42Ae^{At} + 20e^{At} = 0$$

$$A^4 + 13A^3 + 34A^2 + 42A + 20 = 0$$

$$A = -1, -10, -1-j, -1+j$$

$$x_h = C_1 e^{-t} + C_2 e^{-10t} + C_3 e^{-t}\cos(t + C_4)$$

Guess a particular solution, and the solve for the coefficient.

$$x_p = A \qquad \frac{d}{dt}x_p = 0 \qquad \left(\frac{d}{dt}\right)^2 x_p = 0 \qquad \left(\frac{d}{dt}\right)^3 x_p = 0 \qquad \left(\frac{d}{dt}\right)^4 x_p = 0$$

$$0 + 13(0) + 34(0) + 42(0) + 20A = 5 \qquad A = 0.25$$

Figure 57 Solution of a higher order differential equation

Solve for the unknowns, assuming the system starts at rest and undeflected.

$$x(t) = C_1 e^{-t} + C_2 e^{-10t} + C_3 e^{-t}\cos(t + C_4) + 0.25$$

$$0 = C_1 + C_2 + C_3\cos(C_4) + 0.25 \quad (1)$$

$$C_3\cos(C_4) = -C_1 - C_2 - 0.25 \quad (2)$$

$$\frac{d}{dt}x_h = -C_1 e^{-t} - 10C_2 e^{-10t} - C_3 e^{-t}\cos(t + C_4) - C_3 e^{-t}\sin(t + C_4)$$

$$0 = -C_1 - 10C_2 - C_3\cos(C_4) - C_3\sin(C_4) \quad (3)$$

Equations (1) and (3) can be added to get the simplified equation below.

$$0 = -9C_2 - C_3\sin(C_4) + 0.25$$

$$C_3\sin(C_4) = -9C_2 + 0.25 \quad (4)$$

$$\left(\frac{d}{dt}\right)^2 x_h = C_1 e^{-t} + 100C_2 e^{-10t} + C_3 e^{-t}\cos(t + C_4) + C_3 e^{-t}\sin(t + C_4) + C_3 e^{-t}\sin(t + C_4) - C_3 e^{-t}\cos(t + C_4)$$

$$0 = C_1 + 100C_2 + C_3\cos(C_4) + C_3\sin(C_4) + C_3\sin(C_4) - C_3\cos(C_4)$$

$$0 = C_1 + 100C_2 + 2C_3\sin(C_4) \quad (5)$$

Equations (4) and (5) can be combined.

$$0 = C_1 + 100C_2 + 2(-9C_2 + 0.25)$$

$$0 = -17C_1 + 100C_2 + 0.5 \quad (6)$$

$$\left(\frac{d}{dt}\right)^3 x_h = -C_1 e^{-t} + (-1000)C_2 e^{-10t} - 2C_3 e^{-t}\sin(t + C_4) + 2C_3 e^{-t}\cos(t + C_4)$$

$$0 = -C_1 + (-1000)C_2 - 2C_3\sin(C_4) + 2C_3\cos(C_4) \quad (7)$$

Figure 58 Solution of a higher order differential equation

Equations (2 and (4) are substituted into equation (7).

$$0 = -C_1 + (-1000)C_2 - 2(-9C_2 + 0.25) + 2(-C_1 - C_2 - 0.25)$$
$$0 = -3C_1 + (-984)C_2 - 1$$
$$C_1 = \left(-\frac{984}{3}\right)C_2 - \frac{1}{3} \qquad (8)$$

Equations (6) and (8) can be combined.

$$0 = -17\left(\left(-\frac{984}{3}\right)C_2 - \frac{1}{3}\right) + 100C_2 + 0.5$$

$$0 = 5676C_2 + 6.1666667 \qquad C_2 = -0.00109$$

$$C_1 = \left(-\frac{984}{3}\right)(-0.00109) - \frac{1}{3} \qquad C_1 = 0.0242$$

Equations (2) and (4) can be combined.

$$\frac{C_3\sin(C_4)}{C_3\cos(C_4)} = \frac{-9C_2 + 0.25}{-C_1 - C_2 - 0.25}$$

$$\tan(C_4) = \frac{-9(-0.00109) + 0.25}{-(0.0242) - (-0.00109) - 0.25} \qquad C_4 = -0.760$$

Equation (4) can be used the find the remaining unknown.

$$C_3\sin(-0.760) = -9(-0.00109) + 0.25 \qquad C_3 = -0.377$$

The final response function is,

$$x(t) = 0.0242e^{-t} + (-0.00109)e^{-10t} + (-0.377)e^{-t}\cos(t - 0.760) + 0.25$$

Figure 59 Solution of a higher order differential equation (cont’d)

In some cases we will have systems with multiple differential equations, or non-linear terms. In these cases explicit analysis of the equations may not be feasible. In these cases we may turn to other techniques, such as numerical integration which will be covered in later chapters.

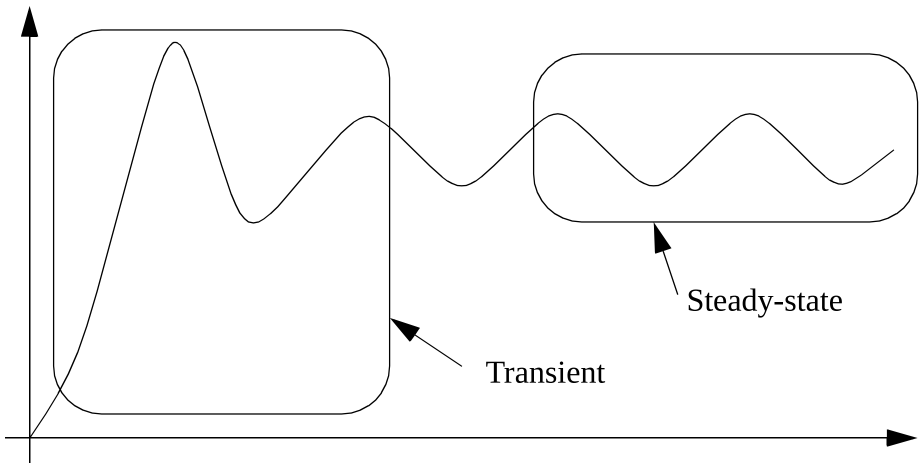
3.4 RESPONSE ANALYSIS

Up to this point we have mostly discussed the process of calculating the system response. As an engineer, obtaining the response is important, but evaluating the results is more important. The most critical design consideration is system stability. In most cases a system should be inherently stable in all situations, such as a car cruise control. In other cases an unstable system may be the objective, such as an explosive munition. Simple methods for determining the stability of a system are listed below:

1. If a step input causes the system to go to infinity, it will be inherently unstable.
2. A ramp input might cause the system to go to infinity; if this is the case, the system might not respond well to constant change.
3. If the response to a sinusoidal input grows with each cycle, the system is probably resonating, and will become unstable.

Beyond establishing the stability of a system, we must also consider general performance. This includes the time constant for a first-order system, or damping coefficient and natural frequency for a second-order system. For example, assume we have designed an elevator that is a second-order system. If it is under damped the elevator will oscillate, possibly leading to motion sickness, or worse. If the elevator is over damped it will take longer to get to floors. If it is critically damped it will reach the floors quickly, without overshoot.

Engineers distinguish between initial setting effects (transient) and long term effects (steady-state). The transient effects are closely related to the homogeneous solution to the differential equations and the initial conditions. The steady-state effects occur after some period of time when the system is acting in a repeatable or non-changing form. Figure 60 shows a system response. The transient effects at the beginning include a quick rise time and an overshoot. The steady-state response settles down to a constant amplitude sine wave.



Note: the transient response is predicted with the homogeneous solution. The steady state response in mainly predicted with the particular solution, although in some cases the homogeneous solution might have steady state effects, such as a non-decaying oscillation.

Figure 60 A system response with transient and steady-state effects

3.5 NON-LINEAR SYSTEMS

Non-linear systems cannot be described with a linear differential equation. A basic linear differential equation has coefficients that are XXXXXXXXXXXXXX

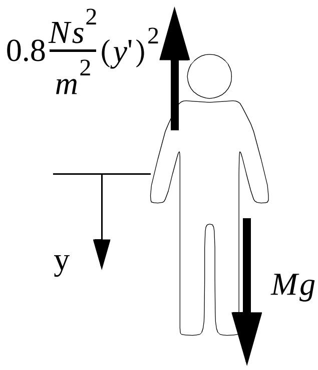
Examples of system conditions that lead to non-linear solutions are,

- XXXXXXXXXXXXXXXXXXXXX

3.5.1 Non-Linear Differential Equations

A non-linear differential equation is presented in Figure 61. It involves a person ejected from an aircraft with a drag coefficient of 0.8. The FBD shows the sum of forces, and the resulting differential equation. The velocity squared term makes the equation non-linear, and so it cannot be analyzed with the previous methods. In this case the terminal velocity is calculated by setting the acceleration to zero. This results in a maximum speed of 126 kph. The equation can also be solved using explicit integration XXXXXXXXXXXXXX

Consider the differential equation for a 100kg human ejected from an airplane. The aerodynamic drag will introduce a squared variable, therefore making the equation non-linear.



$$\sum F_y = 0.8(y')^2 - Mg = -My''$$

$$100kgy'' + 0.8\frac{Ns^2}{m^2}(y')^2 = 100kg9.81\frac{N}{kg}$$

$$100kgy'' + 0.8\frac{Ns^2}{m^2}(y')^2 = 981N$$

$$100kgy'' + 0.8kg\frac{m}{s^2}\frac{s^2}{m^2}(y')^2 = 981kg\frac{m}{s^2}$$

$$100y'' + 0.8m^{-1}(y')^2 = 981ms^{-2}$$

The terminal velocity can be found be setting the acceleration to zero.

$$100(0) + 0.8m^{-1}(y')^2 = 981ms^{-2}$$

$$y' = \sqrt{\frac{981ms^{-2}}{0.8m^{-1}}} = \sqrt{\frac{981}{0.8}m^2s^{-2}} = 35.0\frac{m}{s} = 126\frac{km}{h}$$

Figure 61 Development of a non-linear differential equation

An explicit solution can begin by replacing the position variable with a velocity variable and rewriting the equation as a separable differential equation.

$$100y'' + 0.8m^{-1}(y')^2 = 981ms^{-2}$$

$$100v' + 0.8m^{-1}v^2 = 981ms^{-2}$$

$$100\frac{dv}{dt} + 0.8m^{-1}v^2 = 981ms^{-2}$$

$$100\frac{dv}{dt} = 981ms^{-2} - 0.8m^{-1}v^2$$

$$\frac{100}{981ms^{-2} - 0.8m^{-1}v^2}dv = dt$$

$$\int\frac{\dfrac{100}{-0.8m^{-1}}}{\dfrac{981}{-0.8m^{-1}}ms^{-2} + v^2}dv = \int dt$$

$$\int\frac{-125m}{v^2 - 1226.25m^2s^{-2}}dv = t + C_1$$

$$\int\frac{-125m}{\left(v + 35.02\frac{m}{s}\right)\left(v - 35.02\frac{m}{s}\right)}dv = t + C_1$$

Figure 62 Development of an integral

This can be reduced with a partial fraction expansion.

$$\int\left[\frac{A}{\left(v + 35.02\frac{m}{s}\right)} + \frac{B}{\left(v - 35.02\frac{m}{s}\right)}\right]dv = t + C_1$$

$$Av - A\left(35.02\frac{m}{s}\right) + Bv + B\left(35.02\frac{m}{s}\right) = -125m$$

$$v(A + B) + 35.02\frac{m}{s}(-A + B) = -125m$$

$$A + B = 0 \qquad\qquad A = -B$$

$$35.02\frac{m}{s}(-A + B) = -125m$$

$$(-(-B) + B) = -\frac{125}{35.02}s \qquad\qquad B = -1.785s$$

$$A = 1.785s$$

$$\int\left[\frac{1.785s}{\left(v + 35.02\frac{m}{s}\right)} + \frac{-1.785s}{\left(v - 35.02\frac{m}{s}\right)}\right]dv = t + C_1$$

The integral can then be solved using an identity from the integral table. In this case the integration constants can be left off because they are redundant with the one on the right hand side.

$$1.785s\ln\left|v + 35.02\frac{m}{s}\right| - 1.785s\ln\left|v - 35.02\frac{m}{s}\right| = t + C_1$$

$$1.785s\ln\left|\frac{v + 35.02\frac{m}{s}}{v - 35.02\frac{m}{s}}\right| = t + C_1$$

$$\boxed{\int(a + bx)^{-1}dx = \frac{\ln|a + bx|}{b} + C}$$

$$\left|\frac{v + 35.02\frac{m}{s}}{v - 35.02\frac{m}{s}}\right| = e^{\frac{t}{1.785s} + C_1}$$

$$\left|\frac{v + 35.02\frac{m}{s}}{v - 35.02\frac{m}{s}}\right| = e^{C_1}e^{\frac{t}{1.785s}}$$

Figure 63 Solution of the integral

$$\left|\frac{v + 35.02\frac{m}{s}}{v - 35.02\frac{m}{s}}\right| = C_2 e^{\frac{t}{1.785s}}$$

An initial velocity of zero can be assumed to find the value of the integration constant

$$\left|\frac{0 + 35.02\frac{m}{s}}{0 - 35.02\frac{m}{s}}\right| = C_2 e^{\frac{0}{1.785s}} \qquad 1 = C_2$$

This can then be simplified, and the absolute value sign eliminated.

$$\frac{v + 35.02\frac{m}{s}}{v - 35.02\frac{m}{s}} = \pm e^{\frac{t}{1.785s}}$$

$$v + 35.02\frac{m}{s} = \pm v e^{\frac{t}{1.785s}} \mp 35.02\frac{m}{s} e^{\frac{t}{1.785s}}$$

$$v\left(1 \mp e^{\frac{t}{1.785s}}\right) = \mp 35.02\frac{m}{s} e^{\frac{t}{1.785s}} - 35.02\frac{m}{s}$$

$$v = 35.02\frac{m}{s}\left(\frac{\mp e^{\frac{t}{1.785s}} - 1}{1 \mp e^{\frac{t}{1.785s}}}\right) \qquad 0 = 35.02\frac{m}{s}\left(\frac{\mp 1 - 1}{1 \mp 1}\right) = \left(\frac{1-1}{1+1}\right) = \frac{0}{2}$$

$$v = 35.02\frac{m}{s}\left(\frac{e^{\frac{t}{1.785s}} - 1}{1 + e^{\frac{t}{1.785s}}}\right)$$

Figure 64 Solution of the integral and application of the initial conditions

As evident from the example, non-linear equations are much harder to solve and don't have routine methods. Typically the numerical methods discussed in the next chapter are preferred.

3.5.2 Non-Linear Equation Terms

If our models include a device that is non-linear, we will need to linearize the model before we can proceed. A non-linear system can be approximated with a linear equation using the following method.

1. Pick an operating point or range for the component.
2. Find a constant value that relates a change in the input to a change in the output.
3. Develop a linear equation.
4. Use the linear equation for the analysis.

Consider the non-linear function in Figure 65. The function can be approximated as linear by picking a value XXXXXXXXXXXXXXXXXXXXXXX

In this case the relationship between pressure drop and flow are non-linear. We need to develop an equation that approximates the local operating point.

$$\Delta p \approx R(q - \bar{q})$$

$$R \approx \frac{\Delta(\Delta p)}{\Delta q} \qquad R = 2\frac{q}{K^2}$$

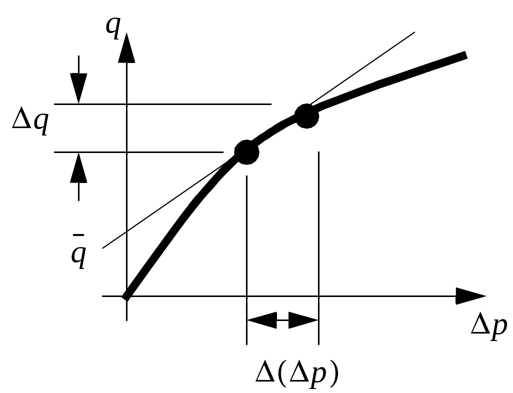


Figure 65 Finding a linear parameter

A linearized differential equation can be approximately solved using known techniques as long as the system doesn't travel too far from the linearized point. The example in Figure 66 shows the linearization of a non-linear equation about a given operating point. This equation will be approximately correct as long as the first derivative doesn't move too far from 100. When this value does the new velocity can be calculated.

Assume we have the non-linear differential equation below. It can be solved by linearizing the value about the operating point

Given,

$$y'^2 + 4y = 200 \qquad y(0) = 10$$

We can make the equation linear by replacing the velocity squared term with the velocity times the actual velocity. As long as the system doesn't vary too much from the given velocity the model should be reasonably accurate.

$$y' = \pm\sqrt{200 - 4y}$$

$$y'(0) = \pm\sqrt{200 - 4(10)} = \pm 12.65$$

$$12.65y' + 4y = 20$$

This system may now be solved as a linear differential equation. If the velocity (first derivative of y) changes significantly, then the differential equation should be changed to reflect this.

Homogeneous:

$$12.65y' + 4y = 0$$

$$12.65A + 4 = 0 \qquad A = -0.316$$

$$y_h = Ce^{-0.316t}$$

Particular:

$$y_p = A$$

$$12.65(0) + 4A = 200 \qquad A = 50$$

Initial conditions:

$$y(t) = Ce^{-0.316t} + 50$$

$$10 = Ce^0 + 50 \qquad C = -40$$

$$y(t) = -40e^{-0.316t} + 50$$

Figure 66 Linearizing a differential equation

If the velocity (first derivative of y) changes significantly, then the differential equation should be changed to reflect this. For example we could decide to recalculate the equation value after 0.1s.

$$y(0.1) = -40e^{-0.316(0.1)} + 50 = 11.24$$

$$\frac{d}{dt}y(0.1) = -40(-0.316)e^{-0.316(0.1)} = 12.25$$ Note: a small change

$$12.25y' + 4y = 20$$

Now recalculate the solution to the differential equation.

Homogeneous:

$$12.25y' + 4y = 0$$

$$12.25A + 4 = 0 \qquad A = -0.327$$

$$y_h = Ce^{-0.327t}$$

Particular:

$$y_p = A$$

$$12.25(0) + 4A = 200 \qquad A = 50$$

Initial conditions:

$$y(t) = Ce^{-0.327t} + 50$$

$$11.24 = Ce^{0.1} + 50 \qquad C = -35.070575$$

$$y(t) = -35.07e^{-0.316t} + 50$$

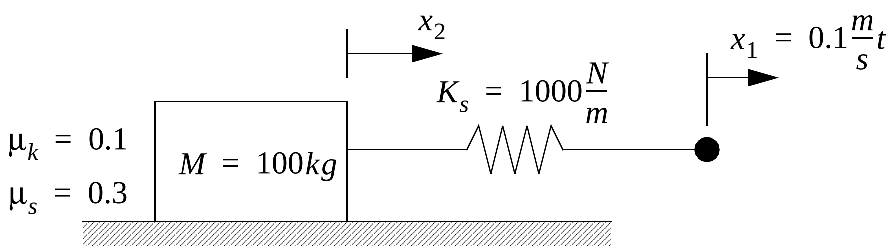
Notice that the values have shifted slightly, and as the analysis progresses the equations will adjust slowly. Higher accuracy can be obtained using smaller steps in time.

Figure 67 Linearizing a differential equation

3.5.3 Changing Systems

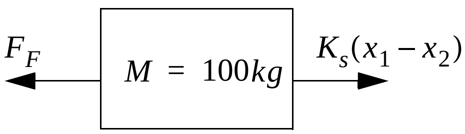
In practical systems, the forces at work are continually changing. For example a system often experiences a static friction force when motion is starting, but once motion starts it is replaced with a smaller kinetic friction. Another example is tension in a cable. When under tension the cable acts as a spring. But, when in compression the force goes to zero.

Consider the example in Figure 68. A mass is pulled by a springy cable. The right hand side of the cable is being pulled at a constant rate, while the block is free to move, only restricted by friction forces and inertia. At the beginning all components are at rest and undeflected.



An FBD and equation can be developed for the system. The friction force will be left as a variable at this point.

For the cable/spring in tension $x_1 - x_2 \geq 0$



$$\sum F_x = -F_F + K_s(x_1 - x_2) = Mx_2''$$

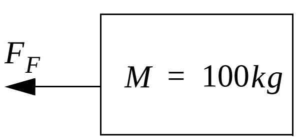
$$-F_F + 1000\frac{N}{m}\left(0.1\frac{m}{s}t - x_2\right) = 100kgx_2''$$

$$100kgx_2'' + 1000\frac{N}{m}x_2 = 1000\frac{N}{m}0.1\frac{m}{s}t - F_F$$

$$x_2'' + 10\frac{N}{kgm}x_2 = 1\frac{N}{kgs}t - \frac{F_F}{100kg}$$

$$x_2'' + 10s^{-2}x_2 = 1\frac{m}{s^3}t - \frac{F_F}{100kg}$$

For the cable/spring in compression $x_1 - x_2 < 0$



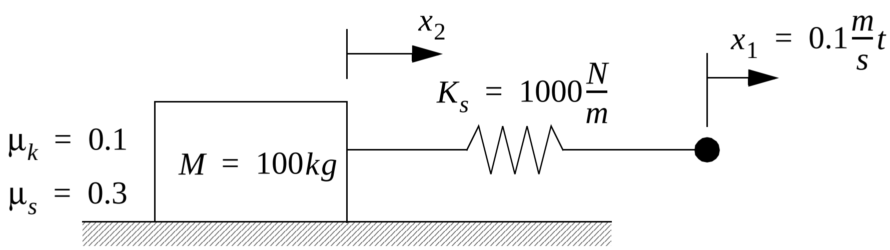
$$\sum F_x = -F_F = Mx_2''$$

$$-F_F = 100kgx_2''$$

$$100kgx_2'' = -F_F$$

$$x_2'' = -\frac{F_F}{100kg}$$

Figure 68 A differential equation for a mass pulled by a springy cable



An FBD and equation can be developed for the system. The friction force will be left as a variable at this point.

$$N = 100kg9.81\frac{N}{kg} = 981N$$

static friction $\qquad \frac{d}{dt}x_2 \le 0\frac{m}{s}$

$$0N \le F_F < \mu_s N < 294.3N$$

kinetic friction $\qquad \frac{d}{dt}x_2 > 0\frac{m}{s}$

$$F_F = \mu_k N = 98.1N$$

Figure 69 Friction forces for the mass

The analysis of the system begins with assuming the system starts at rest and undeflected. In this case the cable/spring will be undeflected with no force, and the mass will be experiencing static friction. Therefore the block will stay in place until the cable stretches enough to overcome the static friction.

$$x_2 = 0 \qquad x_2'' = 0 \qquad F_F = 294.3N$$

$$x_2'' + 10s^{-2}x_2 = 1\frac{m}{s^3}t - \frac{F_F}{100kg}$$

$$0 + 10s^{-2}0 = 1\frac{m}{s^3}t - \frac{294.3N}{100kg}$$

$$1\frac{m}{s^3}t = \frac{294.3kgm}{100kgs^2}$$

$$t = 2.943s$$

Therefore the system is static from 0 to 2.943s

Figure 70 Analysis of the object before motion begins

After motion begins the object will only experience kinetic friction, and continue to accelerate until the cable/spring becomes loose in compression. This stage of motion requires the solution of a differential equation.

$$x_2'' + 10s^{-2}x_2 = 1\frac{m}{s^3}t - \frac{98.1N}{100kg}$$

For the homogeneous,

$$x_2'' + 10s^{-2}x_2 = 0$$

$$A + 10s^{-2} = 0 \qquad A = \pm 3.16js^{-1}$$

$$x_h = C_1 \sin(3.16t + C_2)$$

For the particular,

$$x_p = At + B \qquad x_p' = A \qquad x_p'' = 0$$

$$0 + 10s^{-2}(At + B) = 1\frac{m}{s^3}t - \frac{98.1N}{100kg}$$

$$10s^{-2}A = 1\frac{m}{s^3} \qquad A = 0.1\frac{m}{s}$$

$$10s^{-2}B = -\frac{98.1N}{100kg} \qquad B = -0.0981m$$

Figure 71 Analysis of the object after motion begins

For the initial conditions,

$$x(2.943s) = 0m \qquad \frac{d}{dt}x(2.943s) = 0\frac{m}{s}$$

$$x(t) = C_1 \sin(3.16t + C_2) + 0.1\frac{m}{s}t - 0.0981m$$

$$0 = C_1 \sin(3.16(2.943s) + C_2) + 0.1\frac{m}{s}(2.943s) - 0.0981m$$

$$C_1 \sin(9.29988 + C_2) = -0.1962$$

$$\frac{d}{dt}x(t) = 3.16C_1 \cos(3.16t + C_2) + 0.1\frac{m}{s}$$

$$0 = 3.16C_1 \cos(3.16(2.943) + C_2) + 0.1\frac{m}{s}$$

$$C_1 \cos(9.29988 + C_2) = -0.0316$$

$$\frac{C_1 \sin(9.29988 + C_2)}{C_1 \cos(9.29988 + C_2)} = \frac{-0.1962}{-0.0316}$$

$$\tan(9.29988 + C_2) = 6.209 \qquad C_2 = (-7.889 + \pi n)rad \qquad n \in I$$

$$C_1 = \frac{-0.1962}{\sin(9.29988 - 7.889)} = -0.199m$$

$$x(t) = -0.199m\sin(3.16t - 7.889rad) + 0.1\frac{m}{s}t - 0.0981m$$

$$\frac{d}{dt}x(t) = -0.199(3.16)m\cos(3.16t - 7.889rad) + 0.1\frac{m}{s}$$

Figure 72 Analysis of the object after motion begins

The equation of motion changes after the cable becomes slack. This point in time can be determined when the displacement of the block equals the displacement of the cable/spring end.

$$0.1\frac{m}{s}t = -0.199m\sin(3.16t - 7.889rad) + 0.1\frac{m}{s}t - 0.0981m$$

$$-0.199m\sin(3.16t - 7.889rad) = 0.0981m$$

$$3.16t - 7.889 + \pi n = -0.51549413 \qquad t = 3.328s$$

$$x(3.328) = 0.137m \qquad \frac{d}{dt}x(2.333) = 0.648\frac{m}{s}$$

After this the differential equation without the cable/spring is used.

$$x_2'' = -\frac{98.1N}{100kg} = -0.981\frac{m}{s^2}$$

$$x_2' = \left(-0.981\frac{m}{s^2}\right)t + C_1$$

$$0.648\frac{m}{s} = \left(-0.981\frac{m}{s^2}\right)(3.328s) + C_1$$

$$C_1 = 3.913\frac{m}{s}$$

$$x_2 = \left(-\frac{0.981}{2}\frac{m}{s^2}\right)t^2 + 3.913\frac{m}{s}t + C_2$$

$$0.137m = \left(-\frac{0.981}{2}\frac{m}{s^2}\right)(3.328s)^2 + 3.913\frac{m}{s}(3.328s) + C_2$$

$$C_2 = -7.453m$$

$$x_2(t) = \left(-\frac{0.981}{2}\frac{m}{s^2}\right)t^2 + 3.913\frac{m}{s}t - 7.453m$$

This motion continues until the block stops moving.

$$0 = \left(-0.981\frac{m}{s^2}\right)t + 3.913\frac{m}{s}$$

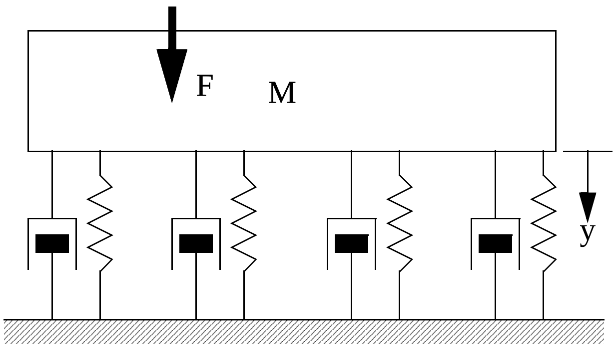
$$t = 3.989s$$

The solution can continue, considering when to switch the analysis conditions.

Figure 73 Determining when the cable become slack

3.6 CASE STUDY

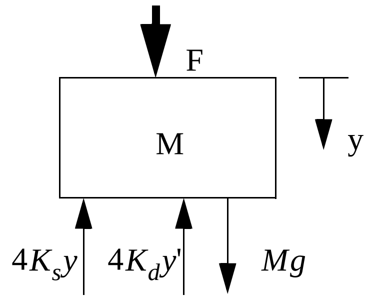
A typical vibration control system design is described in Figure 74.



The model to the left describes a piece of reciprocating industrial equipment. The mass of the equipment is 10000kg. The equipment operates such that a force of 1000N with a frequency of 2Hz is exerted on the mass. We have been asked to design a vibration isolation mounting system. The criteria we are given is that the mounts should be 30cm high when unloaded, and 25cm when loaded with the mass. In addition, the oscillations while the machine is running cannot be more than 2cm total. In total there will be four mounts mounted around the machine. Each isolator will be composed of a spring and a damper.

Figure 74 A vibration control system

There are a number of elements to the design and analysis of this system, but as usual the best place to begin is by developing a free body diagram, and a differential equation. This is done in Figure 75.



$$+\downarrow \sum F_y = F - 4K_s y - 4K_d y' + Mg = My''$$

$$My'' + 4K_d y' + 4K_s y = F + Mg$$

$$y'' + \frac{4K_d y'}{M} + \frac{4K_s y}{M} = \frac{F}{M} + g$$

$$y'' + \frac{4K_d y'}{10000Kg} + \frac{4K_s y}{10000Kg} = \frac{1000N}{10000Kg}\sin(2(2\pi)t) + 9.81ms^{-2}$$

$$y'' + 0.0004Kg^{-1}K_d y' + 0.0004Kg^{-1}K_s y = 0.1ms^{-2}\sin(4\pi t) + 9.81ms^{-2}$$

Figure 75 FBD and derivation of equation

Using the differential equation, the spring values can be found by assuming the machine is at rest. This is done in Figure 76.

When the system is at rest the equation is simplified; the acceleration and velocity terms both become zero. In addition, we will assume that the cyclic force is not applied for the unloaded/loaded case. This simplifies the differential equation by eliminating several terms.

$$0.0004Kg^{-1}K_s y = 9.81ms^{-2}$$

Now we can consider that when unloaded the spring is 0.30m long, and after loading the spring is 0.25m long. This will result in a downward compression of 0.05m, in the positive y direction.

$$0.0004Kg^{-1}K_s(0.05m) = 9.81ms^{-2}$$

$$K_s = \frac{9.81}{0.0004(0.05)}Kgms^{-2}m^{-1}$$

$$\therefore K_s = 491KNm^{-1}$$

Figure 76 Calculation of the spring coefficient

The remaining unknown is the damping coefficient. At this point we have determined the range of motion of the mass. This can be done by developing the particular solution of the differential equation, as it will contain the steady-state oscillations caused by the forces as shown in Figure 77.

$$y'' + 0.0004Kg^{-1}K_d y' + 0.0004Kg^{-1}(491KNm^{-1})y = 0.1ms^{-2}\sin(4\pi t) + 9.81ms^{-2}$$

$$y'' + 0.0004Kg^{-1}K_d y' + 196s^{-2}y = 0.1ms^{-2}\sin(4\pi t) + 9.81ms^{-2}$$

The particular solution can now be found by guessing a value, and solving for the coefficients. (Note: The units in the expression are uniform (i.e., the same in each term) and will be omitted for brevity.)

$$y = A\sin(4\pi t) + B\cos(4\pi t) + C$$

$$y' = 4\pi A\cos(4\pi t) - 4\pi B\sin(4\pi t)$$

$$y'' = -16\pi^2 A\sin(4\pi t) - 16\pi^2 B\cos(4\pi t)$$

$$\therefore (-16\pi^2 A\sin(4\pi t) - 16\pi^2 B\cos(4\pi t)) + 0.0004K_d(4\pi A\cos(4\pi t) - 4\pi B\sin(4\pi t))$$
$$+ 196(A\sin(4\pi t) + B\cos(4\pi t) + C) = 0.1\sin(4\pi t) + 9.81$$

$$-16\pi^2 B + 0.0004K_d 4\pi A + 196A = 0$$
$$B = A(31.8\times 10^{-6}K_d + 1.24)$$
$$-16\pi^2 A + 0.0004K_d(-4\pi B) + 196A = 0.1$$

$$A(-16\pi^2 + 196) + B(-5.0\times 10^{-3}K_d) = 0.1$$
$$A(-16\pi^2 + 196) + A(31.8\times 10^{-6}K_d + 1.24)(-5.0\times 10^{-3}K_d) = 0.1$$
$$A = \frac{0.1}{-16\pi^2 + 196 + (31.8\times 10^{-6}K_d + 1.24)(-5.0\times 10^{-3}K_d)}$$
$$A = \frac{0.1}{K_d^2(-159\times 10^{-9}) + K_d(-6.2\times 10^{-3}) + 38.1}$$
$$B = \frac{3.18\times 10^{-6}K_d - 0.124}{K_d^2(-159\times 10^{-9}) + K_d(-6.2\times 10^{-3}) + 38.1}$$
$$C = 9.81ms^{-2}$$

Figure 77 Particular solution of the differential equation

The particular solution can be used to find a damping coefficient that will give an overall oscillation of 0.02m, as shown in Figure 78. In this case Mathcad was used to find the solution, although it could have also been found by factoring out the algebra, and finding the roots of the resulting polynomial.

In the previous particular solution the values were split into cosine and sine components. The magnitude oscillation can be calculated with the Pythagorean formula.

$$magnitude = \sqrt{A^2 + B^2}$$

$$magnitude = \frac{\sqrt{(0.1)^2 + (3.18 \times 10^{-6} K_d - 0.124)^2}}{K_d^2(-159 \times 10^{-9}) + K_d(-6.2 \times 10^{-3}) + 38.1}$$

The design requirements call for a maximum oscillation of 0.02m, or a magnitude of 0.01m.

$$0.01 = \frac{\sqrt{(0.1)^2 + (3.18 \times 10^{-6} K_d - 0.124)^2}}{K_d^2(-159 \times 10^{-9}) + K_d(-6.2 \times 10^{-3}) + 38.1}$$

A given-find block was used in Mathcad to obtain a damper value of,

$$K_d = 3411 N\frac{s}{m}$$

Aside: the Mathcad solution

$$f(k) := \frac{\sqrt{0.01 + \left[\left(3.18 \cdot 10^{-6} \cdot k\right) - 0.124\right]^2}}{\left[k \cdot k \cdot \left(-159 \cdot 10^{-9}\right)\right] + k \cdot \left(-6.2 \cdot 10^{-3}\right) + 38.1}$$

$$k_d := 1$$

given

$$f(k_d) = 0.01$$

$$find(k_d) = 3.411 \times 10^3$$

Figure 78 Determining the damping coefficient

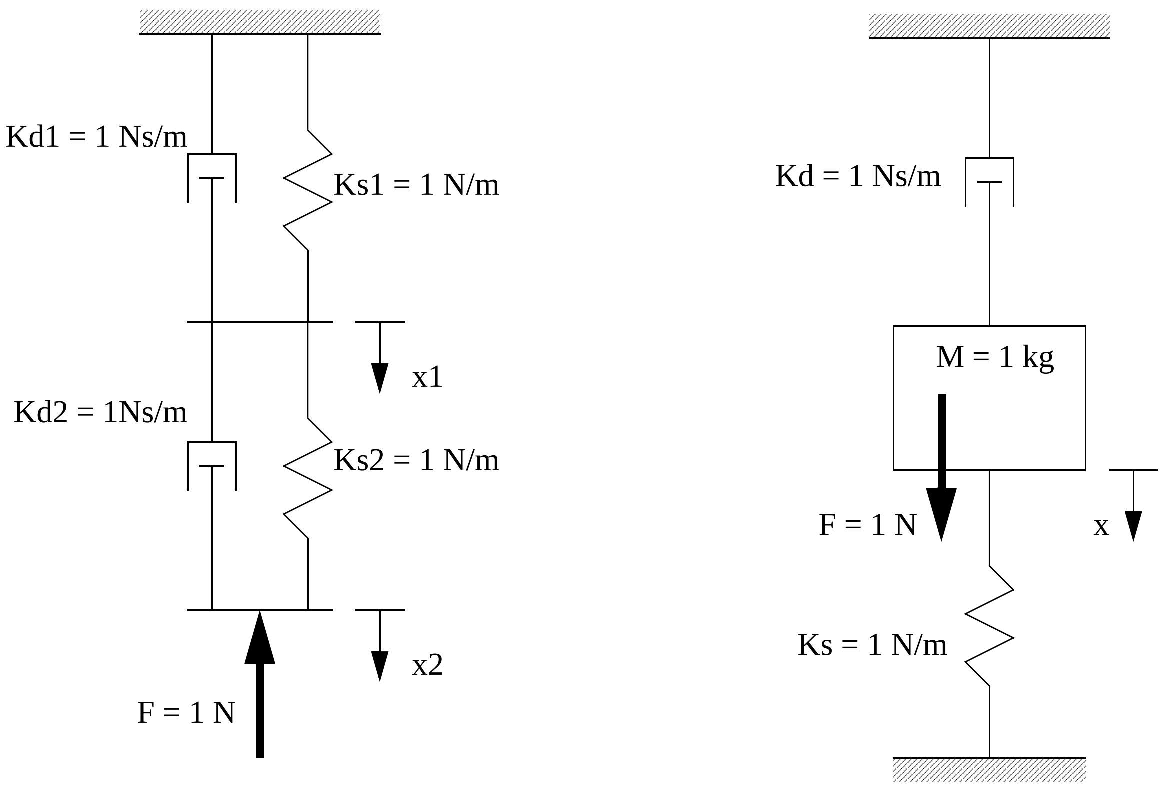
The values of the spring and damping coefficients can be used to select actual components. Some companies will design and build their own components. Components can also be acquired by searching catalogs, or requesting custom designs from other companies.

3.7 SUMMARY

- First and second-order differential equations were analyzed explicitly.
- First and second-order responses were examined.
- The topic of analysis was discussed.
- A case study looked at a second-order system.

3.8 PRACTICE PROBLEMS

1. a) Write the differential equations for the system below. Solve the equations for *x* assuming that the system is at rest and undeflected before t=0. Also assume that gravity is present.



b) State whether the system is first or second-order. If the system if first-order find the time constant. If it is second-order find the natural frequency and damping ratio.

ans.

$$x_1 = 1 - e^{-t}$$

$$x_2 = 2 - 2e^{-t}$$

$$\tau = 1$$

ans.

$$x = -12.485e^{-0.5t}\cos(0.866t - 0.524) + 10.81$$

$$\zeta = 0.5 \qquad \omega_n = 1$$

2. Solve the following differential equation with the given initial conditions and draw a sketch of the first 5 seconds. The input is a step function that turns on at t=0.

$$0.5V_o'' + 0.6V_o' + 2.1V_o = 3V_i + 2 \qquad \text{initial conditions} \quad V_i = 5V, \quad V_o = 0, \quad V_o' = 0$$

3. Solve the following differential equation with the given initial conditions and draw a sketch of the first 5 seconds. The input is a step function that turns on at t=0.

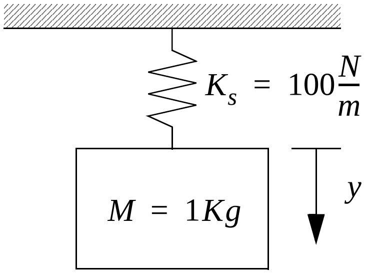
$$0.5V_o'' + 0.6V_o' + 2.1V_o = 3V_i + 2 \qquad \text{initial conditions} \quad V_i = 5V, \quad V_o = 0, \quad V_o' = 1$$

(ans.

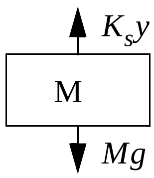
$$V_0(t) = -8.331e^{-0.6t}\cos(1.96t - 0.238) + 8.095$$

4. The following differential equation was derived for a mass suspended with a spring. At time 0s the system is released and allowed to drop. It then oscillates. Solve the differential equation to

find the motion as a function of time.



FBD:



$$+\uparrow \quad \sum F_y = K_s y - Mg = -My''$$

$$\left(100\frac{N}{m}\right)y - (1Kg)\left(9.81\frac{N}{Kg}\right) = (-1Kg)y''$$

$$\left(1\frac{Nm}{s^2}\right)y'' + \left(100\frac{N}{m}\right)y = 9.81N$$

$$(1Kg)y'' + \left(100\frac{Kgm}{ms^2}\right)y = 9.81\frac{Kgm}{s^2}$$

$$y'' + (100s^{-2})y = 9.81ms^{-2}$$

$$y_0 = 0m \qquad y'_0 = 0ms^{-1}$$

(ans. homogeneous: guess $y_h = e^{At} \qquad y_h' = Ae^{At} \qquad y_h'' = A^2e^{At}$

$$A^2e^{At} + (100s^{-2})e^{At} = 0$$

$$A^2 = -100s^{-2} \qquad A = \pm 10js^{-1}$$

$$y_h = C_1\cos(10t + C_2)$$

particular: guess $y_p = At + B \qquad y_p' = A \qquad y_p'' = 0$

$$(0) + (100s^{-2})(At + B) = 9.81ms^{-2}$$

$$(100s^{-2})(At + B) = 9.81ms^{-2}$$

$$A = 0 \qquad B = \frac{9.81ms^{-2}}{100s^{-2}} = 0.0981m$$

$$y_p = 0.0981m$$

initial conditions:

$$y = y_h + y_p = C_1\cos(10t + C_2) + 0.0981m$$

$$y' = -10C_1\sin(10t + C_2)$$

for d/dt y0 = 0m:

$$0 = -10C_1\sin(10(0) + C_2) \qquad C_2 = 0$$

for y0 = 0m:

$$0 = C_1\cos(10(0) + (0)) + 0.0981m$$

$$-0.0981m = C_1\cos(0) \qquad C_1 = -0.0981m$$

$$y = (-0.0981m)\cos(10t) + 0.0981m$$

5. Solve the following differential equation with the three given cases. All of the systems have a step input ’y’ and start undeflected and at rest.

$$x'' + 2\xi\omega_n x' + \omega_n^2 x = y$$

initial conditions $x' = 0$, $x = 0$, $y = 1$

case 1: $\xi = 0.5$ $\omega_n = 10$

case 2: $\xi = 1$ $\omega_n = 10$

case 3: $\xi = 2$ $\omega_n = 10$

6. Solve the following differential equation with the three given cases. All of the systems have a sinusoidal input ’y’ and start undeflected and at rest.

$$x'' + 2\xi\omega_n x' + \omega_n^2 x = y$$

initial conditions $x' = 0$, $x = 0$, $y = \sin(t)$

case 1: $\xi = 0.5$ $\omega_n = 10$

case 2: $\xi = 1$ $\omega_n = 10$

case 3: $\xi = 2$ $\omega_n = 10$

7. A system is to be approximated with a mass-spring-damper model using the following parameters: weight 28N, viscous damping coefficient 6Ns/m and stiffness 36N/m. Calculate the undamped natural frequency (Hz) of the system, the damping ratio and describe the type of response you would expect if the mass were displaced and released. What additional damping would be required to make the system critically damped?

(ans. Given $K_d = 6N\frac{s}{m}$ $K_s = 36\frac{N}{m}$ $M = \frac{28N}{9.81\frac{N}{kg}} = 2.85kg$

The typical transfer function for a mass-spring-damper systems is,

$$\frac{x}{F} = \frac{1}{D^2M + DK_d + K_s} = \frac{A}{D^2 + D2\xi\omega_n + \omega_n^2}$$

The second order parameters can be calculated from this.

$$\omega_n = \sqrt{\frac{K_s}{M}} = \sqrt{\frac{36\frac{N}{m}}{2.85kg}} = \sqrt{\frac{36\frac{kgm}{ms^2}}{2.85kg}} = \sqrt{12.63s^{-2}} = 3.55\frac{rad}{s}$$

$$\xi = \frac{\left(\frac{K_d}{M}\right)}{2\omega_n} = \frac{6N\frac{s}{m}}{2(3.55)\frac{rad}{s}2.85kg} = 0.296$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 3.39\frac{rad}{s}$$

If pulled and released the system would have a decaying oscillation about 1.9Hz

A critically damped system would require a damping coefficient of....

$$\xi = \frac{\left(\frac{K_d}{M}\right)}{2\omega_n} = \frac{K_d}{2(3.55)\frac{rad}{s}2.85kg} = 1 \qquad K_d = 20.2\frac{Ns}{m}$$

8. What is the transfer function for a second-order system that responds to a step input with an overshoot of 20%, with a delay of 0.4 seconds to the first peak?

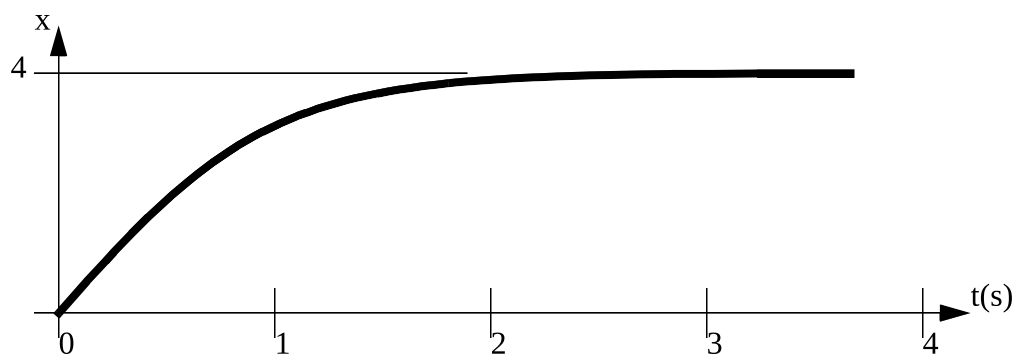
9. For our standard lumped parameter model weight is 36N, stiffness is $2.06*10^3$ N/m and damping coefficient is 100Ns/m. What are the natural frequency (Hz) and damping ratio? (ans. fn=3.77Hz, damp.=.575)

10. What would the displacement amplitude after 100ms for a system having a natural frequency of 13 rads/sec and a damping ratio of 0.20. Assume an initial displacement of 50mm. (0.018m)

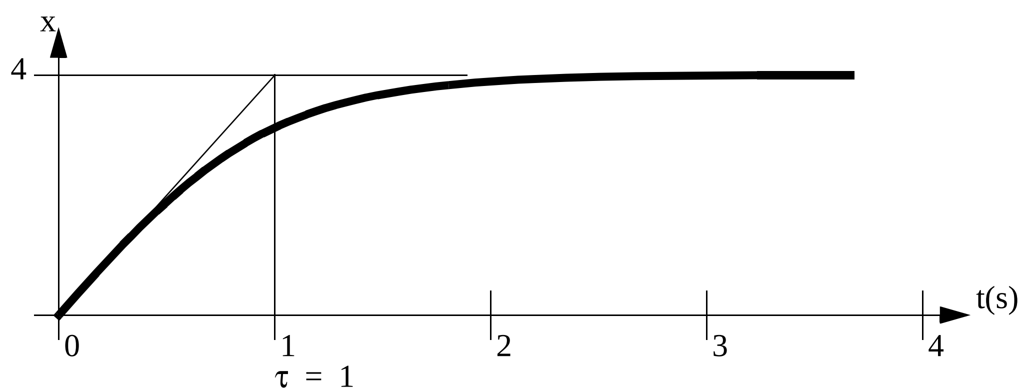
11. A spring damper system supports a mass of 34N. If it has a spring constant of 20.6N/cm, what is the systems natural frequency? (ans. 24.37 rad/sec)

12. Determine the first order differential equation given the graphical response shown below.

Assume the input is a step function.



(ans.



Given the equation form,

$$x' + \frac{1}{\tau}x = A$$

The values at steady state will be

$$x' = 0 \qquad x = 4$$

So the unknown ‘A’ can be calculated.

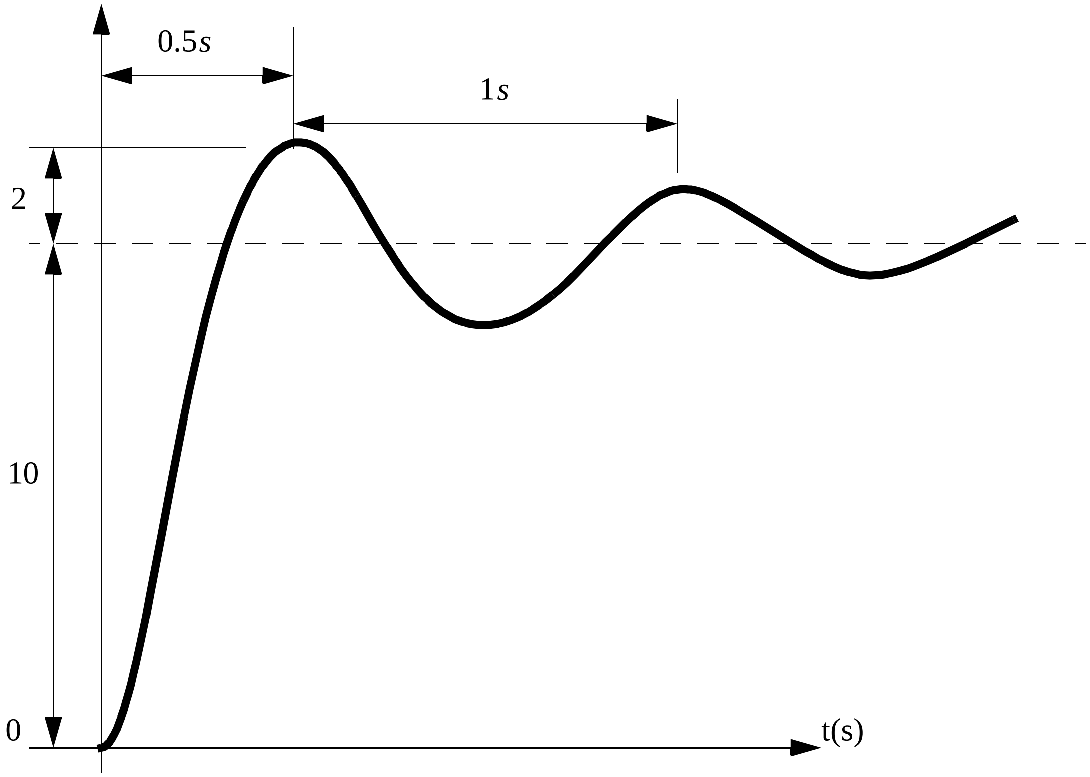
$$0 + \frac{1}{1}4 = A \qquad A = 4$$

$$x' + \frac{1}{1}x = 4$$

$$x' + x = 4$$

13. The second order response below was obtained experimentally. Determine the parameters of

the differential equation that resulted in the response assuming the input was a step function.



(ans. For the first peak:

$$\frac{b}{\Delta x} = e^{-\sigma t_p}$$

$$\frac{2}{10} = e^{-\sigma 0.5}$$

$$\omega_d \approx \frac{\pi}{t_p}$$

$$\ln\left(\frac{2}{10}\right) = -\sigma 0.5$$

$$\sigma = -2\ln\left(\frac{2}{10}\right) = 3.219$$

$$\xi = \frac{1}{\sqrt{\left(\frac{\pi}{t_p\sigma}\right)^2 + 1}}$$

For the damped frequency:

$$\omega_d = \frac{2\pi}{1s} = 2\pi$$

These values can be used to find the damping coefficient and natural frequency

$$\sigma = \xi\omega_n \qquad \omega_n = \frac{3.219}{\xi}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$2\pi = \frac{3.219}{\xi}\sqrt{1-\xi^2}$$

$$\left(\frac{2\pi}{3.219}\right)^2 = \frac{1-\xi^2}{\xi^2}$$

$$\left(\frac{2\pi}{3.219}\right)^2 + 1 = \frac{1}{\xi^2} \qquad \xi = \sqrt{\frac{1}{\left(\frac{2\pi}{3.219}\right)^2 + 1}} = 0.4560$$

$$\omega_n = \frac{3.219}{\xi} = \frac{3.219}{0.4560} = 7.059$$

This leads to the final equation using the steady state value of 10

$$x'' + 2\xi\omega_n x' + \omega_n^2 x = F$$

$$x'' + 2(0.4560)(7.059)x' + (7.059)^2 x = F$$

$$x'' + 6.438x' + 49.83x = F$$

$$(0) + 6.438(0) + 49.83(10) = F \qquad F = 498.3$$

$$x'' + 6.438x' + 49.83x = 498.3$$

14. Explain with graphs how to develop first and second-order equations using experimental data.

(ans. Key points:

First-order:
find initial final values
find time constant with 63% or by slope
use these in standard equation

Second-order:
find damped frequency from graph
find time to first peak
use these in cosine equation

4. NUMERICAL ANALYSIS

Topics:

- State variable form for differential equations
- Numerical integration with Mathcad and calculators
- Numerical integration theory: first-order, Taylor series and Runge-Kutta
- Using tabular data
- A design case

Objectives:

- To be able to solve systems of differential equations using numerical methods.

4.1 INTRODUCTION

For engineering analysis it is always preferable to develop explicit equations that include symbols, but this is not always practical. In cases where the equations are too costly to develop, numerical methods can be used. As their name suggests, numerical methods use numerical calculations (i.e., numbers not symbols) to develop a single solution to a differential equation. The solution is often in the form of a set of numbers, or a graph. This can then be used to analyze a particular design case. The solution is often returned quickly so trial and error design techniques may be used. But, without a symbolic equation the system can be harder to understand and manipulate.

This chapter focuses on techniques that can be used for numerically integrating systems of differential equations.

4.2 THE GENERAL METHOD

The general process of analyzing systems of differential equations involves first putting the equations into standard form, and then integrating these with one of a number of techniques. The most common standard form is state variable form, which reduces all of the equations to first-order differential equations. These first-order equations are then easily integrated to provide a solution for the system of equations.

4.2.1 State Variable Form

At any time a system can be said to have a state. Consider a car for example, the state of the car is described by its position and velocity. Factors that are useful when identifying state variables are:

- The variables should describe energy storing elements (potential or kinetic).
- The variables must be independent.
- They should fully describe the system elements.

After the state variables of a system have been identified, they can be used to write first-order state variable equations. The general form of state variable equations is shown in Figure 79. Notice that the state variable equation is linear, and the value of *x* is used to calculate the derivative. The output equation is not always required, but it can be used to calculate new output values.

$$\left(\frac{d}{dt}\right)x = Ax + Bu \qquad \text{state variable equation}$$

$$y = Cx + Du \qquad \text{output equation}$$

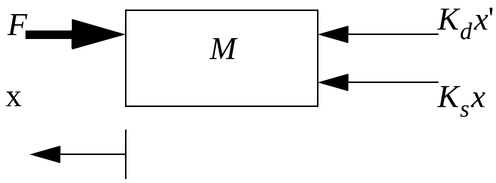
where,

x = state/output vector (variables such as position)
u = input vector (variables such as input forces)
A = transition matrix relating outputs/states
B = matrix relating inputs to outputs/states
y = non-state value that can be found directly (i.e. no integration)
C = transition matrix relating outputs/states
D = matrix relating inputs to outputs/states

Figure 79 The general state variable form

An example of a state variable equation is shown in Figure 80. As always, the FBD is used to develop the differential equation. The resulting differential equation is second-order, but this must be reduced to first-order. Using the velocity variable, ’v’, reduces the second-order differential equation can be reduced to a first-order equation. An equation is also required to define the velocity as the first derivative of the position, ’x’. In the example the two state equations are manipulated into a matrix form. This form can be useful, and may be required for determining a solution. For example, HP calculators require the matrix form, while TI calculators use the equation forms. Software such as Mathcad can use either form. The main disadvantage of the matrix form is that it will only work for linear differential equations.

Given the FBD shown below, the differential equation for the system is,



$$\sum F_x = -F + K_d x' + K_s x = Mx''$$

$$-F + K_d x' + K_s x = Mx''$$

The equation is second-order, so two state variables will be needed. One obvious choice for a state variable in this equation is ’x’. The other choice can be the velocity, ’v’. Equation (1) defines the velocity variable. The velocity variable can then be substituted into the differential equation for the system to reduce it to first-order.

$$x' = v \qquad (1)$$

$$-F + K_d x' + K_s x = Mx''$$

$$-F + K_d v + K_s x = Mv'$$

$$v' = x\left(\frac{K_s}{M}\right) + v\left(\frac{K_d}{M}\right) + \left(\frac{-F}{M}\right) \qquad (2)$$

Equations (1) and (2) can also be put into a matrix form similar to that given in Figure 79.

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{K_s}{M} & \frac{K_d}{M} \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-F}{M} \end{bmatrix}$$

Note: To have a set of differential equations that is solvable, there must be the same number of state equations as variables. If there are too few equations, then an additional equation must be developed using an unexploited relationship. If there are too many equations, a redundancy or over constraint must be eliminated.

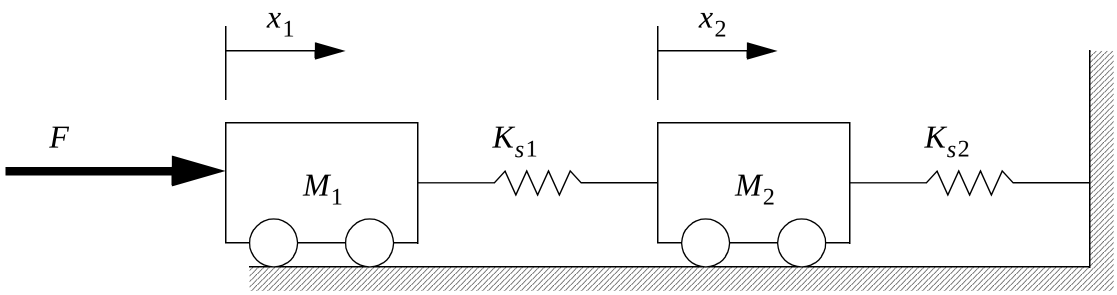
Figure 80 A state variable equation example

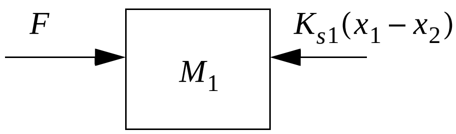
Put the equation into state variable and matrix form.

$$\sum F = F = M\left(\frac{d}{dt}\right)^2 x$$

Figure 81 Drill problem: Put the equation in state variable form

Consider the two cart problem in Figure 82. The carts are separated from each other and the wall by springs, and a force is applied to the left hand side. Free body diagrams are developed for each of the carts, and differential equations developed. For each cart a velocity state variable is created. The equations are then manipulated to convert the second-order differential equations to first-order state equations. The four resulting equations are then put into the state variable matrix form.





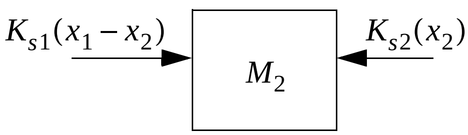
$$\xrightarrow{+}\sum F_x = F - K_{s1}(x_1 - x_2) = M_1 x_1''$$

$$M_1 x_1'' + K_{s1} x_1 - K_{s1} x_2 = F$$

$$x_1' = v_1 \qquad (1)$$

$$M_1 v_1' + K_{s1} x_1 - K_{s1} x_2 = F$$

$$v_1' = \frac{F}{M_1} - \frac{K_{s1}}{M_1} x_1 + \frac{K_{s1}}{M_1} x_2 \qquad (2)$$



$$\xrightarrow{+}\sum F_x = K_{s1}(x_1 - x_2) - K_{s2}(x_2) = M_2 x_2''$$

$$M_2 x_2'' + (K_{s1} + K_{s2}) x_2 - K_{s1} x_1 = 0$$

$$x_2' = v_2 \qquad (3)$$

$$M_2 v_2' + (K_{s1} + K_{s2}) x_2 - K_{s1} x_1 = 0$$

$$v_2' = \frac{K_{s1}}{M_2} x_1 - \left(\frac{K_{s1} + K_{s2}}{M_2}\right) x_2 \qquad (4)$$

The state equations can now be combined in a matrix form.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-K_{s1}}{M_1} & 0 & \frac{K_{s1}}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K_{s1}}{M_2} & 0 & \frac{-K_{s1} - K_{s2}}{M_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{F}{M_1} \\ 0 \\ 0 \end{bmatrix}$$

Figure 82 Two cart state equation example

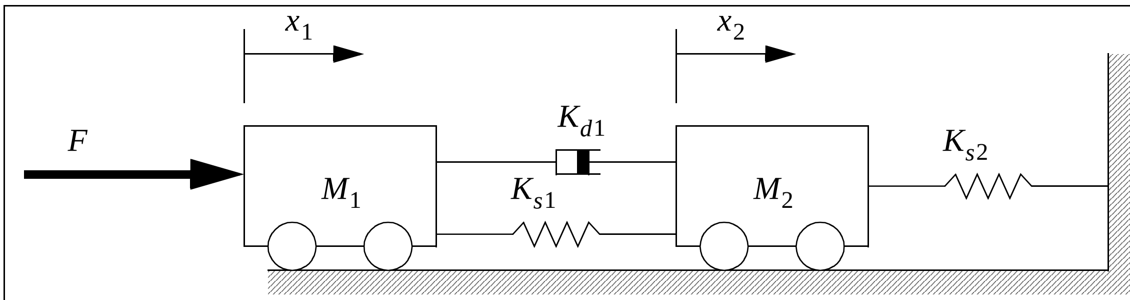


Figure 83 Drill problem: Develop the state equations in matrix form

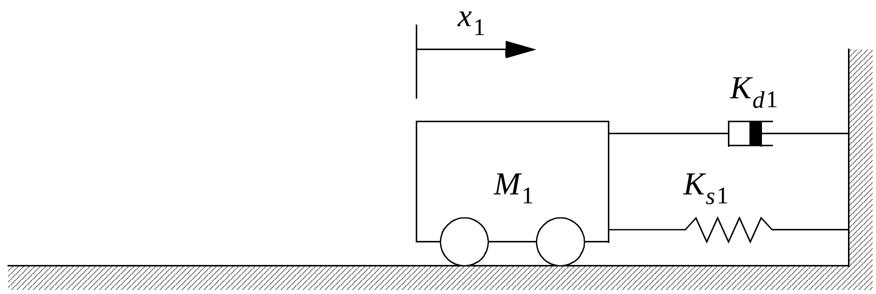


Figure 84 Drill problem: Convert the system to state equations

In some cases we will develop differential equations that cannot be directly reduced because they have more than one term at the highest order. For example, if a second-order equation has two second derivatives it cannot be converted to a state equation in the normal manner. In this case the two high order derivatives can be replaced with a dummy variable. In mechanical systems this often happens when masses are neglected. Consider the example problem in Figure 85, both 'y' and 'u' are first derivatives. To solve this problem, the highest order terms ('y' and 'u') are moved to the left of the equation. A dummy variable, 'q', is then created to replace these two variables with a single variable.

This also creates an output equation as shown in Figure 79.

Given the equation,

$$3y' + 2y = 5u'$$

Step 1: put both the first-order derivatives on the left hand side,

$$3y' - 5u' = -2y$$

Step 2: replace the left hand side with a dummy variable,

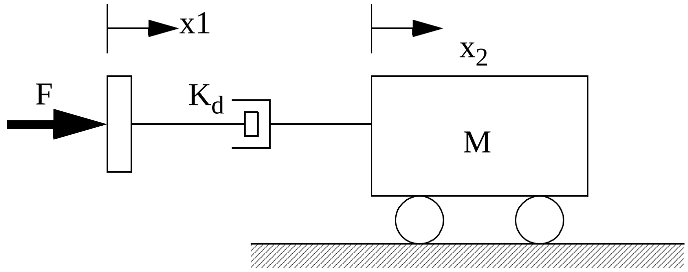
$$q = 3y - 5u \qquad \therefore q' = -2y$$

Step 3: solve the equation using the dummy variable, then solve for y as an output eqn.

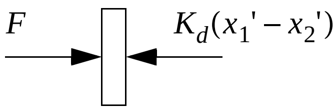
$$q' = -2y \qquad y = \frac{q + 5u}{3}$$

Figure 85 Using dummy variables for multiple high order terms

At other times it is possible to eliminate redundant terms through algebraic manipulation, as shown in Figure 86. In this case the force on both sides of the damper is the same, so it is substituted into the equation for the cart. But, the effects on the damper must also be integrated, so a dummy variable is created for the integration. An output equation needed to be created to calculate the value for x_1.



The FBDs and equations are;



$$\xrightarrow{+} \sum F_x = F - K_d(x_1' - x_2') = 0$$

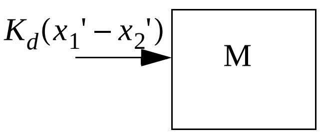
$$K_d(x_1' - x_2') = F$$

$$q = x_1 - x_2$$

$$K_d(q) = F$$

$$q' = \frac{F}{K_d} \qquad (1)$$

$$x_1 = x_2 + q \qquad (2)$$



$$\xrightarrow{+} \sum F_x = K_d(x_1' - x_2') = Mx_2''$$

$$F = Mx_2''$$

$$x_2' = v_2 \qquad (3)$$

$$v_2' = \frac{F}{M} \qquad (4)$$

The state equations (1, 3, 4) can be put in matrix form. The output equation (2) can also be put in matrix form.

$$\frac{d}{dt}\begin{bmatrix} q \\ x_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} q \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} \frac{F}{K_d} \\ 0 \\ \frac{F}{M} \end{bmatrix} \qquad \begin{bmatrix} x_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} q \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}$$

Figure 86 A dummy variable example

4.3 NUMERICAL INTEGRATION

Repetitive calculations can be used to develop an approximate solution to a set of differential equations. Starting from given initial conditions, the equation is solved with small time steps. Smaller time steps result in a higher level of accuracy, while larger time steps give a faster solution.

4.3.1 Numerical Integration With Tools

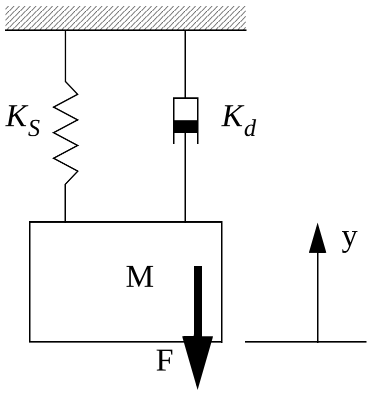
Numerical solutions can be developed with hand calculations, but this is a very time consuming task. In this section we will explore some common tools for solving state variable equations. The analysis process follows the basic steps listed below.

1. Generate the differential equations to model the process.
2. Select the state variables.
3. Rearrange the equations to state variable form.
4. Add additional equations as required.
5. Enter the equations into a computer or calculator and solve.

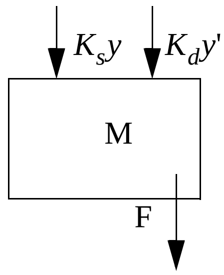
An example in Figure 87 shows the first four steps for a mass-spring-damper combination. The FBD is used to develop the differential equations for the system. The state variables are then selected, in this case the position, y, and velocity, v, of the block. The equations are then rearranged into state equations. The state equations are also put into matrix form, although this is not always necessary. At this point the equations are ready for solution.

Figure 87 Dynamic system example

Figure 88 shows the method for solving state equations on a TI-86 graphing calculator. (Note: this also works on other TI-8x calculators with minor modifications.) In the example a sinusoidal input force, F, is used to make the solution more interesting. The next step is to put the equation in the form expected by the calculator. When solving with the TI calculator the state variables must be replaced with the predefined names Q1, Q2, etc. The steps that follow describe the button sequences required to enter and analyze the equations. The result is a graph that shows the solution of the equation. Points can then be taken from the graph using the cursors. (Note: large solutions can sometimes take a few minutes to solve.)



Step 1: Develop equations



$$\sum F_y = -F - K_d\left(\frac{d}{dt}\right)y - K_s y = M\left(\frac{d}{dt}\right)^2 y$$

$$\therefore F + K_d\left(\frac{d}{dt}\right)y + K_s y + M\left(\frac{d}{dt}\right)^2 y = 0$$

Step 2: We need to identify state variables. In this case the height is clearly a defining variable. We will also need to use the vertical velocity, because the acceleration is a second derivative (we can only have first derivatives). Using the height, y, and velocity, v, as state variables we may now proceed to rewriting the equations. (Note: this is just an algebraic trick, but essential when setting up these matrices.)

Step 3:

$$\left(\frac{d}{dt}\right)y = v$$

$$\left(\frac{d}{dt}\right)v = \frac{-F - K_d v - K_s y}{M}$$

Step 4: We put the equations into a state variable matrix form.

$$\begin{bmatrix} \left(\frac{d}{dt}\right)y \\ \left(\frac{d}{dt}\right)v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-K_S}{M} & \frac{-K_d}{M} \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \frac{F}{M}$$

First, we select some parameter values for the equations of Figure 87. The input force will be a decaying sine wave.

$$\left(\frac{d}{dt}\right)y = v_y$$

$$\left(\frac{d}{dt}\right)v_y = \frac{-F - K_d v_y - K_s y}{M} = -4e^{-0.5t}\sin(t) - 2v_y - 5y$$

Next, the calculator requires that the state variables be Q1, Q2, ..., Q9, so we replace y with Q1 and v with Q2.

$$Q1' = Q2$$

$$Q2' = -4e^{-0.5t}\sin(t) - 2Q2 - 5Q1$$

Now, we enter the equations into the calculator and solve. To do this roughly follow the steps below. Look at the calculator manual for additional details.

1. Put the calculator in differential equation mode [2nd][MODE][DifEq][ENTER]
2. Go to graph mode and enter the equations above [GRAPH][F1]
3. Set up the axis for the graph [GRAPH][F2] so that time and the x-axis is from 0 to 10 with a time step of 0.5, and the y height is from +3 to -3.
4. Enter the initial conditions for the system [GRAPH][F3] as Q1=0, Q2=0
5. Set the axis [GRAPH][F4] as x=t and y=Q
6. (TI-86 only) Set up the format [GRAPH][MORE][F1][Fld-Off][ENTER]
7. Draw the graph [GRAPH][F5]
8. Find points on the graph [GRAPH][MORE][F4]. Move the left/right cursor to move along the trace, use the up/down cursor to move between traces.

Figure 88 Solving state equations with a TI-85 calculator

First, we select some parameter values for the equations of Figure 87. The input force will be a decaying sine wave.

$$\left(\frac{d}{dt}\right)y = v_y$$

$$\left(\frac{d}{dt}\right)v_y = \frac{-F - K_d v_y - K_s y}{M} = -4e^{-0.5t}\sin(t) - 2v_y - 5y$$

Next, the calculator requires that the state variables be Q1, Q2, ..., Q9, so we replace y with Q1 and v with Q2.

$$Q1' = Q2$$

UPDATE FOR TI-89

$$Q2' = -4e^{-0.5t}\sin(t) - 2Q2 - 5Q1$$

Now, we enter the equations into the calculator and solve. To do this roughly follow the steps below. Look at the calculator manual for additional details.

1. Put the calculator in differential equation mode [2nd][MODE][DifEq][ENTER]
2. Go to graph mode and enter the equations above [GRAPH][F1]
3. Set up the axis for the graph [GRAPH][F2] so that time and the x-axis is from 0 to 10 with a time step of 0.5, and the y height is from +3 to -3.
4. Enter the initial conditions for the system [GRAPH][F3] as Q1=0, Q2=0
5. Set the axis [GRAPH][F4] as x=t and y=Q
6. (TI-86 only) Set up the format [GRAPH][MORE][F1][Fld-Off][ENTER]
7. Draw the graph [GRAPH][F5]
8. Find points on the graph [GRAPH][MORE][F4]. Move the left/right cursor to move along the trace, use the up/down cursor to move between traces.

Figure 89 Solving state equations with a TI-89 calculator

State equations can also be solved in Mathcad using built-in functions, as shown in Figure 90. The first step is to enter the state equations as a function, 'D(t, Q)', where 't' is the time and 'Q' is the state variable vector. (Note: the equations are in a vector, but it is not the matrix form.) The state variables in the vector 'Q' replace the original state variables in the equations. The 'rkfixed' function is then used to obtain a solution. The arguments for the function, in sequence are; the state vector, the start time, the end time, the number of steps, and the state equation function. In this case the 10 second time interval is divided into 100 parts each 0.1s in duration. This time is chosen because of the general response time for the system. If the time step is too large the solution may become unstable and go to infinity. A time step that is too small will increase the computation time margnially. When in doubt, run the calculator again using a smaller time step.

$$D(t,Q) := \begin{bmatrix} Q_1 \\ \left(-4\cdot e^{-0.5\cdot t}\cdot \sin(t)\right) - 2\cdot Q_1 - 5\cdot Q_0 \end{bmatrix}$$

$$QI := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

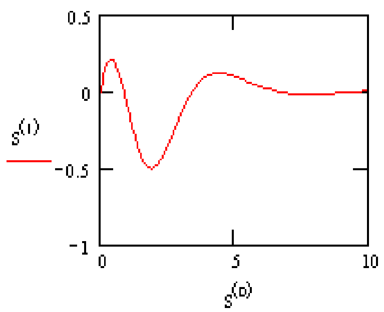
Define state variable equations and initial conditions. Note the origin is at '0' here.

$tMin := 0$ $tMax := 10$ $N := 100$

Define time range and number of steps for the integration.

$S := rkfixed(QI, tMin, tMax, N, D)$

A fixed time step runge-kutta integration



The first state variable S<1> = Q0 plotted against time S<0> = t.

A point in time is,

$i := 40$

$t := \left(S^{\langle 0\rangle}\right)_i$ $Q_0 := \left(S^{\langle 1\rangle}\right)_i$ $Q_1 := \left(S^{\langle 2\rangle}\right)_i$

$t = 4$ $Q_0 = 0.103$ $Q_1 = 0.081$

Figure 90 Solving state variable equations with Mathcad

Note: Notice that for the TI calculators the variables start at Q1, while in Mathcad the arrays start at Q0. Many students encounter problems because they forget this.

4.3.2 Numerical Integration

The simplest form of numerical integration is Euler’s first-order method. Given the current value of a function and the first derivative, we can estimate the function value a short time later, as shown in Figure 91. (Note: Recall that the state equations allow us to calculate first-order derivatives.) The equation shown is known as Euler’s equation. Basically, using a known position and first derivative we can calculate an approximate value a short time, h, later. Notice that the function being integrated curves downward, creating an error between the actual and estimated values at time ’t+h’. If the time step, h, were smaller, the error would decrease.

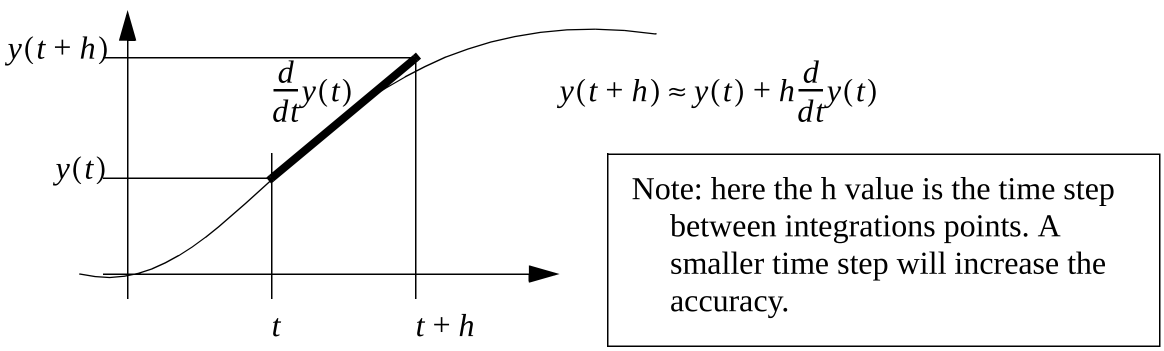


Figure 91 First-order numerical integration

The example in Figure 92 shows the solution of Newton’s equation using Euler’s method. In this example we are determining velocity by integrating the acceleration caused by a force. The acceleration is put directly into Euler’s equation. This is then used to calculate values iteratively in the table. Notice that the values start before zero so that initial conditions can be used. If the system was second-order we would need two previous values for the calculations.

Given the differential equation,

$$F = M\left(\frac{d}{dt}\right)v$$

we can create difference equations using simple methods.

$\left(\frac{d}{dt}\right)v = \frac{F}{M}$	first rearrange equation
$v(t+h) \approx v(t) + h\left(\frac{d}{dt}\right)v(t)$	put this in the Euler equation
$v(t+h) \approx v(t) + h\left(\frac{F(t)}{M}\right)$	finally substitute in known terms

We can now use the equation to estimate the system response. We will assume that the system is initially at rest and that a force of 1N will be applied to the 1kg mass for 4 seconds. After this time the force will rise to 2N. A time step of 2 seconds will be used.

i	t (sec)	F (N)	d/dt v_i	v_i
-1	-2	0	0	0
0	0	1	1	0
1	2	1	1	2
2	4	2	2	4
3	6	2	2	8
4	8	2	2	12
5	10	2	2	16
6	12	2	2	20
7	14	2	etc	etc
8	16	2		

Figure 92 Numerical integration example

Use first-order integration to solve the differential equation from 0 to 10 seconds with time steps of 1 second.

$$x' + 3x = 5$$

Figure 93 Drill problem: Numerically integrate the differential equation

An example of solving the previous example with a traditional programming language is shown in Figure 94. In this example the results will be written to a text file ’out.txt’. The solution iteratively integrates from 0 to 10 seconds with time steps of 0.1s. The force value is varied over the time period with ’if’ statements. The integration is done with a separate function.

```
double step(double, double, double);

int main(){
	double		h = 0.1,
			M = 1.0,
			F;
	FILE	*fp;
	double v,
		t;
	if( ( fp = fopen("out.txt", "w")) != NULL){
		v = 0.0;
		for( t = 0.0; t < 10.0; t += h ){
			if((t >= 0.0) && (t < 4.0)) F = 1.0;
			if(t > 4.0) F = 2.0;
			v = step(v, h, F/M);
			fprintf(fp, "%f, %f, %f\n", t, v, F, M);
		}
	}
	fclose(fp);
}

double step(double v, double h, double slope){
	double		v_new;
	v_new = v + h * slope;
	return v_new;
}
```

Figure 94	Solving state variable equations with a C program

```
double step(double, double, double);

public class Integrate extends Object
    public void main() {
        double      h = 0.1,
                    M = 1.0,
                    F;
        FileOut fp = new FileOut("out.txt");
        if(fp.writeStatus != fp.IO_EXCEPTION){
            double v = 0.0;
            for( double t = 0.0; t < 10.0; t += h ){
                if((t >= 0.0) && (t < 4.0)) F = 1.0;
                if(t > 4.0) F = 2.0;
                v = step(v, h, F/M);
                fp.printf(fp, "%f, %f, %f\n", t, v, F, M);
            }
            fp.close();
        }
        fclose(fp);
    }

    public double step(double v, double h, double slope){
        double      v_new;
        v_new = v + h * slope;
        return v_new;
    }
}
```

Figure 95 Solving state variable equations with a Java program

4.3.3 Taylor Series

First-order integration works well with smooth functions. But, when a highly curved function is encountered we can use a higher order integration equation. Recall the Taylor series equation shown in Figure 96 for approximating a function. Notice that the first part of the equation is identical to Euler's equation, but the higher order terms add accuracy.

$$x(t+h) = x(t) + h\left(\frac{d}{dt}\right)x(t) + \frac{1}{2!}h^2\left(\frac{d}{dt}\right)^2 x(t) + \frac{1}{3!}h^3\left(\frac{d}{dt}\right)^3 x(t) + \frac{1}{4!}h^4\left(\frac{d}{dt}\right)^4 x(t) + \ldots$$

Figure 96 The Taylor series

An example of the application of the Taylor series is shown in Figure 97. Given the differential equation, we must first determine the derivatives and substitute these into Taylor's equation. The resulting equation is then used to iteratively calculate values.

Given $$x' - x = 1 + e^{-20t} + t^3$$

We can write, $$\left(\frac{d}{dt}\right)x = 1 + e^{-20t} + t^3 + x$$
$$\left(\frac{d}{dt}\right)^2 x = -e^{-t} + 3t^2$$
$$\left(\frac{d}{dt}\right)^3 x = e^{-t} + 6t$$

In the Taylor series this becomes,

$$x(t+h) = x(t) + h(1 + e^{-20t} + t^3 + x) + \frac{1}{2!}h^2(-e^{-t} + 3t^2) + \frac{1}{3!}h^3(e^{-t} + 6t)$$

Thus $x_0 = 0$

$h = 0.1$

t (s)	x(t)
0	0
0.1	0
0.2	
0.3	
0.4	
0.5	
0.6	
0.7	
0.8	
0.9	

e.g., for t=0.1 $$x(0+0.1) = 0 + 0.1(2) + \frac{1}{2!}(0.1)^2(-1) + \frac{1}{3!}(0.1)^3(1) =$$

Figure 97 Integration using the Taylor series method

Recall that the state variable equations are first-order equations. But, to obtain accuracy the Taylor method also requires higher order derivatives, thus making is unsuitable for use with state variable equations.

4.3.4 Runge-Kutta Integration

First-order integration provides reasonable solutions to differential equations. That accuracy can be improved by using higher order derivatives to compensate for function curvature. The Runge-Kutta technique uses first-order differential equations (such as state equations) to estimate the higher order derivatives, thus providing higher accuracy without requiring other than first-order differential equations.

The equations in Figure 98 are for fourth order Runge-Kutta integration. The function 'f()' is the state equation or state equation vector. For each time step the values 'F1' to 'F4' are calculated in sequence and then used in the final equation to find the next value. The 'F1' to 'F4' values are calculated at different time steps, and values from previous time steps are used to 'tweak' the estimates of the later states. The final summation equation has a remote similarity to Euler's equation. Notice that the two central values in time are more heavily weighted.

$$F_1 = hf(t, x)$$

$$F_2 = hf\left(t + \frac{h}{2}, x + \frac{F_1}{2}\right)$$

$$F_3 = hf\left(t + \frac{h}{2}, x + \frac{F_2}{2}\right)$$

$$F_4 = hf(t + h, x + F_3)$$

$$x(t + h) = x(t) + \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4)$$

where,

x = the state variables
f = the differential function or (d/dt) x
t = current point in time
h = the time step to the next integration point

Figure 98 Fourth order Runge-Kutta integration

An example of a Runge-Kutta integration calculation is shown in Figure 99. The solution begins by putting the state equations in matrix form and defining initial conditions. After this, the four integrating factors are calculated. Finally, these are combined to get the final value after one time step. The number of calculations for a single time step should make obvious the necessity of computers and calculators.

$$\frac{d}{dt}x = v \qquad y = 2 \text{ (assumed input)}$$

$$\frac{d}{dt}v = 3 + 4v + 5y \qquad v_0 = 1$$

$$x_0 = 3$$

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} y \\ 1 \end{bmatrix} \qquad h = 0.1$$

For the first time step,

$$F_1 = 0.1\left(\begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = 0.1\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 13 \end{bmatrix}\right) = \begin{bmatrix} 0.1 \\ 1.7 \end{bmatrix}$$

$$F_2 = 0.1\left(\begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 3 + \frac{0.1}{2} \\ 1 + \frac{1.7}{2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0.185 \\ 2.04 \end{bmatrix}$$

$$F_3 = 0.1\left(\begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 3 + \frac{0.185}{2} \\ 1 + \frac{2.04}{2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0.202 \\ 2.108 \end{bmatrix}$$

$$F_4 = 0.1\left(\begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 3 + 0.202 \\ 1 + 2.108 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0.3108 \\ 2.5432 \end{bmatrix}$$

$$\begin{bmatrix} x_{i+1} \\ v_{i+1} \end{bmatrix} = \begin{bmatrix} x_i \\ v_i \end{bmatrix} + \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4)$$

$$\begin{bmatrix} x_{i+1} \\ v_{i+1} \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{1}{6}\left(\begin{bmatrix} 0.1 \\ 1.7 \end{bmatrix} + 2\begin{bmatrix} 0.185 \\ 2.04 \end{bmatrix} + 2\begin{bmatrix} 0.202 \\ 2.108 \end{bmatrix} + \begin{bmatrix} 0.3108 \\ 2.5432 \end{bmatrix}\right) = \begin{bmatrix} 3.1974667 \\ 3.0898667 \end{bmatrix}$$

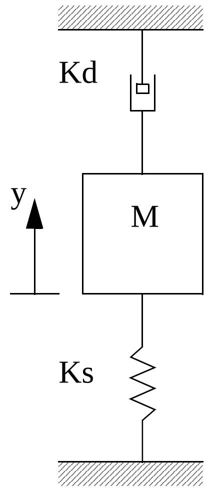
Figure 99 Runge-Kutta integration example

$$F = M\left(\frac{d}{dt}\right)^2 x$$

use,

x(0) = 1
v(0) = 2
h = 0.5 s
F = 10
M = 1

Figure 100 Drill problem: Integrate the acceleration function



Leave the solution in variable form, and then change values to see how the system response changes.

$$K_d = 10Nsm^{-1}$$

$$K_s = 10Nm^{-1}$$

$$M = 1kg$$

$$y_0 = 1 \qquad y_0' = 0$$

Figure 101 Drill problem: Integrate to find the system response

The program below is used to perform a Runge-Kutta integration of a mass-spring-damper system. XXXXXXXXXXXXXXXXXXXXXXXXX

```
//
// A program to do Runge Kutta integration of a mass spring damper system
//
#include <stdio.h>

voidmultiply(double, double[], double[]);
voidadd(double[], double[], double[]);
voidstep(double, double, double[]);
voidderivative(double, double[], double[]);

#defineSIZE      2        // the length of the state vector
#defineKs        1000     // the spring coefficient
#defineKd        10000    // the damping coefficient
#defineMass      10       // the mass coefficient
#defineForce     100      // the applied force

int main(){
    FILE   *fp;

    double h = 0.001;
    double t;
    int          j = 0;

    double X[SIZE];// create state variable list
    X[0] = 0;// set initial condition for x
    X[1] = 0;// set initial condition for v

    if( ( fp = fopen("out.txt", "w")) != NULL){
        fprintf(fp, "   t(s)      x         v \n\n");
        for( t = 0.0; t < 50.0; t += h ){
            step(t, h, X);
            if(j == 0) fprintf(fp, "%9.5f %9.5f %9.5f\n", t, X[0], X[1]);
            j++; if(j >= 10) j = 0;
        }
    }
    fclose(fp);
}
```

Figure 102 Runge-Kutta integration program

```
//
// First order integration done here (could be replaced with runge kutta)
//
void step(double t, double h, double X[]){
	double tmp[SIZE],
			dX[SIZE],
			F1[SIZE],
			F2[SIZE],
			F3[SIZE],
			F4[SIZE];
	// Calculate F1
	derivative(t, X, dX);
	multiply(h, dX, F1);

	// Calculate F2
	multiply(0.5, F1, tmp);
	add(X, tmp, tmp);
	derivative(t+h/2.0, tmp, dX);
	multiply(h, dX, F2);

	// Calculate F3
	multiply(0.5, F2, tmp);
	add(X, tmp, tmp);
	derivative(t+h/2.0, tmp, dX);
	multiply(h, dX, F3);

	// Calculate F4
	add(X, F3, tmp);
	derivative(t+h, tmp, dX);
	multiply(h, dX, F4);

	// Calculate the weighted sum
	add(F2, F3, tmp);
	multiply(2.0, tmp, tmp);
	add(F1, tmp, tmp);
	add(F4, tmp, tmp);
	multiply(1.0/6.0, tmp, tmp);
	add(tmp, X, X);
}
```

Figure 103 Runge-Kutta integration program (continued)

```
//
// State Equations Calculated Here
//
void derivative(double t, double X[], double dX[]){
	dX[0] = X[1];
	dX[1] = (-Ks/Mass)*X[0] + (-Kd/Mass)*X[1] + (Force/Mass);
}

//
// A subroutine to add vectors to simplify other equations
//
void add(double X1[], double X2[], double R[]){
	for(int i = 0; i < SIZE; i++) R[i] = X1[i] + X2[i];
}

//
// A subroutine to multiply a vector by a scalar to simplify other equations
//
void multiply(double X, double V[], double R[]){
	for(int i = 0; i < SIZE; i++) R[i] = X*V[i];
}
```

Figure 104 Runge-Kutta integration program (continued)

4.4 SYSTEM RESPONSE

In most cases the result of numerical analysis is graphical or tabular. In both cases details such as time constants and damped frequencies can be obtained by the same methods used for experimental analysis. In addition to these methods there is a technique that can determine the steady-state response of the system.

4.4.1 Steady-State Response

The state equations can be used to determine the steady-state response of a system by setting the derivatives to zero, and then solving the equations. Consider the example in Figure 105. The solution begins with a state variable matrix. (Note: this can also be done without the matrix also.) The derivatives on the left hand side are set to zero, and the equations are rearranged and solved with Cramer's rule.

Given the state variable form:

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K_s}{M} & -\frac{K_d}{M} \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{F}{M} \end{bmatrix}$$

Set the derivatives to zero

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K_s}{M} & -\frac{K_d}{M} \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{F}{M} \end{bmatrix}$$

Solve for x and v

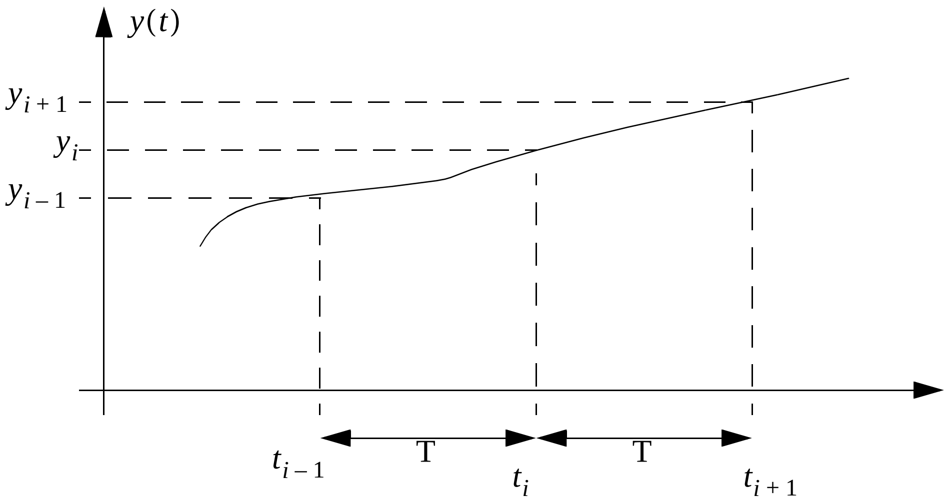
$$\begin{bmatrix} 0 & 1 \\ -\frac{K_s}{M} & -\frac{K_d}{M} \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{F}{M} \end{bmatrix}$$

$$x = \frac{\left|\begin{bmatrix} 0 & 1 \\ -\frac{F}{M} & -\frac{K_d}{M} \end{bmatrix}\right|}{\left|\begin{bmatrix} 0 & 1 \\ -\frac{K_s}{M} & -\frac{K_d}{M} \end{bmatrix}\right|} = \frac{\left(\frac{F}{M}\right)}{\left(\frac{K_s}{M}\right)} = \frac{F}{K_S} \qquad v = \frac{\left|\begin{bmatrix} 0 & 0 \\ -\frac{K_s}{M} & -\frac{F}{M} \end{bmatrix}\right|}{\left|\begin{bmatrix} 0 & 1 \\ -\frac{K_s}{M} & -\frac{K_d}{M} \end{bmatrix}\right|} = \frac{0}{\left(\frac{K_s}{M}\right)} = 0$$

Figure 105 Finding the steady-state response

4.5 DIFFERENTIATION AND INTEGRATION OF EXPERIMENTAL DATA

When doing experiments, data is often collected in the form of individual data points (not as complete functions). It is often necessary to integrate or differente these values. The basic equations for integrating and differentiating are shown in Figure 106. Given data points, y, collected at given times, t, we can integrate and differentiate using the given equations. The integral is basically the average height of the two points multiplied by the width to give an area, or integral. The first derivative is basically the slope between two points. The second derivative is the change in slope values for three points. In a computer based system the time points are often equally spaced in time, so the difference in time can be replaced with a sample period, T. Ideally the time steps would be as small as possible to increase the accuracy of the estimates.



$$\int_{t_{i-1}}^{t_i} y(t_i)dt \approx \left(\frac{y_i + y_{i-1}}{2}\right)(t_i - t_{i-1}) = \frac{T}{2}(y_i + y_{i-1})$$

$$\frac{d}{dt}y(t_i) \approx \left(\frac{y_i - y_{i-1}}{t_i - t_{i-1}}\right) = \left(\frac{y_{i+1} - y_i}{t_{i+1} - t_i}\right) = \frac{1}{T}(y_i - y_{i-1}) = \frac{1}{T}(y_{i+1} - y_i)$$

$$\left(\frac{d}{dt}\right)^2 y(t_i) \approx \frac{\frac{1}{T}(y_{i+1} - y_i) - \frac{1}{T}(y_i - y_{i-1})}{T} = \frac{-2y_i + y_{i-1} + y_{i+1}}{T^2}$$

Figure 106 Integration and differentiation using data points

4.6 ADVANCED TOPICS

4.6.1 Switching Functions

When analyzing a system, we may need to choose an input that is more complex than the inputs such as the step, ramp, sinusoidal and parabolic. The easiest way to do this is to use switching functions. Switching functions turn on (have a value of 1) when their arguments are greater than or equal to zero, or off (a value of 0) when the argument is negative. Examples of the use of switching functions are shown in Figure 107. By changing the values of the arguments we can change when a function turns on and off.

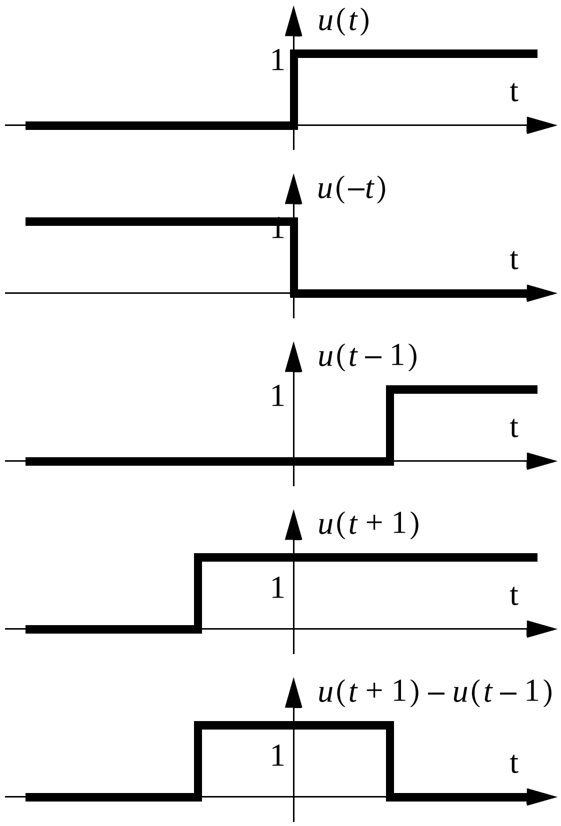
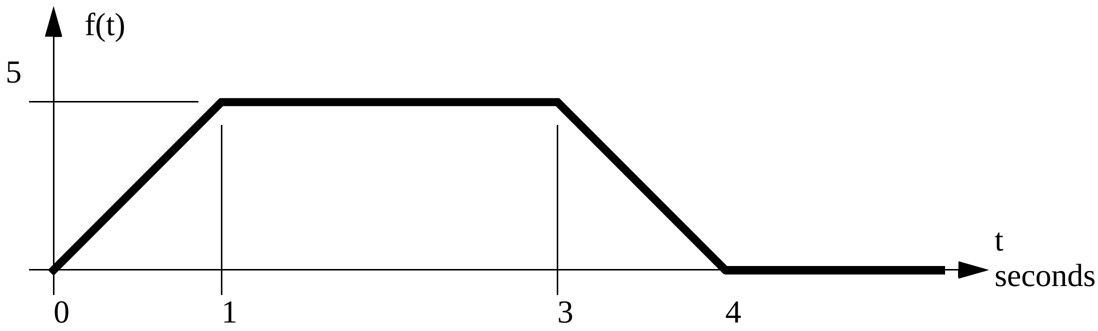


Figure 107 Switching function examples

These switching functions can be multiplied with other functions to create a complex function by turning parts of the function on or off. An example of a curve created with switching functions is shown in Figure 108.



$$f(t) = 5t(u(t)-(t-1)) + 5(u(t-1)-u(t-3)) + (20-5t)(u(t-3)-u(t-4))$$

or

$$f(t) = 5tu(t) - 5(t-1)u(t-1) - 5(t-3)u(t-3) + 5(t-4)u(t-4)$$

Figure 108 Switching functions to create a non-smooth function

The unit step switching function is available in Mathcad and makes creation of complex functions relatively trivial. Step functions are also easy to implement when writing computer programs, as shown in Figure 109.

For the function $f(t) = 5tu(t) - 5(t-1)u(t-1) - 5(t-3)u(t-3) + 5(t-4)u(t-4)$

```
double u(double t){
    if(t >= 0) return 1.0
    return 0.0;
}

double function(double t){
    double      f;

    f =     5.0 * t * u(t)
            - 5.0 * (t - 1.0) * u(t - 1.0)
            - 5.0 * (t - 3.0) * u(t - 3.0)
            + 5.0 * (t - 4.0) * u(t - 4.0);

    return f;
}
```

Figure 109 A subroutine implementing the example in Figure 108

4.6.2 Interpolating Tabular Data

In some cases we are given tables of numbers instead of equations for a system component. These can still be used to do numerical integration by calculating coefficient values as required, in place of an equation.

Tabular data consists of separate data points as seen in Figure 110. But, we may need values between the datapoints. A simple method for finding intermediate values is to interpolate with the "lever law". (Note: it is called this because of its’ similarity to the equation for a lever.) The table in the example only gives flow rates for a valve at 10 degree intervals, but we want flow rates at 46 and 23 degrees. A simple application of the lever law gives approximate values for the flow rates.

valve angle (deg.)	flow rate (gpm)
0	0
10	0.1
20	0.4
30	1.2
40	2.0
50	2.3
60	2.4
70	2.4
80	2.4
90	2.4

Given a valve angle of 46 degrees the flow rate is,

$$Q = 2.0 + (2.3 - 2.0)\left(\frac{46 - 40}{50 - 40}\right) = 2.18$$

Given a valve angle of 23 degrees the flow rate is,

$$Q = 0.4 + (1.2 - 0.4)\left(\frac{23 - 20}{30 - 20}\right) = 0.64$$

Figure 110 Using tables of values to interpolate numerical values using the lever law

The subroutine in Figure 111 was written to return the numerical value for the data table in Figure 110. In the subroutine the tabular data is examined to find the interval that the flow rate value falls between. Once this is found the valve angle is calculated as the ratio between the two known values.

```
#define     SIZE    10;
double      data[SIZE][2] = {{0.0, 0.0},
                          {10.0,  0.1},
                          {20.0,  0.4},
                          {30.0,  1.2},
                          {40.0,  2.0},
                          {50.0,  2.3},
                          {60.0,  2.4},
                          {70.0,  2.4},
                          {80.0,  2.4},
                          {90.0,  2.4}};

double angle(double rate){
    for(int i = 0; i < SIZE-1; i++){
        if((rate >= data[i][0]) && (rate <= data[i+1][0]){
            return  (data[i+1][1] - data[i][1])
                    * (rate - data[i][0]) / (data[i+1][0] - data[i][0])
                    + data[i][1];
        }
    }
    printf("ERROR: rate out of range\n");
    exit(1);
}
```

Figure 111 A tabular interpolation subroutine example

4.6.3 Modeling Functions with Splines

When more accuracy is required smooth curves can be fitted to interpolate points. These curves are known as splines. There are multiple methods for creating splines, but the simplest is to use a polynomial fitted to a set of points.

The example in Figure 112 shows a spline curve being fitted for three data points. In this case a second order polynomial is used. The three data points are written out as equations, and then put into matrix form, using the coefficients as the unknown values. The matrix is then solved to obtain the coefficient values for the final equation. This equation can then be used to build a mathematical model of the system.

The datapoints below might have been measured for the horsepower of an internal combustion engine on a dynamometer.

S (RPM)	P (HP)
1000	105
4000	205
6000	110

In this case there are three datapoints, so we can fit the curve with a second (3-1) order polynomial. The major task is to calculate the coefficients so that the curve passes through all of the given points.

$$P(S) = AS^2 + BS + C$$

Data values can be substituted into the equation,

$$P(1000) = A1000^2 + B1000 + C = 105$$
$$P(4000) = A4000^2 + B4000 + C = 205$$
$$P(6000) = A6000^2 + B6000 + C = 110$$

This can then be put in matrix form to find the coefficients,

$$\begin{bmatrix} 1000^2 & 1000 & 1 \\ 4000^2 & 4000 & 1 \\ 6000^2 & 6000 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 105 \\ 205 \\ 110 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1000000 & 1000 & 1 \\ 16000000 & 4000 & 1 \\ 36000000 & 6000 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 105 \\ 205 \\ 110 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 6.667 \times 10^{-8} & -1.667 \times 10^{-7} & 1.000 \times 10^{-7} \\ -6.667 \times 10^{-4} & 1.167 \times 10^{-3} & -5.000 \times 10^{-4} \\ 1.600 & -1.000 & 0.400 \end{bmatrix} \begin{bmatrix} 105 \\ 205 \\ 110 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} -1.617 \times 10^{-5} \\ 0.114 \\ 7.000 \end{bmatrix}$$

$$P(S) = (-1.617 \times 10^{-5})S^2 + 0.114S + 7.000$$

Note: in this example the inverse matrix is used, but other methods for solving systems of equations are equally valid. If the equations were simpler, substitution might have been a better approach.

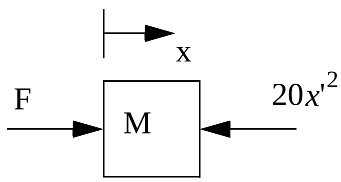
Figure 112 A spline fitting example

The order of the polynomial should match the number of points. Although, as the number of points increases, the shape of the curve will become less smooth. A common way for dealing with this problem is to fit the spline to a smaller number of points and then verify that it matches the remaining points.

4.6.4 Non-Linear Elements

Despite our deepest wishes for simplicity, most systems contain non-linear components. In the last chapter we looked at dealing with non-linearities by linearizing them locally. Numerical techniques will handle non-linearities easily, but smaller time steps are required for accuracy.

Consider the mass and an applied force shown in Figure 113. As the mass moves an aerodynamic resistance force is generated that is proportional to the square of the velocity. This results in a non-linear differential equation. This equation can be numerically integrated using a technique such as Runge-Kutta. Note that the state equation matrix form cannot be used because it requires linear equations.



$$\sum F_x = F - 20x'^2 = Mx''$$

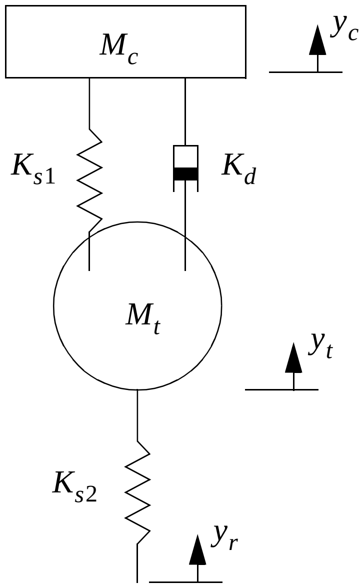
$$x' = v \qquad (1)$$

$$v' = \frac{-20}{M}v^2 + \frac{F}{M} \qquad (2)$$

Figure 113 Developing state equations for a non-linear system

4.7 CASE STUDY

Consider the simplified model of a car suspension shown in Figure 114. The model distributes the vehicle weight over four tires with identical suspensions, so the mass of the vehicle is divided by four. In this model the height of the road will change and drive the tire up, or allow it to drop down. The tire acts as a stiff spring, with little deflection. The upper spring and damper are the vibration isolation units. The damper has been designed to stiffen as the damper is compressed. The given table shows how the damping coefficient varies with the amount of compression.



$$K_{s1} = 50000\frac{N}{m}$$

$$K_{s2} = 200000\frac{N}{m}$$

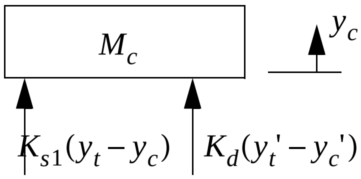
$$M_c = 400Kg$$

$$M_t = 20Kg$$

L (cm)	Kd (Ns/m)
30	1000
0	1600
-20	2000
-30	2250

Figure 114 A model of a car suspension system

For our purposes we will focus only on the translation of the tire, and ignore its rotational motion. The differential equations describing the system are developed in Figure 115.



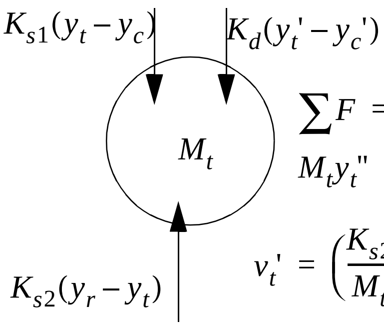
$$\sum F = K_{s1}(y_t - y_c) + K_d(y_t' - y_c') - M_c g = M_c y_c''$$

$$M_c y_c'' = (K_{s1})y_t + (-K_{s1})y_c + (K_d)y_t' + (-K_d)y_c' - M_c g$$

$$\frac{d}{dt}y_c = v_c \qquad (1)$$

$$\frac{d}{dt}y_t = v_t \qquad (2)$$

$$v_c' = \left(\frac{K_{s1}}{M_c}\right)y_t + \left(\frac{-K_{s1}}{M_c}\right)y_c + \left(\frac{K_d}{M_c}\right)v_t + \left(\frac{-K_d}{M_c}\right)v_c + (-g) \qquad (3)$$



$$\sum F = K_{s2}(y_r - y_t) - K_{s1}(y_t - y_c) - K_d(y_t' - y_c') - M_t g = M_t y_t''$$

$$M_t y_t'' = (K_{s2})y_r + (-K_{s2} - K_{s1})y_t + (K_{s1})y_c + (-K_d)y_t' + (K_d)y_c' - M_t g$$

$$v_t' = \left(\frac{K_{s2}}{M_t}\right)y_r + \left(\frac{-K_{s2} - K_{s1}}{M_t}\right)y_t + \left(\frac{K_{s1}}{M_t}\right)y_c + \left(\frac{-K_d}{M_t}\right)v_t + \left(\frac{K_d}{M_t}\right)v_c + (-g) \qquad (4)$$

Figure 115 Differential and state equations for the car suspension system

The damping force must be converted from a tabular form to equation form. This is done in Figure 116.

There are four data points, so a third order polynomial is required.

L (cm)	Kd (Ns/m)
30	1000
0	1600
-20	2000
-30	2250

$$K_d(L) = AL^3 + BL^2 + CL + D$$

The four data points can now be written in equation form, and then put into matrix form.

$$1000 = A(0.3)^3 + B(0.3)^2 + C(0.3) + D$$
$$1600 = A(0)^3 + B(0)^2 + C(0) + D$$
$$2000 = A(-0.2)^3 + B(-0.2)^2 + C(-0.2) + D$$
$$2250 = A(-0.3)^3 + B(-0.3)^2 + C(-0.3) + D$$

$$\begin{bmatrix} 0.027 & 0.09 & 0.3 & 1 \\ 0 & 0 & 0 & 1 \\ -0.008 & 0.04 & -0.2 & 1 \\ -0.027 & 0.09 & -0.3 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 1000 \\ 1600 \\ 2000 \\ 2250 \end{bmatrix}$$

The matrix can be solved to find the coefficients, and the final equation written.

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -2778 \\ 277.8 \\ -1833 \\ 1600 \end{bmatrix}$$

$$K_d(L) = (-2778)L^3 + (277.8)L^2 + (-1833)L + 1600$$

Figure 116 Fitting a spline to the damping values

The system is to be tested for overall deflection when exposed to obstacles on the road. For the initial conditions we need to find the resting heights for the tire and car body. This can be done by setting the accelerations and velocities to zero, and finding the resulting heights.

The initial accelerations and velocities are set to zero, assuming the car has settled to a steady state height. This then yields equations that can be used to calculate the initial deflections. Assume the road height is also zero to begin with.

$$0 = \left(\frac{K_{s1}}{M_c}\right)y_t + \left(\frac{-K_{s1}}{M_c}\right)y_c + \left(\frac{K_d}{M_c}\right)0 + \left(\frac{-K_d}{M_c}\right)0 + (-g)$$

$$y_c = y_t - g\frac{M_c}{K_{s1}}$$

$$0 = \left(\frac{K_{s2}}{M_t}\right)0 + \left(\frac{-K_{s2} - K_{s1}}{M_t}\right)y_t + \left(\frac{K_{s1}}{M_t}\right)y_c + \left(\frac{-K_d}{M_t}\right)0 + \left(\frac{K_d}{M_t}\right)0 + (-g)$$

$$gM_t = (-K_{s2} - K_{s1})y_t + (K_{s1})y_c$$

$$gM_t = (-K_{s2} - K_{s1})y_t + (K_{s1})\left(y_t - g\frac{M_c}{K_{s1}}\right) \qquad y_t = -g\frac{M_c + M_t}{K_{s2}}$$

$$y_c = -g\frac{M_c + M_t}{K_{s2}} - g\frac{M_c}{K_{s1}} \qquad y_c = -g\left(\frac{M_c + M_t}{K_{s2}} + \frac{M_c}{K_{s1}}\right)$$

Figure 117 Calculation of initial deflections

The resulting calculations can then be written in a computer program for analysis, as shown in Figure 118.

```
#include <stdio.h>
#include <math.h>

#defineSIZE4        // define state variables
#definey_c 0
#definey_t 1
#definev_c 2
#definev_t 3

#defineN_step10000// number of steps
#defineh_step0.001// define step size

#defineKs1 50000.0          // define component values
#defineKs2 200000.0
#defineMc  400.0
#defineMt  20.0
#definegrav9.81

void integration_step(double h, double state[], double derivative[]){
      for(int i = 0; i < SIZE; i++) state[i] += h * derivative[i];
}

double damper(double L){
      return (-2778*L*L*L + 277.8*L*L - 1833*L + 1600);
}

double y_r(double t){
      // return 0.0;    // a zero input to test the initial conditions
      // return 0.2 * sin(t);// a sinusoidal oscillation
       return 0.2;// a step function
      // return 0.2 * t;// a ramp function
}
```

Figure 118 Program for numerical analysis of suspension system

```
void d_dt(double t, double state[], double derivative[]){
	double Kd;
	Kd = damper(state[y_c] - state[y_t]);
	derivative[y_c] = state[v_c];
	derivative[y_t] = state[v_t];
	derivative[v_c] = (Ks1/Mc)*state[y_t] - (Ks1/Mc)*state[y_c]
				+ (Kd/Mc)*state[v_t] - (Kd/Mc)*state[v_c] - grav;
	derivative[v_t] = (Ks2/Mt)*y_r(t) - ((Ks2+Ks1)/Mt)*state[y_t]
				+ (Ks1/Mt)*state[y_c] - (Kd/Mt)*state[v_t]
				+ (Kd/Mt)*state[v_c] - grav;
}

main(){
	double state[SIZE];
	double derivative[SIZE];
	FILE		*fp_out;
	double t;
	int		i;

	state[y_c] = - grav *  ( (Mc/Ks1) + (Mt + Mc)/Ks2 ); // initial values
	state[y_t] = - grav * (Mt + Mc) / Ks2;
	state[v_c] = 0.0;
	state[v_t] = 0.0;

	if((fp_out = fopen("out.txt", "w")) != NULL){ // open the file
		fprintf(fp_out, " t   Yc     Yt     Vc     Vt  \n");
		for(t = 0.0, i = 0; i < N_step; i++, t += h_step){
			if((i % 100) == 0) fprintf(fp_out, "%f    %f   %f    %f   %f \n",
				t, state[y_c], state[y_t], state[v_c], state[v_t]);
			d_dt(t, state, derivative);
			integration_step(h_step, state, derivative);
		}
	} else {
		printf("ERROR: Could not open file \n");
	}
	fclose(fp_out);
}
```

Figure 119 Program for numerical analysis of suspension system (continued)

This program was then used to test various design cases by selecting input types for changes in the road height, and then calculating how the tire and vehicle heights would change as a result. Some of these results are seen in Figure 120. These results were obtained by running the program, and then graphing the results in a spreadsheet program. The input of zero for the road height was used to test the program. As shown the height of the vehicle changes, indicating that the initial height calculations are correct, and the model is stable. The step function shows some oscillations that settle out to a stable final value. The oscillation is relatively slow, and is fully transmitted to the automobile. The ramp function shows that the car follows the rise of the slope with small transient effects at the start.

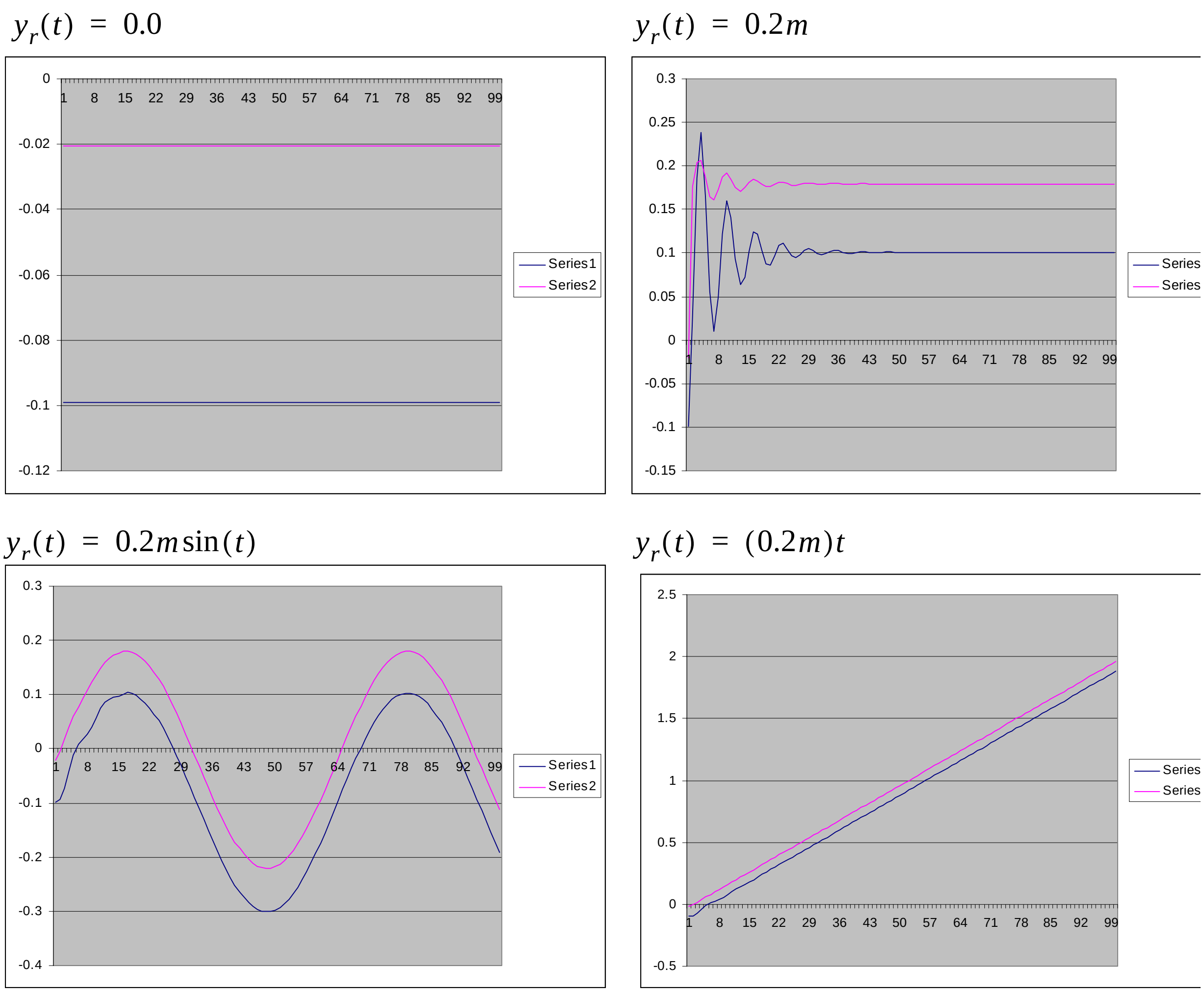


Figure 120 Graphs of simulation results

4.8 SUMMARY

• State variable equations are used to reduced to first order differential equations.
• First order equations can be integrated numerically.
• Higher order integration, such as Runge-Kutta increase the accuracy.
• Switching functions allow functions terms to be turned on and off to provide more complex function.

4.9 PRACTICE PROBLEMS

1. a) Put the differential equations given below in state variable form.
b) Put the state equations in matrix form

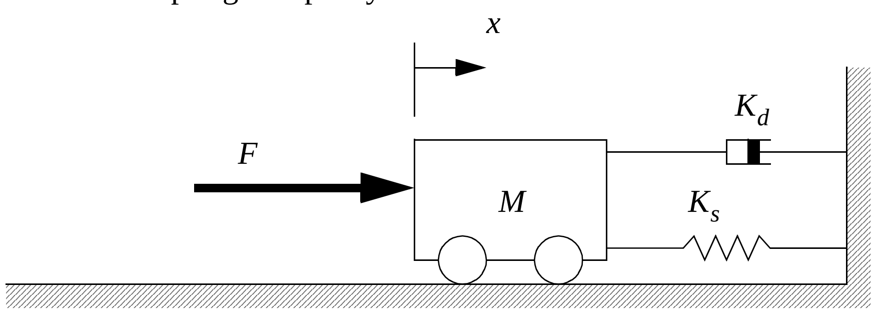
$$y_1'' + 2y_1' + 3y_1 + 4y_2' + 5y_2 + 6y_3' + t + F = 0$$

$$y_1' + 7y_1 + 8y_2' + 9y_2 + 10y_3'' + 11y_3 + 5\cos(5t) = 0$$

$$y_1' + 12y_2'' + 13y_3' = 0$$

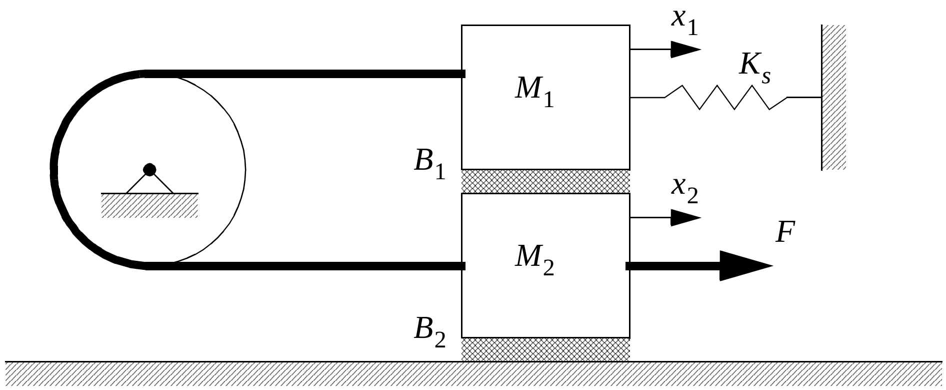
$y_1, y_2, y_3 = $ outputs
$F = $ input

2. Develop state equations for the mass-spring-damper system below.

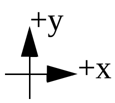


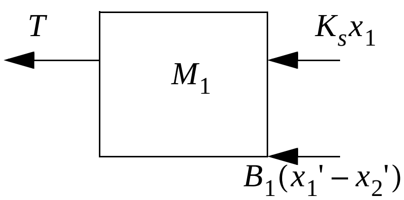
3. The system below is comprised of two masses. There is viscous damping between the masses and between the bottom mass and the floor. The masses are also connected with a cable that is run over a massless and frictionless pulley. Write the differential equations for the system, and

put them in state variable form.



(ans.



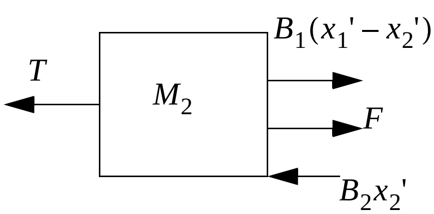


$$\sum F = -T - K_s x_1 - B_1(x_1' - x_2') = M_1 x_1''$$

$$v_1 = x_1'$$

$$v_2 = x_2'$$

$$v_1' = x_1\left(\frac{-K_s}{M_1}\right) + v_1\left(\frac{-B_1}{M_1}\right) + v_2\left(\frac{B_1}{M_1}\right) + \left(\frac{-T}{M_1}\right)$$



$$\sum F = -T + B_1(x_1' - x_2') + F - B_2 x_2' = M_2 x_2''$$

$$v_2' = v_1\left(\frac{B_1}{M_2}\right) + v_2\left(\frac{-B_1 - B_2}{M_2}\right) + \left(\frac{-T + F}{M_2}\right)$$

4. Given the differential equation below integrate the values numerically (show all work) for the first ten seconds in 1 second intervals. Assume the initial value of x is 1. You may use first order or Runge Kutta integration.

$$x' + 0.25x = 3$$

(ans.

$$x' = 3 - 0.25x$$

$$x(t+h) = x(t) + h\left(\frac{d}{dt}x(t)\right)$$

$$x(t+h) = x(t) + 1(3 - 0.25x(t))$$

$$x(t+h) = 0.75x(t) + 3$$

t	x	x'
0	1	2.75
1	3.75	2.06
2	5.81	1.55
3	7.36	1.16
4	8.52	0.870
5	9.39	0.652
6	10.0	0.489
7	10.5	0.367
8	10.9	0.275
9	11.2	0.206
10	11.4	

5. Do a first order numerical integration of the derivative below from 0 to 10 seconds in one second step. Assume the system starts at rest.

$$\frac{d}{dt}x(t) = 5(t-4)^2$$

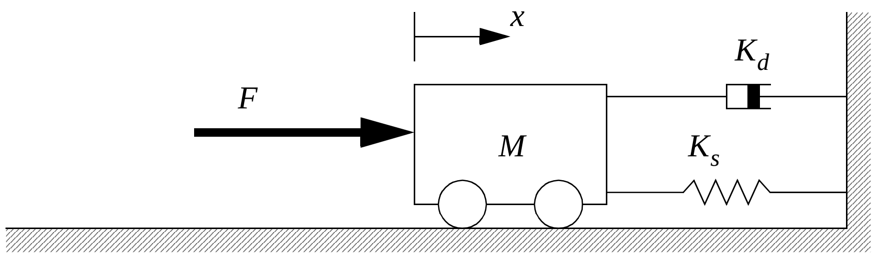
6. Given the following differential equation and initial conditions, draw a sketch of the first 5 seconds of the output response. The input is a step function that turns on at t=0. Use at least two different methods, and compare the results.

$$0.5V_o'' + 0.6V_o' + 2.1V_o = 3V_i + 2$$

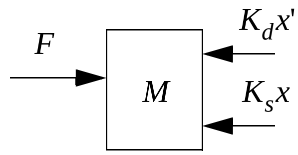
initial conditions $V_i = 5V$, $V_o = 0$, $V_o' = 1$

7. a) For the mass-spring-damper system below solve the differential equation as a function of time. Assume the system starts at rest and undeflected. b) Also solve the problem using your calculator (and state equations) to verify your solution. Sketch the results.

$K_s = 10$
$K_d = 10$
$M = 10$
$F = 10$



(ans.



$$\sum F = F - K_d x' - K_s x = Mx''$$
$$Mx'' + K_d x' + K_s x = F$$
$$10x'' + 10x' + 10x = 10$$
$$x'' + x' + x = 1$$

homogeneous:

$$x'' + x' + x = 0$$
$$A^2 + A + 1 = 0$$
$$A = \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2(1)} = -0.5 \pm j\frac{\sqrt{3}}{2}$$
$$x_h = C_1 e^{-0.5t} \cos\left(\frac{\sqrt{3}}{2}t + C_2\right)$$

particular:

$$x_p = B$$
$$0 + 0 + B = 1$$
$$x_p = 1$$

initial conditions:

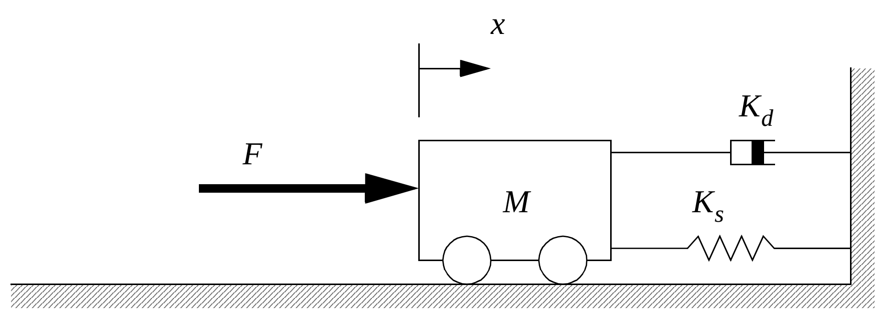
$$x = x_h + x_p = C_1 e^{-0.5t} \cos\left(\frac{\sqrt{3}}{2}t + C_2\right) + 1$$
$$x' = -\frac{\sqrt{3}}{2} C_1 e^{-0.5t} \sin\left(\frac{\sqrt{3}}{2}t + C_2\right) - 0.5 C_1 e^{-0.5t} \cos\left(\frac{\sqrt{3}}{2}t + C_2\right)$$
$$x'(0) = -\frac{\sqrt{3}}{2} C_1 \sin(C_2) - 0.5 C_1 \cos(C_2) = 0$$
$$-\frac{\sqrt{3}}{2} \sin(C_2) = 0.5 \cos(C_2)$$
$$\tan(C_2) = \frac{-1}{\sqrt{3}} \qquad C_2 = \operatorname{atan}\left(\frac{-1}{\sqrt{3}}\right) = -0.5236$$
$$x(0) = C_1 \cos(-0.5236) + 1 = 0$$
$$C_1 = \frac{-1}{\cos(-0.5236)} = -1.155$$

$$x = -1.155 e^{-0.5t} \cos\left(\frac{\sqrt{3}}{2}t - 0.5236\right) + 1$$

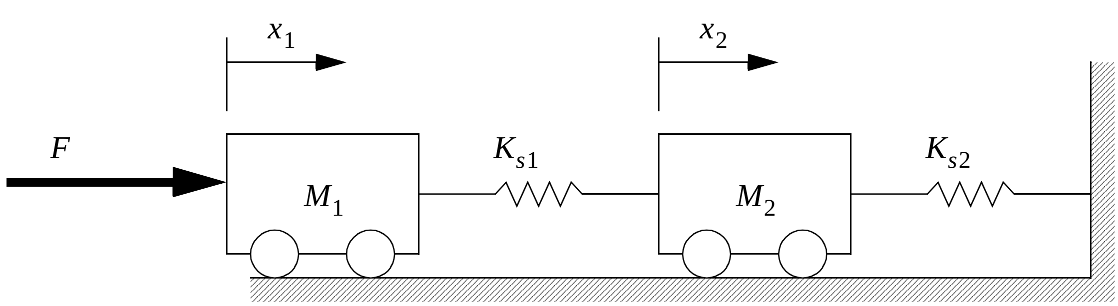
8. The mechanical system below is a mass-spring-damper system. A force ’F’ of 100N is applied to the 10Kg cart at time t=0s. The motion is resisted by the spring and damper. The spring coefficient is 1000N/m, and the damping coefficient is to be determined. Follow the steps below to develop a solution to the problem. Assume the system always starts undeflected and

at rest.

a) Develop the differential equation for the system.
b) Solve the differential equation using damping coefficients of 100Ns/m and 10000Ns/m. Draw a graph of the results.
c) Develop the state equations for the system.
d) Solve the system with a first order numerical analysis using Mathcad for damping coefficients of 100Ns/m and 10000Ns/m. Draw a graph of the results.
e) Solve the system with a Runge Kutta numerical analysis using Mathcad for damping coefficients of 100Ns/m and 10000Ns/m. Draw a graph of the results.
f) Write a computer program (in C, Java or Fortran) to do the Runge Kutta numerical integration in step e). Draw a graph of the results.
g) Compare all of the solutions found in the previous steps.
h) Select a damper value to give an overall system damping coefficient of 1. Verify the results by numerically integrating.



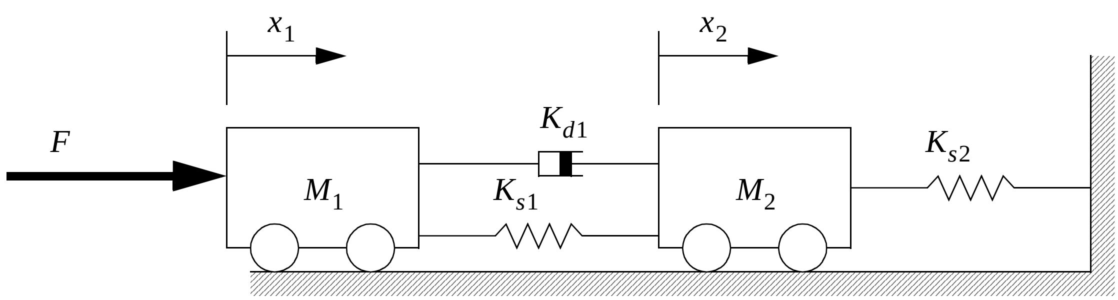
9. For the mechanism illustrated in the figure below:



a. Find the differential equations for the system. Consider the effects of gravity.
b. Put the equations in state variable form.
c. Put the equations in state variable matrices.
d. Use your calculator to find values for x_1 and x_2 over the first 10 seconds using 1/2 second intervals. Use the values, $K_1=K_2=100N/m$, $B=10Nm/s$, $M_1=M_2=1kg$. Assume that the system starts at rest, and the springs are undeformed initially.
e. Use Mathcad to plot the values for the first 10 seconds.
f. Use working model to find the values for the first 10 seconds.
g. Use Mathcad and the Runge-Kutta method to find the first 10 seconds using half second intervals.

h. Repeat step g. using the first-order approximation method.

10. For the mechanism shown in the figure below:



a. Find the differential equations for the system.
b. Put the equations in state variable form.
c. Put the equations in state variable matrices.
d. Use your calculator to find values for x_1 and x_2 over the first 10 seconds using 1/2 second intervals. Use the values, $K_{s1}=K_{s2}=100$N/m, $K_{d1}=10$Nm/s, $M_1=M_2=1$kg. Assume that the system starts at rest, and the springs are undeformed initially.
e. Use Mathcad to plot the values for the first 10 seconds.
f. Use working model to find the values for the first 10 seconds.
g. Use Mathcad and the Runge-Kutta calculations in the notes to find the first 10 seconds using half second intervals.
h. Repeat step g. using the first-order approximation method.

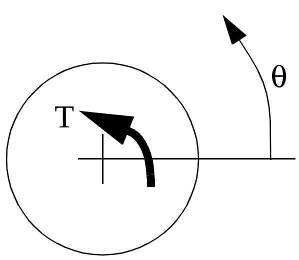
5. ROTATION

Topics:
- Basic laws of motion
- Inertia, springs, dampers, levers, gears and belts
- Design cases

Objectives:
- To be able to develop and analyze differential equations for rotational systems.

5.1 INTRODUCTION

The equations of motion for a rotating mass are shown in Figure 121. Given the angular position, the angular velocity can be found by differentiating once, the angular acceleration can be found by differentiating again. The angular acceleration can be integrated to find the angular velocity, the angular velocity can be integrated to find the angular position. The angular acceleration is proportional to an applied torque, but inversely proportional to the mass moment of inertia.



equations of motion

$$\omega = \left(\frac{d}{dt}\right)\theta \qquad (1)$$

$$\alpha = \left(\frac{d}{dt}\right)\omega = \left(\frac{d}{dt}\right)^2\theta \qquad (2)$$

OR

$$\theta(t) = \int \omega(t)dt = \iint \alpha(t)dtdt \qquad (3)$$

$$\omega(t) = \int \alpha(t)dt \qquad (4)$$

$$\alpha(t) = \frac{T(t)}{J_M} \qquad (5)$$

where,

θ, ω, α = position, velocity and acceleration
J_M = second mass moment of inertia of the body
T = torque applied to body

Figure 121 Basic properties of rotation

Note: A ’torque’ and ’moment’ are equivalent in terms of calculations. The main difference is that ’torque’ normally refers to a rotating moment.

Given the initial state of a rotating mass, find the state 5 seconds later.

$$\theta = 1rad \qquad \omega = 2\frac{rad}{s} \qquad \alpha = 3\frac{rad}{s^2}$$

Figure 122 Drill problem: Find the position with the given conditions

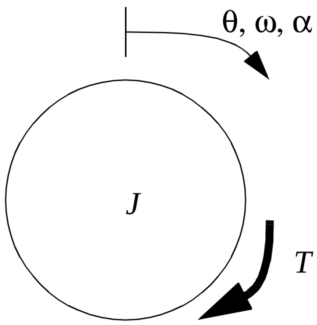
5.2 MODELING

Free Body Diagrams (FBDs) are required when analyzing rotational systems, as they were for translating systems. The force components normally considered in a rotational system include,

- inertia - opposes acceleration and deceleration
- springs - resist deflection
- dampers - oppose velocity
- levers - rotate small angles
- gears and belts - change rotational speeds and torques

5.2.1 Inertia

When unbalanced torques are applied to a mass it will begin to accelerate, in rotation. The sum of applied torques is equal to the inertia forces shown in Figure 123.



$$\sum T = J_M \alpha \qquad (6)$$

$$J_M = I_{xx} + I_{yy} \qquad (7)$$

$$I_{xx} = \int y^2 dM \qquad (8)$$

$$I_{yy} = \int x^2 dM \qquad (9)$$

Note: The 'mass' moment of inertia will be used when dealing with acceleration of a mass. Later we will use the 'area' moment of inertia for torsional springs.

Figure 123 Summing moments and angular inertia

The mass moment of inertia determines the resistance to acceleration. This can be calculated using integration, or found in tables. When dealing with rotational acceleration it is important to use the mass moment of inertia, not the area moment of inertia.

The center of rotation for free body rotation will be the centroid. Moment of inertia values are typically calculated about the centroid. If the object is constrained to rotate about some point, other than the centroid, the moment of inertia value must be recalculated. The parallel axis theorem provides the method to shift a moment of inertia from a centroid to an arbitrary center of rotation, as shown in Figure 124.

$$J_M = \tilde{J}_M + Mr^2$$

where,

J_M = mass moment about the new point

$\tilde{J}_M$ = mass moment about the centroid

M = mass of the object

r = distance from the centroid to the new point

Figure 124 Parallel axis theorem for shifting a mass moment of inertia

$$J_A = \tilde{J}_A + Ar^2$$

where,

J_A = area moment about the new point

$\tilde{J}_A$ = area moment about the centroid

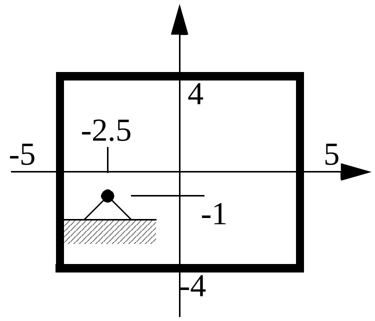
A = mass of the object

r = distance from the centroid to the new point

Figure 125 Parallel axis theorem for shifting a area moment of inertia

Aside: If forces do not pass through the center of an object, it will rotate. If the object is made of a homogeneous material, the area and volume centroids can be used as the center. If the object is made of different materials then the center of mass should be used for the center. If the gravity varies over the length of the (very long) object then the center of gravity should be used.

An example of calculating a mass moment of inertia is shown in Figure 126. In this problem the density of the material is calculated for use in the integrals. The integrals are then developed using slices for the integration element dM. The integrals for the moments about the x and y axes, and then added to give the polar moment of inertia. This is then shifted from the centroid to the new axis using the parallel axis theorem.



The rectangular shape to the right is constrained to rotate about point A. The total mass of the object is 10kg. The given dimensions are in meters. Find the mass moment of inertia.

First find the density and calculate the moments of inertia about the centroid.

$$\rho = \frac{10Kg}{2(5m)2(4m)} = 0.125Kgm^{-2}$$

$$I_{xx} = \int_{-4}^{4} y^2 dM = \int_{-4}^{4} y^2 \rho 2(5m) dy = 1.25Kgm^{-1} \frac{y^3}{3}\bigg|_{-4}^{4}$$

$$\therefore \quad = 1.25Kgm^{-1}\left(\frac{(4m)^3}{3} - \frac{(-4m)^3}{3}\right) = 53.33Kgm^2$$

$$I_{yy} = \int_{-5}^{5} x^2 dM = \int_{-5}^{5} x^2 \rho 2(4m) dx = 1Kgm^{-1} \frac{x^3}{3}\bigg|_{-5}^{5}$$

$$\therefore \quad = 1Kgm^{-1}\left(\frac{(5m)^3}{3} - \frac{(-5m)^3}{3}\right) = 83.33Kgm^2$$

$$J_M = I_{xx} + I_{yy} = 53.33Kgm^2 + 83.33Kgm^2 = 136.67Kgm^2$$

The centroid can now be shifted to the center of rotation using the parallel axis theorem.

$$J_M = \tilde{J}_M + Mr^2 = 136.67Kgm^2 + (10Kg)((-2.5m)^2 + (-1m)^2) = 136.67Kgm^2$$

Figure 126 Mass moment of inertia example

The rectangular shape to the right is constrained to rotate about point A. The total mass of the object is 10kg. The given dimensions are in meters. Find the mass moment of inertia WITHOUT using the parallel axis theorem.

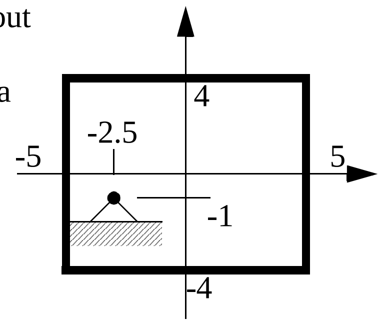
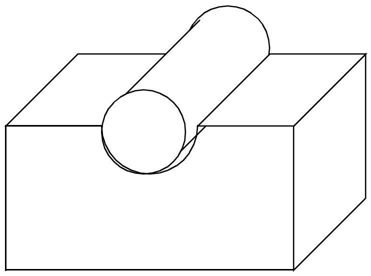


Figure 127 Drill problem: Mass moment of inertia calculation

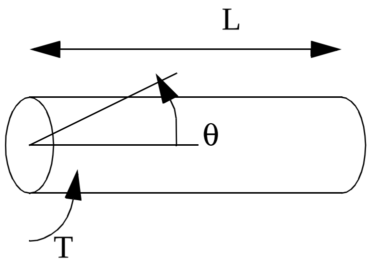


The 20cm diameter 10 kg cylinder to the left is sitting in a depression that is effectively frictionless. If a torque of 10 Nm is applied for 5 seconds, what will the angular velocity be?

Figure 128 Drill problem: Find the velocity of the rotating shaft

5.2.2 Springs

Twisting a rotational spring will produce an opposing torque. This torque increases as the deformation increases. A simple example of a solid rod torsional spring is shown in Figure 129. The angle of rotation is determined by the applied torque, T, the shear modulus, G, the area moment of inertia, JA, and the length, L, of the rod. The constant parameters can be lumped into a single spring coefficient similar to that used for translational springs.



$$T = \left(\frac{J_A G}{L}\right)\theta \quad (8)$$

$$T = -K_S(\Delta\theta) \quad (9)$$

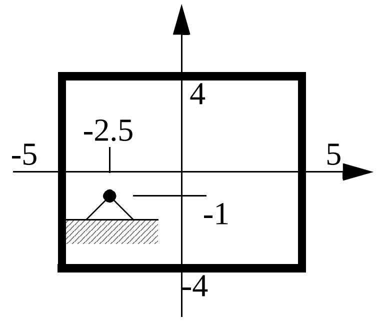
Note: Remember to use radians for these calculations. In fact you are advised to use radians for all calculations. Don’t forget to set your calculator to radians also.

Note: This calculation uses the area moment of inertia.

Figure 129 A solid torsional spring

The spring constant for a torsional spring will be relatively constant, unless the material is deformed outside the linear elastic range, or the geometry of the spring changes significantly.

When dealing with strength of material properties the area moment of inertia is required. The calculation for the area moment of inertia is similar to that for the mass moment of inertia. An example of calculating the area moment of inertia is shown in Figure 130, and based on the previous example in Figure 126. The calculations are similar to those for the mass moments of inertia, except for the formulation of the integration elements. Note the difference between the mass moment of inertia and area moment of inertia for the part. The area moment of inertia can be converted to a mass moment of inertia simply by multiplying by the density. Also note the units.



First, the area moment of inertia is calculated about the centroid by integration. All dimensions are in m.

$$I_{xx} = \int_{-4m}^{4m} y^2 dA = \int_{-4m}^{4m} y^2 2(5m)dy = 10m\frac{y^3}{3}\bigg|_{-4m}^{4m} = 10m\left(\frac{(4m)^3}{3} - \frac{(-4m)^3}{3}\right) = 426.7m^4$$

$$I_{yy} = \int_{-5m}^{5m} x^2 dA = \int_{-5m}^{5m} x^2 2(4m)dx = 8m\frac{x^3}{3}\bigg|_{-5m}^{5m} = 8m\left(\frac{(5m)^3}{3} - \frac{(-5m)^3}{3}\right) = 666.7m^4$$

$$\tilde{J}_A = I_{xx} + I_{yy} = (426.7 + 666.7)m^4 = 1093.4m^4$$

Next, shift the area moment of inertia from the centroid to the other point of rotation.

$$J_A = \tilde{J}_A + Ar^2$$

$$\therefore \quad = 1093.4m^4 + ((4m - (-4m))(5m - (-5m)))((-1m)^2 + (-2.5m)^2)$$

$$\therefore \quad = 1673m^4$$

Note: The basic definitions for the area moment of inertia are shown to the right.

$$I_{xx} = \int y^2 dA \tag{8}$$

$$I_{yy} = \int x^2 dA \tag{9}$$

$$J_A = I_{xx} + I_{yy} \tag{10}$$

$$J_A = \tilde{J}_A + Ar^2 \tag{11}$$

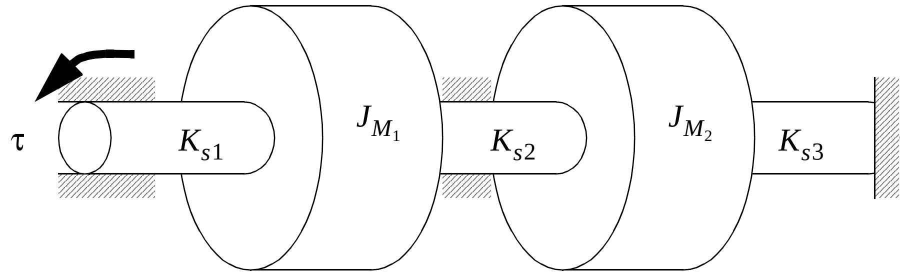
Note: You may notice that when the area moment of inertia is multiplied by the density of the material, the mass moment of inertia is the result. Therefore if you have a table of area moments of inertia, multiplying by density will yield the mass moment of inertia. Keep track of units when doing this.

Figure 130 Area moment of inertia

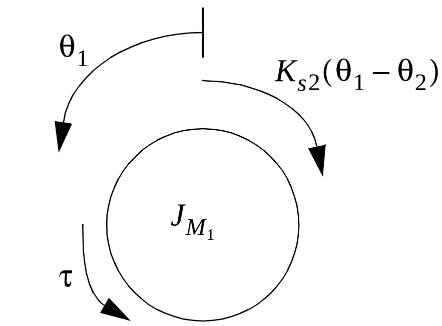
For a 1/2" 1020 steel rod that is 1 yard long, find the torsional spring coefficient.

Figure 131 Drill problem: Find the torsional spring coefficient

An example problem with torsional springs is shown in Figure 132. There are three torsional springs between two rotating masses. The right hand spring is anchored solidly in a wall, and will not move. A torque is applied to the left hand spring. Because the torsional spring is considered massless the tow will be the same at the other end of the spring, at mass J1. FBDs are drawn for both of the masses, and forces are summed. (Note: the similarity in the methods used for torsional, and for translational springs.) These equations are then rearranged into state variable equations, and finally put in matrix form.



Model the system above assuming that the center shaft is a torsional spring, and that a torque is applied to the leftmost disk. Leave the results in state variable form.

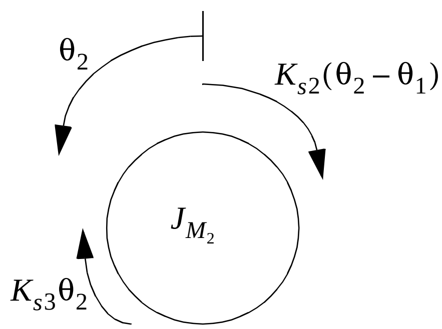


$$+\sum M = \tau - K_{s2}(\theta_1 - \theta_2) = J_{M_1}\theta_1''$$

$$J_{M_1}\theta_1'' = -K_{s2}\theta_1 + K_{s2}\theta_2 + \tau$$

$$\theta_1' = \omega_1 \qquad (1)$$

$$\omega_1' = \left(\frac{-K_{s2}}{J_{M_1}}\right)\theta_1 + \left(\frac{K_{s2}}{J_{M_1}}\right)\theta_2 + \tau \qquad (2)$$



$$+\sum M = -K_{s2}(\theta_2 - \theta_1) - K_{s3}\theta_2 = J_{M_2}\theta_2''$$

$$\theta_2'' = \left(\frac{-K_{s3} - K_{s2}}{J_{M_2}}\right)\theta_2 + \left(\frac{K_{s2}}{J_{M_2}}\right)\theta_1$$

$$\theta_2' = \omega_2 \qquad (3)$$

$$\omega_2' = \left(\frac{-K_{s3} - K_{s2}}{J_{M_2}}\right)\theta_2 + \left(\frac{K_{s2}}{J_{M_2}}\right)\theta_1 \qquad (4)$$

$$\frac{d}{dt}\begin{bmatrix}\theta_1\\ \omega_1\\ \theta_2\\ \omega_2\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ \frac{-K_{s2}}{J_{M_1}} & 0 & \frac{K_{s2}}{J_{M_1}} & 0\\ 0 & 0 & 0 & 1\\ \frac{K_{s2}}{J_{M_2}} & 0 & \frac{-K_{s3} - K_{s2}}{J_{M_2}} & 0\end{bmatrix}\begin{bmatrix}\theta_1\\ \omega_1\\ \theta_2\\ \omega_2\end{bmatrix} + \begin{bmatrix}0\\ \tau\\ 0\\ 0\end{bmatrix}$$

Figure 132 A rotational spring example

5.2.3 Damping

Rotational damping is normally caused by viscous fluids, such as oils, used for lubrication. It opposes angular velocity with the relationships shown in Figure 133. The first equation is used for a system with one rotating and one stationary part. The second equation is used for damping between two rotating parts.

$$T = -K_d \omega$$

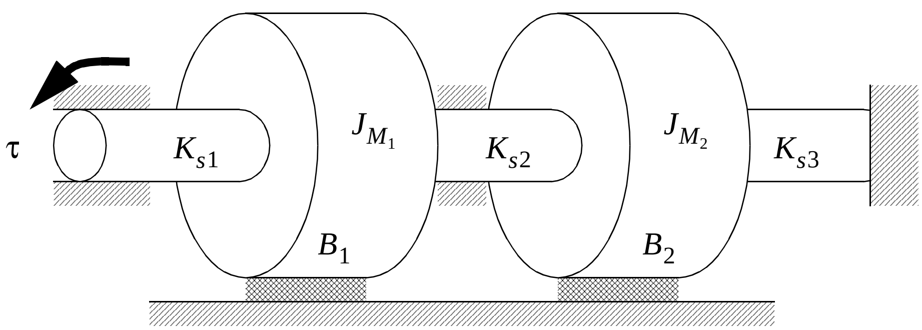
$$T = K_d(\omega_1 - \omega_2)$$

Figure 133 The rotational damping equation

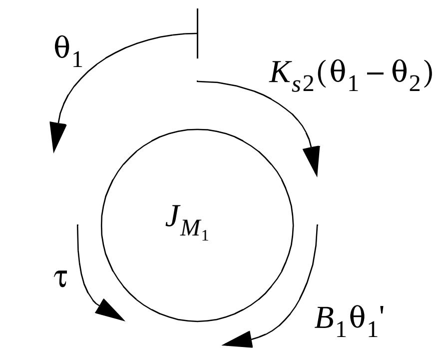
If a wheel (JM=5kg m**2) is turning at 150 rpm and the damping coefficient is 1Nms/rad, what is the deceleration?

Figure 134 Drill problem: Find the deceleration

The example in Figure 132 is extended to include damping in Figure 135. The primary addition from the previous example is the addition of the damping forces to the FBDs. In this case the damping coefficients are indicated with ’B’, but ’Kd’ could have also been used. The state equations were developed in matrix form. Visual comparison of the final matrices in this and the previous example reveal that the damping terms are the only addition.



Model the system above assuming that the center shaft is a torsional spring, and that a torque is applied to the leftmost disk. Leave the results in state variable form.

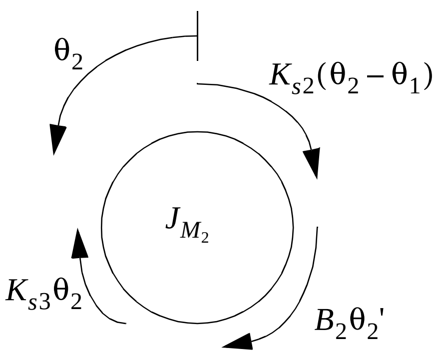


$$+\sum M = \tau - K_{s2}(\theta_1 - \theta_2) - B_1\theta_1' = J_{M_1}\theta_1''$$

$$J_{M_1}\theta_1'' = -B_1\theta_1' - K_{s2}\theta_1 + K_{s2}\theta_2 + \tau$$

$$\theta_1' = \omega_1 \qquad (1)$$

$$\omega_1' = \left(\frac{-B_1}{J_{M_1}}\right)\omega_1 + \left(\frac{-K_{s2}}{J_{M_1}}\right)\theta_1 + \left(\frac{K_{s2}}{J_{M_1}}\right)\theta_2 + \tau \qquad (2)$$



$$+\sum M = -K_{s2}(\theta_2 - \theta_1) - B_2\theta_2' - K_{s3}\theta_2 = J_{M_2}\theta_2''$$

$$\theta_2'' = \left(\frac{-B_2}{J_{M_2}}\right)\theta_2' + \left(\frac{-K_{s3} - K_{s2}}{J_{M_2}}\right)\theta_2 + \left(\frac{K_{s2}}{J_{M_2}}\right)\theta_1$$

$$\theta_2' = \omega_2 \qquad (3)$$

$$\omega_2' = \left(\frac{-B_2}{J_{M_2}}\right)\omega_2 + \left(\frac{-K_{s3} - K_{s2}}{J_{M_2}}\right)\theta_2 + \left(\frac{K_{s2}}{J_{M_2}}\right)\theta_1 \qquad (4)$$

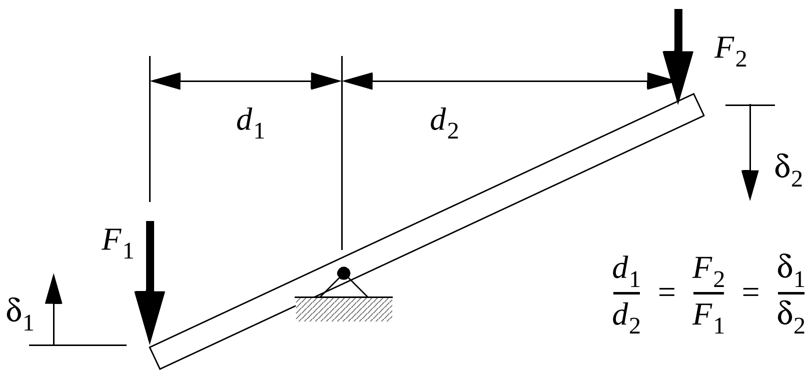
$$\frac{d}{dt}\begin{bmatrix}\theta_1\\ \omega_1\\ \theta_2\\ \omega_2\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-K_{s2}}{J_{M_1}} & \frac{-B_1}{J_{M_1}} & \frac{K_{s2}}{J_{M_1}} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K_{s2}}{J_{M_2}} & 0 & \frac{-K_{s3} - K_{s2}}{J_{M_2}} & \frac{-B_2}{J_{M_2}} \end{bmatrix}\begin{bmatrix}\theta_1\\ \omega_1\\ \theta_2\\ \omega_2\end{bmatrix} + \begin{bmatrix}0\\ \tau\\ 0\\ 0\end{bmatrix}$$

Figure 135 A System Example

5.2.4 Levers

The lever shown in Figure 136 can be used to amplify forces or motion. Although theoretically a lever arm could rotate fully, it typically has a limited range of motion. The amplification is determined by the ratio of arm lengths to the left and right of the center.

- A lever is a simple device used to balance moments in a system.



Note: As the lever rotates the length ratio will be maintained because of similar triangles. This allows the lever to work over a range of angles. The lever above would become ineffective if it was vertical.

Figure 136 Force transmission with a level

Given a lever set to lift a 1000 kg rock - if the lever is 2m long and the distance from the fulcrum to the rock is 10cm, how much force is required to lift it?

Figure 137 Drill problem: Find the required force

5.2.5 Gears and Belts

While levers amplify forces and motions over limited ranges of motion, gears can rotate indefinitely. Some of the basic gear forms are listed below.

Spur - Round gears with teeth parallel to the rotational axis
Rack - A straight gear (used with a small round gear called a pinion
Helical - The teeth follow a helix around the rotational axis.
Bevel - The gear has a conical shape, allowing forces to be transmitted at angles.

Gear teeth are carefully designed so that they will ’mesh’ smoothly as the gears rotate. The forces on gears acts at a tangential distance from the center of rotation called the ’pitch diameter’. The ratio of motions and forces through a pair of gears is proportional to their radii, as shown in Figure 138. The number of teeth on a gear is proportional to the diameter. The gear ratio is used to measure the relative rotations of the shafts. For example a gear ratio of 20:1 would mean the input shaft of the gear box would have to rotate 20 times for the output shaft to rotate once.

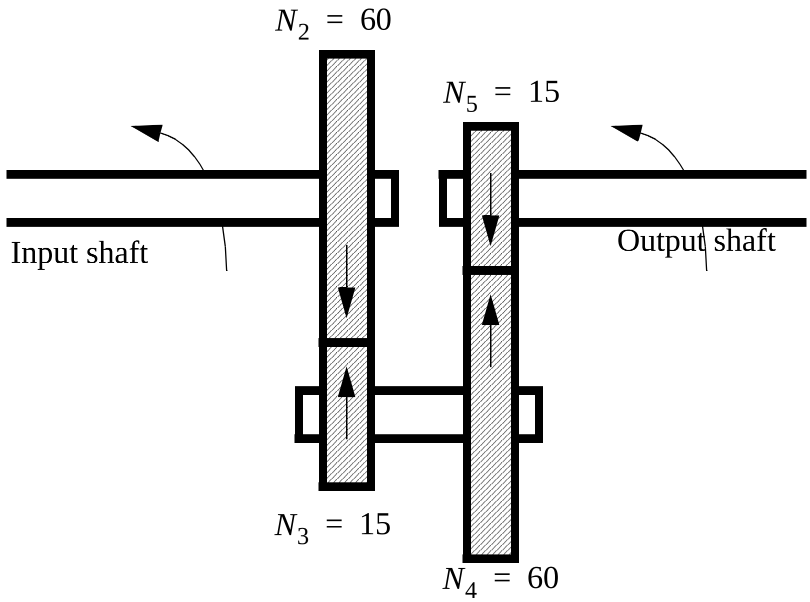
$$T_1 = F_c r_1 \qquad T_2 = -F_c r_2 \qquad \frac{n_1}{r_1} = \frac{n_2}{r_2} \qquad \frac{-T_1}{T_2} = \frac{r_1}{r_2} = \frac{n_1}{n_2}$$

$$V_c = \omega_1 r_1 = -\omega_2 r_2 \qquad \frac{r_2}{r_1} = \frac{-\omega_1}{\omega_2} = \frac{-\alpha_1}{\alpha_2} = \frac{-\Delta\theta_1}{\Delta\theta_2} = \frac{n_2}{n_1}$$

where,
n = number of teeth on respective gears
r = radii of respective gears
Fc = force of contact between gear teeth
Vc = tangential velocity of gear teeth
T = torques on gears

Figure 138 Basic Gear Relationships

For lower gear rations a simple gear box with two gears can be constructed. For higher gear ratios more gears can be added. To do this compound gear sets are required. In a compound gear set two or more gears are connected on a single shaft, as shown in Figure 139. In this example the gear ratio on the left is 4:1, and the ratio for the set on the right is 4:1. Together they give a gear ratio of 16:1.

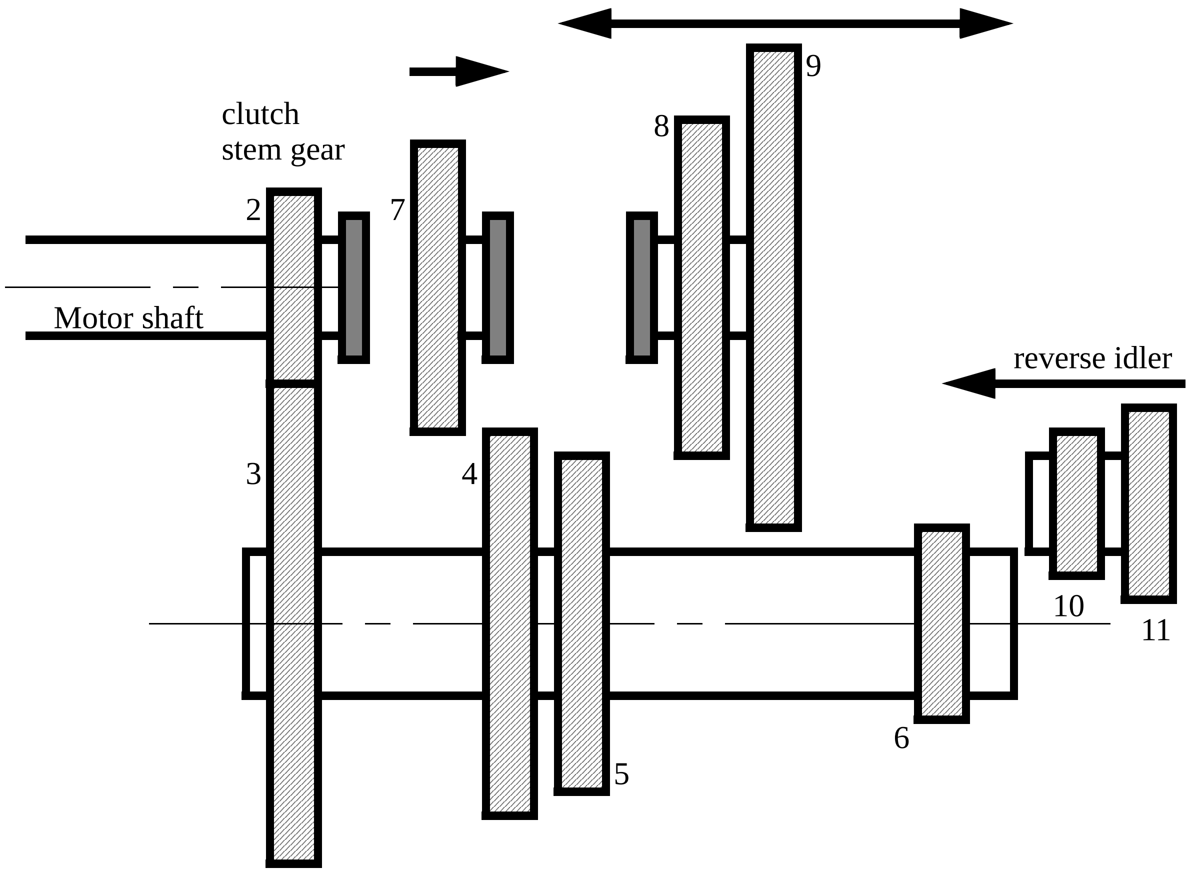


$$e = \frac{N_2 N_4}{N_3 N_5} = 16$$

In this case the output shaft turns 16 times faster than the input shaft. If we reversed directions the output (former input) would now turn 1/16 of the input (former output) shaft speed.

Figure 139 A compound gear set

A manual transmission is shown in Figure 140. In the transmission the gear ratio is changed by sliding some of the gears to change the sequence of gears transmitting the force. Notice that when in reverse an additional compound gear set is added to reverse the direction of rotation.



Speed (gear)	Gear Train
1	2-3-6-9
2	2-3-5-8
3	2-3-4-7
4	bypass gear train
reverse	2-3-6-10-11-9

In this manual transmission the gear shifter will move the gears in and out of contact. At this point all of the needed gears will be meshed and turning. The final step is to engage the last gear in the gear train with the clutch (plate) and this couples the gears to the wheels.

Figure 140 A manual transmission

Rack and pinion gear sets are used for converting rotational to translation. A rack is a long straight gear that is driven by a small mating gear called a pinion. The basic relationships are shown in Figure 141.

$$T = Fr \qquad V_c = \omega r \qquad \Delta l = r\Delta\theta$$

where,

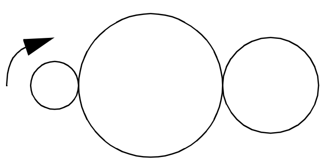
r = radius of pinion
F = force of contact between gear teeth
Vc = tangential velocity of gear teeth and velocity of rack
T = torque on pinion

Figure 141 Relationships for a rack and pinion gear set

Belt based systems can be analyzed with methods similar to gears (with the exception of teeth).

- A belt wound around a drum will act like a rack and pinion gear pair.
- A belt around two or more pulleys will act like gears.

A gear train has an input gear with 20 teeth, a center gear that has 100 teeth, and an output gear that has 40 teeth. If the input shaft is rotating at 5 rad/sec what is the rotation speed of the output shaft?



What if the center gear is removed?

Figure 142 Drill problem: Find the gear speed

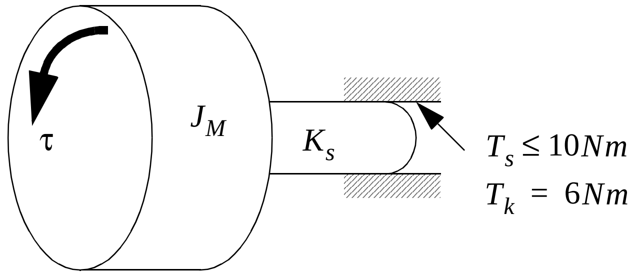
5.2.6 Friction

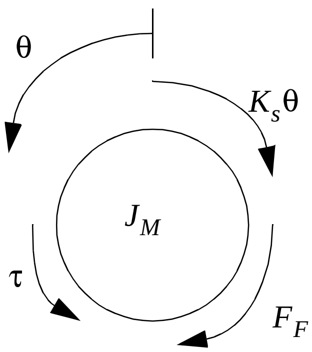
Friction between rotating components is a major source of inefficiency in machines. It is the result of contact surface materials and geometries. Calculating friction values in rotating systems is more difficult than translating systems. Normally rotational friction will be given as a static and kinetic friction torques.

An example problem with rotational friction is shown in Figure 143.

Basically these problems require that the problem be solved as if the friction surface is fixed. If the friction force exceeds the maximum static friction the mechanism is then solved using the dynamic friction torque. There is friction between the shaft and the hole in the wall. The friction force is left as a variable for the derivation of the state equations. The friction value must be calculated using the appropriate state equation. The result of this calculation and the previous static or dynamic condition is then used to determine the new friction value.

Model the system and consider the static and kinetic friction forces on the shaft on the right hand side.





$$\sum M = \tau - K_s\theta - T_F = J_M\theta''$$

$$J_M\theta'' = \tau - K_s\theta - T_F \qquad (1)$$

$$\theta' = \omega \qquad (2)$$

$$\omega' = \left(\frac{\tau - T_F}{J_M}\right) + \left(\frac{K_s}{J_M}\right)\theta \qquad (3)$$

Next, the torque force must be calculated, and then used to determine the new torque force.

$$J_M\omega' = \tau - K_s\theta - T_{test}$$
$$T_{test} = \tau - K_s\theta - J_M\omega' \qquad (4)$$

cases:

Not slipping previously

$T_{test} \leq 10Nm \qquad T_F = T_{test}$

$T_{test} > 10Nm \qquad T_F = 6Nm$

Slipping previously

$T_{test} < 6Nm \qquad T_F = T_{test}$

$T_{test} \geq 6Nm \qquad T_F = 6Nm$

Figure 143 A friction system example

The friction example in Figure 143 can be solved using the C program in Figure 144. For the purposes of the example some component values are selected and the system is assumed to be at rest initially. The program loops to integrate the state equations. Each loop the friction conditions are checked and then used for a first-order solution to the state equations.

```
int main(){
	double		h = 0.1,		// time step
			theta, w,		// the state variables
			acceleration,	// the acceleration
			TF,		// friction force
			Ttest,		// the friction test force
			J = 10,		// the moment of inertia (I picked the value)
			tau = 5,		// the applied torque (I picked the value)
			Ks = 10;		// the spring constant (I picked the value)
	int		slip = 0;		// the system starts with no slip
	FILE	*fp;
	theta = 0; w = 0;	// the initial conditions - starting at rest here
	TF = 0.0;		// set the initial friction to 0.0;
	acceleration = 0.0;	// set the initial acceleration to zero also
	if( ( fp = fopen("out.txt", "w")) != NULL){	// open a file to write the results to
		for( t = 0.0; t < 10.0; t += h ){		// loop
			Ttest = tau - Ks*theta - J*acceleration;
			if(slip == 0){	// not slipping
				if(Ttest >= 10){	TF = 6; slip = 1;
				} else {		TF = Ttest;}
			} else {		// slipping
				if(Ttest < 6){	TF = Ttest; slip = 0;
				} else {		TF = 6;}
			}
			acceleration = (tau - TF + Ks*theta) / J;
			w = w + h * acceleration;
			theta = theta + h * w;
			fprintf(fp, "%f, %f, %f\n", t, theta, w);
		}
	}
	fclose(fp);
}
```

Figure 144 A C program for the friction example in Figure 143

5.2.7 Permanent Magnet Electric Motors

DC motors create a torque between the rotor (inside) and stator (outside) that is related to the applied voltage or current. In a permanent magnet motor there are magnets mounted on the stator, while the rotor consists of wound coils. When a voltage is applied to the coils the motor will accelerate. The differential equation for a motor is shown in Figure 145, and will be derived in a later chapter. The value of the constant 'K' is a function of the motor design and will remain fixed. The value of the coil resistance 'R' can be directly measured from the motor. The moment of inertia 'J' should include the motor shaft, but when a load is added this should be added to the value of 'J'.

$$\therefore \left(\frac{d}{dt}\right)\omega + \omega\left(\frac{K^2}{JR}\right) = V_s\left(\frac{K}{JR}\right) - \frac{T_{load}}{J_M}$$

where,

ω = the angular velocity of the motor

K = the motor speed constant

J_M = the moment of inertia of the motor and attached loads

R = the resistance of the motor coils

T_{load} = a torque applied to a motor shaft

Figure 145 Model of a permanent magnet DC motor

The speed response of a permanent magnet DC motor is first-order. The steady-state velocity will be a straight line function of the torque applied to the motor, as shown in Figure 146. In addition the line shifts outwards as the voltage applied to the motor increases.

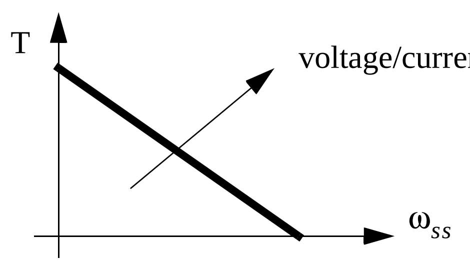


Figure 146 Torque speed curve for a permanent magnet DC motor

5.3 OTHER TOPICS

The energy and power relationships for rotational components are given in Figure

147. These can be useful when designing a system that will store and release energy.

$$E = E_K + E_P \tag{5}$$

$$E_K = J_M \omega^2 \tag{6}$$

$$E_P = T\theta \tag{7}$$

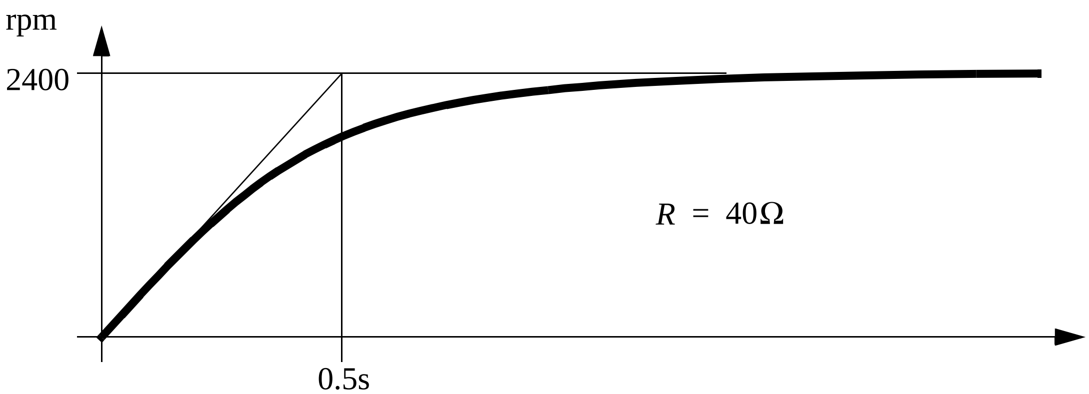
$$P = T\omega \tag{8}$$

Figure 147 Energy and power relations for rotation

5.4 DESIGN CASE

A large machine is to be driven by a permanent magnet electric motor. A 20:1 gear box is used to reduce the speed and increase the torque of the motor. The motor drives a 10000kg mass in translation through a rack and pinion gear set. The pinion has a pitch diameter of 6 inches. A 10 foot long shaft is required between the gear box and the rack and pinion set. The mass moves on rails with static and dynamic coefficients of friction of 0.2 and 0.1 respectively. We want to select a shaft diameter that will keep the system critically damped when a voltage step input of 20V is applied to the motor.

To begin the analysis the velocity curve in Figure 148 was obtained experimentally by applying a voltage of 15V to the motor with no load attached. In addition the resistance of the motor coils was measured and found to be 40 ohms. The steady-state speed and time constant are used to determine the constants for the motor.



$$\left(\frac{d}{dt}\right)\omega_m + \omega_m\left(\frac{K^2}{J_M R}\right) = V_s\left(\frac{K}{J_M R}\right) - \frac{T_{load}}{J_M}$$

The steady-state velocity can be used to find the value of K.

$$(0) + \left(2400\frac{rot}{min}\right)\left(\frac{K^2}{J_M R}\right) = 15V\left(\frac{K}{J_M R}\right) - (0)$$

$$\left(2400\frac{rot}{min}\frac{1min}{60s}\frac{2\pi rad}{1rot}\right)(K) = 15V$$

$$K = \frac{15V}{120\pi rads^{-1}} = 39.8 \times 10^{-3}\frac{Vs}{rad}$$

The time constant can be used to find the remaining parameters.

$$\frac{K^2}{J_M R} = \frac{1}{0.5s} = 2s^{-1}$$

$$J = \frac{\left(39.8 \times 10^{-3}\frac{Vs}{rad}\right)^2}{(40\Omega)(2s^{-1})} = 0.198005{\times}10^{-4} = 19.8 \times 10^{-6} Kgm^2$$

$$\left(\frac{d}{dt}\right)\omega_m + \omega_m 2s^{-1} = V_s(50.3V^{-1}s^{-2}rad) - \frac{T_{load}}{19.8 \times 10^{-6}Kgm^2}$$

$$\theta_m' = \omega_m \qquad (1)$$

$$\omega_m' = V_s 50.3V^{-1}s^{-2}rad - \omega_m 2s^{-1} - 50505Kg^{-1}m^{-2}T_{load} \qquad (2)$$

Figure 148 Motor speed curve and the derived differential equation

The remaining equations describing the system are developed in Figure 149. These calculations are done with the assumption that the inertias of the gears and other components are insignificant.

The long shaft must now be analyzed. This will require that angles at both ends be defined, and the shaft be considered as a spring.

$$\theta_{gear}, \omega_{gear} = \text{angular position and velocity of the shaft at the gear box}$$

$$\theta_{pinion}, \omega_{pinion} = \text{angular position and velocity of the shaft at the pinion}$$

$$\theta_{gear} = \frac{1}{20}\theta_m \qquad \omega_{gear} = \frac{1}{20}\omega_m$$

$$T_{shaft} = K_s(\theta_{gear} - \theta_{pinion})$$

The rotation of the pinion is related to the displacement of the rack through the circumferential travel. This ratio can also be used to find the force applied to the mass.

$$x_{mass} = \theta_{pinion}\pi 6in$$

$$T_{shaft} = F_{mass}\left(\frac{6in}{2}\right)$$

$$K_s(\theta_{gear} - \theta_{pinion}) = F_{mass}\left(\frac{6in}{2}\right)$$

$$\sum F_{mass} = F_{mass} = M_{mass}x_{mass}''$$

$$\frac{K_s(\theta_{gear} - \theta_{pinion})}{\left(\frac{6in}{2}\right)} = M_{mass}\theta_{pinion}''\pi 6in$$

$$\theta_{pinion}'' = (\theta_{gear} - \theta_{pinion})1.768 \times 10^{-6} in^{-2} Kg^{-1} K_s$$

$$\theta_{pinion}'' = \left(\frac{1}{20}\theta_m - \theta_{pinion}\right)1.768 \times 10^{-6} in^{-2} Kg^{-1} K_s \left(\frac{0.0254in}{1.0m}\right)^2$$

$$\theta_{pinion}'' = \left(\frac{1}{20}\theta_m - \theta_{pinion}\right)(1.141 \times 10^{-9}) m^{-2} Kg^{-1} K_s$$

$$\theta_{pinion}' = \omega_{pinion} \tag{3}$$

$$\omega_{pinion}' = 57.1 \times 10^{-12} m^{-2} Kg^{-1} K_s \theta_m - 1.141 \times 10^{-9} m^{-2} Kg^{-1} K_s \theta_{pinion} \tag{4}$$

Figure 149 Additional equations to model the machine

If the gear box is assumed to have relatively small inertia, then we can say that the torque load on the motor is equal to the torque in the shaft. This then allows the equation for the motor shaft to be put into a useful form, as shown in Figure 150. Having this differential equation now allows the numerical analysis to proceed. The analysis involves iteratively solving the equations and determining the point at which the system begins to overshoot, indicating critical damping.

The Tload term is eliminated from equation (2)

$$\omega_m' = V_s 50.3V^{-1}s^{-2}rad - \omega_m 2s^{-1} - 50505Kg^{-1}m^{-2}K_s(\theta_{gear} - \theta_{pinion})$$

$$\omega_m' = V_s 50.3V^{-1}s^{-2}rad - \omega_m 2s^{-1} - 50505Kg^{-1}m^{-2}K_s\left(\frac{1}{20}\theta_m - \theta_{pinion}\right)$$

$$\omega_m' = (V_s 50.3V^{-1}s^{-2}rad) + \theta_{pinion}(50505Kg^{-1}m^{-2}K_s) + \omega_m(-2s^{-1}) + \theta_m(-2525Kg^{-1}m^{-2}K_s)$$

The state equations can then be put in matrix form for clarity. The units will be eliminated for brevity, but acknowledging that they are consistent.

$$\frac{d}{dt}\begin{bmatrix} \theta_m \\ \omega_m \\ \theta_{pinion} \\ \omega_{pinion} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2525K_s & -2 & 50505K_s & 0 \\ 0 & 0 & 0 & 1 \\ 57.1\times10^{-12}K_s & 0 & -1.141\times10^{-9}K_s & 0 \end{bmatrix}\begin{bmatrix} \theta_m \\ \omega_m \\ \theta_{pinion} \\ \omega_{pinion} \end{bmatrix} + \begin{bmatrix} 0 \\ V_s 50.3 \\ 0 \\ 0 \end{bmatrix}$$

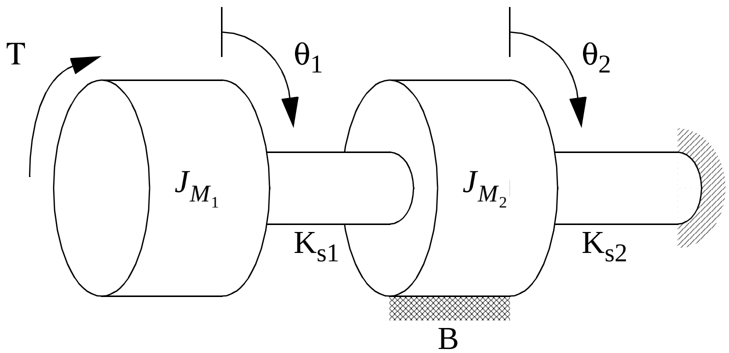
Figure 150 Numerical analysis of system response

5.5 SUMMARY

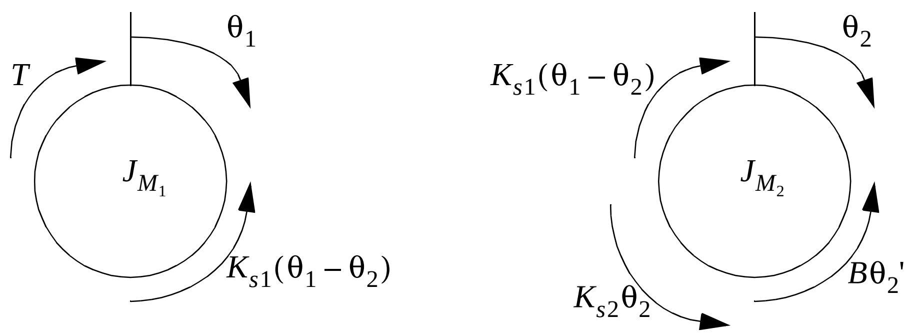
- The basic equations of motion were discussed.
- Mass and area moment of inertia are used for inertia and springs.
- Rotational dampers and springs.
- A design case was presented.

5.6 PRACTICE PROBLEMS

1. Draw the FBDs and write the differential equations for the mechanism below. The right most shaft is fixed in a wall.



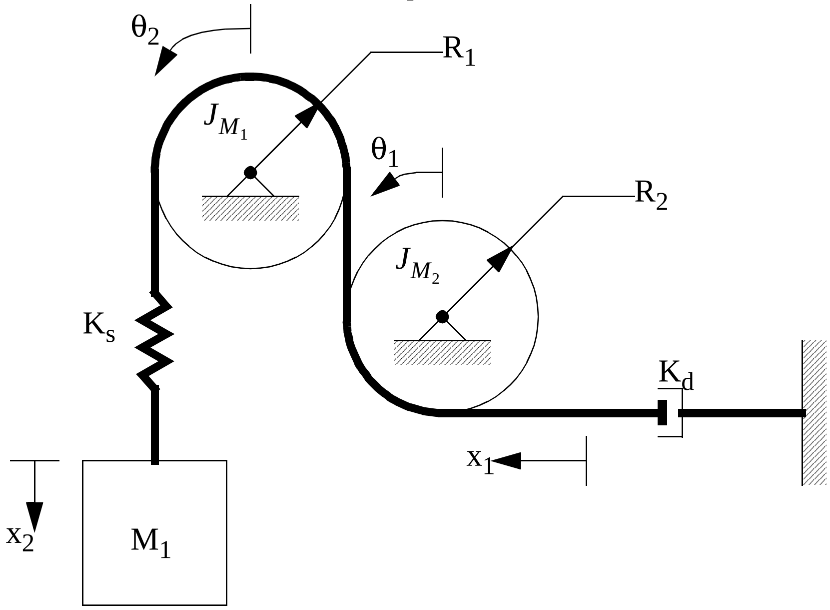
(ans.



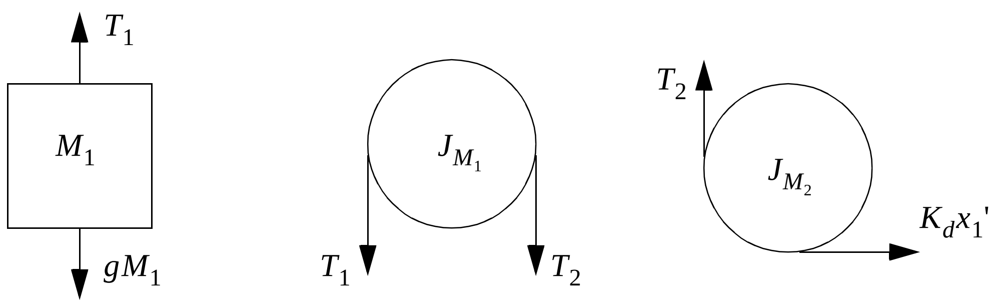
$$+ \sum M_1 = T - K_{s1}(\theta_1 - \theta_2) = J_{M_1}\theta_1''$$

$$+ \sum M_2 = K_{s1}(\theta_1 - \theta_2) - B\theta_2' - K_{s2}\theta_2 = J_{M_2}\theta_2''$$

2. Draw the FBDs and write the differential equations for the mechanism below.



(ans.



$$\theta_1 = \frac{-x_1}{R_2} \qquad \theta_2 = \frac{x_1}{R_1} \qquad T_1 = K_s(x_2 - x_1)$$

$$+\uparrow \quad \sum F_y = T_1 - gM_1 = M_1 x_2''$$

$$+\curvearrowright \quad \sum M_1 = -T_1 R_1 + T_2 R_1 = -J_{M_1}\theta_2''$$

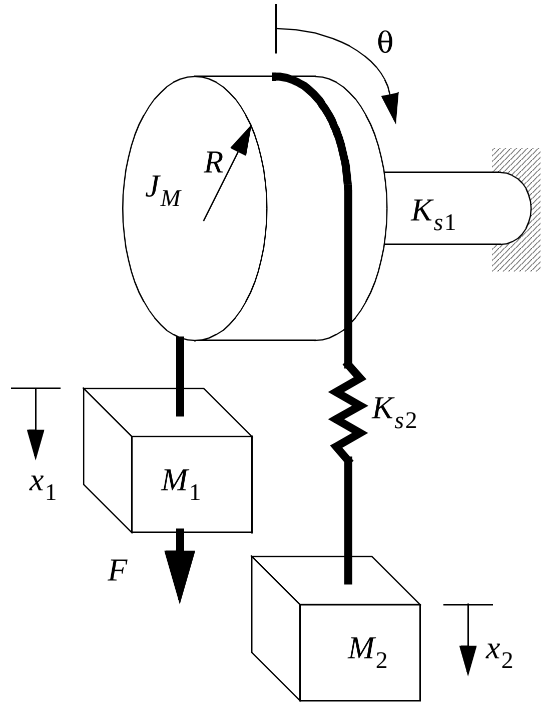
$$+\curvearrowright \quad \sum M_2 = T_2 R_2 - R_2 K_d x_1' = -J_{M_2}\theta_1''$$

6 equations, 6 unknowns

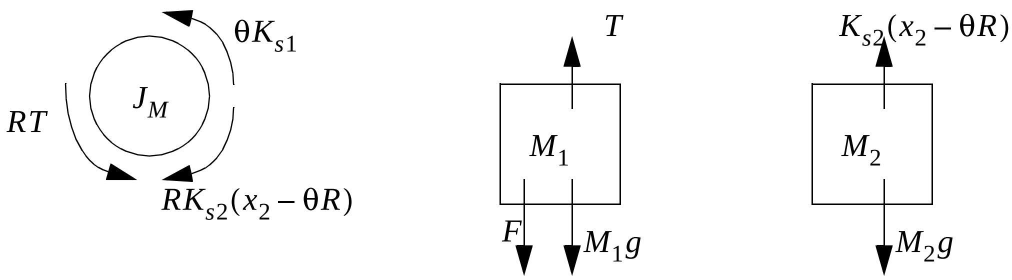
3. The system below consists of two masses hanging by a cable over mass ‘J’. There is a spring in the cable near M2. The cable doesn’t slip on ‘J’.

a) Derive the differential equations for the following system.

b) Convert the differential equations to state variable equations



(ans.
a)



if(T < 0) T=0

if(T >= 0) $R\theta = -x_1$

$$\sum F_{M1} = T - M_1 g - F = -M_1 x_1''$$

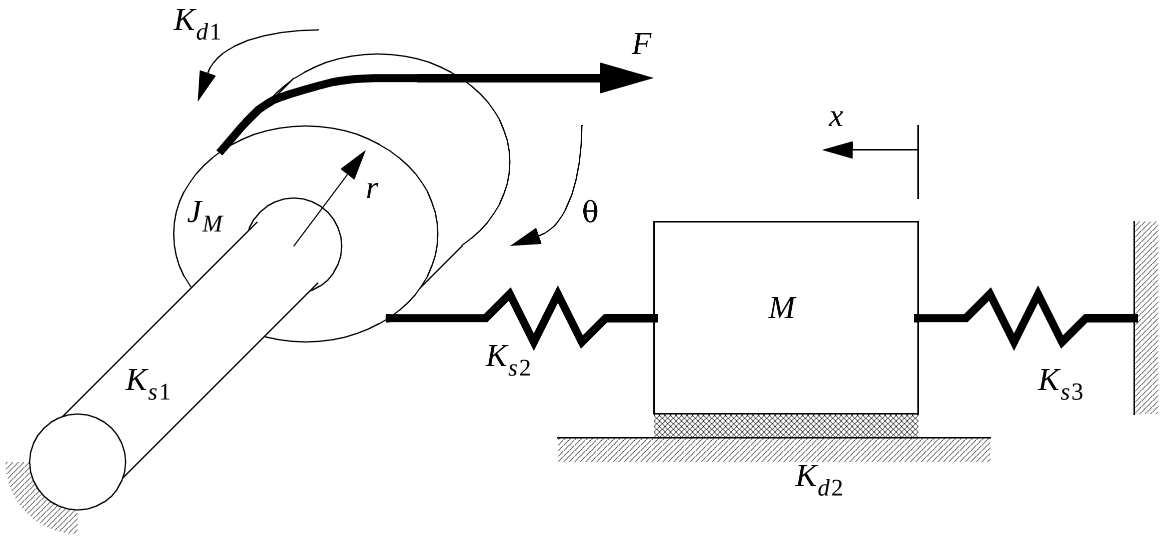
$$T = -M_1 x_1'' + M_1 g + F = -M_1 R\theta'' + M_1 g + F$$

$$\sum M_J = -RT - \theta K_{s1} + RK_{s2}(x_2 - \theta R) = J_M \theta''$$

$$-R(M_1 g + F) - \theta(K_{s1} + R^2 K_{s2}) + (RK_{s2})x_2 = (J_M - R^2 M_1)\theta'' \qquad (1)$$

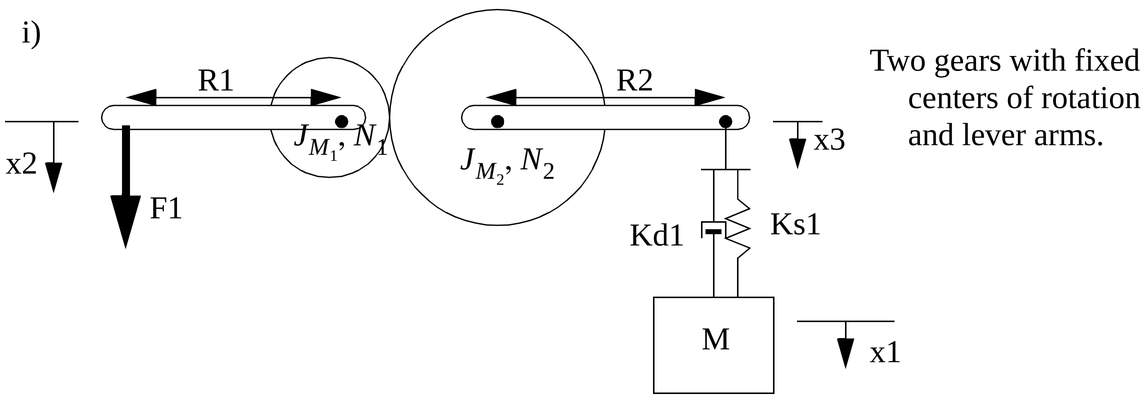
$$\sum F_{M2} = K_{s2}(x_2 - \theta R) - M_2 g = -M_2 x_2'' \qquad (2)$$

4. Write the state equations for the system to relate the applied force 'F' to the displacement 'x'.

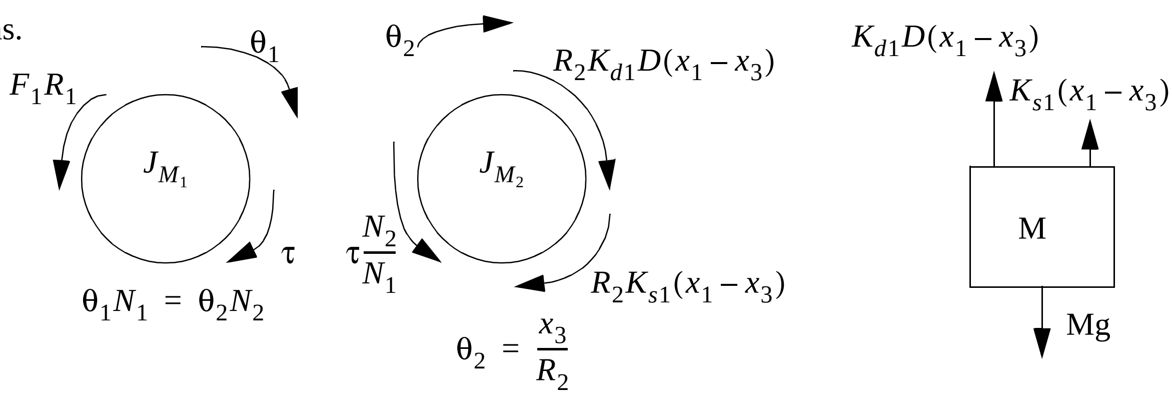


5. for the system pictured below a) write the differential equations (assume small angular deflections) and b) put the equations in state variable form.

i)



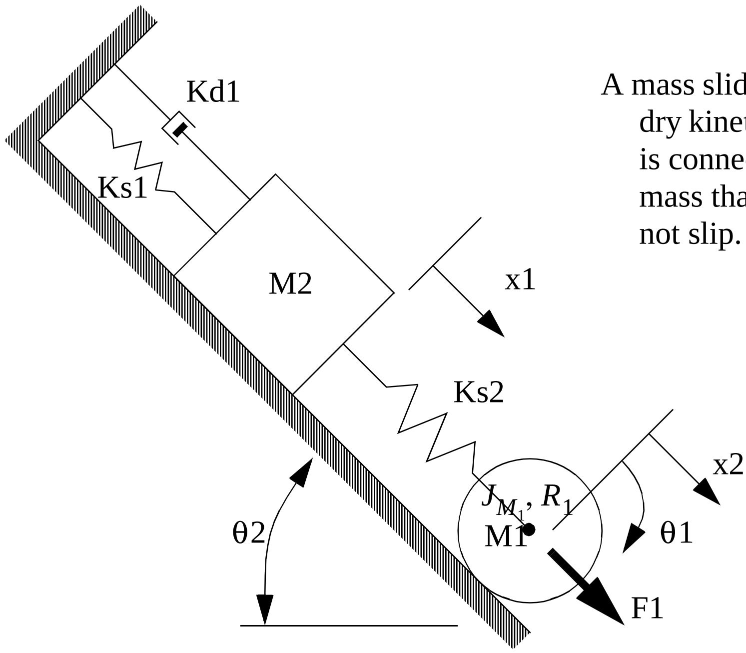
(ans.



6. for the system pictured below a) write the differential equations (assume small angular deflec-

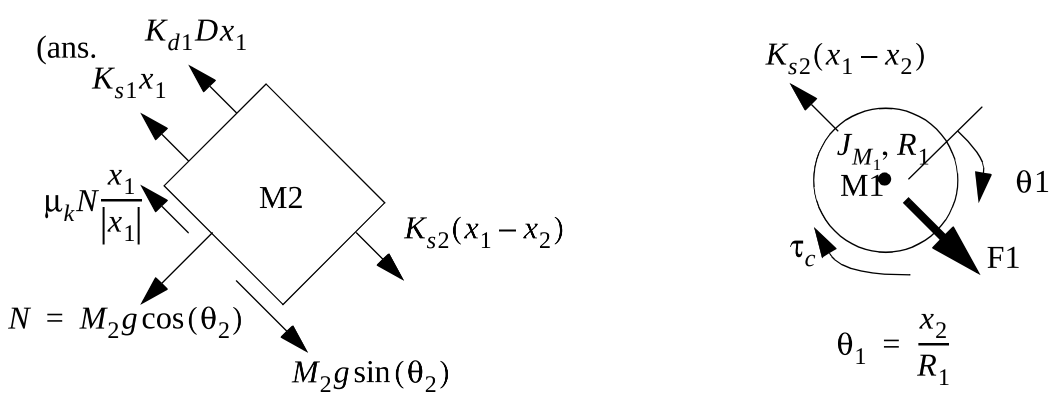
tions) and b) put the equations in state variable form.

ii)



A mass slides on a plane with dry kinetic friction (0.3). It is connected to a round mass that rolls and does not slip.

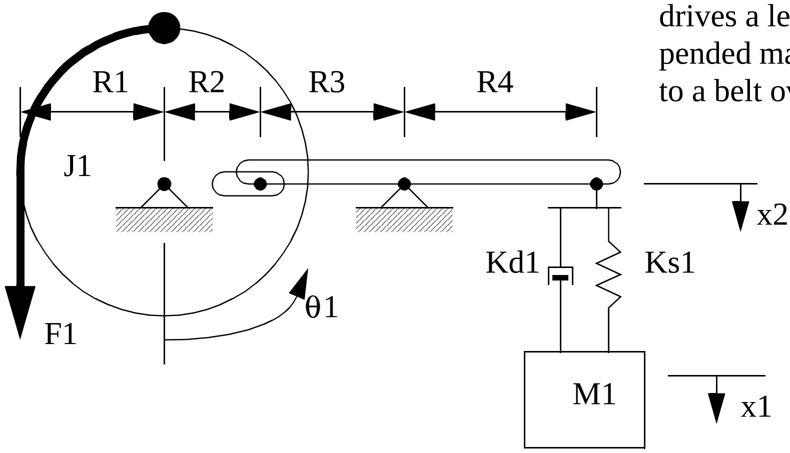
(ans.



7. for the system pictured below a) write the differential equations (assume small angular deflec-

tions) and b) put the equations in state variable form.

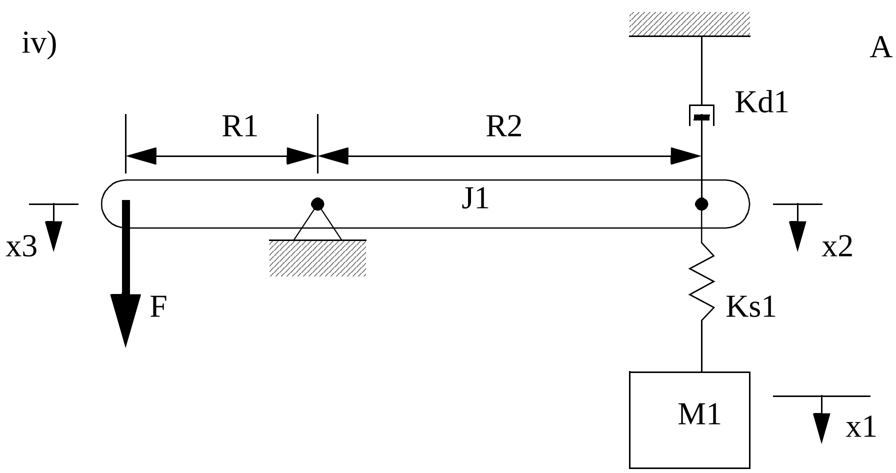
iii)



A round drum with a slot. The slot drives a lever arm with a suspended mass. A force is applied to a belt over the drum.

8. for the system pictured below a) write the differential equations (assume small angular deflections) and b) put the equations in state variable form.

iv)

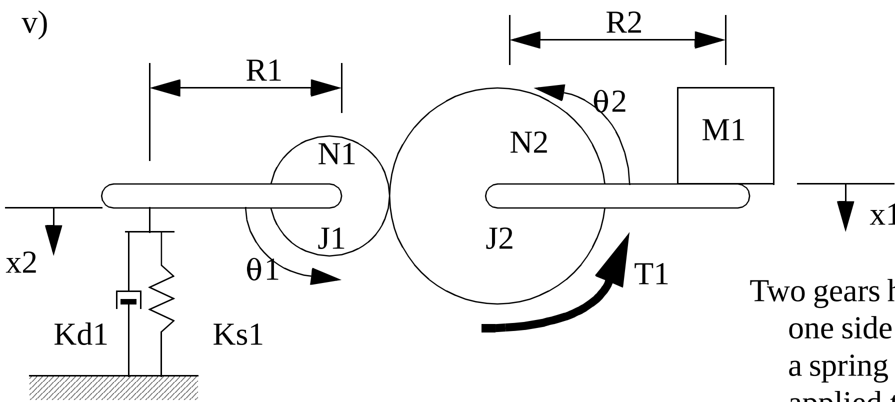


A lever arm has a force on one side, and a spring damper combination on the other side with a suspended mass.

9. for the system pictured below a) write the differential equations (assume small angular deflec-

tions) and b) put the equations in state variable form.

v)

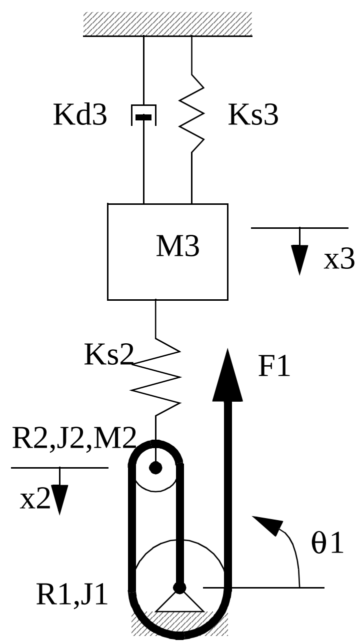


Two gears have levers attached. On one side is a mass, the other side a spring damper pair. A torque is applied to one gear.

10. for the system pictured below a) write the differential equations (assume small angular deflections) and b) put the equations in state variable form.

vi)

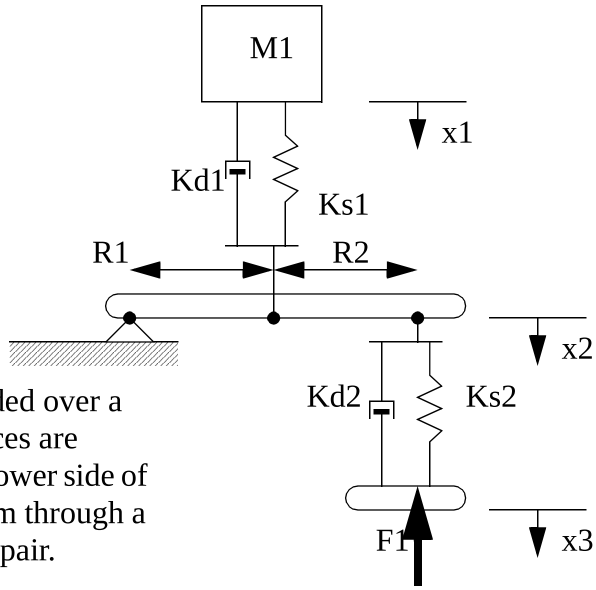
A pulley system has the bottom pulley anchored. A mass is hung in the middle of the arrangement with springs and dampers on either side.



11. for the system pictured below a) write the differential equations (assume small angular deflec-

tions) and b) put the equations in state variable form.

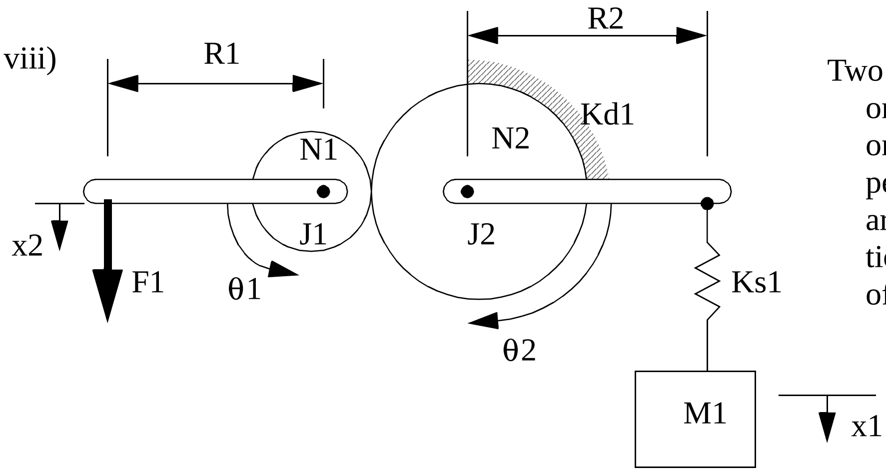
vii)



A mass is suspended over a lever arm. Forces are applied to the lower side of the moment arm through a spring damper pair.

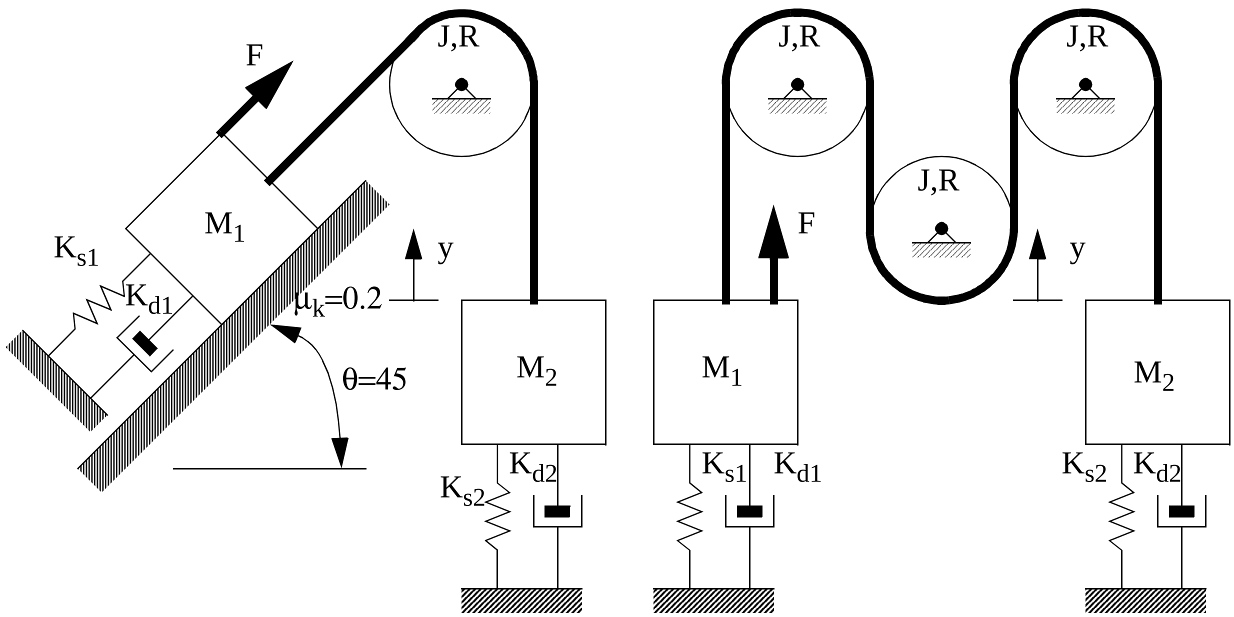
12. for the system pictured below a) write the differential equations (assume small angular deflections) and b) put the equations in state variable form.

viii)



Two gears have a force on one side, and a mass on the other, both suspended from moment arms. There is a rotational damping on one of the gears.

13. For both systems pictured below.



a) Draw FBDs and write the differential equations for the individual masses.
b) Combine the equations in input-output form with y as he output and F as the input.
c) Write the equations in state variable matrix form.
d) Use Runge-Kutta to find the system state after 1 second.

14. Find the polar moments of inertia of area and mass for a round cross section with known radius and mass per unit area. How are they related?

(ans.

For area:

$$J_{area} = \int_0^R r^2 dA = \int_0^R r^2 (2\pi r dr) = 2\pi \int_0^R r^3 dr = 2\pi \frac{r^4}{4}\bigg|_0^R = \frac{\pi R^4}{2}$$

For mass: $\rho = \frac{M}{A} = \frac{M}{\pi R^2}$

$$J_{mass} = \int_0^R r^2 dM = \int_0^R r^2 (\rho 2\pi r dr) = 2\pi\rho \int_0^R r^3 dr = 2\pi\rho \frac{r^4}{4}\bigg|_0^R = \rho\left(\frac{\pi R^4}{2}\right) = \frac{MR^2}{2}$$

The mass moment can be found by multiplying the area moment by the area density.

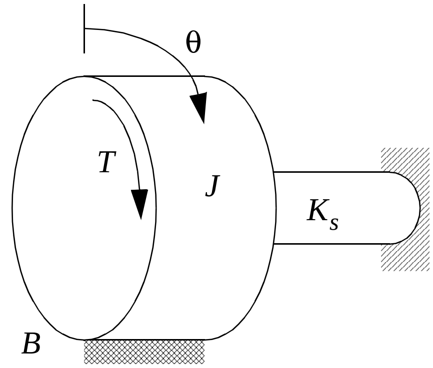
15. The rotational spring is connected between a mass ‘J’, and the wall where it is rigidly held. The mass has an applied torque ‘T’, and also experiences damping ‘B’.

a) Derive the differential equation for the rotational system shown.

b) Put the equation in state variable form (using variables) and then plot the position (not velocity) as a function of time for the first 5 seconds with your calculator using the parameters below. Assume the system starts at rest.

$$K_s = 10\frac{Nm}{rad} \qquad B = 1\frac{Nms}{rad}$$

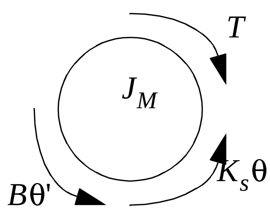
$$J_M = 1Kgm^2 \qquad T = 10Nm$$



c) A differential equation for the rotating mass with a spring and damper is given below. Solve the differential equation to get a function of time. Assume the system starts at rest.

$$\theta'' + (1s^{-1})\theta' + (10s^{-2})\theta = 10s^{-2}$$

(ans. a)



$$+\sum M = T - K_s\theta - B\theta' = J_M\theta''$$

$$J_M\theta'' + B\theta' + K_s\theta = T$$

b)

$$\theta' = \omega$$

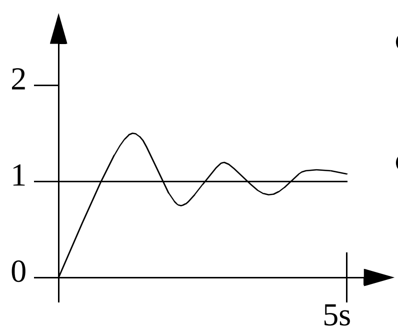
$$\omega' = \frac{T}{J_M} - \frac{K_s}{J_M}\theta - \frac{B}{J_M}\omega$$

$$\omega' = \frac{10Nm}{1Kgm^2} - \frac{10\frac{Nm}{rad}}{1Kgm^2}\theta - \frac{1\frac{Nms}{rad}}{1Kgm^2}\omega$$

$$\omega' = \frac{Nm}{Kgm^2}\left(10 - \frac{10\theta}{rad} - \frac{s}{rad}\omega\right)$$

$$\omega' = \frac{\left(\frac{Kgm}{s^2}\right)m}{Kgm^2}\left(10 - \frac{10\theta}{rad} - \frac{s}{rad}\omega\right)$$

$$\omega' = s^{-2}\left(10 - \frac{10\theta}{rad} - \frac{s}{rad}\omega\right)$$



(c homogeneous:

$$\theta'' + (1s^{-1})\theta' + (10s^{-2})\theta = 0$$

guess: $\theta_h = e^{At}$ $\quad \theta_h' = Ae^{At}$ $\quad \theta_h'' = A^2e^{At}$

$$A^2e^{At} + (1s^{-1})Ae^{At} + (10s^{-2})e^{At} = 0$$

$$A^2 + (1s^{-1})A + 10s^{-2} = 0 \qquad A = \frac{-1s^{-1} \pm \sqrt{(1s^{-1})^2 - 4(1)(10s^{-2})}}{2(1)}$$

$$A = \frac{-1s^{-1} \pm \sqrt{1s^{-2} - 40s^{-2}}}{2(1)} = (-0.5 \pm j3.123)s^{-1}$$

$$\theta_h = C_1e^{-0.5s^{-1}t}\cos(3.123s^{-1}t + C_2)$$

particular:

$$\theta'' + (1s^{-1})\theta' + (10s^{-2})\theta = 10s^{-2}$$

guess: $\theta_p = A$ $\quad \theta_p' = 0$ $\quad \theta_p'' = 0$

$$(0) + (1s^{-1})(0) + (10s^{-2})(A) = 10s^{-2} \qquad A = \frac{10s^{-2}}{10s^{-2}} = 1$$

$$\theta_p = 1$$

Initial conditions:

$$\theta(t) = C_1e^{-0.5s^{-1}t}\cos(3.123s^{-1}t + C_2) + 1$$

$$\theta(0) = C_1e^{-0.5s^{-1}0}\cos(3.123s^{-1}0 + C_2) + 1 = 0$$

$$C_1\cos(C_2) + 1 = 0$$

$$\theta'(t) = -0.5s^{-1}C_1e^{-0.5s^{-1}t}\cos(3.123s^{-1}t + C_2) - 3.123s^{-1}C_1e^{-0.5s^{-1}t}\sin(3.123s^{-1}t + C_2)$$

$$\theta'(0) = -0.5s^{-1}C_1(1)\cos(C_2) - 3.123s^{-1}C_1(1)\sin(C_2) = 0$$

$$-0.5\cos(C_2) - 3.123\sin(C_2) = 0$$

$$\frac{\sin(C_2)}{\cos(C_2)} = \frac{-0.5}{3.123} = \tan(C_2) \qquad C_2 = -0.159$$

$$C_1\cos(-0.159) + 1 = 0 \qquad C_1 = \frac{-1}{\cos(-0.159)} = -1.013$$

$$\theta(t) = -1.013e^{-0.5s^{-1}t}\cos(3.123s^{-1}t - 0.159) + 1$$

b)
$$\theta' = \omega$$
$$\omega' = \theta\left(\frac{-K_{s1} - R^2K_{s2}}{J_M - R^2M_1}\right) + x_2\left(\frac{RK_{s2}}{J_M - R^2M_1}\right) + \left(\frac{-RM_1g - RF}{J_M - R^2M_1}\right)$$
$$x_2' = v_2$$
$$v_2' = \theta\left(\frac{RK_{s2}}{M_2}\right) + x_2\left(\frac{-K_{s2}}{M_2}\right) + g$$

c)
$$K_{s2}(x_2 - \theta R) - M_2g = -M_2x_2D^2$$
$$x_2(K_{s2} + M_2D^2) + \theta(-RK_{s2}) = M_2g$$
$$x_2 = \frac{1}{K_{s2} + M_2D^2}((M_2g) + \theta(RK_{s2})) \qquad (3)$$
$$\theta = \frac{1}{RK_{s2}}((-M_2g) + x_2(K_{s2} + M_2D^2)) \qquad (4)$$
$$-R(M_1g + F) - \theta(K_{s1} + R^2K_{s2}) + (RK_{s2})x_2 = (J_M - R^2M_1)\theta D^2$$
$$-\theta((K_{s1} + R^2K_{s2}) - (J_M - R^2M_1)D^2) + x_2(RK_{s2}) = R(M_1g + F) \quad (5)$$

for x2 put (4) into (5)

$$\frac{1}{K_{s2}}((-M_2g) + x_2(K_{s2} + M_2D^2))\Big)((K_{s1} + R^2K_{s2}) - (J_M - R^2M_1)D^2) + x_2(RK_{s2}) = R(M_1g +$$
$$(M_2g - x_2(K_{s2} + M_2D^2))(K_{s1} + R^2K_{s2} - J_MD^2 + D^2R^2M_1) + x_2(R^2K_{s2}^2) = R^2K_{s2}(M_1g + F$$
$$x_2((-K_{s2} - M_2D^2)(K_{s1} + R^2K_{s2} - J_MD^2 + D^2R^2M_1) + R^2K_{s2}^2)$$
$$= R^2K_{s2}(M_1g + F) - M_2g(K_{s1} + R^2K_{s2} - J_MD^2 + D^2R^2M_1)$$

etc....

6. INPUT-OUTPUT EQUATIONS

Topics:
- The differential operator, input-output equations
- Design case - vibration isolation

Objectives:
- To be able to develop input-output equations for mechanical systems.

6.1 INTRODUCTION

To solve a set of differential equations we have two choices, solve it numerically or symbolically. For a symbolic solution the system of differential equations must be manipulated into a single differential equation. In this chapter we will look at methods for manipulating differential equations into useful forms.

6.2 THE DIFFERENTIAL OPERATOR

The differential operator 'd/dt' can be written in a number of forms. In this book there have been two forms used thus far, d/dt x and x'. For convenience we will add a third, 'D'. The basic definition of this operator, and related operations are shown in Figure 151. In basic terms the operator can be manipulated as if it is a normal variable. Multiplying by 'D' results in a derivative, dividing by 'D' results in an integral. The first-order axiom can be used to help solve a first-order differential equation.

basic definition	$\frac{d}{dt}x = Dx$	$\frac{d^n}{dt^n} = D^n$	$\frac{1}{D}x = \int x dt$
algebraic manipulation	$Dx + Dy = D(x + y)$		
	$Dx + Dy = Dy + Dx$		
	$Dx + (Dy + Dz) = (Dx + Dy) + Dz$		
simplification	$\frac{D^n x}{D^m} = D^{n-m}x$		
	$\frac{x(D + a)}{(D + a)} = x$		
first-order axiom	$\frac{x(t)}{(D + a)} = e^{-at}(\int x(t)e^{at}dt + C)$		

Figure 151 General properties of the differential operator

Note:

$$x' + Ax = y(t)$$

$$xD + Ax = y(t)$$

$$\boxed{\begin{aligned} xD + Ax &= y(t) \\ x(D + A) &= y(t) \\ x &= \frac{y(t)}{D + A} \end{aligned}}$$

$$e^{At}xD + e^{At}Ax = e^{At}y(t)$$

$$e^{At}xD = e^{At}y(t)$$

$$\boxed{\begin{aligned} \frac{d}{dt}(e^{at}x) &= e^{at}\frac{d}{dt}x + ae^{at}x \\ De^{at}x &= e^{at}Dx + ae^{at}x \end{aligned}}$$

$$e^{At}xD = e^{At}y(t)$$

$$e^{At}x = \int e^{At}y(t)dt + C$$

$$x = e^{-At}\left(\int e^{At}y(t)dt + C\right)$$

$$\frac{y(t)}{D + A} = e^{-At}\left(\int e^{At}y(t)dt + C\right)$$

Figure 152 Proof of the first-order axiom

Figure 153 contains an example of the manipulation of a differential equation using the 'D' operator. The solution begins by replacing the 'd/dt' terms with the 'D' operator. After this the equation is rearranged to simplify the expression. Notice that the manipulation follows the normal rules of algebra.

$$\left(\frac{d}{dt}\right)^2 x + \frac{d}{dt}x + 5x = 5t$$

$$D^2 x + Dx + 5x = 5t$$

$$x(D^2 + D + 5) = 5t$$

$$x = \frac{5t}{D^2 + D + 5}$$

$$x = t\left(\frac{5}{D^2 + D + 5}\right)$$

Figure 153 An example of simplification with the differential operator

An example of the solution of a first-order differential equation is given in Figure 154. This begins with replacing the differential operator and rearranging the equation. The first-order axiom is then used to obtain the solution. The initial conditions are then used to find the unknowns.

Given, $\frac{d}{dt}x + 5x = 3t \qquad x(0) = 10$

$$Dx + 5x = 3t$$

$$x = \frac{3t}{D+5}$$

$$x = e^{-5t}\left(\int e^{5t} 3t dt + C\right)$$

$$x = e^{-5t}\left(3\left(\frac{te^{5t} - e^{5t}}{5}\right) + C\right)$$

guess, $\frac{d}{dt}(te^{5t} - e^{5t})$

$$= 5te^{5t} + e^{5t} - e^{5t}$$

$$= 5te^{5t}$$

$$\frac{d}{dt}(te^{5t} - e^{5t}) = 5e^{5t}t$$

$$\frac{te^{5t} - e^{5t}}{5} = \int e^{5t} t dt$$

Initial conditions,

$$x(0) = (1)\left(3\left(\frac{(0)(1) - (1)}{5}\right) + C\right) = 10$$

$$C = 10.6$$

$$x = e^{-5t}\left(3\left(\frac{te^{5t} - e^{5t}}{5}\right) + 10.6\right)$$

$$x = 0.6(t - 1) + 10.6e^{-5t}$$

$$x = 0.6t - 0.6 + 10.6e^{-5t}$$

Figure 154 An example of a solution for a first-order system

6.3 INPUT-OUTPUT EQUATIONS

A typical system will be described by more than one differential equation. These equations can be solved to find a single differential equation that can then be integrated. The basic technique is to arrange the equations into an input-output form, such as that in Figure 155. These equations will have only a single output variable, and these are always shown on the left hand side. The input variables (there can be more than one) are all on the right hand side of the equation, and act as the non-homogeneous forcing function.

e.g.,

$$2y_1''' + y_1'' + y_1' + 4y_1 = u_1' + u_1 + 3u_2 + u_3'' + u_3$$

$$y_2'' + 6y_2' + y_2 = u_1 + 3u_2'' + u_2' + 0.5u_2 + u_3'$$

where,

y = outputs
u = inputs

Figure 155 Developing input-output equations

An example of deriving an input-output equation from a system of differential equations is given in Figure 156. This begins by replacing the differential operator and combining the equations to eliminate one of the output variables. The solution ends by rearranging the equation to input-output form.

Given the differential equations,

$$y_1' = -3y_1 + 2y_2 + u_1 + 2u_2' \qquad (1)$$

$$y_2' = 2y_1 + y_2 + u_1' \qquad (2)$$

Find the input-output equations.

(1) $Dy_1 = -3y_1 + 2y_2 + u_1 + 2Du_2$

$$\therefore y_1(D+3) = 2y_2 + u_1 + 2Du_2$$

$$\therefore y_2 = y_1\left(\frac{D+3}{2}\right) - 0.5u_1 - Du_2$$

(2) $Dy_2 = 2y_1 + y_2 + Du_1$

$$\therefore y_2(D-1) = 2y_1 + Du_1$$

$$\therefore \left(y_1\left(\frac{D+3}{2}\right) - 0.5u_1 - Du_2\right)(D-1) = 2y_1 + Du_1$$

$$\therefore y_1\left(\frac{D^2+2D-3}{2} - 2\right) - 0.5Du_1 + 0.5u_1 - D^2u_2 + Du_2 = Du_1$$

$$\therefore 0.5D^2y_1 + Dy_1 - 3.5y_1 = Du_1 + 0.5Du_1 - 0.5u_1 + D^2u_2 - Du_2$$

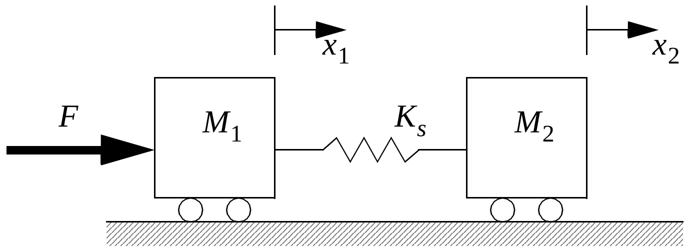
$$\boxed{\therefore 0.5y_1'' + y_1' - 3.5y_1 = u_1' + 0.5u_1' - 0.5u_1 + u_2' - u_2'}$$

Figure 156 An input output equation example

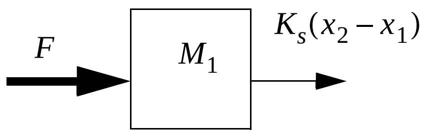
Find the second equation for the example in Figure 156 for the output y2.

Figure 157 Drill problem: Find the second equation in the previous example

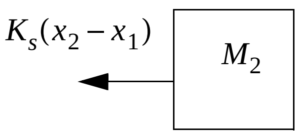
6.3.1 Converting Input-Output Equations to State Equations



Equations of motion can be derived for these masses.



$$\sum F_x = F + K_s(x_2 - x_1) = M_1 D^2 x_1$$
$$x_1(M_1 D^2 + K_s) = F + K_s x_2$$



$$\sum F_x = -K_s(x_2 - x_1) = M_2 D^2 x_2$$
$$K_s x_1 = x_2(M_2 D^2 + K_s)$$

The equations can be combined to eliminate x2.

$$x_1(M_1 D^2 + K_s) = F + K_s\left(\frac{K_s x_1}{M_2 D^2 + K_s}\right)$$
$$x_1((M_1 D^2 + K_s)(M_2 D^2 + K_s) - K_s^2) = F(M_2 D^2 + K_s)$$
$$x_1(D^4 M_1 M_2 + D^2 K_s(M_1 + M_2) + K_s^2 - K_s^2) = F(M_2 D^2 + K_s)$$
$$x_1(D^4 M_1 M_2 + D^2 K_s(M_1 + M_2)) = F(M_2 D^2 + K_s)$$
$$\left(\frac{d}{dt}\right)^4 x_1 M_1 M_2 + \left(\frac{d}{dt}\right)^2 x_1 K_s(M_1 + M_2) = \left(\frac{d}{dt}\right)^2 F M_2 + F K_s$$

This equation can’t be analyzed because of the derivatives on the right hand side. These can be eliminated by integrating the equation twice.

$$\left(\frac{d}{dt}\right)^2 x_1 M_1 M_2 + x_1 K_s(M_1 + M_2) = F M_2 + \left(\iint F(dt)dt\right)K_s$$

Figure 158 Writing an input-output equation as a differential equation

This can then be written in state variable form by creating dummy variables for integrating the function ’F’.

$$\left(\frac{d}{dt}\right)x_1 = v_1$$

$$\left(\frac{d}{dt}\right)q_2 = q_1$$

$$\left(\frac{d}{dt}\right)q_1 = F$$

$$\left(\frac{d}{dt}\right)v_1 M_1 M_2 + x_1 K_s (M_1 + M_2) = F M_2 + q_2 K_s$$

$$\left(\frac{d}{dt}\right)v_1 = -x_1 \frac{K_s (M_1 + M_2)}{M_1 M_2} + \frac{F}{M_1} + \frac{q_2 K_s}{M_1 M_2}$$

These equations can then be written in matrix form.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ q_2 \\ q_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{K_s (M_1 + M_2)}{M_1 M_2} & \frac{K_s}{M_1 M_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ q_2 \\ q_1 \\ v_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ F \\ \frac{F}{M_1} \end{bmatrix}$$

Figure 159 Writing a differential equation with differentials in the non-homogeneous part as a state equation

6.3.2 Integrating Input-Output Eqautions

The solution begins by evaluating the homogeneous equation.

$$\left(\frac{d}{dt}\right)^2 x_1 M_1 M_2 + x_1 K_s (M_1 + M_2) = 0$$

$$A^2 M_1 M_2 + K_s (M_1 + M_2) = 0 \qquad A = \pm\sqrt{\frac{K_s(M_1 + M_2)}{M_1 M_2}} j$$

$$x_{1_h} = C_1 \cos\left(\sqrt{\frac{K_s(M_1 + M_2)}{M_1 M_2}} t + C_2\right)$$

The particular solution can also be found, but in this case the input force must be specified, and then integrated. For this example a step function of magnitude ’P’ is used.

$$\left(\frac{d}{dt}\right)^2 x_1 M_1 M_2 + x_1 K_s (M_1 + M_2) = PM_2 + (\iint P(dt)dt) K_s$$

$$\left(\frac{d}{dt}\right)^2 x_1 M_1 M_2 + x_1 K_s (M_1 + M_2) = PM_2 + P\frac{t^2}{2} K_s$$

Guess,

$$x_{1_p} = At^2 + Bt + C \qquad x_{1_p}' = 2At + B \qquad x_{1_p}'' = 2A$$

$$(2A) M_1 M_2 + (At^2 + Bt + C) K_s (M_1 + M_2) = PM_2 + P\frac{t^2}{2} K_s$$

$$AK_s(M_1 + M_2) = \frac{PK_s}{2} \qquad A = \frac{P}{2(M_1 + M_2)}$$

$$BK_s(M_1 + M_2) = 0 \qquad B = 0$$

$$(2A) M_1 M_2 + CK_s (M_1 + M_2) = PM_2$$

$$CK_s(M_1 + M_2) = PM_2 - \left(2\frac{P}{2(M_1 + M_2)}\right) M_1 M_2$$

$$C = \frac{PM_2}{K_s(M_1 + M_2)}\left(1 - \frac{M_1}{(M_1 + M_2)}\right) \qquad C = \frac{PM_2^2}{K_s(M_1 + M_2)^2}$$

$$x_{1_p} = \left(\frac{P}{2(M_1 + M_2)}\right) t^2 + \left(\frac{PM_2^2}{K_s(M_1 + M_2)^2}\right)$$

Figure 160 Integrating an input-output equation

The initial conditions can then be used to find the values of the coefficients. It will be assumed that the system starts undeflected and at rest.

$$x_1 = C_1 \cos\left(\sqrt{\frac{K_s(M_1+M_2)}{M_1 M_2}}t + C_2\right) + \left(\frac{P}{2(M_1+M_2)}\right)t^2 + \left(\frac{PM_2^2}{K_s(M_1+M_2)^2}\right)$$

$$0 = C_1 \cos(C_2) + \left(\frac{PM_2^2}{K_s(M_1+M_2)^2}\right)$$

$$\frac{d}{dt}x_1 = -\sqrt{\frac{K_s(M_1+M_2)}{M_1 M_2}}C_1 \sin\left(\sqrt{\frac{K_s(M_1+M_2)}{M_1 M_2}}t + C_2\right) + 2\left(\frac{P}{2(M_1+M_2)}\right)t$$

$$0 = -\sqrt{\frac{K_s(M_1+M_2)}{M_1 M_2}}C_1 \sin(C_2) \qquad C_2 = 0$$

$$0 = C_1 \cos(0) + \left(\frac{PM_2^2}{K_s(M_1+M_2)^2}\right) \qquad C_1 = \frac{-PM_2^2}{K_s(M_1+M_2)^2}$$

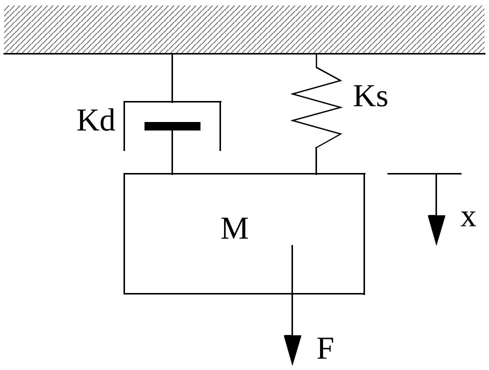
The final equation can then be written.

$$x_1 = \left(\frac{-PM_2^2}{K_s(M_1+M_2)^2}\right)\cos\left(\sqrt{\frac{K_s(M_1+M_2)}{M_1 M_2}}t\right) + \left(\frac{P}{2(M_1+M_2)}\right)t^2 + \left(\frac{PM_2^2}{K_s(M_1+M_2)^2}\right)$$

Figure 161 Integrating an input-output equation (cont’d)

6.4 DESIGN CASE

The classic mass-spring-damper system is shown in Figure 162. In this example the forces are summed to provide an equation. The differential operator is replaced, and the equation is manipulated into transfer function form. The transfer function is given in two different forms because the system is reversible and the output could be either ’F’ or ’x’.



$$\sum F_y = -F + K_d\frac{dx}{dt} + K_s x = -M\frac{d^2x}{dt^2}$$

$$F = M\frac{d^2x}{dt^2} + K_d\frac{dx}{dt} + K_s x$$

$$F = MD^2x + K_dDx + K_sx$$

$$\frac{F}{x} = MD^2x + K_dDx + K_sx$$

OR

$$\frac{x}{F} = \frac{1}{MD^2 + K_dD + K_s}$$

Aside: An important concept that is ubiquitous yet largely unrecognized is the use of functional design. We look at parts of systems as self contained modules that use inputs to produce outputs. Some systems (such a mechanisms) are reversible, others are not (consider a internal combustion engine, turning the crank does not produce gasoline). An input is typically something we can change, an output is the resulting change in a system. For the example above 'F' over 'x' implies that we are changing the input 'x', and there is some change in 'F'. We know this could easily be reversed mathematically and practically.

Figure 162 A transfer function for a mechanical system

Aside: Keep in mind that the mathematical expression 'F/x' is a ratio between input (displacement action) and output (reaction force). When shown with differentials it is obvious that the ratio is not simple, and is a function of time. Also keep in mind that if we were given a force applied to the system it would become the input (action force) and the output would be the displacement (resulting motion). To do this all we need to do is flip the numerators and denominators in the transfer function.

Mass-spring-damper systems are often used when doing vibration analysis and design work. The first stage of such analysis involves finding the actual displacement for a given displacement or force. A system experiencing a sinusoidal oscillating force is given in Figure 163. Numerical values are substituted and the homogeneous solution to the equation is found.

Given the component values input force,

$$M = 1Kg \qquad K_s = 2\frac{N}{m} \qquad K_d = 0.5\frac{Ns}{m} \qquad F = 5\sin(6t)N$$

The differential equation for the mass-spring damper system can be written.

$$1Kg\frac{d^2x}{dt^2} + \left(0.5\frac{Ns}{m}\right)\frac{dx}{dt} + \left(2\frac{N}{m}\right)x = 5\sin(6t)N$$

The homogeneous solution can be found.

$$1Kg\frac{d^2x}{dt^2} + \left(0.5\frac{Ns}{m}\right)\frac{dx}{dt} + \left(2\frac{N}{m}\right)x = 0$$

$$A = \frac{-0.5\frac{Ns}{m} \pm \sqrt{\left(0.5\frac{Ns}{m}\right)^2 - 4(1Kg)\left(2\frac{N}{m}\right)}}{2(1Kg)}$$

$$A = 0.5(-0.5 \pm \sqrt{0.25 - 8})s^{-1}$$

$$A = (-0.25 \pm 1.392j)s^{-1}$$

$$x_h = C_1 e^{-0.25t}\cos(1.392t + C_2)$$

Figure 163 Explicit analysis of a mechanical system

The solution continues in Figure 164 where the particular solution is found and put in phase shift form.

The particular solution can now be found with a guess.

$$1Kg\frac{d^2x}{dt^2} + \left(0.5\frac{Ns}{m}\right)\frac{dx}{dt} + \left(2\frac{N}{m}\right)x = 5\sin(6t)N$$

$$x_p = A\sin 6t + B\cos 6t$$

$$x_p' = 6A\cos 6t - 6B\sin 6t$$

$$x_p'' = -36A\sin 6t - 36B\cos 6t$$

$$-36A\sin 6t - 36B\cos 6t + 0.5(6A\cos 6t - 6B\sin 6t) + 2(A\sin 6t + B\cos 6t) = 5\sin(6t)$$

$$-36B + 3A + 2B = 0 \longrightarrow A = \frac{34}{3}B$$

$$-36A - 3B + 2A = 5$$

$$-34\left(\frac{34}{3}B\right) - 3B = 5$$

$$B = \frac{5}{\frac{-34(34)}{3} - 3} = -0.01288 \qquad A = \frac{34}{3}(-0.01288) = -0.1460$$

$$x_p = (-0.1460)\sin 6t + (-0.01288)\cos 6t$$

$$x_p = \frac{\sqrt{(-0.1460)^2 + (-0.01288)^2}}{\sqrt{(-0.1460)^2 + (-0.01288)^2}}((-0.1460)\sin 6t + (-0.01288)\cos 6t)$$

$$x_p = 0.1466(-0.9961\sin 6t - 0.08788\cos 6t)$$

$$x_p = 0.1466\sin\left(6t + \operatorname{atan}\left(\frac{-0.9961}{-0.08788}\right)\right)$$

$$x_p = 0.1466\sin(6t + 1.483)$$

Figure 164 Explicit analysis of a mechanical system (continued)

The system is assumed to be at rest initially, and this is used to find the constants in the homogeneous solution in Figure 165. Finally the displacement of the mass is used to find the force exerted through the spring on the ground. In this case there are two force frequency components at 1.392rad/s and 6rad/s. The steady-state force at 6rad/s will have a magnitude of .2932N. The transient effects have a time constant of 4 seconds (1/0.25), and should be negligible within a few seconds of starting the machine.

The particular and homogeneous solutions can now be combined.

$$x = x_h + x_p = C_1 e^{-0.25t}\cos(1.392t + C_2) + 0.1466\sin(6t + 1.483)$$

$$x' = -0.25C_1 e^{-0.25t}\cos(1.392t + C_2) - 1.392(C_1 e^{-0.25t}\sin(1.392t + C_2)) + 6(0.1$$

The initial conditions can be used to find the unknown constants.

$$0 = C_1 e^0 \cos(0 + C_2) + 0.1466\sin(0 + 1.483)$$

$$C_1\cos(C_2) = -0.1460$$

$$C_1 = \frac{-0.1460}{\cos(C_2)}$$

$$0 = -0.25C_1 e^0\cos(0 + C_2) - 1.392(C_1 e^0 \sin(0 + C_2)) + 6(0.1466\cos(0 + 1.483$$

$$0 = -0.25C_1\cos(C_2) - 1.392(C_1\sin(C_2)) + 0.07713$$

$$0 = -0.25\left(\frac{-0.1460}{\cos(C_2)}\cos(C_2)\right) - 1.392\left(\frac{-0.1460}{\cos(C_2)}\sin(C_2)\right) + 0.07713$$

$$0 = 0.0365 + (0.2032)\tan(C_2) + 0.07713$$

$$C_2 = \text{atan}\left(\frac{0.0365 + 0.07713}{-0.2032}\right) = -0.5099$$

$$C_1 = \frac{-0.1460}{\cos(-0.5099)} = -0.1673$$

$$x = (-0.1673e^{-0.25t}\cos(1.392t - 0.5099) + 0.1466\sin(6t + 1.483))m$$

The displacement can then be used to find the force transmitted to the ground, assuming the spring is massless.

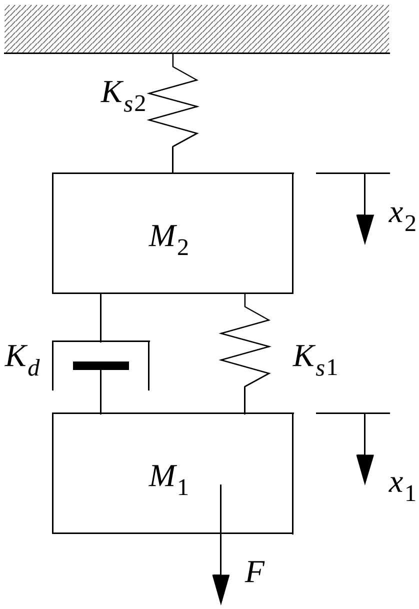
$$F = K_s x$$

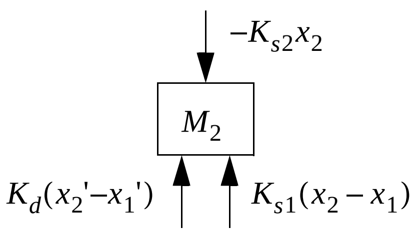
$$F = \left(2\frac{N}{m}\right)(-0.1673e^{-0.25t}\cos(1.392t - 0.5099) + 0.1466\sin(6t + 1.483))m$$

$$F = (-0.3346e^{-0.25t}\cos(1.392t - 0.5099) + 0.2932\sin(6t + 1.483))N$$

Figure 165 Explicit analysis of a mechanical system (continued)

A decision has been made to reduce the vibration transmitted to the ground to 0.1N. This can be done by adding a mass-spring isolator, as shown in Figure 166. In the figure the bottom mass-spring-damper combination is the original system. The mass and spring above have been added to reduce the vibration that will reach the ground. Values must be selected for the mass and spring. The design begins by developing the differential equations for both masses.



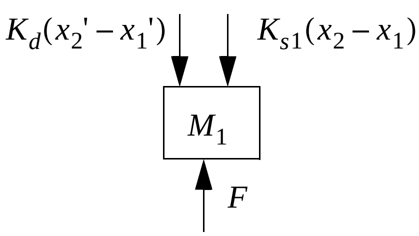


$$\sum F = -K_{s2}x_2 - K_d(x_2' - x_1') - K_{s1}(x_2 - x_1) = M_2 x_2''$$

$$-K_{s2}x_2 - K_d(x_2 D - x_1 D) - K_{s1}(x_2 - x_1) = M_2 x_2 D^2$$

$$x_2(-K_{s2} - K_d D - K_{s1} - M_2 D^2) = x_1(-K_d D - K_{s1})$$

$$x_1 = x_2\left(\frac{K_{s2} + K_d D + K_{s1} + M_2 D^2}{K_d D + K_{s1}}\right) \qquad (1)$$



$$\sum F = K_d(x_2' - x_1') + K_{s1}(x_2 - x_1) - F = M_1 x_1''$$

$$K_d(x_2 D - x_1 D) + K_{s1}(x_2 - x_1) - F = M_1 x_1 D^2$$

$$x_1(-K_d D - K_{s1} - M_1 D^2) + x_2(K_d D + K_{s1}) = F \qquad (2)$$

Figure 166 Vibration isolation system

For the design we are only interested in the upper spring, as it determines the force on the ground. An input-output equation for that spring is developed in Figure 167. The given values for the mass-spring-damper system are used. In addition a value for the upper mass is selected. This is arbitrarily chosen to be the same as the lower mass. This choice may need to be changed later if the resulting spring constant is not practical.

The solution begins by combining equations (1) and (2) and inserting the.numerical values for the lower mass, spring and damper. We can also limit the problem by selecting a mass value for the upper mass.

$$x_2\left(\frac{K_{s2} + K_d D + K_{s1} + M_2 D^2}{K_d D + K_{s1}}\right)(-K_d D - K_{s1} - M_1 D^2) + x_2(K_d D + K_{s1}) = F$$

$$M_1 = 1Kg \qquad K_{s1} = 2\frac{N}{m} \qquad K_d = 0.5\frac{Ns}{m} \qquad M_2 = 1Kg$$

$$x_2\left(\frac{K_{s2} + 0.5D + 2 + D^2}{0.5D + 2}\right)(-0.5D - 2 - D^2) + x_2(0.5D + 2) = F$$

$$x_2(D^2 + 0.5D + 2 + K_{s2})(D^2 - 0.5D - 2) + x_2(0.5D + 2)^2 = F(0.5D + 2)$$

$$x_2(D^4(-1) + D^2(K_{s2}) + D^1(-0.5K_{s2}) + (-2K_{s2})) = F(0.5D + 2)$$

This can now be converted back to a differential equation and combined with the force.

$$-\left(\frac{d}{dt}\right)^4 x_2 + K_{s2}\left(\frac{d}{dt}\right)^2 x_2 - 0.5K_{s2}\left(\frac{d}{dt}\right)x_2 - 2K_{s2}x_2 = 0.5\left(\frac{d}{dt}\right)F + 2F$$

$$-\left(\frac{d}{dt}\right)^4 x_2 + K_{s2}\left(\frac{d}{dt}\right)^2 x_2 - 0.5K_{s2}\left(\frac{d}{dt}\right)x_2 - 2K_{s2}x_2 = 0.5\left(\frac{d}{dt}\right)5\sin(6t) + 2(5)\sin(6t)$$

$$-\left(\frac{d}{dt}\right)^4 x_2 + K_{s2}\left(\frac{d}{dt}\right)^2 x_2 - 0.5K_{s2}\left(\frac{d}{dt}\right)x_2 - 2K_{s2}x_2 = 15\cos(6t) + 10\sin(6t)$$

Figure 167 Developing an input output equation

This particular solution of the differential equation will yield the steady-state displacement of the upper mass. This can then be used to find the needed spring coefficient.

The particular solution begins with a guess.

$$-\left(\frac{d}{dt}\right)^4 x_2 + K_{s2}\left(\frac{d}{dt}\right)^2 x_2 - 0.5K_{s2}\left(\frac{d}{dt}\right)x_2 - 2K_{s2}x_2 = 15\cos(6t) + 10\sin(6t)$$

$$x_p = A\sin 6t + B\cos 6t$$

$$\left(\frac{d}{dt}\right)x_p = 6A\cos 6t - 6B\sin 6t$$

$$\left(\frac{d}{dt}\right)^2 x_p = -36A\sin 6t - 36B\cos 6t$$

$$\left(\frac{d}{dt}\right)^3 x_p = -216A\cos 6t + 216B\sin 6t$$

$$\left(\frac{d}{dt}\right)^4 x_p = 1296A\sin 6t + 1296B\cos 6t$$

$$\sin(6t)(-1296A - 36AK_{s2} + 0.5K_{s2}6B - 2K_{s2}A) = 10\sin(6t)$$

$$A(-1296 - 38K_{s2}) + B(3K_{s2}) = 10$$

$$B = \frac{10 + A(1296 + 38K_{s2})}{3K_{s2}}$$

$$\cos(6t)(-1296B - 36BK_{s2} + (-0.5)K_{s2}6A - 2K_{s2}B) = 15\cos(6t)$$

$$A(-3K_{s2}) + B(-1296 - 38K_{s2}) = 15$$

$$A(-3K_{s2}) + \frac{10 + A(1296 + 38K_{s2})}{3K_{s2}}(-1296 - 38K_{s2}) = 15$$

$$A(-9K_{s2}^2) + (10 + A(1296 + 38K_{s2}))(-1296 - 38K_{s2}) = 45K_{s2}$$

$$A\left(\frac{-9K_{s2}^2}{(-1296 - 38K_{s2})}\right) + A(1296 + 38K_{s2}) = \frac{45K_{s2}}{(-1296 - 38K_{s2})} - 10$$

$$A = \frac{\dfrac{45K_{s2}}{(-1296 - 38K_{s2})} - 10}{\dfrac{-9K_{s2}^2}{(-1296 - 38K_{s2})} + (1296 + 38K_{s2})}$$

$$A = \frac{45K_{s2} + 10(1296 + 38K_{s2})}{-9K_{s2}^2 + (1296 + 38K_{s2})(-1296 - 38K_{s2})}$$

$$A = \frac{425K_{2s} + 12960}{-1453K_{2s}^2 - 98496K_{2s} - 1679616}$$

Figure 168 Finding the particular solution

The value for B can then be found.

$$B = \frac{10 + \left(\dfrac{425K_{2s} + 12960}{-1453K_{2s}^2 - 98496K_{2s} - 1679616}\right)(1296 + 38K_{s2})}{3K_{s2}}$$

$$B = \frac{10(-1453K_{2s}^2 - 98496K_{2s} - 1679616) + (425K_{2s} + 12960)(1296 + 38K_{s2})}{3K_{s2}(-1453K_{2s}^2 - 98496K_{2s} - 1679616)}$$

$$B = \frac{1620K_{2s}^2 + 2.097563\times10^8K_{2s}}{3K_{s2}(-1453K_{2s}^2 - 98496K_{2s} - 1679616)}$$

$$B = \frac{540K_{2s} + 69918768}{-1453K_{2s}^2 - 98496K_{2s} - 1679616}$$

Figure 169 Finding the particular solution (cont'd)

Finally the magnitude of the particular solution is calculated and set to the desired amplitude of 0.1N. This is then used to calculate the spring coefficient.

$$amplitude = \sqrt{A^2 + B^2}$$

$$0.1 = \sqrt{\left(\frac{425K_{2s} + 12960}{-1453K_{2s}^2 - 98496K_{2s} - 1679616}\right)^2 + \left(\frac{540K_{2s} + 69918768}{-1453K_{2s}^2 - 98496K_{2s} - 1679616}\right)^2}$$

A value for the spring coefficient was then found using Mathcad to get a value of 662N/m.

$K := 1$

given

$$\sqrt{\left[\frac{425\cdot K + 12960}{(-1453\cdot K\cdot K) - 98496\cdot K - 1679616}\right]^2 + \left[\frac{540\cdot K + 69918768}{(-1453\cdot K\cdot K) - 98496\cdot K - 1679616}\right]^2} = 0.1$$

$\mathrm{find}(K) = 661.68$

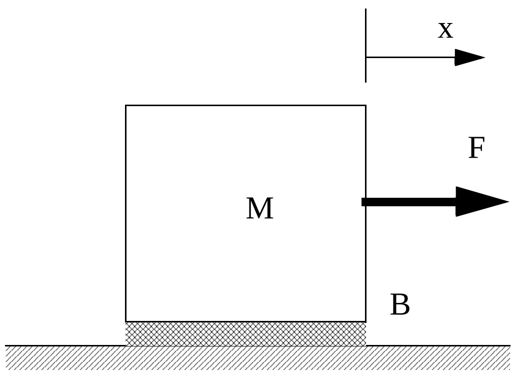
Figure 170 Calculation of the spring coefficient

6.5 SUMMARY

- The differential operator can be manipulated algebraically
- Equations can be manipulated into input-output forms and solved as normal differential equations

6.6 PRACTICE PROBLEMS

1. Develop the input-output equation for the mechanical system below. There is viscous damping between the block and the ground. A force is applied to cause the mass the accelerate.



(ans. $x''(M) + x'(B) = F$

2. Find the input-output form for the following equations.

$$x_1'' + x_1' + 2x_1 - x_2' - x_2 = 0$$

$$-x_1' - x_1 + x_2'' + x_2' + x_2 = F$$

3. The following differential equations were converted to the matrix form shown. Use Cramer's rule to find 'y'.

$$y'' + 2x = \frac{F}{10}$$

$$7y' + 4y + 9x'' + 3x = 0$$

$$\begin{bmatrix} (D^2) & (2) \\ (7D+4) & (9D^2+3) \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} \frac{F}{10} \\ 0 \end{bmatrix}$$

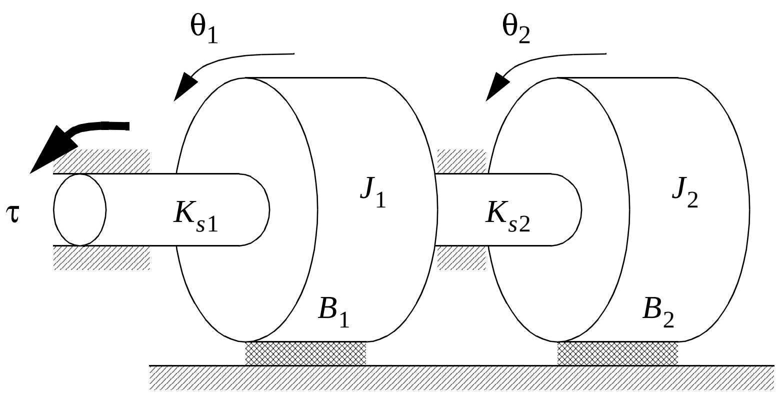
(ans.

$$y = \frac{\left|\begin{bmatrix} \frac{F}{10} & (2) \\ 0 & (9D^2+3) \end{bmatrix}\right|}{\left|\begin{bmatrix} (D^2) & (2) \\ (7D+4) & (9D^2+3) \end{bmatrix}\right|} = \frac{\frac{F}{10}(9D^2+3)}{D^2(9D^2+3)-2(7D+4)} = \frac{F(0.9D^2+0.3)}{9D^4+3D^2-14D-8}$$

$$y(9D^4+3D^2-14D-8) = F(0.9D^2+0.3)$$

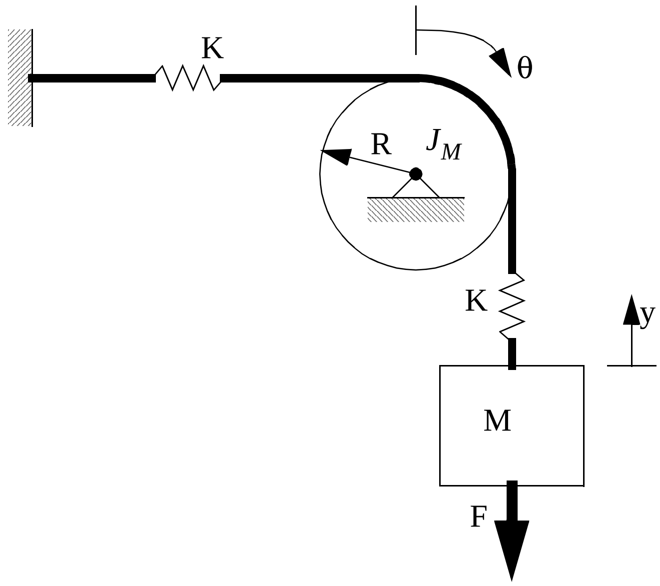
$$\left(\frac{d}{dt}\right)^4 y(9) + \left(\frac{d}{dt}\right)^2 y(3) + \left(\frac{d}{dt}\right)^1 y(-14) + y(-8) = \left(\frac{d}{dt}\right)^2 F(0.9) + F(0.3)$$

4. Find the input-output equations for the systems below. Here the input is the torque on the left hand side.

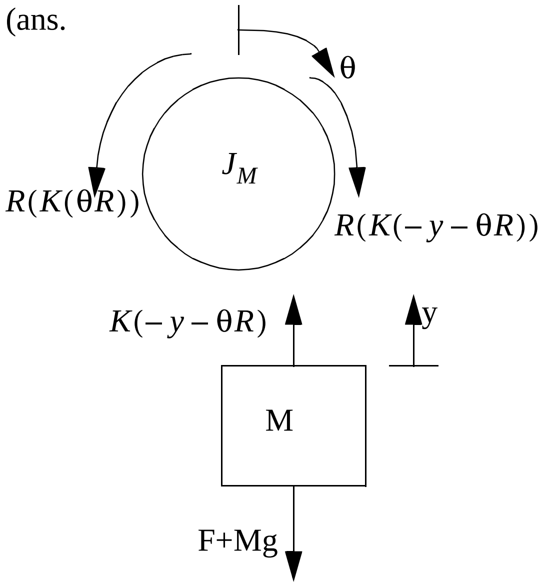


5. Write the input-output equations for the mechanical system below. The input is force ‘F’, and the outputs are ‘y’ and the angle theta. Include the inertia of both masses, and gravity for mass

‘M’.



(ans.



$$\sum M = -R(K(\theta R)) + R(K(-y-\theta R)) = J_M\theta''$$

$$R^2K\theta + RKy + R^2K\theta = -J_M\theta''$$

$$\theta\left(\frac{2R^2K + J_MD^2}{-RK}\right) = y$$

$$\sum F = K(-y-\theta R) - F - Mg = My''$$

$$K(y+\theta R) + F + Mg = -MyD^2$$

$$\theta(KR) + F + Mg = y(-MD^2 - K)$$

$$\theta(KR) + y(MD^2 + K) = -F - Mg$$

for the theta output equation;

$$\theta(KR) + \theta\left(\frac{2R^2K + J_MD^2}{-RK}\right)(MD^2 + K) = -F - Mg$$

$$\theta(-K^2R^2) + \theta(2R^2K + J_MD^2)(MD^2 + K) = FKR + MgKR$$

$$\theta\left(-KR + 2RMD^2 + 2RK + \frac{J_MMD^4}{KR} + \frac{J_MD^2}{R}\right) = F + Mg$$

$$\left(\frac{d}{dt}\right)^4\theta\left(\frac{J_MM}{KR}\right) + \left(\frac{d}{dt}\right)^4\theta\left(2RM + \frac{J_M}{R}\right) + \theta(KR) = FKR + MgKR$$

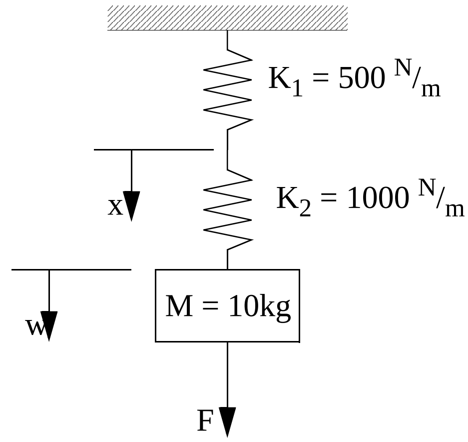
for the y output equation;

$$y\left(\frac{-RK}{2R^2K + J_MD^2}\right)(KR) + y(MD^2 + K) = -F - Mg$$

$$y\left(2R^2MD^2 + \frac{J_MD^4M}{K} + K2R^2 + J_MD^2 - R^2K\right) = -F\left(2R^2 + \frac{J_MD^2}{K}\right) - Mg\left(2R^2 + \frac{JMD^2}{K}\right)$$

$$\left(\frac{d}{dt}\right)^4y\left(\frac{J_MM}{K}\right) + \left(\frac{d}{dt}\right)^2y(2R^2M + J_M) + y(R^2K) = \left(\frac{d}{dt}\right)^2F\left(\frac{J_M}{K}\right) + F(2R^2) + (-2MgR^2)$$

6. The applied force ‘F’ is the input to the system, and the output is the displacement ‘x’.



a) What is the steady-state response for an applied force F(t) = 10cos(t + 1) N ?
b) Find x(t), given F(t) = 10N for t >= 0 seconds.

6.7 REFERENCES

Irwin, J.D., and Graf, E.R., Industrial Noise and Vibration Control, Prentice Hall Publishers, 1979.

Close, C.M. and Frederick, D.K., “Modeling and Analysis of Dynamic Systems, second edition, John Wiley and Sons, Inc., 1995.

7. ELECTRICAL SYSTEMS

Topics:
- Basic components; resistors, power sources, capacitors, inductors and op-amps
- Device impedance
- Example circuits

Objectives:
- To apply analysis techniques to circuits

7.1 INTRODUCTION

A voltage is a pull or push acting on electrons. The voltage will produce a current when the electrons can flow through a conductor. The more freely the electrons can flow, the lower the resistance of a material. Most electrical components are used to control this flow.

7.2 MODELING

Kirchoff's voltage and current laws are shown in Figure 171. The node current law holds true because the current flow in and out of a node must total zero. If the sum of currents was not zero then electrons would be appearing and disappearing at that node, thus violating the law of conservation of matter. The loop voltage law states that the sum of all rises and drops around a loop must total zero.

$$\sum I_{node} = 0 \qquad \text{node current}$$

$$\sum V_{loop} = 0 \qquad \text{loop voltage}$$

Figure 171 Kirchoff's laws

The simplest form of circuit analysis is for DC circuits, typically only requiring algebraic manipulation. In AC circuit analysis we consider the steady-state response to a sinusoidal input. Finally the most complex is transient analysis, often requiring integration, or similar techniques.

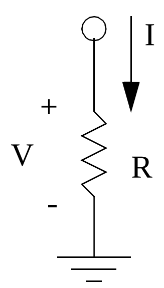
• DC (Direct Current) - find the response for a constant input.
• AC (Alternating Current) - find the steady-state response to an AC input.
• Transient - find the initial response to changes.

There is a wide range of components used in circuits. The simplest are passive, such as resistors, capacitors and inductors. Active components are capable of changing their behaviors, such as op-amps and transistors. A list of components that will be discussed in this chapter are listed below.

• resistors - reduce current flow as described with ohm's law
• voltage/current sources - deliver power to a circuit
• capacitors - pass current based on current flow, these block DC currents
• inductors - resist changes in current flow, these block high frequencies
• op-amps - very high gain amplifiers useful in many forms

7.2.1 Resistors

Resistance is a natural phenomenon found in all materials except superconductors. A resistor will oppose current flow as described by ohm's law in Figure 172. The resistance value is assumed to be linear, but in actuality it varies with conductor temperature.

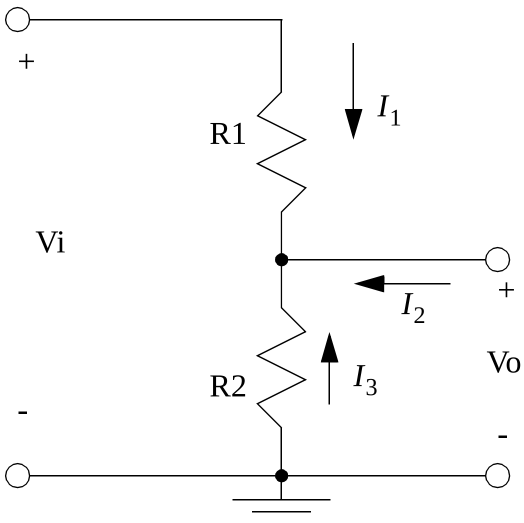


$$I = \frac{V}{R}$$

$$V = IR$$

Figure 172 Ohm's law

The voltage divider example in Figure 173 illustrates the methods for analysis of circuits using resistors. In this circuit an input voltage is supplied on the left hand side. The output voltage on the right hand side will be some fraction of the input voltage. If the output resistance is very large, no current will flow, and the ratio of output to input voltages is determined by the ratio of the resistance between R1 and R2. To prove this the currents into the center node are summed and set equal to zero. The equations are then manipulated to produce the final relationship.



$$I_1 + I_2 + I_3 = 0$$

$$\therefore\left(\frac{V_i - V_o}{R_1}\right) + I_2 + \left(\frac{0 - V_o}{R_2}\right) = 0$$

Assume the output resistance is large, so I2 is negligible.

$$\therefore\left(\frac{V_i - V_o}{R_1}\right) + \left(\frac{-V_o}{R_2}\right) = 0$$

$$\therefore V_o\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = V_i\left(\frac{1}{R_1}\right)$$

$$\therefore V_o\left(\frac{R_1 + R_2}{R_1 R_2}\right) = V_i\left(\frac{1}{R_1}\right)$$

$$\boxed{\therefore\frac{V_o}{V_i} = \frac{R_2}{R_1 + R_2}}$$

Figure 173 A voltage divider circuit

If two resistors are in parallel or series they can be replaced with a single equivalent resistance, as shown in Figure 174.

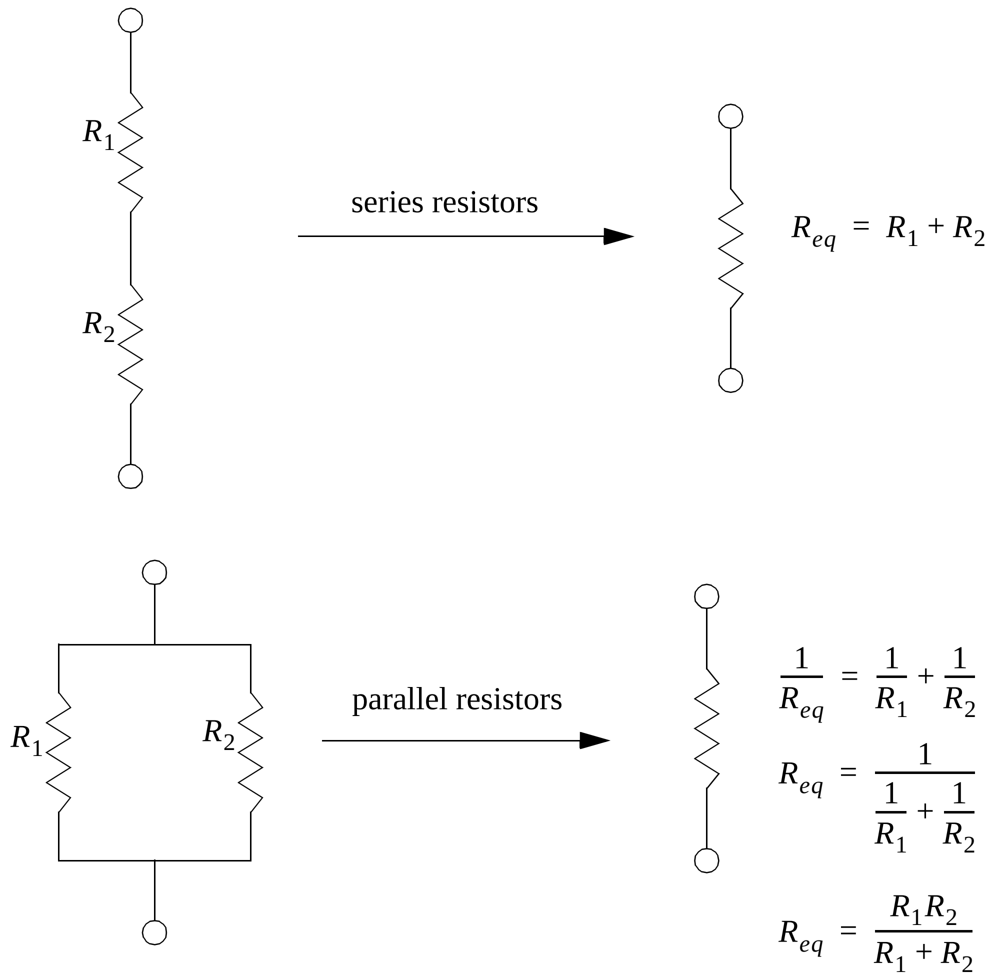


Figure 174 Equivalent resistances for resistors in parallel and series

7.2.2 Voltage and Current Sources

A voltage source will supply a voltage to a circuit, by varying the current as required. A current source will supply a current to a circuit, by varying the voltage as required. The schematic symbols for voltage and current sources are shown in Figure 175. The supplies with ’+’ and ’-’ symbols provide DC voltages, with the symbols indicating polarity. The symbol with two horizontal lines is a battery. The circle with a sine wave is an AC voltage supply. The last symbol with an arrow inside the circle is a current supply. The arrow indicates the direction of positive current flow.

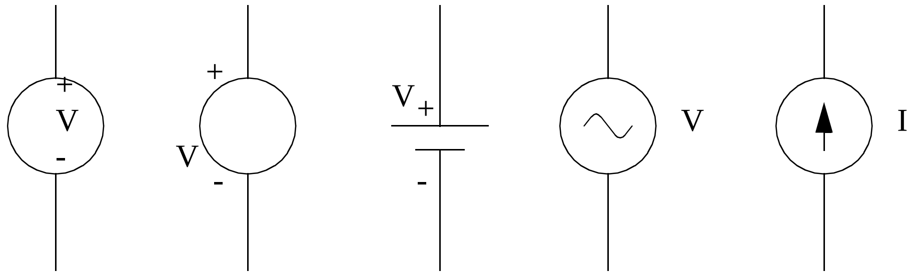
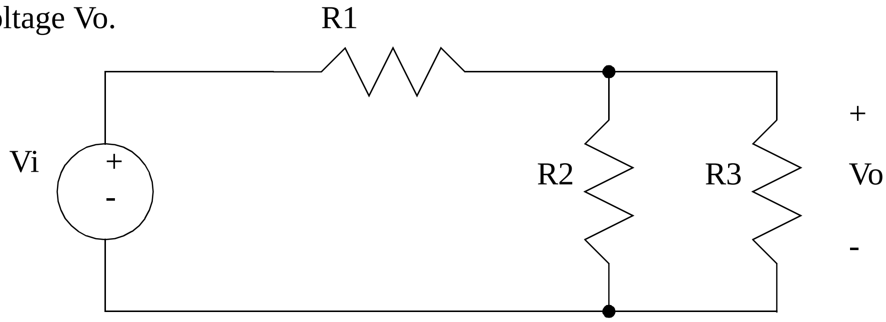


Figure 175 Voltage and current sources

A circuit containing a voltage source and resistors is shown in Figure 176. The circuit is solved using the node voltage method.

Find the output voltage Vo.



Examining the circuit there are two loops, but only one node, so the node current methods is the most suitable for calculations. The currents into the upper right node, Vo, will be solved.

$$\sum I = \frac{V_o - V_i}{R_1} + \frac{V_o}{R_2} + \frac{V_o}{R_3} = 0$$

$$V_o\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) = V_i\left(\frac{1}{R1}\right)$$

$$V_o\left(\frac{R_1 + R_2 + R_3}{R_1 R_2 R_3}\right) = V_i\left(\frac{1}{R1}\right)$$

$$V_o = V_i\left(\frac{R_2 R_3}{R_1 + R_2 + R_3}\right)$$

Aside: when doing node-current methods, select currents out of a node as positive, and in negative. This will reduce the chances of careless mistakes.

Figure 176 A circuit calculation

Solve the circuit in Figure 176 using the loop voltage method.

Figure 177 Drill problem: Mesh solution of voltage divider

Solve the voltage divider problem in Figure 173 using the loop current method. Hint: Put a voltage supply on the left, and an output resistor on the right. Remember that the output resistance should be infinite.

Figure 178 Drill problem: Mesh solution of voltage divider

Dependant (variable) current and voltage sources are shown in Figure 179. The voltage and current values of these supplies are determined by their relationship to some other circuit voltage or current. The dependant voltage source will be accompanied by a '+' and '-' symbol, while the current source has an arrow inside.

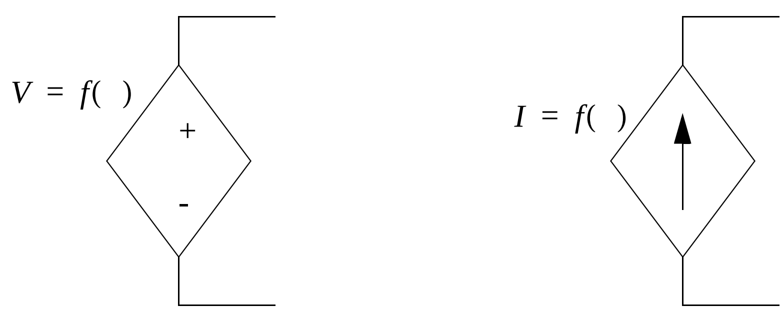
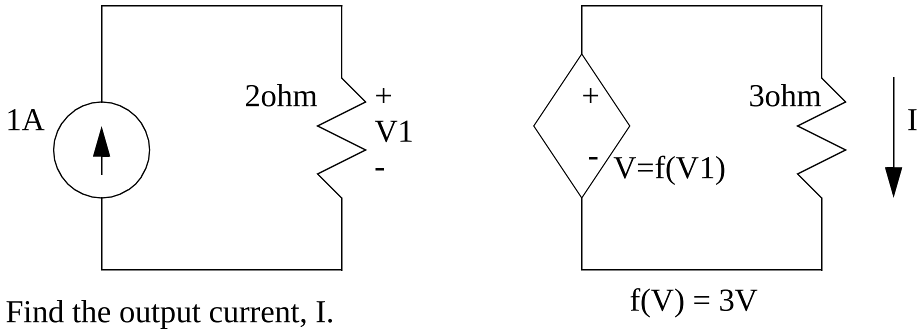


Figure 179 Dependant voltage sources



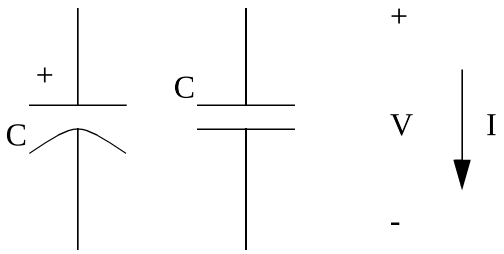
Find the output current, I.

What if the input current is 1sin(2t)A?

Figure 180 Drill problem: Find the currents in the circuit above

7.2.3 Capacitors

Capacitors are composed of two isolated metal plates very close together. When a voltage is applied across the capacitor, electrons will be forced into one plate, and pulled out of the other plate. Temporarily this creates a small current flow until the plates reach equilibrium. So, any voltage change will result in some current flow. In practical terms this means that the capacitor will block any DC voltages, except for transient effects. But, high frequency AC currents will be pass through the device. The equation for a capacitor and schematic symbols are given in Figure 181.



$$I = C\left(\frac{d}{dt}\right)V = CDV$$

Figure 181 Capacitors

The symbol on the left is for an electrolytic capacitor. These contain a special fluid that increases the effective capacitance of the device but requires that the positive and negative sides must be observed in the circuit. (Warning: reversing the polarity on an electrolytic capacitor can make them explode.) The other capacitor symbol is for a regular capacitor.

Find the current as a function of time.

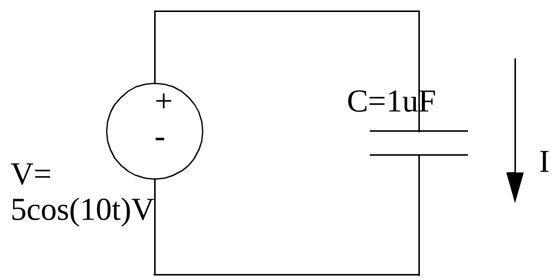
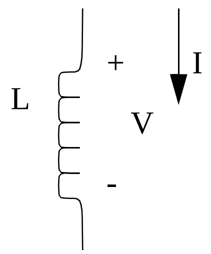


Figure 182 Drill problem: Find the current through the capacitor

7.2.4 Inductors

While a capacitor will block a DC current, and inductor will pass it. Inductors are basically coils of wire. When a current flows through the coils, a magnetic field is generated. If the current through the inductor changes then the magnetic field must change, otherwise the field is maintained without effort (i.e., voltage). Therefore the inductor resists changes in the current. The schematic symbol and relationship for an inductor are shown in Figure 183.



$$V = L\left(\frac{d}{dt}\right)I = LDI$$

Figure 183 An inductor

An inductor is normally constructed by wrapping wire in loops about a core. The core can be hollow, or be made of ferrite to increase the inductance. Inductors usually cost more than capacitors. In addition, inductors are susceptible to interference when metals or other objects disturb their magnetic fields. When possible, designers normally try to avoid using inductors in circuits.

Find the current as a function of time.

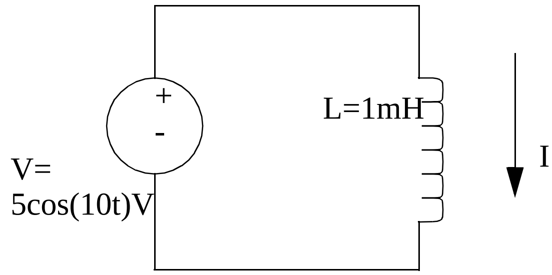
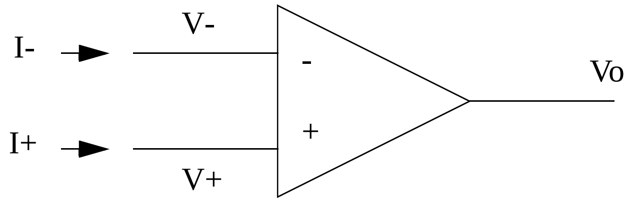


Figure 184 Drill problem: Find the current through the inductor

7.2.5 Op-Amps

The ideal model of an op-amp is shown in Figure 185. On the left hand side are the inverting and non-inverting inputs. Both of these inputs are assumed to have infinite impedance, and so no current will flow. Op-amp application circuits are designed so that the inverting and non-inverting inputs are driven to the same voltage level. The output of the op-amp is shown on the right. In circuits op-amps are used with feedback to perform standard operations such as those listed below.

- adders, subtractors, multipliers, and dividers - simple analog math operations
- amplifiers - increase the amplitude of a signal
- impedance isolators - hide the resistance of a circuit while passing a voltage



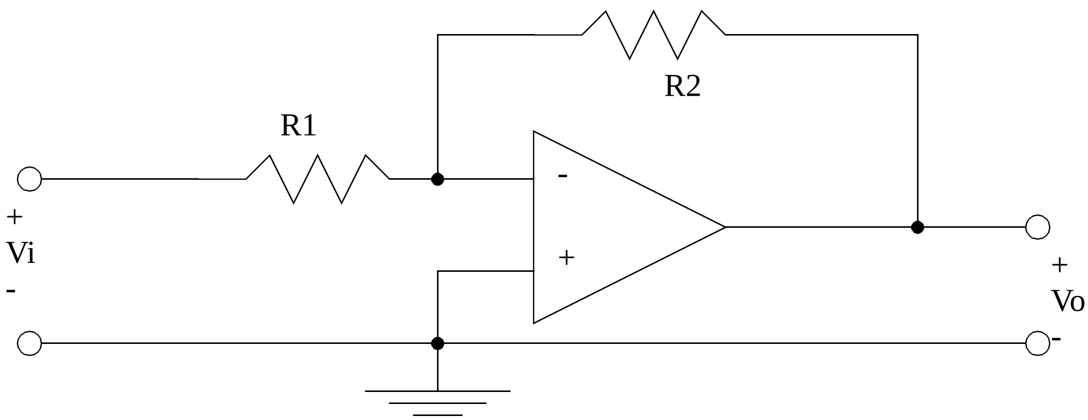
Note: for analysis use,

$$I_- = I_+ = 0$$

$$V_- = V_+$$

Figure 185 An ideal op-amp

A simple op-amp example is given in Figure 186. As expected the voltages on both of the op-amp inputs are the same. This is a function of the circuit design. (Note: most op-amp circuits are designed to force both inputs to have the same voltage, so it is always reasonable to assume they are the same.) The non-inverting input is connected directly to ground, so it will force both of the inputs to 0V. When the currents are summed at the inverting input, an equation with both the input and output voltages is obtained. The final equation shows the system is a simple multiplier, or amplifier. The gain of the amplifier is determined by the ratio of the input and feedback resistors.



The voltage at the non-inverting input will be 0V, by design the voltage at the inverting input will be the same.

$$V_+ = 0V$$

$$V_- = V_+ = 0V$$

The currents at the inverting input can be summed.

$$\sum I_{\mathrm{V-}} = \frac{V_- - V_i}{R_1} + \frac{V_- - V_o}{R_2} = 0$$

$$\frac{0 - V_i}{R_1} + \frac{0 - V_o}{R_2} = 0$$

$$V_o = \frac{-R_2 V_i}{R_1}$$

$$V_o = \left(\frac{-R_2}{R_1}\right) V_i$$

Figure 186 A simple inverting operational amplifier configuration

An op-amp circuit that can subtract signals is shown in Figure 187.

Find the input/output ratio,

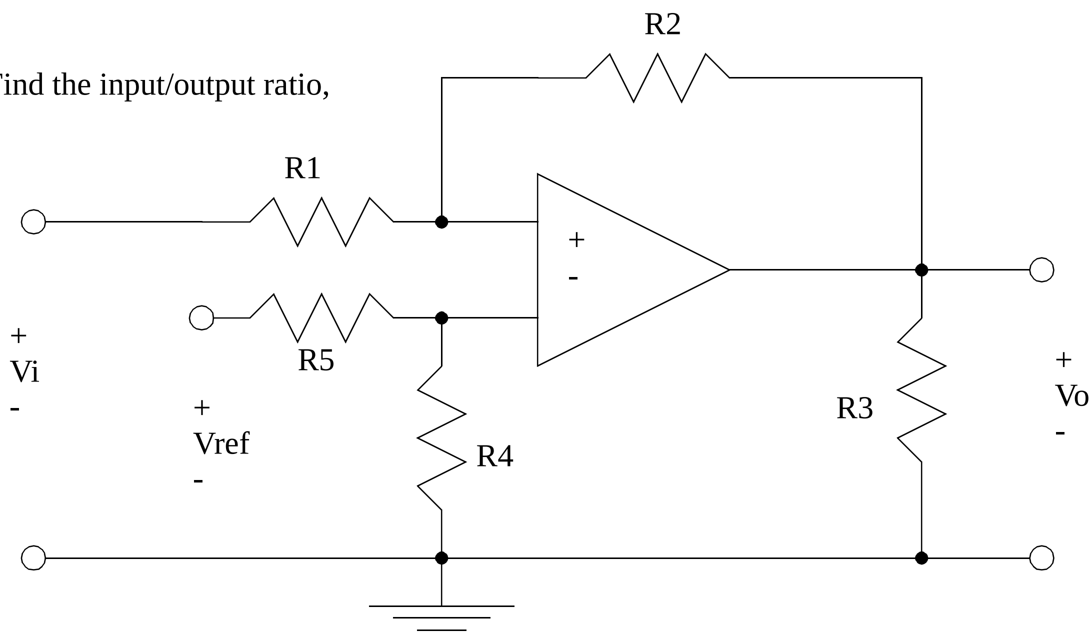


Figure 187 Op-amp example

For ideal op-amp problems the node voltage method is normally the best choice. The equations for the circuit in Figure 187 and derived in Figure 188. The general approach to this solution is to sum the currents into the inverting and non-inverting input nodes. Notice that the current into the op-amp is assumed to be zero. Both the inverting and non-inverting input voltages are then set to be equal. After that algebraic manipulation results in a final expression for the op-amp. Notice that if all of the resistor values are the same then the circuit becomes a simple subtractor.

Note: normally node voltage methods work best with op-amp circuits, although others can be used if the non-ideal op-amp model is used.

First sum the currents at the inverting and non-inverting op-amp terminals.

$$\sum I_{V+} = \frac{V_+ - V_i}{R_1} + \frac{V_+ - V_o}{R_2} = 0$$

$$V_+\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = V_i\left(\frac{1}{R_1}\right) + V_o\left(\frac{1}{R_2}\right)$$

$$V_+\left(\frac{R_1 + R_2}{R_1 R_2}\right) = V_i\left(\frac{1}{R_1}\right) + V_o\left(\frac{1}{R_2}\right)$$

$$V_+ = V_i\left(\frac{R_2}{R_1 + R_2}\right) + V_o\left(\frac{R_1}{R_1 + R_2}\right) \qquad (1)$$

$$\sum I_{V-} = \frac{V_- - V_{ref}}{R_5} + \frac{V_-}{R_4} = 0$$

$$V_-\left(\frac{1}{R_4} + \frac{1}{R_5}\right) = V_{ref}\left(\frac{1}{R_5}\right)$$

$$V_- = V_{ref}\left(\frac{R_4}{R_4 + R_5}\right) \qquad (2)$$

Now the equations can be combined.

$$V_- = V_+$$

$$V_{ref}\left(\frac{R_4}{R_4 + R_5}\right) = V_i\left(\frac{R_2}{R_1 + R_2}\right) + V_o\left(\frac{R_1}{R_1 + R_2}\right)$$

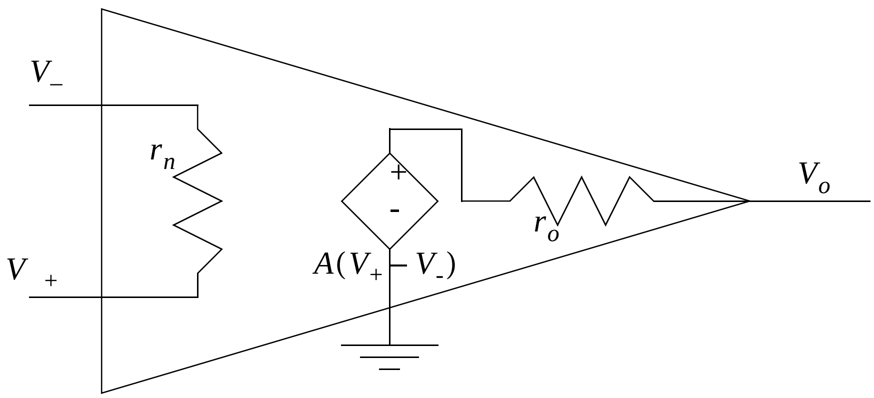
$$V_o\left(\frac{R_1}{R_1 + R_2}\right) = V_i\left(\frac{R_2}{R_1 + R_2}\right) - V_{ref}\left(\frac{R_4}{R_4 + R_5}\right)$$

$$V_o = V_i\left(\frac{R_2}{R_1}\right) - V_{ref}\left(\frac{R_4(R_1 + R_2)}{R_1(R_4 + R_5)}\right)$$

Figure 188 Op-amp example (continued)

An op-amp (operational amplifier) has an extremely high gain, typically 100,000 times. The gain is multiplied by the difference between the inverting and non-inverting terminals to form an output. A typical op-amp will work for signals from DC up to about

100KHz. When the op-amp is being used for high frequencies or large gains, the model of the op-amp in Figure 189 should be used. This model includes a large resistance between the inverting and non-inverting inputs. The voltage difference drives a dependent voltage source with a large gain. The output resistance will limit the maximum current that the device can produce.



typically,

$r_n > 1M\Omega$

$A > 10^5$

$r_o < 100\Omega$

Figure 189 A non-ideal op-amp model

7.3 IMPEDANCE

Circuit components can be represented in impedance form as shown in Figure 190. When represented this way the circuit solutions can focus on impedances, 'Z', instead of resistances, 'R'. Notice that the primary difference is that the differential operator has been replaced. In this form we can use impedances as if they are resistances.

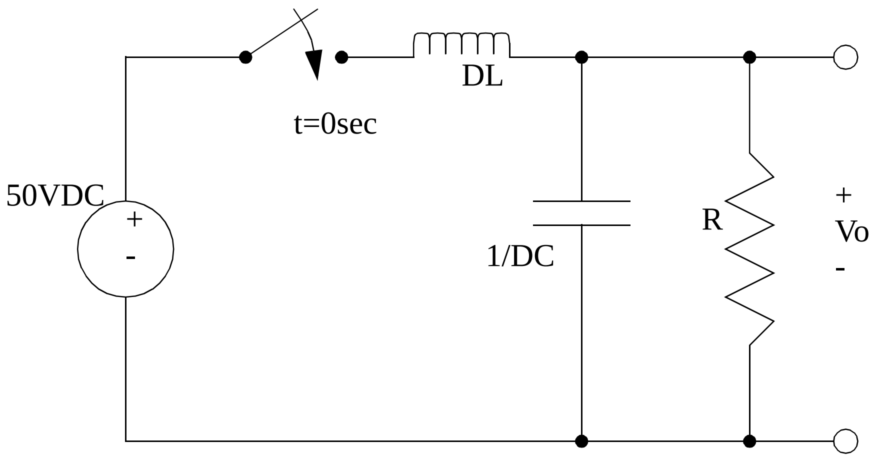
Device	Time domain	Impedance
Resistor	$V(t) = RI(t)$	$Z = R$
Capacitor	$V(t) = \frac{1}{C}\int I(t)dt$	$Z = \frac{1}{DC}$
Inductor	$V(t) = L\frac{d}{dt}I(t)$	$Z = LD$

Note: Impedance is like resistance, except that it includes time variant features also.

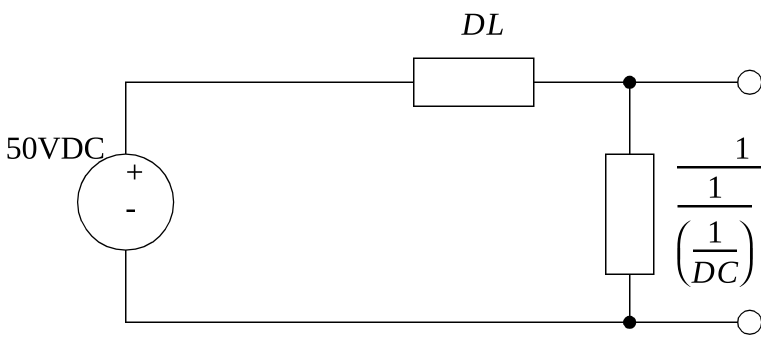
$$V = ZI$$

Figure 190 Impedances for electrical components

When representing component values with impedances the circuit solution is done as if all circuit components are resistors. An example of this is shown in Figure 191. Notice that the two impedances at the right (resistor and capacitor) are equivalent to two resistors in parallel, and the overall circuit is a voltage divider. The impedances are written beside the circuit elements.



Find the equivalent for the capacitor and resistor in parallel.



$$\frac{1}{\frac{1}{\left(\frac{1}{DC}\right)}+\frac{1}{R}} = \frac{1}{DC+\frac{1}{R}} = \frac{R}{RCD+1}$$

Treat the circuit as a voltage divider,

$$V_o = 50V\frac{\left(\frac{R}{1+DCR}\right)}{DL+\left(\frac{R}{1+DCR}\right)} = 50V\left(\frac{R}{D^2RLC+DL+R}\right)$$

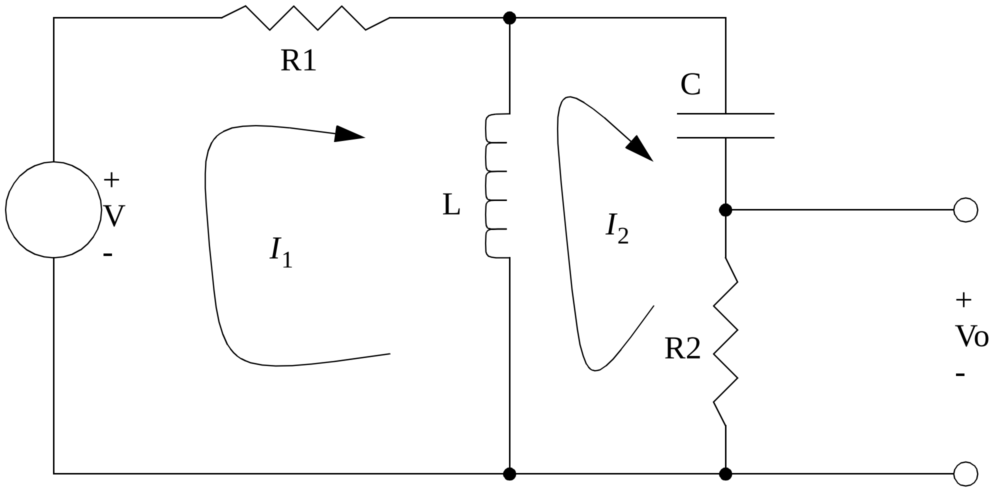
Figure 191 A impedance example for a circuit

7.4 EXAMPLE SYSTEMS

The list of instructions can be useful when approaching a circuits problem. The most important concept to remember is that a minute of thinking about the solution approach will save ten minutes of backtracking and fixing mistakes.

1. Look at the circuit to determine if it is a standard circuit type such as a voltage divider, current divider or an op-amp inverting amplifier. If so, use the standard solution to solve the problem.
2. Otherwise, consider the nodes and loops in the circuit. If the circuit contains fewer loops, select the current loop method. If the circuit contains fewer nodes, select the node voltage method. Before continuing, verify that the select method can be used for the circuit.
3. For the node voltage method define node voltages and current directions. For the current loop method define current loops and indicate voltage rises or drops by adding ’+’ or ’-’ signs.
4. Write the equations for the loops or nodes.
5. Identify the desired value and eliminate unwanted values using algebra techniques.
6. Use numerical values to find a final answer.

The circuit in Figure 192 could be solved with two loops, or two nodes. An arbitrary decision is made to use the current loop method. The voltages around each loop are summed to provide equations for each loop.



Note: when summing voltages in a loop remember to deal with sources that increase the voltage by flipping the sign.

First, sum the voltages around the loops and then eliminate I1.

$$\sum V_{L1} = -V + R_1 I_1 + L(DI_1 - DI_2) = 0$$

$$(R_1 + LD)I_1 = V + (LD)I_2 \qquad (1)$$

$$I_1 = \frac{V}{R_1 + LD} + \left(\frac{LD}{R_1 + LD}\right)I_2 \qquad (2)$$

$$\sum V_{L2} = L(DI_2 - DI_1) + \frac{I_2}{CD} + R_2 I_2 = 0$$

$$L(DI_2 - DI_1) + \frac{I_2}{CD} + R_2 I_2 = 0 \qquad (3)$$

$$(DL)I_1 = \left(LD + \frac{1}{CD} + R_2\right)I_2$$

$$I_1 = \left(1 + \frac{1}{CLD^2} + \frac{R_2}{LD}\right)I_2 \qquad (4)$$

Figure 192 Example problem

The equations in Figure 192 are manipulated further in Figure 193 to develop an input-output equation for the second current loop. This current can be used to find the current through the output resistor R2. The output voltage can then be found by multiplying the R2 and I2.

First, sum the voltages around the loops and then eliminate I1.

$$I_1 = \frac{V}{R_1 + LD} + \left(\frac{LD}{R_1 + LD}\right)I_2 = \left(1 + \frac{1}{CLD^2} + \frac{R_2}{LD}\right)I_2$$

$$\frac{V}{R_1 + LD} = \left(1 + \frac{1}{CLD^2} + \frac{R_2}{LD} - \frac{LD}{R_1 + LD}\right)I_2$$

$$\frac{V}{R_1 + LD} = \left(\frac{(CLD^2 + 1 + CDR_2)(R_1 + LD) - CL^2D^3}{CLD^2}\right)I_2$$

$$I_2 = \left(\frac{CLD^2}{(R_1 + LD)((CLD^2 + 1 + CDR_2)(R_1 + LD) - CL^2D^3)}\right)V$$

$$I_2 = \left(\frac{CLD^2}{CL(R_1 + R_2)D^3 + L(CR_1^2 + 2CR_1R_2 + L)D^2 + R_1(CR_1R_2 + 2L)D + (R_1^2)}\right)V$$

Convert it to a differential equation.

$$CL(R_1 + R_2)I_2''' + L(CR_1^2 + 2CR_1R_2 + L)I_2'' + R_1(CR_1R_2 + 2L)I_2' + (R_1^2)I_2 = CLV''$$

Figure 193 Example problem (continued)

The equations can also be manipulated into state equations, as shown in Figure 194. In this case a dummy variable is required to replace the two first derivatives in the first equation. The dummy variable is used in place of I1, which now becomes an output variable. In the remaining state equations I1 is replaced by q1. In the final matrix form the state equations are in one matrix, and the output variable must be calculated separately.

State equations can also be developed using equations (1) and (3).

(1) becomes $R_1 I_1 + L I_1' = V + L I_2'$

$$L I_1' - L I_2' = V - R_1 I_1$$

$$I_1' - I_2' = \frac{V}{L} - \frac{R_1}{L} I_1$$

$$q_1 = I_1 - I_2 \qquad (10)$$

$$q_1' = \frac{V}{L} - \frac{R_1}{L} I_1$$

$$I_1 = q_1 + I_2 \qquad (11)$$

$$q_1' = \frac{V}{L} - \frac{R_1}{L}(q_1 + I_2)$$

$$q_1' = q_1\left(-\frac{R_1}{L}\right) + I_2\left(-\frac{R_1}{L}\right) + \frac{V}{L} \qquad (12)$$

(3) becomes $L I_2'' - L I_1'' + \frac{I_2}{C} + R_2 I_2' = 0$

$$V - R_1 I_1 + \frac{I_2}{C} + R_2 I_2' = 0$$

$$V - R_1(q_1 + I_2) + \frac{I_2}{C} = -R_2 I_2'$$

$$I_2' = I_2\left(-R_1 + \frac{1}{C}\right) + q_1(-R_1) + V \qquad (13)$$

These can be put in matrix form,

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L} & -\frac{R_1}{L} \\ -R_1 & -R_1 + \frac{1}{C} \end{bmatrix} \begin{bmatrix} q_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} \frac{V}{L} \\ V \end{bmatrix} \qquad \begin{bmatrix} I_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ I_2 \end{bmatrix}$$

Figure 194 Example problem (continued) solve the problem in Figure 192 using the node voltage method.

Figure 195 Drill problem: Use the node voltage method

Find the equation relating the output and input voltages,

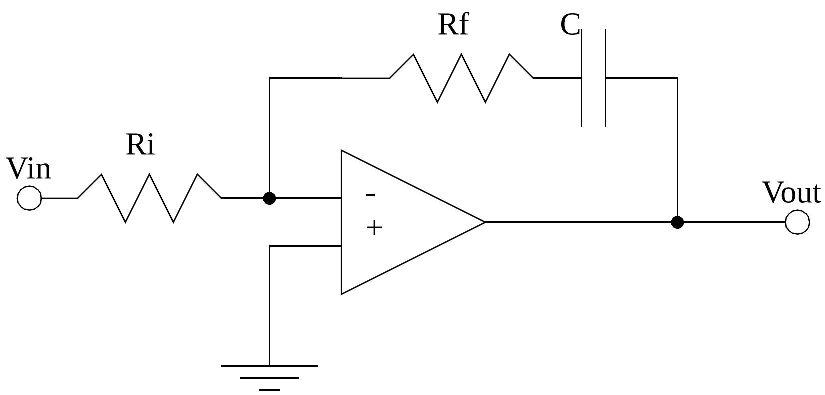
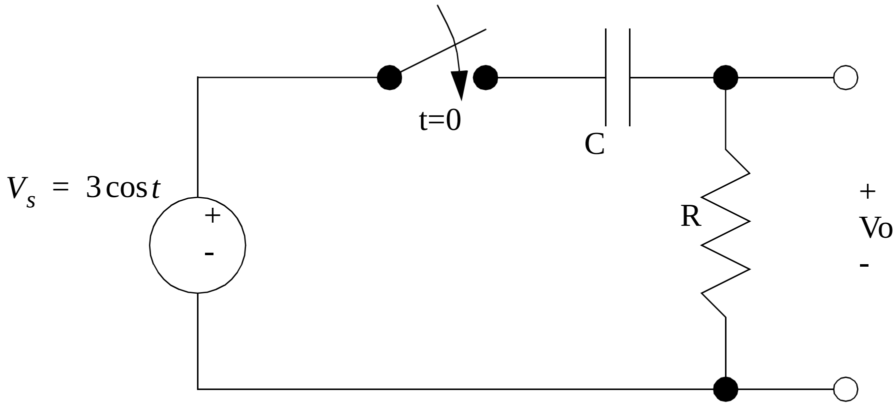


Figure 196 Drill problem: Find the state equation

The circuit in Figure 197 can be solved as a voltage divider when the capacitor is represented as an impedance. In this case the result is a first-order differential equation.



As normal we relate the source voltage to the output voltage. The we find the values for the various terms in the frequency domain.

$$V_o = V_s\left(\frac{Z_R}{Z_R + Z_C}\right) \qquad \text{where,} \qquad Z_R = R \qquad Z_C = \frac{1}{DC}$$

Next, we may combine the equations, and convert it to a differential equation.

$$V_o = V_s\left(\frac{R}{R + \frac{1}{DC}}\right)$$

$$V_o = V_s\left(\frac{CRD}{CRD + 1}\right)$$

$$V_o(CRD + 1) = V_s(CRD)$$

$$V_o'(CR) + V_o = V_s'(CR)$$

$$V_o' + V_o\left(\frac{1}{CR}\right) = V_s'$$

Figure 197 Circuit solution using impedances

The first-order differential equation in Figure 197 is continued in Figure 198 where the equation is integrated. The solution is left in variable form, except for the supply voltage.

First write the homogeneous solution using the known relationship.

$$V_o' + V_o\left(\frac{1}{CR}\right) = 0 \qquad \text{yields} \qquad V_h = C_1 e^{-\frac{t}{CR}}$$

Next, the particular solution can be determined, starting with a guess.

$$V_o' + V_o\left(\frac{1}{CR}\right) = \left(\frac{d}{dt}\right)(3\cos t) = -3\sin t$$

$$V_p = A\sin t + B\cos t$$

$$V_p' = A\cos t - B\sin t$$

$$(A\cos t - B\sin t) + (A\sin t + B\cos t)\left(\frac{1}{CR}\right) = -3\sin t$$

$$A + B\left(\frac{1}{CR}\right) = 0$$

$$A = B\left(\frac{-1}{CR}\right)$$

$$-B + A\left(\frac{1}{CR}\right) = -3$$

$$-B + B\left(\frac{-1}{CR}\right)\left(\frac{1}{CR}\right) = -3$$

$$B\left(\frac{1}{C^2R^2} + 1\right) = 3$$

$$B = \frac{3C^2R^2}{1 + C^2R^2} \qquad A = \left(\frac{3C^2R^2}{1 + C^2R^2}\right)\left(\frac{-1}{CR}\right) = \frac{-3CR}{1 + C^2R^2}$$

$$V_p = \sqrt{A^2 + B^2}\sin\left(t + \operatorname{atan}\left(\frac{B}{A}\right)\right)$$

The homogeneous and particular solutions can now be combined. The system will be assumed to be at rest initially.

$$V_0 = V_h + V_p = C_1 e^{-\frac{t}{CR}} + \sqrt{A^2 + B^2}\sin\left(t + \operatorname{atan}\left(\frac{B}{A}\right)\right)$$

$$0 = C_1 e^0 + \sqrt{A^2 + B^2}\sin\left(0 + \operatorname{atan}\left(\frac{B}{A}\right)\right)$$

$$C_1 = -\sqrt{A^2 + B^2}\sin\left(0 + \operatorname{atan}\left(\frac{B}{A}\right)\right)$$

Figure 198 Circuit solution using impedances (continued)

7.5 PERMANENT MAGNET DC MOTORS

DC motors apply a torque between the rotor and stator that is related to the applied voltage/current. When a voltage is applied the torque will cause the rotor to accelerate. For any voltage and load on the motor there will tend to be a final angular velocity due to friction and drag in the motor. And, for a given voltage the ratio between steady-state torque and speed will be a straight line, as shown in Figure 199.

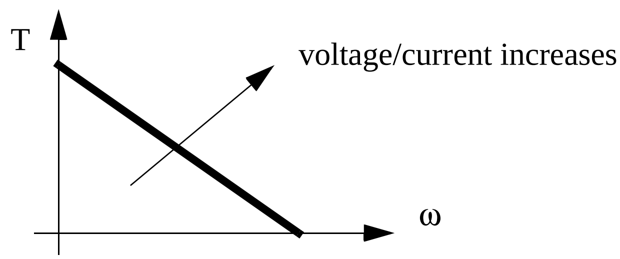
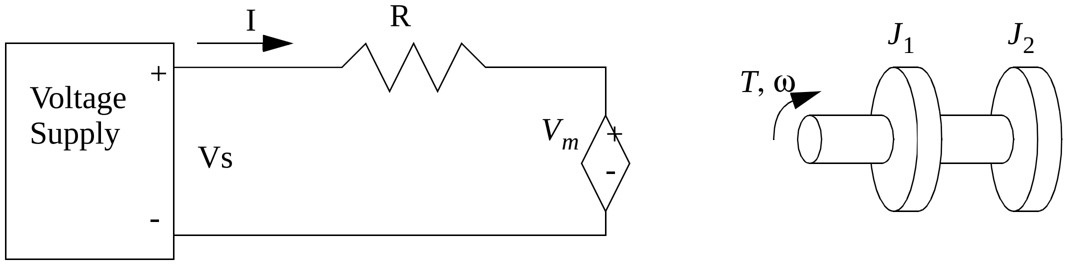


Figure 199 Torque speed curve for a permanent magnet DC motor

The basic equivalent circuit model is shown in Figure 200, includes the rotational inertia of the rotor and any attached loads. On the left hand side is the resistance of the motor and the ’back emf’ dependent voltage source. On the right hand side the inertia components are shown. The rotational inertia J1 is the motor rotor, and the second inertia is an attached disk.



Because a motor is basically wires in a magnetic field, the electron flow (current) in the wire will push against the magnetic field. And, the torque (force) generated will be proportional to the current.

$$T_m = KI \qquad \therefore I = \frac{T_m}{K}$$

Next, consider the power in the motor,

$$P = V_m I = T\omega = KI\omega \qquad \therefore V_m = K\omega$$

Consider the dynamics of the rotating masses by summing moments.

$$\sum M = T_m - T_{load} = J\left(\frac{d}{dt}\right)\omega \qquad \therefore T_m = J\left(\frac{d}{dt}\right)\omega + T_{load}$$

Figure 200 The torque and inertia in a basic motor model

These basic equations can be manipulated into the first-order differential equation in Figure 201.

The current-voltage relationship for the left hand side of the equation can be written and manipulated to relate voltage and angular velocity.

$$I = \frac{V_s - V_m}{R}$$

$$\therefore \frac{T_m}{K} = \frac{V_s - K\omega}{R}$$

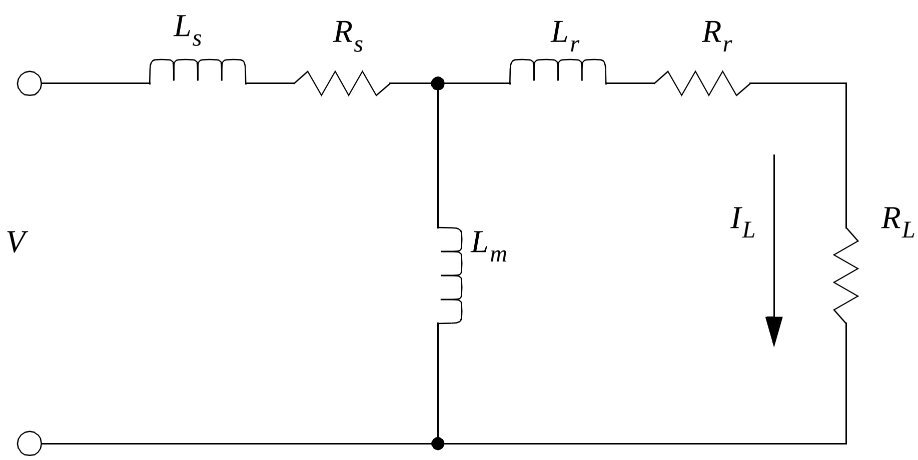
$$\therefore \frac{J\left(\frac{d}{dt}\right)\omega + T_{load}}{K} = \frac{V_s - K\omega}{R}$$

$$\boxed{\therefore \left(\frac{d}{dt}\right)\omega + \omega\left(\frac{K^2}{JR}\right) = V_s\left(\frac{K}{JR}\right) - \frac{T_{load}}{J}}$$

Figure 201 The first-order model of a motor

7.6 INDUCTION MOTORS

• The equivalent circuit for an AC motor is given below.



$$R_L = \frac{1-f}{f} R_r$$

Figure 202 Basic model of an induction motor

• The torque relationships are given below, First the torques on the motor are summed,

$$\sum M = T_{rotor} - T_{load} = J\frac{d}{dt}\omega$$

7.7 BRUSHLESS SERVO MOTORS

- the motors use a permanent magnet on the rotor, and coils on the stator.

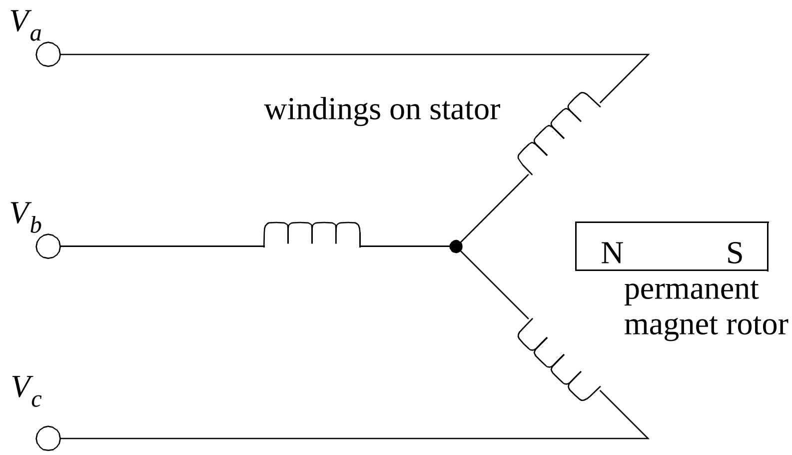


Figure 203 The construction of a brushless servo motor

$$V_t = \left(R_m + \frac{d}{dt}L\right)I_m + E$$

$$E = K_e \omega$$

$$T = K_t I_m$$

where,

V_t = terminal voltage across motor windings
R_m = resistance of a motor winding
L = phase to phase inductance
I_m = current in winding
E = back e.m.f. of motor
K_e = motor speed constant
ω = motor speed
K_t = motor torque constant
T = motor torque

Figure 204 Basic relationships for a brushless motor

$$V_t = \left(R_m + \frac{d}{dt}L\right)\frac{T}{K_t} + K_e\omega$$

$$\sum M = T - T_{load} = J\frac{d}{dt}\omega$$

$$T = J\frac{d}{dt}\omega + T_{load}$$

where,

J = combined moments of inertia for the rotor and external loads

T_{load} = the applied torque in the system

$$V_t = \left(R_m + \frac{d}{dt}L\right)\frac{J\frac{d}{dt}\omega + T_{load}}{K_t} + K_e\omega$$

$$V_t = \frac{JR_m}{K_t}\frac{d}{dt}\omega + \frac{LJ}{K_t}\left(\frac{d}{dt}\right)^2\omega + \frac{R_m}{K_t}T_{load} + \frac{L}{K_t}\frac{d}{dt}T_{load} + K_e\omega$$

$$(LJ)\left(\frac{d}{dt}\right)^2\omega + (JR_m)\frac{d}{dt}\omega + K_eK_t\omega = K_tV_t - L\frac{d}{dt}T_{load} - R_mT_{load}$$

$$\left(\frac{d}{dt}\right)^2\omega + \frac{R_m}{L}\frac{d}{dt}\omega + \frac{K_eK_t}{LJ}\omega = \frac{K_tV_t}{LJ} - \frac{1}{J}\frac{d}{dt}T_{load} - \frac{R_mT_{load}}{LJ}$$

Figure 205 An advanced model of a brushless servo motor

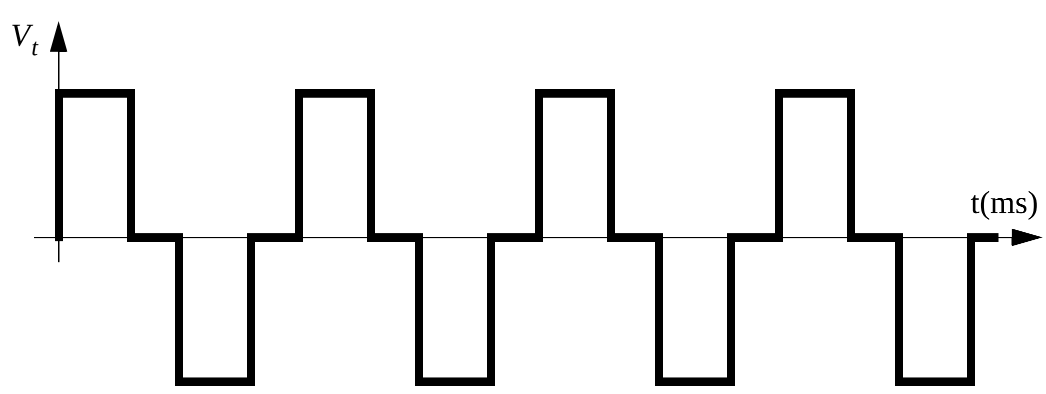


Figure 206 Typical supply voltages

7.8 OTHER TOPICS

The relationships in Figure 207 can be used to calculate the power and energy in a system. Notice that the power calculations focus on resistance, as resistances will dissipate power in the form of heat. Other devices, such as inductors and capacitors, store energy, but don't dissipate it.

$$P = IV = I^2R = \frac{V^2}{R} \qquad\qquad E = Pt$$

Figure 207 Electrical power and energy

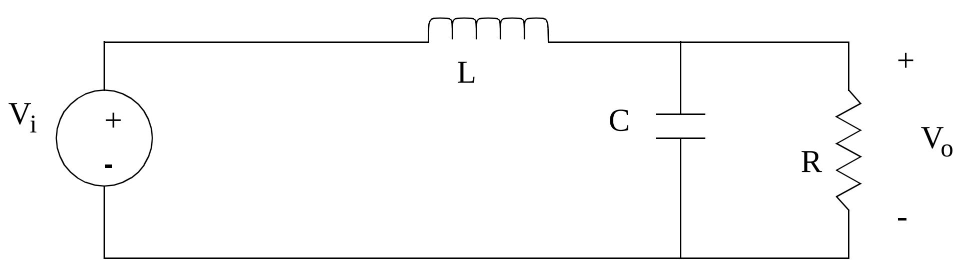
7.9 SUMMARY

• Basic circuit components are resistors, capacitors, inductors op-amps.
• node and loop methods can be used to analyze circuits.
• Capacitor and inductor impedances can be used as resistors in calculations.

7.10 PRACTICE PROBLEMS

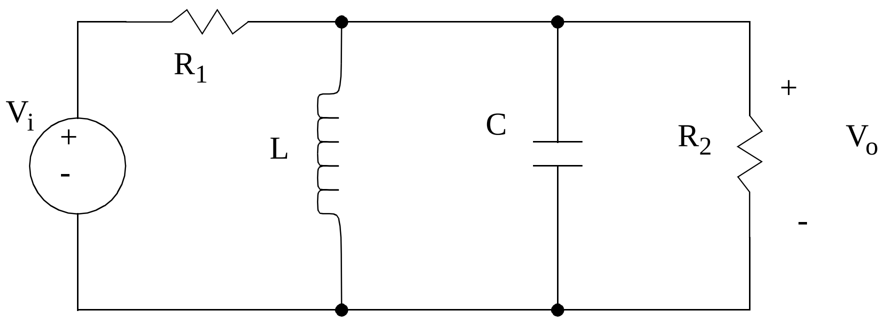
1. Find the combined values for resistors, capacitors and inductors in series and parallel.

2. Consider the following circuit.

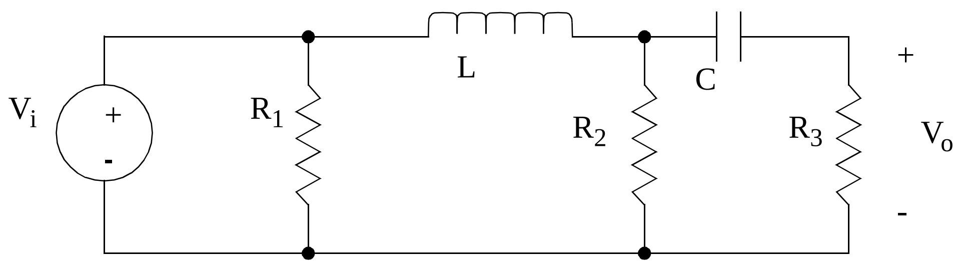


a) Develop a differential equation for the circuit.
b) Put the equation in state variable matrix form.

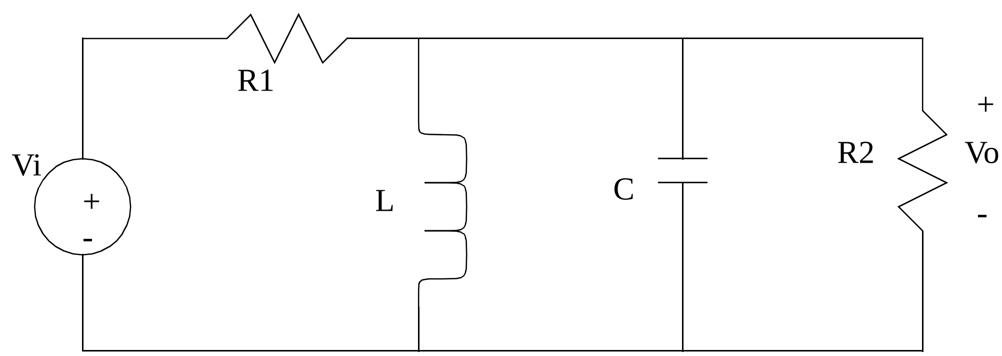
3. Develop differential equations and the input-output equation for the electrical system below.



4. Consider the following circuit. Develop a differential equation for the circuit.



5. Find the input-output equation for the circuit below, and then find the natural frequency and damping coefficient.

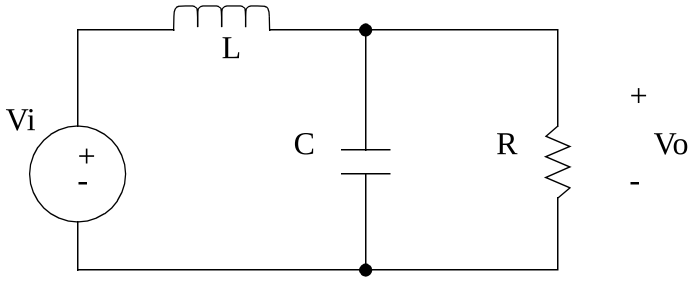


(ans.

a) $$V_o''(C) + V_o'\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + V_o\left(\frac{1}{L}\right) = V_i'\left(\frac{1}{R_1}\right)$$

b) $$\omega_n = \sqrt{\frac{1}{LC}} \qquad \zeta = \frac{\sqrt{L}(R_1 + R_2)}{2\sqrt{C}R_1R_2}$$

6. a) Find the differential equation for the circuit below where the input is Vi, and the output is Vo.



(ans.

Sum currents at node Vo

$$\sum I_{V_o} = \frac{(V_o - V_i)}{DL} + (V_o)DC + \frac{(V_o)}{R} = 0$$

$$V_o - V_i + V_o D^2 LC + \frac{V_o DL}{R} = 0$$

$$V_o''(LC) + V_o'\left(\frac{L}{R}\right) + V_o = V_i$$

b) Convert the equation to an input-output equation.

(ans.

$$V_o''(LC) + V_o'\left(\frac{L}{R}\right) + V_o = V_i$$

c) Solve the differential equation found in part b) using the numerical values given below. Assume at time t=0, the circuit has the voltage Vo and the first derivative shown below.

$L = 10mH$ $\quad C = 1\mu F$ $\quad R = 1K\Omega$ $\quad V_i = 10V$

at t=0s $\quad V_o = 2V$ $\quad V_o' = 3\frac{V}{s}$

(ans.

$$V_o''(LC) + V_o'\left(\frac{L}{R}\right) + V_o = V_i \qquad V_o''(10^{-2}10^{-6}) + V_o'\left(\frac{10^{-2}}{10^{3}}\right) + V_o = 10$$

$$V_o''(10^{-8}) + V_o'(10^{-5}) + V_o = 10$$

$$V_o'' + V_o'(10^{3}) + V_o(10^{8}) = (10^{9})$$

homogeneous;

$$A^2e^{At} + Ae^{At}(10^3) + e^{At}(10^8) = 0$$

$$A^2 + A(10^3) + (10^8) = 0 \qquad A = \frac{-10^3 \pm \sqrt{(10^3)^2 - 4(10^8)}}{2} = -500 \pm 9987j$$

$$V_h = C_1e^{-500t}\cos(9987t + C_2)$$

particular; guess

$$V_p = A \qquad (0) + (0)(10^3) + (A)(10^8) = (10^9) \qquad A = 10$$

$$V_p = 10$$

for initial conditions,

$$V_o = C_1e^{-500t}\cos(9987t + C_2) + 10$$

for t=0, Vo=2V $\qquad 2 = C_1e^{-500(0)}\cos(9987(0) + C_2) + 10$

$$-8 = C_1\cos(C_2) \qquad (1)$$

$$V_o' = -500C_1e^{-500t}\cos(9987t + C_2) - 9987C_1e^{-500t}\sin(9987t + C_2)$$

for t=0, Vo’=3V $\qquad 3 = -500C_1\cos(C_2) - 9987C_1\sin(C_2)$

$$3 = -500C_1\cos(C_2) - 9987C_1\sin(C_2)$$

$$3 = -4000 - 9987\left(\frac{-8}{\cos(C_2)}\right)\sin(C_2)$$

$$\frac{4003}{8(9987)} = \frac{\sin(C_2)}{\cos(C_2)} = \tan(C_2) \qquad C_2 = 0.050$$

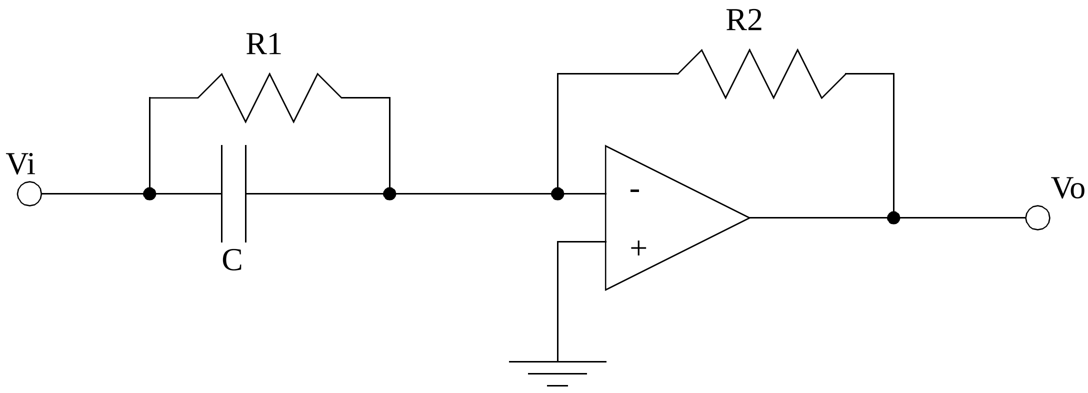
$$C_1 = \frac{-8}{\cos(C_2)} = -8.01$$

$$V_o = -8.01e^{-500t}\cos(9987t + 0.050) + 10$$

7.

a) Write the differential equations for the system pictured below.

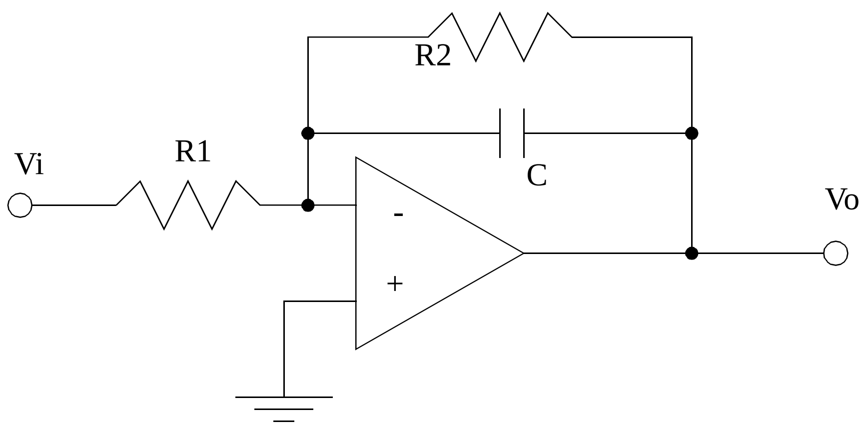
b) Put the equations in input-output form.



ans. a) $$V_i\left(\frac{-1}{R_1}\right) + V_i'(-C) + V_o\left(\frac{-1}{R_2}\right) = 0$$

b) $$V_i\left(\frac{-1}{R_1}\right) + V_i'(-C) = V_o\left(\frac{1}{R_2}\right)$$

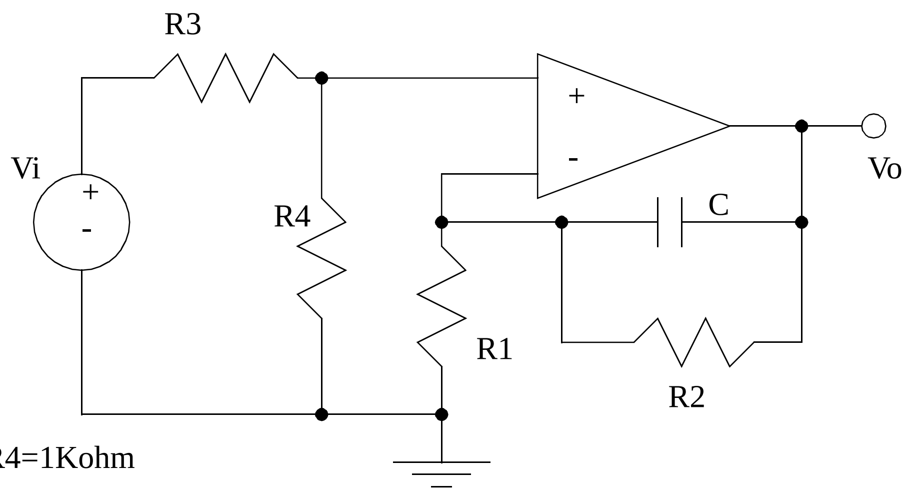
8. Given the circuit below, find the ratio between the output and the input. Simplify the results.



ans. $$\frac{V_o}{V_i} = \frac{-R_2}{R_1 + sR_1R_2C}$$

9. Develop the differential equation(s) for the system below, and use them to find the response to

the following inputs. Assume that the circuit is off initially.



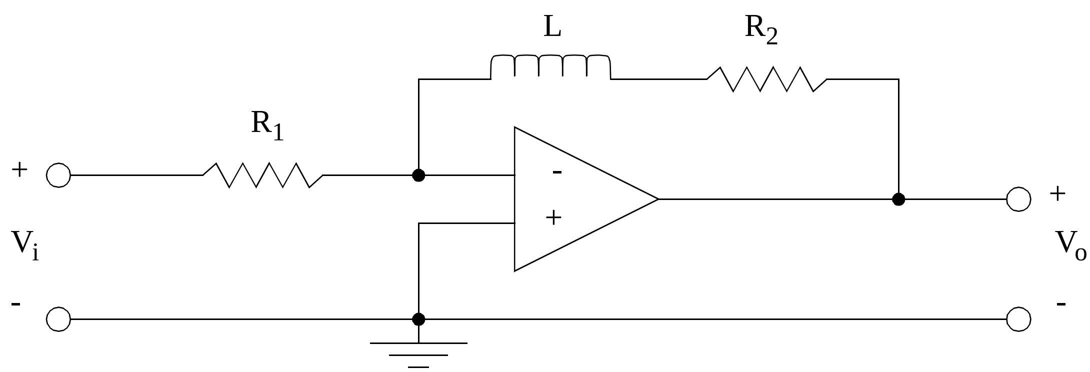
R1=R2=R3=R4=1Kohm
C=1uF

a) $V_i = 5\sin(100t)$

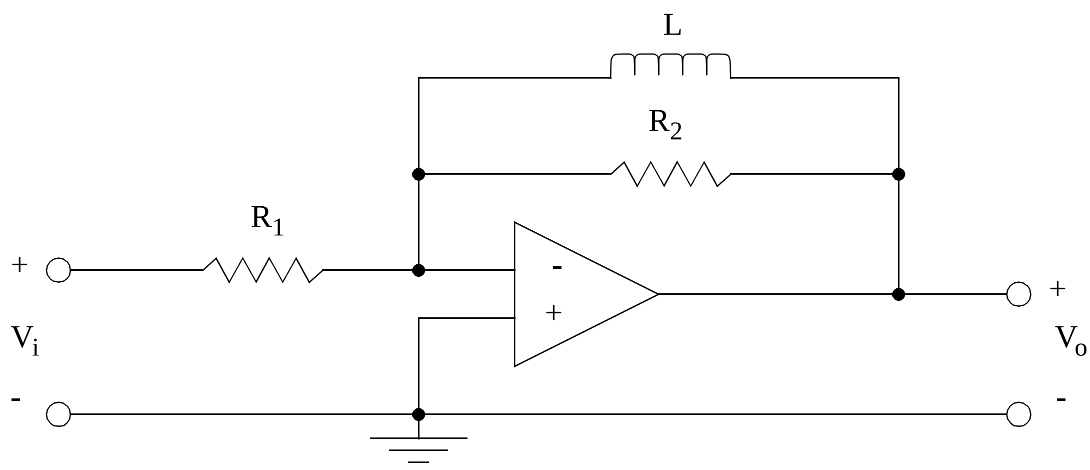
b) $V_i = 5\sin(1000000t)$

c) $V_i = 5$

10. Examine the following circuit and then derive the differential equation.

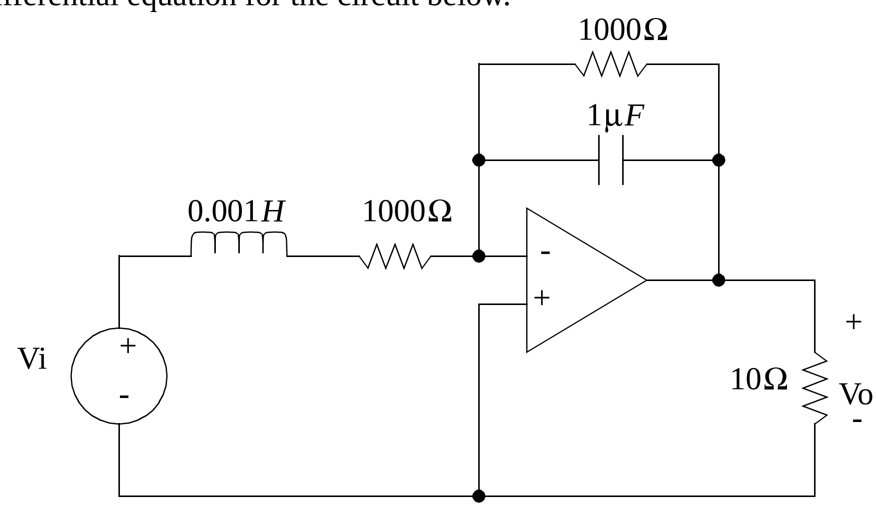


11. Examine the following circuit and then derive the differential equation.



12.

a) Find the differential equation for the circuit below.



(ans.

Create a node between the inductor and resistor Va, and use the node voltage method

$$\sum I_{V_A} = \frac{(V_A - V_i)}{0.001D} + \frac{(V_A - V_-)}{1000} = 0 \qquad V_- = V_+ = 0V$$

$$1000000(V_A - V_i) + V_A D = 0$$

XXXXADD UNITSXXXXX

$$V_A = V_i\left(\frac{1000000}{1000000 + D}\right)$$

$$\sum I_{V_-} = \frac{(V_- - V_A)}{1000} + \frac{(V_- - V_o)}{1000} + (V_- - V_o)(0.000001D) = 0$$

$$\frac{(-1)}{1000}V_i\left(\frac{1000000}{1000000 + D}\right) + \frac{(-V_o)}{1000} + (-V_o)(0.000001D) = 0$$

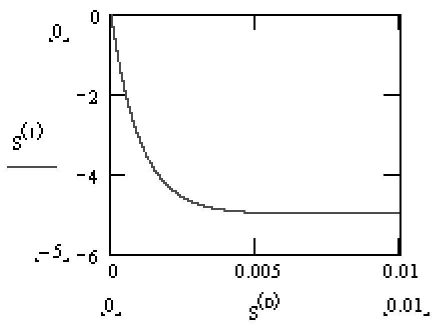
$$V_o(-1 - 0.001D) = V_i\left(\frac{1000000}{1000000 + D}\right)$$

$$V_o(-1000000 - D - 1000D - 0.001D^2) = 1000000V_i$$

$$V_o''(-10^{-9}) + V_o'(-1.001(10^{-3})) + V_o(-1) = V_i$$

b) Put the differential equation in state variable form and use your calculator to produce a detailed sketch of the output voltage Vo. Assume the system starts at rest, and the input is Vi=5V.

(ans. $\frac{d}{dt}V_o = V_o'$ $\qquad \frac{d}{dt}V_o' = -1000000000V_i - 1001000V_o' - 1000000000V_o$



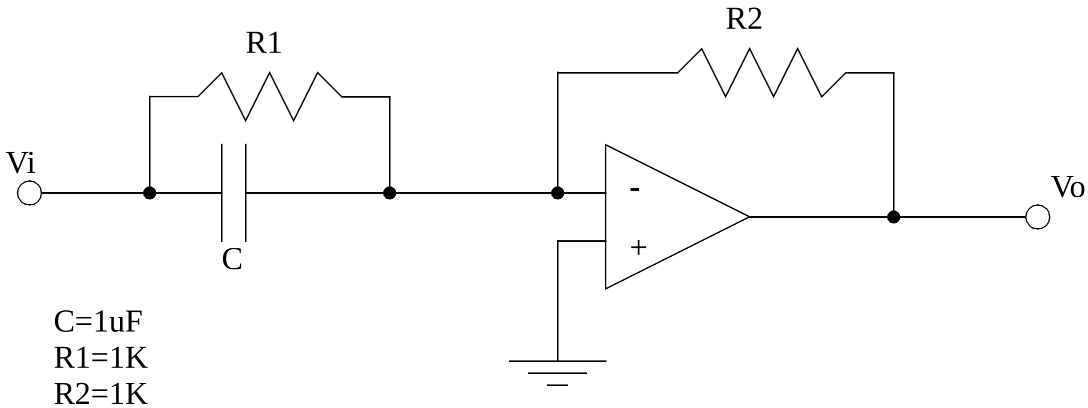
13.

a) Write the differential equations for the system pictured below.

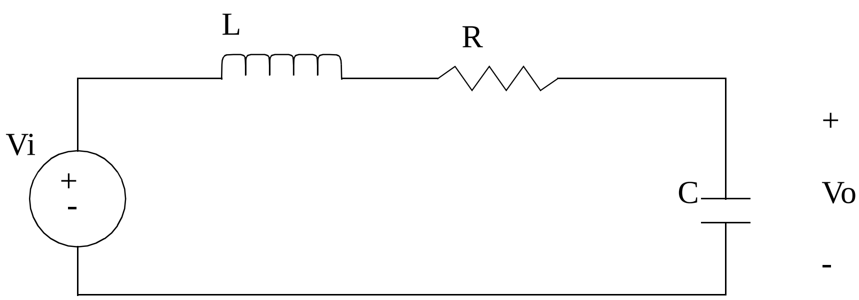
b) Put the equations in state variable form.

c) Use mathcad to find the ratio between input and output voltages for a range of frequencies. The general method is put in a voltage such as Vi=1sin(___t), and see what the magnitude of the output is. Divide the magnitude of the output sine wave by the input magnitude. Note: This should act as a high pass or low pass filter.

d) Plot a graph of gain against the frequency of the input.



14. Write the differential equation for the following circuit.



8. FEEDBACK CONTROL SYSTEMS

Topics:
- Transfer functions, block diagrams and simplification
- Feedback controllers
- Control system design

Objectives:
- To be able to represent a control system with block diagrams.
- To be able to select controller parameters to meet design objectives.

8.1 INTRODUCTION

Every engineered component has some function. A function can be described as a transformation of inputs to outputs. For example this could be an amplifier that accepts a signal from a sensor and amplifies it. Or, consider a mechanical gear box with an input and output shaft. A manual transmission has an input shaft from the motor and from the shifter. When analyzing systems we will often use transfer functions that describe a system as a ratio of output to input.

8.2 TRANSFER FUNCTIONS

Transfer functions are used for equations with one input and one output variable. An example of a transfer function is shown below in Figure 208. The general form calls for output over input on the left hand side. The right hand side is comprised of constants and the ’D’ operator. In the example ’x’ is the output, while ’F’ is the input.

The general form

$$\frac{output}{input} = f(D)$$

An example

$$\frac{x}{F} = \frac{4 + D}{D^2 + 4D + 16}$$

Figure 208 A transfer function example

If both sides of the example were inverted then the output would become ’F’, and the input ’x’. This ability to invert a transfer function is called reversibility. In reality many systems are not reversible.

There is a direct relationship between transfer functions and differential equations. This is shown for the second-order differential equation in Figure 209. The homogeneous equation (the left hand side) ends up as the denominator of the transfer function. The non-homogeneous solution ends up as the numerator of the expression.

$$x'' + 2\zeta\omega_n x' + \omega_n^2 x = \frac{f}{M}$$

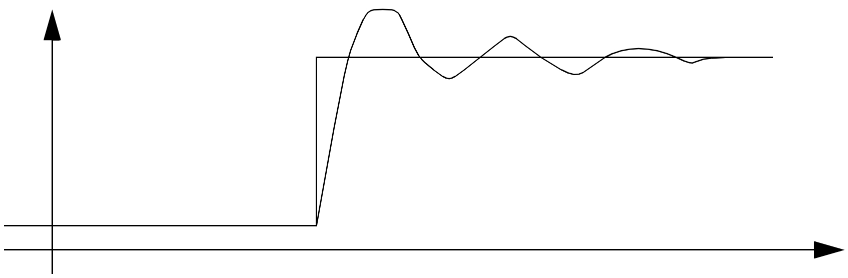
$$xD^2 + 2\zeta\omega_n xD + \omega_n^2 x = \frac{f}{M}$$

$$x(D^2 + 2\zeta\omega_n D + \omega_n^2) = \frac{f}{M}$$

$$\frac{x}{f} = \frac{\left(\frac{1}{M}\right)}{D^2 + 2\zeta\omega_n D + \omega_n^2}$$

particular

homogeneous



ω_n	Natural frequency of system, approximate frequency of control system oscillations.
ξ	Damping factor of system. If < 1 underdamped, and system will oscillate. If =1 critically damped. If < 1 overdamped, and never any oscillation (more like a first-order system). As damping factor approaches 0, the first peak becomes infinite in height.

Figure 209 The relationship between transfer functions and differential equations for a mass-spring-damper example

The transfer function for a first-order differential equation is shown in Figure 210. As before the homogeneous and non-homogeneous parts of the equation becomes the denominator and the numerator of the transfer function.

$$x' + \frac{1}{\tau}x = f$$

$$xD + \frac{1}{\tau}x = f$$

$$x\left(D + \frac{1}{\tau}\right) = f$$

$$\frac{x}{f} = \frac{1}{D + \frac{1}{\tau}}$$

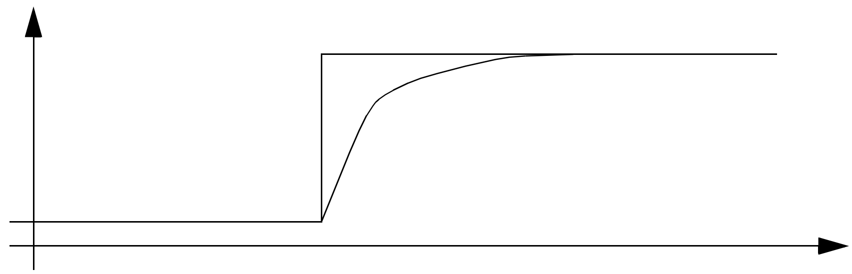
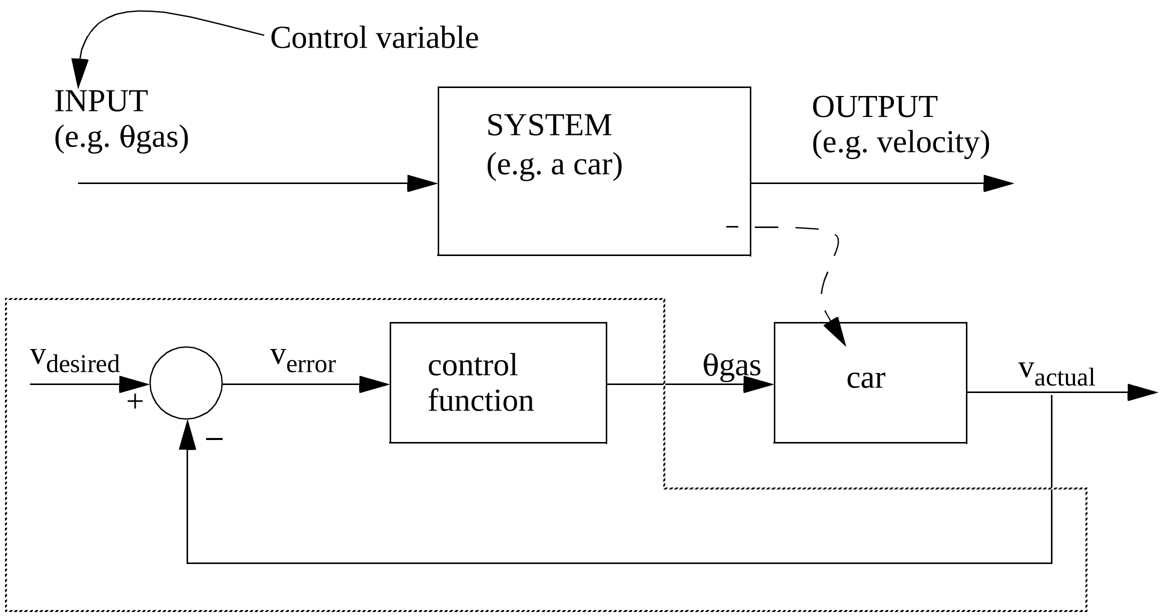


Figure 210 A first-order system response

8.3 CONTROL SYSTEMS

Figure 211 shows a transfer function block for a car. The input, or control variable is the gas pedal angle. The system output, or result, is the velocity of the car. In standard operation the gas pedal angle is controlled by the driver. When a cruise control system is engaged the gas pedal must automatically be adjusted to maintain a desired velocity set-point. To do this a control system is added, in this figure it is shown inside the dashed line. In this control system the output velocity is subtracted from the setpoint to get a system error. The subtraction occurs in the summation block (the circle on the left hand side). This error is used by the controller function to adjust the control variable in the system. Negative feedback is the term used for this type of controller.



Note: The arrows in the diagram indicate directions so that outputs and inputs are unambiguous. Each block in the diagram represents a transfer function.

Figure 211 An automotive cruise control system

There are two main types of feedback control systems: negative feedback and positive feedback. In a positive feedback control system the setpoint and output values are added. In a negative feedback control the setpoint and output values are subtracted. As a rule negative feedback systems are more stable than positive feedback systems. Negative feedback also makes systems more immune to random variations in component values and inputs.

The control function in Figure 211 can be defined many ways. A possible set of rules for controlling the system is given in Figure 212. Recall that the system error is the difference between the setpoint and actual output. When the system output matches the setpoint the error is zero. Larger differences between the setpoint and output will result in larger errors. For example if the desired velocity is 50mph and the actual velocity 60mph, the error is -10mph, and the car should be slowed down. The rules in the figure give a general idea of how a control function might work for a cruise control system.

Human rules to control car (also like expert system/fuzzy logic):

1. If v_{error} is not zero, and has been positive/negative for a while, increase/decrease θ_{gas}
2. If v_{error} is very big/small increase/decrease θ_{gas}
3. If v_{error} is near zero, keep θ_{gas} the same
4. If v_{error} suddenly becomes bigger/smaller, then increase/decrease θ_{gas}.
5. etc.

Figure 212 Example control rules

In following sections we will examine mathematical control functions that are easy to implement in actual control systems.

8.3.1 PID Control Systems

The Proportional Integral Derivative (PID) control function shown in Figure 213 is the most popular choice in industry. In the equation given the 'e' is the system error, and there are three separate gain constants for the three terms. The result is a control variable value.

$$u = K_p e + K_i \int e dt + K_d \left(\frac{de}{dt} \right)$$

Figure 213 A PID controller equation

Figure 214 shows a basic PID controller in block diagram form. In this case the potentiometer on the left is used as a voltage divider, providing a setpoint voltage. At the output the motor shaft drives a potentiometer, also used as a voltage divider. The voltages from the setpoint and output are subtracted at the summation block to calculate the feedback error. The resulting error is used in the PID function. In the proportional branch the error is multiplied by a constant, to provide a longterm output for the motor (a ballpark guess). If an error is largely positive or negative for a while the integral branch value will become large and push the system towards zero. When there is a sudden change occurs in the error value the differential branch will give a quick response. The results of all three branches are added together in the second summation block. This result is then amplified to drive the motor. The overall performance of the system can be changed by adjusting the gains in the three branches of the PID function.

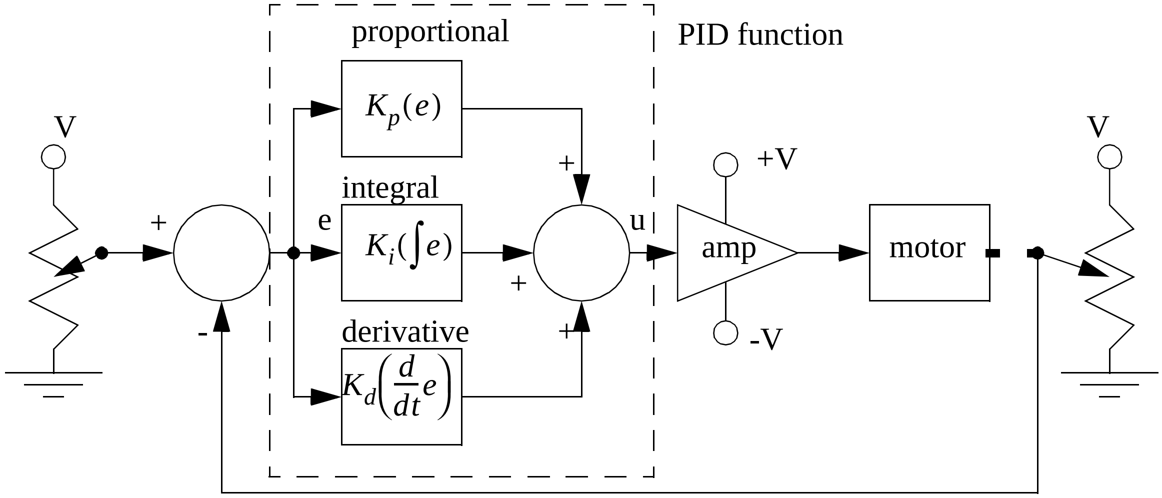


Figure 214 A PID control system

There are other variations on the basic PID controller shown in Figure 215. A PI controller results when the derivative gain is set to zero. Recall the second order response - this controller is generally good for eliminating long term errors, but it is prone to overshoot. I a P controller only the proportional gain in non-zero. This controller will generally work, but often cannot eliminate errors. The PD controller does not deal with longterm errors, but is very responsive to system changes.

For a PI Controller

$$\theta_{gas} = K_p v_{error} + K_i \int v_{error} dt$$

For a P Controller

$$\theta_{gas} = K_p v_{error}$$

For a PD Controller

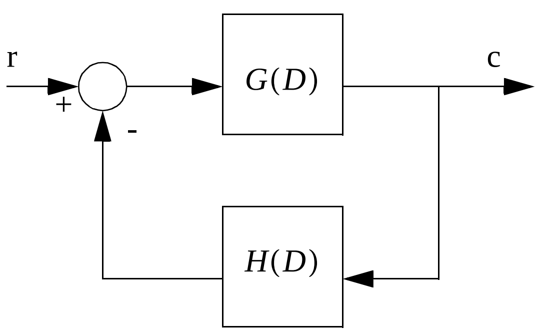
$$\theta_{gas} = K_p v_{error} + K_d\left(\frac{dv_{error}}{dt}\right)$$

Figure 215 Some other control equations

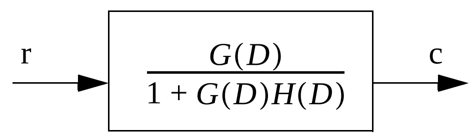
Aside: The manual process for tuning a PID controlled is to set all gains to zero. The proportional gain is then adjusted until the system is responding to input changes without excessive overshoot. After that the integral gain is increased until the longterm errors disappear. The differential gain will be increased last to make the system respond faster.

8.3.2 Manipulating Block Diagrams

A block diagram for a system is not unique, meaning that it may be manipulated into new forms. Typically a block diagram will be developed for a system. The diagram will then be simplified through a process that is both graphical and algebraic. For example, equivalent blocks for a negative feedback loop are shown in Figure 216, along with an algebraic proof.



is equal to



$$c = G(D)e$$

$$c = G(D)(r - H(D)c)$$

$$c(1 + G(D)H(D)) = G(D)r$$

$$c = \left(\frac{G(D)}{1 + G(D)H(D)}\right)r$$

Figure 216 A negative feedback block reduction

Block diagram equivalencies for other block diagram forms are shown in Figure 217 to Figure 223. In all cases these operations are reversible. Proofs are provided, except for the cases where the equivalence is obvious.

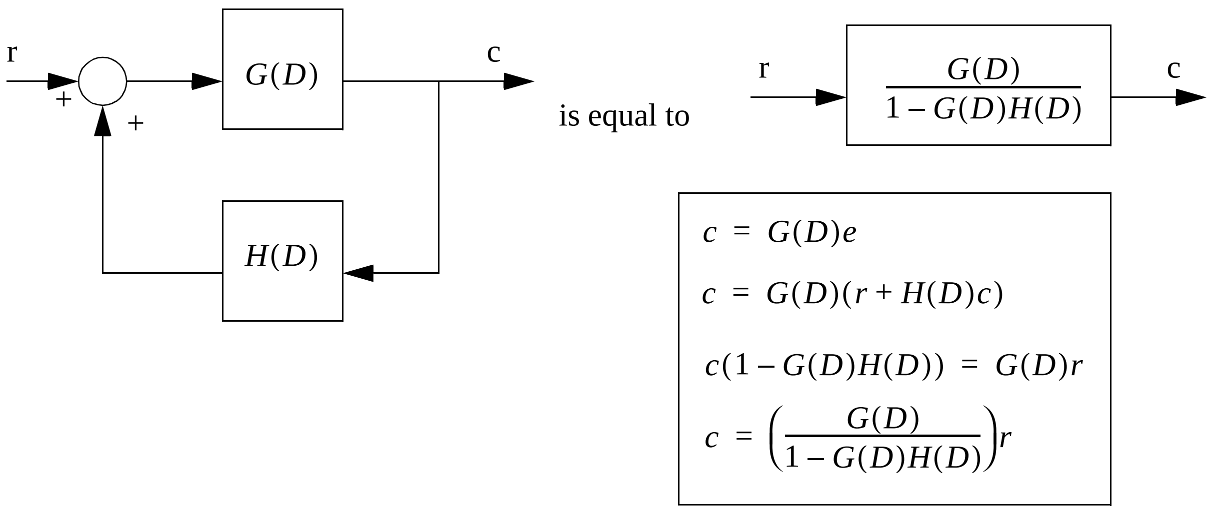


Figure 217 A positive feedback block reduction

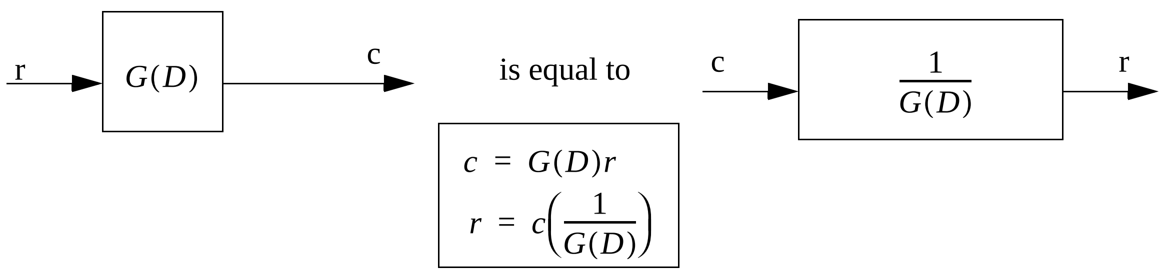


Figure 218 Reversal of function blocks

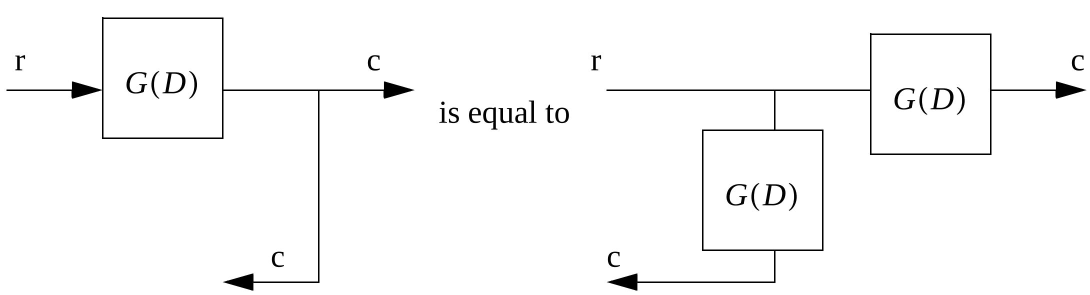
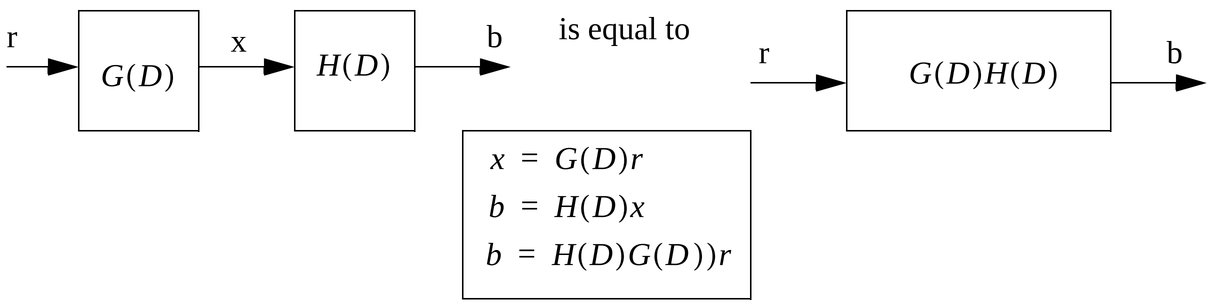


Figure 219 Moving branches before blocks



$$x = G(D)r$$
$$b = H(D)x$$
$$b = H(D)G(D))r$$

Figure 220 Combining sequential function blocks

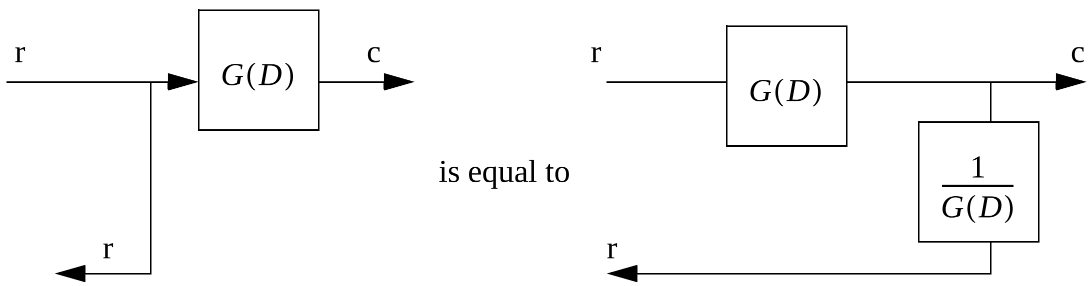
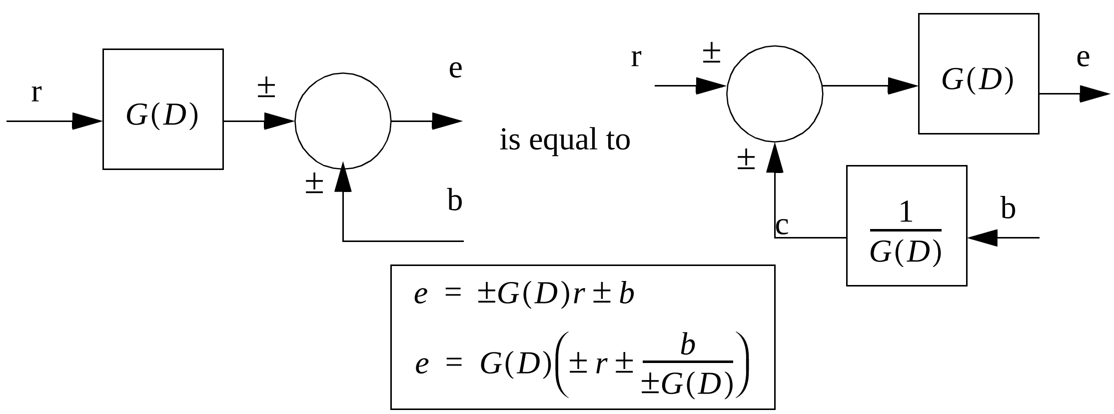


Figure 221 Moving branches after blocks



$$e = \pm G(D)r \pm b$$
$$e = G(D)\left(\pm r \pm \frac{b}{\pm G(D)}\right)$$

Figure 222 Moving summation functions before blocks

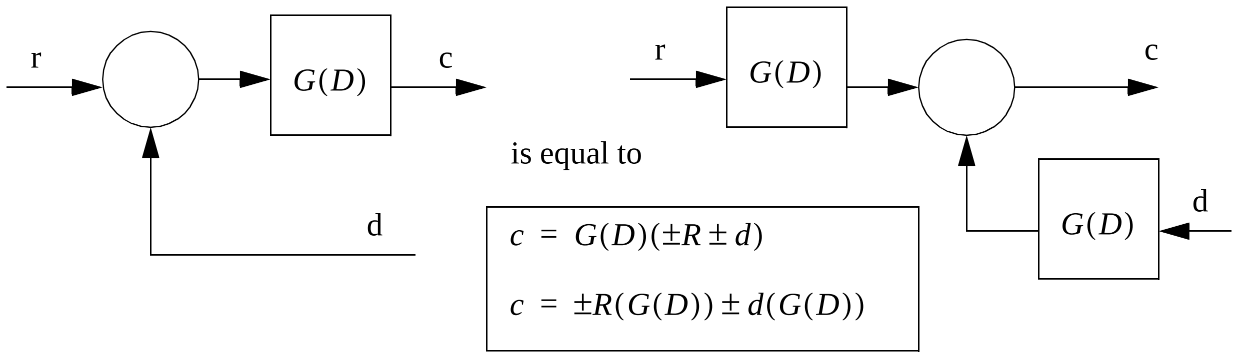


Figure 223 Moving summation function past blocks

Recall the example of a cruise control system for an automobile presented in Figure 211. This example is extended in Figure 224 to include mathematical models for each of the function blocks. This block diagram is first simplified by multiplying the blocks in sequence. The feedback loop is then reduces to a single block. Notice that the feedback line doesn’t have a function block on it, so by default the function is ’1’ - everything that goes in, comes out.

e.g. The block diagram of the car speed control system

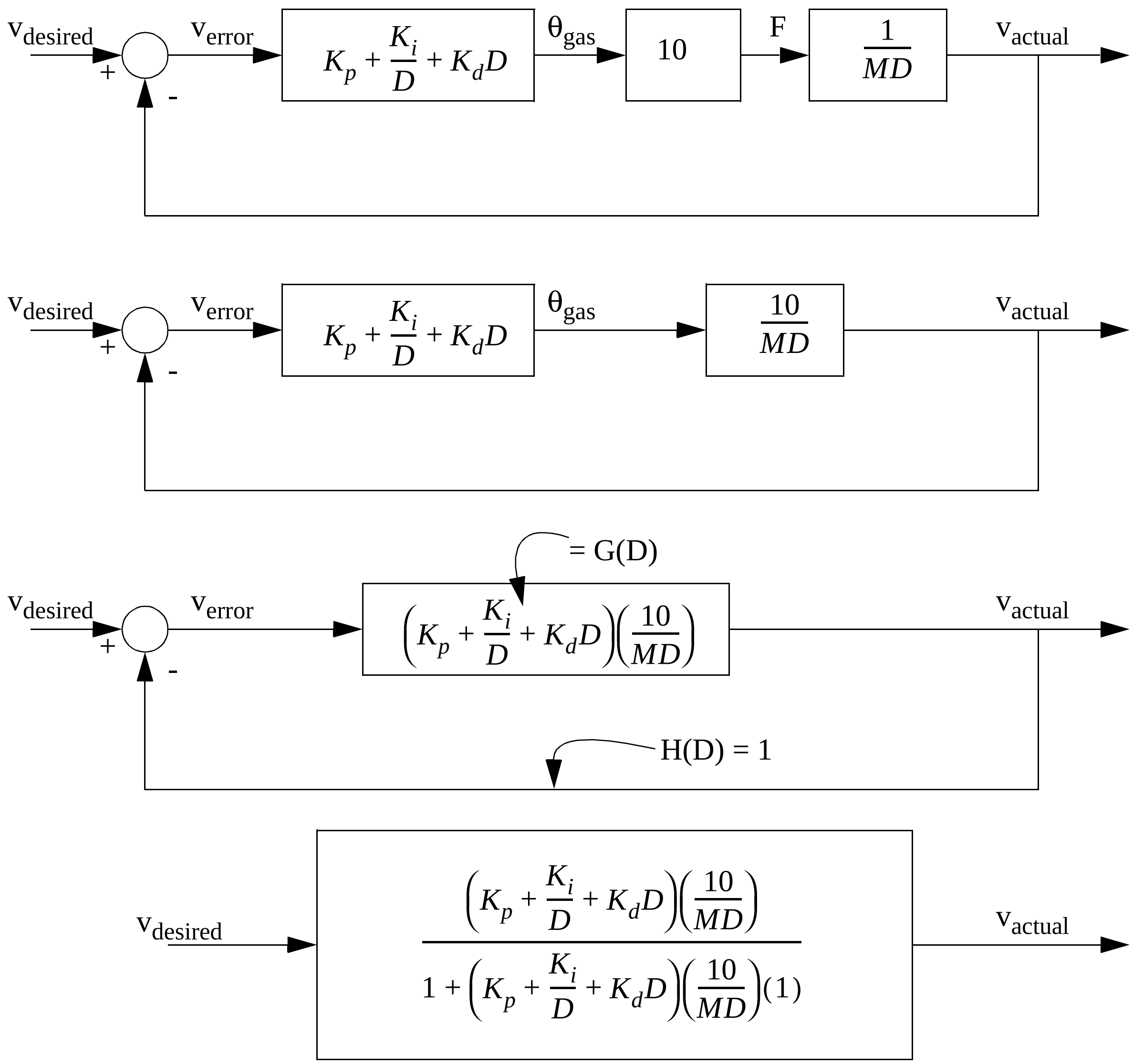
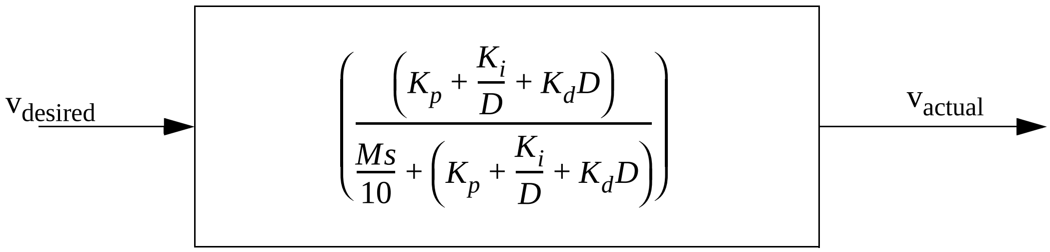
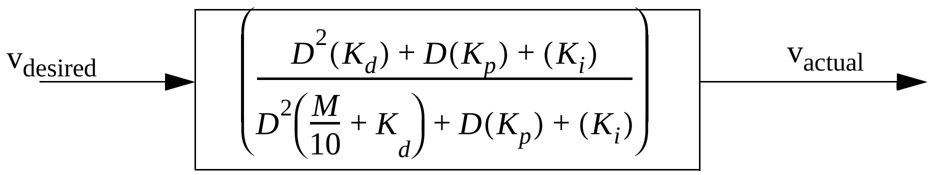


Figure 224 An example of simplifying a block diagram

The function block is further simplified in Figure 225 to a final transfer function for the whole system.





$$\frac{v_{actual}}{v_{desired}} = \left(\frac{D^2(K_d) + D(K_p) + (K_i)}{D^2\left(\frac{M}{10} + K_d\right) + D(K_p) + (K_i)}\right)$$

Figure 225 An example of simplifying a block diagram (continued)

8.3.3 A Motor Control System Example

Consider the example of a DC servo motor controlled by a computer. The purpose of the controller is to position the motor. The system in Figure 226 shows a reasonable control system arrangement. Some elements such as power supplies and commons for voltages are omitted for clarity.

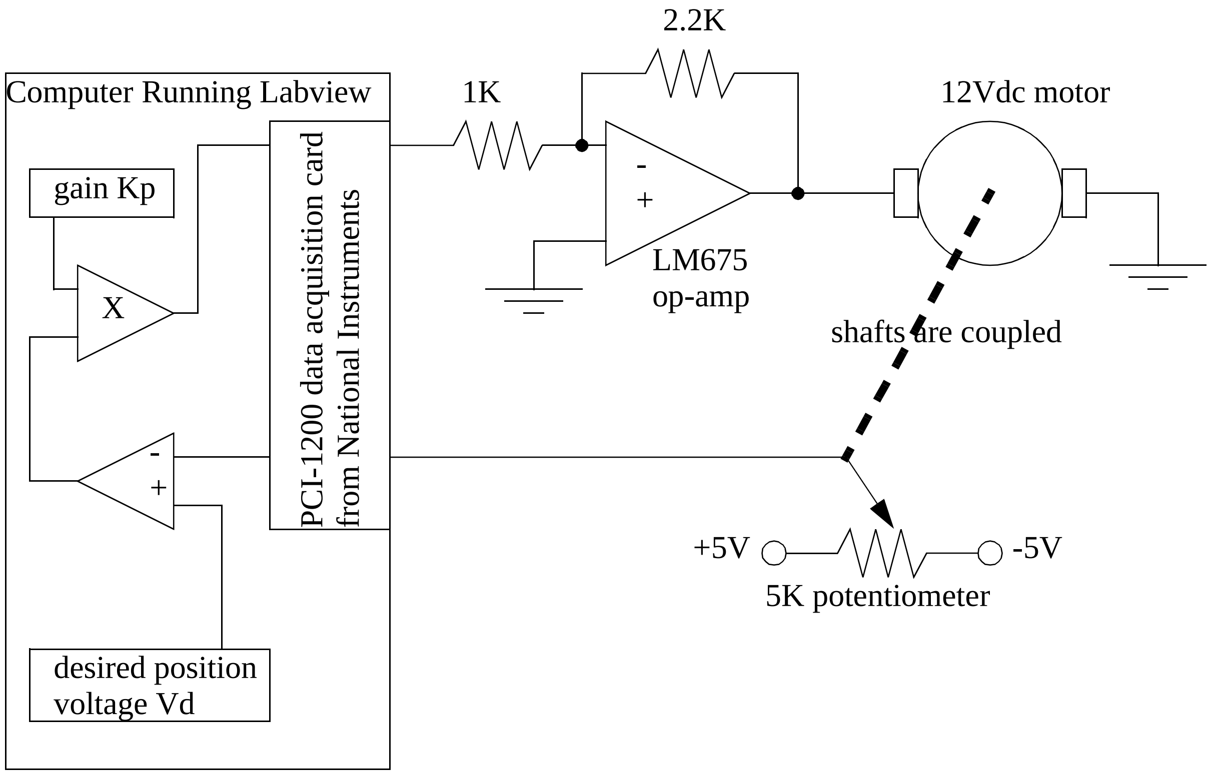


Figure 226 A motor feedback control system

The feedback controller can be represented with the block diagram in Figure 227.

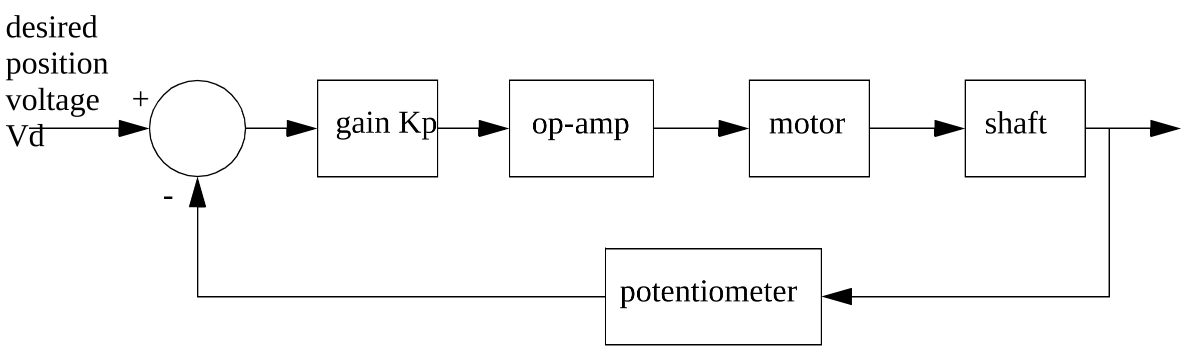


Figure 227 A block diagram for the feedback controller

The transfer functions for each of the blocks are developed in Figure 228. Two of the values must be provided by the system user. The op-amp is basically an inverting amplifier with a fixed gain of -2.2 times. The potentiometer is connected as a voltage divider and the equation relates angle to voltage. Finally the velocity of the shaft is integrated to give position.

Given or selected values:

- desired potentiometer voltage Vd
- gain K

For the op-amp:

$$\sum I_{V+} = \frac{V_+ - V_i}{1K} + \frac{V_+ - V_o}{2.2K} = 0 \qquad V_+ = V_- = 0V$$

$$\frac{-V_i}{1K} + \frac{-V_o}{2.2K} = 0$$

$$\frac{V_o}{V_i} = -2.2$$

For the potentiometer assume that the potentiometer has a range of 10 turns and 0 degrees is in the center of motion. So there are 5 turns in the negative and positive direction.

$$V_o = 5V\left(\frac{\theta}{5(2\pi)}\right)$$

$$\frac{V_o}{\theta} = 0.159 Vrad^{-1}$$

For the shaft, it integrates the angular velocity into position:

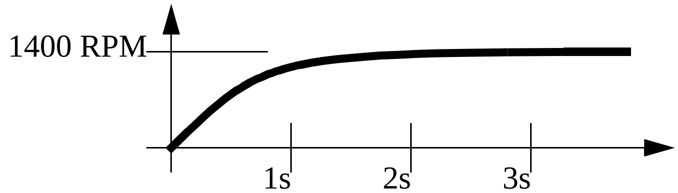
$$\omega = \frac{d}{dt}\theta$$

$$\frac{\theta}{\omega} = \frac{1}{D}$$

Figure 228 Transfer functions for the power amplifier, potentiometer and motor shaft

The basic equation for the motor is derived in Figure 229 using experimental data. In this case the motor was tested with the full inertia on the shaft, so there is no need to calculate ’J’.

For the motor use the differential equation and the speed curve when Vs=10V is applied:



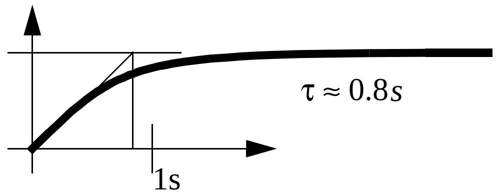
$$\left(\frac{d}{dt}\right)\omega + \left(\frac{K^2}{JR}\right)\omega = \left(\frac{K}{JR}\right)V_s$$

For steady-state

$$\left(\frac{d}{dt}\right)\omega = 0 \qquad \omega = 1400RPM = 146.6rads^{-1}$$

$$0 + \left(\frac{K^2}{JR}\right)146.6 = \left(\frac{K}{JR}\right)10$$

$$K = 0.0682$$



$$\left(\frac{K^2}{JR}\right) = 0.8s$$

$$0.0682\left(\frac{K}{JR}\right) = 0.8s$$

$$\frac{K}{JR} = 11.73$$

$$D\omega + 0.8\omega = 11.73V_s$$

$$\frac{\omega}{V_s} = \frac{11.73}{D + 0.8}$$

Figure 229 Transfer function for the motor

The individual transfer functions for the system are put into the system block diagram in Figure 230. The block diagram is then simplified for the entire system to a single transfer function relating the desired voltage (setpoint) to the angular position (output). The transfer function contains the unknown gain value 'Kp'.

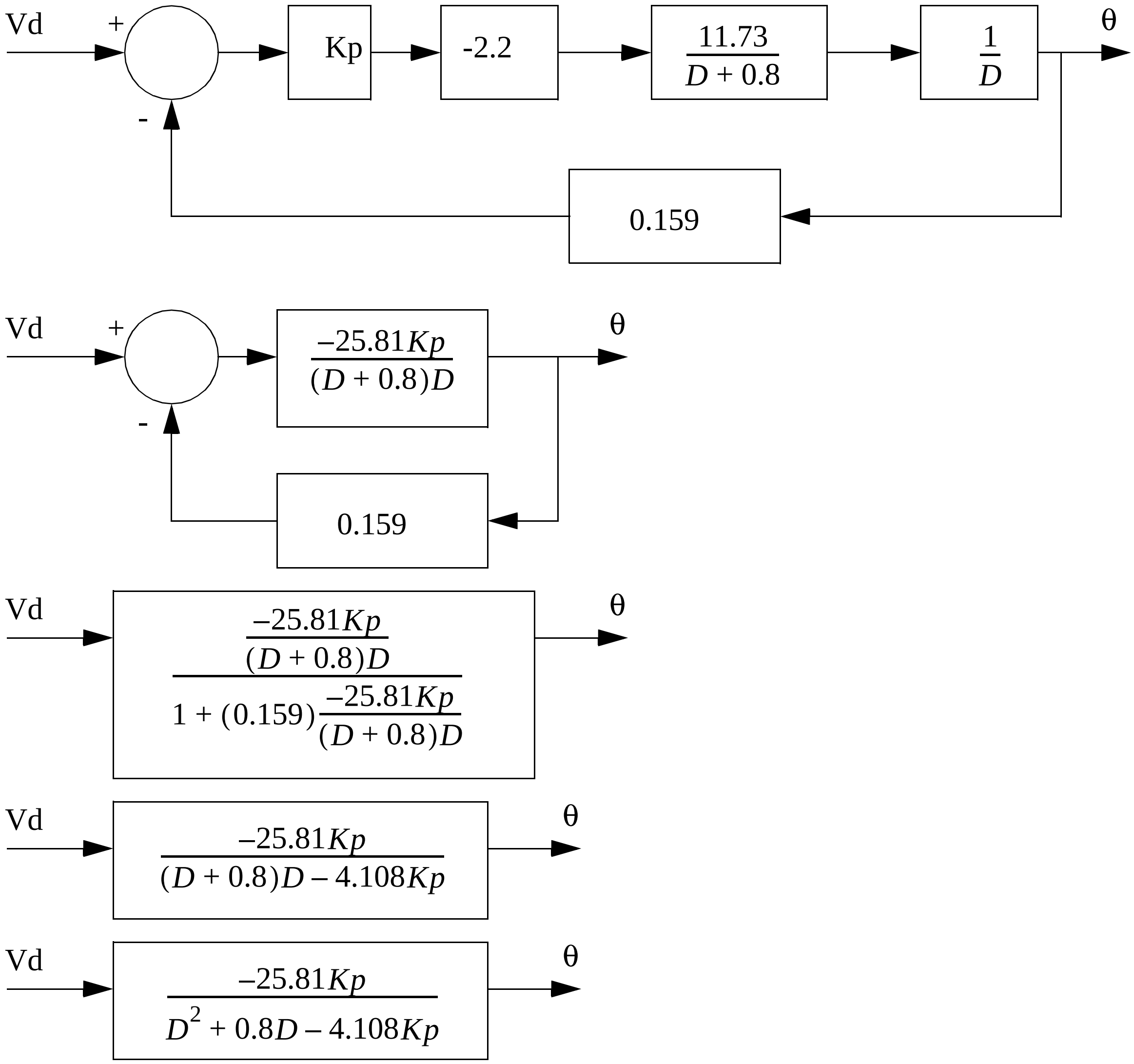


Figure 230 The system block diagram, and simplification

The value of 'Kp' can be selected to 'tune' the system performance. In Figure 231 the gain value is calculated to give the system an overall damping factor of 1.0, or critically damped. This is done by recognizing that the bottom (homogeneous) part of the transfer function is second-order and then extracting the damping factor and natural frequency. The final result of 'Kp' is negative, but this makes sense when the negative gain on the op-amp is considered.

We have specified, or been given the damping coefficient as a design objective.

$$\xi = 1.0$$

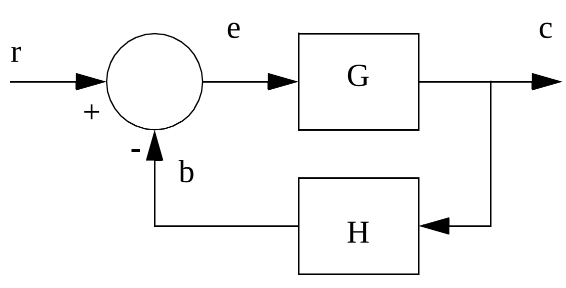
The denominator of the system transfer function can be compared to the standard second-order response.

$$D^2 + 0.8D - 0.0717K = x'' + 2\xi\omega_n x' + \omega_n^2 x$$
$$0.8 = 2\xi\omega_n$$
$$0.8 = 2(1.0)\omega_n$$
$$\omega_n = 0.4$$
$$-0.0717K = \omega_n^2$$
$$-0.0717K = 0.4^2$$
$$K = -2.232$$

Figure 231 Calculating a gain K

8.3.4 System Error

System error is often used when designing control systems. The two common types of error are system error and feedback error. The equations for calculating these errors are shown in Figure 232. If the feedback function 'H' has a value of '1' then these errors will be the same.



System error, $S = r - c$

Feedback error, $e = r - b$

Figure 232 Controller errors

An example of calculating these errors is shown in Figure 233. The system is a simple integrator, with a unity feedback loop. The overall transfer function for the system is calculated and then used to find the system response. The response is then compared to the input to find the system error. In this case the error will go to zero as time approaches infinity.

Given,

$$G(D) = \frac{K_p}{D} \qquad H(D) = 1$$

$$\therefore \frac{c}{r} = \frac{G}{1 + GH} = \frac{K_p}{D + K_p}$$

$$\therefore c' + K_p c = K_p r$$

$$r = A$$

The homogeneous solution is,

$$c_h = C_1 e^{-K_p t}$$

The particular solution is found with a guess,

$$c_p = C_2$$

$$c_p' = 0$$

$$0 + K_p C_2 = K_p A \qquad C_2 = A$$

The solutions can be combined and the remaining unknown found for the system at rest initially.

$$c = c_h + c_p = C_1 e^{-K_p t} + A$$

$$0 = C_1 e^0 + A$$

$$C_1 = -A$$

$$c = -A e^{-K_p t} + A$$

The error can now be calculated.

$$S = r - c$$

$$S = A - (-A e^{-K_p t} + A) = A e^{-K_p t}$$

Figure 233 System error calculation example for a step input

Solve the previous problem for a ramp input,

$$r = At$$

Figure 234 Drill problem: Calculate the system error for a ramp input

Find the system error ’e’ for the given ramp input, R.

$$G(D) = \frac{1}{D^2 + 4D + 5} \qquad H(D) = 5 \qquad r = 4t$$

Figure 235 Drill problem: Calculate the errors

8.3.5 Controller Transfer Functions

The PID controller, and simpler variations were discussed in earlier sections. A more complete table is given in Figure 236.

Type	Transfer Function	
Proportional (P)	$G_c = K$	
Proportional-Integral (PI)	$G_c = K\left(1 + \frac{1}{\tau D}\right)$	
Proportional-Derivative (PD)	$G_c = K(1 + \tau D)$	
Proportional-Integral-Derivative (PID)	$G_c = K\left(1 + \frac{1}{\tau D} + \tau D\right)$	
Lead	$G_c = K\left(\frac{1 + \alpha\tau D}{1 + \tau D}\right)$	$\alpha > 1$
Lag	$G_c = K\left(\frac{1 + \tau D}{1 + \alpha\tau D}\right)$	$\alpha > 1$
Lead-Lag	$G_c = K\left[\left(\frac{1 + \tau_1 D}{1 + \alpha\tau_1 D}\right)\left(\frac{1 + \alpha\tau_2 D}{1 + \tau_2 D}\right)\right]$	$\alpha > 1$, $\tau_1 > \tau_2$

Figure 236 Standard controller types

8.3.6 State Variable Control Systems

State variable matrices were introduced before. These can also be used to form a control system, as shown in Figure 237.

$$\frac{d}{dt}X = AX + BU$$

$$Y = CX + D$$

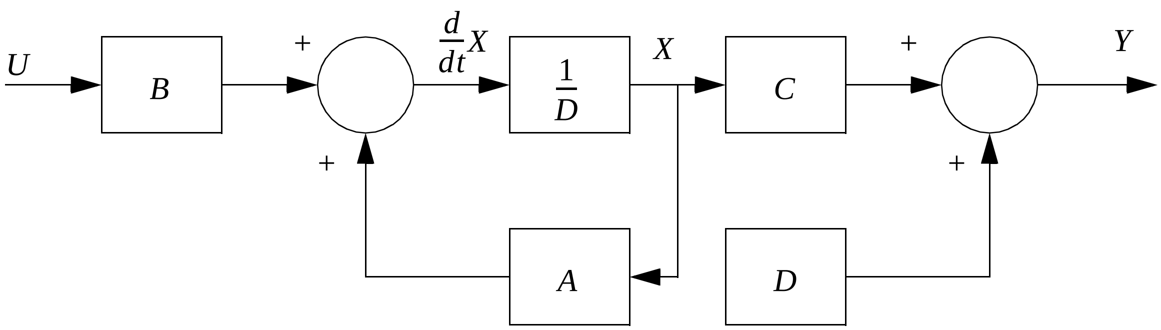


Figure 237 A state variable control system

8.3.7 Feedforward Controllers

When a model of a system is well known it can be used to improve the performance of a control system by adding a feedforward function, as pictured in Figure 238. The feedforward function is basically an inverse model of the process. When this is used together with a more traditional feedback function the overall system can outperform more traditional controllers function, such as PID.

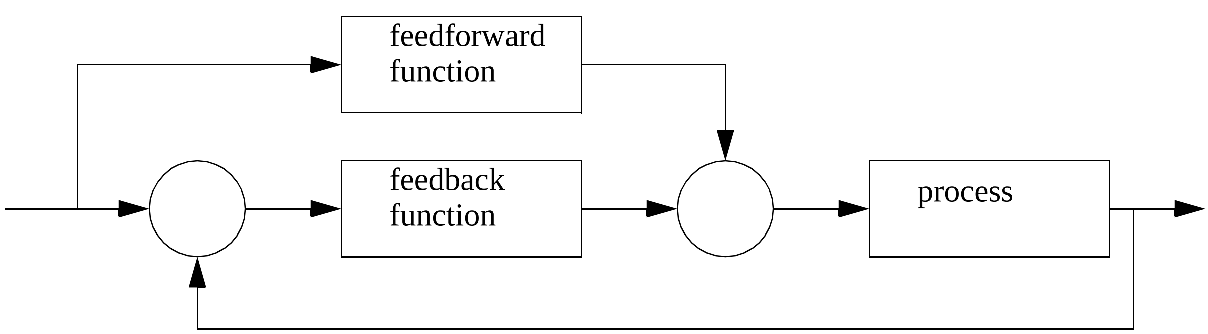


Figure 238 A feed forward controller

8.3.8 Cascade Controllers

When controlling a multistep process a cascade controller can allow refined control of sub-loops within the larger control system.

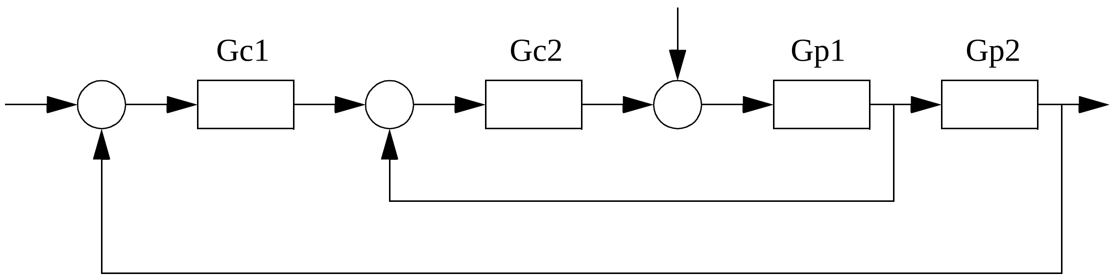


Figure 239 A cascade controller

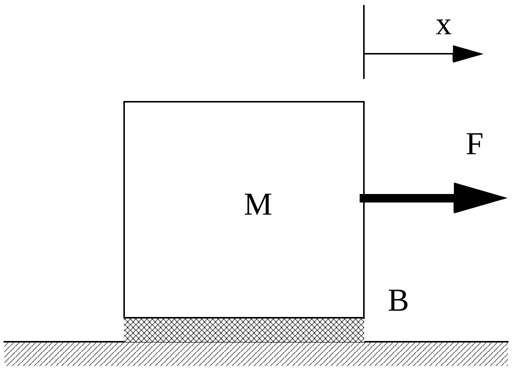
8.4 SUMMARY

- Transfer functions can be used to model the ratio of input to output.
- Block diagrams can be used to describe and simplify systems
- Controllers can be designed to meet criteria, such as damping ratio and natural frequency.
- System errors can be used to determine the long term stability and accuracy of a controlled system.
- Other control types are possible for more advanced systems.

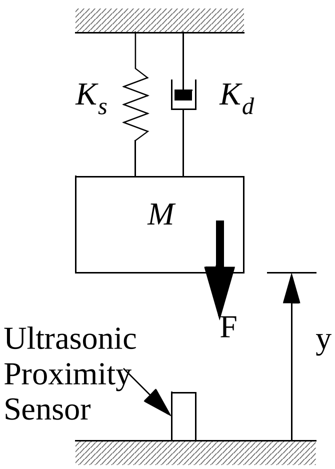
8.5 PRACTICE PROBLEMS

1. Develop differential equations and then transfer functions for the mechanical system below. There is viscous damping between the block and the ground. A force is applied to cause the

mass the accelerate.

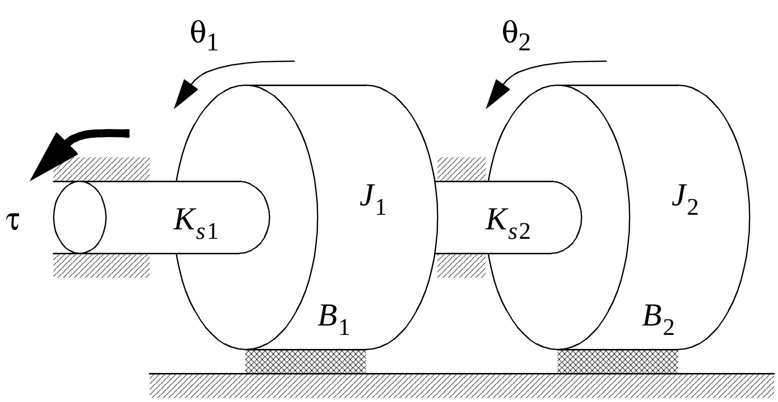


2. Develop a transfer function for the system below. The input is the force ‘F’ and the output is the voltage ‘Vo’. The mass is suspended by a spring and a damper. When the spring is undeflected y=0. The height is measured with an ultrasonic proximity sensor. When y = 0, the output Vo=0V. If y=20cm then Vo=2V and if y=-20cm then Vo=-2V. Neglect gravity.

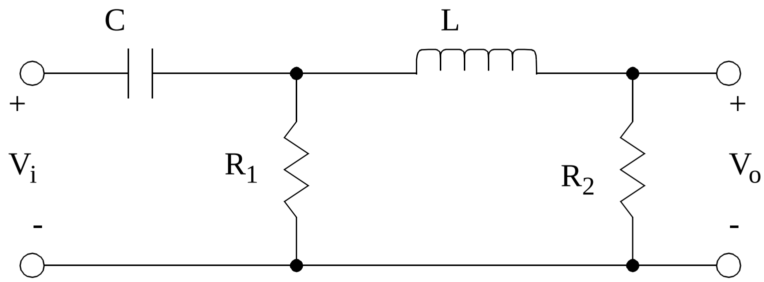


$$K_s = 10\frac{N}{m} \qquad K_d = 5N\frac{s}{m} \qquad M = 0.5Kg$$

3. Find the transfer functions for the systems below. Here the input is a torque, and the output is the angle of the second mass.



4. Find the transfer functions for the system below where Vi is the input and Vo is the output.



5. Given the transfer function, G(s), determine the time response output Y(t) to a step input X(t).

$$G = \frac{4}{D+2} = \frac{Y}{X} \qquad X(t) = 20 \quad \text{When t >= 0}$$

6. Given the transfer function below, develop a mechanical system that it could be for. (Hint: Differential Equations).

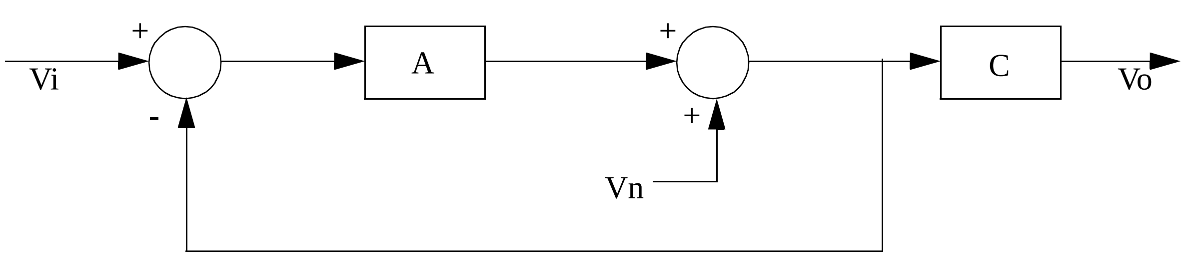
$$\frac{x(D)}{F(D)} = \frac{1}{10 + 20s}$$

where: x = displacement
F = force

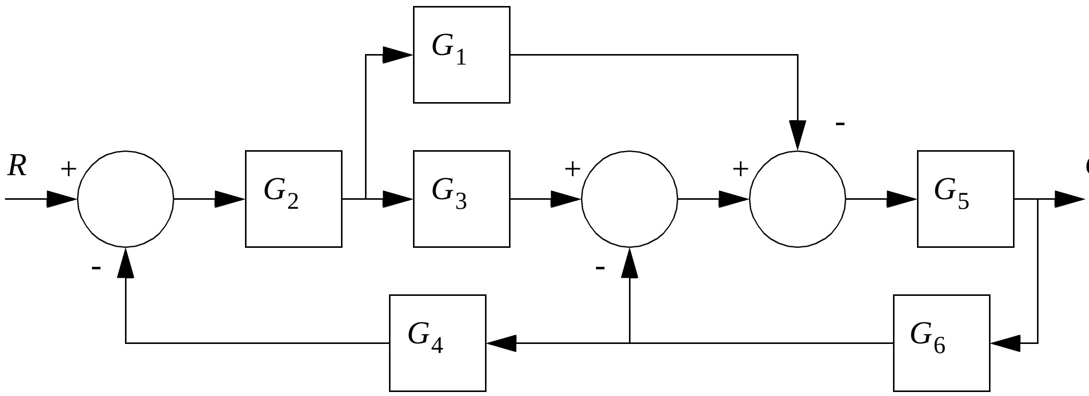
7. Given a mass supported by a spring and damper, find the displacement of the supported mass over time if it is released from neutral at t=0sec, and gravity pulls it downward.
   a) find the transfer function x/F.
   b) find the input function F.
   c) solve the input output equation symbolically to find the position as a function of time for Ks = 10N/m, Kd = 5Ns/m, M=10kg.
   d) solve part c) numerically.

8. a) What is a Setpoint, and what is it used for? b) What does feedback do in control systems?

9. Simplify the block diagram below.

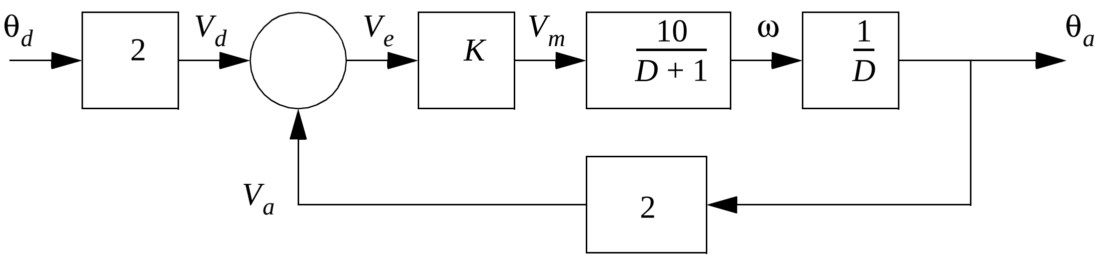


10. Simplify the following block diagram.



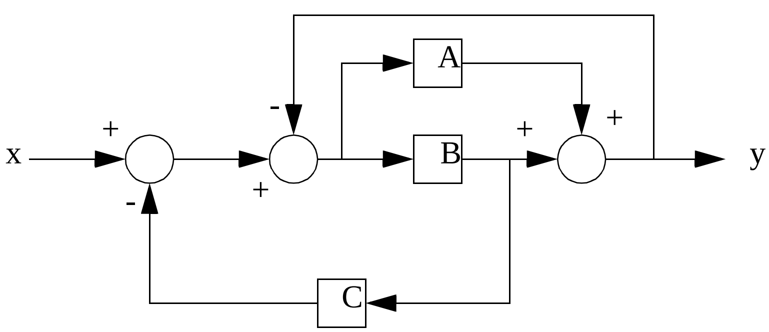
11. The block diagram below is for a servo motor position control system. The system uses a proportional controller.

a) Convert the system to a transfer function.



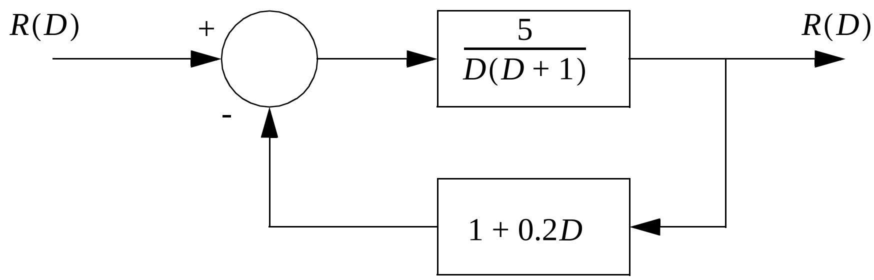
b) draw a sketch of what the actual system might look like. Identify components.

12. Simplify the block diagram below to a transfer function.

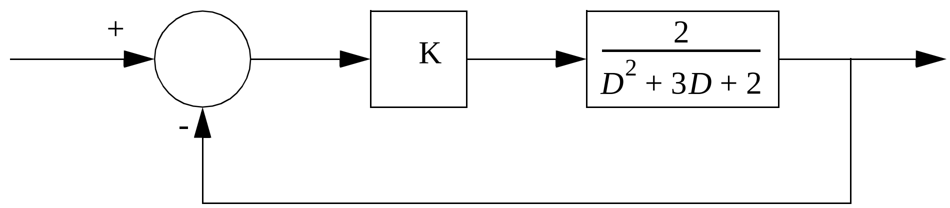


13. Find the system error when the input is a ramp with the function r(t) = 0.5t. Sketch the system

error as a function of time.



14. Given the block diagram below, select a system gain K that will give the overall system a damping ratio of 0.7 (for a step input). What is the resulting undamped natural frequency of the system?



15. The following system is a feedback controller for an elevator. It uses a desired height ‘d’ provided by a user, and the actual height of the elevator ‘h’. The difference between these two is called the error ‘e’. The PID controller will examine the value ‘e’ and then control the speed of the lift motor with a control voltage ‘c’. The elevator and controller are described with transfer functions, as shown below. all of these equations can be combined into a single system transfer

equation as shown.

$$e = d - h \quad \text{error}$$

$$\frac{c}{e} = K_p + \frac{K_i}{D} + K_d D = \frac{2D + 1 + D^2}{D} \quad \text{PID controller} \qquad \frac{h}{c} = \frac{10}{D^2 + D} \quad \text{elevator}$$

combine the transfer functions

$$\left(\frac{c}{e}\right)\left(\frac{h}{c}\right) = \frac{h}{e} = \frac{2D + 1 + D^2}{D}\,\frac{10}{D^2 + D} = \frac{(D+1)^2}{D}\,\frac{10}{D(D+1)} = \frac{10(D+1)}{D^2}$$

$$\frac{h}{d-h} = \frac{10(D+1)}{D^2} \qquad \text{eliminate 'e'}$$

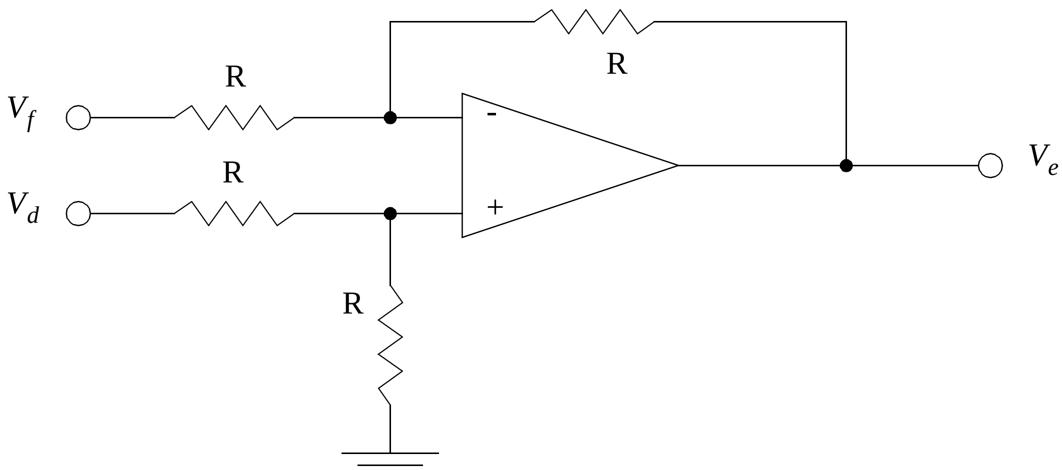
$$h = \left(\frac{10(D+1)}{D^2}\right)(d-h)$$

$$h\left(1 + \frac{10(D+1)}{D^2}\right) = \left(\frac{10(D+1)}{D^2}\right)(d)$$

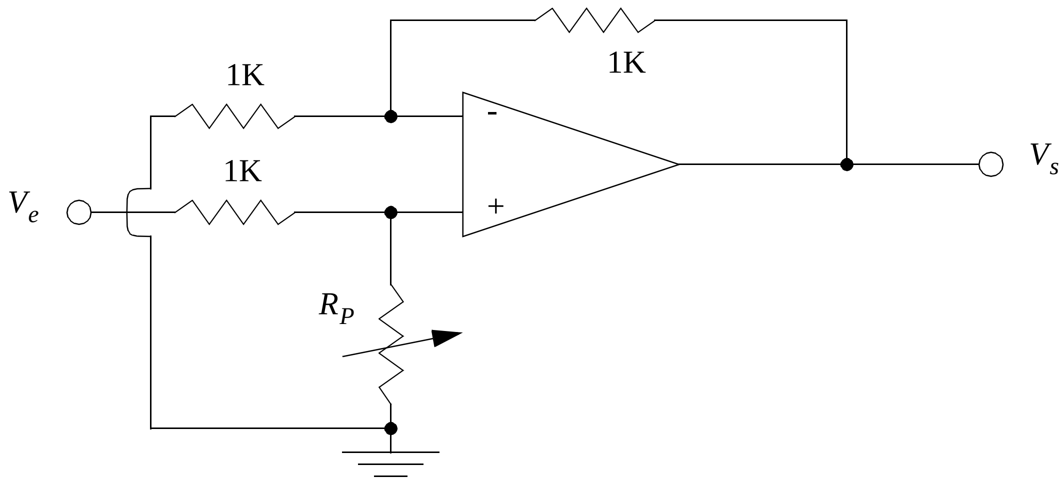
$$\frac{h}{d} = \left(\frac{\frac{10(D+1)}{D^2}}{1 + \frac{10(D+1)}{D^2}}\right) = \frac{10D + 10}{D^2 + 10D + 10} \qquad \text{system transfer function}$$

a) Find the response of the final equation to a step input. The system starts at rest on the ground floor, and the input (desired height) changes to 20 as a step input.
b) Write find the damping coefficient and natural frequency of the results in part a).
c) verify the solution using the initial and final value theorems.

16. a) Develop an equation for the system below relating the two inputs to the output. (Hint: think of a summation block.)

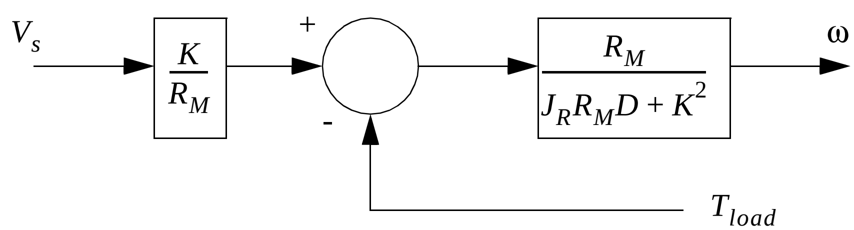


b) Develop an equation for the system below relating the input to the output.

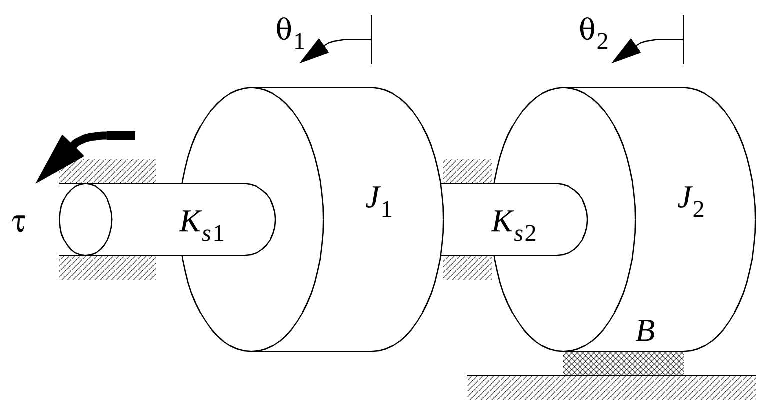


c) The equation below can be used to model a permanent magnet DC motor with an applied torque. An equivalent block diagram is given. Prove that the block diagram is equivalent to the equation.

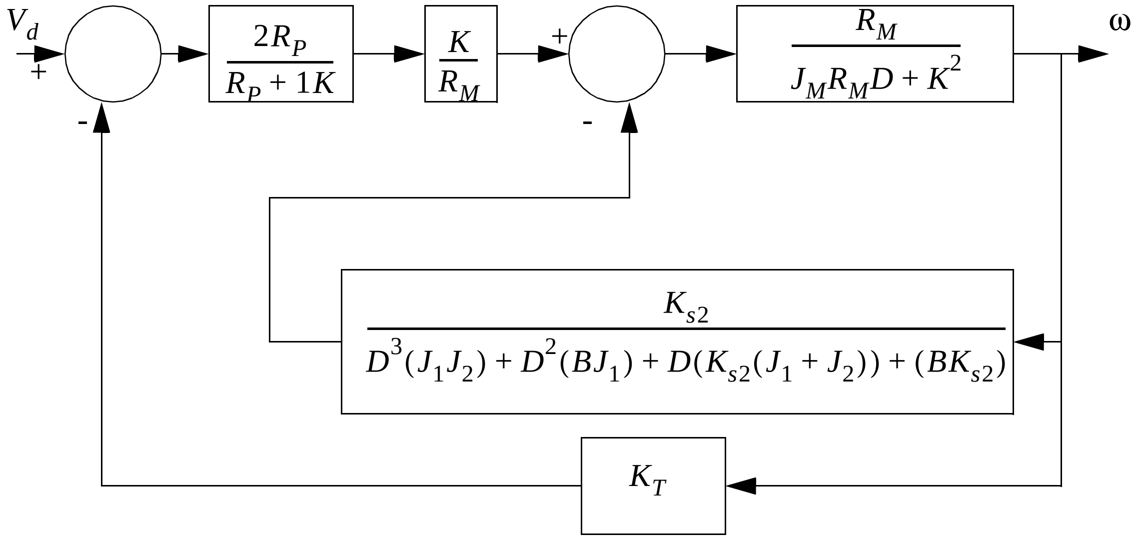
$$\left(\frac{d}{dt}\right)\omega + \omega\left(\frac{K^2}{J_R R_M}\right) = V_s\left(\frac{K}{J_R R_M}\right) - \frac{T_{load}}{J_R}$$



d) Write the transfer function for the system below relating the input torque to the output angle theta2. Then write the transfer function for the angular velocity of mass 2.



e) The system below is a combination of previous components, and a tachometer for velocity feedback. Simplify the block diagram.



(ans.

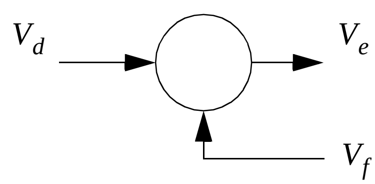
a) $V_+ = V_d\left(\frac{R}{R+R}\right) = 0.5V_d$

$$\sum I_{V_-} = \frac{V_- - V_f}{R} + \frac{V_- - V_e}{R} = 0$$

$$V_e = 2V_- - V_f$$

$$V_e = 2(0.5V_d) - V_f$$

$$V_e = V_d - V_f$$



(ans.

b)

$$\sum I_{V_-} = \frac{V_- - 0}{1K} + \frac{V_- - V_s}{1K} = 0$$

$$V_- = 0.5V_s \qquad (1)$$

$$\sum I_{V_+} = \frac{V_+ - V_e}{1K} + \frac{V_+ - 0}{R_P} = 0$$

$$V_+\left(\frac{1}{1K} + \frac{1}{R_P}\right) = \frac{V_e}{1K}$$

substitute in (1)

$$0.5V_s\left(\frac{R_P + 1K}{R_P(1K)}\right) = \frac{V_e}{1K}$$

$$\frac{V_s}{V_e} = \frac{2R_P}{R_P + 1K}$$

$$V_e \longrightarrow \boxed{\frac{2R_P}{R_P + 1K}} \longrightarrow V_s$$

note: a constant set by the variable resistor Rp

(ans.

c)

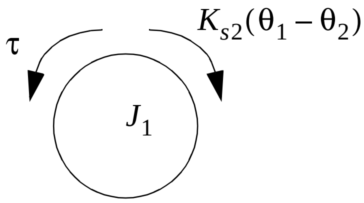
$$\left(\frac{d}{dt}\right)\omega + \omega\left(\frac{K^2}{J_R R_M}\right) = V_s\left(\frac{K}{J_R R_M}\right) - \frac{T_{load}}{J_R}$$

$$\omega\left(D + \frac{K^2}{J_R R_M}\right) = V_s\left(\frac{K}{J_R R_M}\right) - \frac{T_{load}}{J_R}$$

$$\omega = \left(\frac{JR_M}{J_R R_M D + K^2}\right)\left(V_s\left(\frac{K}{J_R R_M}\right) - \frac{T_{load}}{J_R}\right)$$

$$\omega = \left(\frac{R_M}{J_R R_M D + K^2}\right)\left(V_s\left(\frac{K}{R_M}\right) - T_{load}\right)$$

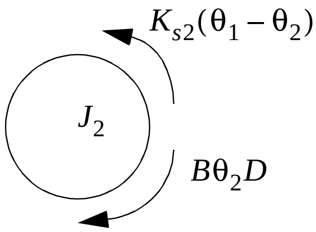
(ans. d)



+ $$\sum M = \tau - K_{s2}(\theta_1 - \theta_2) = J_1\theta_1 D^2$$

$$\tau - K_{s2}(\theta_1 - \theta_2) = J_1\theta_1 D^2$$

$$\theta_1(J_1D^2 + K_{s2}) + \theta_2(-K_{s2}) = \tau \qquad (1)$$



+ $$\sum M = K_{s2}(\theta_1 - \theta_2) - B\theta_2 D = J_2\theta_2 D^2$$

$$K_{s2}(\theta_1 - \theta_2) - B\theta_2 D = J_2\theta_2 D^2$$

$$\theta_2(J_2D^2 + K_{s2} + BD) = \theta_1(K_{s2})$$

$$\theta_1 = \theta_2\left(\frac{J_2D^2 + K_{s2} + BD}{K_{s2}}\right) \qquad (2)$$

substitute (2) into (1)

$$\theta_2\left(\frac{J_2D^2 + BD + K_{s2}}{K_{s2}}\right)(J_1D^2 + K_{s2}) + \theta_2(-K_{s2}) = \tau$$

$$\theta_2\left(\frac{D^4(J_1J_2) + D^3(BJ_1) + D^2(K_{s2}(J_1 + J_2)) + D(BK_{s2})}{K_{s2}}\right) = \tau$$

$$\frac{\theta_2}{\tau} = \frac{K_{s2}}{D^4(J_1J_2) + D^3(BJ_1) + D^2(K_{s2}(J_1 + J_2)) + D(BK_{s2})}$$

$$\omega_2 = D\theta_2$$

$$\frac{\omega_2}{\tau} = \frac{K_{s2}}{D^3(J_1J_2) + D^2(BJ_1) + D(K_{s2}(J_1 + J_2)) + (BK_{s2})}$$

9. PHASOR ANALYSIS

Topics:
- The Fourier transform
- Complex and polar calculation of steady state system responses
- Vibration analysis

Objectives:
- To be able to analyze steady state responses using the Fourier transform

9.1 INTRODUCTION

When a system is stimulated by an input it will respond. Initially there is a substantial transient response, that is eventually replaced by a steady state response. Techniques for finding the combined steady state and transient responses were covered in earlier chapters. These include the integration of differential equations, and numerical solutions. Phasor analysis can be used to find the steady state response only. These techniques involve using the Fourier transform on the system transfer function, input and output.

9.2 FOURIER TRANSFORMS FOR STEADY-STATE ANALYSIS

When considering the differential operator we can think of it as a complex number, as in Figure 240. The real component of the number corresponds to the natural decay (e-to-the-t) of the system. But, the complex part corresponds to the oscillations of the system. In other words the real part of the number will represent the transient effects of the system, while the complex part will represent the sinusoidal steady-state. Therefore to do a steady-state sinusoidal analysis we can replace the 'D' operator with $j\omega$, this is the Fourier transform.

$$D = \sigma + j\omega$$

where,

D = differential operator
σ = decay constant
ω = oscillation frequency

Fourier transform

$$\boxed{D = j\omega}$$

Figure 240 Transient and steady-state parts of the differential operator

An example of the Fourier transform is given in Figure 241. We start with a transfer function for a mass-spring-damper system. In this example numerical values are assumed to put the equation in a numerical form. The differential operator is replaced with 'jw', and the equation is simplified to a complex number in the denominator. This equation then described the overall response of the system to an input based upon the frequency of the input. A generic form of sinusoidal input for the system is defined, and also converted to phasor (complex) form. (Note: the frequency of the input does not show up in the complex form of the input, but it will be used later.) The general of the system is then obtained by multiplying the transfer function by the input, to obtain the output.

A Fourier transform can be applied to a transfer function for a mass-spring-damper system. Some component values are assumed.

$$M = 1000kg \qquad K_s = 2000\frac{N}{m} \qquad K_d = 3000\frac{Ns}{m}$$

$$\frac{x(D)}{F(D)} = \frac{1}{MD^2 + K_d D + K_s} = \frac{1}{1000D^2 + 3000D + 2000}$$

$$\frac{x(\omega)}{F(\omega)} = \frac{1}{1000j^2\omega^2 + 3000j\omega + 2000} = \frac{1}{(2000 - 1000\omega^2) + j(3000\omega)}$$

A given input function can also be converted to Fourier (phasor) form.

$$F(t) = A\sin(\omega_{input}t + \theta_{input})$$

$$F(\omega) = A(\cos\theta_{input} + j\sin\theta_{input})$$

$$F(\omega) = A\cos\theta_{input} + jA\sin\theta_{input}$$

Note: the frequency is not used when converting an oscillating signal to complex form. But it is needed for the transfer function.

The response of the steady state output 'x' can now be found for the given input.

$$x(\omega) = \frac{x(\omega)}{F(\omega)}F(\omega) = \frac{1}{(2000 - 1000\omega^2) + j(3000\omega)}(A\cos\theta_{input} + jA\sin\theta_{input})$$

Figure 241 A Fourier transform example

To continue the example in Figure 241 values for the sinusoidal input force are assumed. After this the method only requires the simplification of the complex expression. In particular having a complex denominator makes analysis difficult and is undesirable. To simplify this expression it is multiplied by the complex conjugate. After this, the expression is quickly reduced to a simple complex number. The complex number is then converted to polar form, and then finally back into a function of time.

Assume the input to the system is, $F(t) = 10\sin(100t + 0.5)N$

$$A = 10N \qquad \omega = 100\frac{rad}{s} \qquad \theta_{input} = 0.5rad$$

These can be applied to find the steady state output response,

$$x(\omega) = \frac{1}{(2000 - 1000(100)^2) + j(3000(100))}(10\cos 0.5 + j10\sin 0.5)$$

$$x(\omega) = \frac{1}{(-9998000) + j(300000)}(8.776 + j4.794)$$

$$x(\omega) = \frac{8.776 + j4.794}{-9998000 + j300000}\left(\frac{(-9998000 - j300000)}{(-9998000 - j300000)}\right)$$

Note: This is known as the complex conjugate
1. The value is obviously 1 so it does not change the value of the expression
2. The complex component is now negative
3. Only the denominator is used top and bottom

$$x(\omega) = \frac{(-86304248) + j(-50563212)}{1.0005 \times 10^{14}}$$

$$x(\omega) = (-0.863 \times 10^{-6}) + j(-0.505 \times 10^{-6})$$

$$x(\omega) = \sqrt{(-0.863 \times 10^{-6})^2 + (-0.505 \times 10^{-6})^2}\angle\operatorname{atan}\left(\left(\frac{-0.505 \times 10^{-6}}{-0.863 \times 10^{-6}}\right) + \pi\right)$$

Note: the signs of the components indicate that the angle is in the bottom left quadrant of the complex plane, so the angle should be between 180 and 270 degrees. To correct for this pi radians are added to the result of the calculation.

$$x(\omega) = 0.9999 \times 10^{-6}\angle 3.671$$

This can then be converted to a function of time.

$$x(t) = 0.9999 \times 10^{-6}\sin(100t + 3.671)m$$

Figure 242 A Fourier transform example (cont'd)

Note: when dividing and multiplying complex numbers in polar for the magnitudes can be multiplied or divided, and the angles added or subtracted. XXXXX

Unfortunately when the numbers are only added or subtracted they need to be converted back to cartesian form to perform the operations. This method eliminates the need to multiply by the complex conjugate.

$$\frac{A + jB}{C + jD} = \frac{\sqrt{A^2 + B^2}\angle\operatorname{atan}\left(\frac{B}{A}\right)}{\sqrt{C^2 + D^2}\angle\operatorname{atan}\left(\frac{D}{C}\right)} = \frac{\sqrt{A^2 + B^2}}{\sqrt{C^2 + D^2}}\angle\left(\operatorname{atan}\left(\frac{B}{A}\right) - \operatorname{atan}\left(\frac{D}{C}\right)\right)$$

$$\frac{A\angle\theta_1}{B\angle\theta_2} = \frac{A}{B}\angle(\theta_1 - \theta_2)$$

$$(A\angle\theta_1)(B\angle\theta_2) = AB\angle(\theta_1 + \theta_2)$$

For example,

$$(1 + j)\left(\frac{2 + j}{3 + 4j}\right)$$

$$= \sqrt{2}\angle 0.7854\frac{\sqrt{5}\angle 0.4636}{\sqrt{25}\angle 0.9273}$$

$$= \frac{\sqrt{2}\sqrt{5}}{\sqrt{25}}\angle 0.7854 + 0.4636 - 0.9273$$

$$= 0.6325\angle 0.3217$$

$$= 0.6 + j0.2$$

Figure 243 Calculations in polar notation

The cartesian form of complex numbers seen in the last section are well suited to operations where complex numbers are added and subtracted. But, when complex numbers are to be multiplied and divided these become tedious and bulky. The polar form for complex numbers simplifies many calculations. The previous example started in Figure 241 is redone using polar notation in Figure 244. In this example the input is directly converted to polar form, without the need for calculation. The input frequency is substituted into the transfer function and it is then converted to polar form. After this the output is found by multiplying the transfer function by the input. The calculations for magnitudes involve simple multiplications. The angles are simply added. After this the polar form of the result is converted directly back to a function of time.

Consider the input function from the previous example in polar form it becomes,

$$F(t) = 10\sin(100t + 0.5)N \qquad F(\omega) = 10\angle 0.5$$

The transfer function can also be put in polar form.

$$\frac{x(\omega)}{F(\omega)} = \frac{1}{(2000 - 1000(100)^2) + j(3000(100))} = \frac{1}{-9998000 + j300000}$$

$$\frac{x(\omega)}{F(\omega)} = \frac{1\angle 0}{\sqrt{(-9998000)^2 + (300000)^2}\angle\left(\operatorname{atan}\left(\frac{300000}{-9998000}\right) + \pi\right)}$$

$$\frac{x(\omega)}{F(\omega)} = \frac{1\angle 0}{10002500\angle 3.112} = \frac{1}{10002500}\angle(0 - 3.112) = 0.9998\times 10^{-7}\angle -3.112$$

The output can now be calculated.

$$x(\omega) = \frac{x(\omega)}{F(\omega)}F(\omega) = (0.9998\times 10^{-7}\angle -3.112)(10\angle 0.5)$$

$$x(\omega) = 0.9998\times 10^{-7}(10)\angle(-3.112 + 0.5) = 0.9998\times 10^{-6}\angle -2.612$$

The output function can be written from this result.

$$x(\omega) = 0.9998\times 10^{-6}\sin(100t - 2.612)$$

Note: recall that $\tan\theta = \frac{Re}{Im}$ but the $\operatorname{atan}\theta$ function in calculators and software only returns values between -90 to 90 degrees. To compensate for this the sign of the real and imaginary components must be considered to determine where the angle lies. If it lies beyond the -90 to 90 degree range the correct angle can be obtained by adding or subtracting 180 degrees.

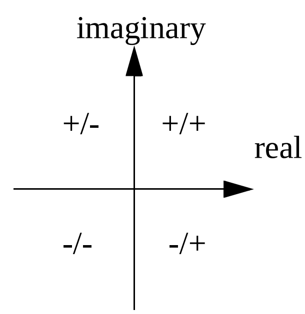
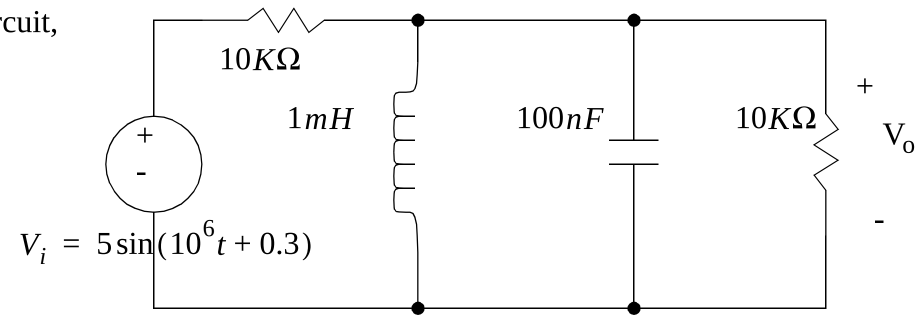


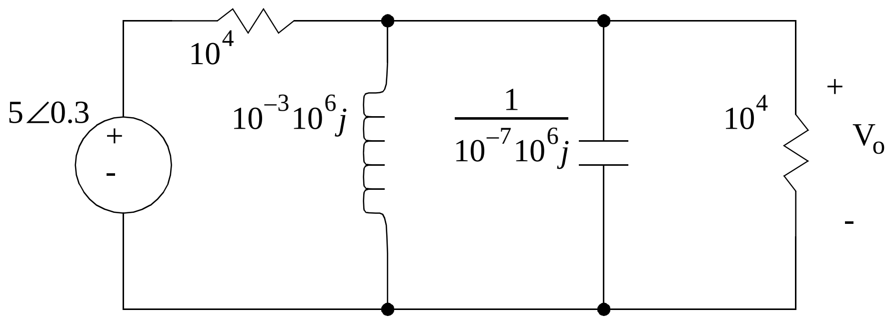
Figure 244 Correcting quadrants for calculated angles

Consider the circuit analysis example in Figure 245. In this example the component values are converted to their impedances, and the input voltage is converted to phasor form. (Note: this is a useful point to convert all magnitudes to powers of 10.) After this the three output impedances are combined to a single impedance. In this case the calculations were simpler in the cartesian form.

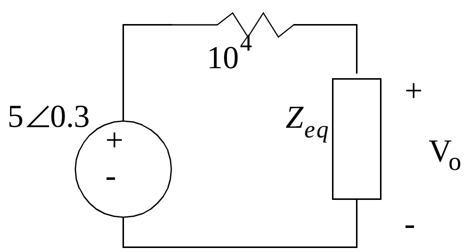
Given the circuit,



the impedances and input voltage can be written in phasor form.



The three output impedances in parallel can then be combined.



$$\frac{1}{Z_{eq}} = \frac{1}{10^{-3}10^{6}j} + \frac{1}{\left(\frac{1}{10^{-7}10^{6}j}\right)} + \frac{1}{10^{4}}$$

$$\frac{1}{Z_{eq}} = -10^{-3}j + 10^{-1}j + 10^{-4} = (10^{-4}) + j(0.099)$$

$$Z_{eq} = \frac{1}{(10^{-4}) + j(0.099)}$$

Figure 245 Phasor analysis of a circuit

The analysis continues in Figure 246 as the output is found using a voltage divider. In this case a combination of cartesian and polar forms are used to simplify the calculations. The final result is then converted back from phasor form to a function of time.

The output can be found using the voltage divider form.

$$V_o(\omega) = (5\angle 0.3)\left(\frac{Z_{eq}}{10^4 + Z_{eq}}\right)$$

$$V_o(\omega) = (5\angle 0.3)\left(\frac{\left(\frac{1}{(10^{-4}) + j(0.099)}\right)}{10^4 + \left(\frac{1}{(10^{-4}) + j(0.099)}\right)}\right)$$

$$V_o(\omega) = (5\angle 0.3)\left(\frac{1}{(10^4)((10^{-4}) + j(0.099)) + 1}\right)$$

$$V_o(\omega) = (5\angle 0.3)\left(\frac{1}{2 + j990}\right) = \frac{5\angle 0.3}{\sqrt{2^2 + 990^2}\angle\operatorname{atan}\frac{990}{2}} = \frac{5\angle 0.3}{990\angle 1.5687761}$$

$$V_o(\omega) = \frac{5}{990}\angle(0.3 - 1.5687761) = 5.05 \times 10^{-3}\angle -1.269$$

Finally, the output voltage can be written.

$$V_o(t) = 5.05\sin(10^6 t - 1.269)mV$$

Figure 246 Phasor analysis of a circuit (cont'd)

Phasor analysis is applicable to systems that are linear. This means that the principle of superposition applies. Therefore, if an input signal has more than one frequency component then the system can be analyzed for each component, and then the results simply added. The example considered in Figure 241 is extended in Figure 247. In this example the input has a static component, as well as frequencies at 0.5 and 20 rad/s. The transfer function is analyzed for each of these frequencies components. The output components are found by multiplying the inputs by the response at the corresponding frequency. The results are then converted back to functions of time, and added together.

Given the transfer function for the system,

$$\frac{x(\omega)}{F(\omega)} = \frac{1}{(2000 - 1000\omega^2) + j(3000\omega)}$$

and an input with multiple frequency components,

$$F(t) = 1000 + 20\sin(20t) + 10\sin((0.5t)(N))$$

the transfer function for each frequency can be calculated,

$$\frac{x(0)}{F(0)} = \frac{1}{(2000 - 1000(0)^2) + j(3000(0))} = \frac{1}{2000} = 0.0005\angle 0$$

$$\frac{x(20)}{F(20)} = \frac{1}{(2000 - 1000(20)^2) + j(3000(20))} = \frac{1}{-398000 + j60000} = \frac{1\angle 0}{402497\angle 2.992}$$

$$\frac{x(0.5)}{F(0.5)} = \frac{1}{(2000 - 1000(0.5)^2) + j(3000(0.5))} = \frac{1}{1750 + j1500} = \frac{1\angle 0}{2305\angle 0.709}$$

Note: these are gains and phase shifts that will be used heavily in Bode plots later.

These can then be multiplied by the input components to find output components.

$$x(0) = (0.0005\angle 0)1000\angle 0 = 0.5\angle 0$$

$$x(20) = \left(\frac{1\angle 0}{402497\angle 2.992}\right)20\angle 0 = 0.497\times 10^{-4}\angle -2.992$$

$$x(0.5) = \left(\frac{1\angle 0}{2305\angle 0.709}\right)10\angle 0 = 4.34\times 10^{-3}\angle -0.709$$

Therefore the output is,

$$x(t) = 0.5 + 49.7\times 10^{-6}\sin(20t - 2.992) + 4.34\times 10^{-3}\sin(0.5t - 0.709)$$

Figure 247 A example for a signal with multiple frequency components (based on the example in Figure 241)

9.3 VIBRATIONS

Oscillating displacements and forces in mechanical systems will cause vibrations. In some cases these become a nuisance, or possibly lead to premature wear and failure in mechanisms. A common approach to dealing with these problems is to design vibration isolators. The equations for transmissibility and isolation is shown in Figure 248. These equations can compare the ratio of forces or displacements through an isolator. The calculation is easy to perform with a transfer function or Bode plot.

Given a vibration force in, to a force out,

$$T = \frac{F_{out}(\omega)}{F_{in}(\omega)} = \frac{x_{out}(\omega)}{x_{in}(\omega)}$$ The gain of a transfer function gives transmissibility

$$\%I = (1 - T)100\%$$

Figure 248 Transmissibility

Given the transfer function for a vibration isolator below, find a value of K that will give 50% isolation for a 10Hz vibration.

$$\frac{x_{out}}{x_{in}} = \frac{5D + 10}{4D^2 + 20KD + 4}$$

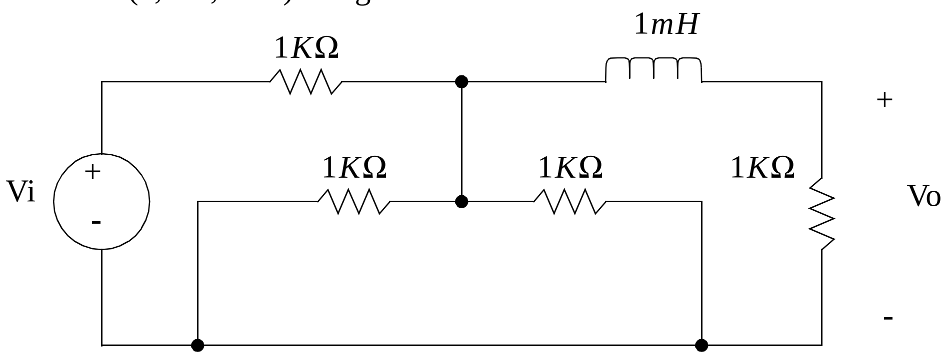
Figure 249 Drill problem: Select a K value

9.4 SUMMARY

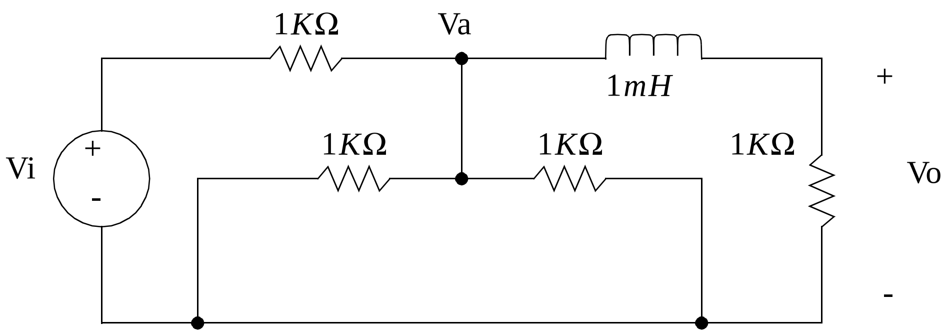
- Fourier transforms and phasor representations can be used to find the steady state response of a system to a given input.
- Vibration analysis determines frequency components in mechanical systems.

9.5 PRACTICE PROBLEMS

1. Develop a transfer function for the system pictured below and then find the response to an input voltage of Vi = 10cos(1,000,000 t) using Fourier transforms.



(ans.



$$\sum I_{Va} = \frac{V_a - V_i}{1K\Omega} + \frac{V_a}{1K\Omega} + \frac{V_a}{1K\Omega} + \frac{V_a - V_o}{0.001D} = 0$$

$$V_a\left(\frac{3}{1K\Omega} + \frac{1}{0.001D}\right) + V_i\left(\frac{-1}{1K\Omega}\right) = V_o\left(\frac{1}{0.001D}\right) \qquad (1)$$

$$\sum I_{Vo} = \frac{V_o - V_a}{0.001D} + \frac{V_o}{1K\Omega} = 0$$

$$V_o\left(\frac{1}{0.001D} + \frac{1}{1K\Omega}\right) = V_a\left(\frac{1}{0.001D}\right)$$

$$V_o\left(\frac{0.001D + 1K\Omega}{1K\Omega}\right) = V_a \qquad (2)$$

substitute (2) into (1)

$$V_o\left(\frac{0.001D + 1K\Omega}{1K\Omega}\right)\left(\frac{3}{1K\Omega} + \frac{1}{0.001D}\right) + V_i\left(\frac{-1}{1K\Omega}\right) = V_o\left(\frac{1}{0.001D}\right)$$

$$\frac{V_o}{V_i} = \frac{\frac{1}{1K\Omega}}{\left(\frac{0.001D + 1K\Omega}{1K\Omega}\right)\left(\frac{3}{1K\Omega} + \frac{1}{0.001D}\right) - \left(\frac{1}{0.001D}\right)}$$

for the given input of Vi = 10cos(1,000,000 t).

$$\frac{V_o}{10 + 0j} = \frac{\frac{1}{1K\Omega}}{\left(\frac{0.001j10^6 + 1K\Omega}{1K\Omega}\right)\left(\frac{3}{1K\Omega} + \frac{1}{0.001(j10^6)}\right) - \left(\frac{1}{0.001(j10^6)}\right)}$$

$$V_o = \frac{10}{(j10^3 + 1K\Omega)\left(3 + \frac{1}{j}\right) - \left(\frac{1}{j}\right)} = \frac{10j}{(j10^3 + 1K\Omega)(3j + 1) - (1)}$$

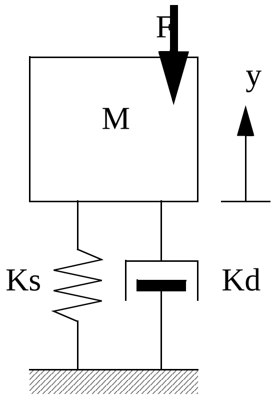
$$V_o = \frac{10j}{(-2001) + j(4000)} = \frac{10\angle 1.570795}{4472.6\angle 2.035} = 0.00224\angle -0.464$$

$$V_o = 2.24\cos(1,000,000t - 0.464)mV$$

2. A single d.o.f. model with a weight of 1.2 kN and a stiffness of 340 N/m has a steady-state harmonic excitation force applied at 95 rpm (revolutions per minute). What damper value will

give a vibration isolation of 92%?

(ans.



$$\sum F_y = -K_s y - K_d y D - F = M y D^2$$

$$\frac{y}{F} = \frac{-1}{D^2 M + D K_d + K_s}$$

$$F_{floor} = K_s y + K_d y D$$

$$\frac{F_{floor}}{y} = K_s + K_d D$$

$$\frac{F_{floor}}{F} = \left(\frac{F_{floor}}{y}\right)\left(\frac{y}{F}\right) = \frac{-(K_s + K_d D)}{D^2 M + D K_d + K_s}$$

$$\frac{F_{floor}}{F} = \frac{-(K_s + K_d \omega j)}{-\omega^2 M + \omega j K_d + K_s}$$

$$\left|\frac{F_{floor}}{F}\right| = \frac{\sqrt{K_s^2 + (K_d \omega)^2}}{\sqrt{(K_s - \omega^2 M)^2 + (\omega K_d)^2}}$$

(ans. cont'd

For 92% isolation, there is 100-92 = 8% transmission, at 95rpm.

$$K_s = 340\frac{N}{m} \qquad M = \frac{1200N}{9.81\frac{N}{kg}} = 122kg$$

$$\omega = \left(95\frac{rev}{min}\right)\left(\frac{1min}{60\sec}\right)\left(\frac{2\pi rad}{rev}\right) = 9.95\frac{rad}{s}$$

$$0.08 = \frac{\sqrt{\left(340\frac{N}{m}\right)^2 + \left(K_d 9.95\frac{rad}{s}\right)^2}}{\sqrt{\left(340\frac{N}{m} - \left(9.95\frac{rad}{s}\right)^2 122kg\right)^2 + \left(9.95\frac{rad}{s}K_d\right)^2}}$$

$$\left(340\frac{N}{m} - \left(9.95\frac{rad}{s}\right)^2 122kg\right)^2 + \left(9.95\frac{rad}{s}K_d\right)^2 = \frac{\left(340\frac{N}{m}\right)^2 + \left(K_d 9.95\frac{rad}{s}\right)^2}{0.08^2}$$

$$(1.377878\times10^8)\frac{N^2}{m^2} + 99.0025\frac{rad^2}{s^2}K_d^2 = \frac{115600}{0.0064}\frac{N^2}{m^2} + K_d^2\frac{99.0025}{0.0064}\frac{rad^2}{s^2}$$

$$\left(\frac{1.197253\times10^8}{15370.138}\right)\frac{N^2s^2}{m^2rad^2} = K_d^2 \qquad K_d = 88.3\frac{Ns}{m}$$

3. Four helical compression springs are used at each corner of a piece of equipment. The spring rate is 240 N/m for each spring and the vertical static deflection of the equipment is 10mm. Calculate the weight of the equipment and determine the amount of isolation the springs would afford if the equipment operating frequency is twice the natural frequency of the system.

10. BODE PLOTS

Topics:
- Bode plots

Objectives:
- To be able to describe the response of a system using Bode plots

10.1 INTRODUCTION

When a Fourier transform is applied to a transfer function the result can be expressed as a magnitude and angle that are functions of frequency. The magnitude is the gain, and the angle is the phase shift. In the previous chapter these values were calculated for a single frequency and then multiplied by the input values to get an output value. At different frequencies the transfer function value will change. The transfer function gain and phase angle can be plotted as a function of frequency to give an overall picture of system response.

Aside: Consider a 'graphic equalizer' commonly found on home stereo equipment. The spectrum can be adjusted so that high or low tones are emphasized or muted. The position of the sliders adjusts the envelope that the audio signal is filtered through. The sliders trace out a Bode gain plot. In theoretical terms the equalizer can be described with a transfer function. As the slides are moved the transfer function is changed, and the bode plot shifts. In the example below the slides are positioned to pass more of the lower frequencies. The high frequencies would not be passed clearly, and might sound somewhat muffled.

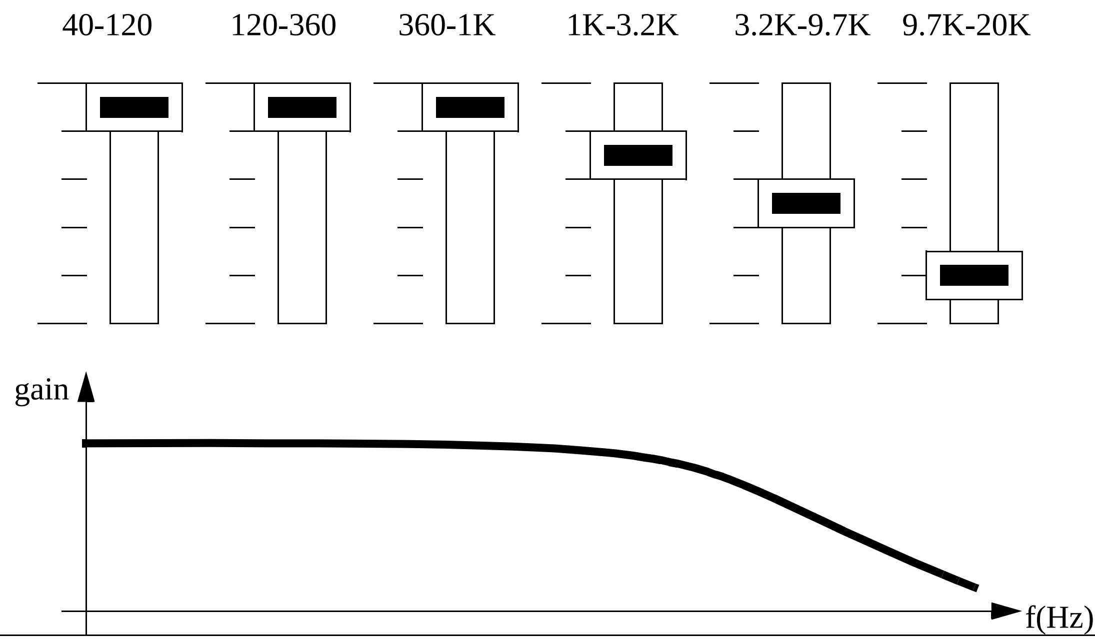


Figure 250 Commonly seen Bode plot

The mass-spring-damper transfer function from the previous chapter is expanded in Figure 251. In this example the transfer function is multiplied by the complex conjugate to eliminate the complex number in the denominator. The magnitude of the resulting transfer function is the gain, and the phase shift is the angle. Note that to correct for the quadrant of the phase shift angles pi radians is subtracted for certain frequency values.

$$\frac{x(\omega)}{F(\omega)} = \frac{1}{j(3000\omega) + (2000 - 1000\omega^2)}$$

$$\frac{x(\omega)}{F(\omega)} = \left[\frac{1}{j(3000\omega) + (2000 - 1000\omega^2)}\right]\left[\frac{(-j)(3000\omega) + (2000 - 1000\omega^2)}{(-j)(3000\omega) + (2000 - 1000\omega^2)}\right]$$

$$\frac{x(\omega)}{F(\omega)} = \frac{(2000 - 1000\omega^2) - j(3000\omega)}{(2000 - 1000\omega^2)^2 + (3000\omega)^2}$$

$$\left|\frac{x(\omega)}{F(\omega)}\right| = \frac{\sqrt{(2000 - 1000\omega^2)^2 + (3000\omega)^2}}{(2000 - 1000\omega^2)^2 + (3000\omega)^2} = \frac{1}{\sqrt{(2000 - 1000\omega^2)^2 + (3000\omega)^2}}$$

$$\theta = \operatorname{atan}\left(\frac{-3000\omega}{-1000\omega^2 + 2000}\right) = \operatorname{atan}\left(\frac{-3\omega}{2-\omega^2}\right) \quad \text{for } (\omega \le 2)$$

$$\theta = \operatorname{atan}\left(\left(\frac{-3\omega}{2-\omega^2}\right) - \pi\right) \quad \text{for } (\omega > 2)$$

Figure 251 A Fourier transform example

The results in Figure 251 are normally left in variable form so that they may be analyzed for a range of frequencies. An example of this type of analysis is done in Figure 252. A set of frequencies is used for calculations. These need to be converted from Hz to rad/s before use. For each one of these the gain and phase angle is calculated. The gain gives a ratio between the input sine wave and output sine wave of the system. The magnitude of the output wave can be calculated by multiplying the input wave magnitude by the gain. (Note: recall this example was used in the previous chapter) The phase angle can be added to the input wave to get the phase of the output wave. Gain is normally converted to 'dB' so that it may cover a larger range of values while still remaining similar numerically. Also note that the frequencies are changed in multiples of tens, or magnitudes.

f(Hz)	(rad/sec)	Gain	Gain (dB)	θ (rad.)	θ (deg.)
0	0				
0.001	0.006283				
0.01	0.06283				
0.1	0.6283				
1	6.283				
10	62.83				
100	628.3				
1000	6283				

Note: the gain values will cover many magnitudes of values. To help keep the graphs rational we will "compress" the values by converting them to dB (decibels) using the following formula.

$$gain_{db} = 20\log(gain)$$

Note: negative phase angles mean that the mass motion lags the force.

Note: The frequencies chosen should be chosen to cover the points with the greatest amount of change.

Figure 252 A Fourier transform example (continued)

In this example gain is defined as x/F. Therefore F is the input to the system, and x is the resulting output. The gain means that for each unit of F in, there will be gain*F=x out. The input and output are sinusoidal and there is a difference in phase between the input and output wave of θ (the phase angle). This is shown in Figure 253, where an input waveform is supplied with three sinusoidal components. For each of the frequencies a gain and phase shift are calculated. These are then used to calculate the resulting output wave. The resulting output represents the steady-state response to the sinusoidal output.

Assuming there is excitation from two sinusoidal sources in addition to the static load, as defined by the equation,

$$F(t) = \underbrace{1000}\underbrace{+ 20\sin(20t)}\underbrace{+ 10\sin(0.5t)}(N)$$

The result for each component can be evaluated separately

$$x(t) = 1000(\_\_\_\_\_\_\_\_)\sin\left(0t + \_\_\_\_\_\_\left(\frac{2\pi}{360}\right)\right)$$
$$+ 20(\_\_\_\_\_\_\_\_)\sin\left(20t + \_\_\_\_\_\_\left(\frac{2\pi}{360}\right)\right)$$
$$+ 10(\_\_\_\_\_\_\_\_)\sin\left(0.5t - \_\_\_\_\_\_\left(\frac{2\pi}{360}\right)\right)$$

gain from calculations or Bode plot

phase angle from calculations or Bode plot

$$x(t) = 0.5 + \_\_\_\_\_\sin(20t + \_\_\_\_\_) + \_\_\_\_\_\_\sin(0.5t + \_\_\_\_\_)$$

Figure 253 A Fourier transform example (continued)

10.2 BODE PLOTS

In the previous section we calculated a table of gains and phase angles over a range of frequencies. Graphs of these values are called Bode plots. These plots are normally done on semi-log graph paper, such as that seen in Figure 254. Along the longer axis of this paper the scale is base 10 logarithmic. This means that if the paper started at 0.1 on one side, the next major division would be 1, then 10, then 100, and finally 1000 on the other side of the paper. The basic nature of logarithmic scales prevents the frequency from being zero. Along the linear axis (the short one) the gains and phase angles are plotted, normally with two graphs side-by-side on a single sheet of paper.

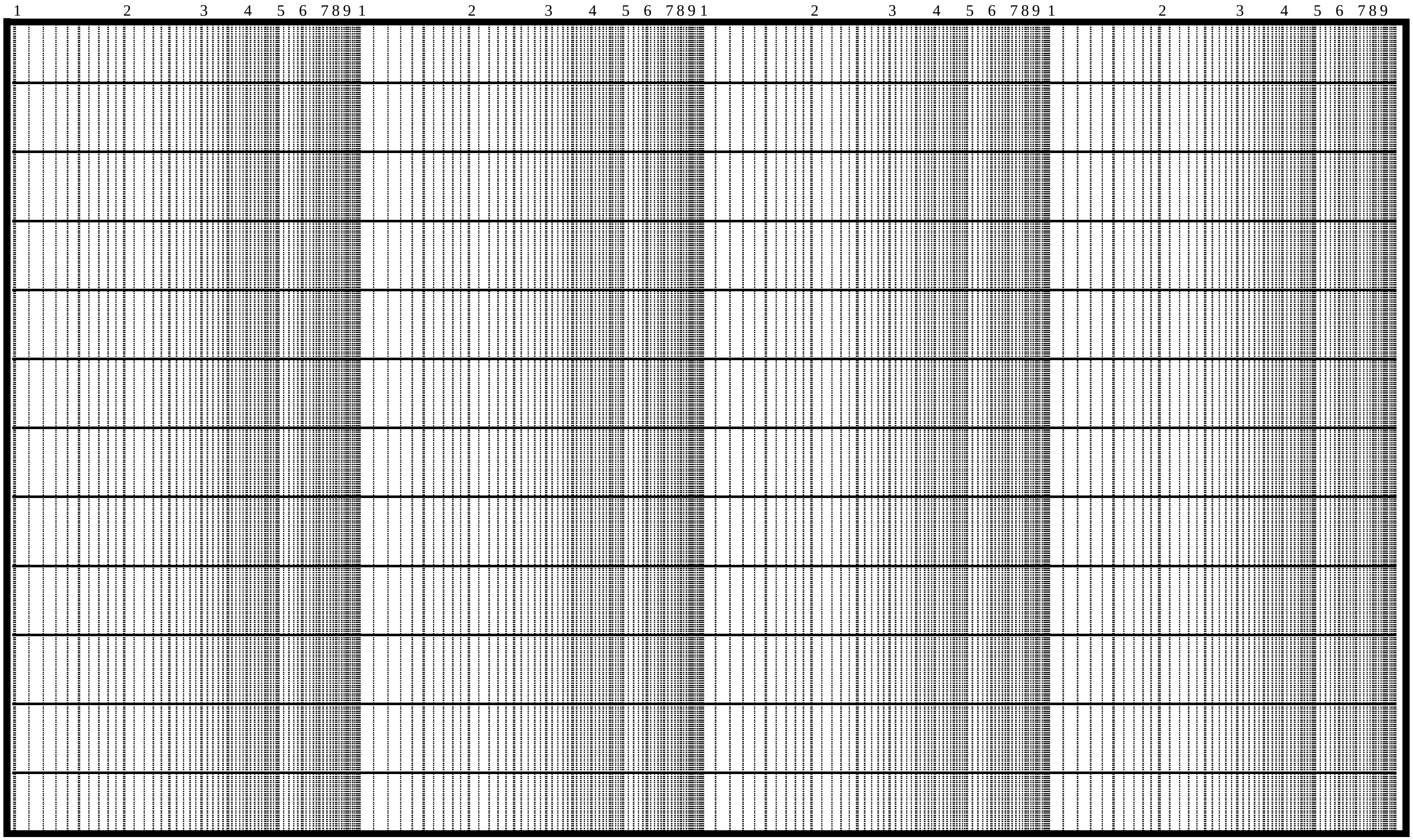

1 2 3 4 5 6 7 8 9 1 2 3 4 5 6 7 8 9 1 2 3 4 5 6 7 8 9 1 2 3 4 5 6 7 8 9

Figure 254 4 cycle semi-log graph paper

Plot the points from Figure 252 in the semi-log graph paper in Figure 254. The general layout is pictured below.

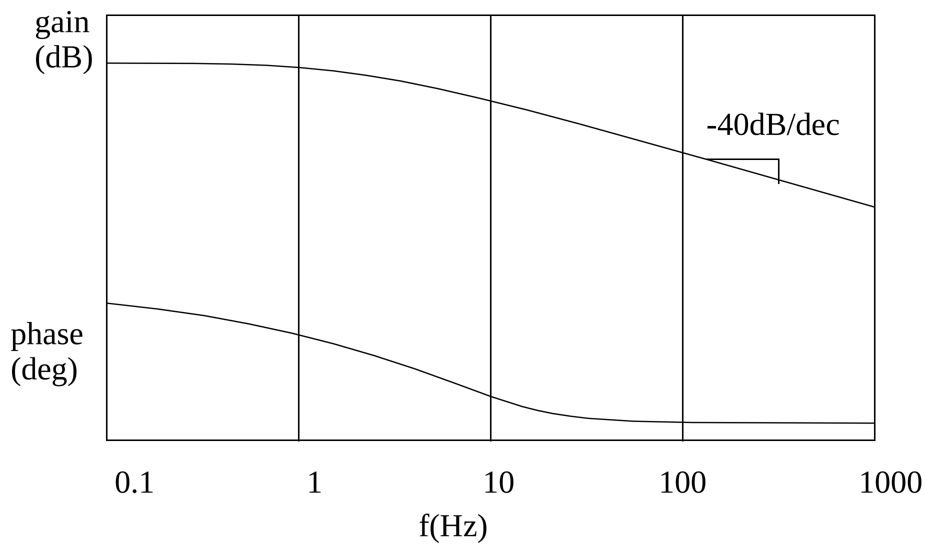


Figure 255 Drill problem: Plot the points from Figure 252 on graph paper

Use computer software, such as Mathcad or a spreadsheet, to calculate the points in Figure 251, and then draw Bode plots. Most software will offer options for making one axis use a log10 scale.

Figure 256 Drill problem: Plot the points from Figure 251 with a computer

Draw the Bode plot for the transfer function by hand or with computer.

$$\frac{D + 3}{D^2 + 10000D + 10000}$$

Figure 257 Drill problem: Draw the Bode plot for gain and phase

An approximate technique for constructing a gain Bode plot is shown in Figure 258. This method involves looking at the transfer function and reducing it to roots in the numerator and denominator. Once in that form a straight line approximation for each term can be drawn on the graph. An initial gain is also calculated to shift the results up or down. When done, the straight line segments are added to produce a more complex straight line curve. A smooth curve is then drawn over top of this curve.

Bode plots for transfer functions can be approximated with the following steps.

1. Plot the straight line pieces.
   a) The gain at 0rad/sec is calculated and used to find an initial offset. For example this transfer function starts at 10(D+1)/(D+1000)=10(0+1)/(0+1000)=0.01=-40dB.
   b) Put the transfer function in root form to identify corner frequencies. For example (D+1)/(D+1000) will have corner frequencies at 1 and 1000 rad/sec.
   c) Curves that turn up or down are drawn for each corner frequency. At each corner frequency a numerator term causes the graph to turn up, each term in the dominator causes the graph to turn down. The slope up or down is generally +/- 20dB/decade for each term. Also note that squared (second-order) terms would have a slope of +/-40dB/decade.
2. The effect of each term is added up to give the resulting straight line approximation.
3. When the smooth curve is drawn, there should commonly be a 3dB difference at the corner frequencies. In second-order systems the damping coefficient make may the corner flatter or peaked.

Figure 258 The method for Bode graph straight line gain approximation

Note: Some of the straight line approximation issues are discussed below.

Why is there 3dB between a first order corner and the smooth plot, and the phase angle is 45 degrees of the way to +/- 90 degrees??

$$G(j\omega) = \frac{1}{\omega_c + j\omega}$$

the initial gain is

$$G(0) = \frac{1}{\omega_c + j0} = \frac{1}{\omega_c}$$

at the corner frequency $\omega_c = \omega$

$$G(j\omega) = \frac{1}{\omega_c + j\omega_c} = \frac{1}{\omega_c\sqrt{2}\angle\frac{\pi}{4}} = \frac{1}{\omega_c\sqrt{2}}\angle-\frac{\pi}{4}$$

Therefore the difference is

$$diff = \frac{G(j\omega)}{G(0)} = \frac{1}{\sqrt{2}}\angle-\frac{\pi}{4} = 20\log\left(\frac{1}{\sqrt{2}}\right)\angle-\frac{\pi}{4} = -3.01dB\angle-\frac{\pi}{4}$$

Why does a first order pole go down at 20dB/dec?

$$|G(j\omega)| = \left|\frac{1}{\omega_c + j\omega}\right|$$

before the corner frequency, $\omega_c > j\omega$

$$|G(j\omega)| = \left|\frac{1}{\omega_c}\right| = 20\log(\omega_c^{-1}) = -20\log(\omega_c)$$

after the corner frequency, $\omega_c < j\omega$

$$|G(j\omega)| = \left|\frac{1}{j\omega}\right| = 20\log(\omega^{-1}) = -20\log(\omega)$$

each time the frequency increases by a multiple of 10, the log value becomes 1 larger, thus resulting in a gain change of -20 dB.

Figure 259 Why the straight line method works

Why does a pole make the phase angle move by -90deg after the corner frequency?

$$G(j\omega) = \frac{1}{\omega_c + j\omega} = \frac{1\angle 0}{\sqrt{\omega_c^2 + \omega^2}\angle \operatorname{atan}\left(\frac{\omega}{\omega_c}\right)} = \frac{1}{\sqrt{\omega_c^2 + \omega^2}}\angle -\operatorname{atan}\left(\frac{\omega}{\omega_c}\right)$$

before the corner frequency, $\omega_c > j\omega$

$$angle(G(j\omega)) = -\operatorname{atan}\left(\frac{\omega}{\omega_c}\right) = -\operatorname{atan}\left(\frac{0}{\omega_c}\right) = 0$$

after the corner frequency, $\omega_c < j\omega$

$$angle(G(j\omega)) = -\operatorname{atan}\left(\frac{\omega}{\omega_c}\right) = -\operatorname{atan}(\infty) = -\frac{\pi}{2}$$

Figure 260 Why the straight line method works (cont’d)

Show that the second order transfer function below would result in a slope of +/- 40 dB/decade.

$$G(j\omega) = \frac{1}{(\omega_c + j\omega)^2}$$

Figure 261 Drill problem: Slope of second order transfer functions.

An example of the straight line plotting technique is shown in Figure 262. In this example the transfer function is first put into a root form. In total there are three roots, 1, 10 and 100 rad/sec. The single root in the numerator will cause the curve to start upward with a slope of 20dB/dec after 1rad/sec. The two roots will cause two curves downwards at -20dB/dec starting at 10 and 100 rad/sec. The initial gain of the transfer function is also calculated, and converted to decibels. The frequency axis is rad/sec by default, but if Hz are used then it is necessary to convert the values.

$$G(D) = \frac{100D + 100}{0.01D^2 + 0.11D + 10} = \frac{10^4(D+1)}{(D+10)(D+100)}$$ (the equation is put in root form)

Step 1: Draw lines for each of the terms in the transfer function,

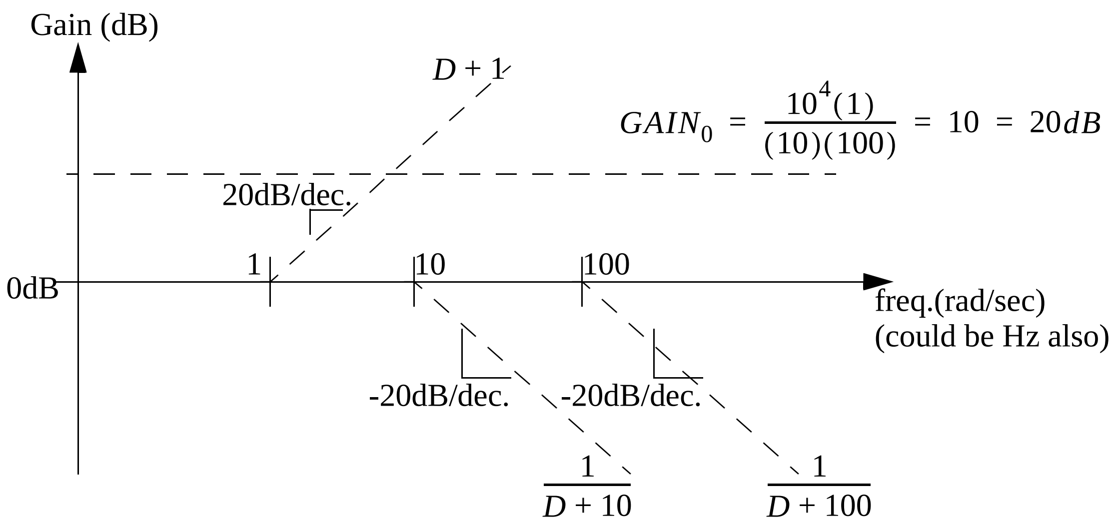


Figure 262 An approximate gain plot example

The example is continued in Figure 263 where the straight line segments are added to produce a combined straight line curve.

Step 2: Sum the individual lines, and get the straight line approximation,

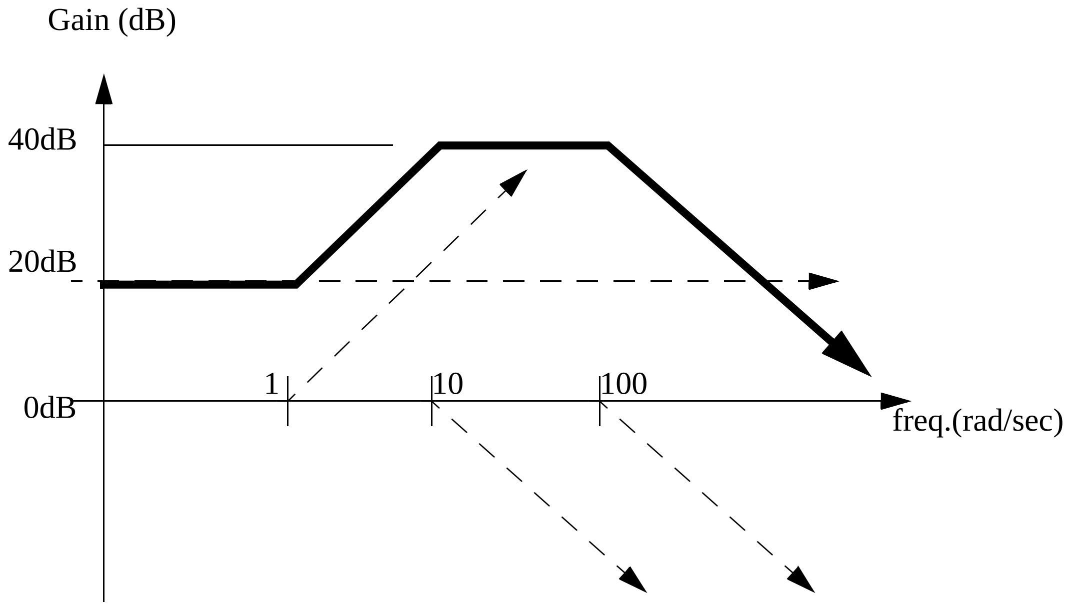


Figure 263 An approximate gain plot example (continued)

Finally a smooth curve is fitted to the straight line approximation. When drawing the curve imagine that there are rubber bands at the corners that pull slightly and smooth out. For a simple first-order term there is a 3dB gap between the sharp corner and the function. Higher order functions will be discussed later.

Step 3: Draw the smooth curve (leaving 3dB at the corners),

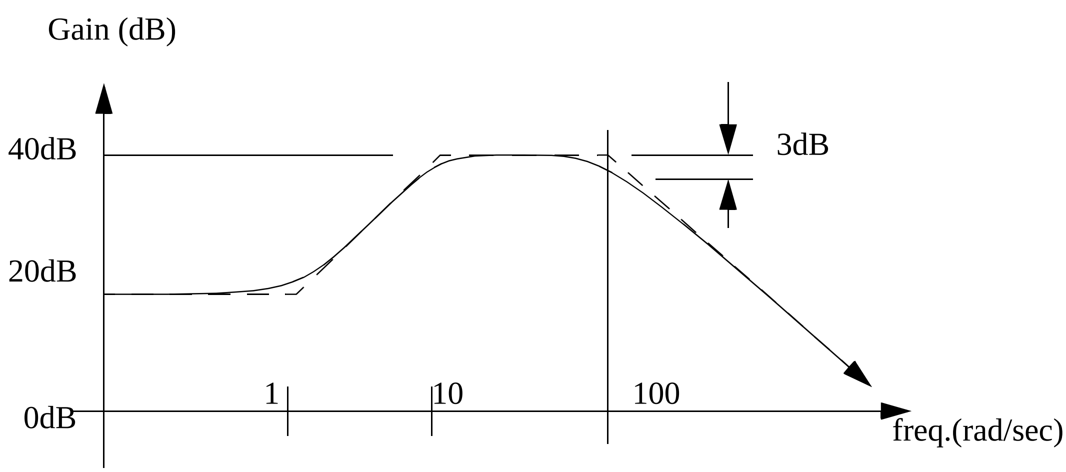


Figure 264 An approximate gain plot example (continued)

The process for constructing phase plots is similar to that of gain plots, as seen in Figure 266. The transfer function is put into root form, and then straight line phase shifts are drawn for each of the terms. Each term in the numerator will cause a positive shift of 90 degrees, while terms in the denominator cause negative shifts of 90 degrees. The phase shift occur over two decades, meaning that for a center frequency of 100, the shift would start at 10 and end at 1000. If there are any lone 'D' terms on the top or bottom, they will each shift the initial value by 90 degrees, otherwise the phase should start at 0degrees.

Gain plots for transfer functions can be approximated with the following steps.

1. Plot the straight line segments.
    a) Put the transfer function in root form to identify center frequencies. For example (D+1)/(D+1000) will have center frequencies at 1 and 1000 rad/sec. This should have already been done for the gain plot.
    b) The phase at 0rad/sec is determined by looking for any individual D terms. Effectively they have a root of 0rad/sec. Each of these in the numerator will shift the starting phase angle up by 90deg. Each in the denominator will shift the start down by 90 deg. For example the transfer function 10(D+1)/(D+1000) would start at 0 deg while 10D(D+1)/(D+1000) would start at +90deg.
    c) Curves that turn up or down are drawn around each center frequency. Again terms in the numerator cause the curve to go up 90 deg, terms in the denominator cause the curves to go down 90 deg. Curves begin to shift one decade before the center frequency, and finish one decade after.
2. The effect of each term is added up to give the resulting straight line approximation.
3. The smooth curve is drawn.

Figure 265 The method for Bode graph straight line gain approximation

The previous example started in Figure 262 is continued in Figure 266 to develop a phase plot using the approximate technique. There are three roots for the transfer function. None of these are zero, so the phase plot starts at zero degrees. The root in the numerator causes a shift of positive 90 deg, starting one decade before 1rad/sec and ending one decade later. The two roots in the denominator cause a shift downward.

$$G(D) = \frac{10^4(D+1)}{(D+10)(D+100)}$$

Step 1: Draw lines for each of the terms in the transfer function

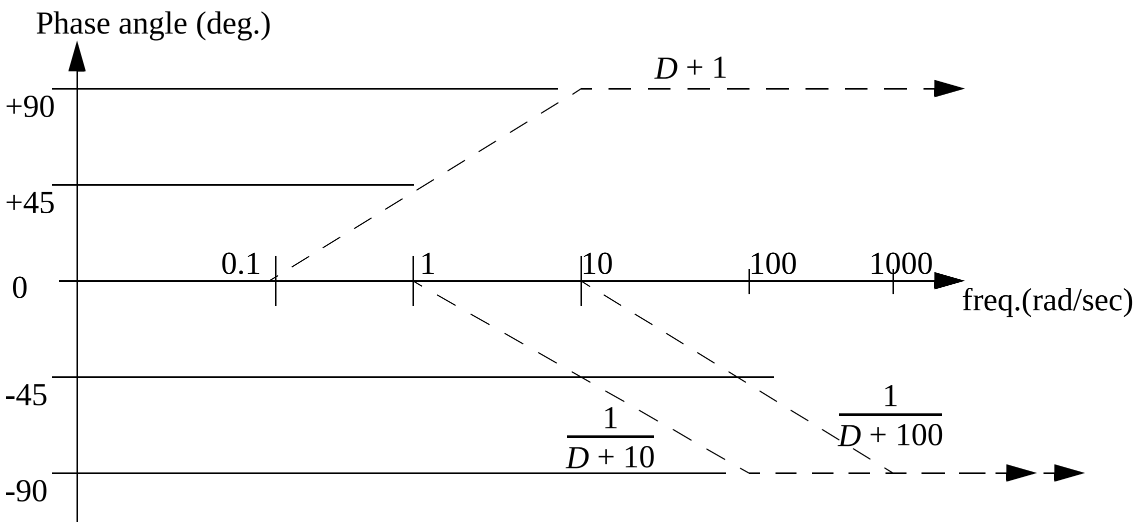


Figure 266 An approximate phase plot example

The straight line segments for the phase plot are added in Figure 267 to produce a straight line approximation of the final plot. A smooth line approximation is drawn using the straight line as a guide. Again, the concept of an rubber band will smooth the curve.

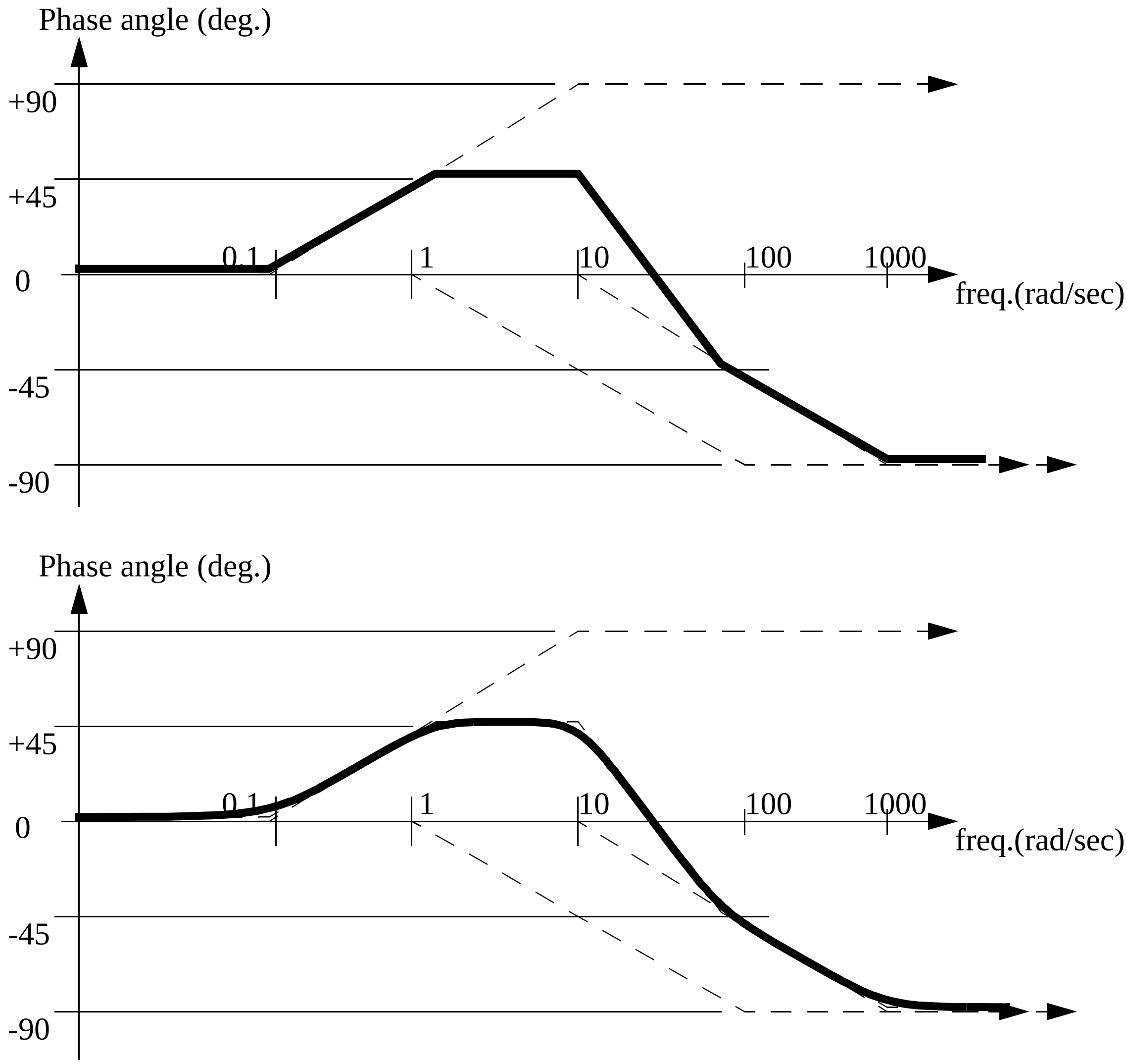
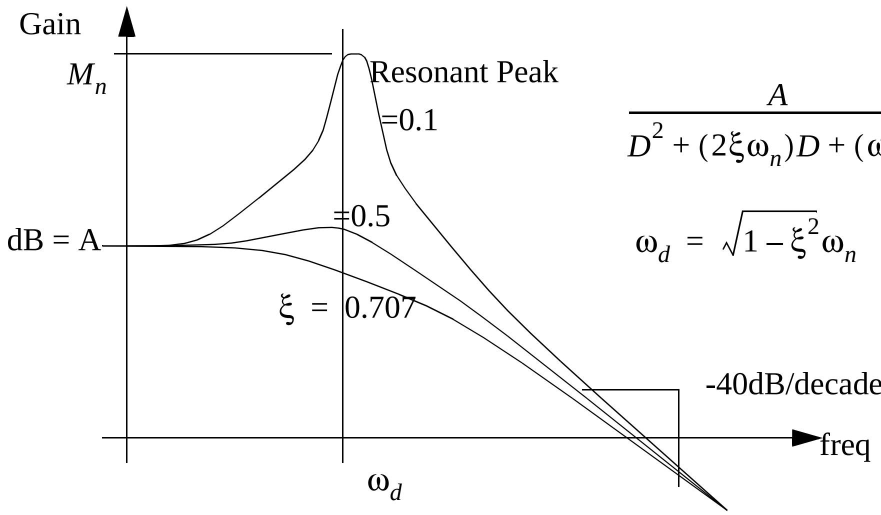


Figure 267 An approximate phase plot example (continued)

The previous example used a transfer function with real roots. In a second-order system with double real roots (overdamped) the curve can be drawn with two overlapping straight line approximations. If the roots for the transfer function are complex (under-damped the corner frequencies will become peaked. This can be handled by determining the damping coefficient and natural frequency as shown in Figure 268. The peak will occur at the damped frequency. The peaking effect will become more pronounced as the damping coefficient goes from 0.707 to 0 where the peak will be infinite.



$$\frac{A}{D^2 + (2\xi\omega_n)D + (\omega_n)^2}$$

$$\omega_d = \sqrt{1-\xi^2}\,\omega_n$$

Note: If ζ is less than 1 the roots become complex, and the Bode plots get a peak. This can be seen mathematically because the roots of the transfer function become complex.

Figure 268 Resonant peaks

The approximate techniques do decrease the accuracy of the final solution, but they can be calculated quickly. In addition these curves provide an understanding of the system that makes design easier. For example, a designer will often describe a system with a Bode plot, and then convert this to a desired transfer function.

Draw the straight line approximation for the transfer function.

$$\frac{D + 3}{D^2 + 10000D + 10000}$$

Figure 269 Drill problem: Draw the straight line approximation

10.3 SIGNAL SPECTRUMS

If a vibration signal is measured and displayed it might look like Figure 270. The overall sinusoidal shape is visible, along with a significant amount of ’noise’. When this is considered in greater detail it can be described with the given function. To determine the function other tools are needed to determine the frequencies, and magnitudes of the frequency components.

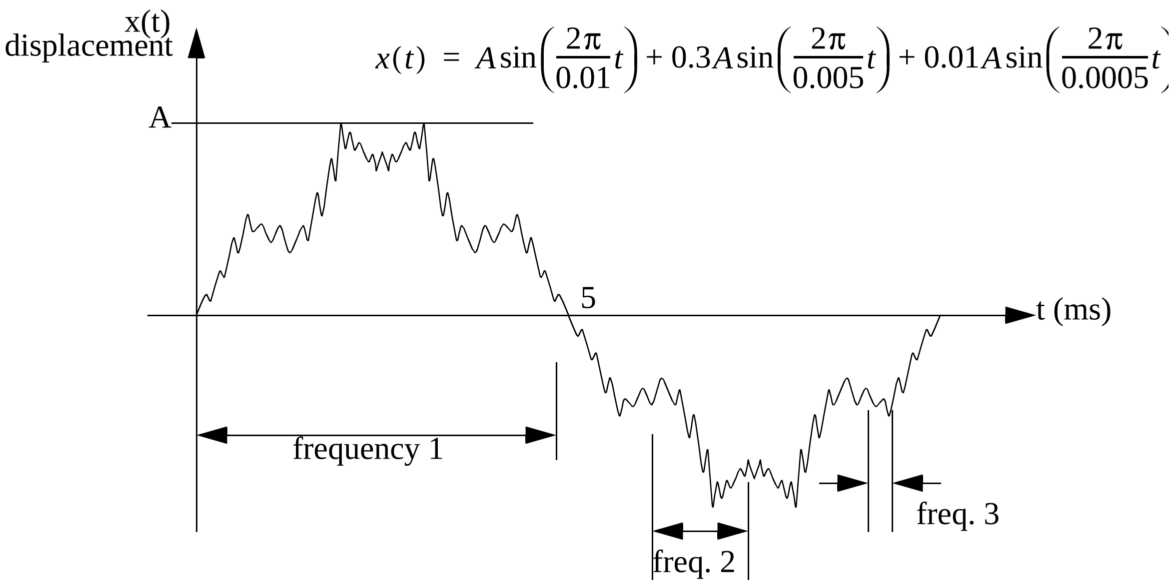


Figure 270 A vibration signal as a function of time

A signal spectrum displays signal magnitude as a function of frequency, instead of time. The time based signal in Figure 270 is shown in the spectrum in Figure 271. The three frequency components are clearly identifiable spikes. The height of the peaks indicates the relative signal magnitude.

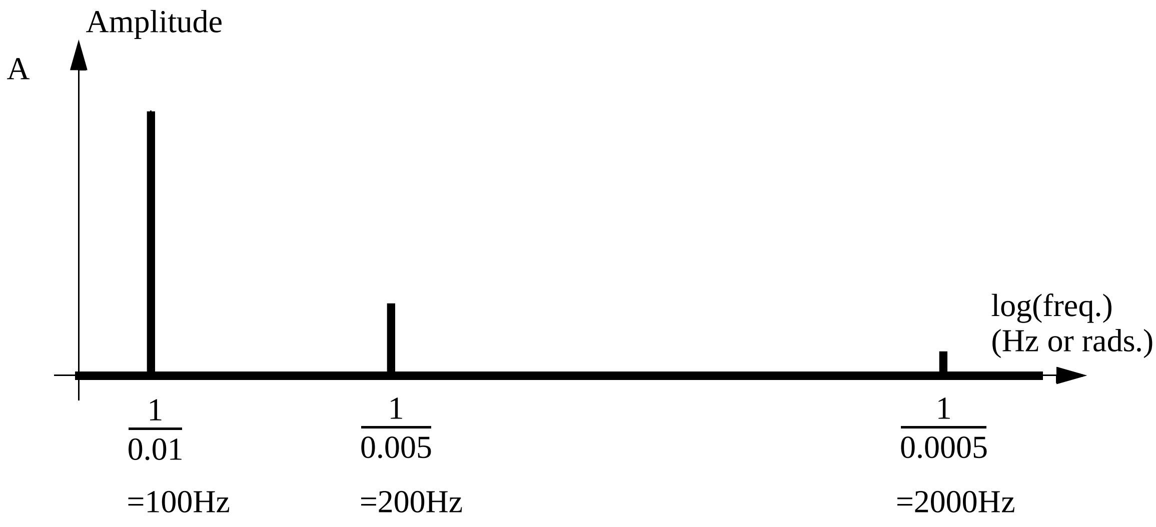


Figure 271 The spectrum for the signal in Figure 270

10.4 SUMMARY

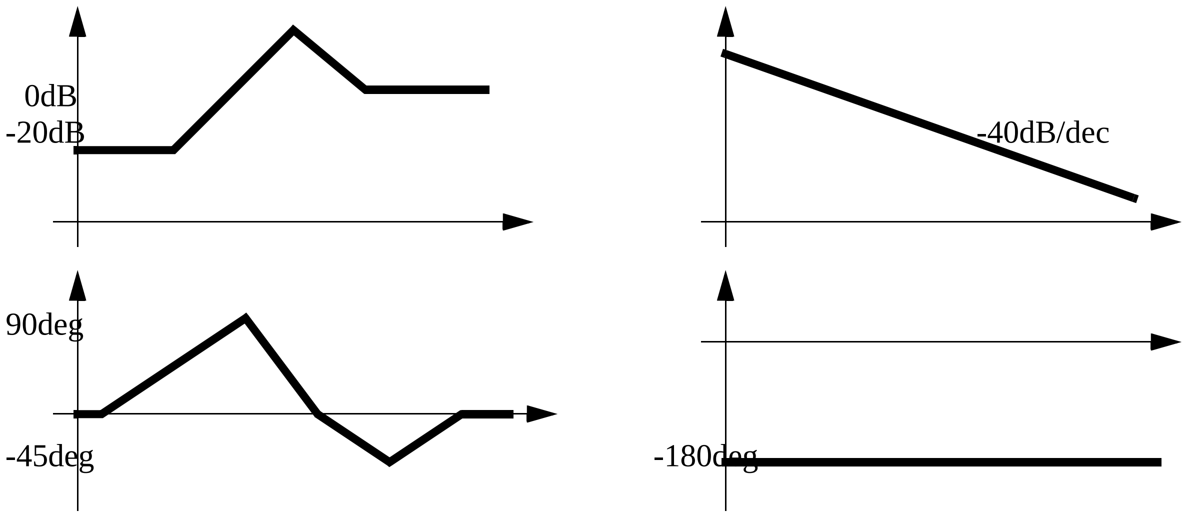
- Bode plots show gain and phase angle as a function of frequency.
- Bode plots can be constructed by calculating point or with straight line approximations.
- A signal spectrum shows the relative strengths of components at different frequencies.

10.5 PRACTICE PROBLEMS

1. Draw a Bode Plot for both of the transfer functions below.

$$\frac{(D+1)(D+1000)}{(D+100)^2} \qquad \text{AND} \qquad \frac{5}{D^2}$$

(ans.



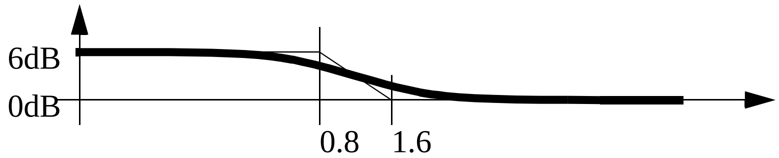
2. Given the transfer function below,

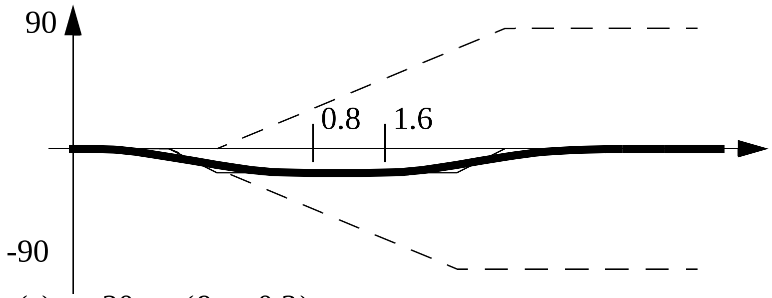
$$\frac{y(D)}{x(D)} = \frac{(D+10)(D+5)}{(D+5)^2}$$

a) draw the straight line approximation of the bode and phase shift plots.
b) determine the steady-state output if the input is x(t) = 20 sin(9t+0.3).

(ans.

$$\frac{y(D)}{x(D)} = \frac{(D+10)(D+5)}{(D+5)^2} = \frac{(D+10)}{(D+5)}$$





$$x(t) = 20\cos(9t + 0.3)$$

Aside: the numbers should be obtained from the graphs, but I have calculated them

$$\frac{y}{x} = \frac{(D+10)}{(D+5)} = \left(\frac{9j+10}{9j+5}\right)\left(\frac{5-9j}{5-9j}\right) = \frac{50+81-45j}{25+81} = 1.236 - 0.425j$$

$$\frac{y}{x} = \sqrt{1.236^2 + 0.425^2}\angle\operatorname{atan}\left(\frac{-0.425}{1.236}\right) = 1.307\angle -0.3312rad$$

$$\frac{y}{x} = 2.33dB\angle -9.49°$$

$$y(t) = 20(1.307)\sin(9t + 0.3 + (-0.3312))$$

$$y(t) = 26.1\sin(9t - 0.031)$$

Aside: This can also be done entirely with phasors in cartesian notation

$$\frac{y}{x} = \frac{(D+10)}{(D+5)} = \left(\frac{9j+10}{9j+5}\right)\left(\frac{5-9j}{5-9j}\right) = \frac{50+81-45j}{25+81} = 1.236 - 0.425j$$

$$x = 20(\cos(0.3rad) + j\sin(0.3rad)) = 19.1 + 5.91j$$

$$\frac{y}{19.1 + 5.91j} = (1.236 - 0.425j)(19.1 + 5.91j) = 26.1 - 0.813j = 26.1\angle -0.031$$

$$y(t) = 26.11\sin(9t - 0.031)$$

(ans. cont'd

Aside: This can also be done entirely with phasors in polar notation

$$\frac{y}{x} = \frac{(D+10)}{(D+5)} = \left(\frac{9j+10}{9j+5}\right) = \frac{13.45\angle 0.733}{10.30\angle 1.064} = \frac{13.45}{10.30}\angle 0.733 - 1.064 = 1.31\angle -0.331$$

$$x = 20\angle 0.3$$

$$\frac{y}{20\angle 0.3} = 1.31\angle -0.331$$

$$y = 1.31(20)\angle(-0.331 + 0.3) = 26.2\angle -0.031$$

$$y(t) = 26.2\sin(9t - 0.031)$$

3. For the transfer functions below, draw the root locus plots, and draw an approximate time response for each.

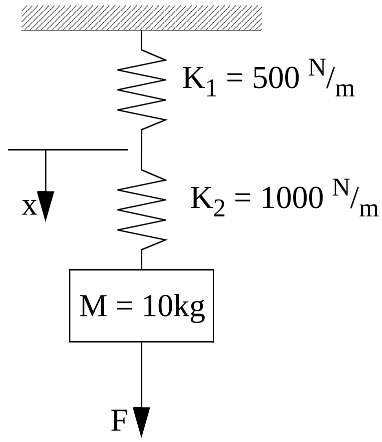
$$\frac{1}{D+1} \qquad \frac{1}{D^2+1} \qquad \frac{1}{(D+1)^2} \qquad \frac{1}{D^2+2D+2}$$

4. Use the straightline approximation techniques to draw the Bode plot for the transfer function below.

$$G = \frac{F}{x} = \frac{D+1000}{D^2+5D+100}$$

5. The applied force ‘F’ is the input to the system, and the output is the displacement ‘x’.

a) find the transfer function.



b) What is the steady-state response for an applied force F(t) = 10cos(t + 1) N ?
c) Give the transfer function if ‘x’ is the input.
d) Draw the bode plots.
e) Find x(t), given F(t) = 10N for t >= 0 seconds.

6. The following differential equation is supplied, with initial conditions.

$$y'' + y' + 7y = F \qquad y(0) = 1 \qquad y'(0) = 0$$

$$F(t) = 10 \qquad t > 0$$

a) Write the equation in state variable form.
b) Solve the differential equation numerically.
c) Solve the differential equation using calculus techniques.
d) Find the frequency response (gain and phase) for the transfer function using the Fourier transform. Roughly sketch the bode plots.

7. You are given the following differential equation for a spring damper pair.

$$5x + \left(\frac{d}{dt}\right)x = F \qquad F(t) = 10\sin(100t)$$

a) Write the transfer function for the differential equation if the input is F.
b) Apply the Fourier transform to the transfer function to find magnitude and phase as functions of frequency.
c) Draw a Bode plot for the system using either approximate or exact techniques on semi-log graph paper
.d) Use the Bode plot to find the response to;
$F(t) = 10\sin(100t)$

e) Put the differential equation in state variable form and use a calculator to find values in time for the given input.

$$F = 10\sin(100t)$$

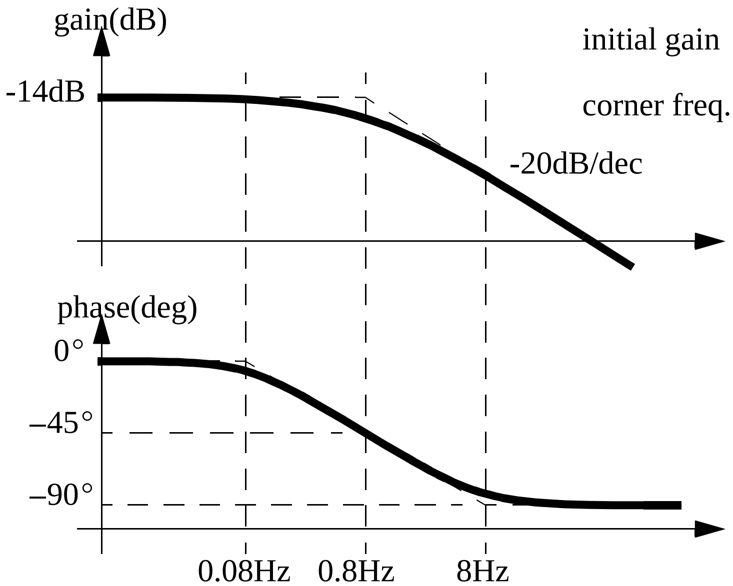
t	x
0.0	
0.002	
0.004	
0.006	
0.008	
0.010	

f) Give the expected 'x' response of this first-order system to a step function input for force F = 1N for t > 0 if the system starts at rest. Hint: Use the canonical form.

(ans. a) $\frac{x}{F} = \frac{1}{5+D}$

b) $\frac{x}{F} = \frac{1}{5+D} = \frac{1}{5+j\omega} = \frac{1\angle 0}{\sqrt{5^2+\omega^2}\angle \operatorname{atan}\left(\frac{\omega}{5}\right)} = \frac{1}{\sqrt{5^2+\omega^2}}\angle -\operatorname{atan}\left(\frac{\omega}{5}\right)$

c)



$$\text{initial gain} = 20\log\left(\frac{1}{5+0}\right) = -14dB$$

$$\text{corner freq.} = \frac{5}{2\pi} = 0.8Hz$$

d) $f = \frac{100}{2\pi} = 16Hz$

From the Bode plot, $gain = -40dB = 0.01$

$$phase = -90° = -\frac{\pi}{2}rad$$

$$x(t) = 10(0.01)\sin\left(100t - \frac{\pi}{2}\right)$$

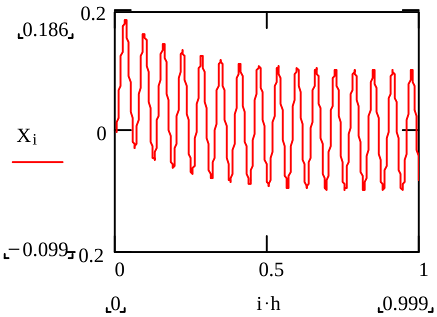
Verified by calculations,

$$\frac{x}{10\angle 0} = \frac{1}{\sqrt{5^2+100^2}}\angle -\operatorname{atan}\left(\frac{100}{5}\right) = 0.00999\angle -1.521$$

$$x = (10\angle 0)(0.00999\angle -1.521) = 0.0999\angle -1.521$$

$$x(t) = 0.0999\sin(100t - 1.521)$$

e)

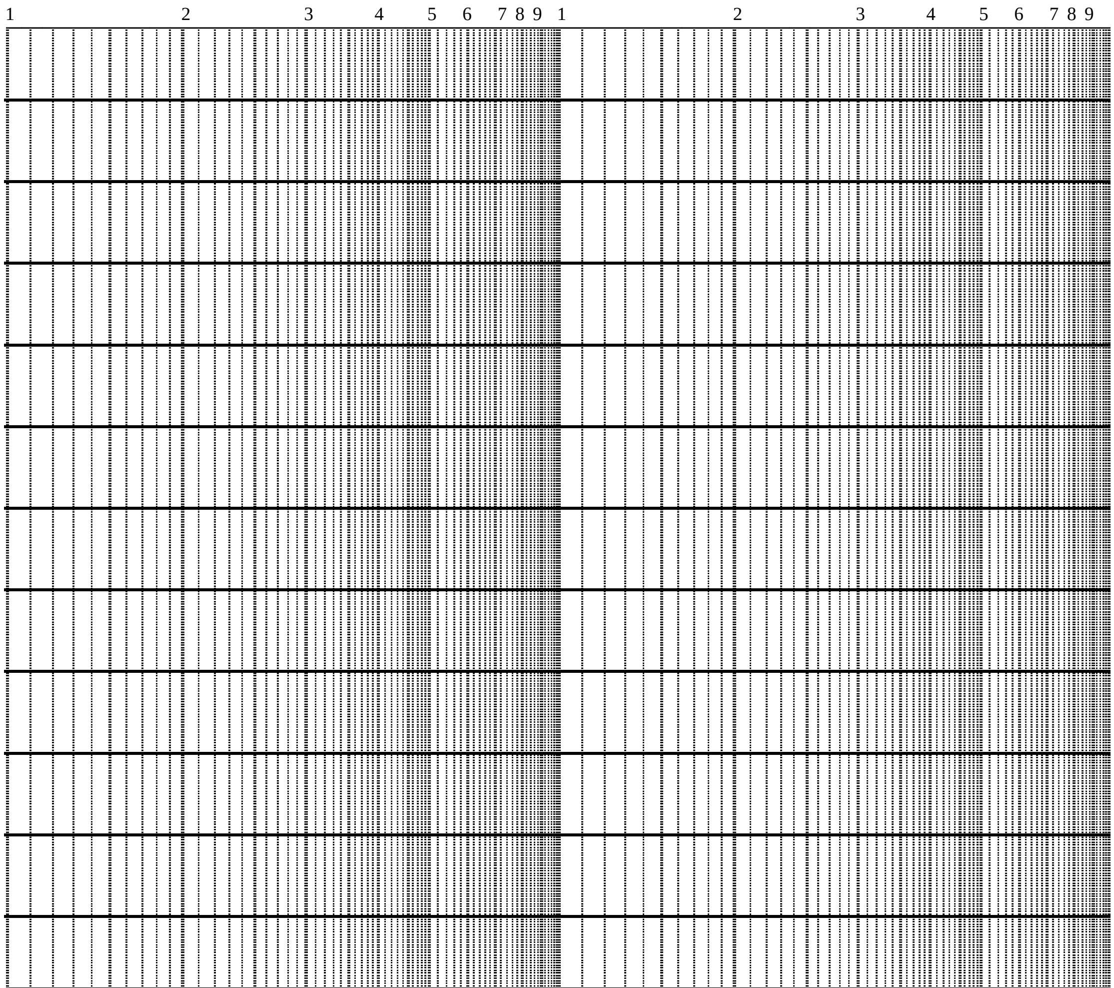


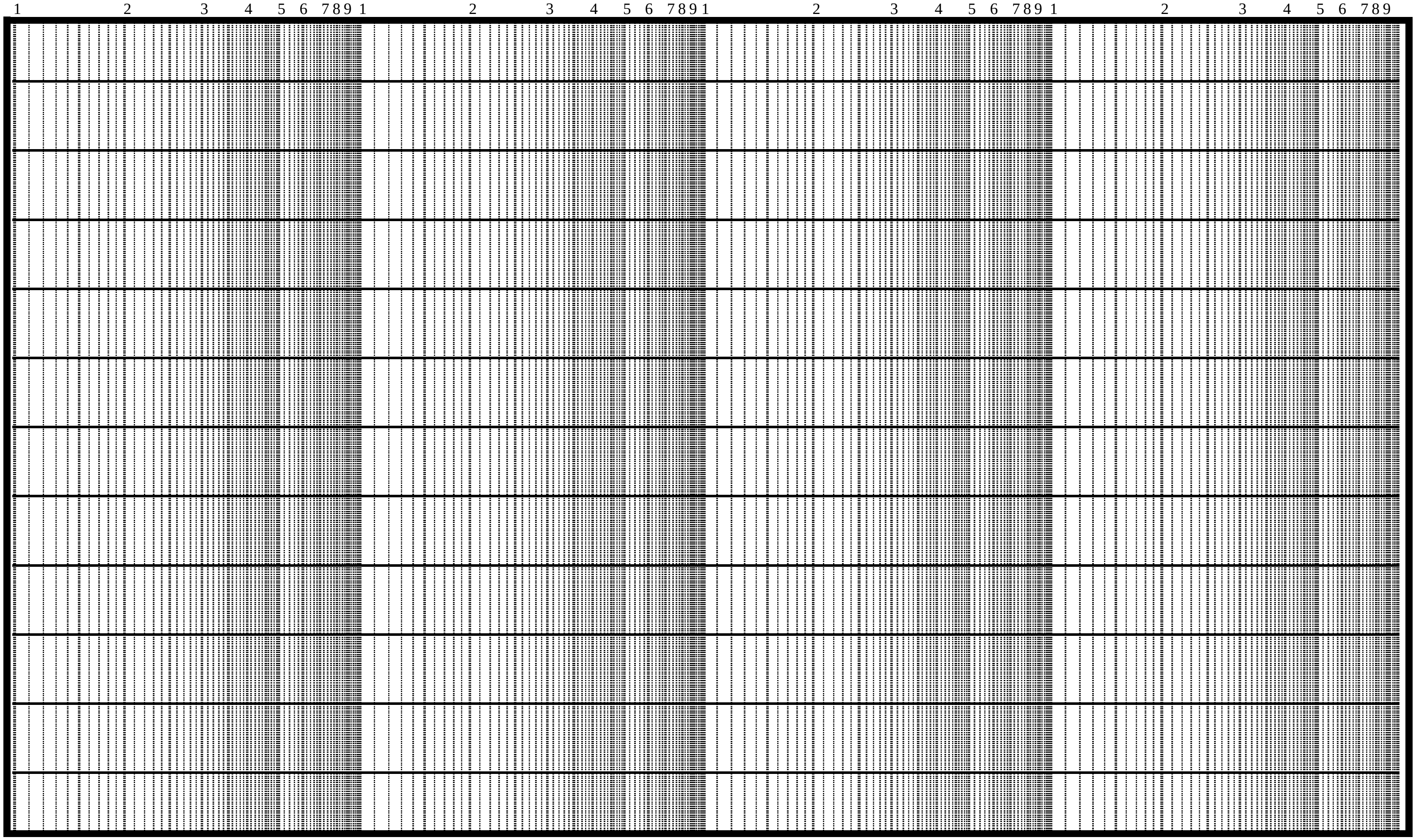
t	x
0.0	0
0.002	9.98e-4
0.004	5.92e-3
0.006	0.015
0.008	0.026
0.010	0.041

g) $$x = \frac{1}{5} - \frac{e^{-5t}}{5}$$

10.6 LOG SCALE GRAPH PAPER

Please notice that there are a few sheets of 2 and 4 cycle log paper attached, make additional copies if required, and if more cycles are required, sheets can be cut and pasted together. Also note that better semi-log paper can be purchased at technical bookstores, as well at most large office supply stores.


1 2 3 4 5 6 7 8 9 1 2 3 4 5 6 7 8 9


1 2 3 4 5 6 7 8 9 1 2 3 4 5 6 7 8 9 1 2 3 4 5 6 7 8 9 1 2 3 4 5 6 7 8 9

11. ROOT LOCUS ANALYSIS

Topics:
• Root-locus plots

Objectives:
• To be able to predict and control system stability.

11.1 INTRODUCTION

The system can also be checked for general stability when controller parameters are varied using root-locus plots.

11.2 ROOT-LOCUS ANALYSIS

In a engineered system we may typically have one or more design parameters, adjustments, or user settings. It is important to determine if any of these will make the system unstable. This is generally undesirable and possibly unsafe. For example, think of a washing machine that vibrates so much that it 'walks' across a floor, or a high speed aircraft that fails due to resonant vibrations. Root-locus plots are used to plot the system roots over the range of a variable to determine if the system will become unstable.

Recall the general solution to a homogeneous differential equation. Complex roots will result in a sinusoidal oscillation. If the roots are real the result will be e-to-the-t terms. If the real roots are negative then the terms will tend to decay to zero and be stable, while positive roots will result in terms that grow exponentially and become unstable. Consider the roots of a second-order homogeneous differential equation, as shown in Figure 272 to Figure 278. These roots are shown on the complex planes on the left, and a time response is shown to the right. Notice that in these figures (negative real) roots on the left hand side of the complex plane cause the response to decrease while roots on the right hand side cause it to increase. The rule is that any roots on the right hand side of the plane make a system unstable. Also note that the complex roots cause some amount of oscillation.

$R = -A, -B$

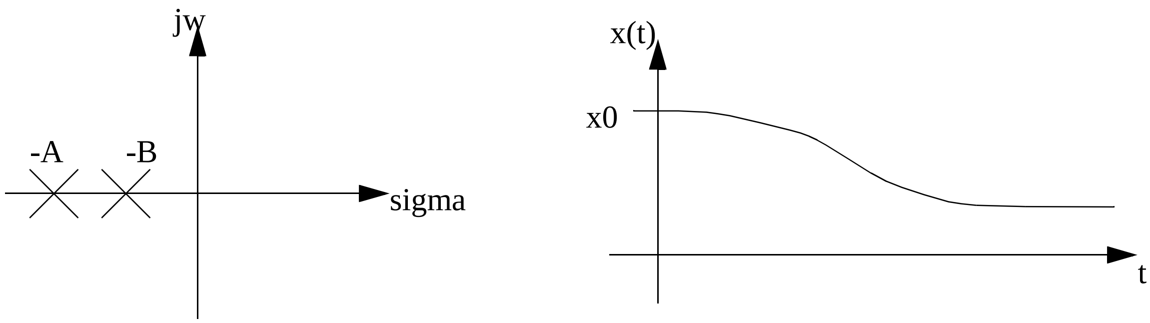


Figure 272 Negative real roots make a system stable

$R = \pm Aj$

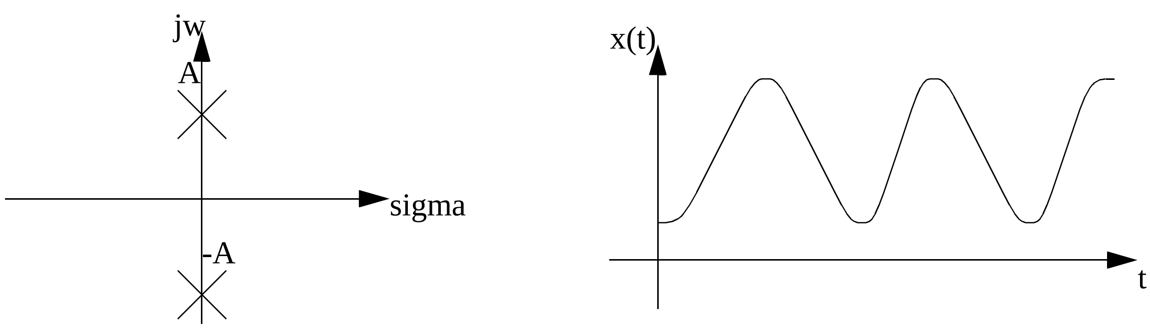


Figure 273 Complex roots make a system oscillate

$R = -A \pm Bj$

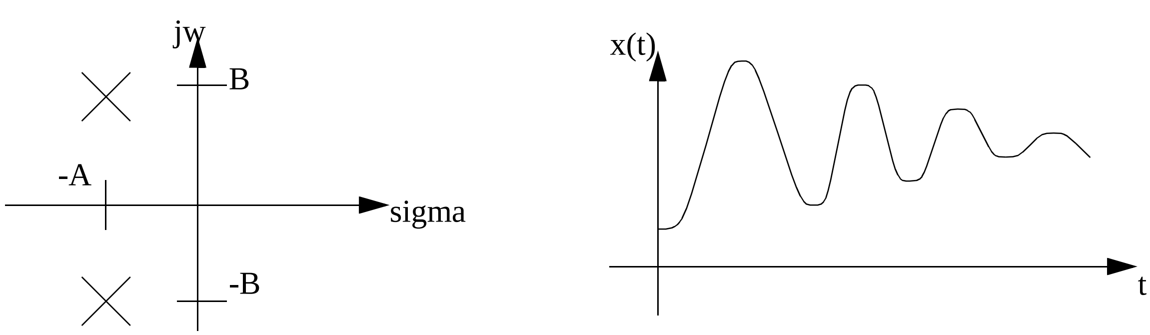


Figure 274 Negative real and complex roots cause decaying oscillation

$$R = -A \pm Bj$$

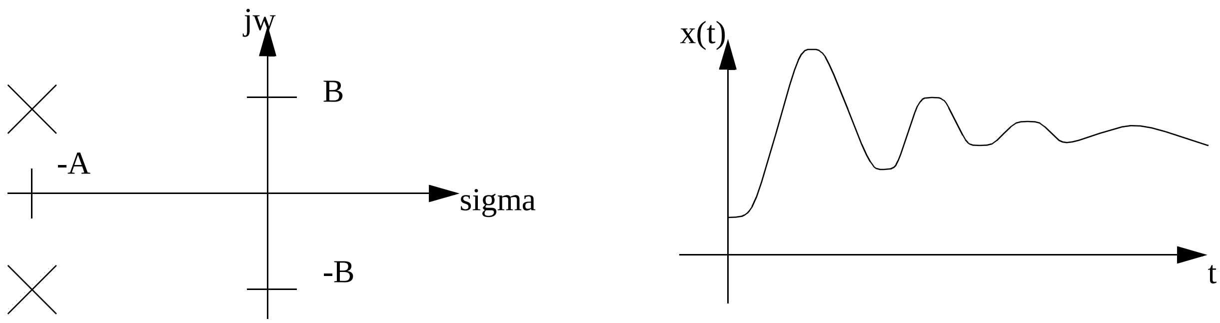


Figure 275 More negative real and complex roots cause a faster decaying oscillation

$$R = -A, -A$$

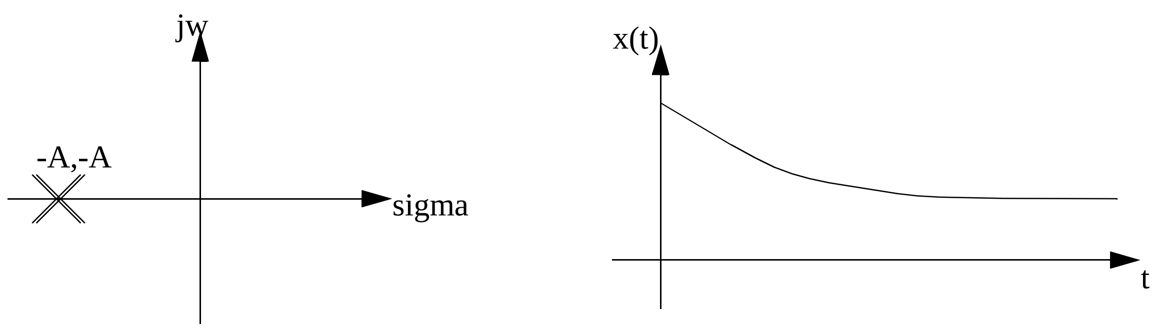


Figure 276 Overlapped roots are possible

$$R = A, A$$

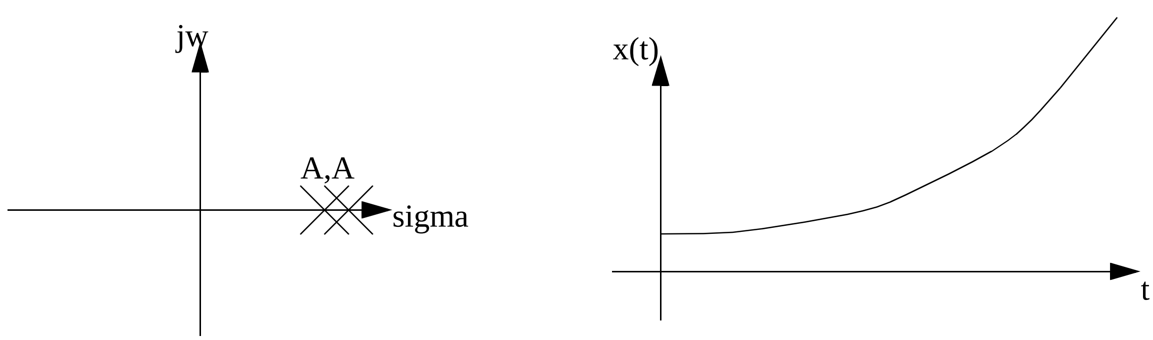


Figure 277 Positive real roots cause exponential growth and are unstable

$$R = A \pm Bj$$

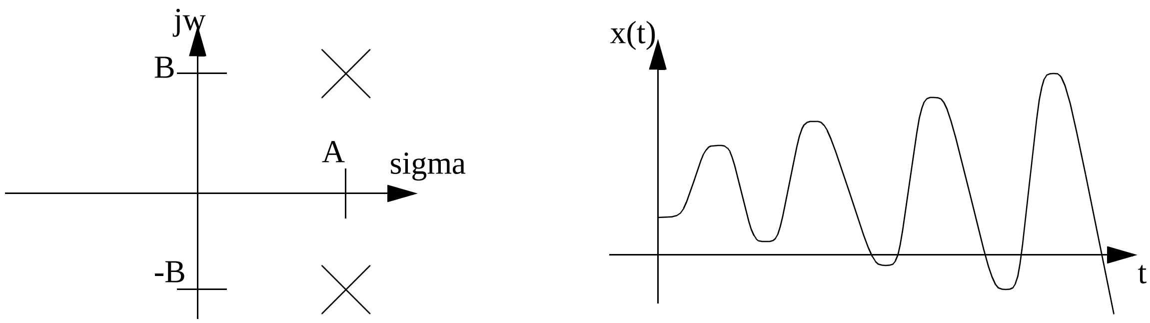


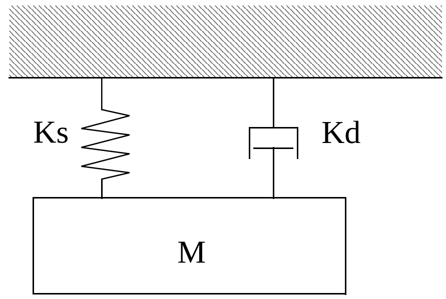
Figure 278 Complex roots with positive real parts have growing oscillations and are unstable

Next, recall that the denominator of a transfer function is the homogeneous equation. By analyzing the function in the denominator of a transfer function the general system response can be found. An example of root-locus analysis for a mass-spring-damper system is given in Figure 279. In this example the transfer function is found and the roots of the equation are written with the quadratic equation. At this point there are three unspecified values that can be manipulated to change the roots. The mass and damper values are fixed, and the spring value will be varied. The range of values for the spring coefficient should be determined by practical and design limitations. For example, the spring coefficient should not be zero or negative.

$$\frac{x(D)}{F(D)} = \frac{1}{MD^2 + K_d D + K_s}$$

Note: We want the form below

$$\frac{A}{(D+B)(D+C)}$$



$$\frac{x(D)}{F(D)} = \frac{\frac{1}{M}}{D^2 + \frac{K_d}{M}D + \frac{K_s}{M}}$$

$$B, C = \frac{\frac{-K_d}{M} \pm \sqrt{\frac{K_d^2}{M^2} - 4\frac{K_s}{M}}}{2}$$

Aside:

$$ax^2 + bx + c = (x+A)(x+B)$$

$$A, B = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

M	Kd	Ks	B	C
0	0	0	0	0
100	100	100		
100	100	1000		
100	100	10000		

Figure 279 A mass-spring-damper system equation

The roots of the equation can then be plotted to provide a root locus diagram. These will show how the values of the roots change as the design parameter is varied. If any of these roots pass into the right hand plane we will know that the system is unstable. In addition complex roots will indicate oscillation.

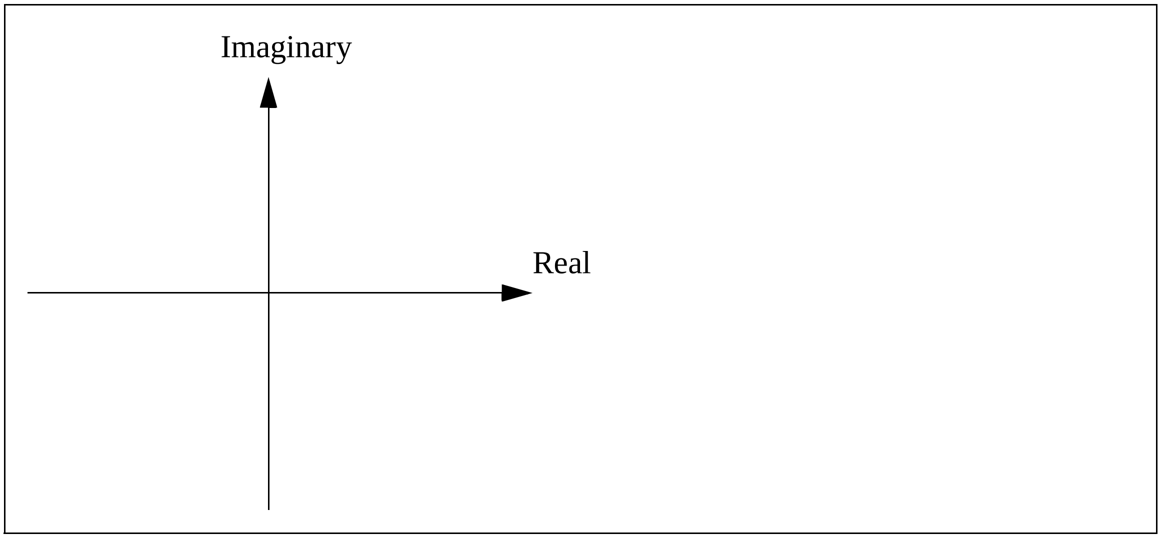
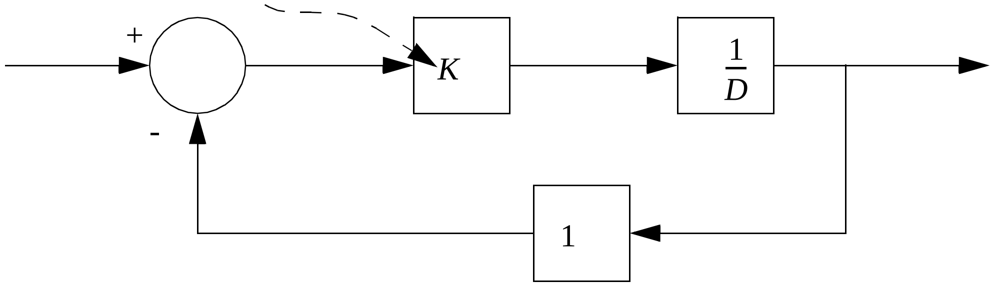


Figure 280 Drill problem: Plot the calculated roots on the axes above

A feedback controller with a variable control function gain is shown in Figure 281. The variable gain ’K’ calls for determining controller stability over the range of values. This analysis begins by developing a transfer function for the overall system. The root of the denominator is then calculated and plotted for a range of ’K’ values. In this case all of the roots are in the left side of the plane, so the system is stable and doesn’t oscillate. Keep in mind that gain values near zero put the control system close to the right hand plane. In real terms this will mean that the controller becomes unresponsive, and the system can go where it pleases. It would be advisable to keep the system gain greater than zero to avoid this region.

Note: This controller has adjustable gain. After this design is built we must anticipate that all values of K will be used. It is our responsibility to make sure that none of the possible K values will lead to instability.



$$G(D) = \frac{K}{D} \qquad H(D) = 1$$

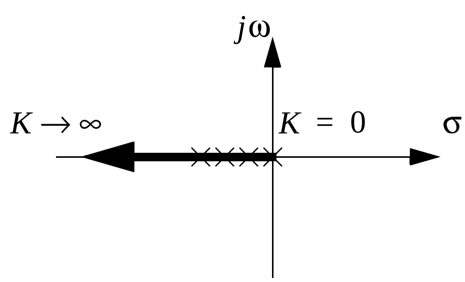
First, we must develop a transfer function for the entire control system.

$$G_S(D) = \frac{G(D)}{1 + G(D)H(D)} = \frac{\left(\frac{K}{D}\right)}{1 + \left(\frac{K}{D}\right)(1)} = \frac{K}{D + K}$$

Next, we use the characteristic equation of the denominator to find the roots as the value of K varies. These can then be plotted on a complex plane. Note: the value of gain 'K' is normally found from 0 to +infinity.

$$D + K = 0$$

K	root
0	-0
1	-1
2	-2
3	-3
etc...	



Note: This system will always be stable because all of the roots for all values of K are negative real, and it will always have a damped response. Also, larger values of K, make the system more stable.

Figure 281 Root-locus analysis in controller design

• Consider the example,

Given the system elements (assume a negative feedback controller),

$$G(D) = \frac{K}{D^2 + 3D + 2} \qquad H(D) = 1$$

First, find the characteristic equation,. and an equation for the roots,

$$1 + \left(\frac{K}{D^2 + 3D + 2}\right)(1) = 0$$

$$D^2 + 3D + 2 + K = 0$$

Note: For a negative feedback controller the denominator is,

$$1 + G(D)H(D)$$

$$\text{roots} = \frac{-3 \pm \sqrt{9 - 4(2 + K)}}{2} = -1.5 \pm \frac{\sqrt{1 - 4K}}{2}$$

Next, find values for the roots and plot the values,

K	roots
0	
1	
2	
3	

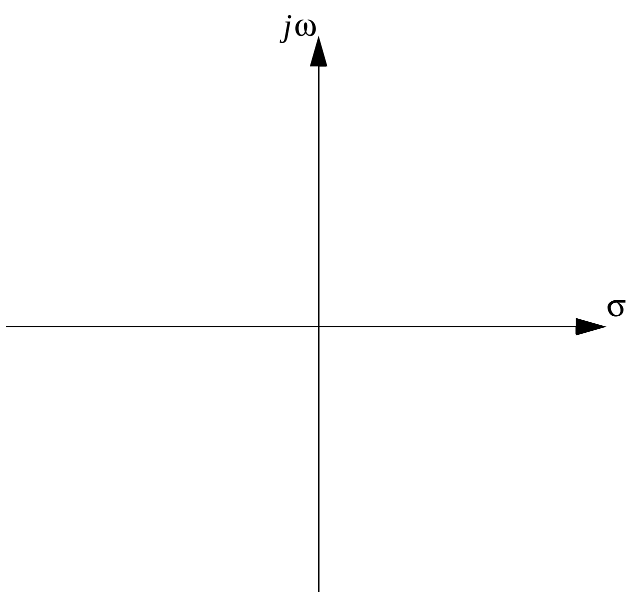


Figure 282 Drill problem: Complete the root-locus analysis

$$G(D)H(D) = \frac{K(D+5)}{D(D^2+4D+8)}$$

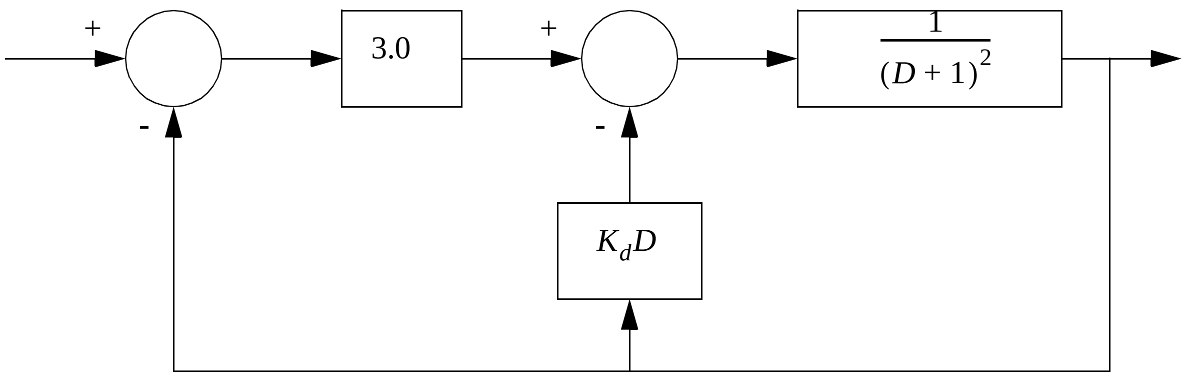
Figure 283 Drill problem: Draw a root locus plot

11.3 SUMMARY

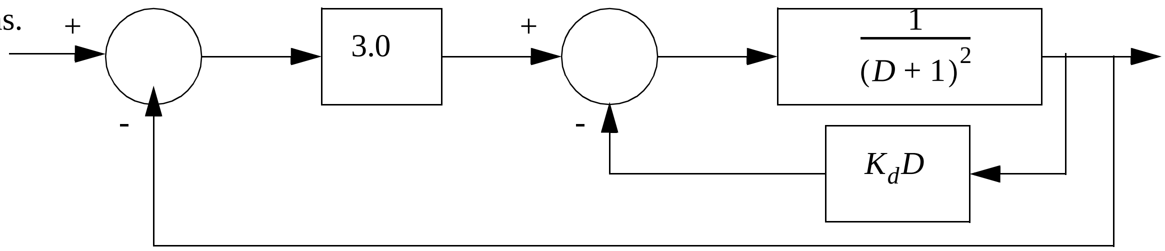
- Root-locus plots show the roots of a transfer function denominator to determine stability

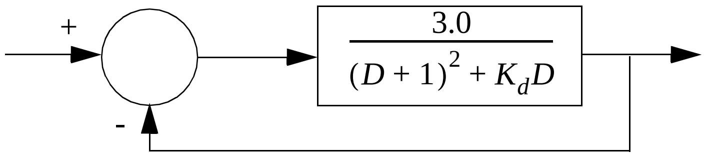
11.4 PRACTICE PROBLEMS

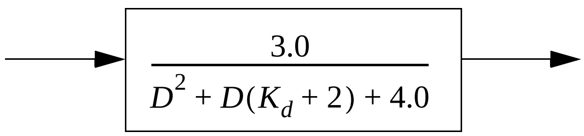
1. Draw the root locus diagram for the system below. specify all points and values.



(ans.







$$D^2 + D(K_d + 2) + 4.0 = 0$$

$$D = \frac{-K_d - 2 \pm \sqrt{(K_d + 2)^2 - 4(4.0)}}{2}$$

$$D = \frac{-K_d - 2 \pm \sqrt{K_d^2 + 4K_d - 12}}{2}$$

Kd	roots
-10	7.464, 0.536
-6	2.000, 2.000
-2	0 +/- 2.000j
-1	-0.5 +/- 1.936j
0	-1 +/- 1.732j
1	-1.5 +/- 1.323j
2	-2.000, -2.000
5	-0.628, -6.372
10	-0.343, -11.657
100	-0.039, -102.0
1000	-0.004, -1000

Critical points: (this is simple for a quadratic)

The roots becomes positive when

$$0 > -K_d - 2 \pm \sqrt{K_d^2 + 4K_d - 12}$$

$$2 + K_d > \pm\sqrt{K_d^2 + 4K_d - 12}$$

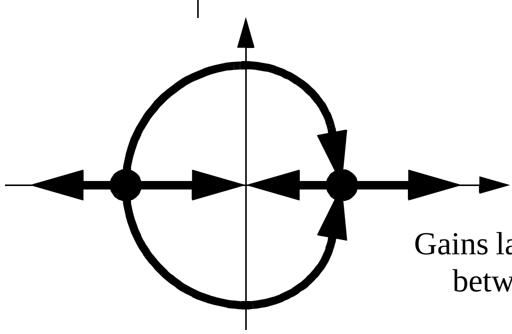
$$16 > 0$$

$$0 > -K_d - 2 \qquad\qquad K_d > -2$$

The roots becomes complex when

$$0 > K_d^2 + 4K_d - 12$$

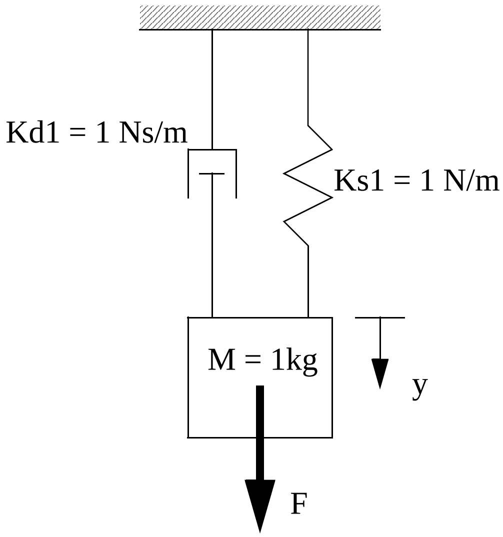
$$K_d = \frac{-4 \pm \sqrt{16 - 4(-12)}}{2} \qquad K_d = -6, 2$$

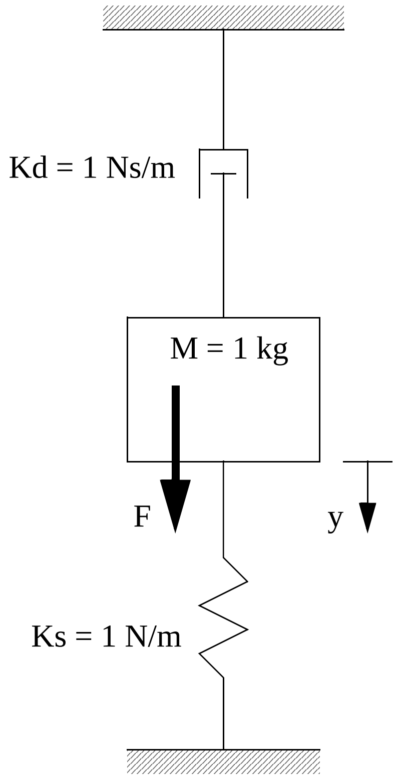


Gains larger than -2 will result in a stable system. Any gains between -4 and -2 will result in oscillations.

2. For each of the systems below,

a) write the differential equation and convert it to a transfer function.





b) If the input force is a step function of magnitude 1N, convert the input to a transfer function, and use it to find the time response for ‘y’ by solving a differential equation.
c) Draw the poles on a real-complex plane.
d) Apply the Fourier transform function and make it a function of frequency. Plot the Gain and magnitude as a function of frequency.

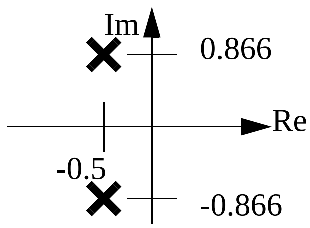
ans. (the same for both)

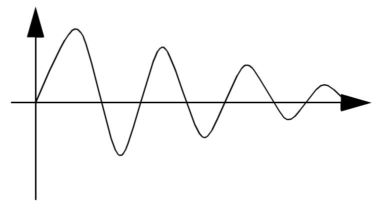
a) $$\frac{y}{F} = \frac{1}{D^2 + D + 1}$$

b) $$y = 1 - e^{-0.5t}\left[\cos(\sqrt{0.75}t) + \left(\frac{1 - 0.5(1)}{\sqrt{0.75}}\right)\sin(\sqrt{0.75}t)\right]$$

Note: a similar answer can be found with a single sin or cos

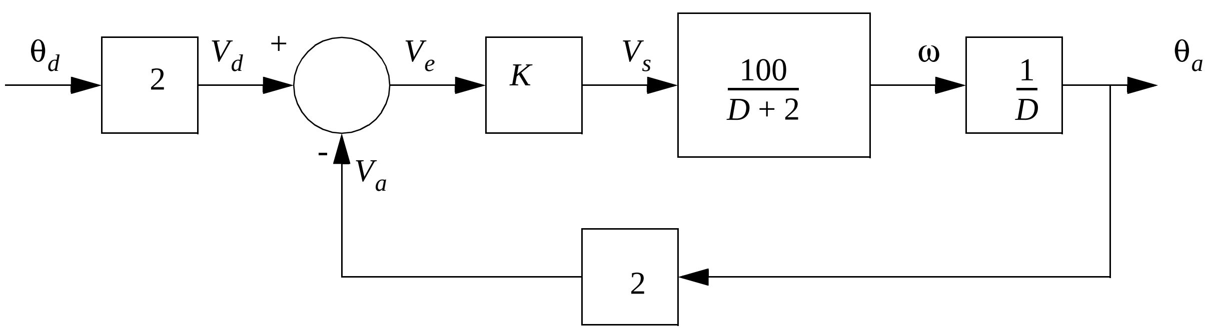
c)





d) $$\left|\frac{y}{F}\right| = \frac{1}{\sqrt{1 - \omega^2 + \omega^4}}$$ $$\theta = angle(1 - \omega^2, -\omega)$$

3. The block diagram below is for a motor position control system. The system has a proportional controller with a variable gain K.



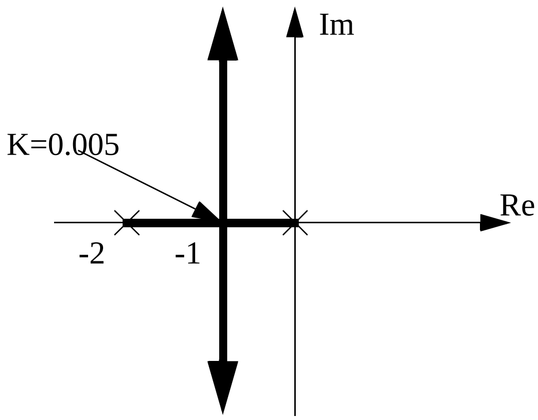
a) Simplify the block diagram to a single transfer function.

ans. $$\frac{200K}{D^2 + 2D + 200K}$$

b) Draw the Root-Locus diagram for the system (as K varies). Use either the approximate or exact techniques.

ans. $$roots = \frac{-2 \pm \sqrt{4 - 4(200K)}}{2} = -1 \pm \sqrt{1 - 200K}$$

K	roots
0	0,-2
0.001	-0.1,-1.9
0.005	-1,-1
0.1	etc.
1	
5	
10	



c) Select a K value that will result in an overall damping coefficient of 1. State if the Root-Locus diagram shows that the system is stable for the chosen K.

ans. $D^2 + 2D + 200K = D^2 + 2\zeta\omega_n D + \omega_n^2$ $\quad \therefore \omega_n = 1$ $\quad \therefore K = 0.005$

From the root locus graph this value is critically stable.

4. Given the system transfer function below.

$$\frac{\theta_o}{\theta_d} = \frac{20K}{D^2 + D + 20K}$$

a) Draw the root locus diagram and state what values of K are acceptable.
b) Select a gain value for K that has either a damping factor of 0.707 or a natural frequency of 3 rad/sec.
c) Given a gain of K=10 find the steady-state response to an input step of 1 rad.
d) Given a gain of K=0.01 find the response of the system to an input step of 0.1rad.

(ans. a) $D^2 + D + 20K = 0$

$$D = \frac{-1 \pm \sqrt{1 - 4(20K)}}{2}$$

For complex roots

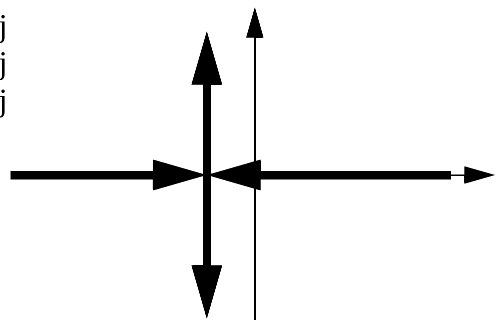
$$1 - 80K < 0 \qquad K > \frac{1}{80}$$

K	roots
-10	13.65, -14.65
-1	4.000, -5.000
0	0.000, -1.000
1/80	-0.500, -0.500
1	-0.5 +/- 4.444j
10	-0.5 +/- 14.13j
1000	-0.5 +/- 141.4j

For negative real roots (stable)

$$\frac{-1 \pm \sqrt{1 - 80K}}{2} < 0$$

$$\pm\sqrt{1 - 80K} < 1 \qquad K > 0$$



b) Matching the second order forms,

$$2\omega_n \xi = 1 \qquad \omega_n^2 = 20K$$

The gain can only be used for the natural frequency

$$K = \frac{20}{\omega_n^2} = \frac{20}{3^2} = 2.22$$

c) $$\frac{\theta_o}{\theta_d} = \frac{20(10)}{D^2 + D + 20(10)}$$

$$\theta_o'' + \theta_d' + \theta_d 200 = 200\theta_d$$

Homogeneous:

$$A^2 + A + 200 = 0$$

$$A = \frac{-1 \pm \sqrt{1 - 4(200)}}{2} \qquad A = -0.5 \pm 14.1j$$

$$\theta_o(t) = C_1 e^{-0.5t} \sin(14.1t + C_2)$$

Particular:

$$\theta = A$$

$$0 + 0 + A200 = 200(1rad) \qquad A = 1rad$$

$$\theta_o(t) = 1rad$$

Initial Conditions (assume at rest):

$$\theta_o(t) = C_1 e^{-0.5t} \sin(14.1t + C_2) + 1rad$$

$$\theta_o(0) = C_1(1)\sin(14.1(0) + C_2) + 1rad = 0$$

$$C_1 \sin(C_2) = -1rad \qquad (1)$$

$$\theta'_o(t) = -0.5C_1 e^{-0.5t} \sin(14.1t + C_2) - 14.1C_1 e^{-0.5t} \cos(14.1t + C_2)$$

$$0 = -0.5C_1 \sin(C_2) - 14.1C_1 \cos(C_2)$$

$$14.1\cos(C_2) = -0.5\sin(C_2)$$

$$\frac{14.1}{-0.5} = \tan(C_2) \qquad C_2 = -1.54$$

$$C_1 = \frac{-1rad}{\sin(C_2)} = \frac{-1rad}{\sin(-1.54)} = 1.000rad$$

$$\theta_o(t) = (e^{-0.5t} \sin(14.1t - 1.54) + 1)(rad)$$

d) $$\frac{\theta_o}{\theta_d} = \frac{20(0.01)}{D^2 + D + 20(0.01)}$$

$$\theta_o'' + \theta_d' + \theta_d 0.2 = 0.2\theta_d$$

Homogeneous:

$$A^2 + A + 0.2 = 0$$

$$A = \frac{-1 \pm \sqrt{1 - 4(0.2)}}{2} \qquad A = -0.7236068, -0.2763932$$

$$\theta_o(t) = C_1 e^{-0.724t} + C_2 e^{-0.276t}$$

Particular:

$$\theta = A$$

$$0 + 0 + A0.2 = 0.2(1rad) \qquad A = 1rad$$

$$\theta_o(t) = 1rad$$

Initial Conditions (assume at rest):

$$\theta_o(t) = C_1 e^{-0.724t} + C_2 e^{-0.276t} + 1rad$$

$$\theta_o(0) = C_1 e^{-0.724t} + C_2 e^{-0.276t} + 1rad = 0$$

$$C_1 + C_2 = -1rad \qquad (1)$$

$$\theta'_o(t) = -0.724(C_1 e^{-0.724t}) - 0.276(C_2 e^{-0.276t})$$

$$C_1 = -0.381C_2$$

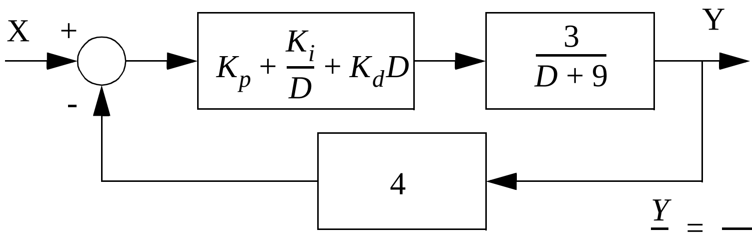
$$-0.381C_2 + C_2 = -1rad \qquad C_2 = -1.616rad$$

$$C_1 = -0.381(-1.616rad) = 0.616rad$$

$$\theta_o(t) = (0.616)e^{-0.724t} + (-1.616)e^{-0.276t} + 1rad$$

5. A feedback control system is shown below. The system incorporates a PID controller. The

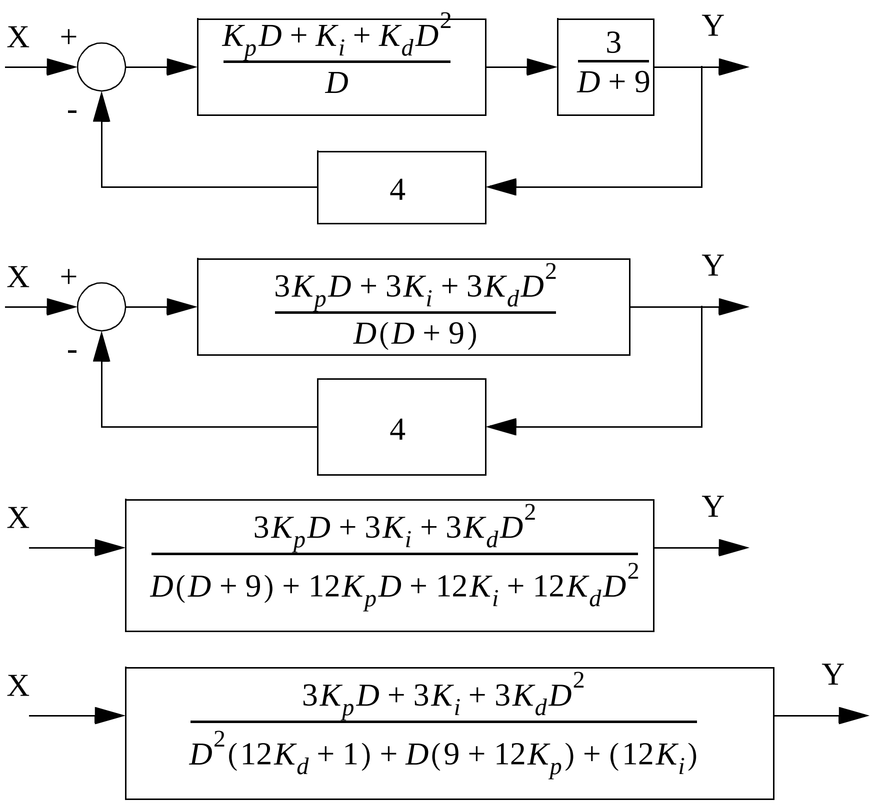
closed loop transfer function is given.



$$\frac{Y}{X} = \frac{D^2(3K_d) + D(3K_p) + (3K_i)}{D^2(12K_d + 1) + D(9 + 12K_p) + (12K_i)}$$

a) Verify the close loop controller function given.
b) Draw a root locus plot for the controller if Kp=1 and Ki=1. Identify any values of Kd that would leave the system unstable.
c) Draw a Bode plot for the feedback system if Kd=Kp=Ki=1.
d) Select controller values that will result in a natural frequency of 2 rad/sec and damping coefficient of 0.5. Verify that the controller will be stable.
e) For the parameters found in the last step can the initial values be found?
f) If the values of Kd=1 and Ki=Kp=0, find the response to a unit ramp input as a function of time.

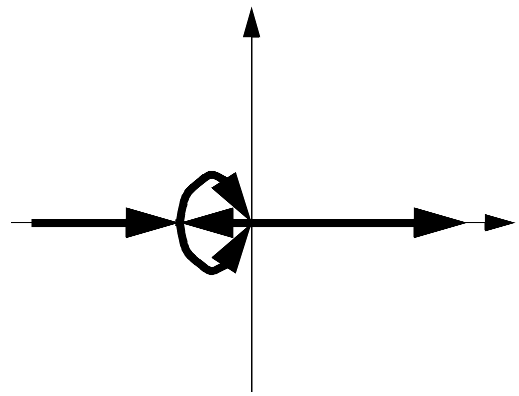
(ans.



b) $D^2(12K_d + 1) + D(9 + 12K_p) + (12K_i) = 0$

$$D = \frac{-9 - 12K_p \pm \sqrt{(9 + 12K_p)^2 - 4(12K_d + 1)12K_i}}{2(12K_d + 1)}$$

Kd	roots
-100	-0.092, 0.109
-10	-0.241, 0.418
-1	-0.46, 2.369
-0.1	-0.57, 105.6
0	-0.588, -20.41
1	-0.808 +/- 0.52j
10	-0.087 +/- 0.303j
100	-0.0087 +/- 0.1j



Stable for, $-9 - 12K_p \pm \sqrt{(9 + 12K_p)^2 - 4(12K_d + 1)12K_i} < 0$

$$\pm\sqrt{(9 + 12K_p)^2 - 4(12K_d + 1)12K_i} < 9 + 12K_p$$

$$(9 + 12K_p)^2 - 4(12K_d + 1)12K_i < (9 + 12K_p)^2$$

$$-4(12K_d + 1)12K_i < 0$$

$$K_d > \frac{-1}{12}$$

Becomes complex at,

$$0 > (9 + 12K_p)^2 - 4(12K_d + 1)12K_i$$

$$576K_dK_i > (9 + 12K_p)^2 - 48K_i$$

$$K_d > \frac{(9 + 12K_p)^2 - 48K_i}{576K_dK_i} \qquad K_d > 0.682$$

c) $K_p = 1 \qquad K_i = 1 \qquad K_d = 1$

$$\frac{Y}{X} = \frac{3K_pD + 3K_i + 3K_dD^2}{D^2(12K_d + 1) + D(9 + 12K_p) + (12K_i)}$$

$$\frac{Y}{X} = \frac{3D^2 + 3D + 3}{D^2 13 + D21 + 12} = \left(\frac{3}{13}\right)\left(\frac{D^2 + D + 1}{D^2 + D1.615 + 0.923}\right)$$

$$\text{final gain} = 20\log\left(\frac{3}{13}\right) = -12.7$$

$$\text{initial gain} = 20\log\left(\frac{3}{12}\right) = -12.0$$

for the numerator,

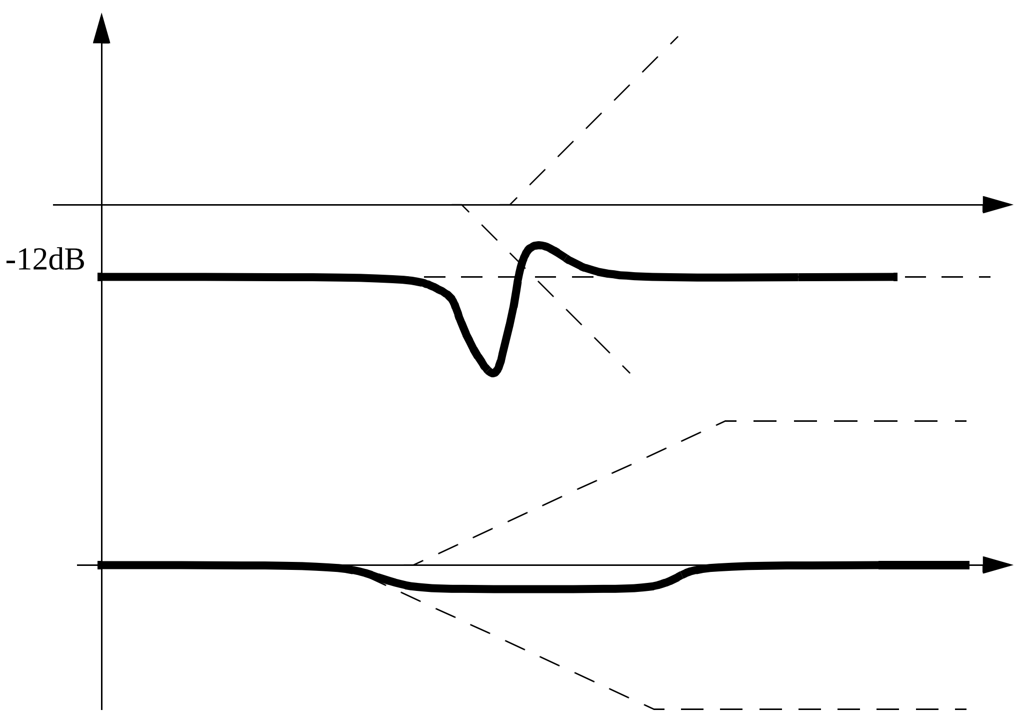
$$\omega_n = \sqrt{1} = 1 \qquad \xi = \frac{1}{2\omega_n} = 0.5$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = \sqrt{1-0.5^2} = 0.866$$

for the denominator,

$$\omega_n = \sqrt{0.923} = 0.961 \qquad \xi = \frac{1.615}{2\omega_n} = 0.840$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 0.961\sqrt{1-0.840^2} = 0.521$$



$$\frac{Y}{X} = \frac{3K_pD + 3K_i + 3K_dD^2}{D^2(12K_d + 1) + D(9 + 12K_p) + (12K_i)}$$

$$\omega_n = \sqrt{\frac{12K_i}{12K_d + 1}} = 2 \qquad 12K_i = 48K_d + 4$$

$$2\xi\omega_n = \frac{9 + 12K_p}{12K_d + 1} = 20.5(2) \qquad 24K_d = 7 + 12K_p$$

At this point there are two equations and two unknowns, one value must be selected to continue, therefore,

$$K_p = 10$$

$$24K_d = 7 + 12K_p = 7 + 12(10) = 127 \qquad K_d = 5.292$$

$$12K_i = 48K_d + 4 = 48(5.292) + 4 = 258.0 \qquad K_i = 21.5$$

Now to check for stability

$$D^2(12(5.292) + 1) + D(9 + 12(10)) + (12(21.5)) = 0$$

$$64.504D^2 + 129D + 258 = 0$$

$$D = \frac{-129 \pm \sqrt{129^2 - 4(64.5)258}}{2(64.5)} = -1 \pm 1.73j$$

e) Cannot be found without an assumed input and initial conditions

f)

$$\frac{Y}{X} = \frac{3(0)D + 3(0) + 3(1)D^2}{D^2(12(1) + 1) + D(9 + 12(0)) + (12(0))}$$

$$\frac{Y}{X} = \frac{3D^2}{13D^2 + 9D}$$

$$Y(13D^2 + 9D) = X(3D^2)$$

$$Y''13 + Y'9 = X''3 \qquad X = t \qquad X' = 1 \qquad X'' = 0$$

$$Y'' + Y'\frac{9}{13} = 0$$

It is a first order system,

$$Y(t) = C_1 e^{-\frac{9}{13}t} + C_2$$

$$Y(0) = 0 \qquad Y'(0) = 0$$ starts at rest/undeflected

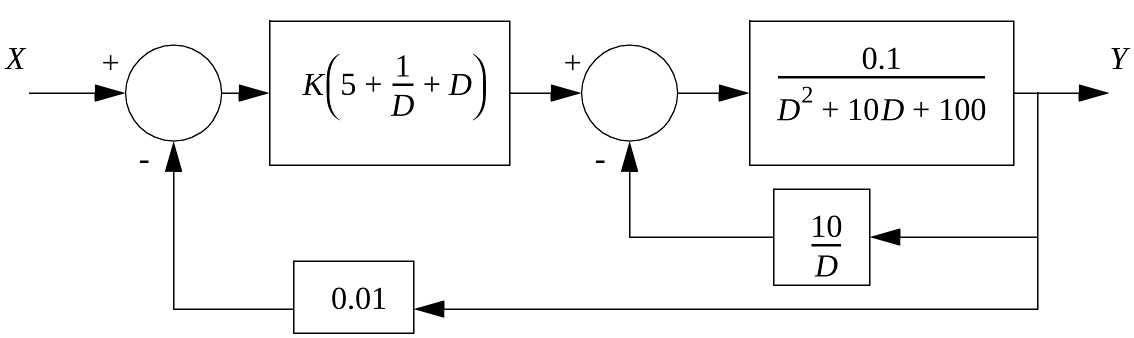
$$0 = C_1 1 + C_2 \qquad C_1 = -C_2$$

$$Y'(t) = -\frac{9}{13}C_1 e^{-\frac{9}{13}t}$$

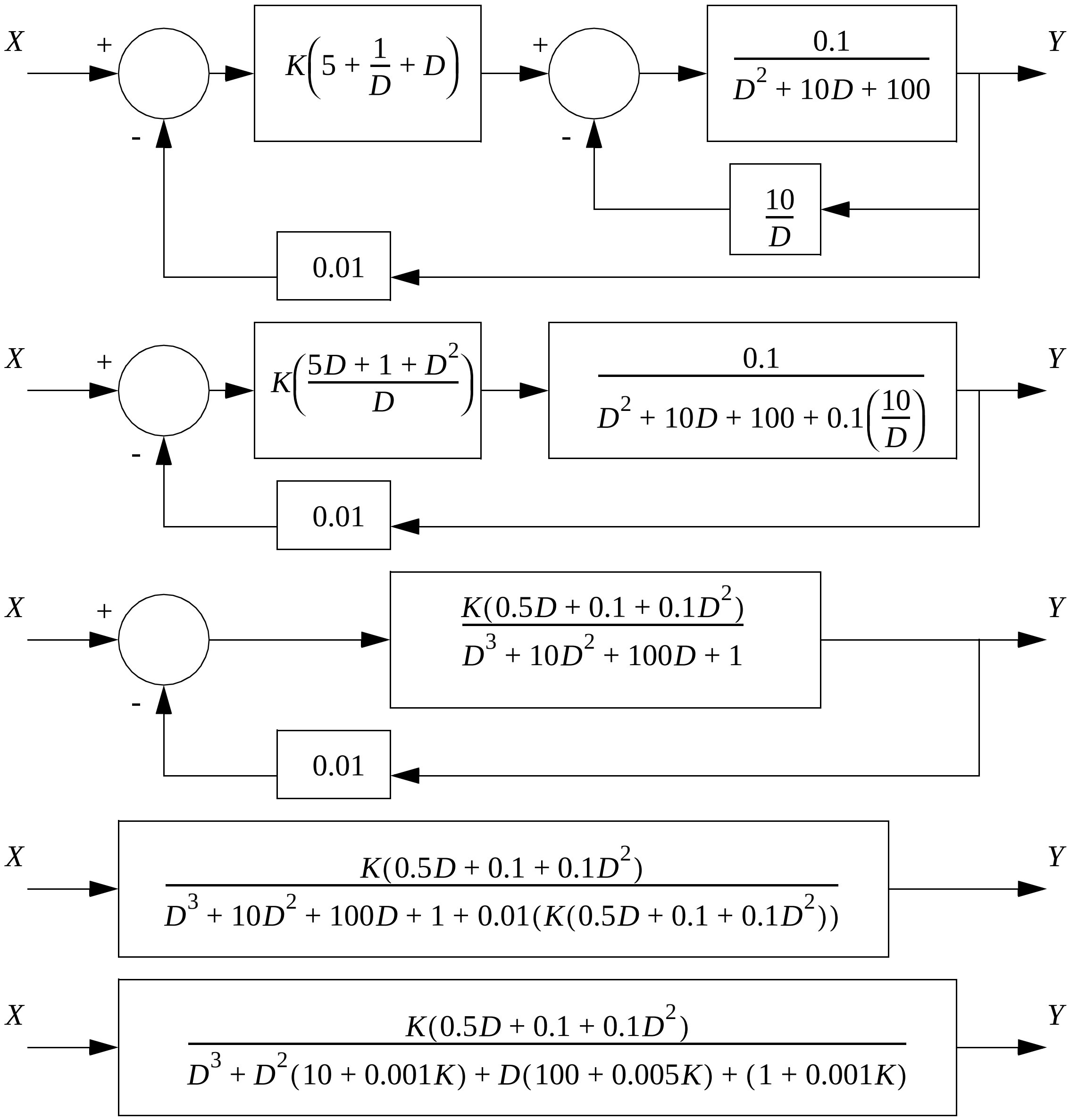
$$0 = -\frac{9}{13}C_1 1 \qquad C_1 = 0$$

$$C_2 = 0$$ no response

6. Draw a root locus plot for the control system below and determine acceptable values of K, including critical points.



(ans.


X
+
-
$K\left(5 + \frac{1}{D} + D\right)$
+
-
$\frac{0.1}{D^2 + 10D + 100}$
Y
$\frac{10}{D}$
0.01
X
+
-
$K\left(\frac{5D + 1 + D^2}{D}\right)$
$\frac{0.1}{D^2 + 10D + 100 + 0.1\left(\frac{10}{D}\right)}$
Y
0.01
X
+
-
$\frac{K(0.5D + 0.1 + 0.1D^2)}{D^3 + 10D^2 + 100D + 1}$
Y
0.01
X
$\frac{K(0.5D + 0.1 + 0.1D^2)}{D^3 + 10D^2 + 100D + 1 + 0.01(K(0.5D + 0.1 + 0.1D^2))}$
Y
X
$\frac{K(0.5D + 0.1 + 0.1D^2)}{D^3 + D^2(10 + 0.001K) + D(100 + 0.005K) + (1 + 0.001K)}$
Y

Given the homogeneous equation for the system,

$$D^3 + D^2(10 + 0.001K) + D(100 + 0.005K) + (1 + 0.001K) = 0$$

The roots can be found with a calculator, Mathcad, or equivalent.

K	roots	notes
-100,000	94.3, -3.992, -0.263	
-1000	0, -4.5+/-8.65j	roots become negative
-10	-0.0099, -4.99+/-8.66j	
0	-0.01, -4.995+/-8.657j	
10	-0.01, -5+/-8.66j	
1000	-0.019, -5.49+/-8.64j	
17165.12	-0.099, -13.52, -13.546	roots become real
100,000	-0.0174, -104.3, -5.572	

12. NONLINEAR SYSTEMS

Topics:

-

Objectives:

-

12.1 INTRODUCTION

• how they are different from linear

- no transfer functions

- new elements needed in block diagrams

12.2 SYSTEM DIAGRAMS

12.3 SUMMARY

-

12.4 PRACTICE PROBLEMS

1.

13. ANALOG INPUTS AND OUTPUTS

Topics:
- Analog inputs and outputs
- Sampling issues; aliasing, quantization error, resolution

Objectives:
- To understand the basics of conversion to and from analog values.

13.1 INTRODUCTION

An analog value is continuous, not discrete, as shown in Figure 284. In the previous chapters, techniques were discussed for designing logical control systems that had inputs and outputs that could only be on or off. These systems are less common than the logical control systems, but they are very important. In this chapter we will examine analog inputs and outputs so that we may design continuous control systems in a later chapter.

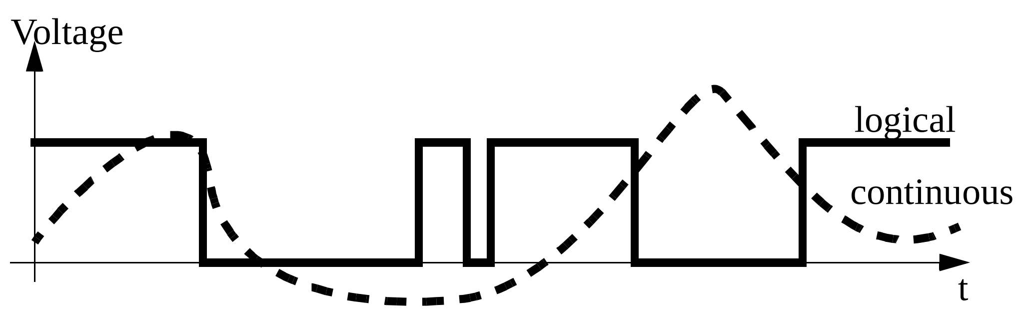


Figure 284 Logical and Continuous Values

Typical analog inputs and outputs for PLCs are listed below. Actuators and sensors that can be used with analog inputs and outputs will be discussed in later chapters.

Inputs:
- oven temperature
- fluid pressure
- fluid flow rate

Outputs:
- fluid valve position
- motor position
- motor velocity

This chapter will focus on the general principles behind digital-to-analog (D/A) and analog-to-digital (A/D) conversion.

13.2 ANALOG INPUTS

To input an analog voltage (into a computer) the continuous voltage value must be *sampled* and then converted to a numerical value by an A/D converter. Figure 285 shows a continuous voltage changing over time. There are three samples shown on the figure. The process of sampling the data is not instantaneous, so each sample has a start and stop time. The time required to acquire the sample is called the *sampling time*. A/D converters can only acquire a limited number of samples per second. The time between samples is called the sampling period *T*, and the inverse of the sampling period is the sampling frequency (also called sampling rate). The sampling time is often much smaller than the sampling period. The sampling frequency is specified when buying hardware, but for a PLC a maximum sampling rate might be 20Hz.

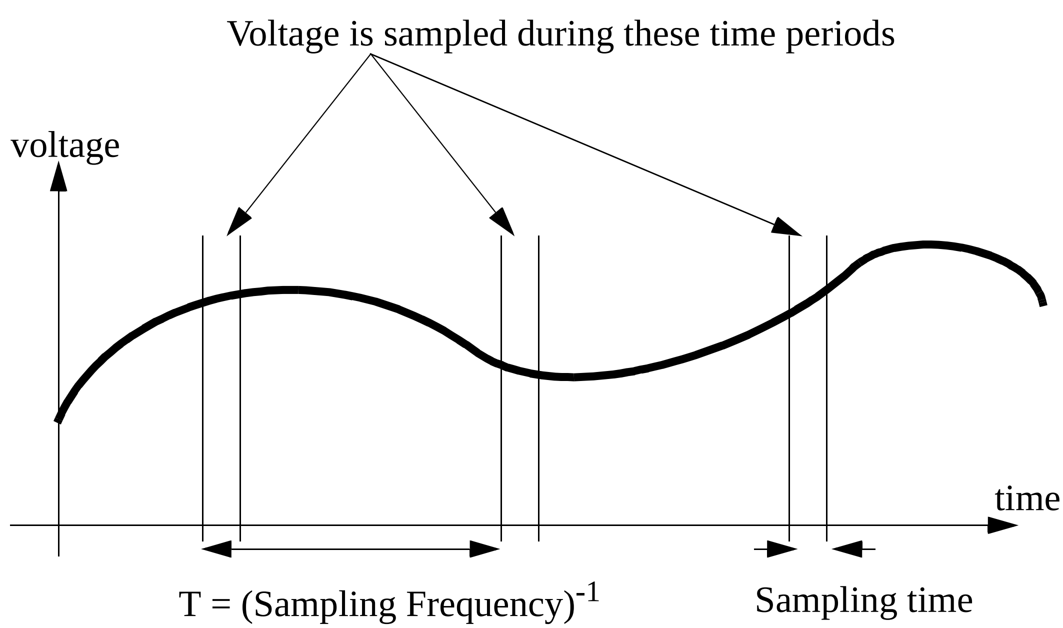
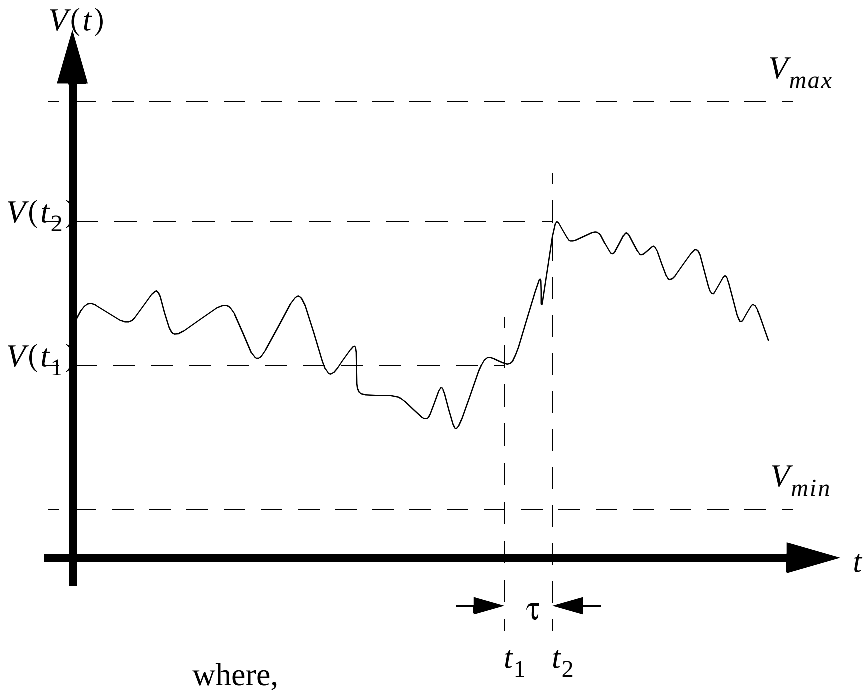


Figure 285 Sampling an Analog Voltage

A more realistic drawing of sampled data is shown in Figure 286. This data is noisier, and even between the start and end of the data sample there is a significant change in the voltage value. The data value sampled will be somewhere between the voltage at the start and end of the sample. The maximum (Vmax) and minimum (Vmin) voltages are a function of the control hardware. These are often specified when purchasing hardware, but reasonable ranges are;

0V to 5V
0V to 10V
-5V to 5V
-10V to 10V

The number of bits of the A/D converter is the number of bits in the result word. If the A/D converter is *8 bit* then the result can read up to 256 different voltage levels. Most A/D converters have 12 bits, 16 bit converters are used for precision measurements.



where,

$V(t)$ = the actual voltage over time

τ = sample interval for A/D converter

t = time

t_1, t_2 = time at start,end of sample

$V(t_1), V(t_2)$ = voltage at start, end of sample

V_{min}, V_{max} = input voltage range of A/D converter

N = number of bits in the A/D converter

Figure 286 Parameters for an A/D Conversion

The parameters defined in Figure 286 can be used to calculate values for A/D converters. These equations are summarized in Figure 287. Equation 1 relates the number of bits of an A/D converter to the resolution. Equation 2 gives the error that can be expected with an A/D converter given the range between the minimum and maximum voltages, and the resolution (this is commonly called the quantization error). Equation 3 relates the voltage range and resolution to the voltage input to estimate the integer that the A/D converter will record. Finally, equation 4 allows a conversion between the integer value from the A/D converter, and a voltage in the computer.

$$R = 2^N \qquad (1)$$

$$V_{ERROR} = \left(\frac{V_{max} - V_{min}}{2R}\right) \qquad (2)$$

$$V_I = INT\left[\left(\frac{V_{in} - V_{min}}{V_{max} - V_{min}}\right)(R-1)\right] \qquad (3)$$

$$V_C = \left(\frac{V_I}{R-1}\right)(V_{max} - V_{min}) + V_{min} \qquad (4)$$

where,

R = resolution of A/D converter

V_I = the integer value representing the input voltage

V_C = the voltage calculated from the integer value

V_{ERROR} = the maximum quantization error

Figure 287 A/D Converter Equations

Consider a simple example, a 10 bit A/D converter can read voltages between -10V and 10V. This gives a resolution of 1024, where 0 is -10V and 1023 is +10V. Because there are only 1024 steps there is a maximum error of ±9.8mV. If a voltage of 4.564V is input into the PLC, the A/D converter converts the voltage to an integer value of 746. When we convert this back to a voltage the result is 4.570V. The resulting quantization error is 4.570V-4.564V=+0.006V. This error can be reduced by selecting an A/D converter with more bits. Each bit halves the quantization error.

Given,

$$N = 10$$
$$V_{max} = 10V$$
$$V_{min} = -10V$$
$$V_{in} = 4.564V$$

Calculate,

$$R = 2^N = 1024$$

$$V_{ERROR} = \left(\frac{V_{max} - V_{min}}{2R}\right) = 0.0098V$$

$$V_I = INT\left[\left(\frac{V_{in} - V_{min}}{V_{max} - V_{min}}\right)R\right] = 746$$

$$V_C = \left(\frac{V_I}{R}\right)(V_{max} - V_{min}) + V_{min} = 4.570V$$

Figure 288 Sample Calculation of A/D Values

If the voltage being sampled is changing too fast we may get false readings, as shown in Figure 289. In the upper graph the waveform completes seven cycles, and 9 samples are taken. The bottom graph plots out the values read. The sampling frequency was too low, so the signal read appears to be different that it actually is, this is called aliasing.

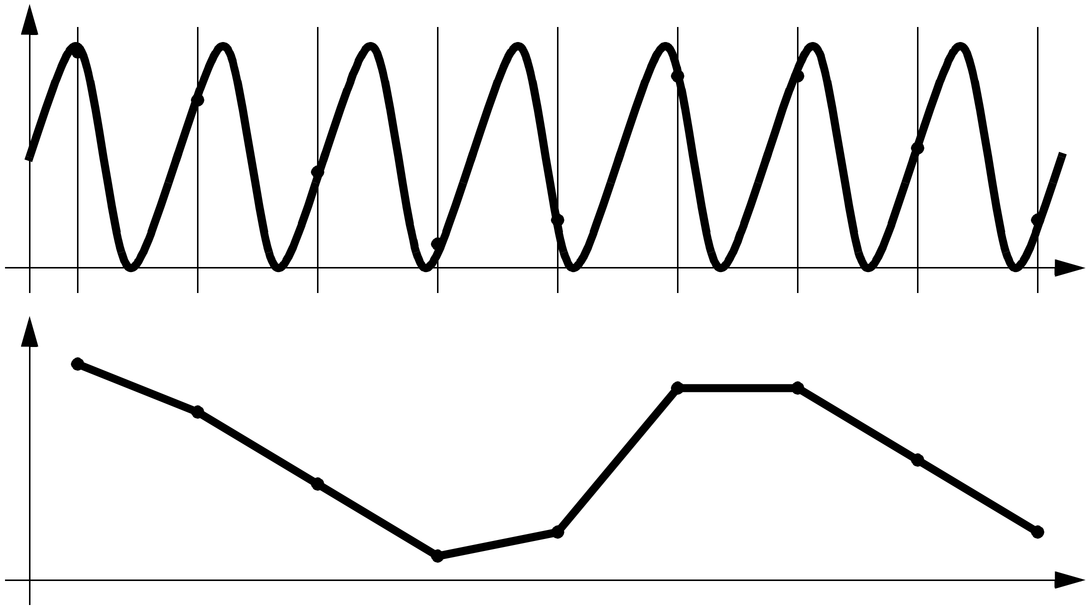


Figure 289 Low Sampling Frequencies Cause Aliasing

The Nyquist criterion specifies that sampling frequencies should be at least twice the frequency of the signal being measured, otherwise aliasing will occur. The example in Figure 289 violated this principle, so the signal was aliased. If this happens in real applications the process will appear to operate erratically. In practice the sample frequency should be 4 or more times faster than the system frequency.

$$f_{AD} > 2f_{signal}$$

where,

f_{AD} = sampling frequency
f_{signal} = maximum frequency of the input

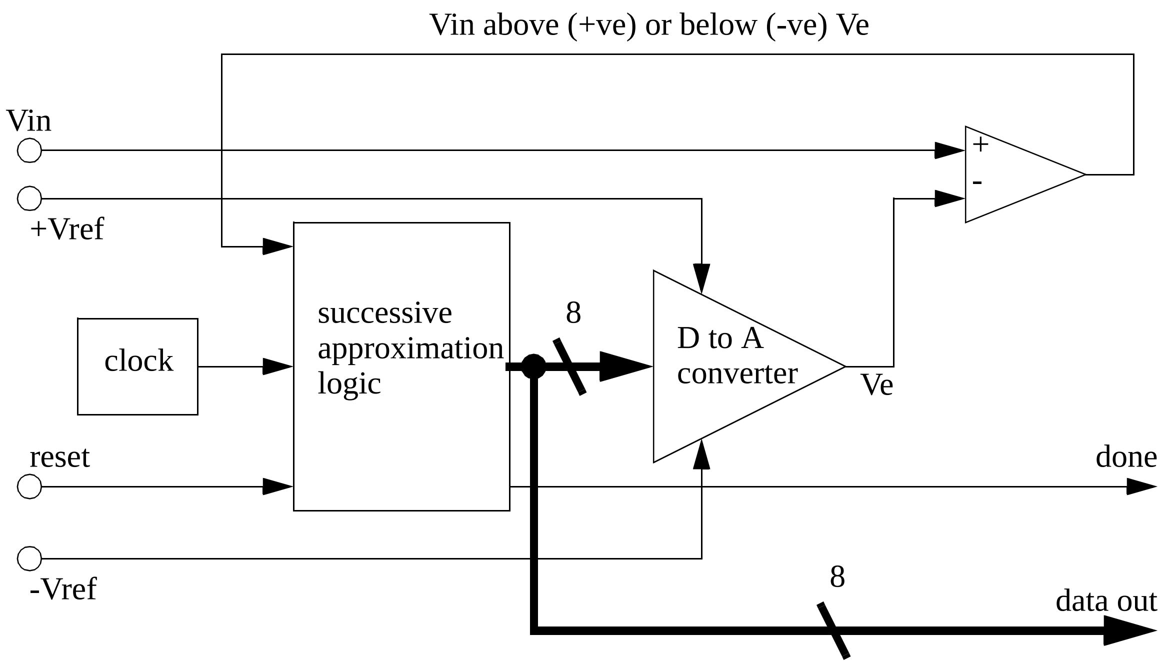
There are other practical details that should be considered when designing applications with analog inputs;

- Noise - Since the sampling window for a signal is short, noise will have added effect on the signal read. For example, a momentary voltage spike might result in a higher than normal reading. Shielded data cables are commonly used to reduce the noise levels.
- Delay - When the sample is requested, a short period of time passes before the final sample value is obtained.
- Multiplexing - Most analog input cards allow multiple inputs. These may share the A/D converter using a technique called multiplexing. If there are 4 channels

using an A/D converter with a maximum sampling rate of 100Hz, the maximum sampling rate per channel is 25Hz.

• Signal Conditioners - Signal conditioners are used to amplify, or filter signals coming from transducers, before they are read by the A/D converter.
• Resistance - A/D converters normally have high input impedance (resistance), so they affect circuits they are measuring.
• Single Ended Inputs - Voltage inputs to a PLC can use a single common for multiple inputs, these types of inputs are called *single* ended inputs. These tend to be more prone to noise.
• Double Ended Inputs - Each double ended input has its own common. This reduces problems with electrical noise, but also tends to reduce the number of inputs by half.

ASIDE: This device is an 8 bit A/D converter. The main concept behind this is the successive approximation logic. Once the reset is toggled the converter will start by setting the most significant bit of the 8 bit number. This will be converted to a voltage *Ve* that is a function of the *+/-Vref* values. The value of *Ve* is compared to *Vin* and a simple logic check determines which is larger. If the value of *Ve* is larger the bit is turned off. The logic then repeats similar steps from the most to least significant bits. Once the last bit has been set on/off and checked the conversion will be complete, and a done bit can be set to indicate a valid conversion value.



Quite often an A/D converter will multiplex between various inputs. As it switches the voltage will be sampled by a *sample and hold circuit*. This will then be converted to a digital value. The sample and hold circuits can be used before the multiplexer to collect data values at the same instant in time.

Figure 290 A Successive Approximation A/D Converter

13.3 ANALOG OUTPUTS

Analog outputs are much simpler than analog inputs. To set an analog output an integer is converted to a voltage. This process is very fast, and does not experience the timing problems with analog inputs. But, analog outputs are subject to quantization errors. Figure 291 gives a summary of the important relationships. These relationships are almost identical to those of the A/D converter.

$$R = 2^N \tag{18.5}$$

$$V_{ERROR} = \left(\frac{V_{max} - V_{min}}{2R}\right) \tag{18.6}$$

$$V_I = INT\left[\left(\frac{V_{desired} - V_{min}}{V_{max} - V_{min}}\right)R\right] \tag{18.7}$$

$$V_{output} = \left(\frac{V_I}{R}\right)(V_{max} - V_{min}) + V_{min} \tag{18.8}$$

where,

R = resolution of A/D converter
V_{ERROR} = the maximum quantization error
V_I = the integer value representing the desired voltage
V_{output} = the voltage output using the integer value
$V_{desired}$ = the desired output voltage

Figure 291 Analog Output Relationships

Assume we are using an 8 bit D/A converter that outputs values between 0V and 10V. We have a resolution of 256, where 0 results in an output of 0V and 255 results in 10V. The quantization error will be 20mV. If we want to output a voltage of 6.234V, we would specify an output integer of 160, this would result in an output voltage of 6.250V. The quantization error would be 6.250V-6.234V=0.016V.

Given,

$$N = 8$$
$$V_{max} = 10V$$
$$V_{min} = 0V$$
$$V_{desired} = 6.234V$$

Calculate,

$$R = 2^N = 256$$

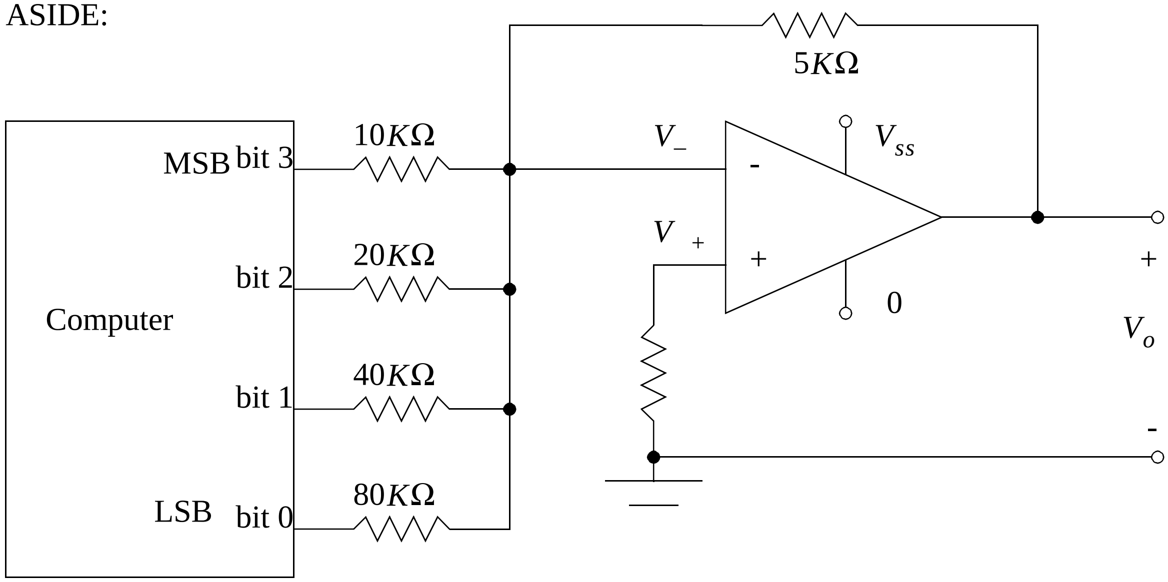
$$V_{ERROR} = \left(\frac{V_{max} - V_{min}}{2R}\right) = 0.020V$$

$$V_I = INT\left[\left(\frac{V_{in} - V_{min}}{V_{max} - V_{min}}\right)R\right] = 160$$

$$V_C = \left(\frac{V_I}{R}\right)(V_{max} - V_{min}) + V_{min} = 6.250V$$

The current output from a D/A converter is normally limited to a small value, typically less than 20mA. This is enough for instrumentation, but for high current loads, such as motors, a current amplifier is needed. This type of interface will be discussed later. If the current limit is exceeded for 5V output, the voltage will decrease (so don't exceed the rated voltage). If the current limit is exceeded for long periods of time the D/A output may be damaged.

ASIDE:



First we write the obvious,

$$V_+ = 0 = V_-$$

Next, sum the currents into the inverting input as a function of the output voltage and the input voltages from the computer,

$$\frac{V_{b_3}}{10K\Omega} + \frac{V_{b_2}}{20K\Omega} + \frac{V_{b_1}}{40K\Omega} + \frac{V_{b_0}}{80K\Omega} = \frac{V_o}{5K\Omega}$$

$$\therefore V_o = 0.5V_{b_3} + 0.25V_{b_2} + 0.125V_{b_1} + 0.0625V_{b_0}$$

Consider an example where the binary output is 1110, with 5V for on,

$$\therefore V_o = 0.5(5V) + 0.25(5V) + 0.125(5V) + 0.625(0V) = 4.375V$$

Figure 292 A Digital-To-Analog Converter

13.4 SHIELDING

When a changing magnetic field cuts across a conductor, it will induce a current flow. The resistance in the circuits will convert this to a voltage. These unwanted voltages result in erroneous readings from sensors, and signal to outputs. Shielding will reduce the effects of the interference. When shielding and grounding are done properly, the effects of electrical noise will be negligible. Shielding is normally used for; all logical signals in noisy environments, high speed counters or high speed circuitry, and all analog signals.

There are two major approaches to reducing noise; shielding and twisted pairs. Shielding involves encasing conductors and electrical equipment with metal. As a result electrical equipment is normally housed in metal cases. Wires are normally put in cables with a metal sheath surrounding both wires. The metal sheath may be a thin film, or a woven metal mesh. Shielded wires are connected at one end to "drain" the unwanted signals into the cases of the instruments. Figure 293 shows a thermocouple connected with a thermocouple. The cross section of the wire contains two insulated conductors. Both of the wires are covered with a metal foil, and final covering of insulation finishes the cable. The wires are connected to the thermocouple as expected, but the shield is only connected on the amplifier end to the case. The case is then connected to the shielding ground, shown here as three diagonal lines.

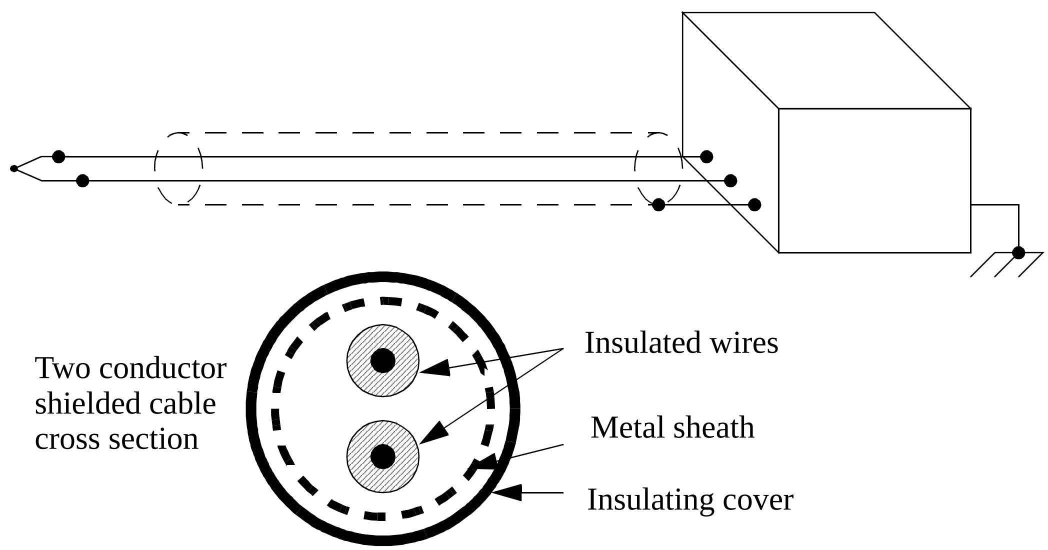


Figure 293 Shielding for a Thermocouple

A twisted pair is shown in Figure 294. The two wires are twisted at regular intervals, effectively forming small loops. In this case the small loops reverse every twist, so any induced currents are cancel out for every two twists.

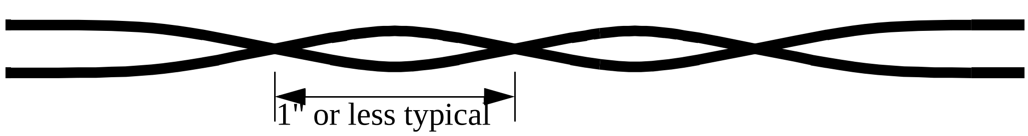


Figure 294 A Twisted Pair

When designing shielding, the following design points will reduce the effects of electromagnetic interference.

• Avoid “noisy” equipment when possible.
• Choose a metal cabinet that will shield the control electronics.
• Use shielded cables and twisted pair wires.
• Separate high current, and AC/DC wires from each other when possible.
• Use current oriented methods such as sourcing and sinking for logical I/O.
• Use high frequency filters to eliminate high frequency noise.
• Use power line filters to eliminate noise from the power supply.

13.5 SUMMARY

• A/D conversion will convert a continuous value to an integer value.
• D/A conversion is easier and faster and will convert a digital value to an analog value.
• Resolution limits the accuracy of A/D and D/A converters.
• Sampling too slowly will alias the real signal.
• Analog inputs are sensitive to noise.
• Analog shielding should be used to improve the quality of electrical signals.

13.6 PRACTICE PROBLEMS

1. Analog inputs require:
   a) A Digital to Analog conversion at the PLC input interface module
   b) Analog to Digital conversion at the PLC input interface module
   c) No conversion is required
   d) None of the above

(ans. b)

2. You need to read an analog voltage that has a range of -10V to 10V to a precision of +/-0.05V. What resolution of A/D converter is needed?

(ans.

$$R = \frac{10V - (-10V)}{0.1V} = 200$$

7 bits = 128
8 bits = 256

The minimum number of bits is 8.

3. We are given a 12 bit analog input with a range of -10V to 10V. If we put in 2.735V, what will

the integer value be after the A/D conversion? What is the error? What voltage can we calculate?

(ans.

$$N = 12 \quad R = 4096 \qquad V_{min} = -10V \qquad V_{max} = 10V \qquad V_{in} = 2.735V$$

$$V_I = INT\left[\left(\frac{V_{in} - V_{min}}{V_{max} - V_{min}}\right)R\right] = 2608$$

$$V_C = \left(\frac{V_I}{R}\right)(V_{max} - V_{min}) + V_{min} = 2.734V$$

14. CONTINUOUS SENSORS

Topics:
- Continuous sensor issues; accuracy, resolution, etc.
- Angular measurement; potentiometers, encoders and tachometers
- Linear measurement; potentiometers, LVDTs, Moire fringes and accelerometers
- Force measurement; strain gages and piezoelectric
- Liquid and fluid measurement; pressure and flow
- Temperature measurement; RTDs, thermocouples and thermistors
- Other sensors
- Continuous signal inputs and wiring
- Glossary

Objectives:
- To understand the common continuous sensor types.
- To understand interfacing issues.

14.1 INTRODUCTION

Continuous sensors convert physical phenomena to measurable signals, typically voltages or currents. Consider a simple temperature measuring device, there will be an increase in output voltage proportional to a temperature rise. A computer could measure the voltage, and convert it to a temperature. The basic physical phenomena typically measured with sensors include;

- angular or linear position
- acceleration
- temperature
- pressure or flow rates
- stress, strain or force
- light intensity
- sound

Most of these sensors are based on subtle electrical properties of materials and devices. As a result the signals often require *signal conditioners*. These are often amplifiers that boost currents and voltages to larger voltages.

Sensors are also called transducers. This is because they convert an input phenomena to an output in a different form. This transformation relies upon a manufactured device with limitations and imperfection. As a result sensor limitations are often characterized with;

Accuracy - This is the maximum difference between the indicated and actual reading. For example, if a sensor reads a force of 100N with a ±1% accuracy, then the force could be anywhere from 99N to 101N.
Resolution - Used for systems that *step* through readings. This is the smallest increment that the sensor can detect, this may also be incorporated into the accuracy value. For example if a sensor measures up to 10 inches of linear displacements, and it outputs a number between 0 and 100, then the resolution of the device is 0.1 inches.
Repeatability - When a single sensor condition is made and repeated, there will be a small variation for that particular reading. If we take a statistical range for repeated readings (e.g., ±3 standard deviations) this will be the repeatability. For example, if a flow rate sensor has a repeatability of 0.5cfm, readings for an actual flow of 100cfm should rarely be outside 99.5cfm to 100.5cfm.
Linearity - In a linear sensor the input phenomenon has a linear relationship with the output signal. In most sensors this is a desirable feature. When the relationship is not linear, the conversion from the sensor output (e.g., voltage) to a calculated quantity (e.g., force) becomes more complex.
Precision - This considers accuracy, resolution and repeatability or one device relative to another.
Range - Natural limits for the sensor. For example, a sensor for reading angular rotation may only rotate 200 degrees.
Dynamic Response - The frequency range for regular operation of the sensor. Typically sensors will have an upper operation frequency, occasionally there will be lower frequency limits. For example, our ears hear best between 10Hz and 16KHz.
Environmental - Sensors all have some limitations over factors such as temperature, humidity, dirt/oil, corrosives and pressures. For example many sensors will work in relative humidities (RH) from 10% to 80%.
Calibration - When manufactured or installed, many sensors will need some calibration to determine or set the relationship between the input phenomena, and output. For example, a temperature reading sensor may need to be *zeroed* or adjusted so that the measured temperature matches the actual temperature. This may require special equipment, and need to be performed frequently.
Cost - Generally more precision costs more. Some sensors are very inexpensive, but the signal conditioning equipment costs are significant.

14.2 INDUSTRIAL SENSORS

This section describes sensors that will be of use for industrial measurements. The sections have been divided by the phenomena to be measured. Where possible details are provided.

14.2.1 Angular Displacement

14.2.1.1 - Potentiometers

Potentiometers measure the angular position of a shaft using a variable resistor. A potentiometer is shown in Figure 295. The potentiometer is resistor, normally made with a thin film of resistive material. A wiper can be moved along the surface of the resistive film. As the wiper moves toward one end there will be a change in resistance proportional to the distance moved. If a voltage is applied across the resistor, the voltage at the wiper interpolate the voltages at the ends of the resistor.

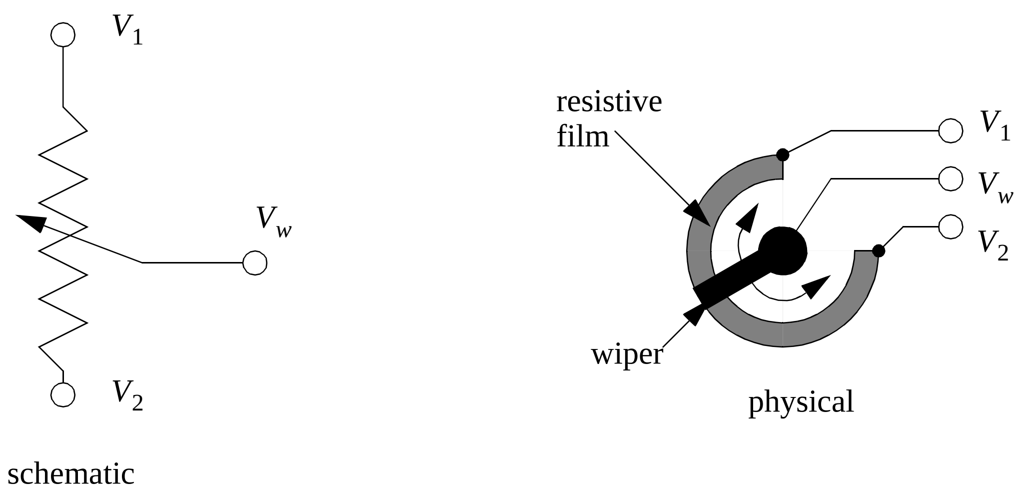
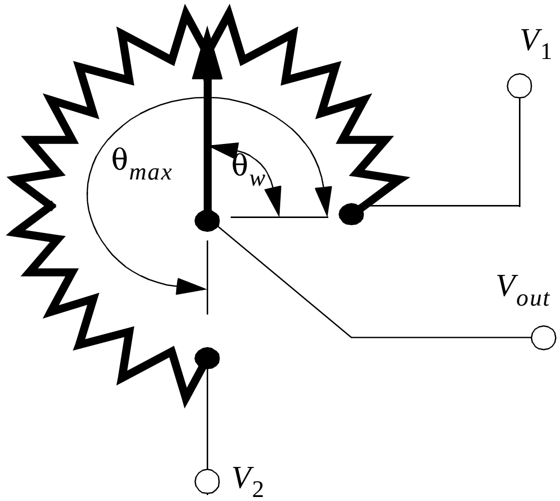


Figure 295 A Potentiometer

The potentiometer in Figure 296 is being used as a voltage divider. As the wiper rotates the output voltage will be proportional to the angle of rotation.



$$V_{out} = (V_2 - V_1)\left(\frac{\theta_w}{\theta_{max}}\right) + V_1$$

Figure 296 A Potentiometer as a Voltage Divider

Potentiometers are popular because they are inexpensive, and don't require special signal conditioners. But, they have limited accuracy, normally in the range of 1% and they are subject to mechanical wear.

Potentiometers measure absolute position, and they are calibrated by rotating them in their mounting brackets, and then tightening them in place. The range of rotation is normally limited to less than 360 degrees or multiples of 360 degrees. Some potentiometers can rotate without limits, and the wiper will jump from one end of the resistor to the other.

Faults in potentiometers can be detected by designing the potentiometer to never reach the ends of the range of motion. If an output voltage from the potentiometer ever reaches either end of the range, then a problem has occurred, and the machine can be shut down. Two examples of problems that might cause this are wires that fall off, or the potentiometer rotates in its mounting.

14.2.2 Encoders

Encoders use rotating disks with optical windows, as shown in Figure 297. The encoder contains an optical disk with fine windows etched into it. Light from emitters passes through the openings in the disk to detectors. As the encoder shaft is rotated, the light beams are broken. The encoder shown here is a quadrature encode, and it will be discussed later.

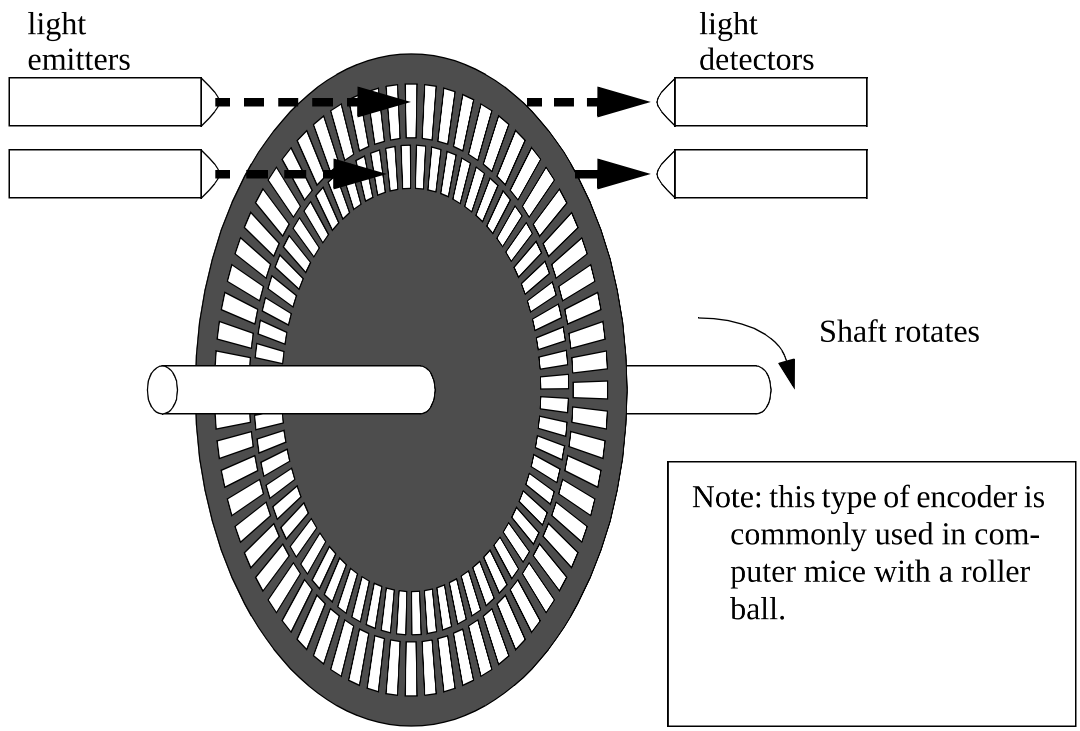


Figure 297 An Encoder Disk

There are two fundamental types of encoders; absolute and incremental. An absolute encoder will measure the position of the shaft for a single rotation. The same shaft angle will always produce the same reading. The output is normally a binary or grey code number. An incremental (or relative) encoder will output two pulses that can be used to determine displacement. Logic circuits or software is used to determine the direction of rotation, and count pulses to determine the displacement. The velocity can be determined by measuring the time between pulses.

Encoder disks are shown in Figure 298. The absolute encoder has two rings, the outer ring is the most significant digit of the encoder, the inner ring is the least significant digit. The relative encoder has two rings, with one ring rotated a few degrees ahead of the other, but otherwise the same. Both rings detect position to a quarter of the disk. To add accuracy to the absolute encoder more rings must be added to the disk, and more emitters and detectors. To add accuracy to the relative encoder we only need to add more windows to the existing two rings. Typical encoders will have from 2 to thousands of windows per ring.

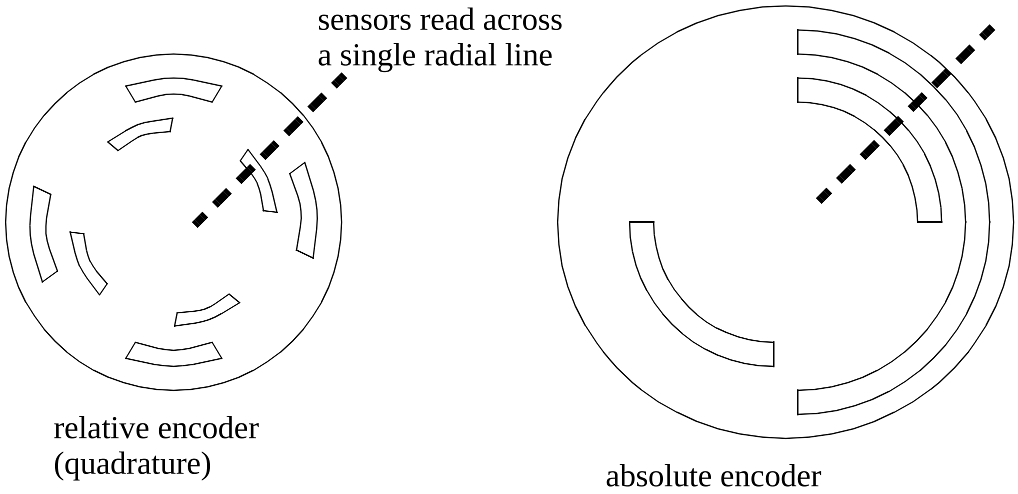
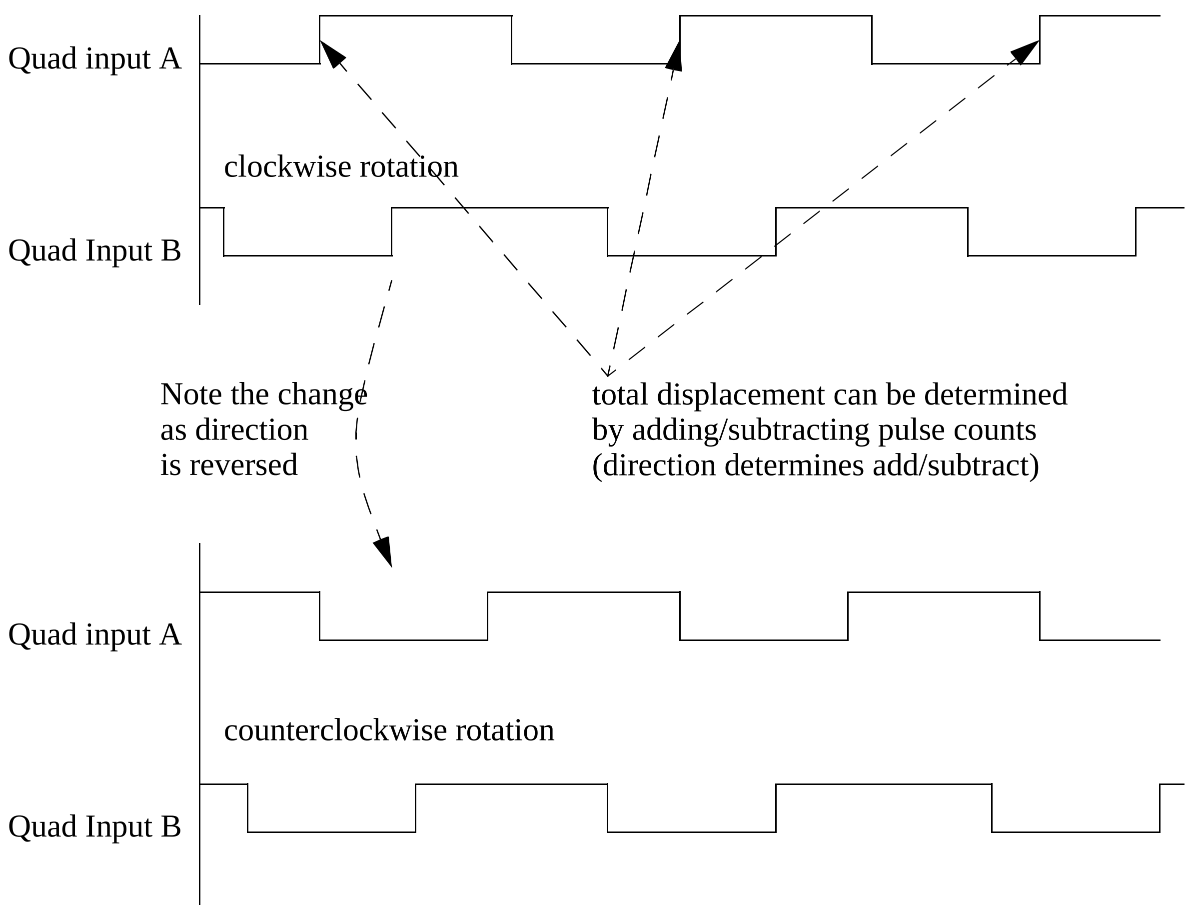


Figure 298 Encoder Disks

When using absolute encoders, the position during a single rotation is measured directly. If the encoder rotates multiple times then the total number of rotations must be counted separately.

When using a relative encoder, the distance of rotation is determined by counting the pulses from one of the rings. If the encoder only rotates in one direction then a simple count of pulses from one ring will determine the total distance. If the encoder can rotate both directions a second ring must be used to determine when to subtract pulses. The quadrature scheme, using two rings, is shown in Figure 299. The signals are set up so that one is out of phase with the other. Notice that for different directions of rotation, input *B* either leads or lags *A*.



Note: To determine direction we can do a simple check. If both are off or on, the first to change state determines direction. Consider a point in the graphs above where both A and B are off. If A is the first input to turn on the encoder is rotating clockwise. If B is the first to turn on the rotation is counterclockwise.

Aside: A circuit (or program) can be built for this circuit using an up/down counter. If the positive edge of input *A* is used to trigger the clock, and input *B* is used to drive the up/down count, the counter will keep track of the encoder position.

Figure 299 Quadrature Encoders

Interfaces for encoders are commonly available for PLCs and as purchased units. Newer PLCs will also allow two normal inputs to be used to decode encoder inputs.

Normally absolute and relative encoders require a calibration phase when a controller is turned on. This normally involves moving an axis until it reaches a logical sensor that marks the end of the range. The end of range is then used as the zero position. Machines using encoders, and other relative sensors, are noticeable in that they normally move to some extreme position before use.

14.2.2.1 - Tachometers

Tachometers measure the velocity of a rotating shaft. A common technique is to mount a magnet to a rotating shaft. When the magnetic moves past a stationary pick-up coil, current is induced. For each rotation of the shaft there is a pulse in the coil, as shown in Figure 300. When the time between the pulses is measured the period for one rotation can be found, and the frequency calculated. This technique often requires some signal conditioning circuitry.

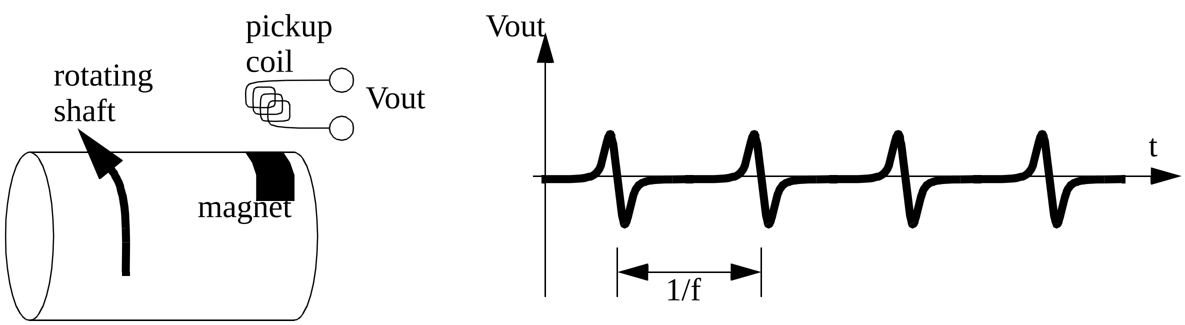


Figure 300 A Magnetic Tachometer

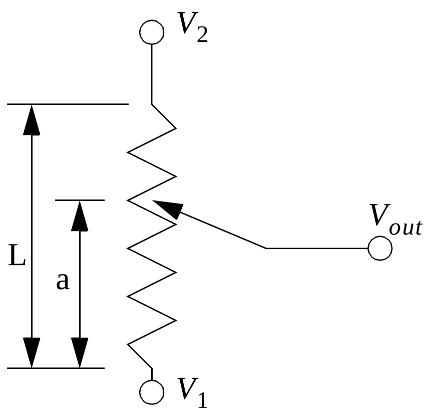
Another common technique uses a simple permanent magnet DC generator (note: you can also use a small DC motor). The generator is hooked to the rotating shaft. The rotation of a shaft will induce a voltage proportional to the angular velocity. This technique will introduce some drag into the system, and is used where efficiency is not an issue.

Both of these techniques are common, and inexpensive.

14.2.3 Linear Position

14.2.3.1 - Potentiometers

Rotational potentiometers were discussed before, but potentiometers are also available in linear/sliding form. These are capable of measuring linear displacement over long distances. Figure 301 shows the output voltage when using the potentiometer as a voltage divider.



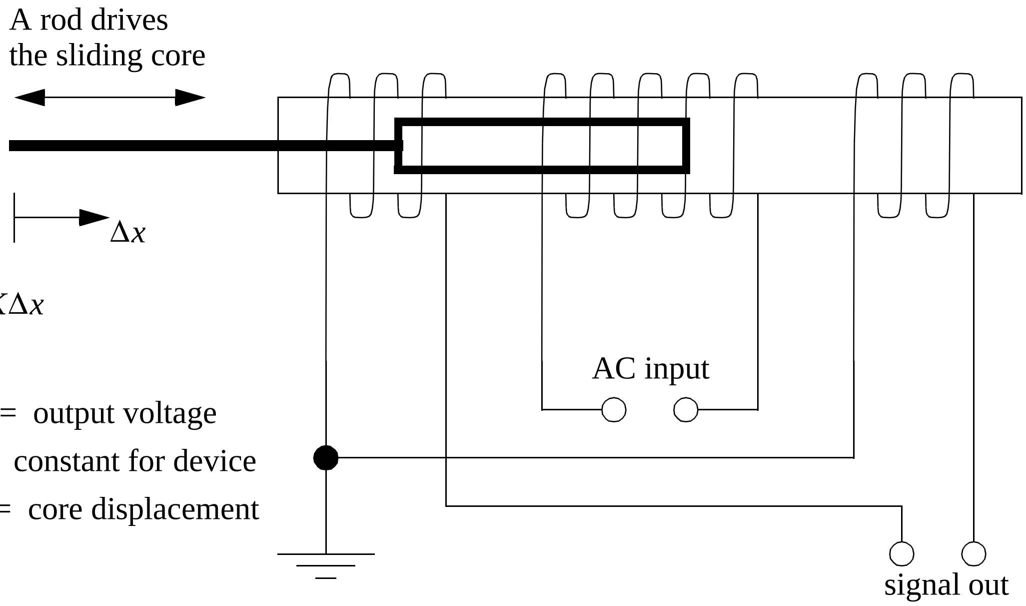
$$V_{out} = V_1 + (V_2 - V_1)\left(\frac{a}{L}\right)$$

Figure 301 Linear Potentiometer

Linear/sliding potentiometers have the same general advantages and disadvantages of rotating potentiometers.

14.2.3.2 - Linear Variable Differential Transformers (LVDT)

Linear Variable Differential Transformers (LVDTs) measure linear displacements over a limited range. The basic device is shown in Figure 302. It consists of outer coils with an inner moving magnetic core. High frequency alternating current (AC) is applied to the center coil. This generates a magnetic field that induces a current in the two outside coils. The core will pull the magnetic field towards it, so in the figure more current will be induced in the left hand coil. The outside coils are wound in opposite directions so that when the core is in the center the induced currents cancel, and the signal out is zero (0Vac). The magnitude of the *signal out* voltage on either line indicates the position of the core. Near the center of motion the change in voltage is proportional to the displacement. But, further from the center the relationship becomes nonlinear.



$\Delta V = K \Delta x$

where,

ΔV = output voltage

K = constant for device

Δx = core displacement

Figure 302 An LVDT

Aside: The circuit below can be used to produce a voltage that is proportional to position. The two diodes convert the AC wave to a half wave DC wave. The capacitor and resistor values can be selected to act as a low pass filter. The final capacitor should be large enough to smooth out the voltage ripple on the output.

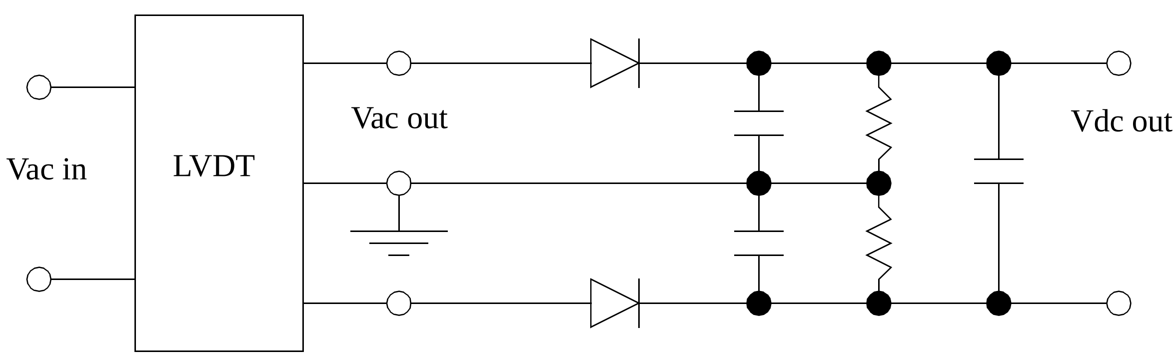


Figure 303 A Simple Signal Conditioner for an LVDT

These devices are more accurate than linear potentiometers, and have less friction. Typical applications for these devices include measuring dimensions on parts for quality control. They are often used for pressure measurements with Bourdon tubes and bellows/diaphragms. A major disadvantage of these sensors is the high cost, often in the thousands.

14.2.3.3 - Moire Fringes

High precision linear displacement measurements can be made with Moire Fringes, as shown in Figure 304. Both of the strips are transparent (or reflective), with black lines at measured intervals. The spacing of the lines determines the accuracy of the position measurements. The stationary strip is offset at an angle so that the strips interfere to give irregular patterns. As the moving strip travels by a stationary strip the patterns will move up, or down, depending upon the speed and direction of motion.

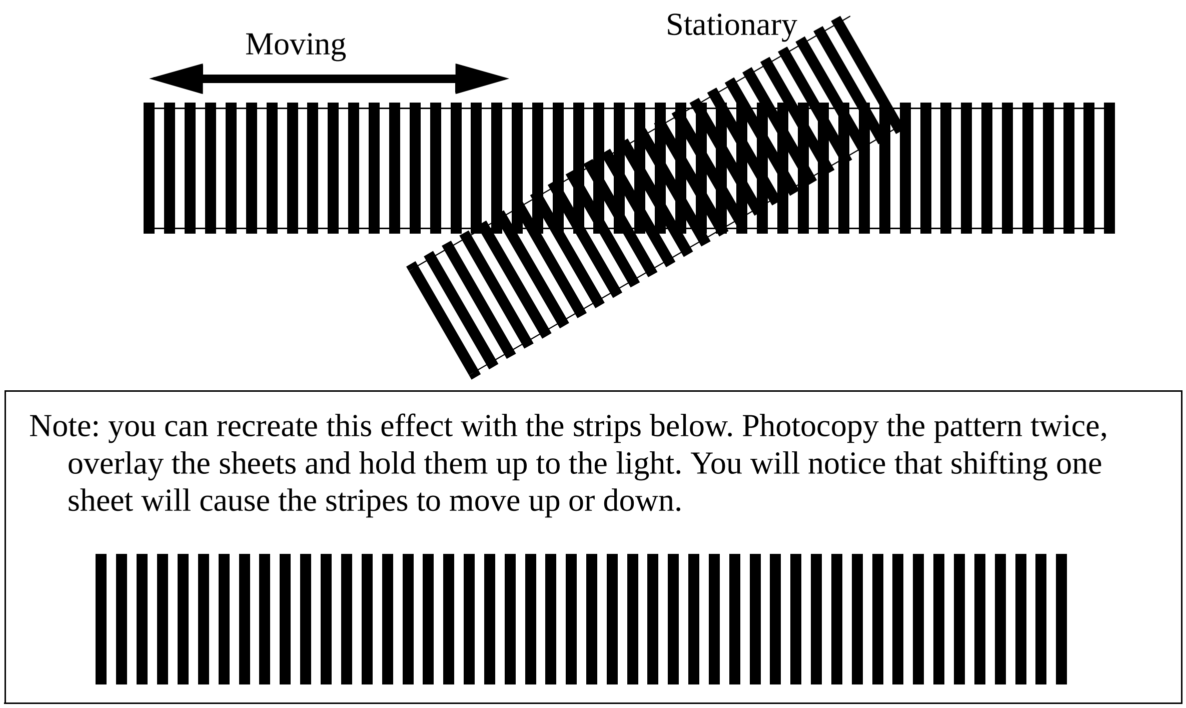


Figure 304 The Moire Fringe Effect

A device to measure the motion of the moire fringes is shown in Figure 305. A light source is collimated by passing it through a narrow slit to make it one slit width. This is then passed through the fringes to be detected by light sensors. At least two light sensors are needed to detect the bright and dark locations. Two sensors, close enough, can act as a quadrature pair, and the same method used for quadrature encoders can be used to determine direction and distance of motion.

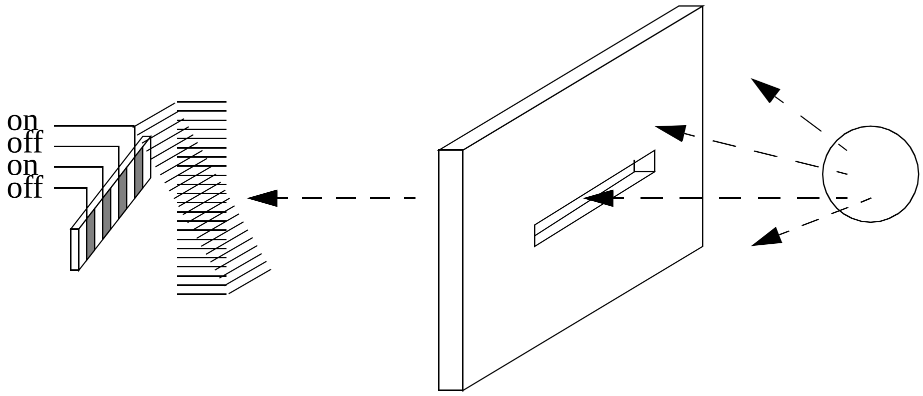


Figure 305 Measuring Motion with Moire Fringes

These are used in high precision applications over long distances, often meters. They can be purchased from a number of suppliers, but the cost will be high. Typical applications include Coordinate Measuring Machines (CMMs).

14.2.3.4 - Accelerometers

Accelerometers measure acceleration using a mass suspended on a force sensor, as shown in Figure 306. When the sensor accelerates, the inertial resistance of the mass will cause the force sensor to deflect. By measuring the deflection the acceleration can be determined. In this case the mass is cantilevered on the force sensor. A base and housing enclose the sensor. A small mounting stud (a threaded shaft) is used to mount the accelerometer.

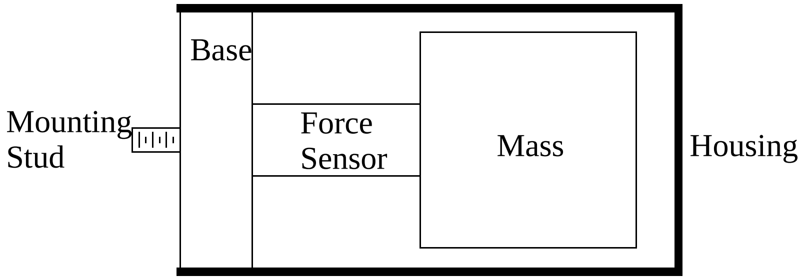


Figure 306 A Cross Section of an Accelerometer

Accelerometers are dynamic sensors, typically used for measuring vibrations between 10Hz to 10KHz. Temperature variations will affect the accuracy of the sensors. Standard accelerometers can be linear up to 100,000 m/s**2: high shock designs can be used up to 1,000,000 m/s**2. There is often a trade-off between a wide frequency range and device sensitivity (note: higher sensitivity requires a larger mass). Figure 307 shows the sensitivity of two accelerometers with different resonant frequencies. A smaller resonant frequency limits the maximum frequency for the reading. The smaller frequency results in a smaller sensitivity. The units for sensitivity is charge per m/s**2.

resonant freq. (Hz)	sensitivity
22 KHz	4.5 pC/(m/s**2)
180KHz	.004

Figure 307 Piezoelectric Accelerometer Sensitivities

The force sensor is often a small piece of piezoelectric material (discussed later in this chapter). The piezoelectic material can be used to measure the force in shear or compression. Piezoelectric based accelerometers typically have parameters such as,

-100to250°C operating range
1mV/g to 30V/g sensitivity
operate well below one forth of the natural frequency

The accelerometer is mounted on the vibration source as shown in Figure 308. The accelerometer is electrically isolated from the vibration source so that the sensor may be grounded at the amplifier (to reduce electrical noise). Cables are fixed to the surface of the vibration source, close to the accelerometer, and are fixed to the surface as often as possible to prevent noise from the cable striking the surface. Background vibrations can be detected by attaching control electrodes to *non-vibrating* surfaces. Each accelerometer is different, but some general application guidelines are;

• The control vibrations should be less than 1/3 of the signal for the error to be less than 12%).
• Mass of the accelerometers should be less than a tenth of the measurement mass.
• These devices can be calibrated with shakers, for example a 1g shaker will hit a peak velocity of 9.81 m/s**2.

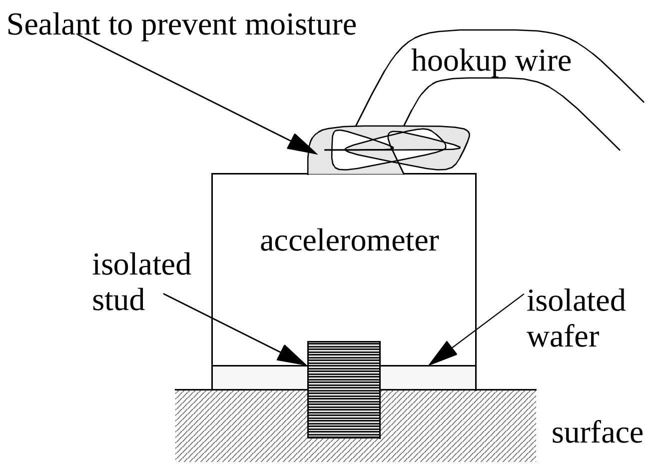


Figure 308 Mounting an Accelerometer

Equipment normally used when doing vibration testing is shown in Figure 309. The sensor needs to be mounted on the equipment to be tested. A pre-amplifier normally converts the charge generated by the accelerometer to a voltage. The voltage can then be analyzed to determine the vibration frequencies.

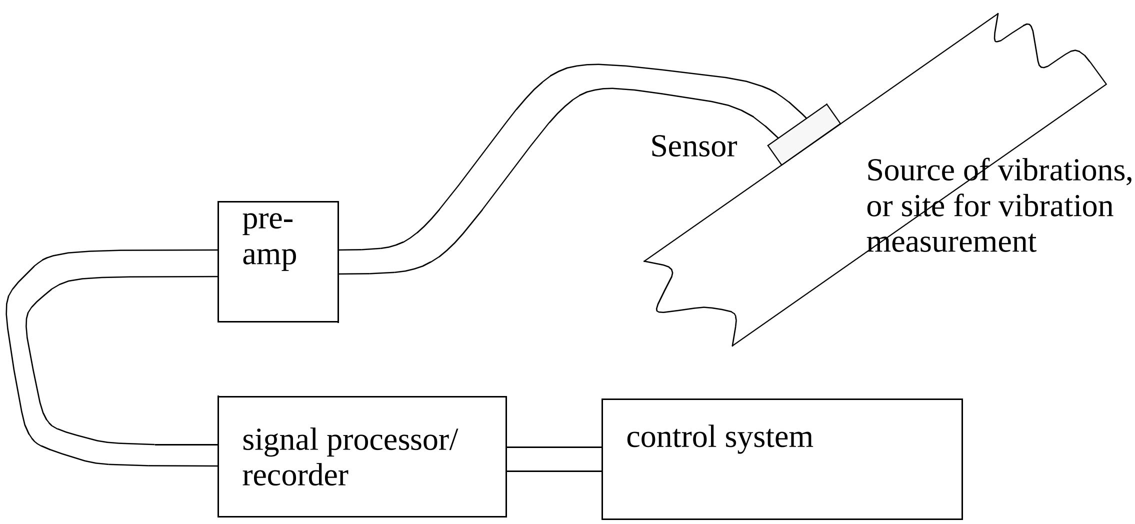


Figure 309 Typical Connection for Accelerometers

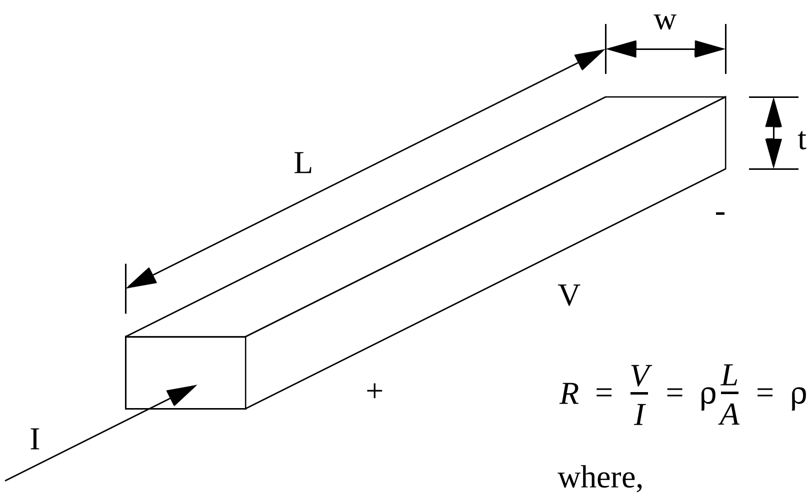
Accelerometers are commonly used for control systems that adjust speeds to reduce vibration and noise. Computer Controlled Milling machines now use these sensors to actively eliminate chatter, and detect tool failure. The signal from accelerometers can be integrated to find velocity and acceleration.

Currently accelerometers cost hundreds or thousands per channel. But, advances in micromachining are already beginning to provide integrated circuit accelerometers at a low cost. Their current use is for airbag deployment systems in automobiles.

14.2.4 Forces and Moments

14.2.4.1 - Strain Gages

Strain gages measure strain in materials using the change in resistance of a wire. The wire is glued to the surface of a part, so that it undergoes the same strain as the part (at the mount point). Figure 310 shows the basic properties of the undeformed wire. Basically, the resistance of the wire is a function of the resistivity, length, and cross sectional area.



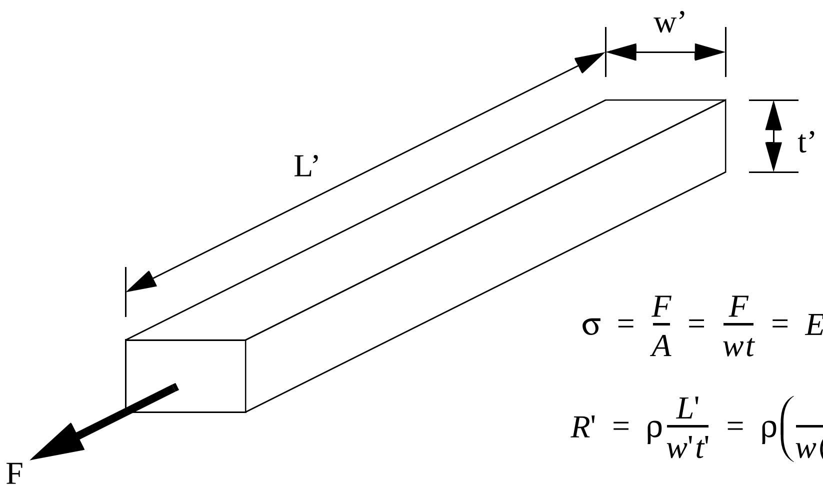
$$R = \frac{V}{I} = \rho\frac{L}{A} = \rho\frac{L}{wt}$$

where,

R = resistance of wire
V, I = voltage and current
L = length of wire
w, t = width and thickness
A = cross sectional area of conductor
ρ = resistivity of material

Figure 310 The Electrical Properties of a Wire

After the wire in Figure 310 has been deformed it will take on the new dimensions and resistance shown in Figure 311. If a force is applied as shown, the wire will become longer, as predicted by Young's modulus. But, the cross sectional area will decrease, as predicted by Poison's ratio. The new length and cross sectional area can then be used to find a new resistance.



$$\sigma = \frac{F}{A} = \frac{F}{wt} = E\varepsilon \qquad \therefore \varepsilon = \frac{F}{Ewt}$$

$$R' = \rho\frac{L'}{w't'} = \rho\left(\frac{L(1+\varepsilon)}{w(1-\nu\varepsilon)t(1-\nu\varepsilon)}\right)$$

$$\therefore \Delta R = R' - R = R\left[\frac{(1+\varepsilon)}{(1-\nu\varepsilon)(1-\nu\varepsilon)} - 1\right]$$

where,

ν = poissons ratio for the material
F = applied force
E = Youngs modulus for the material
σ, ε = stress and strain of material

Figure 311 The Electrical and Mechanical Properties of the Deformed Wire

Aside: Changes in strain gauge resistance are typically small (large values would require strains that would cause the gauges to plastically deform). As a result, Wheatstone bridges are used to amplify the small change. In this circuit the variable resistor R2 would be tuned until Vo = 0V. Then the resistance of the strain gage can be calculated using the given equation.

$$R_{strain} = \frac{R_2 R_1}{R_3} \quad \text{when Vo = 0V}$$

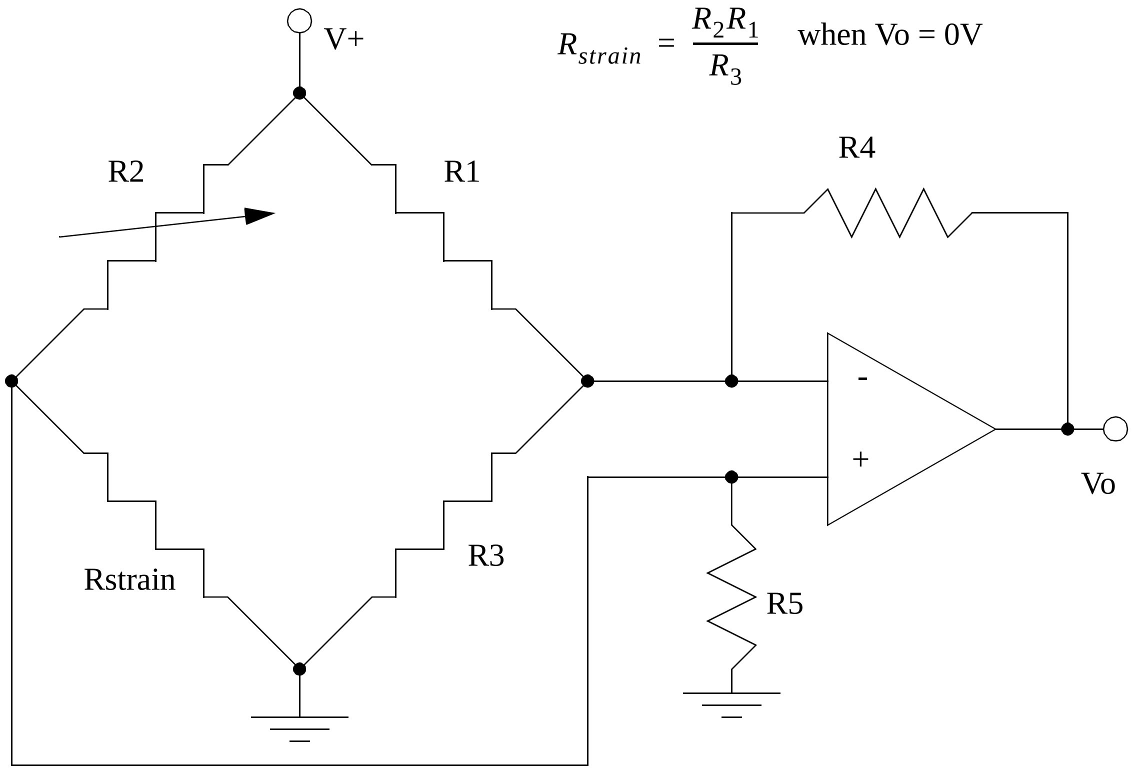


Figure 312 Measuring Strain with a Wheatstone Bridge

A strain gage must be small for accurate readings, so the wire is actually wound in a uniaxial or rosette pattern, as shown in Figure 313. When using uniaxial gages the direction is important, it must be placed in the direction of the normal stress. (Note: the gages cannot read shear stress.) Rosette gages are less sensitive to direction, and if a shear force is present the gage will measure the resulting normal force at 45 degrees. These gauges are sold on thin films that are glued to the surface of a part. The process of mounting strain gages involves surface cleaning. application of adhesives, and soldering leads to the strain gages.

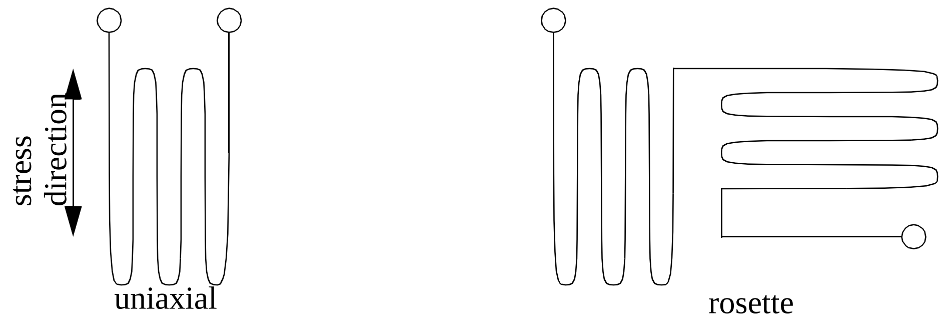


Figure 313 Wire Arrangements in Strain Gages

A design techniques using strain gages is to design a part with a narrowed neck to mount the strain gage on, as shown in Figure 314. In the narrow neck the strain is proportional to the load on the member, so it may be used to measure force. These parts are often called *load cells*.

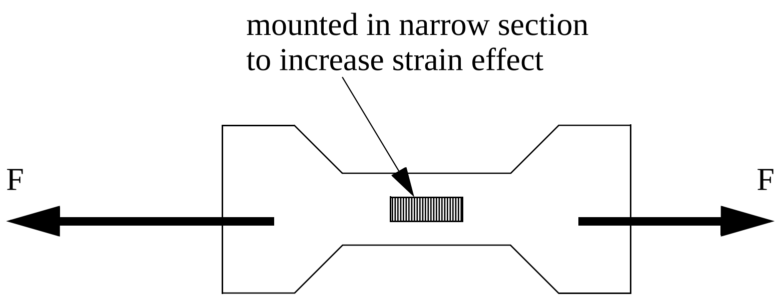
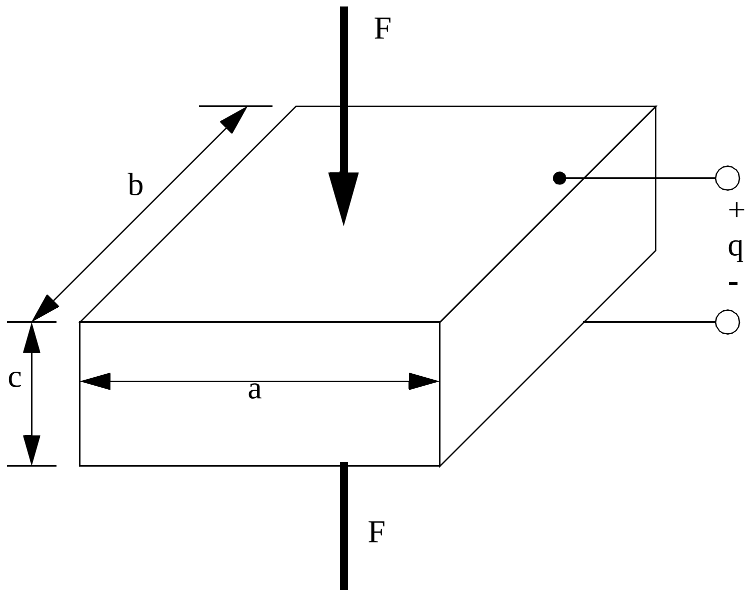


Figure 314 Using a Narrow to Increase Strain

Strain gauges are inexpensive, and can be used to measure a wide range of stresses with accuracies under 1%. Gages require calibration before each use. This often involves making a reading with no load, or a known load applied. An example application includes using strain gages to measure die forces during stamping to estimate when maintenance is needed.

14.2.4.2 - Piezoelectric

When a crystal undergoes strain it displaces a small amount of charge. In other words, when the distance between atoms in the crystal lattice changes some electrons are forced out or drawn in. This also changes the capacitance of the crystal. This is known as the Piezoelectric effect. Figure 315 shows the relationships for a crystal undergoing a linear deformation. The charge generated is a function of the force applied, the strain in the material, and a constant specific to the material. The change in capacitance is proportional to the change in the thickness.



$$C = \frac{\varepsilon ab}{c} \qquad i = \varepsilon g \frac{d}{dt} F$$

where,

C = capacitance change

a, b, c = geometry of material

ε = dielectric constant (quartz typ. 4.06*10**-11 F/m)

i = current generated

F = force applied

g = constant for material (quartz typ. 50*10**-3 Vm/N)

E = Youngs modulus (quartz typ. 8.6*10**10 N/m**2)

Figure 315 The Piezoelectric Effect

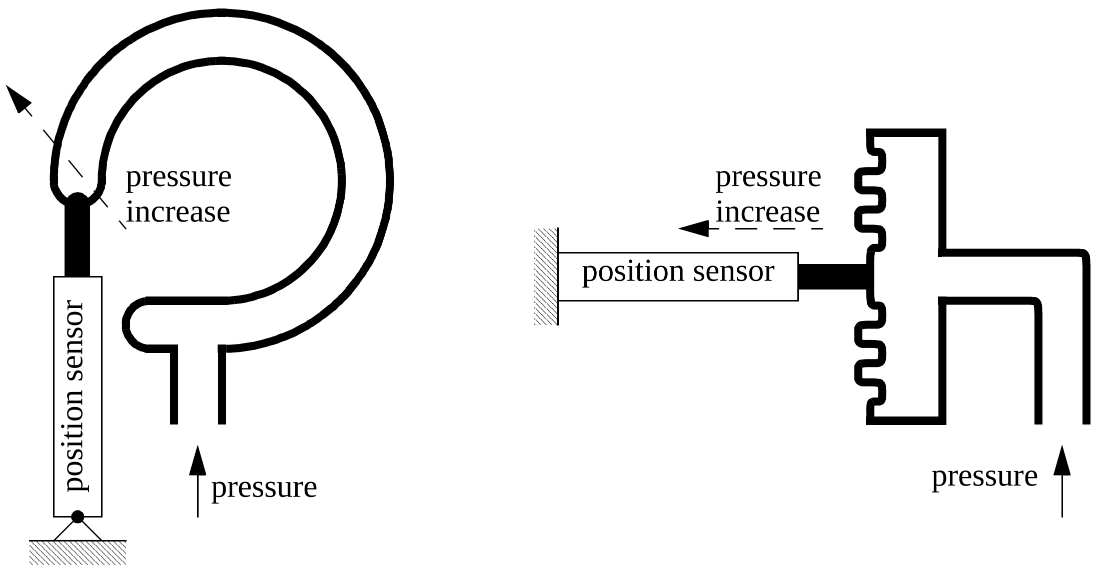
These crystals are used for force sensors, but they are also used for applications such as microphones and pressure sensors. Applying an electrical charge can induce strain, allowing them to be used as actuators, such as audio speakers.

When using piezoelectric sensors charge amplifiers are needed to convert the small amount of charge to a larger voltage. These sensors are best suited to dynamic measurements, when used for static measurements they tend to *drift* or slowly lose charge, and the signal value will change.

14.2.5 Liquids and Gases

14.2.5.1 - Pressure

Figure 316 shows different two mechanisms for pressure measurement. The Bourdon tube uses a circular pressure tube. When the pressure inside is higher than the surrounding air pressure (14.7psi approx.) the tube will straighten. A position sensor, connected to the end of the tube, will be elongated when the pressure increases.



a) Bourdon Tube b) Baffle

Figure 316 Pressure Transducers

These sensors are very common and have typical accuracies of 0.5%.

14.2.5.2 - Venturi Valves

When a flowing fluid or gas passes through a narrow pipe section (neck) the pressure drops. If there is no flow the pressure before and after the neck will be the same. The faster the fluid flow, the greater the pressure difference before and after the neck. This is known as a Venturi valve. Figure 317 shows a Venturi valve being used to measure a fluid flow rate. The fluid flow rate will be proportional to the pressure difference before and at the neck of the valve.

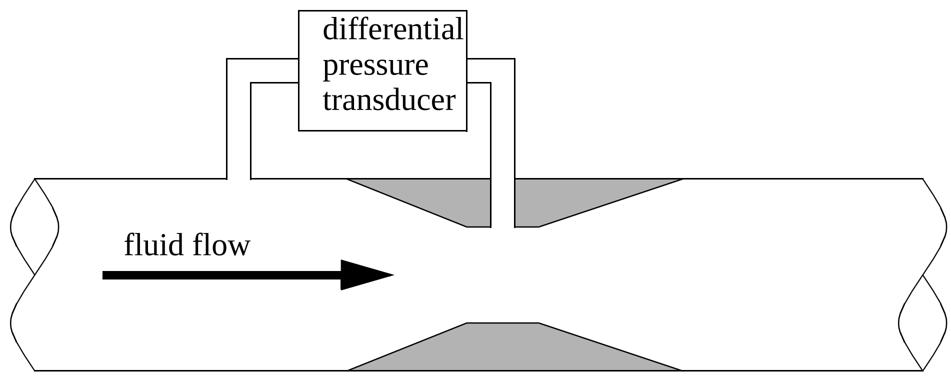


Figure 317 A Venturi Valve

Aside: Bernoulli's equation can be used to relate the pressure drop in a venturi valve.

$$\frac{p}{\rho} + \frac{v^2}{2} + gz = C$$

where,

p = pressure
ρ = density
v = velocity
g = gravitational constant
z = height above a reference
C = constant

Consider the centerline of the fluid flow through the valve. Assume the fluid is incompressible, so the density does not change. And, assume that the center line of the valve does not change. This gives us a simpler equation, as shown below, that relates the velocit and pressure before and after it is compressed.

$$\frac{p_{before}}{\rho} + \frac{v_{before}^2}{2} + gz = C = \frac{p_{after}}{\rho} + \frac{v_{after}^2}{2} + gz$$

$$\frac{p_{before}}{\rho} + \frac{v_{before}^2}{2} = \frac{p_{after}}{\rho} + \frac{v_{after}^2}{2}$$

$$p_{before} - p_{after} = \rho\left(\frac{v_{after}^2}{2} - \frac{v_{before}^2}{2}\right)$$

The flow velocity v in the valve will be larger than the velocity in the larger pipe section before. So, the right hand side of the expression will be positive. This will mean that the pressure before will always be higher than the pressure after, and the difference will be proportional to the velocity squared.

Figure 318 The Pressure Relationship for a Venturi Valve

Venturi valves allow pressures to be read without moving parts, which makes them very reliable and durable. They work well for both fluids and gases. It is also common to use Venturi valves to generate vacuums for actuators, such as suction cups.

14.2.5.3 - Coriolis Flow Meter

Fluid passes through thin tubes, causing them to vibrate. As the fluid approaches the point of maximum vibration it accelerates. When leaving the point it decelerates. The result is a distributed force that causes a bending moment, and hence twisting of the pipe. The amount of bending is proportional to the velocity of the fluid flow. These devices typically have a large constriction on the flow, and result is significant loses. Some of the devices also use bent tubes to increase the sensitivity, but this also increases the flow resistance. The typical accuracy for a corriolis flowmeter is 0.1%.

14.2.5.4 - Magnetic Flow Meter

A magnetic sensor applies a magnetic field perpendicular to the flow of a conductive fluid. As the fluid moves, the electrons in the fluid experience an electromotive force. The result is that a potential (voltage) can be measured perpendicular to the direction of the flow and the magnetic field. The higher the flow rate, the greater the voltage. The typical accuracy for these sensors is 0.5%.

These flowmeters don’t oppose fluid flow, and so they don’t result in pressure drops.

14.2.5.5 - Ultrasonic Flow Meter

A transmitter emits a high frequency sound at point on a tube. The signal must then pass through the fluid to a detector where it is picked up. If the fluid is flowing in the same direction as the sound it will arrive sooner. If the sound is against the flow it will take longer to arrive. In a transit time flow meter two sounds are used, one traveling forward, and the other in the opposite direction. The difference in travel time for the sounds is used to determine the flow velocity.

A doppler flowmeter bounces a soundwave off particle in a flow. If the particle is moving away from the emitter and detector pair, then the detected frequency will be lowered, if it is moving towards them the frequency will be higher.

The transmitter and receiver have a minimal impact on the fluid flow, and therefore don’t result in pressure drops.

14.2.5.6 - Vortex Flow Meter

Fluid flowing past a large (typically flat) obstacle will shed vortices. The frequency of the vortices will be proportional to the flow rate. Measuring the frequency allows an estimate of the flow rate. These sensors tend be low cost and are popular for low accuracy applications.

14.2.5.7 - Pitot Tubes

Gas flow rates can be measured using Pitot tubes, as shown in Figure 319. These are small tubes that project into a flow. The diameter of the tube is small (typically less than 1/8") so that it doesn't affect the flow.

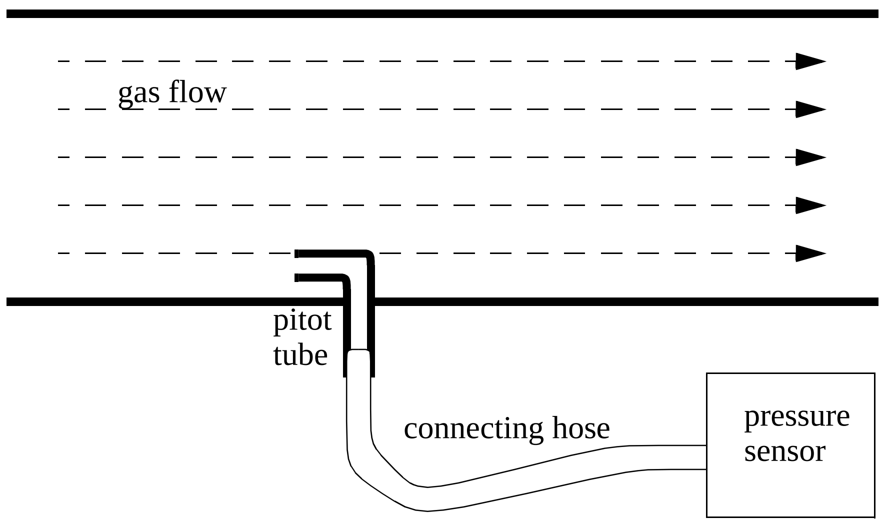


Figure 319 Pitot Tubes for Measuring Gas Flow Rates

14.2.6 Temperature

Temperature measurements are very common with control systems. The temperature ranges are normally described with the following classifications.

very low temperatures <-60 deg C - e.g. superconductors in MRI units
low temperature measurement -60 to 0 deg C - e.g. freezer controls
fine temperature measurements 0 to 100 deg C - e.g. environmental controls
high temperature measurements <3000 deg F - e.g. metal refining/processing
very high temperatures > 2000 deg C - e.g. plasma systems

14.2.6.1 - Resistive Temperature Detectors (RTDs)

When a metal wire is heated the resistance increases. So, a temperature can be measured using the resistance of a wire. Resistive Temperature Detectors (RTDs) normally use a wire or film of platinum, nickel, copper or nickel-iron alloys. The metals are wound or wrapped over an insulator, and covered for protection. The resistances of these alloys are shown in Figure 320.

Material	Temperature Range C (F)	Typical Resistance (ohms)
Platinum	-200 - 850 (-328 - 1562)	100
Nickel	-80 - 300 (-112 - 572)	120
Copper	-200 - 260 (-328 - 500)	10

Figure 320 RTD Properties

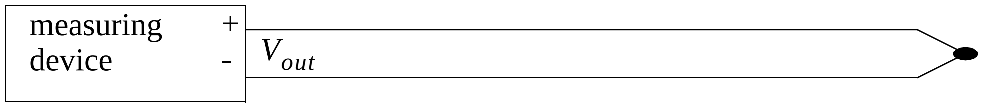
These devices have positive temperature coefficients that cause resistance to increase linearly with temperature. A platinum RTD might have a resistance of 100 ohms at 0C, that will increase by 0.4 ohms/°C. The total resistance of an RTD might double over the temperature range.

A current must be passed through the RTD to measure the resistance. (Note: a voltage divider can be used to convert the resistance to a voltage.) The current through the RTD should be kept to a minimum to prevent self heating. These devices are more linear than thermocouples, and can have accuracies of 0.05%. But, they can be expensive

14.2.6.2 - Thermocouples

Each metal has a natural potential level, and when two different metals touch there is a small potential difference, a voltage. (Note: when designing assemblies, dissimilar metals should not touch, this will lead to corrosion.) Thermocouples use a junction of dissimilar metals to generate a voltage proportional to temperature. This principle was discovered by T.J. Seebeck.

The basic calculations for thermocouples are shown in Figure 321. This calculation provides the measured voltage using a reference temperature and a constant specific to the device. The equation can also be rearranged to provide a temperature given a voltage.



$$V_{out} = \alpha(T - T_{ref})$$

$$\therefore T = \frac{V_{out}}{\alpha} + T_{ref}$$

where,

α = constant (V/C) $\quad 50\frac{\mu V}{°C}$ (typical)

T, T_{ref} = current and reference temperatures

Figure 321 Thermocouple Calculations

The list in Table 1 shows different junction types, and the normal temperature ranges. Both thermocouples, and signal conditioners are commonly available, and relatively inexpensive. For example, most PLC vendors sell thermocouple input cards that will allow multiple inputs into the PLC.

Table 1: Thermocouple Types

ANSI Type	Materials	Temperature Range (°F)	Voltage Range (mV)
T	copper/constantan	-200 to 400	-5.60 to 17.82
J	iron/constantan	0 to 870	0 to 42.28
E	chromel/constantan	-200 to 900	-8.82 to 68.78
K	chromel/aluminum	-200 to 1250	-5.97 to 50.63
R	platinum-13%rhodium/platinum	0 to 1450	0 to 16.74
S	platinum-10%rhodium/platinum	0 to 1450	0 to 14.97
C	tungsten-5%rhenium/tungsten-26%rhenium	0 to 2760	0 to 37.07

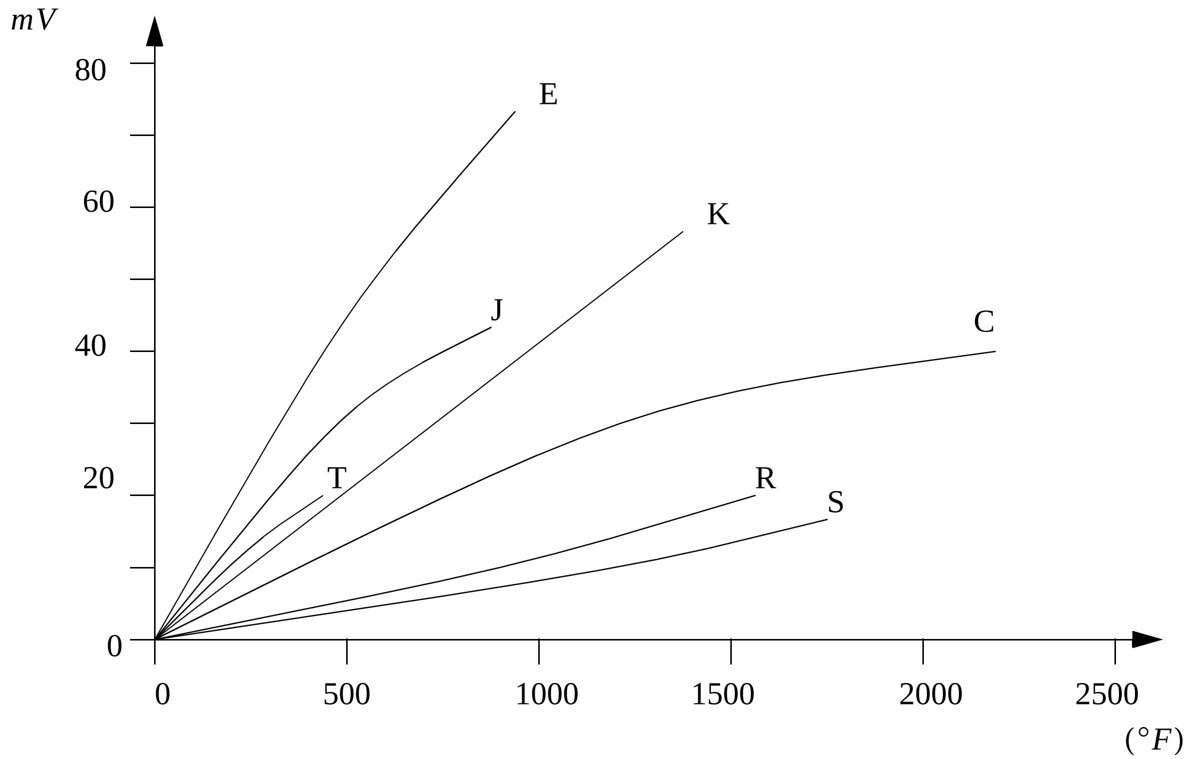


Figure 322 Thermocouple Temperature Voltage Relationships (Approximate)

The junction where the thermocouple is connected to the measurement instrument is normally cooled to reduce the thermocouple effects at those junctions. When using a thermocouple for precision measurement, a second thermocouple can be kept at a known temperature for reference. A series of thermocouples connected together in series produces a higher voltage and is called a thermopile. Readings can approach an accuracy of 0.5%.

14.2.6.3 - Thermistors

Thermistors are non-linear devices, their resistance will decrease with an increase in temperature. (Note: this is because the extra heat reduces electron mobility in the semiconductor.) The resistance can change by more than 1000 times. The basic calculation is shown in Figure 323.

often metal oxide semiconductors The calculation uses a reference temperature and resistance, with a constant for the device, to predict the resistance at another temperature. The expression can be rearranged to calculate the temperature given the resistance.

$$R_t = R_o e^{\beta\left(\frac{1}{T} - \frac{1}{T_o}\right)}$$

$$\therefore T = \frac{\beta T_o}{T_o \ln\left(\frac{R_t}{R_o}\right) + \beta}$$

where,

R_o, R_t = resistances at reference and measured temps.

T_o, T = reference and actual temperatures

β = constant for device

Figure 323 Thermistor Calculations

Aside: The circuit below can be used to convert the resistance of the thermistor to a voltage using a Wheatstone bridge and an inverting amplifier.

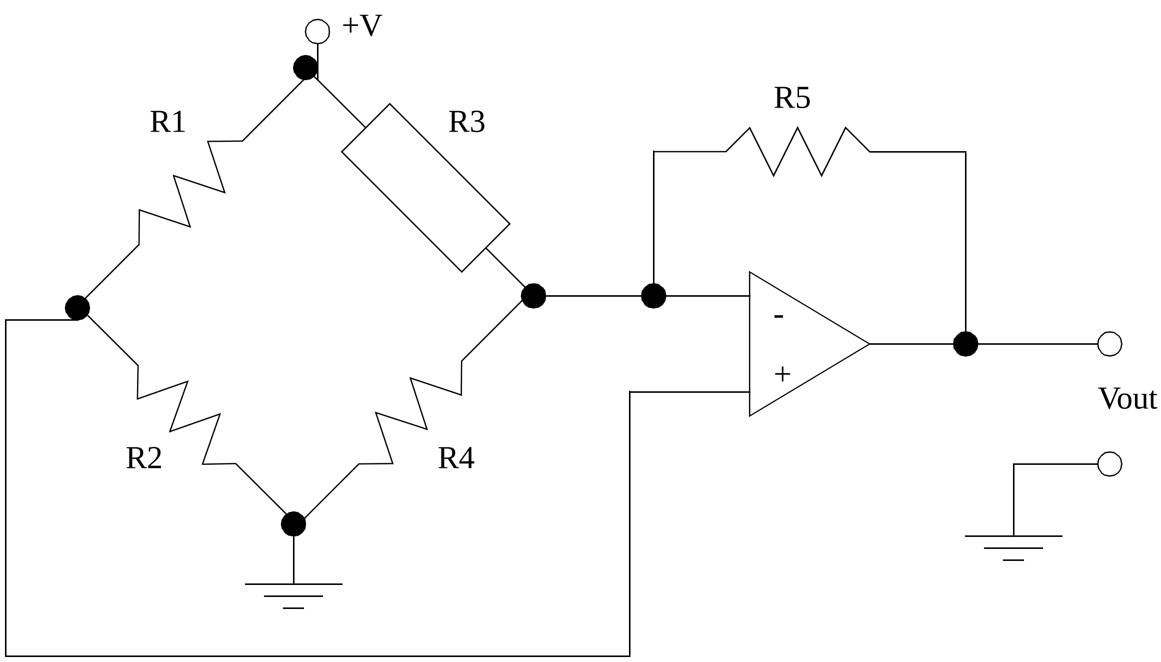


Figure 324 Thermistor Signal Conditioning Circuit

Thermistors are small, inexpensive devices that are often made as beads, or metallized surfaces. The devices respond quickly to temperature changes, and they have a higher resistance, so junction effects are not an issue. Typical accuracies are 1%, but the devices are not linear, have a limited temperature/resistance range and can be self heating.

14.2.6.4 - Other Sensors

IC sensors are becoming more popular. They output a digital reading and can have accuracies better than 0.01%. But, they have limited temperature ranges, and require some knowledge of interfacing methods for serial or parallel data.

Pyrometers are non-contact temperature measuring devices that use radiated heat. These are normally used for high temperature applications, or for production lines where it is not possible to mount other sensors to the material.

14.2.7 Light

14.2.7.1 - Light Dependant Resistors (LDR)

Light dependant resistors (LDRs) change from high resistance (>Mohms) in bright light to low resistance (<Kohms) in the dark. The change in resistance is non-linear, and is also relatively slow (ms).

Aside: an LDR can be used in a voltage divider to convert the change in resistance to a measurable voltage.

These are common in low cost night lights.

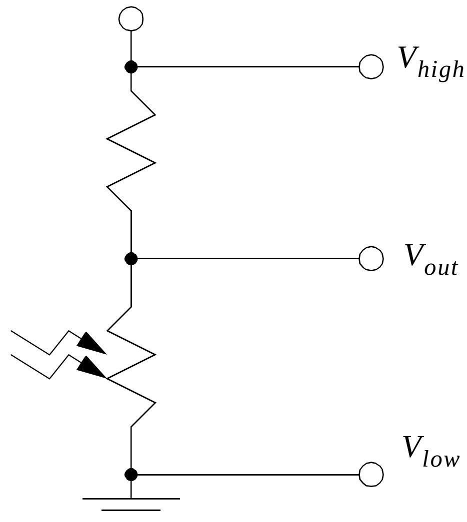


Figure 325 A Light Level Detector Circuit

14.3 INPUT ISSUES

Signals from sensors are often not in a form that can be directly input to a controller. In these cases it may be necessary to buy or build signal conditioners. Normally, a signal conditioner is an amplifier, but it may also include noise filters, and circuitry to convert from current to voltage. This section will discuss the electrical and electronic interfaces between sensors and controllers.

Analog signal are prone to electrical noise problems. This is often caused by electromagnetic fields on the factory floor inducing currents in exposed conductors. Some of the techniques for dealing with electrical noise include;

twisted pairs - the wires are twisted to reduce the noise induced by magnetic fields.
shielding - shielding is used to reduce the effects of electromagnetic interference.
single/double ended inputs - shared or isolated reference voltages (commons).

When a signal is transmitted through a wire, it must return along another path. If the wires have an area between them the magnetic flux enclosed in the loop can induce current flow and voltages. If the wires are twisted, a few times per inch, then the amount of noise induced is reduced. This technique is common in signal wires and network cables.

A shielded cable has a metal sheath, as shown in Figure 326. This sheath needs to be connected to the measuring device to allow induced currents to be passed to ground. This prevents electromagnetic waves to induce voltages in the signal wires.

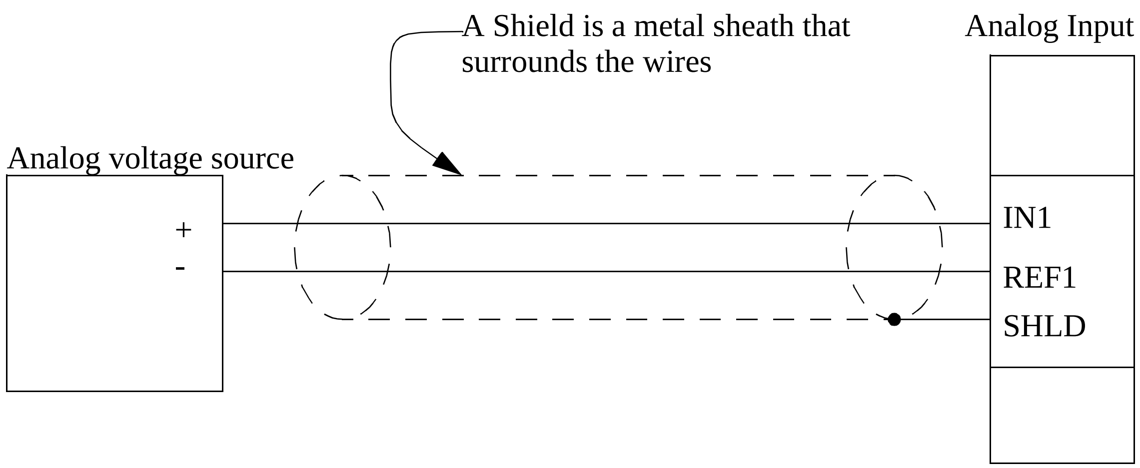
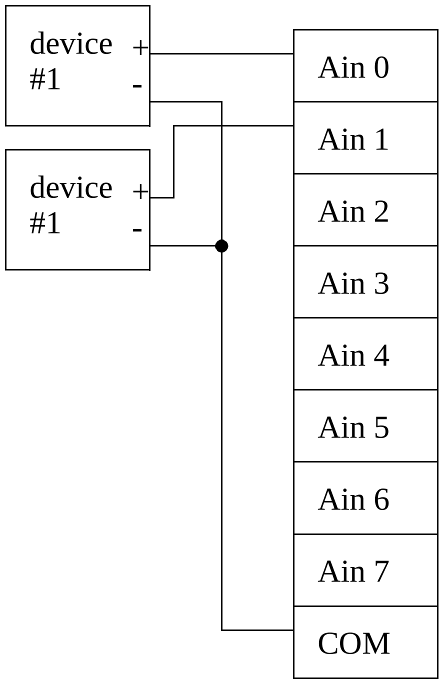
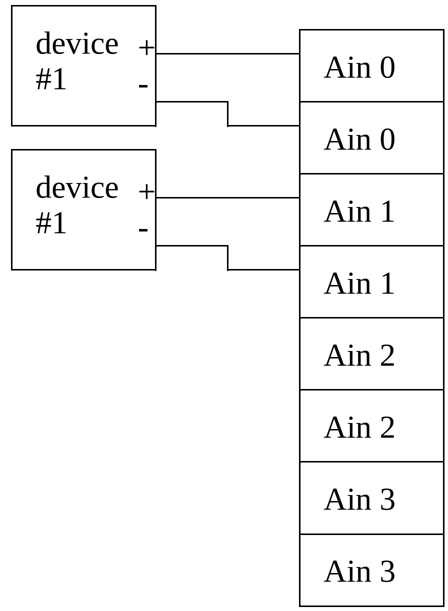


Figure 326 Cable Shielding

When connecting analog voltage sources to a controller the common, or reference voltage can be connected different ways, as shown in Figure 327. The least expensive method uses one shared common for all analog signals, this is called single ended. The more accurate method is to use separate commons for each signal, this is called double ended. Most analog input cards allow a choice between one or the other. But, when double ended inputs are used the number of available inputs is halved. Most analog output cards are double ended.



Single ended - with this arrangement the signal quality can be poorer, but more inputs are available.

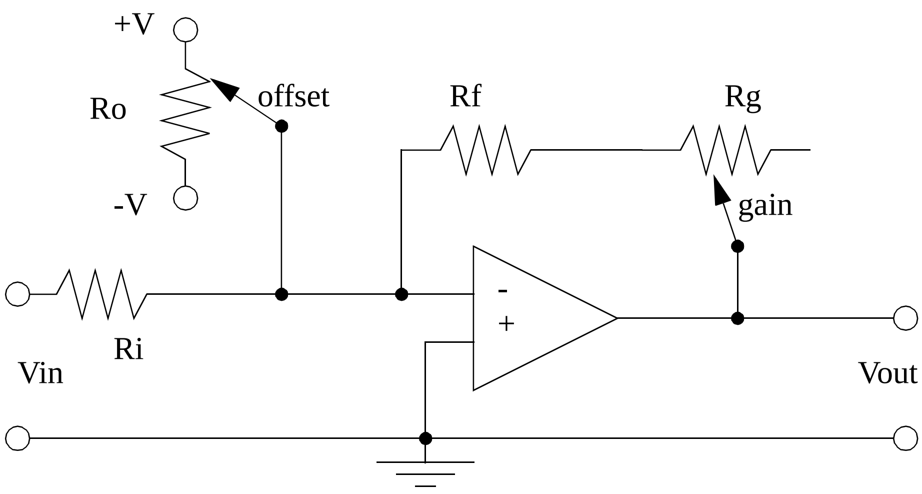


Double ended - with this arrangement the signal quality can be better, but fewer inputs are available.

Figure 327 Single and Double Ended Inputs

Signals from transducers are typically too small to be read by a normal analog input card. Amplifiers are used to increase the magnitude of these signals. An example of a single ended signal amplifier is shown in Figure 328. The amplifier is in an inverting configuration, so the output will have an opposite sign from the input. Adjustments are provided for *gain* and *offset* adjustments.

Note: op-amps are used in this section to implement the amplifiers because they are inexpensive, common, and well suited to simple design and construction projects. When purchasing a commercial signal conditioner, the circuitry will be more complex, and include other circuitry for other factors such as temperature compensation.



$$V_{out} = \left(\frac{R_f + R_g}{R_i}\right)V_{in} + offset$$

Figure 328 A Single Ended Signal Amplifier

A differential amplifier with a current input is shown in Figure 329. Note that Rc converts a current to a voltage. The voltage is then amplified to a larger voltage.

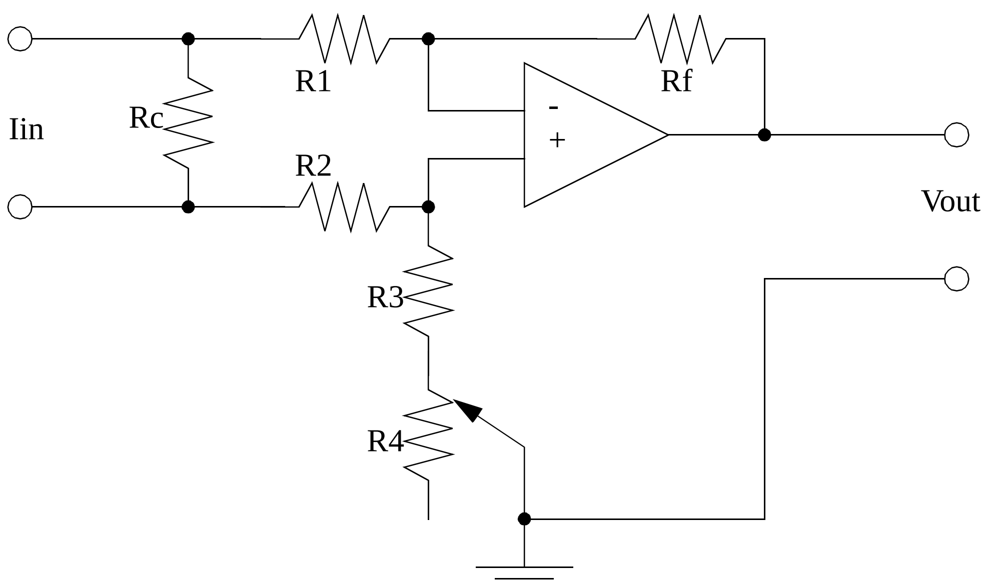


Figure 329 A Current Amplifier

The circuit in Figure 330 will convert a differential (double ended) signal to a single ended signal. The two input op-amps are used as unity gain followers, to create a high input impedance. The following amplifier amplifies the voltage difference.

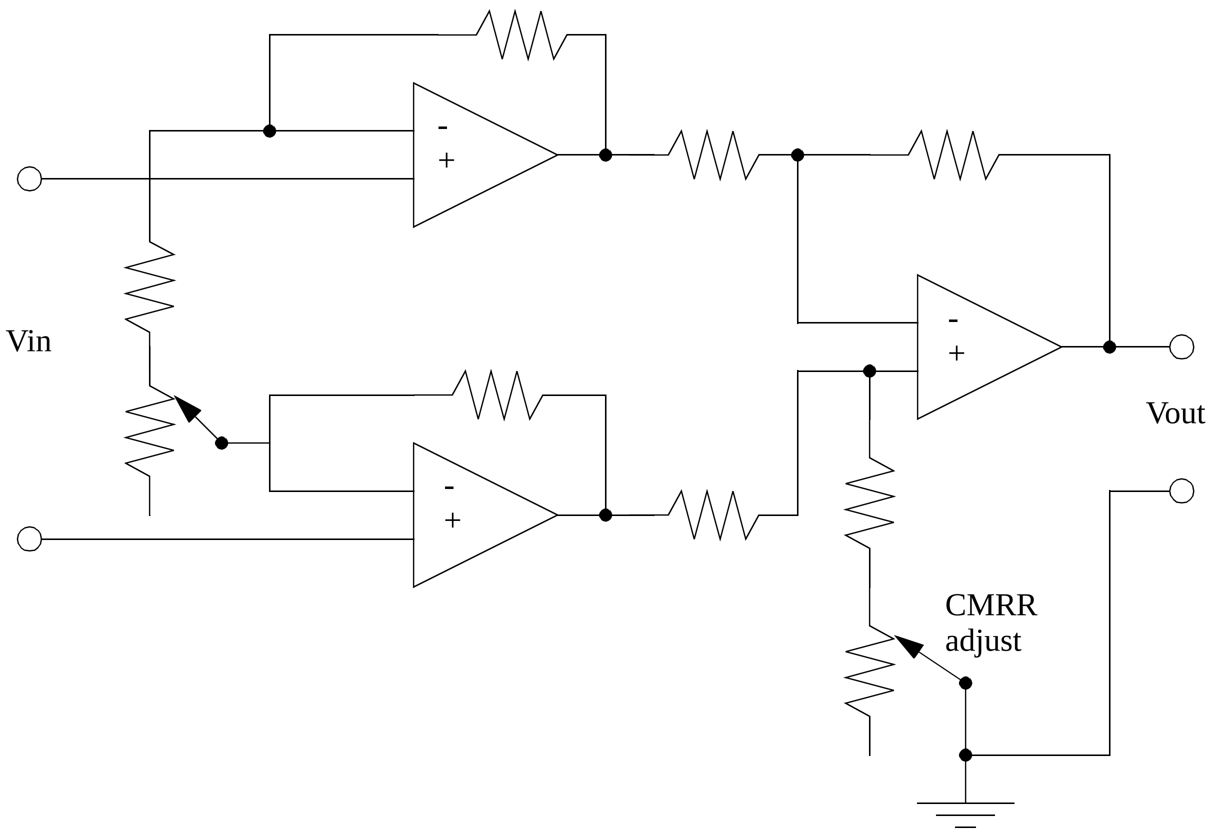
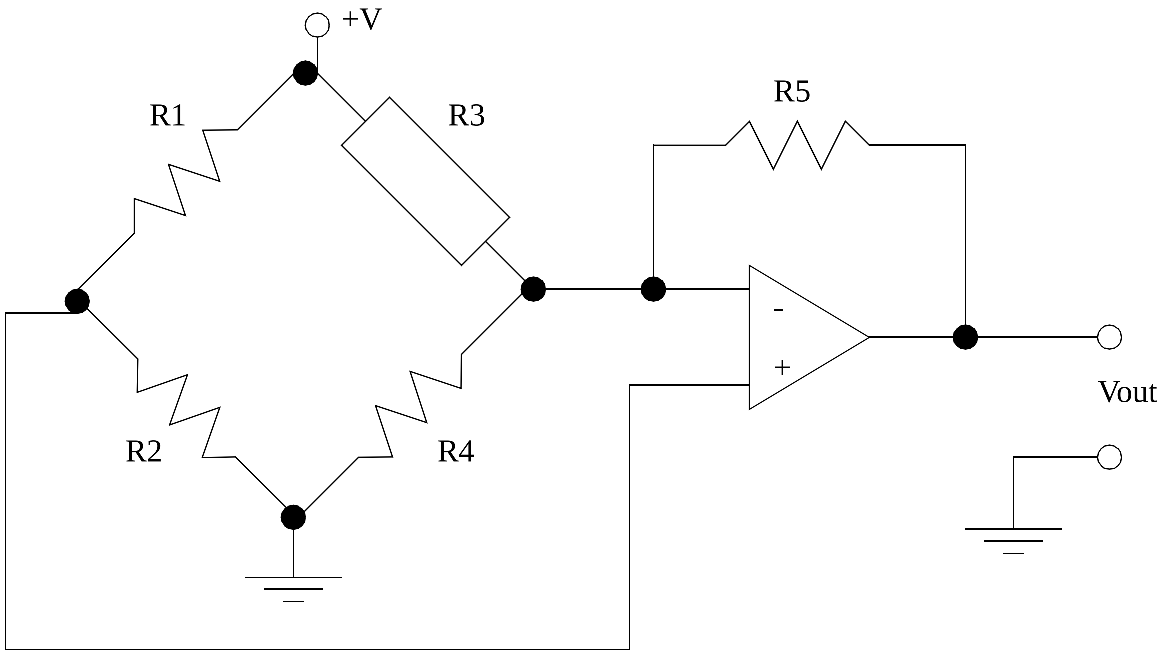


Figure 330 A Differential Input to Single Ended Output Amplifier

The Wheatstone bridge can be used to convert a resistance to a voltage output, as shown in Figure 331. If the resistor values are all made the same (and close to the value of R3) then the equation can be simplified.



$$V_{out} = V(R_5)\left(\left(\frac{R_2}{R_1+R_2}\right)\left(\frac{1}{R_3}+\frac{1}{R_4}+\frac{1}{R_5}\right)-\frac{1}{R_3}\right)$$

or if $R = R_1 = R_2 = R_4 = R_5$

$$V_{out} = V\left(\frac{R}{2R_3}\right)$$

Figure 331 A Resistance to Voltage Amplifier

14.4 SENSOR GLOSSARY

Ammeter - A meter to indicate electrical current. It is normally part of a DMM

Bellows - This is a flexible volumed that will expand or contract with a pressure change. This often looks like a cylinder with a large radius (typ. 2") but it is very thin (type 1/4"). It can be set up so that when pressure changes, the displacement of one side can be measured to determine pressure.

Bourdon tube - Widely used industrial gage to measure pressure and vacuum. It resembles a crescent moon. When the pressure inside changes the moon shape will tend to straighten out. By measuring the displacement of the tip the pressure can be measured.

Chromatographic instruments - laboratory-type instruments used to analyze chemical compounds and gases.

Inductance-coil pulse generator - transducer used to measure rotational speed. Output is pulse train.

Interferometers - These use the interference of light waves 180 degrees out of phase to determine distances. Typical sources of the monochromatic light required are lasers.

Linear-Variable-Differential transformer (LVDT) electromechanical transducer used to measure angular or linear displacement. Output is Voltage

Manometer - liquid column gage used widely in industry to measure pressure.

Ohmmeter - meter to indicate electrical resistance

Optical Pyrometer - device to measure temperature of an object at high temperatures by sensing the brightness of an objects surface.

Orifice Plate - widely used flowmeter to indicate fluid flow rates

Photometric Transducers - a class of transducers used to sense light, including phototubes, photodiodes, phototransistors, and photoconductors.

Piezoelectric Accelerometer - Transducer used to measure vibration. Output is emf.

Pitot Tube - Laboratory device used to measure flow.

Positive displacement Flowmeter - Variety of transducers used to measure flow. Typical output is pulse train.

Potentiometer - instrument used to measure voltage

Pressure Transducers - A class of transducers used to measure pressure. Typical output is voltage. Operation of the transducer can be based on strain gages or other devices.

Radiation pyrometer - device to measure temperature by sensing the thermal radiation emitted from the object.

Resolver - this device is similar to an incremental encoder, except that it uses coils to generate magnetic fields. This is like a rotary transformer.

Strain Gage - Widely used to indicate torque, force, pressure, and other variables. Output is change in resistance due to strain, which can be converted into voltage.

Thermistor - Also called a resistance thermometer; an instrument used to measure temperature. Operation is based on change in resistance as a function of temperature.

Thermocouple - widely used temperature transducer based on the Seebeck effect, in which a junction of two dissimilar metals emits emf related to temperature.

Turbine Flowmeter - transducer to measure flow rate. Output is pulse train.

Venturi Tube - device used to measure flow rates.

14.5 SUMMARY

- Selection of continuous sensors must include issues such as accuracy and resolution.
- Angular positions can be measured with potentiometers and encoders (more accurate).
- Tachometers are useful for measuring angular velocity.

- Linear positions can be measured with potentiometers (limited accuracy), LVDTs (limited range), moire fringes (high accuracy).
- Accelerometers measure acceleration of masses.
- Strain gauges and piezoelectric elements measure force.
- Pressure can be measured indirectly with bellows and Bourdon tubes.
- Flow rates can be measured with Venturi valves and pitot tubes.
- Temperatures can be measured with RTDs, thermocouples, and thermistors.
- Input signals can be single ended for more inputs or double ended for more accuracy.

14.6 REFERENCES

Bryan, L.A. and Bryan, E.A., Programmable Controllers; Theory and Implementation, Industrial Text Co., 1988.

Swainston, F., A Systems Approach to Programmable Controllers, Delmar Publishers Inc., 1992.

14.7 PRACTICE PROBLEMS

1. Name two types of inputs that would be analog input values (versus a digital value).

2. Search the web for common sensor manufacturers for 5 different types of continuous sensors. If possible identify prices for the units. Sensor manufacturers include (hyde park, banner, allen bradley, omron, etc.)

3. What is the resolution of an absolute optical encoder that has six tracks? nine tracks? twelve tracks?

4. Suggest a couple of methods for collecting data on the factory floor

5. If a thermocouple generates a voltage of 30mV at 800F and 40mV at 1000F, what voltage will be generated at 1200F?

6. A potentiometer is to be used to measure the position of a rotating robot link (as a voltage divider). The power supply connected across the potentiometer is 5.0 V, and the total wiper travel is 300 degrees. The wiper arm is directly connected to the rotational joint so that a given rotation of the joint corresponds to an equal rotation of the wiper arm.
   - a) To position the joint at 42 degrees, what voltage is required from the potentiometer?
   - b) If the joint has been moved, and the potentiometer reads 2.765V, what is the position of the potentiometer?

7. A motor has an encoder mounted on it. The motor is driving a reducing gear box with a 50:1 ratio. If the position of the geared down shaft needs to be positioned to 0.1 degrees, how many divisions are needed on the encoder?

8. What is the difference between a strain gauge and an accelerometer? How do they work?

9. Use the model or equations for a permanent magnet DC motor to explain how it can be used as a tachometer.

10. What are the trade-offs between encoders and potentiometers?

11. A potentiometer is connected to a PLC analog input card. The potentiometer can rotate 300 degrees, and the voltage supply for the potentiometer is +/-10V. Write a ladder logic program to read the voltage from the potentiometer and convert it to an angle in radians stored in F8:0.

14.8 PRACTICE PROBLEM SOLUTIONS

1. temperature and displacement

2. Sensors can be found at www.ab.com, www.omron.com, etc

3. 360°/64steps, 360°/512seps, 360°/4096 steps

4. data bucket, smart machines, PLCs with analog inputs and network connections

5.

$$V_{out} = \alpha(T - T_{ref}) \qquad 0.030 = \alpha(800 - T_{ref}) \qquad 0.040 = \alpha(1000 - T_{ref})$$

$$\frac{1}{\alpha} = \frac{800 - T_{ref}}{0.030} = \frac{1000 - T_{ref}}{0.040}$$

$$800 - T_{ref} = 750 - 0.75T_{ref}$$

$$50 = 0.25T_{ref} \qquad T_{ref} = 200C \qquad \alpha = \frac{0.040}{1000 - 200} = \frac{50\mu V}{C}$$

$$V_{out} = 0.00005(1200 - 200) = 0.050V$$

6.

a) $$V_{out} = (V_2 - V_1)\left(\frac{\theta_w}{\theta_{max}}\right) + V_1 = (5V - 0V)\left(\frac{42deg}{300deg}\right) + 0V = 0.7V$$

b) $$2.765V = (5V - 0V)\left(\frac{\theta_w}{300deg}\right) + 0V$$

$$2.765V = (5V - 0V)\left(\frac{\theta_w}{300deg}\right) + 0V$$

$$\theta_w = 165.9deg$$

7.

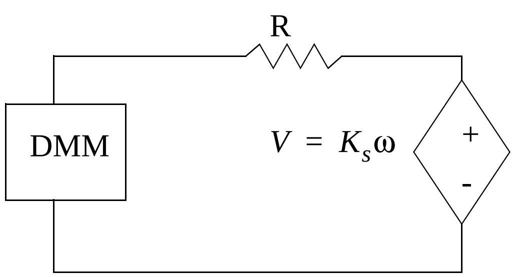
$$\theta_{output} = 0.1\frac{deg}{count} \qquad \frac{\theta_{input}}{\theta_{output}} = \frac{50}{1} \qquad \theta_{input} = 50\left(0.1\frac{deg}{count}\right) = 5\frac{deg}{count}$$

$$R = \frac{360\frac{deg}{rot}}{5\frac{deg}{count}} = 72\frac{count}{rot}$$

8.

strain gauge measures strain in a material using a stretching wire that increases resistance - accelerometers measure acceleration with a cantilevered mass on a piezoelectric element.

9.

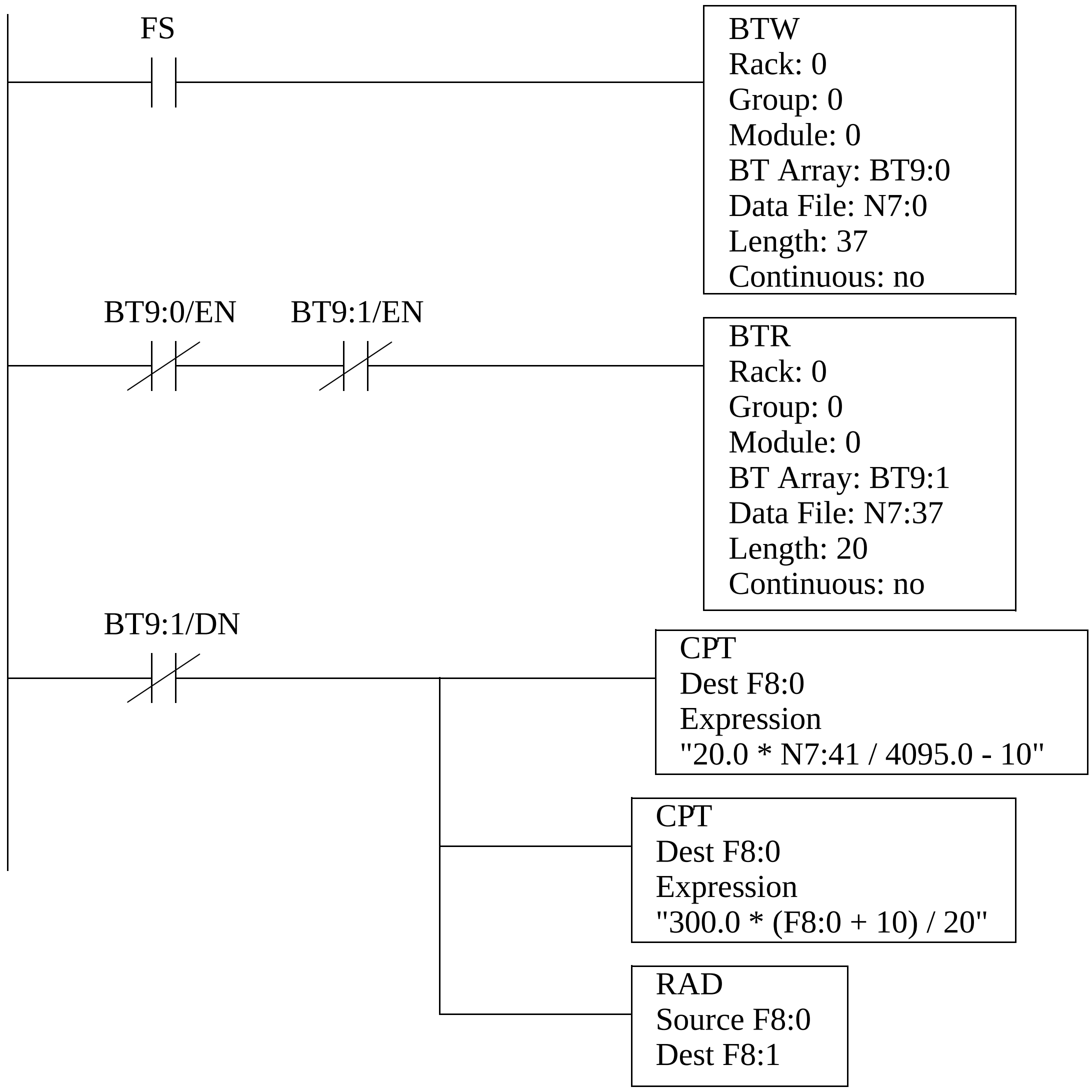


When the motor shaft is turned by another torque source a voltage is generated that is proportional to the angular velocity. This is the reverse emf. A dmm, or other high impedance instrument can be used to measure this, thus minizing the loses in resistor R.

10.

encoders cost more but have higher costs. Potentiometers have limited ranges of motion

11.



14.9 ASSIGNMENT PROBLEMS

15. CONTINUOUS ACTUATORS

Topics:
- Servo Motors; AC and DC
- Stepper motors
- Single axis motion control
- Hydraulic actuators

Objectives:
- To understand the main differences between continuous actuators
- Be able to select a continuous actuator
- To be able to plan a motion for a single servo actuator

15.1 INTRODUCTION

Continuous actuators allow a system to position or adjust outputs over a wide range of values. Even in their simplest form, continuous actuators tend to be mechanically complex devices. For example, a linear slide system might be composed of a motor with an electronic controller driving a mechanical slide with a ball screw. The cost for such an actuators can easily be higher than for the control system itself. These actuators also require sophisticated control techniques that will be discussed in later chapters. In general, when there is a choice, it is better to use discrete actuators to reduce costs and complexity.

15.2 ELECTRIC MOTORS

An electric motor is composed of a rotating center, called the rotor, and a stationary outside, called the stator. These motors use the attraction and repulsion of magnetic fields to induce forces, and hence motion. Typical electric motors use at least one electromagnetic coil, and sometimes permanent magnets to set up opposing fields. When a voltage is applied to these coils the result is a torque and rotation of an output shaft. There are a variety of motor configuration the yields motors suitable for different applications. Most notably, as the voltages supplied to the motors will vary the speeds and torques that they will provide.

A control system is required when a motor is used for an application that requires continuous position or velocity. A typical controller is shown in Figure 332. In any controlled system a command generator is required to specify a desired position. The controller will compare the feedback from the encoder to the desired position or velocity to determine the system error. The controller with then generate an output, based on the system error. The output is then passed through a power amplifier, which in turn drives the motor. The encoder is connected directly to the motor shaft to provide feedback of position.

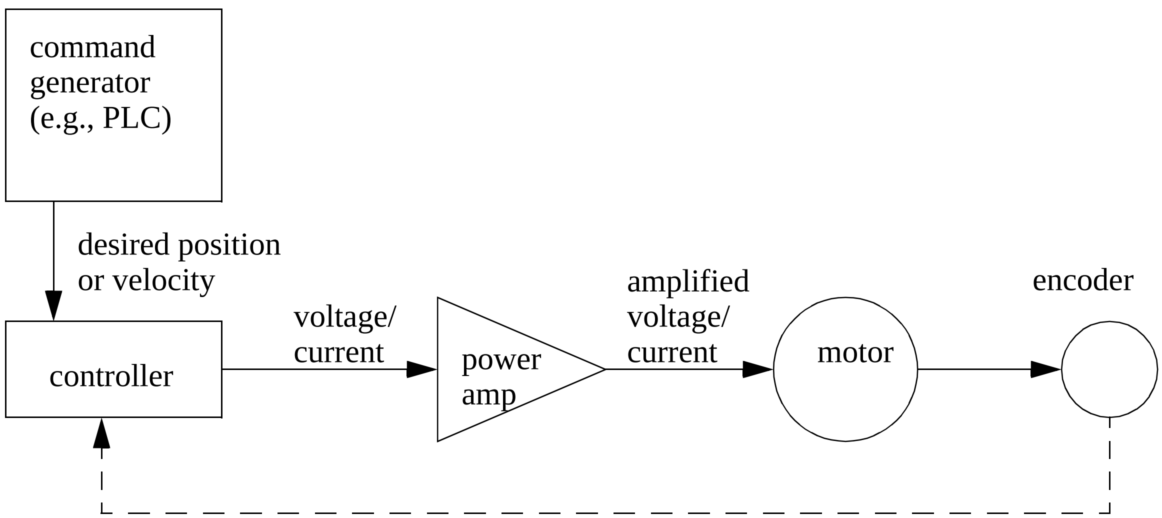


Figure 332 A Typical Feedback Motor Controller

15.2.1 Brushed DC Motors

In a DC motor there is normally a set of coils on the rotor that turn inside a stator populated with permanent magnets. Figure 333 shows a simplified model of a motor. The magnetics provide a permanent magnetic field for the rotor to push against. When current is run through the wire loop it creates a magnetic field.

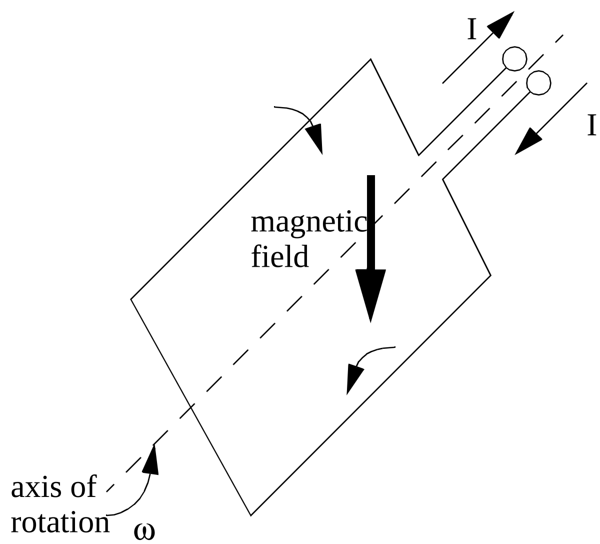


Figure 333 A Simplified Rotor

The power is delivered to the rotor using a commutator and brushes, as shown in Figure 334. In the figure the power is supplied to the rotor through graphite brushes rubbing against the commutator. The commutator is split so that every half revolution the polarity of the voltage on the rotor, and the induced magnetic field reverses to push against the permanent magnets.

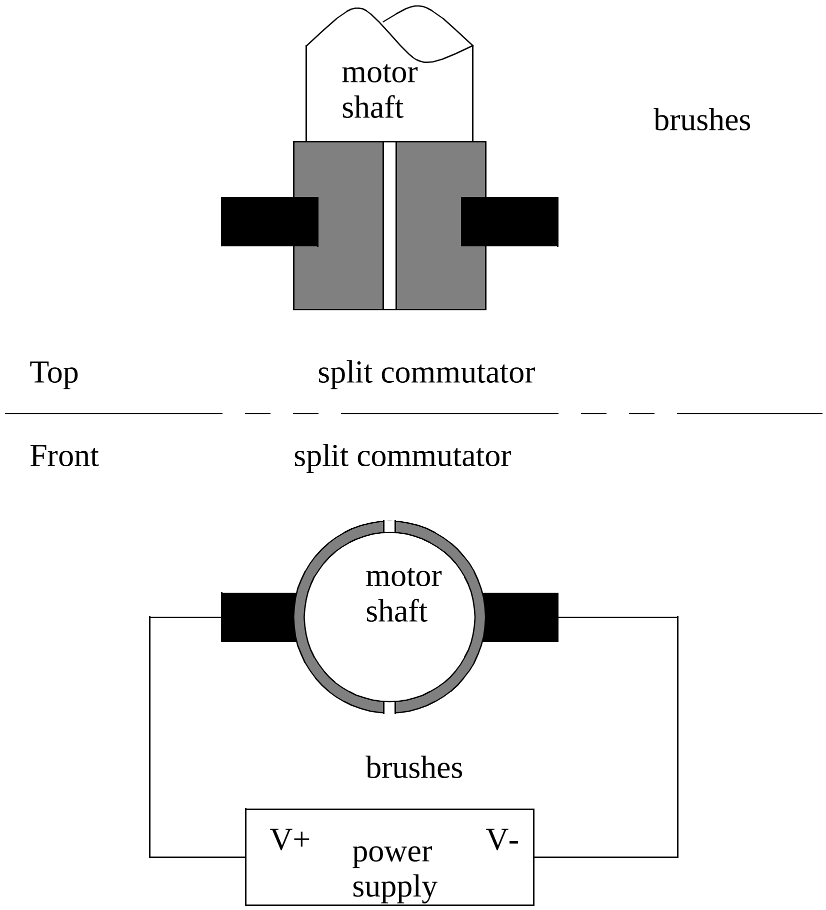


Figure 334 A Split Ring Commutator

The direction of rotation will be determined by the polarity of the applied voltage, and the speed is proportional to the voltage. A feedback controller is used with these motors to provide motor positioning and velocity control.

These motors are losing popularity to brushless motors. The brushes are subject to wear, which increases maintenance costs. In addition, the use of brushes increases resistance, and lowers the motors efficiency.

ASIDE: The controller to drive a servo motor normally uses a Pulse Width Modulated (PWM) signal. As shown below the signal produces an effective voltage that is relative to the time that the signal is on. The percentage of time that the signal is on is called the *duty cycle*. When the voltage is on all the time the effective voltage delivered is the maximum voltage. So, if the voltage is only on half the time, the effective voltage is half the maximum voltage. This method is popular because it can produce a variable effective voltage efficiently. The frequency of these waves is normally above 20KHz, above the range of human hearing.

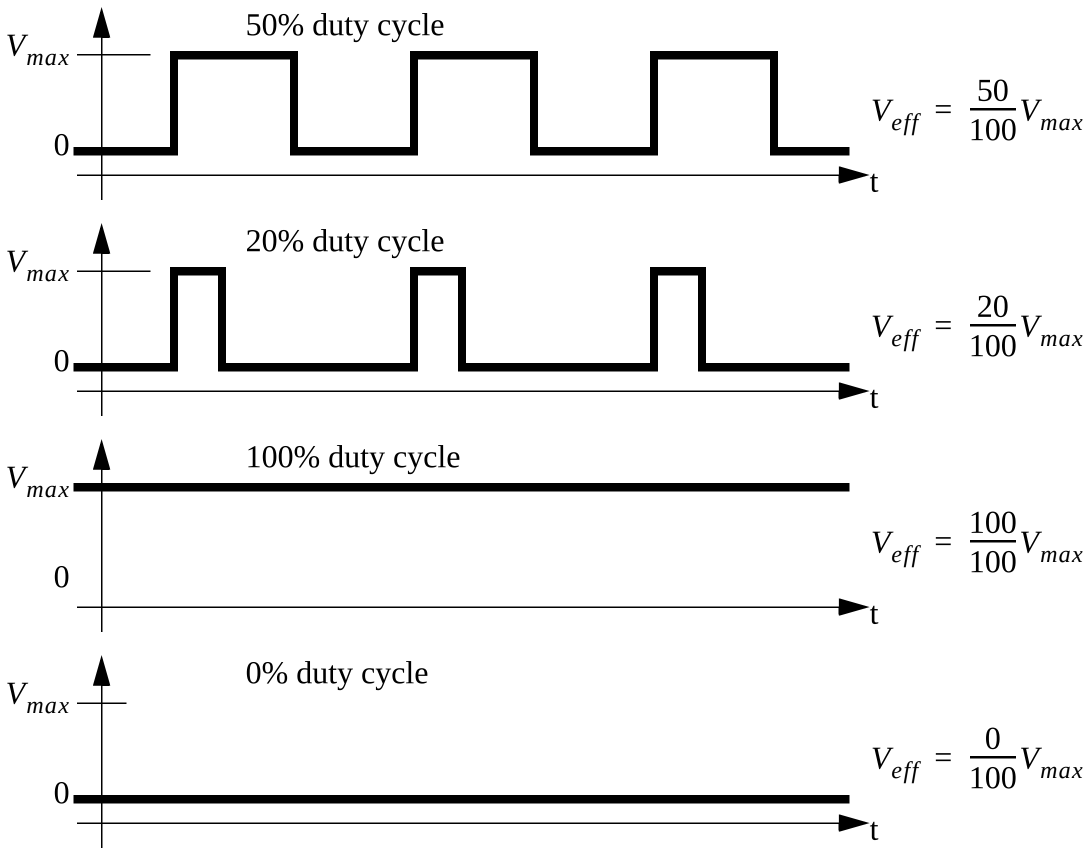


Figure 335 Pulse Width Modulation (PWM) For Control

ASIDE: A PWM signal can be used to drive a motor with the circuit shown below. The PWM signal switches the NPN transistor, thus switching power to the motor. In this case the voltage polarity on the motor will always be the same direction, so the motor may only turn in one direction.

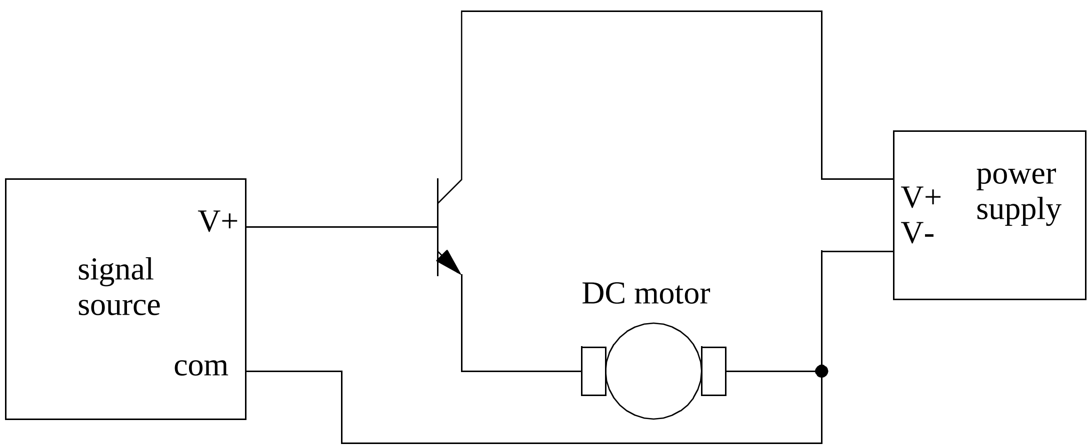


Figure 336 PWM Unidirectional Motor Control Circuit

ASIDE: When a motor is to be controlled with PWM in two directions the H-bridge circuit (shown below) is a popular choice. These can be built with individual components, or purchased as integrated circuits for smaller motors. To turn the motor in one direction the PWM signal is applied to the Va inputs, while the Vb inputs are held low. In this arrangement the positive voltage is at the left side of the motor. To reverse the direction the PWM signal is applied to the Vb inputs, while the Va inputs are held low. This applies the positive voltage to the right side of the motor.

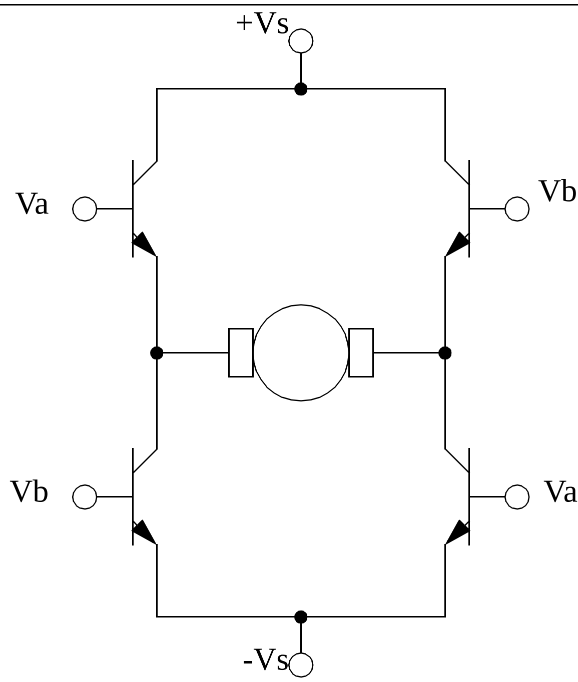


Figure 337 PWM Bidirectional Motor Control Circuit

15.2.2 AC Synchronous Motors

A synchronous motor has the windings on the stator. The rotor is normally a squirrel cage design. The squirrel cage is a cast aluminum core that when exposed to a changing magnetic field will set up an opposing field. When an AC voltage is applied to the stator coils an AC magnetic field is created, the squirrel cage sets up an opposing magnetic field and the resulting torque causes the motor to turn.

The motor is called synchronous because the rotor will turn at a frequency close to that of the applied voltage, but there is always some slip. It is possible to control the speed of the motor by controlling the frequency of the AC voltage. Synchronous motor drives control the speed of the motors by synthesizing a variable frequency AC waveform, as shown in Figure 338.

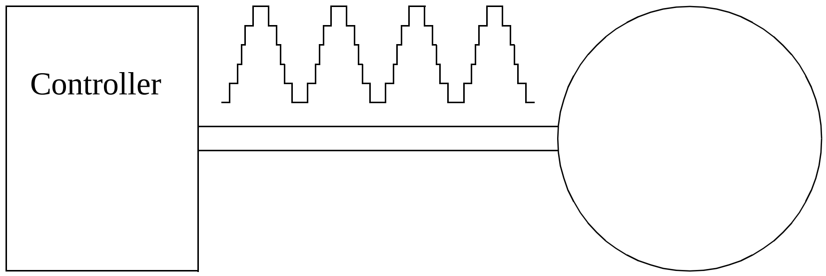


Figure 338 Synchronous AC Motor Speed Control

These drives should be used for applications that only require a single rotational direction. The torque speed curve for a typical induction motor is shown in Figure 339. When the motor is used with a fixed frequency AC source the synchronous speed of the motor will be the frequency of AC voltage divided by the number of poles in the motor. The motor actually has the maximum torque below the synchronous speed. For example a motor 2 pole motor might have a synchronous speed of (2*60*60/2) 3600 RPM, but be rated for 3520 RPM. When a feedback controller is used the issue of slip becomes insignificant.

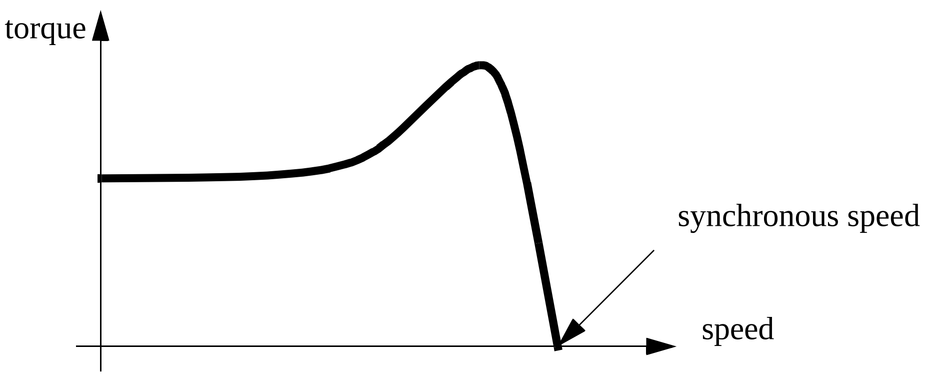


Figure 339 Torque Speed Curve for an Induction Motor

$$RPM = \frac{f120}{p}$$

where,

f = power frequency (60Hz typ.)

p = number of poles (2, 4, 6, etc...)

RPM = ideal motor speed in rotations per minute

15.2.3 Brushless DC Motors

Brushless motors use a permanent magnet on the rotor, and user wire windings on the stator. Therefore there is no need to use brushes and a commutator to switch the polarity of the voltage on the coil. The lack of brushes means that these motors require less maintenance than the brushed DC motors.

To continuously rotate these motors the current in the outer coils must alternate continuously. If the power supplied to the coils is an AC sinusoidal waveform, the motor will behave like an AC motor. The applied voltage can also be trapezoidal, which will give a similar effect. The changing waveforms are controller using position feedback from the motor to select switching times. The speed of the motor is proportional to the frequency of the signal.

A typical torque speed curve for a brushless motor is shown in Figure 340.

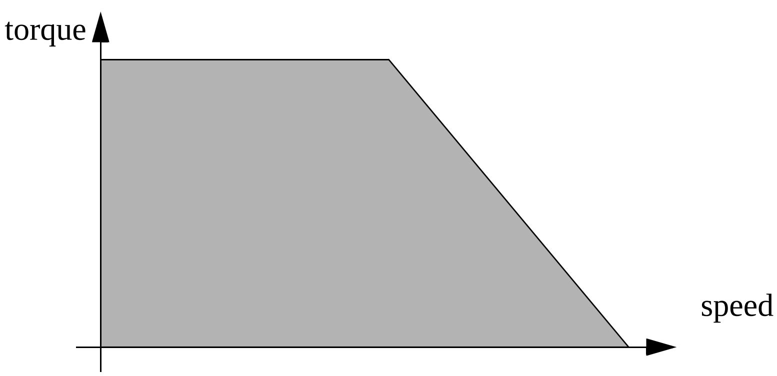


Figure 340 Torque Speed Curve for a Brushless DC Motor

15.2.4 Stepper Motors

Stepper motors are designed for positioning. They move one step at a time with a typical step size of 1.8 degrees giving 200 steps per revolution. Other motors are designed for step sizes of 2, 2.5, 5, 15 and 30 degrees.

There are two basic types of stepper motors, unipolar and bipolar, as shown in Figure 341. The unipolar uses center tapped windings and can use a single power supply. The bipolar motor is simpler but requires a positive and negative supply and more complex switching circuitry.

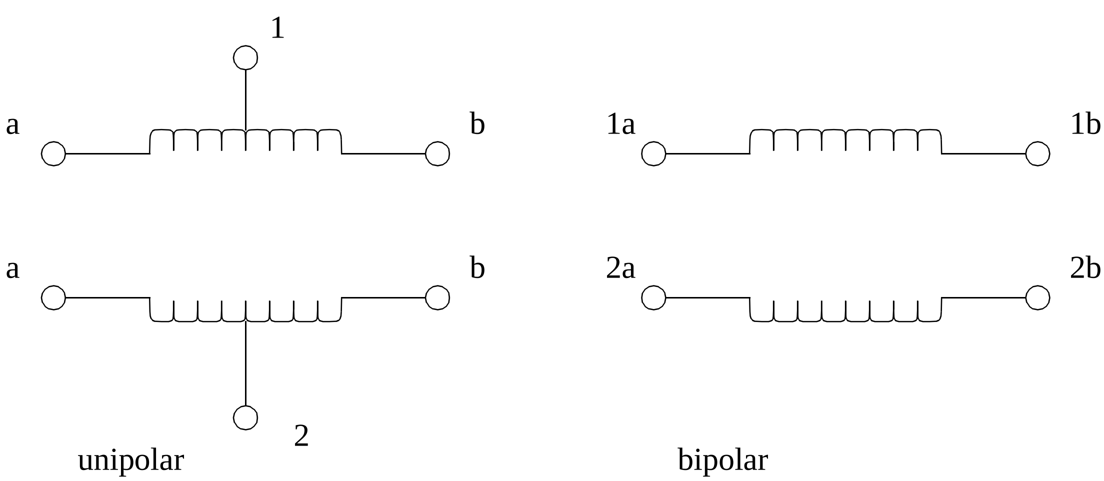
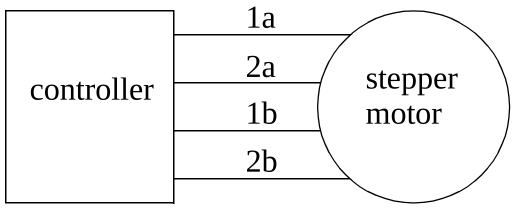


Figure 341 Unipolar and Bipolar Stepper Motor Windings

The motors are turned by applying different voltages at the motor terminals. The voltage change patterns for a unipolar motor are shown in Figure 342. For example, when the motor is turned on we might apply the voltages as shown in line 1. To rotate the motor we would then output the voltages on line 2, then 3, then 4, then 1, etc. Reversing the sequence causes the motor to turn in the opposite direction. The dynamics of the motor and load limit the maximum speed of switching, this is normally a few thousand steps per second. When not turning the output voltages are held to keep the motor in position.



Step	1a	2a	1b	2b
1	1	0	1	0
2	0	1	1	0
3	0	1	0	1
4	1	0	0	1

To turn the motor the phases are stepped through 1, 2, 3, 4, and then back to 1. To reverse the direction of the motor the sequence of steps can be reversed, eg. 4, 3, 2, 1, 4, ... If a set of outputs is kept on constantly the motor will be held in position.

Figure 342 Stepper Motor Control Sequence for a Unipolar Motor

Stepper motors do not require feedback except when used in high reliability applications and when the dynamic conditions could lead to slip. A stepper motor slips when the holding torque is overcome, or it is accelerated too fast. When the motor slips it will move a number of degrees from the current position. The slip cannot be detected without position feedback.

Stepper motors are relatively weak compared to other motor types. The torque speed curve for the motors is shown in Figure 343. In addition they have different static and dynamic holding torques. These motors are also prone to resonant conditions because of the stepped motion control.

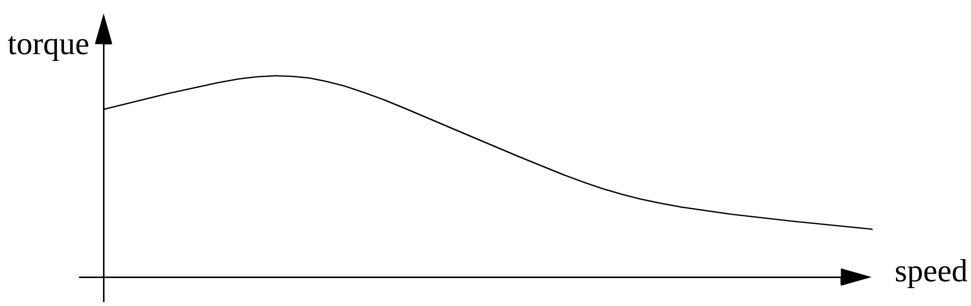


Figure 343 Stepper Motor Torque Speed Curve

The motors are used with controllers that perform many of the basic control functions. At the minimum a *translator* controller will take care of switching the coil voltages.

A more sophisticated *indexing* controller will accept motion parameters, such as distance, and convert them to individual steps. Other types of controllers also provide finer step resolutions with a process known as *microstepping*. This effectively divides the logical steps described in Figure 342 and converts them to sinusoidal steps.

translators - the user indicates maximum velocity and acceleration and a distance to move
indexer - the user indicates direction and number of steps to take
microstepping - each step is subdivided into smaller steps to give more resolution

15.3 HYDRAULICS

Hydraulic systems are used in applications requiring a large amount of force and slow speeds. When used for continuous actuation they are mainly used with position feedback. An example system is shown in Figure 344. The controller examines the position of the hydraulic system, and drivers a servo valve. This controls the flow of fluid to the actuator. The remainder of the provides the hydraulic power to drive the system.

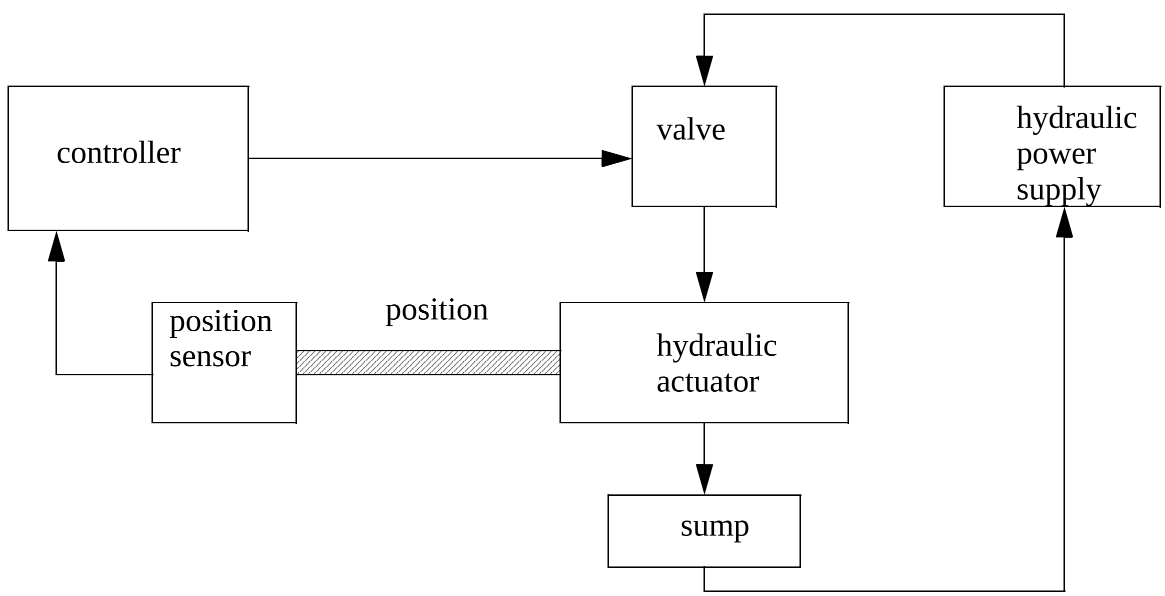


Figure 344 Hydraulic Servo System

The valve used in a hydraulic system is typically a solenoid controlled valve that is simply opened or closed. Newer, more expensive, valve designs use a scheme like pulse with modulation (PWM) which open/close the valve quickly to adjust the flow rate.

15.4 OTHER SYSTEMS

The continuous actuators discussed earlier in the chapter are the more common types. For the purposes of completeness additional actuators are listed and described briefly below.

Heaters - to control a heater with a continuous temperature a PWM scheme can be used to limit a DC voltage, or an SCR can be used to supply part of an AC waveform.
Pneumatics - air controlled systems can be used for positioning with suitable feedback. Velocities can also be controlled using fast acting valves.
Linear Motors - a linear motor works on the same principles as a normal rotary motor. The primary difference is that they have a limited travel and their cost is typically much higher than other linear actuators.
Ball Screws - rotation is converted to linear motion using balls screws. These are low friction screws that drive nuts filled with ball bearings. These are normally used with slides to bear mechanical loads.

15.5 SUMMARY

- AC motors work at higher speeds
- DC motors work over a range of speeds
- Motion control introduces velocity and acceleration limits to servo control
- Hydraulics make positioning easy

15.6 PRACTICE PROBLEMS

1. A stepping motor is to be used to drive each of the three linear axes of a cartesian coordinate robot. The motor output shaft will be connected to a screw thread with a screw pitch of 0.125”. It is desired that the control resolution of each of the axes be 0.025”
   a) to achieve this control resolution how many step angles are required on the stepper motor?
   b) What is the corresponding step angle?
   c) Determine the pulse rate that will be required to drive a given joint at a velocity of 3.0”/sec.

2. For the stepper motor in the previous question, a pulse train is to be generated by the robot controller.

a) How many pulses are required to rotate the motor through three complete revolutions?
b) If it is desired to rotate the motor at a speed of 25 rev/min, what pulse rate must be generated by the robot controller?

3. Explain the differences between stepper motors, variable frequency induction motors and DC motors using tables.

15.7 PRACTICE PROBLEM SOLUTIONS

1.

a) $$P = 0.125\left(\frac{in}{rot}\right) \qquad R = 0.025\frac{in}{step}$$

$$\theta = \frac{R}{P} = \frac{0.025\frac{in}{step}}{0.125\left(\frac{in}{rot}\right)} = 0.2\frac{rot}{step} \qquad \text{Thus} \qquad \frac{1}{0.2\frac{rot}{step}} = 5\frac{step}{rot}$$

b) $$\theta = 0.2\frac{rot}{step} = 72\frac{deg}{step}$$

c) $$PPS = \frac{3\frac{in}{s}}{0.025\frac{in}{step}} = 120\frac{steps}{s}$$

2.

a) $$pulses = (3rot)\left(5\frac{step}{rot}\right) = 15steps$$

b) $$\frac{pulses}{s} = \left(25\frac{rot}{min}\right)\left(5\frac{step}{rot}\right) = 125\frac{steps}{min} = 125\left(\frac{1min}{60s}\right)\frac{steps}{min} = 2.08\frac{step}{s}$$

3.

	speed	torque
stepper motor	very low speeds	low torque
vfd	limited speed range	good at rated speed
dc motor	wide range	decreases at higher speeds

15.8 ASSIGNMENT PROBLEMS

1. A stepper motor is to be used to actuate one joint of a robot arm in a light duty pick and place application. The step angle of the motor is 10 degrees. For each pulse received from the pulse train source the motor rotates through a distance of one step angle.
    a) What is the resolution of the stepper motor?
    b) Relate this value to the definitions of control resolution, spatial resolution, and accuracy, as discussed in class.
    c) For the stepper motor, a pulse train is to be generated by a motion controller. How many pulses are required to rotate the motor through three complete revolutions? If it is desired to rotate the motor at a speed of 25 rev/min, what pulse rate must be generated by the robot controller?

16. MOTION CONTROL

Topics:
- Motion controllers
- Motion profiles, trapezoidal and smooth
- Gain schedulers

Objectives:
- To understand single and multi axis motion control systems.

16.1 INTRODUCTION

A system with a feedback controller will attempt to drive the system to a state described by the desired input, such as a velocity. In earlier chapters we simply chose step inputs, ramp inputs and other simple inputs to determine the system response. In practical applications this setpoint needs to be generated automatically. A simple motion control system is used to generate setpoints over time.

An example of a motion control system is shown in Figure 345. The motion controller will accept commands or other inputs to generate a motion profile using parameters such as distance to move, maximum acceleration and maximum velocity. The motion profile is then used to generate a set of setpoints, and times they should be output. The setpoint scheduler will then use a realtime clock to output these setpoints to the motor drive.

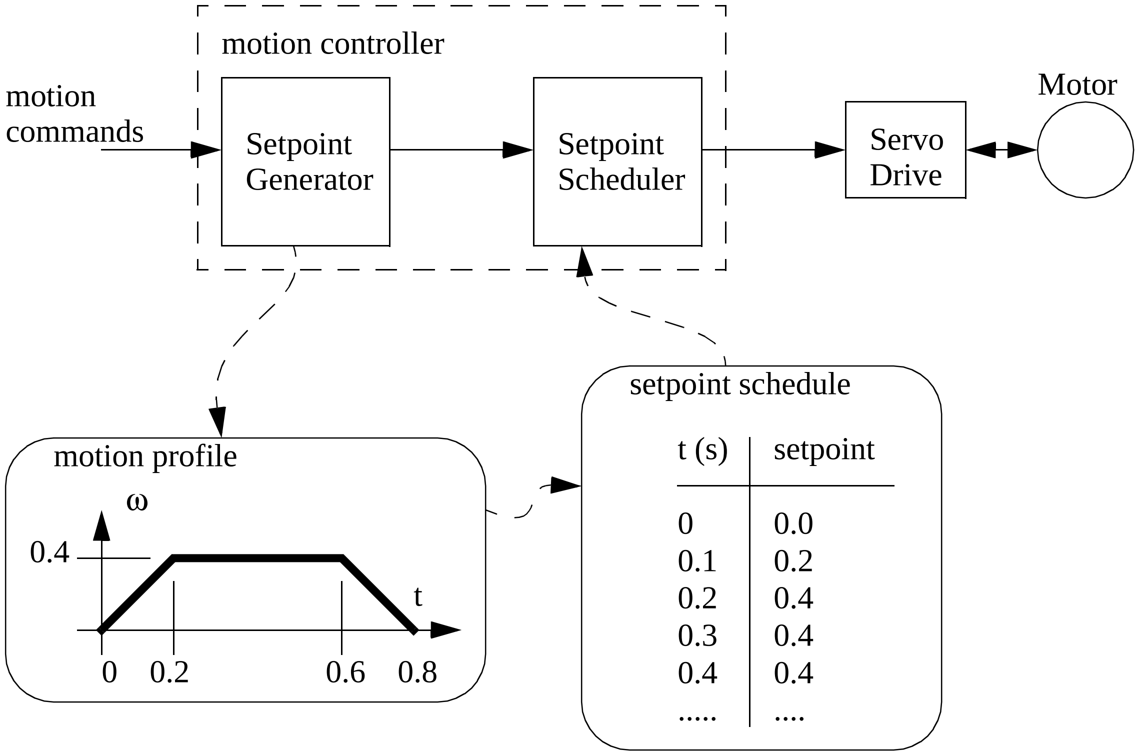


Figure 345 A motion controller

The combination of a motion controller, drive and actuator is called an axis. When there is more than one drive and actuator the system is said to have multiple axes. Complex motion control systems such as computer controlled milling machines (CNC) and robots have 3 to 6 axes which must be moved in coordination.

16.2 MOTION PROFILES

A simple example of a motion profile for a point-to-point motion is shown in Figure 346. In this example the motion starts at 20 deg and ends at 100 deg. (Note: in motion controllers it is more common to used encoder pulses, instead of degrees, for positions velocities, etc.) For position control we want a motion that has a velocity of zero at the start and end of the motion, and accelerates and decelerates smoothly.

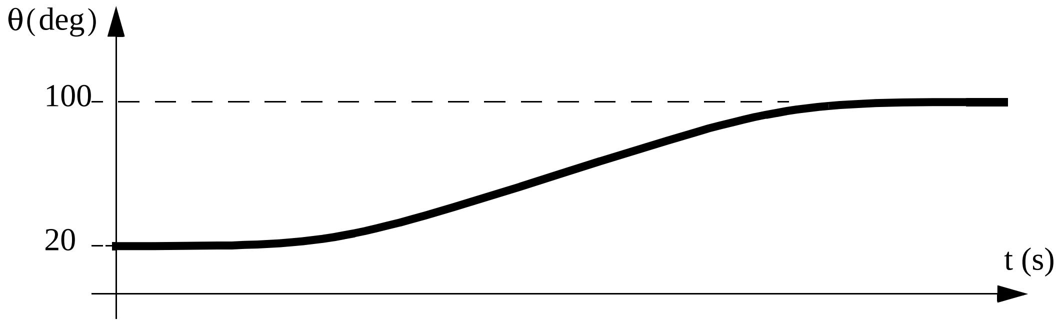
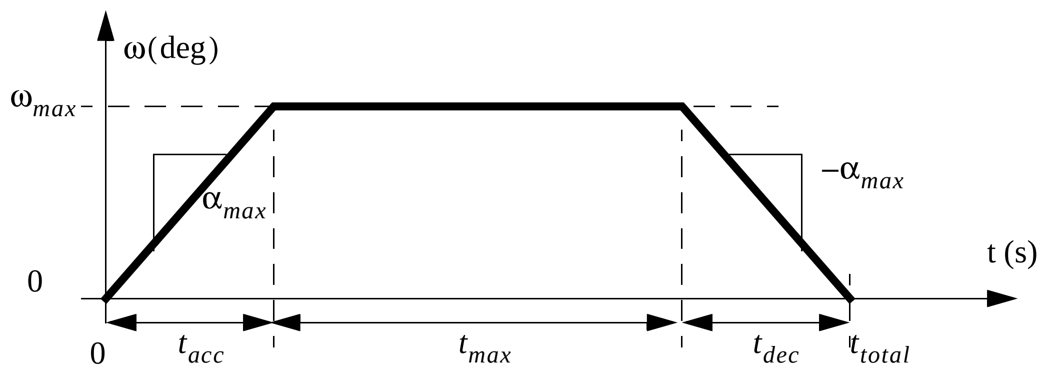


Figure 346 An example of a desired motion (position)

A trapezoidal velocity profile is shown in Figure 347. The area under the curve is the total distance moved. The slope of the initial and final ramp are the maximum acceleration and deceleration. The top level of the trapezoid is the maximum velocity. Some controllers allow the user to use the acceleration and deceleration times instead of the maximum acceleration and deceleration. This profile gives a continuous acceleration, but there will be a jerk (third order derivative) at the four sharp corners.



where,

ω_{max} = the maximum velocity

α_{max} = the maximum acceleration

t_{acc}, t_{dec} = the acceleration and deceleration times

t_{max} = the times at the maximum velocity

t_{total} = the total motion time

Figure 347 An example of a velocity profile

The basic relationships for these variables are shown in Figure 348. The equations can be used to find the acceleration and deceleration times. These equations can also be used to find the time at the maximum velocity. If this time is negative it indicates that the axis will not reach the maximum velocity, and the acceleration and deceleration times must be decreased. The resulting velocity profile will be a triangle.

$$t_{acc} = t_{dec} = \frac{\omega_{max}}{\alpha_{max}} \qquad (1)$$

$$t_{total} = t_{acc} + t_{max} + t_{dec} \qquad (2)$$

$$\theta = \frac{1}{2}t_{acc}\omega_{max} + t_{max}\omega_{max} + \frac{1}{2}t_{dec}\omega_{max} = \omega_{max}\left(\frac{t_{acc}}{2} + t_{max} + \frac{t_{dec}}{2}\right) \qquad (3)$$

$$t_{max} = \frac{|\theta|}{\omega_{max}} - \frac{|t_{acc}|}{2} - \frac{|t_{dec}|}{2} \qquad (4)$$

Note: if the time calculated in equation 4 is negative then the axis never reaches maximum velocity, and the velocity profile becomes a triangle.

Figure 348 Velocity profile basic relationships

For the example in Figure 349 the move starts at 100deg and ends at 20 deg. The acceleration and decelerations are completed in half a second. The system moves for 7.5 seconds at the maximum velocity.

Given, $\theta_{start} = 100deg$ $\theta_{end} = 20deg$

$$\omega_{max} = 10\frac{deg}{s} \qquad \alpha_{max} = 20\frac{deg}{s^2}$$

The times can be calculated as,

$$t_{acc} = t_{dec} = \frac{\omega_{max}}{\alpha_{max}} = \frac{10\frac{deg}{s}}{20\frac{deg}{s^2}} = 0.5s$$

$$\theta = \theta_{end} - \theta_{start} = 20deg - 100deg = -80deg$$

$$t_{max} = \frac{|\theta|}{\omega_{max}} - \frac{|t_{acc}|}{2} - \frac{|t_{dec}|}{2} = \frac{80deg}{10\frac{deg}{s}} - \frac{0.5s}{2} - \frac{0.5s}{2} = 7.5s$$

$$t_{total} = t_{acc} + t_{max} + t_{dec} = 0.5s + 7.5s + 0.5s = 8.5s$$

Figure 349 Velocity profile example

The motion example in Figure 350 is so short the axis never reaches the maximum velocity. This is made obvious by the negative time at maximum velocity. In place of this the acceleration and deceleration times can be calculated by using the basic acceleration position relationship. The result in this example is a motion that accelerates for 0.316s and then decelerates for the same time.

Given, $\theta_{start} = 20deg$ $\theta_{end} = 22deg$

$$\omega_{max} = 10\frac{deg}{s} \qquad \alpha_{max} = 20\frac{deg}{s^2}$$

The times can be calculated as,

$$t_{acc} = t_{dec} = \frac{\omega_{max}}{\alpha_{max}} = \frac{10\frac{deg}{s}}{20\frac{deg}{s^2}} = 0.5s$$

$$\theta = \theta_{end} - \theta_{start} = 22deg - 20deg = 2deg$$

$$t_{max} = \frac{|\theta|}{\omega_{max}} - \frac{|t_{acc}|}{2} - \frac{|t_{dec}|}{2} = \frac{2deg}{10\frac{deg}{s}} - \frac{0.5s}{2} - \frac{0.5s}{2} = -0.3s$$

The time was negative so the acceleration and deceleration times become,

$$\frac{\theta}{2} = \frac{1}{2}\alpha_{max}t_{acc}^2$$

$$t_{acc} = \sqrt{\frac{\theta}{\alpha_{max}}} = \sqrt{\frac{2deg}{20\frac{deg}{s^2}}} = \sqrt{0.1s^2} = 0.316s$$

$$t_{max} = 0s$$

Figure 350 Velocity profile example without reaching maximum velocity

Given the parameters calculated for the motion, the setpoints for motion can be calculated with the equations in Figure 351.

Assuming the motion starts at 0s,

$0s \le t < t_{acc}$

$$\theta(t) = \frac{1}{2}\alpha_{max}t^2 + \theta_{start}$$

$t_{acc} \le t < t_{acc} + t_{max}$

$$\theta(t) = \frac{1}{2}\alpha_{max}t_{acc}^{\ 2} + \omega_{max}(t - t_{acc}) + \theta_{start}$$

$t_{acc} + t_{max} \le t < t_{acc} + t_{max} + t_{dec}$

$$\theta(t) = \frac{1}{2}\alpha_{max}t_{acc}^{\ 2} + \omega_{max}t_{max} + \frac{1}{2}\alpha_{max}(t - t_{max} - t_{acc})^2 + \theta_{start}$$

$t_{acc} + t_{max} + t_{dec} \le t$

$$\theta(t) = \theta_{end}$$

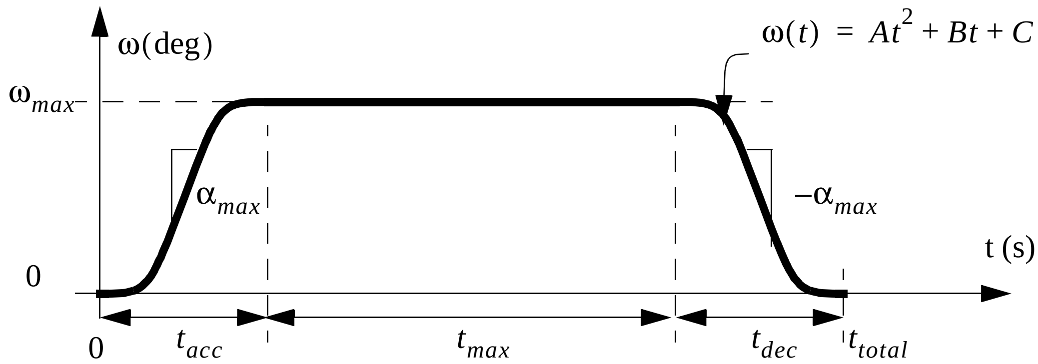
Figure 351 Generating points given motion parameters

A subroutine that implements these is shown in Figure 352. In this subroutine the time is looped with fixed time steps. The position setpoint values are put into the setpoint array, which will then be used elsewhere to guide the mechanism.

```
void generate_setpoint_table(double t_acc, double t_max, double t_step,
                              double vel_max, double acc_max,
                              double theta_start, double theta_end,
                              double setpoint[], int *count){
    double t, t_1, t_2, t_total;
    t_1 = t_acc;
    t_2 = t_acc + t_max;
    t_total = t_acc + t_max + t_acc;
    *count = 0;
    for(t = 0.0; t <= t_total; t += t_step){
        if( t < t_1){
            setpoint[*count] = 0.5*acc_max*t*t + theta_start;
        } else if ( (t >= t_1) && (t < t_2)){
            setpoint[*count] = 0.5*acc_max*t_acc*t_acc
                + vel_max*(t - t_1) + theta_start;
        } else if ( (t >= t_2) && (t < t_total)){
            setpoint[*count] = 0.5*acc_max*t_acc*t_acc
                + vel_max*(t_max)
                + 0.5*acc_max*(t-t_2)*(t-t_2) + theta_start;
        } else {
                setpoint[*count] = theta_end;
        }
        *count++;
    }
    setpoint[*count] = theta_end;
    *count++;
}
```

Figure 352 Subroutine for calculating motion setpoints

In some cases the jerk should be minimized. This can be achieved by replacing the ulceration ramps with a smooth polynomial, as shown in Figure 353. In this case two quadratic polynomials will be used for the acceleration, and another two for the deceleration.



where,

ω_{max} = the maximum velocity

α_{max} = the maximum acceleration

t_{acc}, t_{dec} = the acceleration and deceleration times

t_{max} = the times at the maximum velocity

t_{total} = the total motion time

Figure 353 A smooth velocity profile

An example of calculating the polynomial coefficients is given in Figure 354. The curve found is for the first half of the acceleration. It can then be used for the three other required curves.

Given, θ_{start} θ_{end} ω_{max} α_{max}

The constraints for the polynomial are,

$$\omega(0) = 0 \qquad \omega\left(\frac{t_{acc}}{2}\right) = \frac{\omega_{max}}{2}$$

$$\frac{d}{dt}\omega(0) = 0 \qquad \frac{d}{dt}\omega\left(\frac{t_{acc}}{2}\right) = \alpha_{max}$$

These can be used to calculate the polynomial coefficients,

$$0 = A0^2 + B0 + C \qquad \therefore C = 0$$

$$0 = 2A0 + B \qquad \therefore B = 0$$

$$\omega_{max} = At_{acc}^{\ 2} \qquad A = \frac{\omega_{max}}{t_{acc}^{\ 2}}$$

$$\alpha_{max} = 2At_{acc} \qquad A = \frac{\alpha_{max}}{2t_{acc}}$$

$$A = \frac{\omega_{max}}{t_{acc}^{\ 2}} = \frac{\alpha_{max}}{2t_{acc}} \qquad t_{acc} = \frac{2\omega_{max}}{\alpha_{max}}$$

$$A = \frac{\alpha_{max}}{2t_{acc}} = \frac{\alpha_{max}}{2\left(\frac{2\omega_{max}}{\alpha_{max}}\right)} = \frac{\alpha_{max}^2}{4\omega_{max}}$$

The equation for the first segment is,

$$\omega(t) = \frac{\alpha_{max}^2}{4\omega_{max}}t^2 \qquad 0 \le t < \frac{t_{acc}}{2}$$

The equation for the second segment can be found using the first segment,

$$\omega(t) = \omega_{max} - \frac{\alpha_{max}^2}{4\omega_{max}}(t_{acc} - t)^2$$

$$\omega(t) = \omega_{max} - \frac{\alpha_{max}^2}{4\omega_{max}}(t^2 - 2t_{acc}t + t_{acc}^2) \qquad \frac{t_{acc}}{2} \le t < t_{acc}$$

Figure 354 A smooth velocity profile example

The distance covered during acceleration, the area under the curves, is,

$$\theta_{acc} = \int_0^{\frac{t_{acc}}{2}} \frac{\alpha_{max}^2}{4\omega_{max}} t^2 dt + \int_{\frac{t_{acc}}{2}}^{t_{acc}} \left(\omega_{max} - \frac{\alpha_{max}^2}{4\omega_{max}} (t^2 - 2t_{acc}t + t_{acc}^2) \right) dt$$

$$\theta_{acc} = \frac{\alpha_{max}^2}{12\omega_{max}} t^3 \Bigg|_0^{\frac{t_{acc}}{2}} + \left(\omega_{max}t - \frac{\alpha_{max}^2}{4\omega_{max}} \left(\frac{t^3}{3} - t_{acc}t^2 + t_{acc}^2 t \right) \right) \Bigg|_{\frac{t_{acc}}{2}}^{t_{acc}}$$

$${}_{acc} = \frac{\alpha_{max}^2}{12\omega_{max}} \frac{t_{acc}^3}{8} + \omega_{max}t_{acc} - \frac{\alpha_{max}^2}{4\omega_{max}} \left(\frac{t_{acc}^3}{3} - t_{acc}^3 + t_{acc}^3 \right) - \omega_{max}\frac{t_{acc}}{2} + \frac{\alpha_{max}^2}{4\omega_{max}} \left(\frac{t_{acc}^3}{24} - \frac{t_{acc}^3}{4} + \frac{t_{acc}^3}{2} \right)$$

$$\theta_{acc} = \frac{\alpha_{max}^2}{96\omega_{max}} t_{acc}^3 + \frac{\omega_{max}t_{acc}}{2} - \frac{\alpha_{max}^2}{12\omega_{max}} t_{acc}^3 - \frac{7\alpha_{max}^2}{96\omega_{max}} t_{acc}^3$$

$$\theta_{acc} = \frac{-14\alpha_{max}^2}{96\omega_{max}} t_{acc}^3 + \frac{\omega_{max}t_{acc}}{2}$$

so the time required at the maximum velocity is,

$$t_{max} = \frac{(\theta - 2\theta_{acc})}{\omega_{max}}$$

Figure 355 A smooth velocity profile example (cont'd)

16.3 MULTI AXIS MOTION

In a machine with multiple axes the motions of individual axes must often be coordinated. A simple example would be a robot that needed to move two joints to reach a new position. We could extend the motion of the slower joints so that the motion of each joint would begin and end together.

16.3.1 Slew Motion

When the individual axis of a machine are not coordinated this is known as slew motion. Each of the axes will start moving at the same time, but finish at separate times. Consider the example in Figure 356. A three axis motion is required from the starting angles of (40, 80, -40)deg, and must end at (120, 0, 0)deg. The maximum absolute accelerations and decelerations are (50, 100, 150) degrees/sec/sec, and the maximum velocities are (20, 40, 50) degrees/sec.

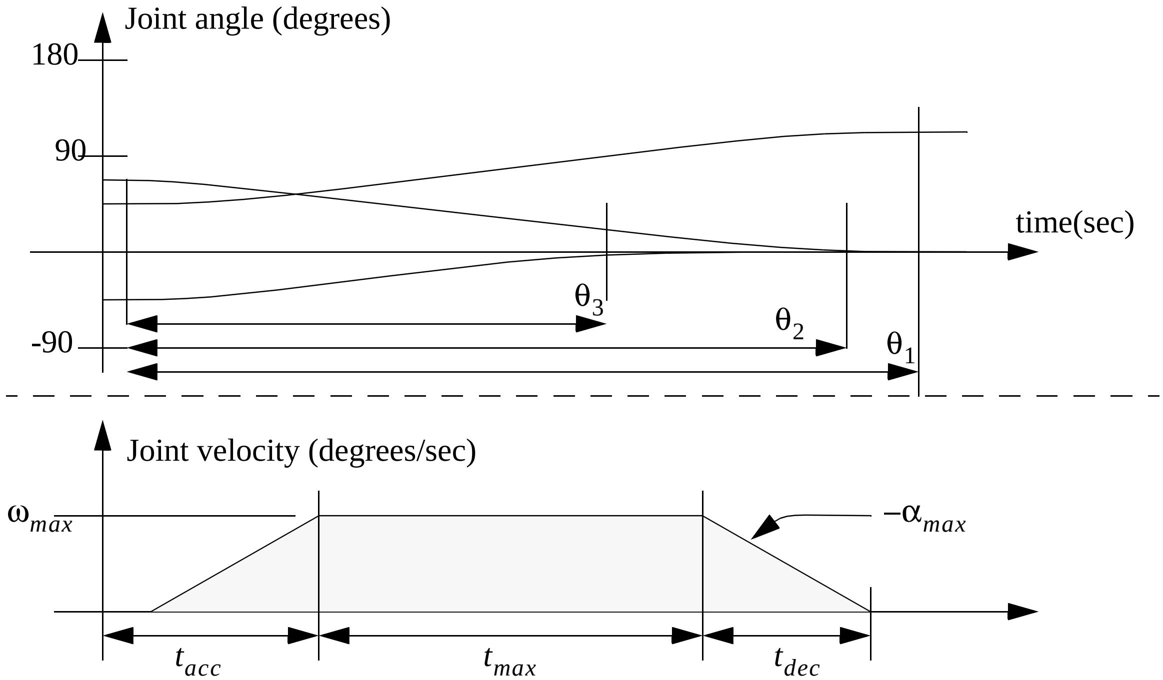


Figure 356 Multi-axis slew motion

The calculations for the motion parameters are shown in Figure 357. These are done in vector format for simplicity. All of the joints reach the maximum acceleration. The fastest motion is complete in 1.13s, while the longest motion takes 4.4s.

The area under the velocity curve is the distance (angle in this case) travelled. First we can determine the distance covered during acceleration, and deceleration and the time during acceleration, and deceleration.

$$t_{acc} = t_{dec} = \frac{\omega_{max}}{\alpha_{max}} = \left(\frac{20}{50}, \frac{40}{100}, \frac{50}{150}\right) = (0.4, 0.4, 0.333)\text{sec.}$$

$$\theta_{acc.} = \theta_{dec.} = \frac{t_{acc}\omega_{max.vel.}}{2} = \left(\frac{0.4(20)}{2}, \frac{0.4(40)}{2}, \frac{0.333(50)}{2}\right) = (4, 8, 8.33)\text{deg.}$$

The next step is to examine the moves specified,

$$\theta_{move} = \theta_{end} - \theta_{start} = (120 - 40, 0 - 80, 0 - (-40)) = (80, -80, 40)\text{deg.}$$

Remove the angles covered during accel./deccel., and find the travel time at maximum velocity.

$$t_{max} = \frac{|\theta_{move}| - 2\theta_{acc}}{\omega_{max}} = \left(\frac{80 - 2(4)}{20}, \frac{80 - 2(8)}{40}, \frac{40 - 2(8.333)}{50}\right)$$

$$t_{max} = (3.6, 1.6, 0.46668)\text{sec.}$$

$$t_{total} = t_{acc} + t_{max} + t_{dec} = (4.4, 2.4, 1.13)s$$

Figure 357 Calculated times for the slew motion

16.3.1.1 - Interpolated Motion

In interpolated motion the faster joints are slowed so that they finish in coordination with the slowest. This is essential in devices such as CNC milling machines. If this did not occur a straight line cut in the x-y plane would actually be two straight lines. The slew motion example can be extended to be slew motion where all joints finish their motion at 4.4s. This can be done for each joint by finding a multiplying factor to reduce accelerations and velocities, and increase times, as shown in Figure 358.

The longest time is 4.4, and this is used to calculate adjustment factors.

$$F = \frac{(4.4s, 2.4s, 1.13s)}{4.4s} = (1, 0.55, 0.26)$$

These can be used to calculate new maximum accelerations, velocities and times.

$$\omega_{max} = (1(20), 0.55(40), 0.26(50)) = (20, 22, 13)$$

$$\alpha_{max} = (1^2(50), 0.55^2(100), 0.26^2(150)) = (50, 30.25, 10.14)$$

$$t_{acc} = \left(\frac{0.4}{1}, \frac{0.4}{0.55}, \frac{0.333}{0.26}\right) = (0.4, 0.73, 1.28)$$

$$t_{max} = \left(\frac{3.6}{1}, \frac{1.6}{0.55}, \frac{0.467}{0.26}\right) = (3.6, 2.91, 1.80)$$

Figure 358 Interpolated motion

16.3.2 Motion Scheduling

After the setpoint schedule has been developed, it is executed by the setpoint scheduler. The setpoint scheduler will use a clock to determine when an output setpoint should be updated. A diagram of a scheduler is shown in Figure 359. In this system the setpoint scheduler is an interrupt driven subroutine that compares the system clock to the total motion time. When enough time has elapsed the routine will move to the next value in the setpoint table. The frequency of the interrupt clock should be smaller than or equal to the time steps used to calculate the setpoints. The servo drive is implemented with an algorithm such a PID control.

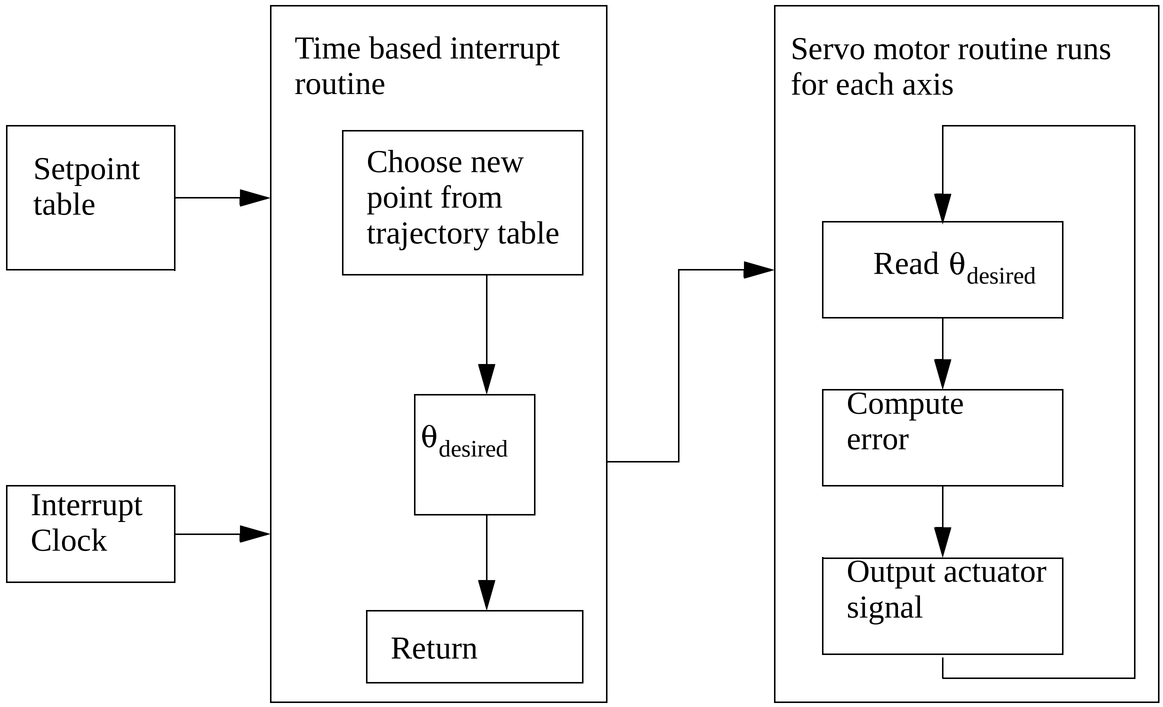


Figure 359 A setpoint scheduler

The output from the scheduler updates every time step. This then leads to a situation where the axis is always chasing the target value. This leads to small errors, as shown in Figure 360.

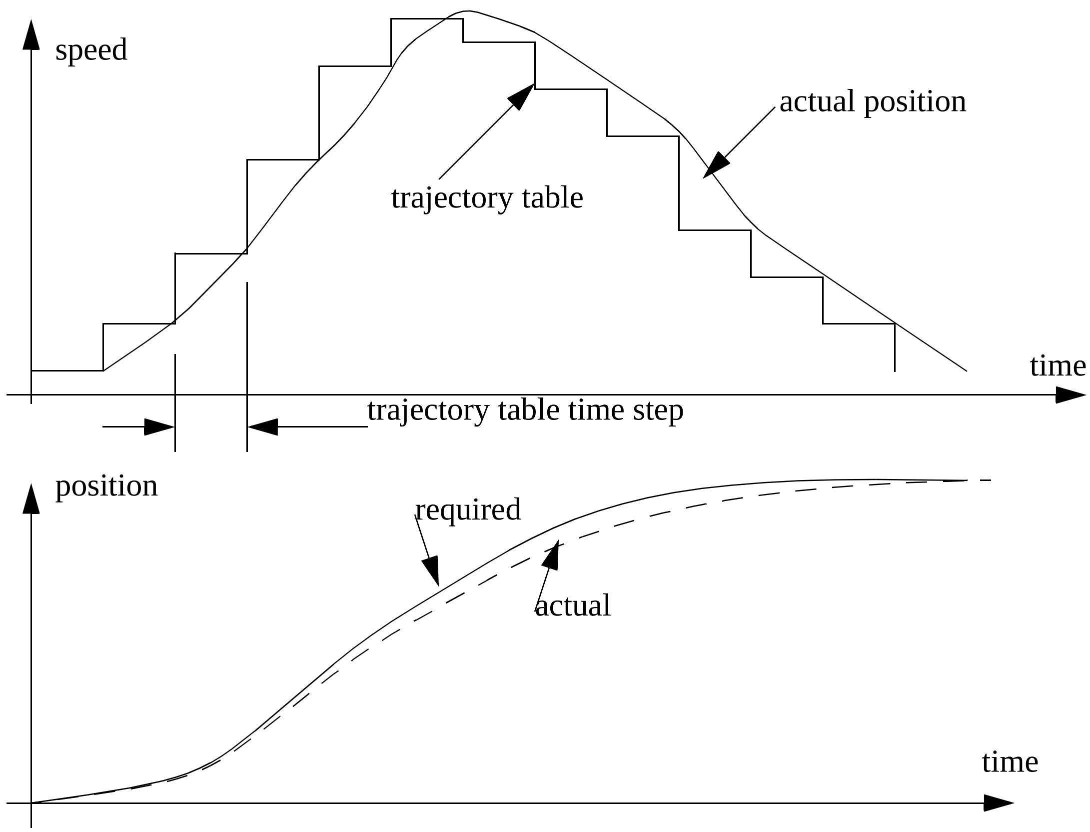


Figure 360 Errors in path following

16.4 SUMMARY

- Axis limits can be used to calculate motion profiles.
- Trapezoidal and smooth motion profiles were presented.
- Motion profiles can be used to generate setpoint tables.
- Values from the setpoints can then be output by a scheduler to drive an axis.

16.5 PRACTICE PROBLEMS

1. a) Develop a motion profile for a joint that moves from -100 degrees to 100 degrees with a maximum velocity of 20 deg/s and a maximum acceleration of 100deg/s/s. b) Develop a set-point table that has values for positions every 0.5 seconds for the entire motion.

(ans. Given,

$$\omega_{max} = 20\frac{deg}{s} \qquad \alpha_{max} = 100\frac{deg}{s^2} \qquad \Delta\theta = 100-(-100) = 200°$$

The motion times can be calculated.

$$t_{acc} = t_{dec} = \frac{\omega_{max}}{\alpha_{max}} = \frac{20\frac{deg}{s}}{100\frac{deg}{s^2}} = 0.2s$$

$$t_{max} = \frac{\Delta\theta - \omega_{max}t_{acc}}{\omega_{max}} = \frac{200deg - 20\frac{deg}{s}0.2s}{20\frac{deg}{s}} = 9.8s$$

$$t_{total} = t_{acc} + t_{max} + t_{dec} = 0.2s + 9.8s + 0.2s = 10.2s$$

t (s)	angle (deg)
0.0	-100
0.5	-92
1.0	-82
1.5	-72
2.0	-62
2.5	-52
3.0	-42
3.5	-32
4.0	-22
4.5	-12
5.0	-2
5.5	8
6.0	18
6.5	28
7.0	38
7.5	48
8.0	58
8.5	68
9.0	78
9.5	88
10.0	98
10.5	100
11.0	100

$$\theta_{0.5s} = \frac{1}{2}100\frac{deg}{s^2}(0.2s)^2 + 20\frac{deg}{s}(0.5s - 0.2s) - 100deg$$

$$\theta_{0.5s} = -92deg$$

$$\theta_{1.0s} = \theta_{0.5s} + 20\frac{deg}{s}(0.5s) = -92deg + 10deg$$

2. Find a smooth path for a robot joint that will turn from θ= 75° to θ = -35° in 10 seconds. Do this

by developing an equation then calculating points every 1.0 seconds along the path for a total motion time of 10 seconds.

3. Paths are to be planned for a three axis motion controller. Each of the joints has a maximum velocity of 20 deg/s, and a maximum acceleration of 30 deg/s/s. Assuming all of the joints start at an angle of 0 degrees. Joints 1, 2 and 3 move to 40 deg, 100deg and -50deg respectively. Develop the motion profiles assuming,
    a) slew motion
    b) interpolated motion

4. Develop a smooth velocity profile for moving a cutting tool that starts at 1000 inches and moves to -1000 inches. The maximum velocity is 100 in/s and the maximum acceleration is 50in/s/s.

17. ELECTROMECHANICAL SYSTEMS

Topics:

Objectives:

17.1 INTRODUCTION

- Magnetic fields and forces are extremely useful. The fields can allow energy storage, or transmit forces.

-

17.2 MATHEMATICAL PROPERTIES

- Magnetic fields have direction. As a result we must pay special attention to directions, and vector calculations.

17.2.1 Induction

- Magnetic fields pass through space.

• resistivity of materials decreases with temperature

• Flux density,

$$B = \frac{I}{2\pi r}$$ For an infinitely long straight conductor

where,

B = Flux density$\left(\frac{Wb}{m^2} \text{ or } Tesla\right)$

I = Current in the conductor (A)

r = radial distance from the conductor

• Flux and flux density,

$$\Phi = \frac{B}{A}$$

where,

Φ = Flux density (Wb)

A = Cross section area

• Permeability,

$$\mu = \frac{B}{H}$$

$$\mu_0 = 4\pi 10^{-7}\frac{H}{m}$$

$$\mu = \mu_r \mu_0$$

where,

H = Magnetic field intensity$\left(\frac{A}{m}\right)$

μ_0 = permeability of free space

μ_r = relative permeability of a material

μ = permeability of a material

• Permeability is approximately linear for smaller electric fields, but with larger magnetic fields the materials saturate and the value of B reaches a maximum value.

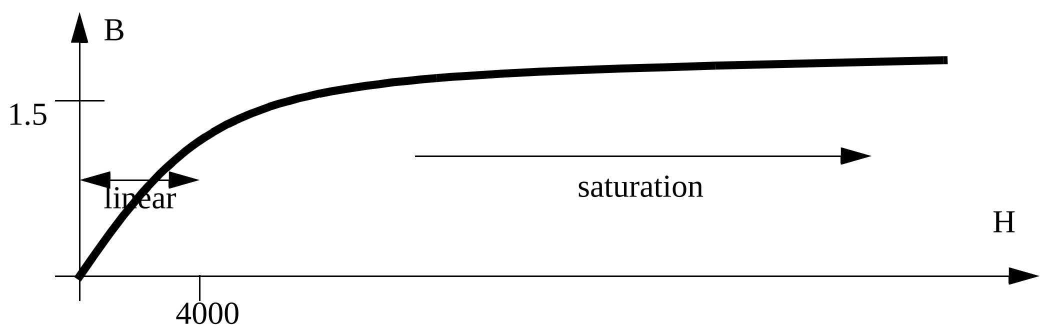


Figure 361 Saturation for a mild steel (approximately)

• When a material is used out of the saturation region the permeabilities may be written as reluctances,

$$R = \frac{L}{\mu A}$$

where,

R = reluctance of a magnetic path

L = length of a magnetic path

A = cross section area of a magnetic path

• Faraday's law,

$$e = N\frac{d}{dt}\Phi$$ For a coil

where,

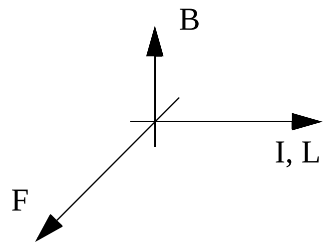
e = the potential voltage across the coil

N = the number of turns in the coil

• The basic property of induction is that it will (in the presence of a magnetic field) convert a changing current flowing in a conductor to a force or convert a force to a current flow from a change in the current or the path.

$$F = (I \times B)L$$

$$F = (L \times B)I$$



NOTE: As with all cross products we can use the right hand rule here.

where,

L = conductor length

F = force (N)

The FBD/schematic equivalent is,

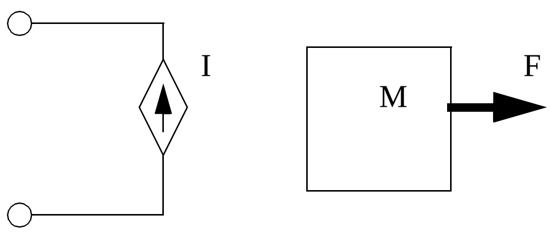
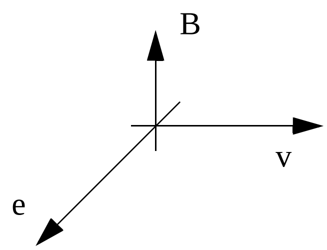


Figure 362 The current and force relationship

• We will also experience an induced current caused by a conductor moving in a magnetic field. This is also called emf (Electro-Motive Force)

$$e_m = (v \times B)L$$



where,

e_m = electromotive force (V)
ϕ = magnetic flux (Wb - webers)
v = velocity of conductor

The FBD/schematic equivalent

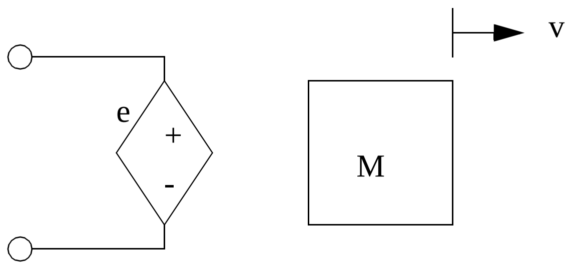


Figure 363 Electromagnetically induced voltage

• Hysteresis

17.3 EXAMPLE SYSTEMS

• These systems are very common, take for example a DC motor. The simplest motor has a square conductor loop rotating in a magnetic field. By applying voltage the wires push back against the magnetic field.

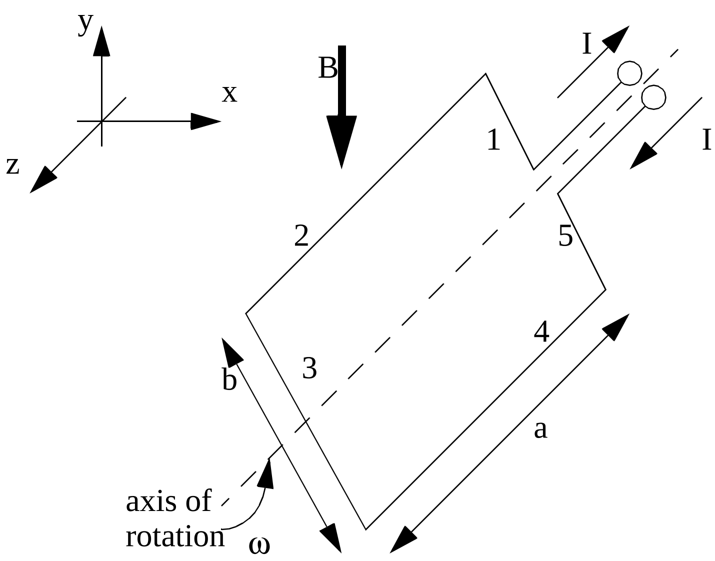


Figure 364 A motor winding in a magnetic field

For wire 3,

$$P_3 = \begin{bmatrix} r\cos(\omega t) \\ r\sin(\omega t) \\ -\frac{b}{2}\text{to}\frac{b}{2} \end{bmatrix} \qquad V_3 = \begin{bmatrix} -r\omega\sin(\omega t) \\ r\omega\cos(\omega t) \\ 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ -B \\ 0 \end{bmatrix}$$

$$de_{m3} = (V \times B)dL$$

$$e_{m3} = \int_{-\frac{b}{2}}^{\frac{b}{2}} \begin{bmatrix} -r\omega\sin(\omega t) \\ r\omega\cos(\omega t) \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ -B \\ 0 \end{bmatrix} dr$$

$$e_{m3} = \int_{-\frac{b}{2}}^{\frac{b}{2}} \begin{bmatrix} 0 \\ 0 \\ Br\omega\sin(\omega t) \end{bmatrix} dr$$

$$e_{m3} = B\frac{r^2}{2}\omega\sin(\omega t)\Big|_{-\frac{b}{2}}^{\frac{b}{2}} = B\omega\sin(\omega t)\left[\left(\frac{b}{2}\right) - \left(-\frac{b}{2}\right)^2\right] = 0$$

For wires 1 and 5,

By symmetry, the two wires together will act like wire 3. Therefore they both have an emf (voltage) of 0V.

$$e_{m1} = e_{m5} = 0V$$

Figure 365 Calculation of the motor torque

For wire 2 (and 4 by symmetry),

$$P_2 = \begin{bmatrix} \frac{b}{2}\cos(\omega t) \\ \frac{b}{2}\sin(\omega t) \\ 0 \text{to} a \end{bmatrix} \qquad V_2 = \begin{bmatrix} -\frac{b}{2}\omega\sin(\omega t) \\ \frac{b}{2}\omega\cos(\omega t) \\ 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ -B \\ 0 \end{bmatrix}$$

$$de_{m2} = (V \times B)dL$$

$$e_{m2} = \int_0^a \begin{bmatrix} -\frac{b}{2}\omega\sin(\omega t) \\ \frac{b}{2}\omega\cos(\omega t) \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ -B \\ 0 \end{bmatrix} dl$$

$$e_{m2} = \int_0^a \begin{bmatrix} 0 \\ 0 \\ B\frac{b}{2}\omega\cos(\omega t) \end{bmatrix} dl = aB\frac{b}{2}\cos(\omega t)$$

$$e_{m4} = e_{m2}$$

For the total loop,

$$e_m = e_{m1} + e_{m2} + e_{m3} + e_{m4} + e_{m5}$$

$$e_m = 0 + aB\frac{b}{2}\omega\cos(\omega t) + 0 + aB\frac{b}{2}\omega\cos(\omega t) + 0$$

$$e_m = aBb\omega\cos(\omega t)$$

Figure 366 Calculation of the motor torque (continued)

• As can be seen in the previous equation, as the loop is rotated a voltage will be generated (a generator), or a given voltage will cause the loop to rotate (motor).

• In this arrangement we have to change the polarity on the coil every 180 deg of rotation. If we didn't do this the loop the torque on the loop would reverse for half the motion. The result would be that the motor would swing back and forth, but not rotate fully. To make the torque push consistently in the same direction we need to reverse the applied voltage for half the cycle. The device that does this is called a commutator. It is basically a split ring with brushes.

$$e_m = aBb\omega|\cos(\omega t)|$$

• Real motors also have more than a single winding (loop of wire). To add this into the equation we only need to multiply by the number of loops in the winding.

$$e_m = NaBb\omega|\cos(\omega t)|$$

• As with most devices the motor is coupled. This means that one change, say in torque/force will change the velocity and hence the voltage. But a change in voltage will also change the current in the windings, and hence the force, etc.

• Consider a motor that is braked with a constant friction load of Tf.

$$F_w = (I \times B)L = IBa$$

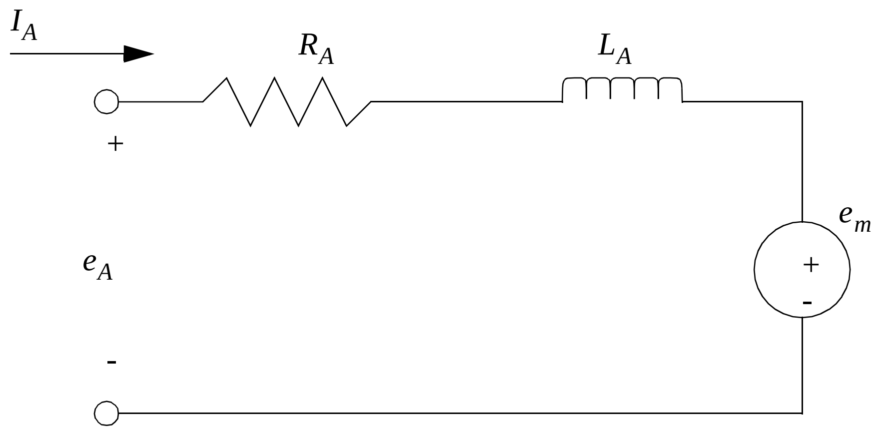
$$T_w = 2\left(F \times \frac{b}{2}\right) = Fb = IBab$$

$$\sum M = T_w - T_f = J\alpha$$

$$IBab - T_f = J\alpha$$

Figure 367 Calculation of the motor torque (continued)

• We still need to relate the voltage and current on the motor. The equivalent circuit for a motor shows the related components.



where,

I_A, e_A = voltage and current applied to the armature (motor supply)
R_A, L_A = equivalent resistance and inductance of windings

$$\sum V = e_A - I_A R_A - L_A \frac{d}{dt} I_A - e_m = 0$$

we can now add in the other equations,

$$e_A - I_A R_A - L_A \frac{d}{dt} I_A - NaBb\omega|\cos(\omega t)| = 0$$

and recall the previous equation,

$$IBab - T_f = J\alpha$$

Figure 368 Calculation of the motor torque (continued)

• Practice problem,

Write the transfer function relating the displacement ’x’ to the current ’I’

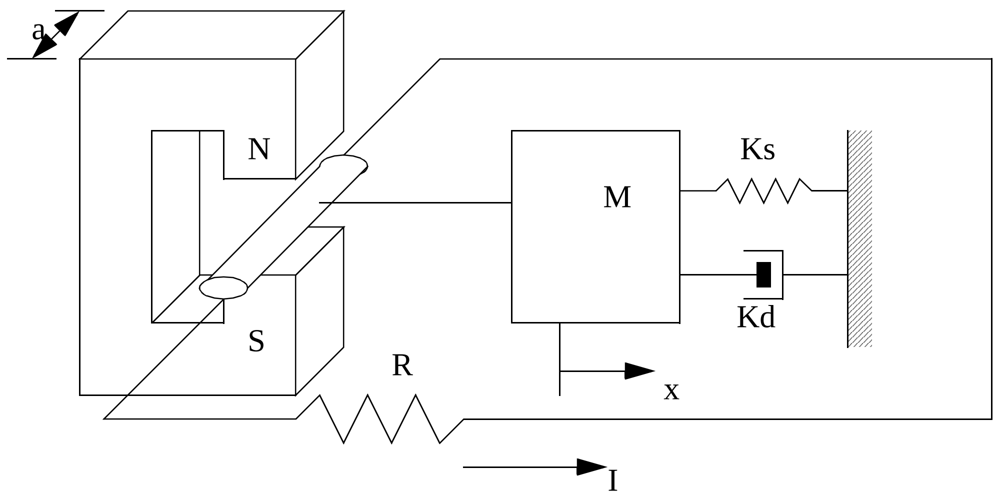


Figure 369 Drill problem: Electromotive force

• Consider a motor with a separately excited magnetic field (instead of a permanent magnet there is a coil that needs a voltage to create a magnetic field). The model is similar to the previous motor models, but the second coil makes the model highly nonlinear.

17.4 SUMMARY

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17.5 PRACTICE PROBLEMS

1.

18. FLUID SYSTEMS

Topics:

Objectives:

18.1 SUMMARY

• Fluids are a popular method for transmitting power (hydraulics). Basically, by applying a pressure at one point, we can induce flow through a pipe/value/orifice.

18.2 MATHEMATICAL PROPERTIES

• Fluids do work when we have a differential pressure on a surface. The pressure may be expressed as an absolute value. More correctly we should consider atmospheric pressure (gauge pressure).

• When we deal with fluids we approximate them as incompressible.

• Fluids observe some basic laws,

w = flow rate

p = pressure

18.2.1 Resistance

• If fluid flows freely, we say it is without resistance. In reality, every fluid flow experiences some resistance. Even a simple pipe has resistance. Of similar interest is the resistance of a value.

$$w = K\sqrt{\Delta p}$$

where,

w = flow rate through the pipe

Δp = pressure difference across valve/pipe

K = a constant specific to the pipe/valve/orifice

NOTE: In this case the relationship between pressure drop and flow are non-linear. We have two choices if we want to analyze this system.
1. We can do a non-linear analysis (e.g. integration)
2. We can approximate the equation with a linear equation. This is only good for operation near the chosen valve position. As the flow rate changes significantly the accuracy of the equation will decrease.

$$\Delta p \approx R(w - \bar{w})$$

$$R \approx \frac{\Delta(\Delta p)}{\Delta w} \qquad R = 2\frac{w}{K^2}$$

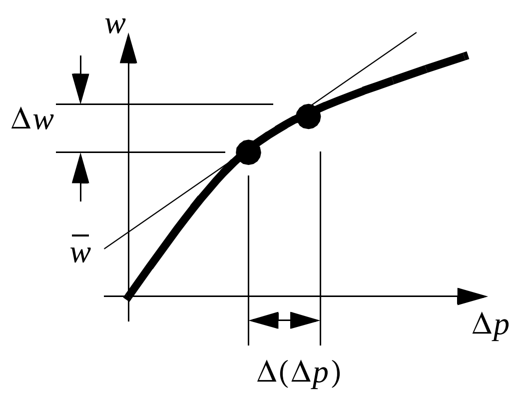
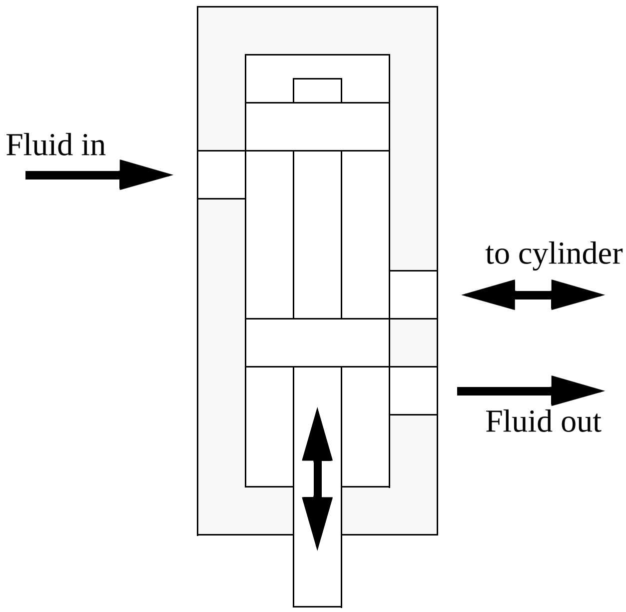


Figure 370 Fluid flow resistance

• Resistance may also result from valves. Valves usually restrict flow by reducing an area for fluid to flow through.

• A simple form of valve is a sliding plunger. The valve below is called a two way valve because it will allow fluid to flow in or out.



NOTE: For this type of valve the fluid in is typically under pressure. When the valve slide is up the pressure is applied to the cylinder, and causes it to actuate. The fluid out is typically not under pressure, and just returns fluid to a reservoir tank. When used this way a cylinder needs a spring, or some other force to return it and force the fluid out.

Figure 371 Fluid servo valve

• Four way valve allow fluid force to be applied in both directions of a cylinder motion.

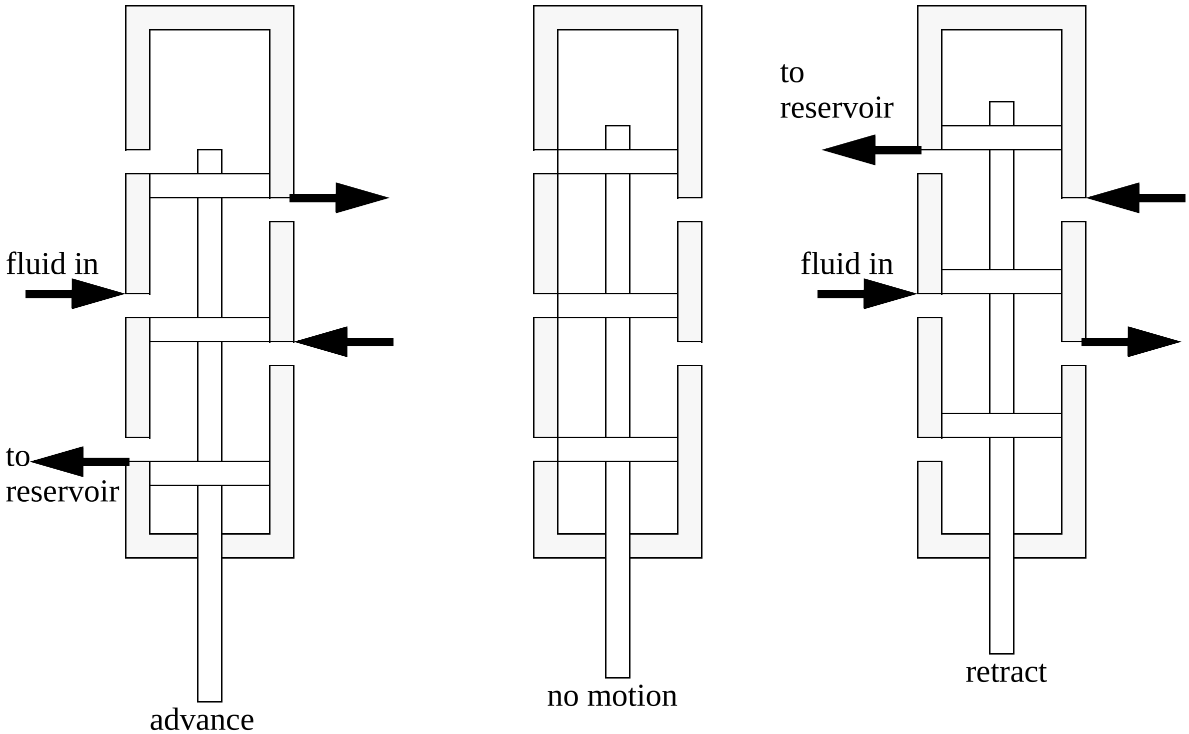
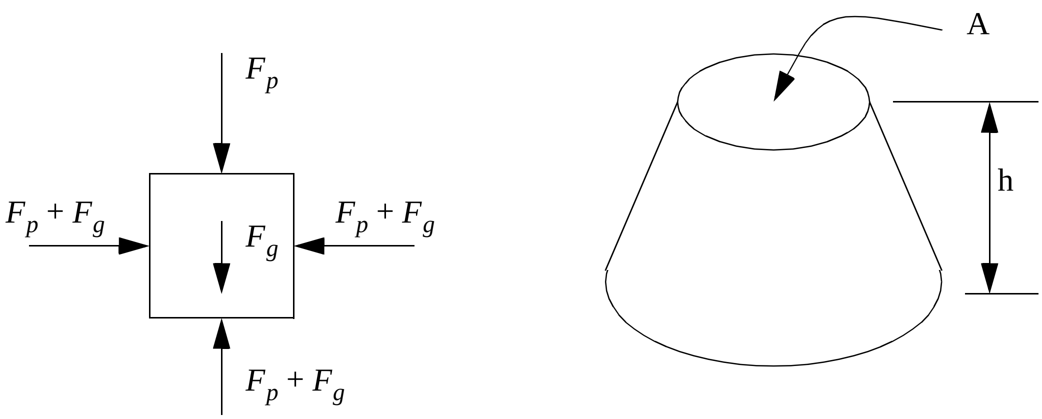


Figure 372 Fluid flow control valves

18.2.2 Capacitance

• Fluids are often stored in reservoirs or tanks. In a tank we have little pressure near the top, but at the bottom the mass of the fluid above creates a hydrostatic pressure. Other factors also affect the pressure, such as the shape of the tank, or whether or not the top of the tank is open.

• To calculate the pressure we need to integrate over the height of the fluid.



$$P = \frac{F_g}{A}$$

$$F_g = \rho V = \rho g A h$$

$$P = \frac{\rho g A h}{A} = \rho g h$$

Figure 373 Pressures on fluid elements

• Consider a tank as a capacitor. As fluid is added the height of the fluid rises, and the hydrostatic pressure increases. Hence we can pump fluid into a tank to store energy, and letting fluid out recovers the energy. A very common application of this principle is a municipal water tower. Water is pumped into these tanks. As consumers draw water through the system these tanks provide pressure to the system. When designing these tanks we should be careful to keep the cross section constant (e.g. a cylinder). If the cross section varies then the fluid pressure will not drop at a linear rate and you won't be able to use linear analysis techniques (i.e., Laplace and State Space).

• The mathematical equations for a constant cross section tank are,

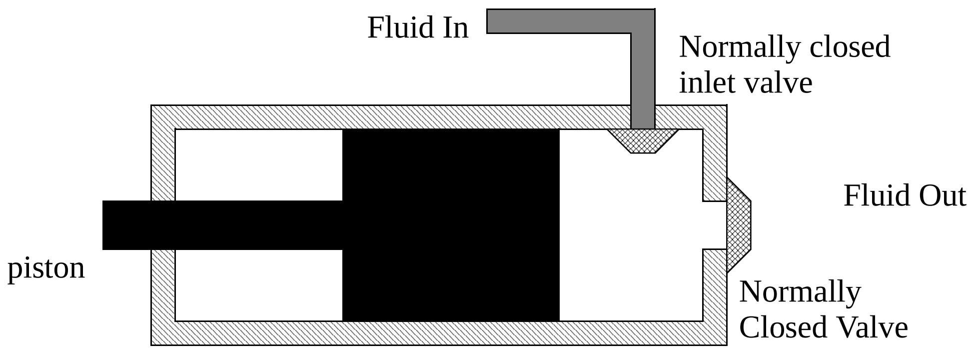
$$C = \frac{A}{\rho g}$$

$$w = C\frac{d}{dt}p$$

18.2.3 Power Sources

• As with most systems we need power sources. In hydraulics these are pumps that will provide pressure and/or flow to the system.

• One type of pump uses a piston.



In this common form of piston pump, the piston rod is drawn back creating suction that holds the valve closed, and pulls fluid into the chamber. When the cylinder is full of fluid the piston motion is reversed, creating a pressure, and forcing the inlet vale closed, and the outlet valve open, and the fluid is pumped out. The fluid volume can be controlled by using the cylinder size, and piston strokes

Figure 374 A piston driven hydraulic pump

• A geared hydraulic pump is pictured below. Other types use vanes and pistons.

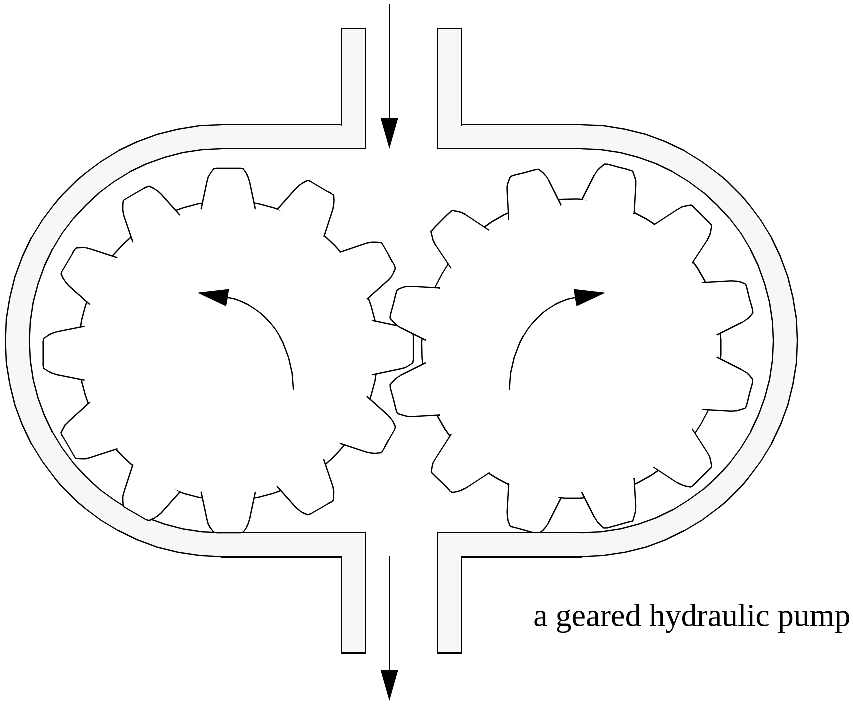


Figure 375 A gear driven hydraulic pump

• Vane based pumps can be used to create fluid flow. As the pump rotates the vanes move to keep a good seal with the outer pump wall. The displacement on the advance and return sides are unequal (aided by the sliding vanes). The relative displacement across the pump determines the fluid flow.

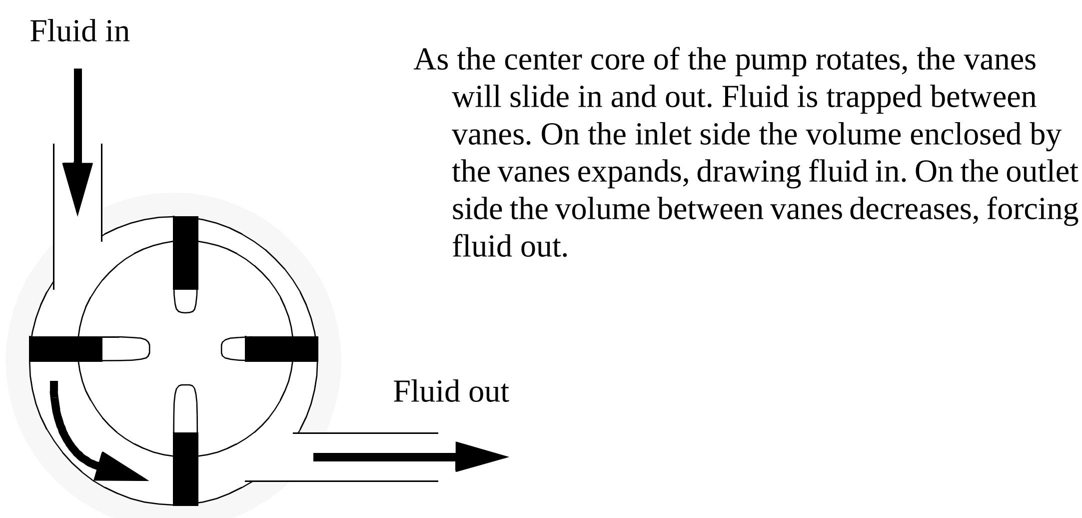


Figure 376 A vaned hydraulic pump

• As with the resistance of valves, these are not linear devices. It is essential that we linearize the devices. To do this we look at the pressure flow curves. (Note: most motors and engines have this problem)

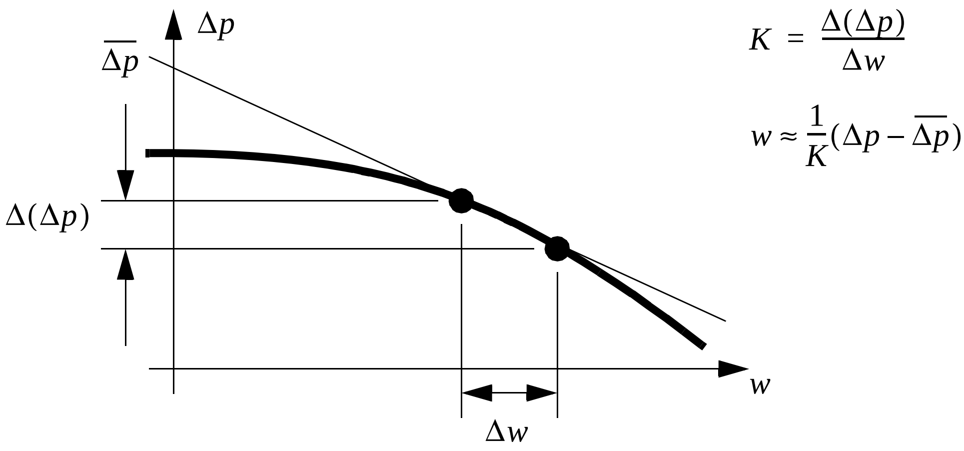


Figure 377 Linearizing a hydraulic valve ?????

18.3 EXAMPLE SYSTEMS

• We can model a simple hydraulic system using the elements from before. Consider the example below,

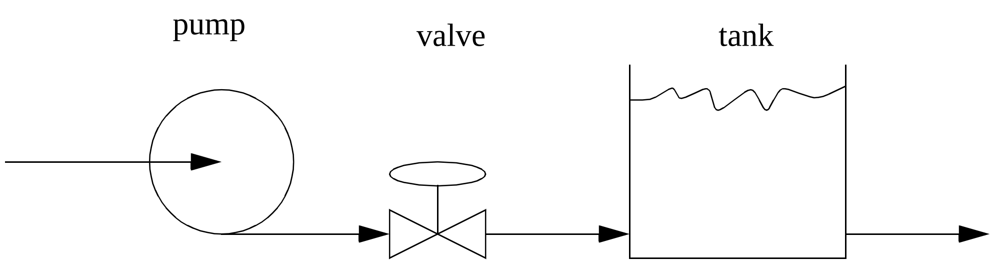
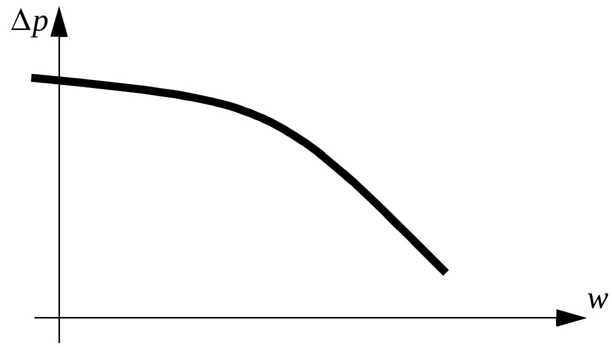
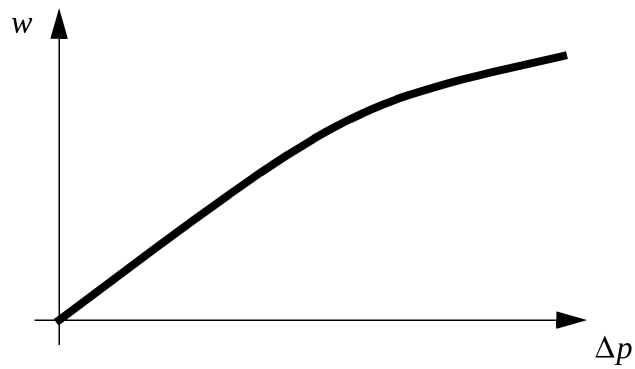


Figure 378 A hydraulic system example

For the pump the relationship is shown in the graph, and it will operate at the point marked.



For the valve the relationship is given below



The pipes are all equal length and have the relationships shown below.

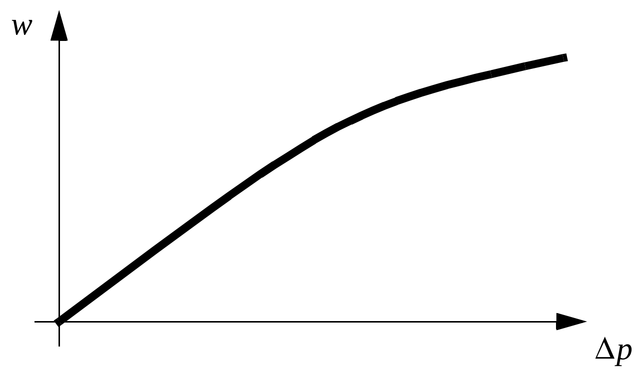


Figure 379 A hydraulic system example (continued)

18.4 SUMMARY

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18.5 PRACTICE PROBLEMS

1.

19. THERMAL SYSTEMS

Topics:

Objectives:

19.1 INTRODUCTION

• Energy can be stored and transferred in materials in a number of forms. Thermal energy (heat) is stored and transmitted through all forms of matter.

19.2 MATHEMATICAL PROPERTIES

19.2.1 Resistance

• A universal property of energy is that it constantly strives for equilibrium. This means that a concentration of energy will tend to dissipate. When material separates regions of different temperatures it will conduct heat energy at a rate proportional to the difference. Materials have a measurable conductivity or resistance (note: this is different than electrical conductivity and resistance.)

$$q = \frac{1}{R}(\theta_j - \theta_i) \qquad \frac{dq}{dl} = \frac{1}{R}\left(\frac{d\theta}{dl}\right)$$

where,

q = heat flow rate from j to i (J/s or watts)

R = thermal resistance

θ = temperature

$$R = \frac{d}{A\alpha}$$

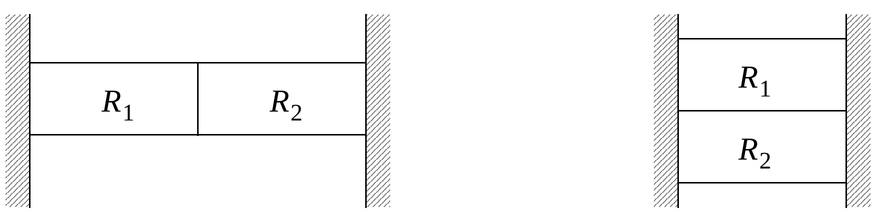
d = length of thermal conductor

A = cross section area of thermal conductor

α = thermal conductivity (W/mK)

Figure 380 Thermal resistance

• When dealing with thermal resistance there are many parallels to electrical resistance. The flow of heat (current) is proportional to the thermal difference (voltage). If we have thermal systems in parallel or series they add as normal resistors do.



$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} \qquad R_T = R_1 + R_2$$

Figure 381 Parallel and series thermal resistance

19.2.2 Capacitance

• Heat energy is absorbed at different rates in different materials - this rate is referred to as thermal capacitance. And as long as the material has no major changes in structure (i.e., gas-solid or phase transition) this number is relatively constant.

$$\Delta\theta = \frac{1}{C}(q_{in} - q_{out}) \qquad \frac{d\theta}{dt} = \frac{1}{C}\frac{dq}{dt}$$

where,

C = thermal capacitance

$$C = M\sigma$$

where,

M = mass of thermal body
σ = specific heat of material in mass

Figure 382 Thermal capacitance

• One consideration when dealing with heating capacitance is that the heat will not instantly disperse throughout the mass. When we want to increase the rate of heat absorption we can use a mixer (with a gas or fluid). A mixer is shown in the figure below. This mixer is just a rotating propeller that will cause the liquid to circulate.

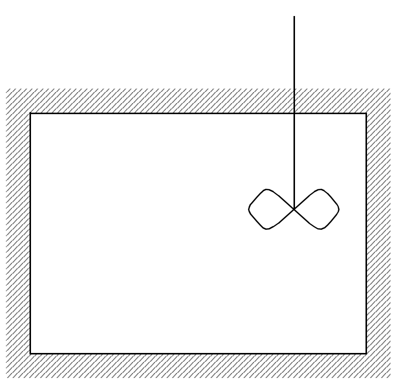


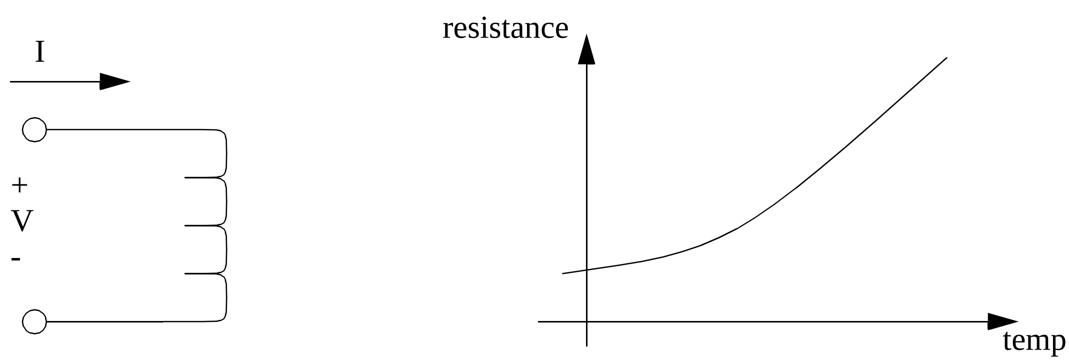
Figure 383 A mixer to prevent uneven temperature distribution in thermal storage

19.2.3 Sources

• If we plan to design a thermal system we will need some sort of heat source. One popular heating source is an electrical heating element.

• heating coils are normally made out of some high resistance metal/alloy such as nichrome (nickel and chrome) or tungsten.

• If we run a current through the wire the resistance will generate heat. As these metals get hotter the resistance rises, hence the temperature is self regulating. If we control the voltage and current we can control the amount of heat delivered by the coil.



$$P = IV$$

where,

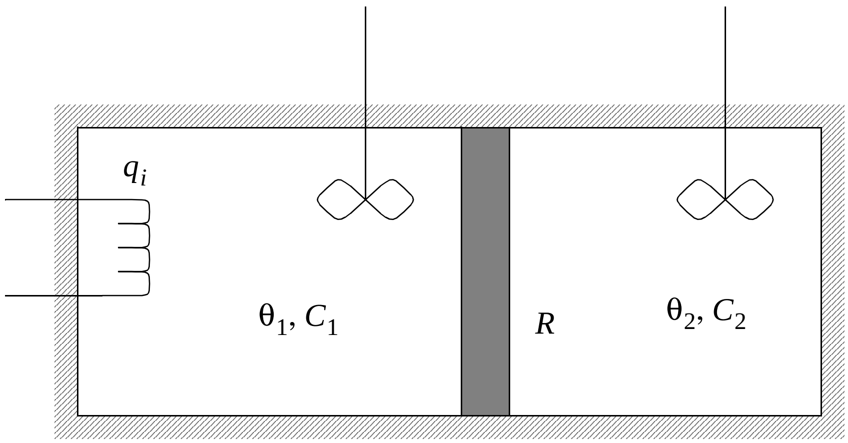
P = power generated as heat
I = current into coil
V = voltage across coil

Figure 384 Electrical heating elements

19.3 EXAMPLE SYSTEMS

• We know that as we heat materials at one point, they do not instantly heat at all points, there is some delay (consider a spoon in a hot bowl of soup). This delay is a function of heat capacitance (yes there is a parallel to the electrical description).

• Consider an insulated sealed chamber with a resistive barrier between sides, and a heating element in one side.



Given

q_0 = heat flow into the left chamber from the heating element

θ_1, C_1 = initial temperature and heat capacitance of left side

θ_2, C_2 = initial temperature and heat capacitance of right side

Next,

q_1 = heat flow through the center barrier

$$q_1 = \frac{1}{R}(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\theta_1 = \frac{1}{C_1}(q_0 - q_1) \qquad \frac{d}{dt}\theta_2 = \frac{1}{C_2}(q_1)$$

$$s\theta_1 = \frac{1}{C_1}\left(q_0 - \frac{1}{R}(\theta_1 - \theta_2)\right) \qquad s\theta_2 = \frac{1}{RC_2}(\theta_1 - \theta_2)$$

$$Rq_o - sRC_1\theta_1 = \theta_1 - \theta_2 \qquad \theta_1 = \theta_2 + sRC_2\theta_2$$

$$\theta_1(1 + sRC_1) = \theta_2 + Rq_o$$

$$\theta_1 = \frac{\theta_2 + Rq_o}{1 + sRC_1}$$

Figure 385 A two chamber thermal system

$$\theta_1 = \frac{\theta_2 + Rq_o}{1 + sRC_1} = \theta_2 + sRC_2\theta_2$$

$$\frac{\theta_2 + Rq_o}{\theta_2 + sRC_2\theta_2} = 1 + sRC_1$$

$$\frac{1}{1 + sRC_2} + \frac{\left(\frac{Rq_o}{\theta_2}\right)}{1 + sRC_2} = 1 + sRC_1$$

$$1 + \left(\frac{Rq_o}{\theta_2}\right) = 1 + s(RC_1 + RC_2) + s^2R^2C_1C_2$$

$$\frac{Rq_o}{\theta_2} = s(RC_1 + RC_2) + s^2R^2C_1C_2$$

$$\frac{q_o}{\theta_2} = s(C_1 + C_2) + s^2RC_1C_2$$

$$\theta_2 = \frac{\left(\frac{q_o}{RC_1C_2}\right)}{s\left(\frac{C_1 + C_2}{RC_1C_2}\right) + s^2} = \frac{A}{s} + \frac{B}{s + \frac{C_1 + C_2}{RC_1C_2}}$$

$$A = \lim_{s \to 0}\left(\frac{\left(\frac{q_o}{RC_1C_2}\right)}{s\left(\frac{C_1 + C_2}{RC_1C_2}\right) + s^2}s\right) = \frac{\left(\frac{q_o}{RC_1C_2}\right)}{\left(\frac{C_1 + C_2}{RC_1C_2}\right) + (0)} = \frac{q_o}{C_1 + C_2}$$

$$B = \lim_{s \to -\frac{C_1 + C_2}{RC_1C_2}}\left(\frac{\left(\frac{q_o}{RC_1C_2}\right)}{s\left(\frac{C_1 + C_2}{RC_1C_2}\right) + s^2}\left(s + \frac{C_1 + C_2}{RC_1C_2}\right)\right) = \frac{\left(\frac{q_o}{RC_1C_2}\right)}{-\left(\frac{C_1 + C_2}{RC_1C_2}\right)} = \frac{-q_o}{C_1 + C_2}$$

$$\theta_2 = \frac{\frac{q_o}{C_1 + C_2}}{s} + \frac{\frac{-q_o}{C_1 + C_2}}{s + \frac{C_1 + C_2}{RC_1C_2}}$$

Figure 386 A two chamber thermal system (continued)

$$L^{-1}(\theta_2) = \frac{q_o}{C_1 + C_2}\left[L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s + \frac{C_1 + C_2}{RC_1C_2}}\right]\right]$$

$$\theta_2(t) = \frac{q_o}{C_1 + C_2}\left[1 - e^{-\left(\frac{C_1 + C_2}{RC_1C_2}\right)t}\right]$$

Figure 387 A two chamber thermal system (continued)

19.4 SUMMARY

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19.5 PRACTICE PROBLEMS

1.

20. LAPLACE TRANSFER FUNCTIONS

Topics:

Objectives:

20.1 INTRODUCTION

• We can model systems as a ratio between output and input. This allows powerful mathematical manipulation.

20.2 THE LAPLACE TRANSFORM

• The Laplace transform allows us to reverse time. And, as you recall from before the inverse of time is frequency. Because we are normally concerned with response, the Laplace transform is much more useful in system analysis.

• The basic Laplace transform equations is shown below,

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

where,

$f(t)$ = the function in terms of time t

$F(s)$ = the function in terms of the Laplace s

Figure 388 The Laplace transform

• Consider the examples below,

ASIDE:

- Recall that,

$$F(s) = \int_0^\infty f(t)e^{-st}dt = L[f(t)]$$

- so for f(t) = 5,

$$F(s) = \int_0^\infty 5e^{-st}dt = -\frac{5}{s}e^{-st}\Big|_0^\infty = \left[-\frac{5}{s}e^{-s\infty}\right] - \left[-\frac{5e^{-s0}}{s}\right] = \frac{5}{s}$$

- for the derivatives of a function g(t)=df(t)/dt,

$$G(s) = L[g(t)] = L\left[\frac{d}{dt}f(t)\right] = \int_0^\infty (\mathrm{d/dt})f(t)e^{-st}dt$$

we can use integration by parts to go backwards,

$$\int_a^b u\,dv = uv\big|_a^b - \boxed{\int_a^b v\,du}$$

$$\int_0^\infty (\mathrm{d/dt})f(t)e^{-st}dt$$

therefore,

$$du = df(t) \qquad v = e^{-st}$$

$$u = f(t) \qquad dv = -se^{-st}dt$$

$$\therefore \int_0^\infty f(t)(-s)e^{-st}dt = f(t)e^{-st}\Big|_0^\infty - \int_0^\infty (\mathrm{d/dt})f(t)e^{-st}dt$$

$$\therefore \int_0^\infty (\mathrm{d/dt})f(t)e^{-st}dt = [f(t)e^{-\infty s} - f(t)e^{-0s}] + s\int_0^\infty f(t)e^{-st}dt$$

$$\boxed{\therefore L\left[\frac{d}{dt}f(t)\right] = -f(0) + sL[f(t)]}$$

Figure 389 Some example Laplace transforms

20.2.1 A Few Transforms

• The basic properties Laplace Transforms for are given below,

TIME DOMAIN	FREQUENCY DOMAIN
$f(t)$	$f(s)$
$Kf(t)$	$KL[f(t)]$
$f_1(t) + f_2(t) - f_3(t) + \ldots$	$f_1(s) + f_2(s) - f_3(s) + \ldots$
$\frac{df(t)}{dt}$	$sL[f(t)] - f(0^-)$
$\frac{d^2f(t)}{dt^2}$	$s^2L[f(t)] - sf(0^-) - \frac{df(0^-)}{dt}$
$\frac{d^nf(t)}{dt^n}$	$s^nL[f(t)] - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt} - \ldots - \frac{d^nf(0^-)}{dt^n}$
$\int_0^t f(t)dt$	$\frac{L[f(t)]}{s}$
$f(t-a)u(t-a), a > 0$	$e^{-as}L[f(t)]$
$e^{-at}f(t)$	$f(s-a)$
$f(at), a > 0$	$\frac{1}{a}f\left(\frac{s}{a}\right)$
$tf(t)$	$\frac{-df(s)}{ds}$
$t^nf(t)$	$(-1)^n\frac{d^nf(s)}{ds^n}$
$\frac{f(t)}{t}$	$\int_s^{\infty} f(u)du$

Figure 390 Laplace transform tables

• A set of useful functional Laplace transforms are given below. These are mainly used for converting to and from time ’t’ to the Laplace ’s’.

TIME DOMAIN		FREQUENCY DOMAIN
$\delta(t)$	unit impulse	1
A	step	$\frac{A}{s}$
t	ramp	$\frac{1}{s^2}$
t^2		$\frac{2}{s^3}$
$t^n, n > 0$		$\frac{n!}{s^{n+1}}$
e^{-at}	exponential decay	$\frac{1}{s+a}$
$\sin(\omega t)$		$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$		$\frac{s}{s^2+\omega^2}$
te^{-at}		$\frac{1}{(s+a)^2}$
t^2e^{-at}		$\frac{2!}{(s+a)^3}$

Figure 391 Laplace transform tables (continued)

TIME DOMAIN	FREQUENCY DOMAIN
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\left[B\cos\omega t+\left(\frac{C-aB}{\omega}\right)\sin\omega t\right]$	$\frac{Bs+C}{(s+a)^2+\omega^2}$
$2\|A\|e^{-\alpha t}\cos(\beta t+\theta)$	$\frac{A}{s+\alpha-\beta j}+\frac{A^{\text{complex conjugate}}}{s+\alpha+\beta j}$
$2t\|A\|e^{-\alpha t}\cos(\beta t+\theta)$	$\frac{A}{(s+\alpha-\beta j)^2}+\frac{A^{\text{complex conjugate}}}{(s+\alpha+\beta j)^2}$
$\frac{(c-a)e^{-at}-(c-b)e^{-bt}}{b-a}$	$\frac{s+c}{(s+a)(s+b)}$
$\frac{e^{-at}-e^{-bt}}{b-a}$	$\frac{1}{(s+a)(s+b)}$

Figure 392 Laplace transform tables (continued)

20.2.2 Impulse Response (or Why Laplace Transforms Work)

• Consider a system model. That model can be said to have an input (forcing function) and an output (resulting response function).

$$F(t) = \left(\frac{d}{dt}\right)^2 x(t)$$

where,

$F(t)$ = force (forcing function or input)

$x(t)$ = displacement (resulting/output function)

$$\frac{x(t)}{F(t)} = g(t)$$

where,

$g(t)$ = a function that is the ratio of input and output

Figure 393 A transfer function example

• If we look at an input signal (force here) we can break it into very small segments in time. As the time becomes small we call it an impulse function.

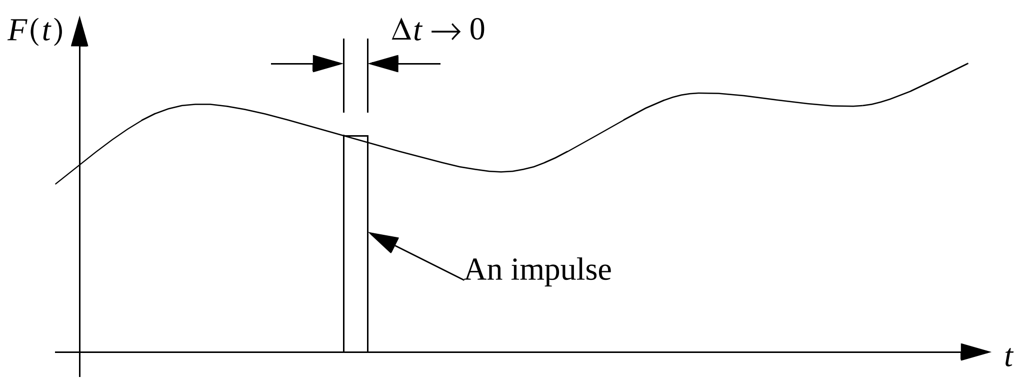


Figure 394 An impulse as a brief duration pulse

• If we put an impulse into a system the output will be an impulse response.

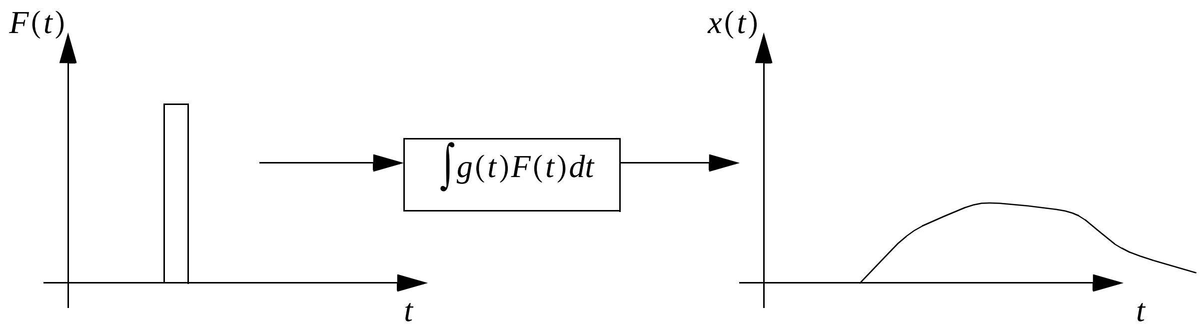
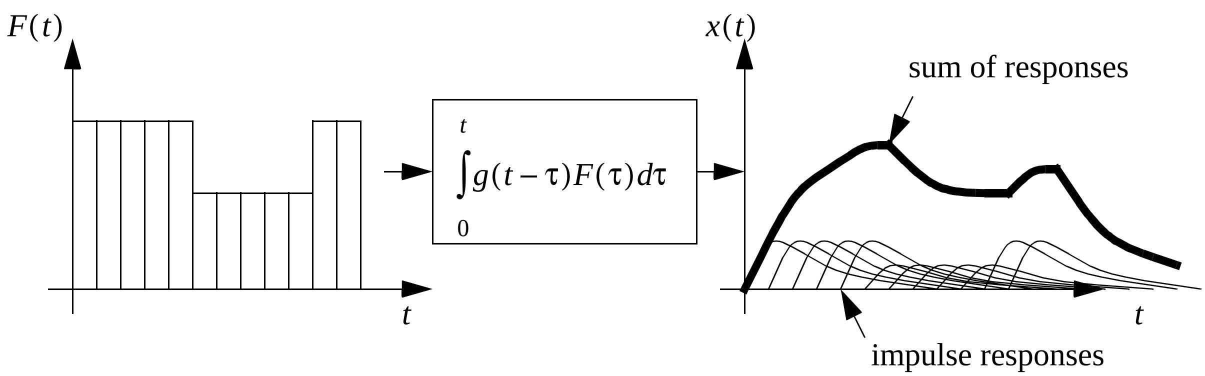


Figure 395 Response of the system to a single pulse

• If we add all of the impulse responses together we will get a total system response. This operation is called convolution.



The convolution integral

$$c(t) = \int_0^t g(t-\tau) r(\tau) d\tau$$

Figure 396 A set of pulses for a system gives summed responses to give the output

• The convolution integral can be difficult to deal with because of the time shift. But, the Laplace transform for the convolution integral turns it into a simple multiplication.

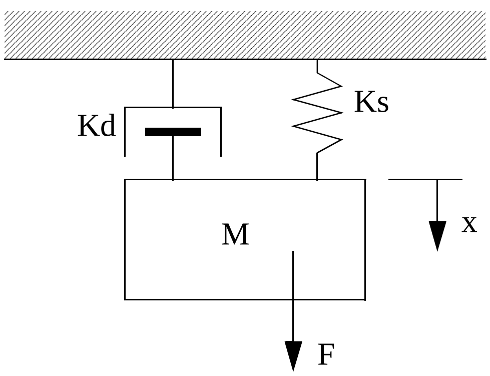
$$c(t) = \int_0^t g(t-\tau)r(\tau)d\tau$$

$$C(s) = G(s)R(s)$$

Figure 397 The convolution integral

20.3 MODELING MECHANICAL SYSTEMS

• Before doing any sort of analysis of a vibrating system, a system model must be developed. The obvious traditional approach is with differential equations.



$$F = M\frac{d^2x}{dt^2} + K_d\frac{dx}{dt} + K_s x$$

$$\frac{F(t)}{x(t)} = M\frac{d^2}{dt^2} + K_d\frac{d}{dt} + K_s$$

$$L\left[\frac{F(t)}{x(t)}\right] = \frac{F(s)}{x(s)} = Ms^2 + K_d s + K_s$$

ASIDE: An important concept that is ubiquitous yet largely unrecognized is the use of functional design. We look at parts of systems as self contained modules that use inputs to produce outputs. Some systems (such a mechanisms) are reversible, others are not (consider a worm gear). An input is typically something we can change, an output is the resulting change in a system. For the example above ‘F’ over ‘x’ implies that we are changing the input ‘x’, and there is some change in ‘F’. We know this could easily be reversed mathematically and practically.

Figure 398 A mass-spring-damper example

ANOTHER ASIDE: Keep in mind that the mathematical expression 'F/x' is a ratio between input (displacement action) and output (reaction force). When shown with differentials it is obvious that the ratio is not simple, and is a function of time. Also keep in mind that if we were given a force applied to the system it would become the input (action force) and the output would be the displacement (resulting motion). To do this all we need to do is flip the numerators and denominators in the transfer function.

20.4 MODELING ELECTRICAL SYSTEMS

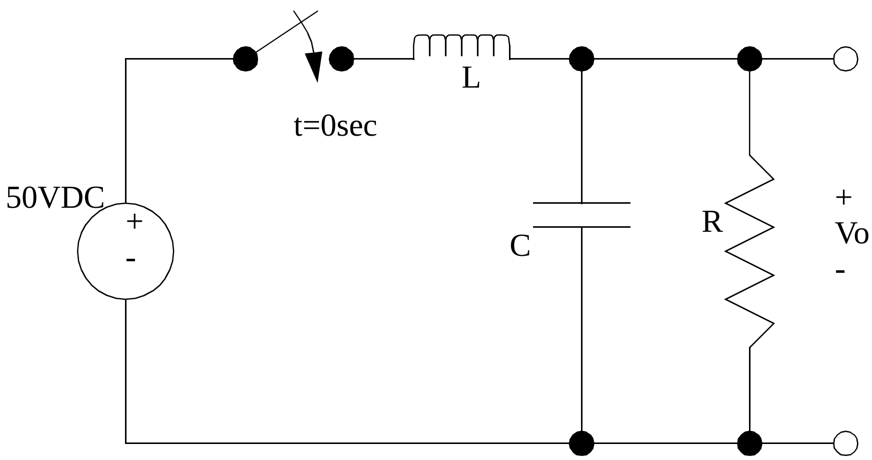
• Consider the basic equations for capacitors, inductors and resistors.

Device	Time domain	Frequency domain	Impedance
Resistor	$V(t) = RI(t)$	$V(s) = RI(s)$	$Z = R$
Capacitor	$V(t) = \frac{1}{C}\int I(t)dt$	$V(s) = \left(\frac{1}{C}\right)\frac{I(s)}{s}$	$Z = \frac{1}{sC}$
Inductor	$V(t) = L\frac{d}{dt}I(t)$	$V(s) = LsI(s)$	$Z = Ls$

Note: Impedance is like resistance, except that it includes time variant features also. $V = ZI$

Figure 399 Impedances of electrical components

• For the circuit below, the switch is closed at t=0sec,



Treat the circuit as a voltage divider,

$$V_o = \frac{50V\left(\frac{1}{sC + \frac{1}{R}}\right)}{sL + \left(\frac{1}{sC + \frac{1}{R}}\right)} = \frac{50V\left(\frac{R}{1 + sCR}\right)}{sLR + \left(\frac{R}{1 + sCR}\right)} = 50V\left(\frac{R}{s^2R^2LC + sLR + R}\right)$$

Figure 400 A circuit example

20.5 USING LAPLACE TRANSFORMS

• The differential equation is in the time domain. By doing a Laplace transform we can move the system into the frequency domain. This makes it much easier to solve complex convolution problems. Without this method, very complex integrals would be required.

• Inputs must in the time domain must also be converted to the frequency domain.

e.g., Apply a constant force of A, starting at time t=0 sec.
(*Note: a force applied instantly is impossible)

F(t) = 0 for t < 0
= A for t >= 0

Perform Laplace transform using tables

$$F(s) = L[F(t)] = \frac{A}{s}$$

Figure 401 An input function

• Normally at this point we would have the input to the system, and the system differential equation. The convolution integral would be used to find the time response, but using Laplace transforms this becomes a simple substitution.

$$\frac{F(s)}{x(s)} = Ms^2 + K_d s + K_s = \frac{\left(\frac{A}{s}\right)}{x(s)}$$

$$\therefore x(s) = \frac{A}{(Ms^2 + K_d s + K_s)s}$$

Assume some values such as,

$$K_d = 3000\frac{Ns}{m}$$

$$K_s = 2000\frac{N}{m}$$

$$M = 1000kg$$

$$A = 1000N$$

$$\therefore x(s) = \frac{1}{(s^2 + 3s + 2)s}$$

Figure 402 A transfer function multiplied by the input function

• At this point we have 'x' as a function of 's' (later we will see 's' is equivalent to frequency). We can find the initial, and final values (steady state) of 'x' using the final value theorem.

$$\boxed{x(t \to \infty) = \lim_{s \to 0} [sx(s)]}$$ Final value theorem

$$\therefore x(t \to \infty) = \lim_{s \to 0}\left[\frac{1s}{(s^2 + 3s + 2)s}\right] = \lim_{s \to 0}\left[\frac{1}{s^2 + 3s + 2}\right] = \frac{1}{(0)^2 + 3(0) + 2} = \frac{1}{2}$$

$$\boxed{x(t \to 0) = \lim_{s \to \infty} [sx(s)]}$$ Initial value theorem

$$\therefore x(t \to 0) = \lim_{s \to \infty}\left[\frac{1(s)}{(s^2 + 3s + 2)s}\right] = \frac{1}{((\infty)^2 + 3(\infty) + 2)} = \frac{1}{\infty} = 0$$

Figure 403 Final and initial values theorems

• All that is needed to get the time domain function is an inverse Laplace transform. This is quite often done by using partial fraction expansion of the equations, followed by Inverse Laplace transforms of the simpler parts.

$$x(s) = \frac{1}{(s^2 + 3s + 2)s} = \frac{1}{(s+1)(s+2)s} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$A = \lim_{s \to 0}\left[s\left(\frac{1}{(s+1)(s+2)s}\right)\right] = \frac{1}{2}$$

$$B = \lim_{s \to -1}\left[(s+1)\left(\frac{1}{(s+1)(s+2)s}\right)\right] = -1$$

$$C = \lim_{s \to -2}\left[(s+2)\left(\frac{1}{(s+1)(s+2)s}\right)\right] = \frac{1}{2}$$

Aside: the short cut above can reduce time for simple partial fraction expansions. A simple proof for finding ‘B’ above is given in this box.

$$\frac{1}{(s+1)(s+2)s} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$(s+1)\left[\frac{1}{(s+1)(s+2)s}\right] = (s+1)\left[\frac{A}{s}\right] + (s+1)\left[\frac{B}{s+1}\right] + (s+1)\left[\frac{C}{s+2}\right]$$

$$\frac{1}{(s+2)s} = (s+1)\left[\frac{A}{s}\right] + B + (s+1)\left[\frac{C}{s+2}\right]$$

$$\lim_{s \to -1}\left[\frac{1}{(s+2)s}\right] = \lim_{s \to -1}\left[(s+1)\left[\frac{A}{s}\right]\right] + \lim_{s \to -1} B + \lim_{s \to -1}\left[(s+1)\left[\frac{C}{s+2}\right]\right]$$

$$\lim_{s \to -1}\left[\frac{1}{(s+2)s}\right] = \lim_{s \to -1} B = B$$

$$x(s) = \frac{1}{(s^2 + 3s + 2)s} = \frac{0.5}{s} + \frac{-1}{s+1} + \frac{0.5}{s+2}$$

Figure 404 Partial fractions to reduce an output function

• The terms from the partial fraction expansion are put through an inverse Laplace transform using a lookup table. (A sample table is given later)

$$x(t) = L^{-1}[x(s)] = L^{-1}\left[\frac{0.5}{s} + \frac{-1}{s+1} + \frac{0.5}{s+2}\right]$$

$$x(t) = L^{-1}\left[\frac{0.5}{s}\right] + L^{-1}\left[\frac{-1}{s+1}\right] + L^{-1}\left[\frac{0.5}{s+2}\right]$$

$$x(t) = [0.5] + [(-1)e^{-t}] + [(0.5)e^{-2t}]$$

$$\boxed{x(t) = 0.5 - e^{-t} + 0.5e^{-2t}}$$ Actual response of the system in time

Figure 405 Partial fractions to reduce an output function (continued)

• Some numbers can be calculated to verify this,

t (sec.)	x(t)
0.0	0.00000
0.1	0.00453
0.2	0.01643
0.3	0.03359
0.4	0.05434
0.5	0.07741
0.6	0.10179
0.7	0.12671
0.8	0.15162
0.9	0.17608
1.0	0.19979
10.0	0.49995
100.0	0.50000
1000.0	0.50000

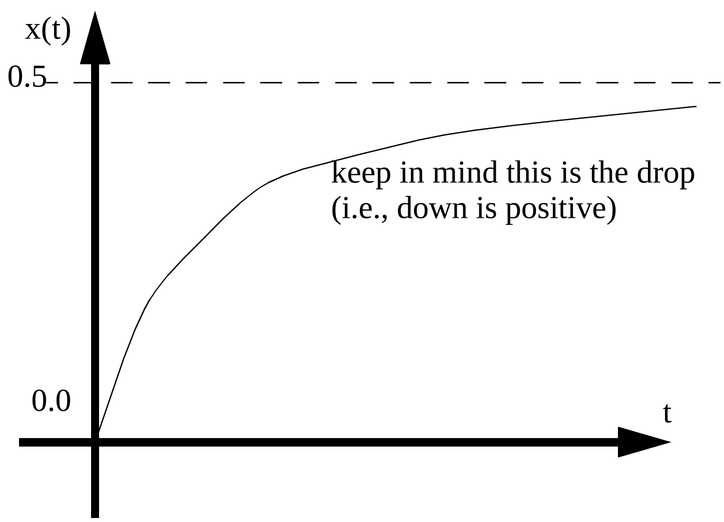


Figure 406 Partial fractions to reduce an output function (continued)

• Note that the damper is relatively larger than the spring, therefore no “oscillation”.

• What if the damping approaches 0?

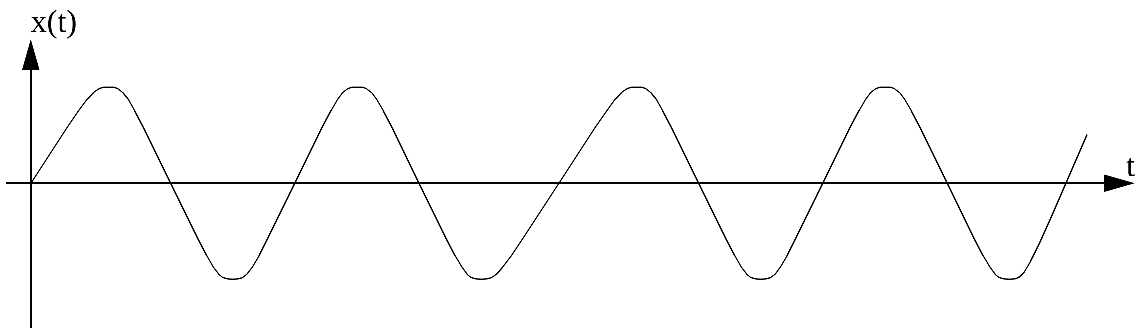


Figure 407 a system with no damping tends to oscillate

20.5.1 Solving Partial Fractions

• The following is a flowchart that shows the general method for doing inverse Laplace transforms.

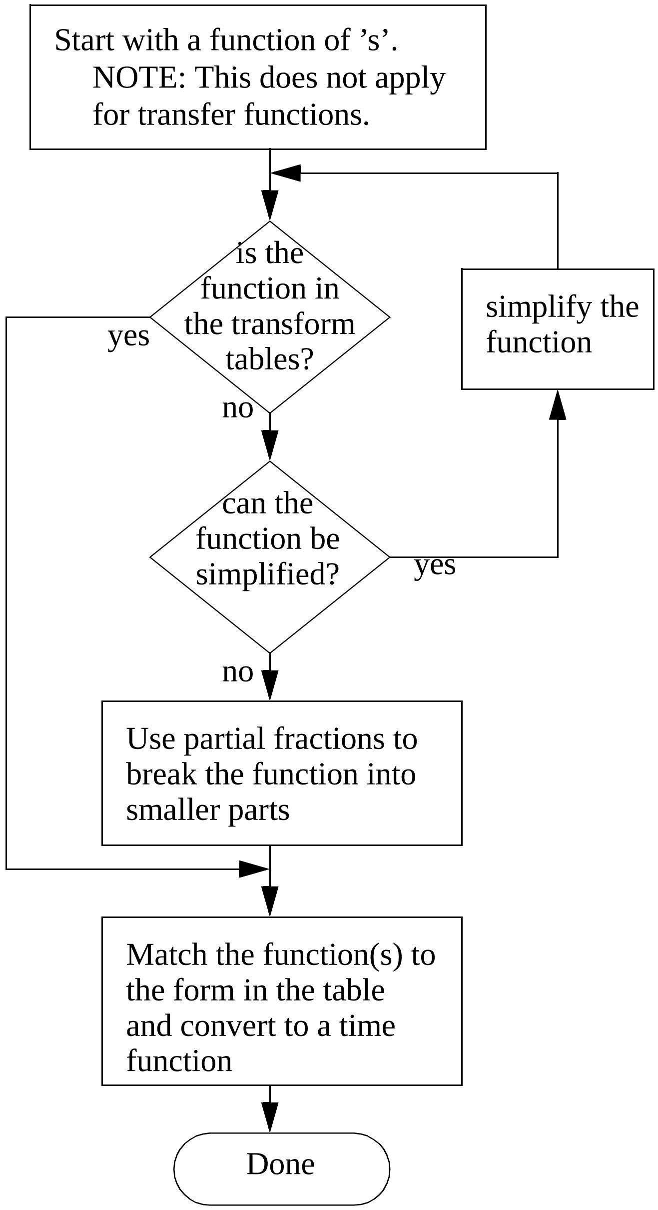


Figure 408 The methodology for doing inverse transforms

• The next is a flowchart for partial fraction expansions.

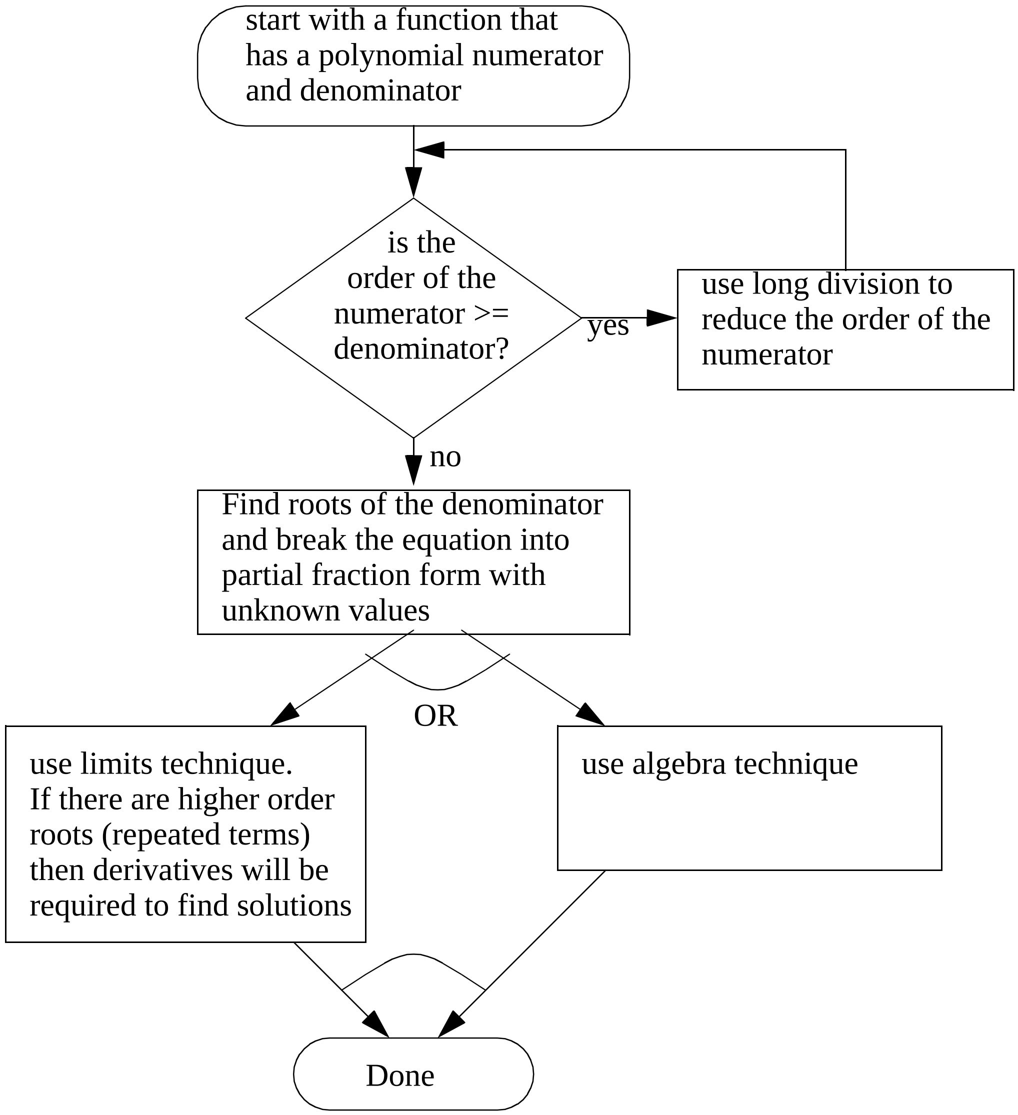


Figure 409 The methodology for solving partial fractions

• The partial fraction expansion for,

$$x(s) = \frac{1}{s^2(s+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}$$

$$C = \lim_{s \to -1}\left[(s+1)\left(\frac{1}{s^2(s+1)}\right)\right] = 1$$

$$A = \lim_{s \to 0}\left[s^2\left(\frac{1}{s^2(s+1)}\right)\right] = \lim_{s \to 0}\left[\frac{1}{s+1}\right] = 1$$

$$B = \lim_{s \to 0}\left[\frac{d}{ds}\left[s^2\left(\frac{1}{s^2(s+1)}\right)\right]\right] = \lim_{s \to 0}\left[\frac{d}{ds}\left(\frac{1}{s+1}\right)\right] = \lim_{s \to 0}[-(s+1)^{-2}] = -1$$

Figure 410 A partial fraction example

• Consider the example below where the order of the numerator is larger than the denominator.

$$x(s) = \frac{5s^3 + 3s^2 + 8s + 6}{s^2 + 4}$$

This cannot be solved using partial fractions because the numerator is 3rd order and the denominator is only 2nd order. Therefore long division can be used to reduce the order of the equation.

$$\begin{array}{r|l} & 5s + 3 \\ \hline s^2 + 4 & 5s^3 + 3s^2 + 8s + 6 \\ & \underline{5s^3 + 20s} \\ & \quad 3s^2 - 12s + 6 \\ & \quad \underline{3s^2 + 12} \\ & \qquad -12s - 6 \end{array}$$

This can now be used to write a new function that has a reduced portion that can be solved with partial fractions.

$$x(s) = 5s + 3 + \frac{-12s - 6}{s^2 + 4} \qquad \text{solve} \qquad \frac{-12s - 6}{s^2 + 4} = \frac{A}{s + 2j} + \frac{B}{s - 2j}$$

Figure 411 Partial fractions when the numerator is larger than the denominator

• When the order of the denominator terms is greater than 1 it requires an expanded partial fraction form, as shown below.

$$F(s) = \frac{5}{s^2(s + 1)^3}$$

$$\frac{5}{s^2(s + 1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s + 1)^3} + \frac{D}{(s + 1)^2} + \frac{E}{(s + 1)}$$

Figure 412 Partial fractions with repeated roots

• We can solve the previous problem using the algebra technique.

$$\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}$$

$$= \frac{A(s+1)^3 + Bs(s+1)^3 + Cs^2 + Ds^2(s+1) + Es^2(s+1)^2}{s^2(s+1)^3}$$

$$= \frac{s^4(B+E) + s^3(A+3B+D+2E) + s^2(3A+3B+C+D+E) + s(3A+B) + (A)}{s^2(s+1)^3}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 & 2 \\ 3 & 3 & 1 & 1 & 1 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 & 2 \\ 3 & 3 & 1 & 1 & 1 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ -15 \\ 5 \\ 10 \\ 15 \end{bmatrix}$$

$$\frac{5}{s^2(s+1)^3} = \frac{5}{s^2} + \frac{-15}{s} + \frac{5}{(s+1)^3} + \frac{10}{(s+1)^2} + \frac{15}{(s+1)}$$

Figure 413 Solving partial fractions with algebra

• This problem can also be solved using the limits technique. But, because of the repeated roots we will need to differentiate to find the repeated roots.

$$\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}$$

$$A = \lim_{s \to 0}\left[\left(\frac{5}{s^2(s+1)^3}\right)s^2\right] = \lim_{s \to 0}\left[\frac{5}{(s+1)^3}\right] = 5$$

$$B = \lim_{s \to 0}\left[\frac{d}{ds}\left(\frac{5}{s^2(s+1)^3}\right)s^2\right] = \lim_{s \to 0}\left[\frac{d}{ds}\left(\frac{5}{(s+1)^3}\right)\right] = \lim_{s \to 0}\left[\frac{5(-3)}{(s+1)^4}\right] = -15$$

$$C = \lim_{s \to -1}\left[\left(\frac{5}{s^2(s+1)^3}\right)(s+1)^3\right] = \lim_{s \to -1}\left[\frac{5}{s^2}\right] = 5$$

$$D = \lim_{s \to -1}\left[\frac{1}{1!}\frac{d}{ds}\left(\frac{5}{s^2(s+1)^3}\right)(s+1)^3\right] = \lim_{s \to -1}\left[\frac{1}{1!}\frac{d}{ds}\frac{5}{s^2}\right] = \lim_{s \to -1}\left[\frac{1}{1!}\frac{-2(5)}{s^3}\right] = 10$$

$$E = \lim_{s \to -1}\left[\frac{1}{2!}\frac{d}{ds}^2\left(\frac{5}{s^2(s+1)^3}\right)(s+1)^3\right] = \lim_{s \to -1}\left[\frac{1}{2!}\frac{d}{ds}^2\frac{5}{s^2}\right] = \lim_{s \to -1}\left[\frac{1}{2!}\frac{30}{s^4}\right] = 15$$

$$\frac{5}{s^2(s+1)^3} = \frac{5}{s^2} + \frac{-15}{s} + \frac{5}{(s+1)^3} + \frac{10}{(s+1)^2} + \frac{15}{(s+1)}$$

Figure 414 Solving partial fractions with limits

• We can prove the technique for the derivatives of the functions.

$$\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}$$

$$\lim_{s \to -1}\left[\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}\right]$$

$$\lim_{s \to -1}\left[(s+1)^3\left(\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}\right)\right]$$

$$\lim_{s \to -1}\left[\frac{5}{s^2} = \frac{A(s+1)^3}{s^2} + \frac{B(s+1)^3}{s} + C + D(s+1) + E(s+1)^2\right]$$

For C, evaluate now,

$$\frac{5}{(-1)^2} = \frac{A(-1+1)^3}{(-1)^2} + \frac{B(-1+1)^3}{-1} + C + D(-1+1) + E(-1+1)^2$$

$$\frac{5}{(-1)^2} = \frac{A(0)^3}{(-1)^2} + \frac{B(0)^3}{-1} + C + D(0) + E(0)^2 \qquad \boxed{C = 5}$$

For D, differentiate once, then evaluate

$$\lim_{s \to -1}\left[\frac{d}{dt}\left(\frac{5}{s^2} = \frac{A(s+1)^3}{s^2} + \frac{B(s+1)^3}{s} + C + D(s+1) + E(s+1)^2\right)\right]$$

$$\lim_{s \to -1}\left[\frac{-2(5)}{s^3} = A\left(-\frac{2(s+1)^3}{s^3} + \frac{3(s+1)^2}{s^2}\right) + B\left(-\frac{(s+1)^3}{s^2} + \frac{3(s+1)^2}{s}\right) + D + 2E(s+1)\right]$$

$$\frac{-2(5)}{(-1)^3} = \boxed{D = 10}$$

For E, differentiate twice, then evaluate (the terms for A and B will be ignored to save space, but these will drop out anyway).

$$\lim_{s \to -1}\left[\left(\frac{d}{dt}\right)^2\left(\frac{5}{s^2} = \frac{A(s+1)^3}{s^2} + \frac{B(s+1)^3}{s} + C + D(s+1) + E(s+1)^2\right)\right]$$

$$\lim_{s \to -1}\left[\left(\frac{d}{dt}\right)\left(\frac{-2(5)}{s^3} = A(\ldots) + B(\ldots) + D + 2E(s+1)\right)\right]$$

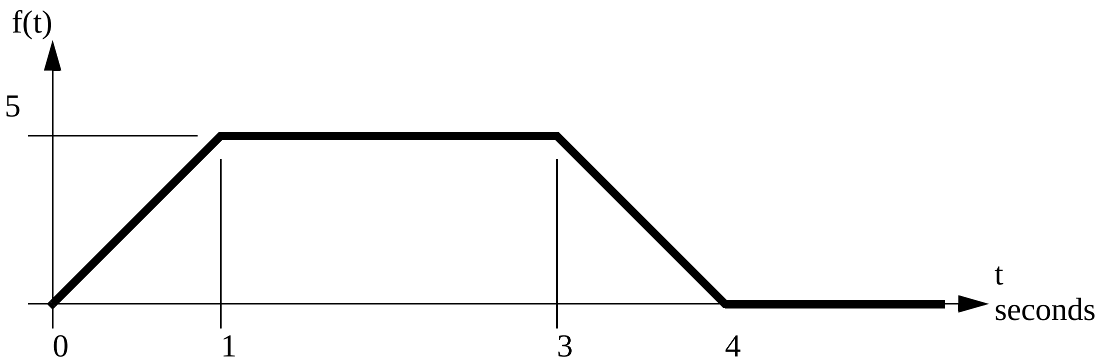
$$\lim_{s \to -1}\left[\frac{-3(-2(5))}{s^4} = A(\ldots) + B(\ldots) + 2E\right]$$

$$\frac{-3(-2(5))}{(-1)^4} = A(0) + B(0) + 2E \qquad \boxed{E = 15}$$

Figure 415 A proof of the need for differentiation for repeated roots

20.5.2 Input Functions

• An example of a complex time function is,



$$f(t) = 5tu(t) - 5(t-1)u(t-1) - 5(t-3)u(t-3) + 5(t-4)u(t-4)$$

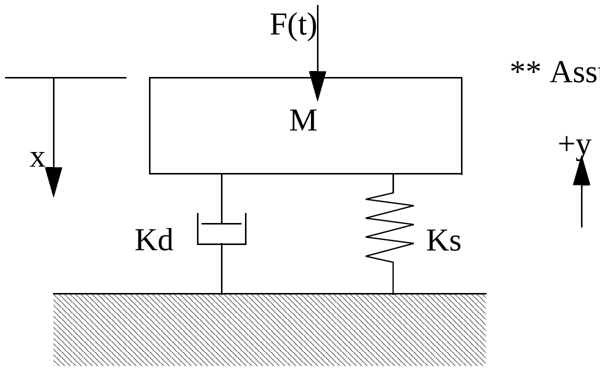
$$f(s) = \frac{5}{s^2} - \frac{5e^{-s}}{s^2} - \frac{5e^{-3s}}{s^2} + \frac{5e^{-4s}}{s^2}$$

Figure 416 Switching on and off function parts

20.5.3 Examples

• These systems tend to vibrate simply. This vibration will often decay naturally. The contrast is the first order system that tends to move towards new equilibrium points without any sort of resonance or vibration.

•



** Assume gravity is negligible

$$\sum F_y = -F(t) + K_d \frac{dx}{dt} + K_s x = -M\frac{d^2x}{dt^2}$$

$$\therefore -F(s) + K_d s x + K_s x + M s^2 x = 0$$

$$\therefore \frac{F(s)}{x} = K_d s + K_s + M s^2$$

$$\therefore \frac{x}{F(s)} = \frac{1}{Ms^2 + K_d s + K_s}$$

$$\therefore \frac{x}{F(s)} = \frac{\frac{1}{M}}{s^2 + \frac{K_d}{M}s + \frac{K_s}{M}}$$

Figure 417 A mass-spring-damper example

• To continue the example with numerical values, Assuming component values of,

$$M = 1kg \qquad K_s = 2\frac{N}{m} \qquad K_d = 0.5\frac{Ns}{m}$$

Assuming an input of,

$$F(t) = 5\cos(6t)N$$

$$\therefore F(s) = \frac{5s}{s^2 + 6^2}$$

This means,

$$x(s) = F(s)\left(\frac{x(s)}{F(s)}\right) = \left(\frac{5s}{s^2 + 6^2}\right)\left(\frac{\frac{1}{1}}{s^2 + 0.5s + 2}\right)$$

$$\therefore x(s) = \frac{5s}{(s^2 + 36)(s^2 + 0.5s + 2)}$$

$$\therefore x(s) = \frac{A}{s + 6j} + \frac{B}{s - 6j} + \frac{C}{s - 0.5 + 1.39j} + \frac{D}{s - 0.5 - 1.39j}$$

Figure 418 A mass-spring-damper example (continued)

$$A = \lim_{s \to -6j}\left[\frac{(s + 6j)(5s)}{(s - 6j)(s^2 + 36)(s^2 + 0.5s + 2)}\right] = \frac{-30j}{(-12j)(36 - 3j + 2)}$$

$$\therefore A = \frac{-30j}{-432j - 36 - 24j} = \frac{30j}{36 + 456j} \times \frac{36 - 456j}{36 - 456j} = \frac{13680 + 1080j}{209,232} = 0.0654 + 0.00516j$$

--Continue on to find B, C, D same way

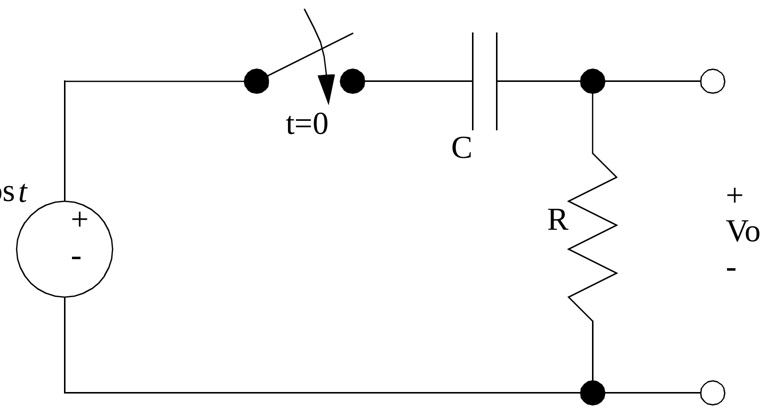
$$\therefore x(s) = \frac{0.0654 + 0.00516j}{s + 6j} + \frac{0.0654 - 0.00516j}{s - 6j} + \ldots$$

Do inverse Laplace transform

$$x(t) = 2\sqrt{0.0654^2 + 0.00516^2}e^{-0t}\cos\left(0.00516t + \operatorname{atan}\left(-\frac{0.00516}{0.0654}\right)\right) + \ldots$$

Figure 419   A mass-spring-damper example (continued)

• Consider the circuit below,



$V_s = 3\cos t$

As normal we relate the source voltage to the output voltage. The we find the values for the various terms in the frequency domain.

$$V_o = V_s\left(\frac{Z_R}{Z_R + Z_C}\right) \qquad V_s(s) = \frac{3s^2}{s^2+1}$$

$$Z_R = R$$

$$Z_C = \frac{1}{sC}$$

Next, we may combine the equations, and find a partial fraction equivalent.

$$V_o = \frac{3s^2}{s^2+1}\left(\frac{R}{R+\frac{1}{sC}}\right) = \frac{3s^3RC}{(s^2+1)(1+RsC)} = \frac{3s^3}{(s^2+1)\left(s+\frac{1}{RC}\right)}$$

$$V_o = \frac{A}{\ } + \frac{B}{\ } + \frac{C}{\ }$$

$$A = \lim_{s \to} [\quad] =$$

Finally we do the inverse transform to get the function back into the time domain.

Figure 420 A circuit example

20.6 A MAP OF TECHNIQUES FOR LAPLACE ANALYSIS

• The following map is to be used to organize the various topics covered in the course.

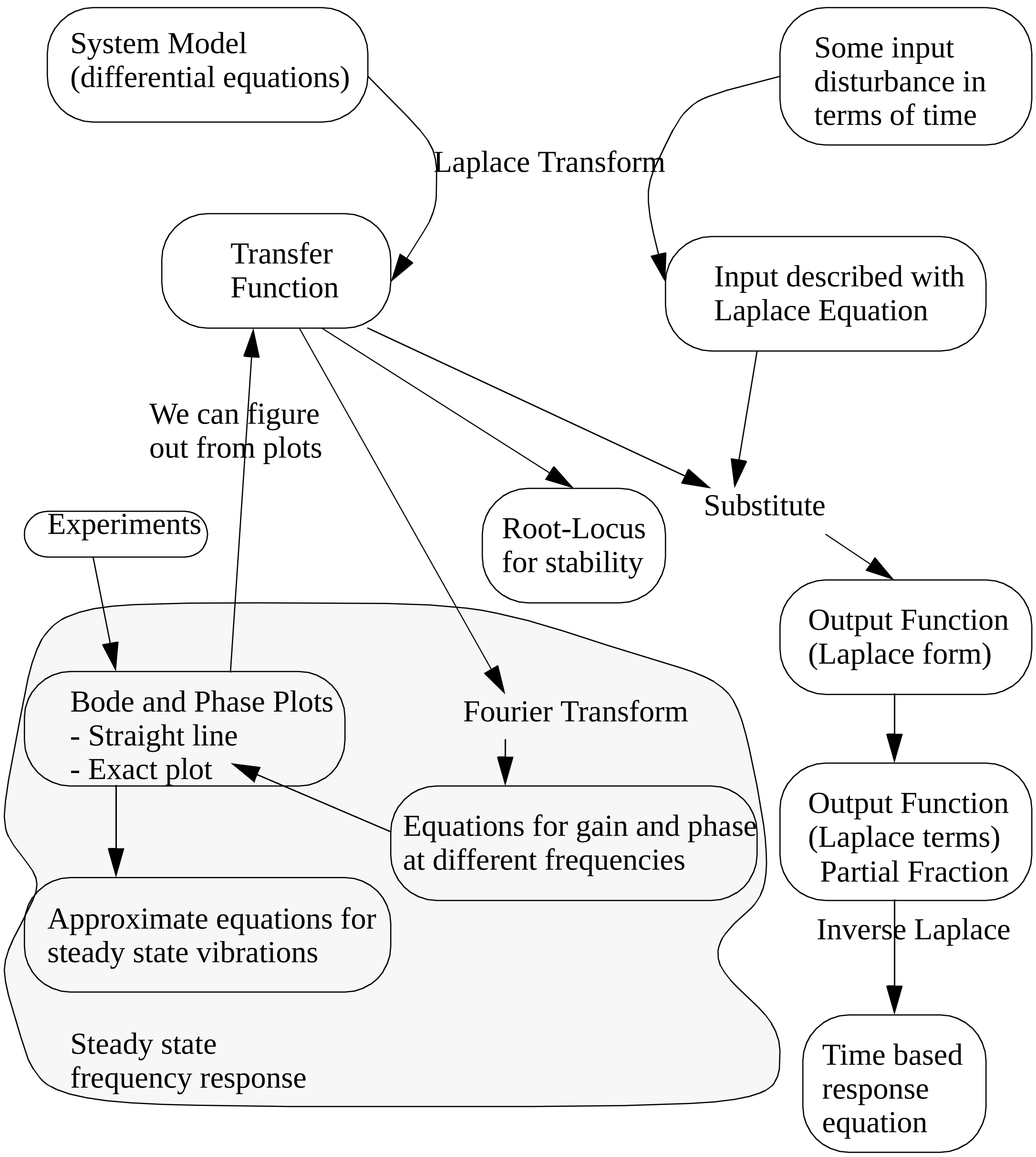


Figure 421 A map of Laplace analysis techniques

20.7 NON-LINEAR ELEMENTS

• If our models include a device that is non linear we will need to linearize the model before we can proceed.

• A non-linear system can be approximated with a linear equation using the following method.

1. Pick an operating point or range for the component.
2. Find a constant value that relates a change in the input to a change in the output.
3. Develop a linear equation.
4. Use the linear equation in the analysis (Laplace or other)

• Consider the example below,

In this case the relationship between pressure drop and flow are non-linear. We need to develop an equation that approximates the local operating point.

$$\Delta p \approx R(q - \bar{q})$$

$$R \approx \frac{\Delta(\Delta p)}{\Delta q} \qquad R = 2\frac{q}{K^2}$$

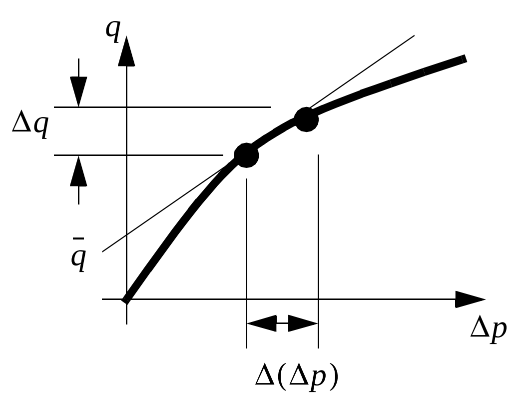


Figure 422 Linearizing non-linear elements

20.8 SUMMARY

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20.9 PRACTICE PROBLEMS

1. Convert the following functions from time to laplace functions.

a) $L[5]$

b) $L[e^{-3t}]$

c) $L[5e^{-3t}]$

d) $L[5te^{-3t}]$

e) $L[5t]$

f) $L[4t^2]$

g) $L[\cos(5t)]$

h) $L[\cos(5t+1)]$

i) $L[5e^{-3t}\cos(5t)]$

j) $L[5e^{-3t}\cos(5t+1)]$

k) $L[\sin(5t)]$

l) $L[\sinh(3t)]$

m) $L[t^2\sin(2t)]$

n) $L\left[\frac{d}{dt}t^2e^{-3t}\right]$

o) $L\left[\int_0^t x^2e^{-x}\ dx\right]$

p) $L\left[\frac{d}{dt}\sin(6t)\right]$

q) $L\left[\left(\frac{d}{dt}\right)^3 t^2\right]$

r) $L\left[\int_0^t ydy\right]$

s) $L[4\sin(2t)]$, for 0 < t < pi

t) $L[u(t-1)-u(t-2)]$

u) $L[e^{-2t}u(t-2)]$

v) $L[e^{-(t-3)}u(t-1)]$

w) $L[5e^{-3t}+u(t-1)-u(t-2)]$

x) $L[\cos(7t+2)+e^{t-3}]$

y) $L[3(t-1)+e^{-(t+1)}]$

z) $L[3t^3(t-1)+e^{-5t}]$

aa) $L[6e^{-2.7t}\cos(9.2t+3)]$

ans.

w) $\frac{5}{s+3} + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$

x) $\frac{\cos(2)s - \sin(2)7}{s^2 + 49} + \frac{e^{-3}}{s-1} = \frac{-0.416s - 6.37}{s^2 + 49} + \frac{e^{-3}}{s-1}$

y) $\frac{3}{s^2} - \frac{3}{s} + \frac{e^{-1}}{s+1}$

2. Convert the following functions below from the laplace to time domains.

a) $L^{-1}\left[\frac{1}{s+1}\right]$

b) $L^{-1}\left[\frac{5}{s+1}\right]$

c) $L^{-1}\left[\frac{6}{s^2}\right]$

d) $L^{-1}\left[\frac{6}{s^3}\right]$

e) $L^{-1}\left[\frac{s+2}{(s+3)(s+4)}\right]$

f) $L^{-1}\left[\frac{6}{s^2 + 5s + 6}\right]$

g) $L^{-1}\left[\frac{6}{4s^2 + 20s + 24}\right]$

h) $L^{-1}\left[\frac{6}{s^2 + 6}\right]$

i) $L^{-1}[5(1 - e^{-4.5s})]$

j) $L^{-1}\left[\frac{4+3j}{s+1-2j} + \frac{4-3j}{s+1+2j}\right]$

k) $L^{-1}\left[\frac{6}{s^4} + \frac{6}{s^2 + 9}\right]$

l) $L^{-1}[\ ]$

m) $L^{-1}[\ ]$

n) $L^{-1}[\ ]$

o) $L^{-1}[\ ]$

p) $L^{-1}[\ ]$

q) $L^{-1}[\ ]$

r) $L^{-1}[\ ]$

s) $L^{-1}[\ ]$

t) $L^{-1}[\ ]$

ans. a) e^{-t}

3. Convert the following functions below from the laplace to time domains using partial fractions.

a) $L^{-1}\left[\frac{s+2}{(s+3)(s+4)}\right]$

b) $L^{-1}\left[\frac{6}{s^2+5s+6}\right]$

c) $L^{-1}\left[\frac{6}{4s^2+20s+24}\right]$

d) $L^{-1}\left[\frac{6}{s^2+6}\right]$

e) $L^{-1}\left[\frac{6}{s^2+5s}\right]$

f) $L^{-1}\left[\frac{9s^2+6s+3}{s^3+5s^2+4s+6}\right]$

g) $L^{-1}\left[\frac{s^3+9s^2+6s+3}{s^3+5s^2+4s+6}\right]$

h) $L^{-1}\left[\frac{9s+4}{(s+3)^3}\right]$

i) $L^{-1}\left[\frac{9s+4}{s^3(s+3)^3}\right]$

j) $L^{-1}\left[\frac{s^2+2s+1}{s^2+3s+2}\right]$

k) $L^{-1}\left[\frac{s^2+3s+5}{6s^2+6}\right]$

l) $L^{-1}\left[\frac{s^2+2s+3}{s^2+2s+1}\right]$

m) $L^{-1}[\ \ ]$

n) $L^{-1}[\ \ ]$

o) $L^{-1}[\ \ ]$

p) $L^{-1}[\ \ ]$

q) $L^{-1}[\ \ ]$

r) $L^{-1}[\ \ ]$

s) $L^{-1}[\ \ ]$

t) $L^{-1}[\ \ ]$

ans.

j) $\delta(t) - e^{-2t}$

k) $\frac{\delta(t)}{6} + 0.834\cos(t + 0.927)$

l) $\delta(t) + 2te^{-t}$

4. Convert the following differential equations to transfer functions.

a) $5x'' + 6x' + 2x = 5F$

b) $y' + 8y = 3x$

c) $y' - y + 5x = 0$

d)

e)

f)

g)

h)

i)

j)

5. Given the following input functions and transfer functions, find the response in time.

	Transfer Function	Input
a)	$\frac{x(s)}{F(s)} = \frac{s+2}{(s+3)(s+4)}\left(\frac{m}{n}\right)$	$F(t) = 5N$
b)	$\frac{x(s)}{F(s)} = \frac{s+2}{(s+3)(s+4)}\left(\frac{m}{n}\right)$	$x(t) = 5m$
c)	$\frac{x(s)}{F(s)} =$	$F(t) =$
d)	$\frac{x(s)}{F(s)} =$	$x(t) =$
e)	$\frac{x(s)}{F(s)} =$	$F(t) =$
f)	$\frac{x(s)}{F(s)} =$	$x(t) =$

6. Prove the following relationships.

a) $L\left[f\left(\frac{t}{a}\right)\right] = aF(as)$

b) $L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$

c) $L[e^{-at}f(t)] = F(s+a)$

d) $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

e) $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

f) $L[tf(t)] = -\frac{d}{dt}F(s)$

g)

h)

i)

j)

k)

l)

V2. Given the transfer function, G(s), determine the time response output Y(t) to a step input X(t).

$$G(s) = \frac{4}{s+2} = \frac{Y(s)}{X(s)} \qquad X(t) = 20 \quad \text{When t >= 0}$$

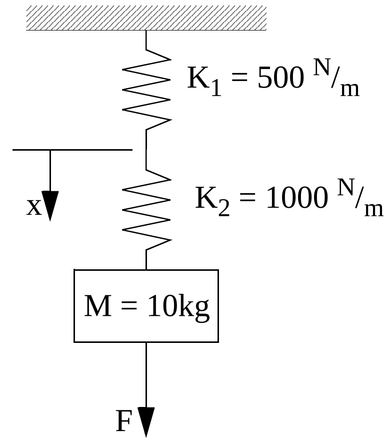
V3. Given a mass supported by a spring and damper, find the displacement of the supported mass over time if it is released from neutral at t=0sec, and gravity pulls it downward.

a) find the transfer function x/F (as a Laplace function of time).

b) find the input function F.
c) find the position as a function of ‘s’.
d) do the inverse Laplace transform to find the position as a function of time for Ks = 10N/m, Kd = 5Ns/m, M=10kg.

V4. The applied force ‘F’ is the input to the system, and the output is the displacement ‘x’.

a) find the transfer function.



b) What is the steady state response for an applied force F(t) = 10cos(t + 1) N ?
c) Give the transfer function if ‘x’ is the input.
d) Draw the bode plots.
e) Find x(t), given F(t) = 10N for t >= 0 seconds.

V5. Given the transfer function below,

$$\frac{y(s)}{x(s)} = \frac{(s+10)(s+5)}{(s+5)^2}$$

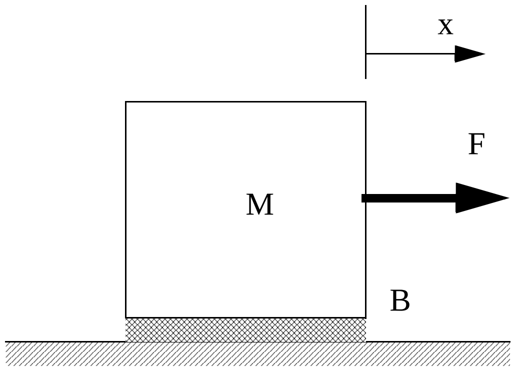
a) draw the straight line approximation of the bode and phase shift plots.
b) determine the steady state output if the input is x(s) = 20 cos(9t+.3)

V6. Convert the Laplace function below Y(s) to the time domain Y(t).

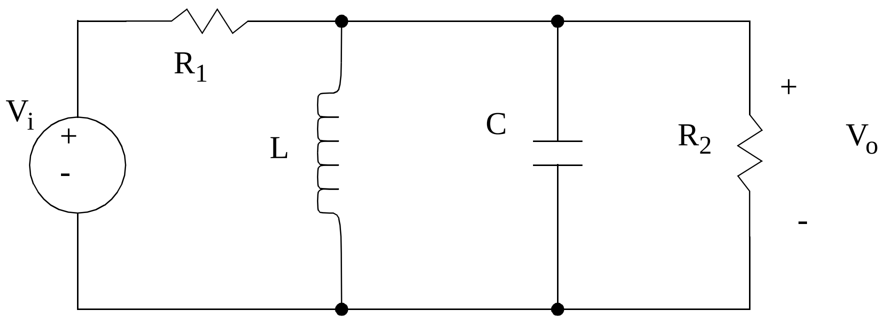
$$Y(s) = \frac{5}{s} + \frac{12}{s^2+4} + \frac{3}{s+2-3j} + \frac{3}{s+2+3j}$$

1. Develop differential equations and then transfer functions for the mechanical and electrical systems below.
a) There is viscous damping between the block and the ground. A force is applied

to cause the mass the accelerate.



b)



2. The following differential equation is supplied, with initial conditions.

$$y'' + y' + 7y = F \qquad y(0) = 1 \qquad y'(0) = 0$$

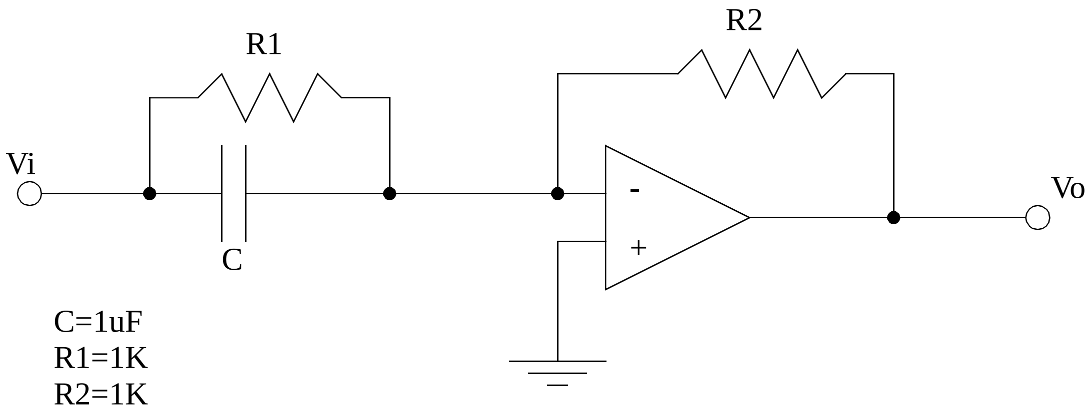
$$F(t) = 10 \qquad t > 0$$

a) Write the equation in state variable form.
b) Convert the differential equation to the Laplace domain, including initial conditions. Solve to find the time response to the given input using Laplace transforms.
c) Solve the differential equation using calculus techniques.
d) Find the frequency response (gain and phase) for the transfer function using the Fourier transform. Roughly sketch the bode plots.

3.

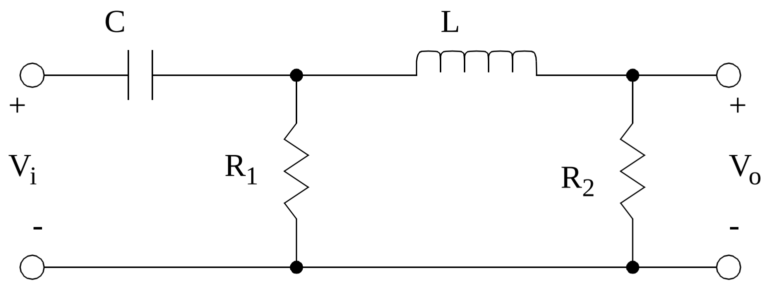
a) Write the differential equations for the system pictured below.
b) Put the equations in state variable form.
c) Use mathcad to find the ratio between input and output voltages for a range of frequencies. The general method is put in a voltage such as Vi=1sin(___t), and see what the magnitude of the output is. Divide the magnitude of the output sine wave by the input magnitude. Note: This should act as a high pass or low pass filter.

d) Plot a graph of gain against the frequency of the input.

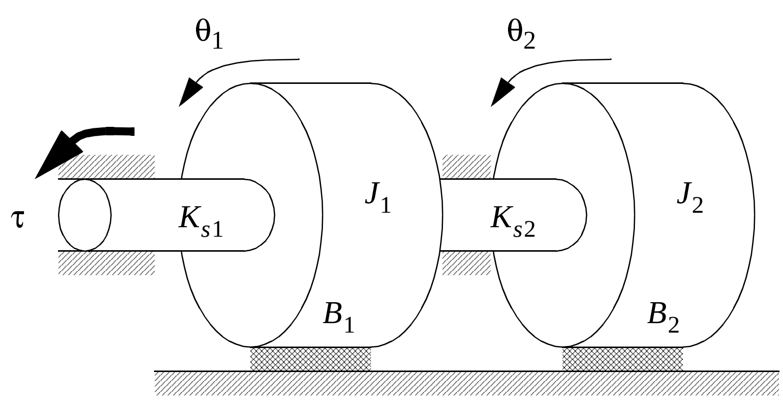


4. Find the transfer functions for the systems below.

a) Vi is the input and Vo is the output.

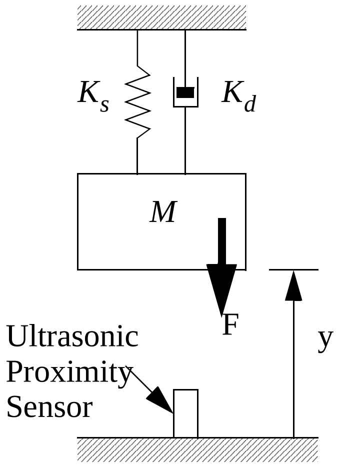


b) Here the input is a torque, and the output is the angle of the second mass.



5. Develop a transfer function for the system below. The input is the force ‘F’ and the output is the voltage ‘Vo’. The mass is suspended by a spring and a damper. When the spring is undeflected y=0. The height is measured with an ultrasonic proximity sensor. When y = 0, the output

Vo=0V. If y=20cm then Vo=2V and if y=-20cm then Vo=-2V. Neglect gravity.



$$K_s = 10\frac{N}{m} \qquad K_d = 5N\frac{s}{m} \qquad M = 0.5Kg$$

6. Do the following conversions as indicated.

a)

a) $L[5e^{-4t}\cos(3t+2)] =$

b) $L[e^{-2t} + 5t(u(t-2) - u(t))] =$

c) $L\left[\left(\frac{d}{dt}\right)^3 y + 2\left(\frac{d}{dt}\right)y + y\right] =$ where at t=0 $y_0 = 1$, $y_0' = 2$, $y_0'' = 3$, $y_0''' = 4$

d) $L^{-1}\left[\frac{1+j}{s+3+4j} + \frac{1-j}{s+3-4j}\right] =$

e) $L^{-1}\left[s + \frac{1}{s+2} + \frac{3}{s^2+4s+40}\right] =$

(ans. a)

$$L[5e^{-4t}\cos(3t+2)] = L[2|A|e^{-\alpha t}\cos(\beta t + \theta)]$$

$$\alpha = 4 \qquad \beta = 3$$
$$|A| = 2.5 \qquad \theta = 2$$

$$A = 2.5\cos 2 + 2.5j\sin 2 = -1.040 + 2.273j$$

$$\frac{A}{s+\alpha-\beta j} + \frac{A^{\text{complex conjugate}}}{s+\alpha+\beta j} = \frac{-1.040+2.273j}{s+4-3j} + \frac{-1.040-2.273j}{s+4+3j}$$

(ans. b)

$$L[e^{-2t} + 5t(u(t-2) - u(t))] = L[e^{-2t}] + L[5tu(t-2)] - L[5tu(t)]$$

$$= \frac{1}{s+2} + 5L[tu(t-2)] - \frac{5}{s^2} = \frac{1}{s+2} + 5L[(t-2)u(t-2) + 2u(t-2)] - \frac{5}{s^2}$$

$$= \frac{1}{s+2} + 5L[(t-2)u(t-2)] + 10L[u(t-2)] - \frac{5}{s^2}$$

$$= \frac{1}{s+2} + 5e^{-2s}L[t] + 10e^{-2s}L[1] - \frac{5}{s^2}$$

$$= \frac{1}{s+2} + \frac{5e^{-2s}}{s^2} + \frac{10e^{-2s}}{s} - \frac{5}{s^2}$$

(ans. c)

$$\left(\frac{d}{dt}\right)^3 y = s^3 y + 1s^2 + 2s^1 + 3s^0$$

$$\left(\frac{d}{dt}\right) y = s^1 y + s^0 1$$

$$L\left[\left(\frac{d}{dt}\right)^3 y + 2\left(\frac{d}{dt}\right) y + y\right] = (s^3 y + 1s^2 + 2s + 3) + (sy + 1) + (y)$$

$$= y(s^3 + s + 1) + (s^2 + 2s + 4)$$

(ans. d)

$$L^{-1}\left[\frac{1+j}{s+3+4j} + \frac{1-j}{s+3-4j}\right] = L^{-1}\left[\frac{A}{s+\alpha-\beta j} + \frac{A^{\text{complex conjugate}}}{s+\alpha+\beta j}\right]$$

$$|A| = \sqrt{1^2 + 1^2} = 1.141 \qquad \theta = \operatorname{atan}\left(\frac{-1}{1}\right) = -\frac{\pi}{4} \qquad \alpha = 3 \qquad \beta = 4$$

$$= 2|A|e^{-\alpha t}\cos(\beta t + \theta) = 2.282e^{-3t}\cos\left(4t - \frac{\pi}{4}\right)$$

(ans. e)

$$L^{-1}\left[s + \frac{1}{s+2} + \frac{3}{s^2 + 4s + 40}\right] = L[s] + L\left[\frac{1}{s+2}\right] + L\left[\frac{3}{s^2 + 4s + 40}\right]$$

$$= \frac{d}{dt}\delta(t) + e^{-2t} + L\left[\frac{3}{(s+2)^2 + 36}\right] = \frac{d}{dt}\delta(t) + e^{-2t} + 0.5L\left[\frac{6}{(s+2)^2 + 36}\right]$$

$$= \frac{d}{dt}\delta(t) + e^{-2t} + 0.5e^{-2t}\sin(6t)$$

7. Solve the following partial fractions by hand, and convert them back to functions of time. You may use your calculator to find roots of equations, and to verify the solutions.

a)
$$\frac{s^3 + 4s^2 + 4s + 4}{s^3 + 4s}$$

b)
$$\frac{s^2 + 4}{s^4 + 10s^3 + 35s^2 + 50s + 24}$$

(ans. a)
$$\frac{s^3 + 4s^2 + 4s + 4}{s^3 + 4s}$$

$$\begin{array}{r|l} & \quad 1 \\ \hline s^3 + 4s & s^3 + 4s^2 + 4s + 4 \\ & -(s^3 + 4s) \\ \hline & \quad 4s^2 + 4 \end{array}$$

$$= 1 + \frac{4s^2 + 4}{s^3 + 4s} = 1 + \frac{A}{s} + \frac{Bs + C}{s^2 + 4} = 1 + \frac{s^2(A + B) + s(C) + (4A)}{s^3 + 4s}$$

$$A = 1 \qquad C = 0 \qquad B = 3$$

$$= 1 + \frac{1}{s} + \frac{3s}{s^2 + 4}$$

$$= \delta(t) + 1 + 3\cos(2t)$$

(ans. b)
$$\frac{s^2 + 4}{s^4 + 10s^3 + 35s^2 + 50s + 24} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} + \frac{D}{s+4}$$

$$A = \lim_{s \to -1}\left(\frac{s^2 + 4}{(s+2)(s+3)(s+4)}\right) = \frac{5}{6}$$

$$B = \lim_{s \to -2}\left(\frac{s^2 + 4}{(s+1)(s+3)(s+4)}\right) = \frac{8}{-2}$$

$$C = \lim_{s \to -3}\left(\frac{s^2 + 4}{(s+1)(s+2)(s+4)}\right) = \frac{13}{2}$$

$$D = \lim_{s \to -4}\left(\frac{s^2 + 4}{(s+1)(s+2)(s+3)}\right) = \frac{20}{-6}$$

$$\frac{5}{6}e^{-t} - 4e^{-2t} + \frac{13}{2}e^{-3t} - \frac{10}{3}e^{-4t}$$

20.10 REFERENCES

Irwin, J.D., and Graf, E.R., Industrial Noise and Vibration Control, Prentice Hall Publishers, 1979.

Close, C.M. and Frederick, D.K., “Modeling and Analysis of Dynamic Systems, second edition, John Wiley and Sons, Inc., 1995.

21. CONTROL SYSTEM ANALYSIS

Topics:

Objectives:

21.1 INTRODUCTION

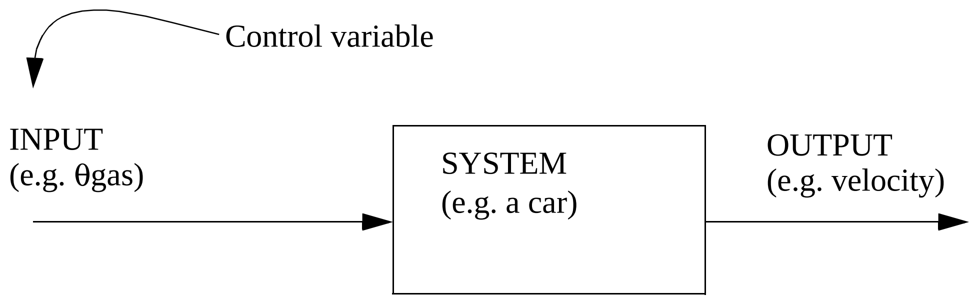
21.2 CONTROL SYSTEMS

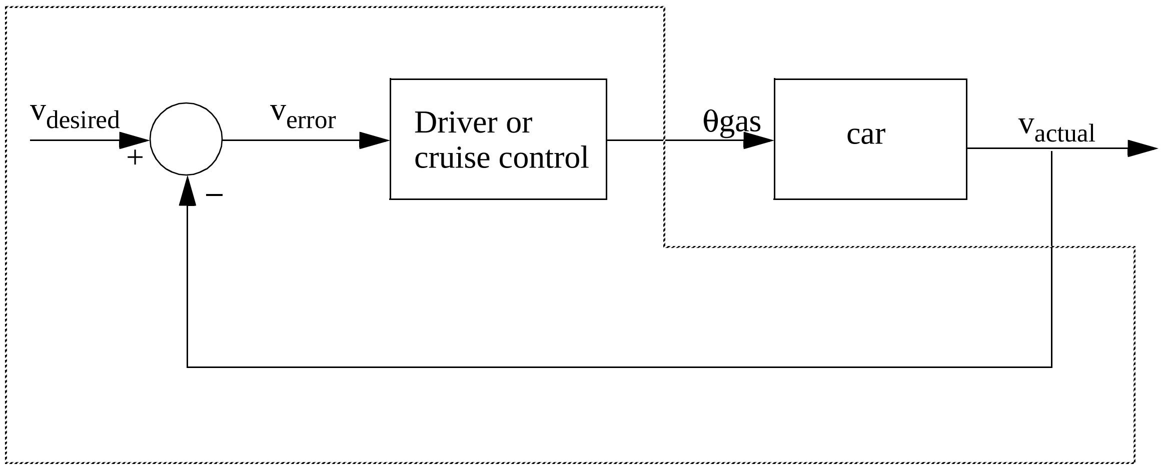
• Control systems use some output state of a system and a desired state to make control decisions.

• In general we use negative feedback systems because,

- they typically become more stable
- they become less sensitive to variation in component values
- it makes systems more immune to noise

• Consider the system below, and how it is enhanced by the addition of a control system.





The control system is in the box and could be a driver or a cruise control (this type is known as a feedback control system)

Figure 423 An example of a feedback controller

Human rules to control car (also like expert system/fuzzy logic):

1. If v_{error} is not zero, and has been positive/negative for a while, increase/decrease θ_{gas}
2. If v_{error} is very big/small increase/decrease θ_{gas}
3. If v_{error} is near zero, keep θ_{gas} the same
4. If v_{error} suddenly becomes bigger/smaller, then increase/decrease θ_{gas}.
5. etc.

Figure 424 Rules for a feedback controller

• Some of the things we do naturally (like the rules above) can be done with mathematics

21.2.1 PID Control Systems

• The basic equation for a PID controller is shown below. This function will try to compensate for error in a controlled system (the difference between desired and actual output values).

$$u = K_c e + K_i \int e dt + K_d \left(\frac{de}{dt} \right)$$

Figure 425 The PID control equation

• The figure below shows a basic PID controller in block diagram form.

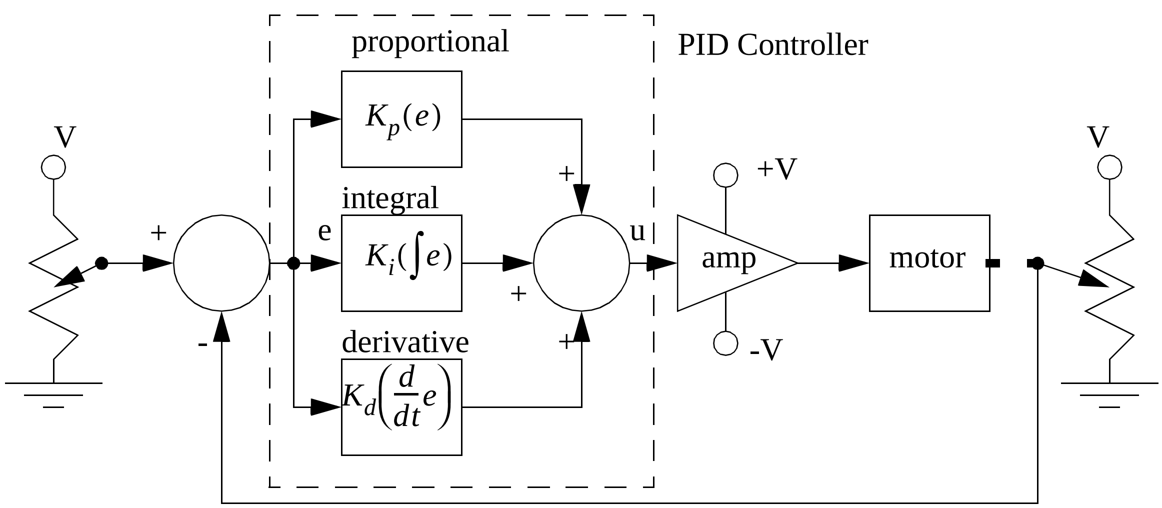


Figure 426 A block diagram of a feedback controller

e.g.

$$\theta_{gas} = K_c v_{error} + K_i \int v_{error} dt + K_d \left(\frac{dv_{error}}{dt} \right)$$

Rules 2 & 3
(general difference)

Rule 1
(Long term error)

Rule 4
(Immediate error)

Kc
Ki
Kd

Relative weights of components

This is a PID Controller

Proportional
Integral
Derivative

For a PI Controller

$$\theta_{gas} = K_c v_{error} + K_i \int v_{error} dt$$

For a P Controller

$$\theta_{gas} = K_c v_{error}$$

For a PD Controller

$$\theta_{gas} = K_c v_{error} + K_d \left(\frac{dv_{error}}{dt} \right)$$

• The PID controller is the most common controller on the market.

21.2.2 Analysis of PID Controlled Systems With Laplace Transforms

1. We can rewrite the control equation as a ratio of output to input.

$$\theta_{gas} = K_c v_{error} + K_i \int v_{error} dt + K_d \left(\frac{dv_{error}}{dt} \right)$$

$$\frac{\theta_{gas}}{v_{error}} = K_c + K_i \int dt + K_d \left(\frac{d}{dt} \right)$$

Then do a Laplace transform

$$\frac{d}{dt} \rightarrow s \qquad \frac{dx}{dt} \rightarrow sx$$

$$\int dt = \frac{1}{s} \qquad \int x dt = \frac{x}{s}$$

$$\boxed{L\left[\frac{\theta_{gas}}{v_{error}} \right] = K_c + \frac{K_i}{s} + K_d s}$$

The transfer function

2. We can also develop a transfer function for the car.

$$F = A\theta_{gas} = 10\theta_{gas}$$

$$\boxed{\frac{F}{\theta_{gas}} = 10}$$

Transfer function for engine and transmission. (Laplace transform would be the same as initial value.)

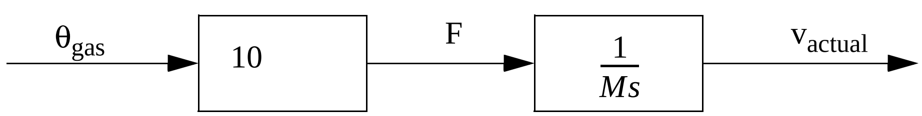
$$F = Ma = M\frac{d^2x}{dt^2} = M\frac{dv}{dt}$$

$$\frac{F}{v} = M\frac{d}{dt}$$

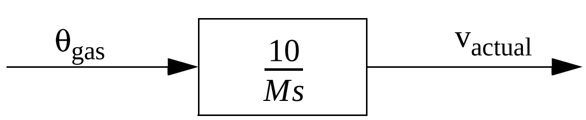
$$\boxed{L\left[\frac{v}{F}\right] = \frac{1}{Ms}}$$

Transfer function for acceleration of car mass

3. We want to draw the system model for the car.



- The 'system model' is shown above.
- If θ_{gas} is specified directly, this is called 'open loop control'. This is not desirable, but much simpler.
- The two blocks above can be replaced with a single one.



4. If we have an objective speed, and an actual speed, the difference is the 'system error'

$$v_{error} = v_{desired} - v_{actual}$$

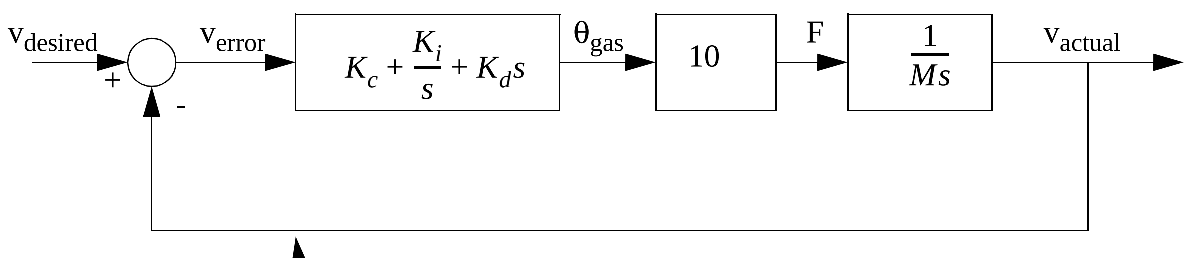
'set-point' - desired system operating point

5. Finally, knowing the error is v_{error}, and we can control θ_{gas} (the control variable), we can select a control system.

$$L\left[\frac{\theta_{gas}}{v_{error}}\right] = K_c + \frac{K_i}{s} + K_d s$$

*The coefficients can be calculated using classical techniques, but they are more commonly approximated by trial and error.

6. For all the components we can now draw a 'block diagram'

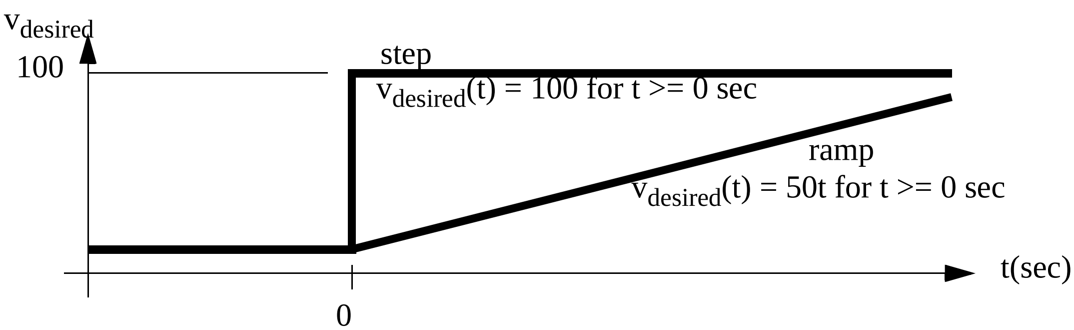


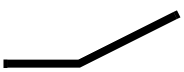
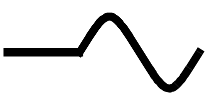
• A 'negative feedback loop that is the fundamental part of this 'closed loop control system'

21.2.3 Finding The System Response To An Input

• Even though the transfer function uses the Laplace 's', it is still a ratio of input to output.

• Find an input in terms of the Laplace 's'



	Input type	Time function	Laplace function
	STEP	$f(t) = Au(t)$	$f(s) = \frac{A}{s}$
	RAMP	$f(t) = Atu(t)$	$f(s) = \frac{A}{s^2}$
	SINUSOID	$f(t) = A\sin(\omega t)u(t)$	$f(s) = \frac{A\omega^2}{s^2 + \omega^2}$
	PULSE	$f(t) = A(u(t) - u(t - t_1))$	$f(s) =$
	etc......		

Therefore to continue the car example, lets assume the input below,

$$v_{desired}(t) = 100 \qquad\qquad t \geq 0 \sec$$

$$v_{desired}(s) = L[v_{desired}(t)] = \frac{100}{s}$$

Next, lets use the input, and transfer function to find the output of the system.

$$v_{actual} = \left(\frac{v_{actual}}{v_{desired}}\right) v_{desired}$$

$$v_{actual} = \left(\frac{s^2(K_d) + s(K_c) + K_i}{s^2\left(\frac{M}{10} + K_d\right) + s(K_c) + K_i}\right)\left(\frac{100}{s}\right)$$

To go further, some numbers will be selected for the values.

$K_d = 10000$
$K_c = 10000$
$K_i = 1000$
$M = 1000$

$$v_{actual} = \left(\frac{s^2(10000) + s(10000) + 1000}{s^2(10100) + s(10000) + 1000}\right)\left(\frac{100}{s}\right)$$

At this point we have the output function, but not in terms of time yet. To do this we break up the function into partial fractions, and then find inverse Laplace transforms for each.

$$v_{actual} = 10^2\left(\frac{s^2 + s + 0.1}{s(s^2(1.01) + s + 0.1)}\right)$$

Aside: We must find the roots of the equation, before we can continue with the partial fraction expansion.

recall the quadratic formula,

$$ax^2 + bx + c = 0 \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{-1 \pm \sqrt{1^2 - 4(1.01)(0.1)}}{2(1.01)} = -0.113, -0.877$$

$$v_{actual} = \frac{10^2}{1.01}\left(\frac{s^2 + s + 0.1}{s(s + 0.113)s + 0.877}\right)$$

$$v_{actual} = \frac{A}{s} + \frac{B}{s + 0.114} + \frac{C}{s + 0.795}$$

$$A = \lim_{s \to 0}\left[s\left(\frac{10^2}{1.01}\left(\frac{s^2 + s + 0.1}{s(s + 0.113)(s + 0.877)}\right)\right)\right] = \frac{10^2}{1.01}\left(\frac{0.1}{(0.113)(0.877)}\right)$$

$$\therefore A = 99.9$$

$$B = \lim_{s \to -0.113}\left[\left(\frac{10^2}{1.01}\left(\frac{s^2 + s + 0.1}{s(s + 0.113)(s + 0.877)}\right)\right)(s + 0.113)\right]$$

$$\therefore B = \left(\frac{10^2}{1.01}\left(\frac{(-0.113)^2 + (-0.113) + 0.1}{(-0.113)(-0.113 + 0.877)}\right)\right) = 0.264$$

$$C = \lim_{s \to -0.877}\left[\left(\frac{10^2}{1.01}\left(\frac{s^2 + s + 0.1}{s(s + 0.113)(s + 0.877)}\right)\right)(s + 0.877)\right]$$

$$\therefore C = \left(\frac{10^2}{1.01}\left(\frac{(-0.877)^2 + (-0.877) + 0.1}{(-0.877)(-0.877 + 0.113)}\right)\right) = -1.16$$

$$v_{actual} = \frac{99.9}{s} + \frac{0.264}{s + 0.113} - \frac{1.16}{s + 0.877}$$

Next we use a list of forward/inverse transforms to replace the terms in the partial fraction expansion.

$f(t)$	$f(s)$
A	$\frac{A}{s}$
At	$\frac{A}{s^2}$
$Ae^{-\alpha t}$	$\frac{A}{s+\alpha}$
$A\sin(\omega t)$	$\frac{A\omega}{s^2+\omega^2}$
$e^{-\xi\omega_n t}\sin(\omega_n t\sqrt{1-\xi^2})$	$\frac{\omega_n\sqrt{1-\xi^2}}{s^2+2\xi\omega_n s+\omega_n^2}$ $\quad for(\xi < 1)$

etc.

To finish the problem, we simply convert each term of the partial fraction back to the time domain.

$$v_{actual} = \frac{99.9}{s} + \frac{0.264}{s+0.113} - \frac{1.16}{s+0.877}$$

$$\boxed{v_{actual} = 99.9 + 0.264e^{-0.113t} - 1.16e^{-0.877t}}$$

21.2.4 Controller Transfer Functions

• The table below is for typical control system types,

Type	Transfer Function
Proportional (P)	$G_c = K$
Proportional-Integral (PI)	$G_c = K\left(1 + \frac{1}{\tau s}\right)$
Proportional-Derivative (PD)	$G_c = K(1 + \tau s)$
Proportional-Integral-Derivative (PID)	$G_c = K\left(1 + \frac{1}{\tau s} + \tau s\right)$
Lead	$G_c = K\left(\frac{1 + \alpha\tau s}{1 + \tau s}\right) \qquad \alpha > 1$
Lag	$G_c = K\left(\frac{1 + \tau s}{1 + \alpha\tau s}\right) \qquad \alpha > 1$
Lead-Lag	$G_c = K\left[\left(\frac{1 + \tau_1 s}{1 + \alpha\tau_1 s}\right)\left(\frac{1 + \alpha\tau_2 s}{1 + \tau_2 s}\right)\right] \qquad \alpha > 1, \; \tau_1 > \tau_2$

21.3 ROOT-LOCUS PLOTS

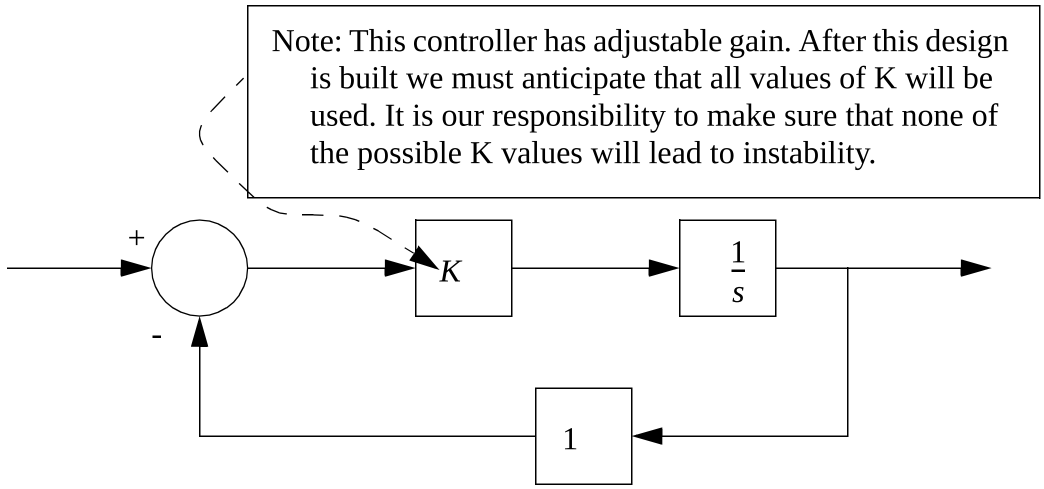
• Consider the basic transform tables. A superficial examination will show that the denominator (bottom terms) are the main factor in determining the final form of the solution. To explore this further, consider that the roots of the denominator directly impact the partial fraction expansion and the following inverse Laplace transfer.

• When designing a controller with variable parameters (typically variable gain), we need to determine if any of the adjustable gains will lead to an unstable system.

• Root locus plots allow us to determine instabilities (poles on the right hand side of the plane), overdamped systems (negative real roots) and oscillations (complex roots).

• Note: this procedure can take some time to do, but the results are very important when designing a control system.

• Consider the example below,



$$G(s) = \frac{K}{s} \qquad H(s) = 1$$

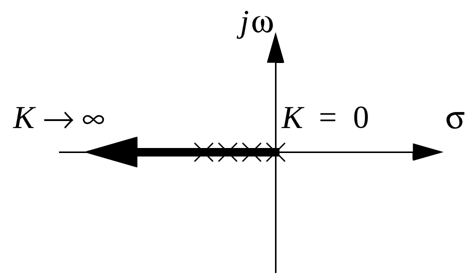
First, we must develop a transfer function for the entire control system.

$$G_S(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{\left(\frac{K}{s}\right)}{1 + \left(\frac{K}{s}\right)(1)} = \frac{K}{s + K}$$

Next, we use the characteristic equation of the denominator to find the roots as the value of K varies. These can then be plotted on a complex plane. Note: the value of gain ’K’ is normally found from 0 to +infinity.

$$s + K = 0$$

K	root
0	
1	
2	
3	
etc..	



Note: because all of the roots for all values of K are real negative this system will always be stable, and it will always tend to have a damped response. The large the value of K, the more stable the system becomes.

• Consider the previous example, the transfer function for the whole system was found, but then only the denominator was used to determine stability. So in general we do not need to find the transfer function for the whole system.

Consider the general form for a negative feedback system.

$$G_S(s) = \frac{G(s)}{1 + G(s)H(s)}$$

Note: two assumptions that are not often clearly stated are that we are assuming that the control system is a negative feedback controller, and that when not given the feedback gain is 1.

The system response is a function of the denominator, and it's roots.

$$1 + G(s)H(s) = 0$$

It is typical, (especially in textbook problems) to be given only G(s) or G(s)H(s).

The transfer function values will often be supplied in a pole zero form.

$$G(s)H(s) = \frac{K(s + z_0)(s + z_1)\ldots(s + z_m)}{(s + p_0)(s + p_1)\ldots(s + p_n)}$$

• Consider the example,

Given the system elements (you should assume negative feedback),

$$G(s) = \frac{K}{s^2 + 3s + 2} \qquad H(s) = 1$$

First, find the characteristic equation,. and an equation for the roots,

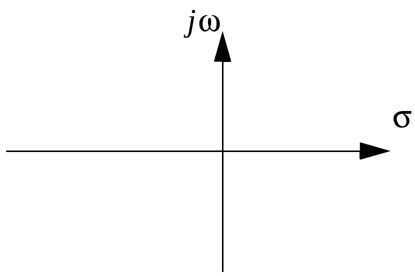
$$1 + \left(\frac{K}{s^2 + 3s + 2}\right)(1) = 0$$

$$s^2 + 3s + 2 + K = 0$$

$$\text{roots} = \frac{-3 \pm \sqrt{9 - 4(2 + K)}}{2} = -1.5 \pm \frac{\sqrt{1 - 4K}}{2}$$

Next, find values for the roots and plot the values,

K	root
0	0
1	-1
2	-2
3	-3



******CALCULATE AND PUT IN NUMBERS

21.3.1 Approximate Plotting Techniques

• The basic procedure for creating root locus plots is,

1. write the characteristic equation. This includes writing the poles and zeros of the

equation.

$$1 + G(s)H(s) = 1 + K\frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)} = 0$$

2. count the number of poles and zeros. The difference (n-m) will indicate how many root loci lines end at infinity (used later).
3. plot the root loci that lie on the real axis. Points will be on a root locus line if they have an odd number of poles and zeros to the right. Draw these lines in.
4. determine the asymptotes for the loci that go to infinity using the formula below. Next, determine where the asymptotes intersect the real axis using the second formula. Finally, draw the asymptotes on the graph.

$$\beta(k) = \frac{\pm 180°(2k+1)}{n-m} \qquad k \in [0, n-m-1]$$

$$\sigma = \frac{(p_1 + p_2 + \ldots + p_n)(z_1 + z_2 + \ldots + z_m)}{n-m}$$

5. the breakaway and breakin points are found next. Breakaway points exist between two poles on the real axis. Breakin points exist between zeros. to calculate these the following polynomial must be solved. The resulting roots are the breakin/breakout points.

$$A = (s+p_1)(s+p_2)\ldots(s+p_n) \qquad B = (s+z_1)(s+z_2)\ldots(s+z_m)$$

$$\left(\frac{d}{ds}A\right)B - A\left(\frac{d}{ds}B\right) = 0$$

6. Find the points where the loci lines intersect the imaginary axis. To do this substitute the fourier frequency for the laplace variable, and solve for the frequencies. Plot the asymptotic curves to pass through the imaginary axis at this point.

$$1 + K\frac{(j\omega+z_1)(j\omega+z_2)\ldots(j\omega+z_m)}{(j\omega+p_1)(j\omega+p_2)\ldots(j\omega+p_n)} = 0$$

• Consider the example in the previous section, Given the system elements (you should assume negative feedback),

$$G(s) = \frac{K}{s^2 + 3s + 2} \qquad H(s) = 1$$

Step 1: (put equation in standard form)

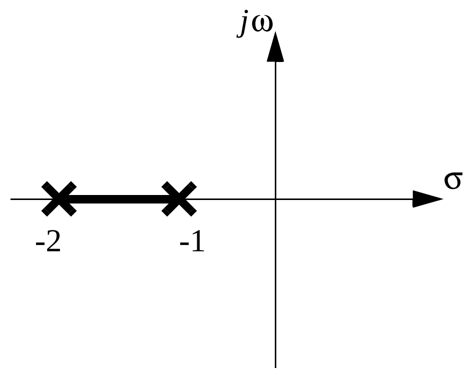
$$1 + G(s)H(s) = 1 + \left(\frac{K}{s^2 + 3s + 2}\right)(1) = 1 + K\frac{1}{(s+1)(s+2)}$$

Step 2: (find loci ending at infinity)

$m = 0 \qquad n = 2$ (from the poles and zeros of the previous step)

$n - m = 2$ (loci end at infinity)

Step 3: (plot roots)



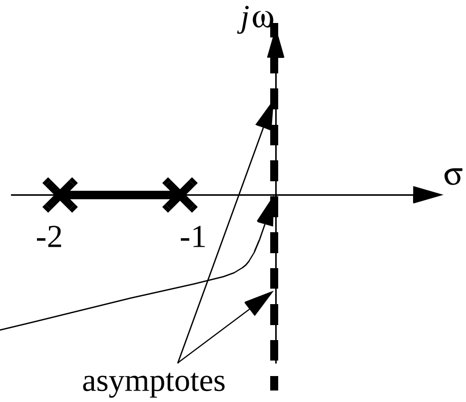
Step 4: (find asymptotes angles and real axis intersection)

$$\beta(k) = \frac{180°(2k+1)}{2} \qquad k \in I[0, 1]$$

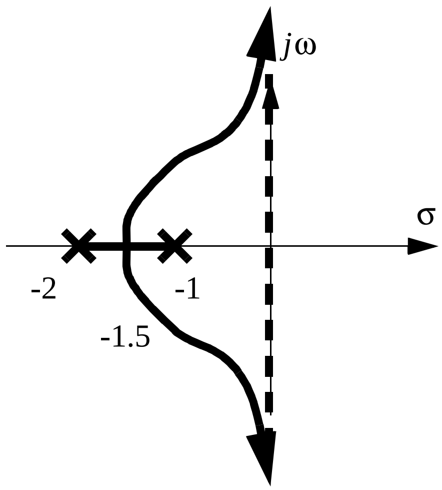
$$\beta(0) = \frac{180°(2(0)+1)}{2} = 90°$$

$$\beta(1) = \frac{180°(2(1)+1)}{2} = 270°$$

$$\sigma = \frac{(0)(-1-2)}{2} = 0$$



Step 5: (find the breakout points for the roots)



$$A = 1 \qquad B = s^2 + 3s + 2$$

$$\frac{d}{ds}A = 0 \qquad \frac{d}{ds}B = 2s + 3$$

$$A\left(\frac{d}{ds}B\right) - B\left(\frac{d}{ds}A\right) = 0$$

$$1(2s+3) - (s^2 + 3s + 2)(0) = 0$$

$$2s + 3 = 0$$

$$s = -1.5$$

Note: because the loci do not intersect the imaginary axis, we know the system will be stable, so step 6 is not necessary, but we it will be done for illustrative purposes.

Step 6: (find the imaginary intercepts)

$$1 + G(s)H(s) = 0$$

$$1 + K\frac{1}{s^2 + 3s + 2} = 0$$

$$s^2 + 3s + 2 + K = 0$$

$$(j\omega)^2 + 3(j\omega) + 2 + K = 0$$

$$-\omega^2 + 3j\omega + 2 + K = 0$$

$$\omega^2 + \omega(-3j) + (-2 - K) = 0$$

$$\omega = \frac{3j \pm \sqrt{(-3j)^2 - 4(-2-K)}}{2} = \frac{3j \pm \sqrt{-9 + 8 + 4K}}{2} = \frac{3j \pm \sqrt{4K - 1}}{2}$$

In this case the frequency has an imaginary value. This means that there will be no frequency that will intercept the imaginary axis.

- Plot the root locus diagram for the function below,

$$G(s)H(s) = \frac{K(s+5)}{s(s^2+4s+8)}$$

21.4 DESIGN OF CONTINUOUS CONTROLLERS

21.5 SUMMARY

•

21.6 PRACTICE PROBLEMS

3. Given the transfer function below, and the input 'x(s)', find the output 'y(t)' as a function of time.

$$\frac{y(s)}{x(s)} = \frac{5}{s+2} \qquad\qquad x(t) = 5 \qquad t \geq 0\,\text{sec}$$

8. Draw a detailed root locus diagram for the transfer function below. Be careful to specify angles of departure, ranges for breakout/breakin points, and gains and frequency at stability limits.

$$G(s) = \frac{2K(s+0.5)(s^2+2s+2)}{s^3(s+1)(s+2)}$$

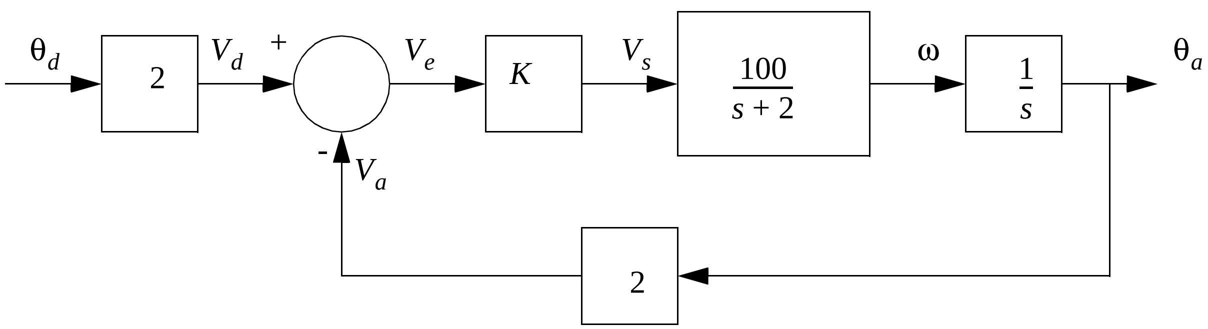
10. Draw the root locus diagram for the transfer function below,

$$G(s) = \frac{K(s+4)^2}{s^2(s+1)}$$

11. Draw the root locus diagram for the transfer function below,

$$G(s) = \frac{K(s+1)(s+2)}{s^3}$$

12. The block diagram below is for a motor position control system. The system has a proportional controller with a variable gain K.



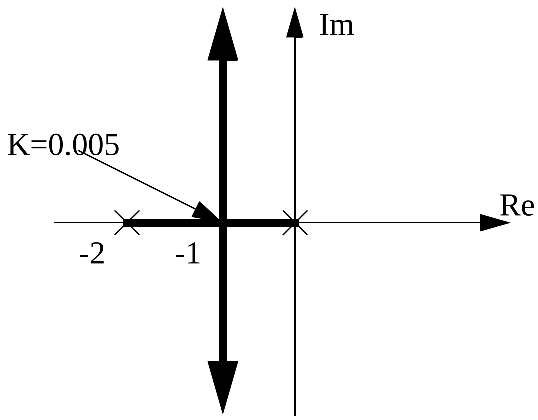
a) Simplify the block diagram to a single transfer function.

ans. $$\frac{200K}{s^2+2s+200K}$$

b) Draw the Root-Locus diagram for the system (as K varies). Use either the approximate or exact techniques.

ans.
$$roots = \frac{-2 \pm \sqrt{4 - 4(200K)}}{2} = -1 \pm \sqrt{1 - 200K}$$

K	roots
0	
0	0



c) Select a K value that will result in an overall damping coefficient of 1. State if the Root-Locus diagram shows that the system is stable for the chosen K.

(ans. $s^2 + 2s + 200K = s^2 + 2\zeta\omega_n s + \omega_n^2$ $\quad \therefore \omega_n = 1$ $\quad \therefore K = 0.005$

From the root locus graph this value is critically stable.

13. Draw a Bode Plot for either one of the two transfer functions below.

$$\frac{(s+1)(s+1000)}{(s+100)^2} \qquad \text{OR} \qquad \frac{5}{s^2}$$

15. Given the system transfer function below.

$$\frac{\theta_o}{\theta_d} = \frac{20K}{s^2 + s + 20K}$$

a) Draw the root locus diagram and state what values of K are acceptable.
b) Select a gain value for K that has either a damping factor of 0.707 or a natural frequency of 3 rad/sec.
c) Given a gain of K=10 find the steady state response to an input step of 1 rad.
d) Given a gain of K=10 find the response of the system as

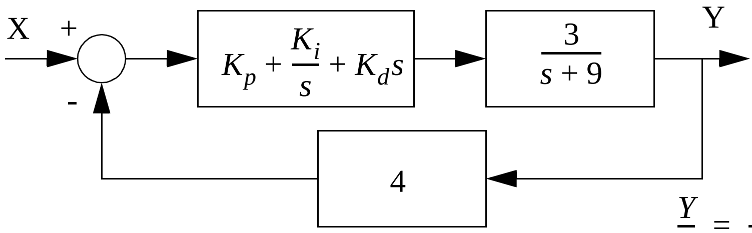
17. The equation below describes a dynamic system. The input is ‘F’ and the output is ‘V’. It has the initial values specified. The following questions ask you to find the system response to a

unit step input using various techniques.

$$V'' + 10V' + 20V = -20F \qquad V(0) = 1 \qquad V'(0) = 2$$

a) Find the response using Laplace transforms.
b) Find the response using the homogenous and particular solutions.
c) Put the equation is state variable form, and solve it using your calculator. Sketch the result accurately below.

18. A feedback control system is shown below. The system incorporates a PID controller. The closed loop transfer function is given.



$$\frac{Y}{X} = \frac{s^2(3K_d) + s(3K_p) + (3K_i)}{s^2(4 + 3K_d) + s(36 + 3K_p) + (3K_i)}$$

a) Verify the close loop controller function given.
b) Draw a root locus plot for the controller if Kp=1 and Ki=1. Identify any values of Kd that would leave the system unstable.
c) Draw a Bode plot for the feedback system if Kd=Kp=Ki=1.
d) Select controller values that will result in a natural frequency of 2 rad/sec and damping coefficient of 0.5. Verify that the controller will be stable.
e) For the parameters found in the last step find the initial and final values.
f) If the values of Kd=1 and Ki=Kd=0, find the response to a ramp input as a function of time.

19. The following system is a feedback controller for an elevator. It uses a desired heigh ‘d’ provided by a user, and the actual height of the elevator ‘h’. The difference between these two is called the error ‘e’. The PID controller will examine the value ‘e’ and then control the speed of the lift motor with a control voltage ‘c’. The elevator and controller are described with transfer functions, as shown below. all of these equations can be combined into a single system transfer

equation as shown.

$$e = d - h \qquad \text{error}$$

$$\frac{c}{e} = K_p + \frac{K_i}{s} + K_d s = \frac{2s + 1 + s^2}{s} \qquad \text{PID controller} \qquad\qquad \frac{h}{c} = \frac{10}{s^2 + s} \qquad \text{elevator}$$

combine the transfer functions

$$\left(\frac{c}{e}\right)\left(\frac{h}{c}\right) = \frac{h}{e} = \frac{2s + 1 + s^2}{s}\frac{10}{s^2 + s} = \frac{(s+1)^2}{s}\frac{10}{s(s+1)} = \frac{10(s+1)}{s^2}$$

$$\frac{h}{d - h} = \frac{10(s+1)}{s^2} \qquad \text{eliminate 'e'}$$

$$h = \left(\frac{10(s+1)}{s^2}\right)(d - h)$$

$$h\left(1 + \frac{10(s+1)}{s^2}\right) = \left(\frac{10(s+1)}{s^2}\right)(d)$$

$$\frac{h}{d} = \left(\frac{\frac{10(s+1)}{s^2}}{1 + \frac{10(s+1)}{s^2}}\right) = \frac{10s + 10}{s^2 + 10s + 10} \qquad \text{system transfer function}$$

a) Find the response of the final equation to a step input. The system starts at rest on the ground floor, and the input (desired height) changes to 20 as a step input.
b) Write find the damping coefficient and natural frequency of the results in part a).
c) verify the solution using the initial and final value theorems.

(ans. a) $\frac{h}{d} = \frac{10s + 10}{s^2 + 10s + 10}$ $\quad d(t) = 20u(t) \quad d(s) = \frac{20}{s}$

$$h = \left(\frac{10s + 10}{s^2 + 10s + 10}\right)\frac{20}{s} = \frac{A}{s} + \frac{B}{s + 5 - 3.873j} + \frac{C}{s + 5 + 3.873j}$$

$$A = \lim_{s \to 0}\left(\frac{200s + 200}{s^2 + 10s + 10}\right) = 20$$

$$B = \lim_{s \to -5 + 3.873j}\left(\frac{200s + 200}{s(s + 5 + 3.873j)}\right) = -2.5 - 22.6j$$

$$C = -2.5 + 22.6j$$

$$h(t) = 20 + L^{-1}\left[\frac{-2.5 - 22.6j}{s + 5 - 3.873j} + \frac{-2.5 + 22.6j}{s + 5 + 3.873j}\right]$$

$$|A| = \sqrt{2.5^2 + 22.6^2} = 22.73$$

$$\theta = 4.602$$

$$\alpha = 5$$

$$\beta = 3.873$$

$$h(t) = 20 + 2(22.73)e^{-5t}\cos(3.873t + 4.602)$$

b)

$$-5 = -\omega_n\zeta \qquad \zeta = \frac{5}{\omega_n}$$

$$3.873 = \sqrt{1 - \zeta^2}\,\omega_n = \sqrt{1 - \frac{25}{\omega_n^2}}\,\omega_n$$

$$15 = \left(1 - \frac{25}{\omega_n^2}\right)\omega_n^2 = \omega_n^2 - 25$$

$$\omega_n = \sqrt{35} = 5.916$$

$$\zeta = \frac{5}{\sqrt{35}} = 0.845$$

c)

$$h(0) = \lim_{s \to \infty}\left[s\left(\frac{10s + 10}{s^2 + 10s + 10}\right)\frac{20}{s}\right] = \lim_{s \to \infty}\left[\left(\frac{10s}{s^2}\right)20\right] = 0$$

$$h(\infty) = \lim_{s \to 0}\left[s\left(\frac{10s + 10}{s^2 + 10s + 10}\right)\frac{20}{s}\right] = \frac{(10)20}{10} = 20$$

22. LABORATORY GUIDE

• The laboratory work will help enforce the concepts presented in this and previous courses.

• Various labs will require pre- or post-lab work.

• General rules include:

- Unless specified, work is to be done individually.
- All written work is to be clear and accurate.

22.1 Lab 1 - Introduction to Resources and Tutorials

• These tutorials prepare you to use computer and other resources throughout the semester.

22.1.1 Tutorial 1a - Creating Web Pages

• The general steps are:

1. Get a computer account on ‘claymore.engineer.gvsu.edu’ from Prof. Jack. This account will have a prototype web page that you can edit.
2. Go to a laboratory (EC 616), or home computer and run ‘Netscape Communicator’. Go to ‘claymore.engineer.gvsu.edu’ and look for your account under ‘students’. You should be able to find a page that starts with ‘YOUR_NAME_GOES_HERE’.
3. In Netscape (with your home page showing), select ‘edit’ from the tool bar, or under ‘file’ select ‘edit’ or ‘edit page’. You will be asked if you want to save the page. Create a ‘temp’ directory on the computer. This directory will be used to temporarily hold your web page files. Make sure that the files will be saved in the ‘temp’ directory, and then ‘save’ the files. An editor will start on the screen.
5. The editor behaves much like Microsoft Word, with some subtle differences. At this point add your name, and change your email address to your river account. You can change your email address by clicking on the email link, and then clicking on the chain link near the top of the screen.
6. To upload the changes you have made to the website, select ‘publish’. You will need to indicate the file name as ‘index.html’, the destination as ‘ftp://claymore.engineer.gvsu.edu/home/YOUR_NAME/public_html’. You will also

need to enter your user name and your password (DO NOT SAVE THE PASSWORD - SOMEBODY ELSE CAN GET ACCESS TO YOUR ACCOUNT). You should see a message that indicates files have been uploaded successfully.

7. Use Netscape, not the editor, to see if the changes have occurred. Your changes may not show up on the browser. This is because Netscape does not reload pages every time to look at them. Pages are often stored for up to 1 month on the PCs hard drive, and reused when you look at them. There are two ways to update the screen before this time limit - click on the reload button.
8. Next we will add links to your home page. First, run Mathcad, and create a simple file, and then save it in the same folder/directory you saved. Use a file name that is all lower case such as 'test.mcd' - any upper case letters cause problems in Windows 95.
9. Get your home page back in the Netscape editor. Someplace type the word 'GVSU'. Use the mouse to select what you just typed, and then click on the link button. For the link name enter 'http://www.gvsu.edu', and apply the change. This will now be a link to the Grand Valley home page. For your Mathcad file type something like 'Mathcad file', highlight it, and add a link to 'test.mcd'. This link will connect to your Mathcad file.
10. Publish the file, but first add the Mathcad file to the list of files at the bottom of the screen.
11. Test the page.

• Some tips are,

- Windows will not allow multiple applications to open the same file at the same time. If you seem to be having trouble opening a file, make sure it is not open in another application.
- As you add other files to your homepage, put them in the 'temp' directory. This will make all of the procedures simpler.
- Try to make your web pages small, and link them together. This will decrease download time and make browsers happier.
- Avoid using excessive images. Anything over 10K will make it very slow downloading over modem. Anything over 100K makes modem downloading painfully slow.
- When putting images on the web page use 'jpg' for photographic images, and 'gif' for line images. 'jpg' images can be compressed more than 'gif', but lines will become blurred.
- To link to other files or web pages there will be a 'link' command. If you want to add a file that is in your 'temp' directory, just put the name of the file in the 'URL' field.
- Watch upper/lower case. This is a major cause of web page problems. It is best to keep to lower case for all file names.

22.1.2 Tutorial 1b - Introduction to Mathcad, Working Model 2D and The Internet

• Objective:

Working Model 2D, Mathcad and the Internet will be used in this course. In some cases students have not been exposed to one or more of these software packages in the past. This session will be used as a refresher for those with little prior exposure, and as a tutorial for those with no experience.

• Theory:

Mathcad is a software package that allows us to do complex calculations both numerically and symbolically. To learn it initially will require a time investment. But, when doing calculations later, it will save a significant amount of time and reduce calculation errors.

Working Model 2D is a software package that allows us to set up systems of multiple rigid bodies. We can then apply forces, moments, etc. and then see how the system dynamics are effected. In comparison, Mathcad will allow longer, precise calculations, whereas Working Model allows faster results with reduced accuracy. Working Model also presents a visual simulation - this allows a more intuitive understanding of a dynamic system.

The Internet is a huge collection of computers providing information and connection on an unprecedented scale. It has become a standard business tool, and continues to evolve.

• Procedure

1. (If needed) Go over the Mathcad tutorial provided.
2. Use Mathcad to calculate the position of a ball that has been held then released just above the surface of the earth, and add the file to your home page.
3. (If needed) Go over the Working Model tutorial provided.
4. Repeat the problem solved in Mathcad with Working Model and add the file to your home page.
5. (If needed) Get a computer account set up, and create a home page.
6. Go to a search engine and find a website for a major business that is related to your co-op position and add a link to it on your home page. Explain how the business is related to your co-op position.

• Post-lab:

None

• Submit:

1. A Mathcad file linked to your home page.
2. A Working Model file linked to your home page.
3. An explanation and a link to a company on your home page.

22.1.3 Presentation 1a - Introduction to Library Searches

• Objective:

To prepare students to use the libraries resources in typical research studies.

• Theory:

The essential purpose of engineering is to apply principles of the arts and sciences to solve real problems. Scientific principles tend to evolve over time, but the essential principles and written works are valid for a number of decades or centuries. As a result, books can be excellent resources for this knowledge. The applications that make use of the basic principles tend to be more revolutionary. As a result, printed books have a value for teaching the fundamentals, but the ’state of the art’ must often be found in magazines, journals, etc. To put this in simpler terms, when we look for scientific resources, we will often use sources over a decade old. When using engineering resources, most will be less than five years old. Until recently, print has been the major means of exchanging information, and libraries have been the traditional repositories of printed materials. To deal with the extensive number of publications available in a library, we need to learn how to search for needed information, and what resources are available.

New technology has changed access to library materials. Libraries pool resources and share materials. Internet technology has also helped increase accessibility. In particular internet tools allow the entire library catalog to be examined without visiting the library. There are also a number of resources that can be searched and retrieved over the internet.

• Procedure

1. A presentation will be made by Mr. Lee Lebin, the University Library Director.
2. Use the library resources to identify an application of systems modeling.

• Post-lab:

1. Search for library resource.

• Submit:

1. A copy of the material referenced.

22.2 Lab 2 - Computer Based Data Collection

22.2.1 Prelab 2a - Tutorial for LabVIEW Programming

Objective:

To learn the basic use of LabVIEW.

Theory:

To obtain the greatest computing power and flexibility we need to write computer programs. But, traditional programming languages are not well suited to designing user interfaces and dealing with data flows.

Most computer programs are written with lines of program and compiled to execute. LabVIEW allows you to “write” programs using graphical symbols. This graphical programming approach allows systems to be designed by connecting the symbols with "wires" (i.e., lines).

Equipment:

PC with LabVIEW software

Procedure:

1. Go through the LabVIEW QuickStart Guide provided in the laboratory. This will also be good review for those who have used LabVIEW in previous courses.
2. Write a Labview program that will count from 1 to 100, square the values, and print the results on a strip chart.

Marking:

1. The VI created should be posted to the web.

22.2.2 Prelab 2b - Overview of Labview and the DAQ Cards

• To obtain the greatest computing power and flexibility we need to write computer programs. But, traditional programming languages are not well suited to designing user interfaces and dealing with data flows.

• LabVIEW allows you to “write” programs using graphical symbols. This graphical programming approach allows systems to be designed by connecting the symbols with "wires" (i.e., lines).

• The remainder of the labs will focus on using LabVIEW to write programs to allow a computer to interact with the environment outside the computer.

• The computers we will use all have DAQ (Data AcQuisition) boards - National Instruments PCI-1200 DAQ cards. These cards have capabilities that include:

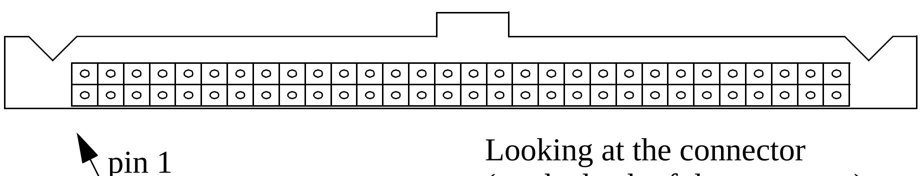
24 I/O bits - TTL 0,5VDC, 20mA max.
8 single ended or 4 double ended analog inputs - 12 bits
3 counters - 16 bits
2 analog outputs - 12 bits

• The connector for the card can be found on the back of the computer. It will have a connector with pinouts like the one shown below. A ribbon cable will be used to make electrical connection to the connector in the back of the computer.

Signal	Pin	Pin	Signal
ACH1	2	1	ACH0
ACH3	4	3	ACH2
ACH5	6	5	ACH4
ACH7	8	7	ACH6
DAC0OUT	10	9	AISENSE/AIGND
DAC1OUT	12	11	AGND
PA0	14	13	DGND
PA2	16	15	PA1
PA4	18	17	PA3
PA6	20	19	PA5
PB0	22	21	PA7
PB2	24	23	PB1
PB4	26	25	PB3
PB6	28	27	PB5
PC0	30	29	PB7
PC2	32	31	PC1
PC4	34	33	PC3
PC6	36	35	PC5
EXTTRIG	38	37	PC7
EXTCONV	40	39	EXTUPDATE
GATB0	42	41	OUTB1
GATB1	44	43	OUTB2
OUTB2	46	45	CLKB1
CLKB2	48	47	GATB2
DGND	50	49	+5V

LEGEND:

Analog inputs - ACHx
Analog input ground - AISENSE/AIGND
Analog outputs - DACxOUT
Analog output ground - AGND
Digital inputs and outputs - PAx, PBx, PCx
Digital input/output ground - DGND
Control handshaking - EXTTRIG, EXTUPDATE, EXTCONV
Counter inputs/outputs - OUTBx, GATBx, CLKBx



pin 1

Looking at the connector
(on the back of the computer)

NOTE: LABVIEW MANUALS ARE AVAILABLE ON-LINE, AND CAN BE FOUND ON THE COURSE HOME PAGE - LEAVE THE PAPER MANUALS IN THE LAB.

22.2.3 Experiment 2 - Introduction to LabVIEW and the DAQ Cards

Objective:

Learn to use computers equipped for A/D and digital inputs.

Theory:

The computer reads data at discrete points in time (like a strobe light). We can read the data into the computer and then do calculations with it.

To read the data into a computer we write programs, and use "canned" software to help with the task. LabVIEW allows us to write programs for data collection, but instead of typing instructions we draw function blocks and connect them. How we connect them determines how the data (numbers) flow. The functions are things like data reads and calculations.

In this lab we will be using Labview to connect to a data acquisition (DAQ) board in the computer. This will allow us to collect data from the world outside the computer, and make changes to the world outside with outputs.

When interfacing to the card using a program such as Labview, there must be ways to address or request information for a specific input or output (recall memory addresses in EGR226). The first important piece of information is the board number. There can be multiple DAQ boards installed in the computer. In our case there is only one, and it is designated device ’1’. There are also many inputs and outputs available on the card. For analog outputs there are two channels so we need to specify which one when using the output with 0 or 1. For analog inputs there are 8 channels, and as before, we must specify which one we plan to read from using 0 to 7. For digital I/O there are a total of 24 pin distributed across 3 ports (1 byte each). Therefore when connecting inputs and output we must specify the port (PA=0, PB=1, PC=2) and the channel from 0 to 7. Note is that we can make the ports inputs or outputs, but not mixed - in other words we must pick whether a port will only be used for inputs or for outputs.

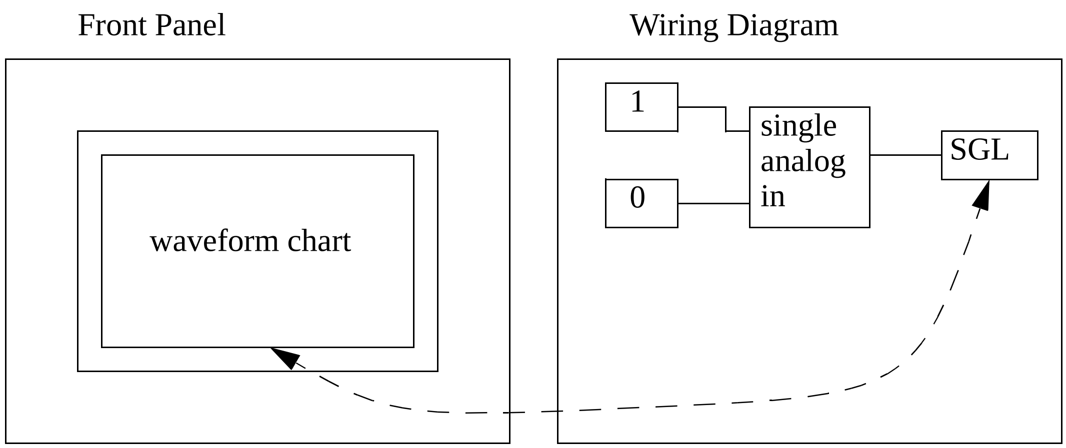
The voltage levels for the inputs and outputs are important, and you will need to be aware of these. For the digital outputs they will only ever be 0V or 5V. But the analog inputs and outputs will vary from -5V to 5V. This is build into the board. If we exceed these voltage limits by a few volts on the inputs, the boards have built in protection and should be undamaged. If we exceed the input voltages significantly, there is a potential to permanently damage the board.

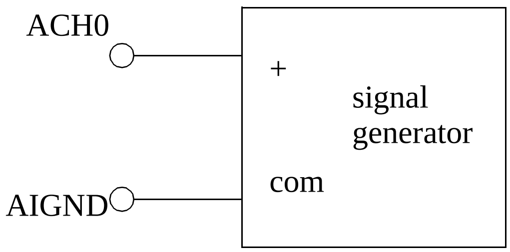
Equipment:

PC with LabVIEW software and PCI-1200 DAQ card
Interface cable
PLC trainer boards
Signal generator
Digital multimeter

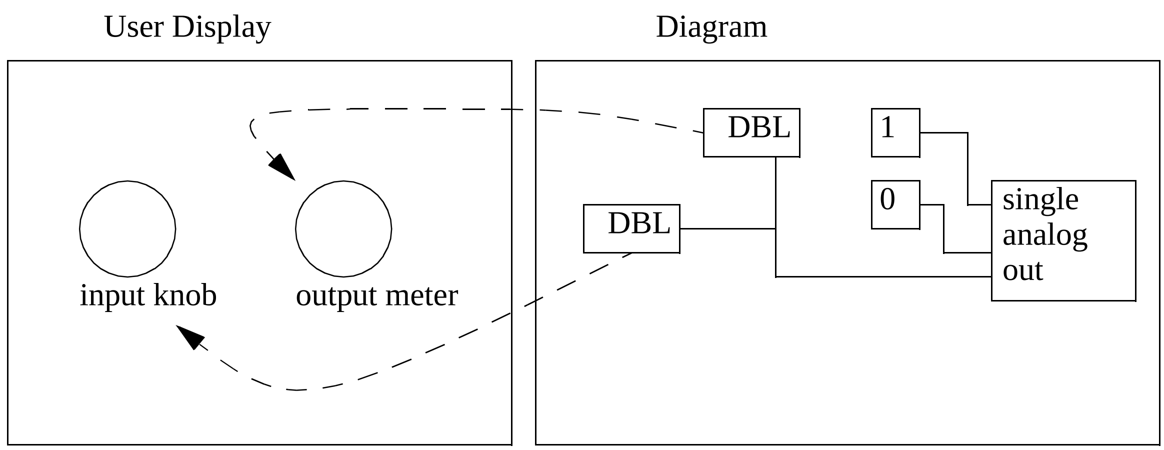
Procedure:

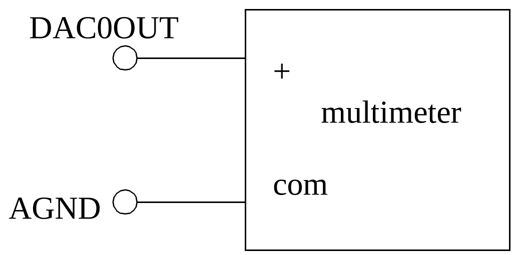
1. Go through the LabVIEW QuickStart Guide provided in the laboratory. This will also be good review for those who have used LabVIEW in previous courses.
2. Enter the LabVIEW program (layout) schematically shown below and connect a signal generator to the analog input (ACH0). (Note: there is a pin diagram for the connector in the Labview tutorial section.) Start the signal generator with a low frequency sinusoidal wave. Use the ‘DAQ Configure’ software to test the circuitry and verify that your hardware is operational. Then run your Labview program. Record the observations seen on the screen.



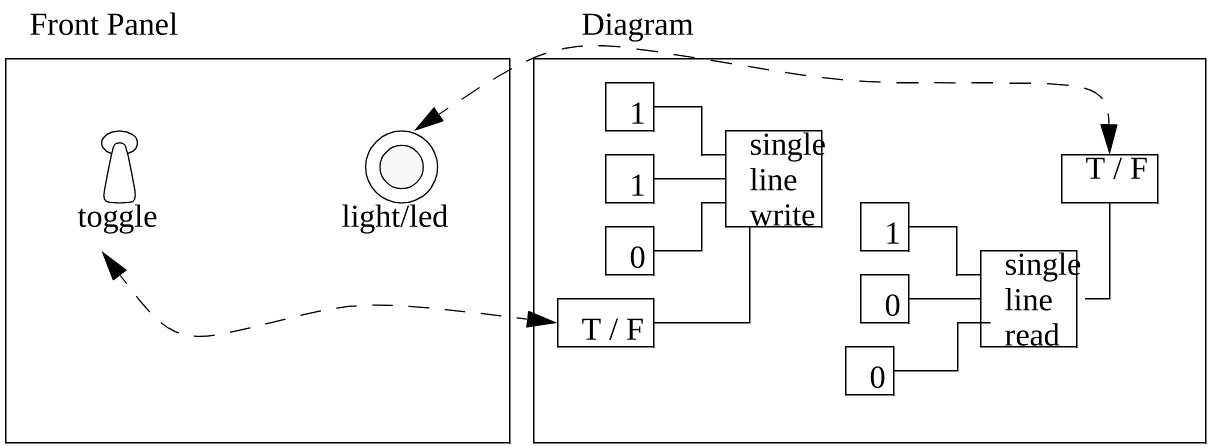


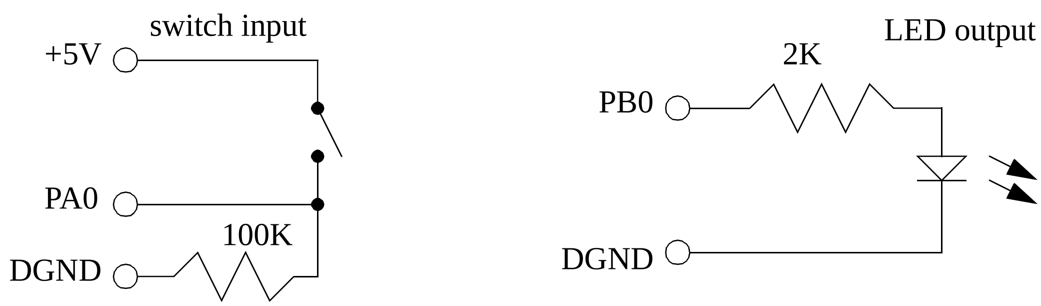
2b. Use a stop watch to determine the average number of samples per second. Run additional programs such as browsers, spreadsheets, etc. and see how this affects the data colection speed.
3. Connect the multimeter as shown below. Test the circuit using the ‘DAQ Configure’ utility. Enter the LabVIEW program schematically illustrated below and then run it. You should be able to control the output voltage from the screen using the mouse. Record your observations.





4. Connect the digital input and output circuits to the DAQ card and use the test panel to test the circuits. To do this, run the ‘DAQ Configure’ utility, double click on the ‘PCI-1200’, run the test panel window and ensure that the inputs and outputs are working correctly. Create the LabVIEW screen schematically illustrated below. This should allow you to scan an input switch and set an output light. When done, quit the program and run your LabVIEW program.





Marking:

1. A laboratory report should be written, including observations, and posted to the web.
2. The programs (VIs) that use the DAQ card should be posted to the web.

22.3 Lab 3 - Sensors and More Labview

22.3.1 Prelab 3 - Sensors

Theory:

Sensors allow us to convert physical phenomenon to measurable signals, normally voltage or current. These tend to fall into one of two categories, discrete or continuous. Discrete sensors will only switch on or off. Examples of these include,

Inductive Proximity Sensors - use magnetic fields to detect presence of metals

Capacitive Proximity Sensors - use capacitance to detect most objects

Optical Proximity Sensors - use light to detect presence

Contact Switches - require physical contact

Continuous sensors output values over a range. Examples of these are,

Potentiometers - provide a resistance proportional to an angle or displacement

Ultrasonic range sensors - provides a voltage output proportional to distance

Strain Gauges - their resistance changes as they are stretched

Accelerometers - output a voltage proportional to acceleration

Thermocouples - output small voltages proportional to temperature

In both cases these sensors will have ranges of operation, maximum/minimum resolutions and sensitivities.

Prelab:

1. Prepare a Mathcad sheet to relate sensor outputs to the physical phenomenon they are measuring.

22.3.2 Experiment 3 - Measurement of Sensor Properties

Objective:

To investigate popular industrial and laboratory sensors.

Procedure:

1. Sensors will be set up in the laboratory at multiple stations. You and your team should circulate to each station and collect results as needed. Instructions will be provided at each station to clarify the setup. The stations might include,
   - a mass on a spring will be made to oscillate. The mass will be observe by measuring position and acceleration.
   - a signal generator with an oscilloscope to read voltages phenomenon observe should include sampling rates and clipping.
2. Enter the data into Mathcad and develop a graph for each of the sensors relating input and output.

Submit:

1. A full laboratory report with graphs and mathematical functions for each sensor.

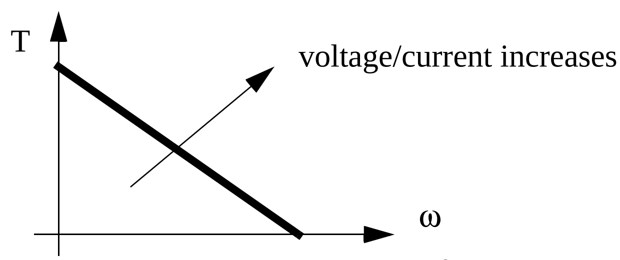
22.4 Lab 4 - Motors

• This set of labs will examine devices that have multiple phenomenon occurring.

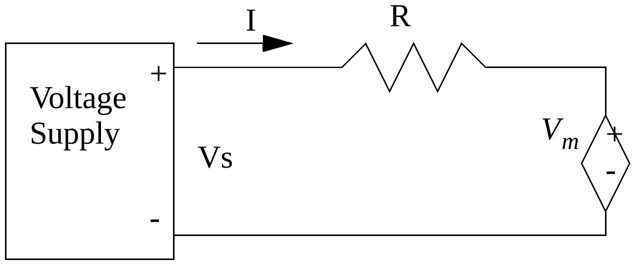
22.4.1 Prelab 4a - Permanent Magnet DC Motors

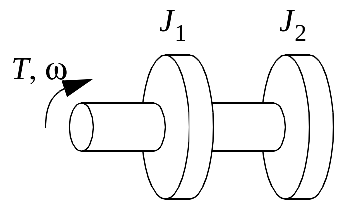
• Theory:

DC motors will apply a torque between the rotor and stator that is related to the applied voltage or current. When a voltage is applied the torque will cause the rotor to accelerate. For any voltage and load on the motor there will tend to be a final angular velocity due to friction and drag in the motor. And, for a given voltage the ratio between steady-state torque and speed will be a straight line.



The basic equivalent circuit model for the motor is shown below. We can develop equations for this model. This model must also include the rotational inertia of the rotor and any attached loads. On the left hand side is the resistance of the motor and the 'back emf' dependent voltage source. On the right hand side the inertia components are shown. The rotational inertia J1 is the motor rotor, and the second inertia is an attached disk.





Because a motor is basically wires in a magnetic field, the electron flow (current) in the wire will push against the magnetic field. And the torque (force) generated will be proportional to the current.

$$T = KI \qquad \therefore I = \frac{T}{K}$$

Next, consider the power in the motor,

$$P = V_m I = T\omega = KI\omega \qquad \therefore V_m = K\omega$$

Consider the dynamics of the rotating masses by summing moments.

$$\sum M = T = J_M\left(\frac{d}{dt}\right)\omega \qquad \therefore T = J_M\left(\frac{d}{dt}\right)\omega$$

The model can now be considered as a complete system.

The current-voltage relationship for the left hand side of the equation can be written and manipulated to relate voltage and angular velocity.

$$I = \frac{V_s - V_m}{R}$$

$$\therefore \frac{T}{K} = \frac{V_s - K\omega}{R}$$

XXXXXXAdd friction to the model

$$\therefore \frac{J_M\left(\frac{d}{dt}\right)\omega}{K} = \frac{V_s - K\omega}{R}$$

$$\boxed{\therefore \left(\frac{d}{dt}\right)\omega + \omega\left(\frac{K^2}{J_M R}\right) = V_s\left(\frac{K}{J_M R}\right)}$$

Looking at this relationship we see a basic first-order differential equation. We can measure motor properties using some basic measurements.

• Prelab:

1. Integrate the differential equation to find an explicit function of speed as a function of time.

2. Develop a Mathcad document that will accept values for time constant, supplied voltage and steady-state speed and calculate the coefficients in the differential equation for the motor.
3. In the same Mathcad sheet add a calculation that will accept the motor resistance and calculate values for K and J.
4. Get the data sheets for an LM675 from the web (www.national.com).

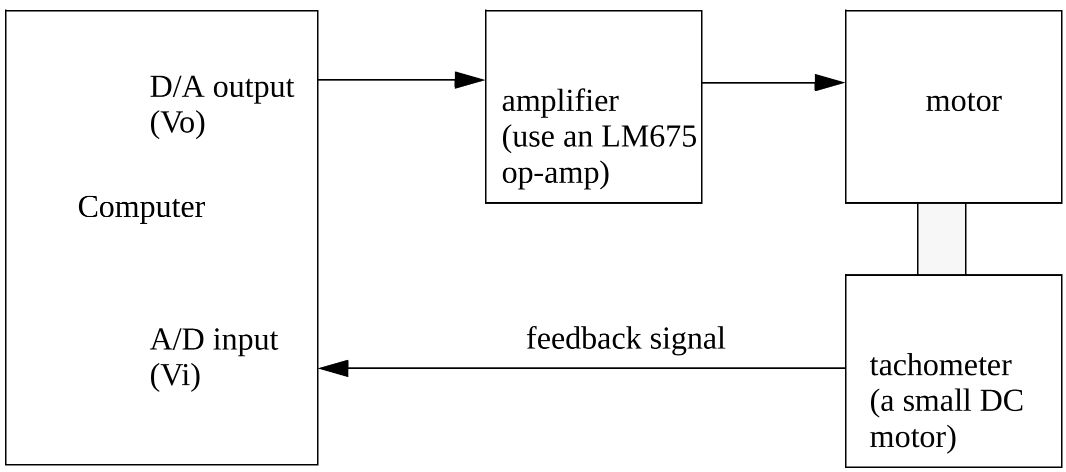
22.4.2 Experiment 4a - Modeling of a DC Motor

• Objective:

To investigate a permanent magnet DC motor with the intention of determining a descriptive equation.

• Procedure

1. With the motor disconnected from all other parts of the circuit, measure the resistance across the motor terminals.
2. Connect the motor amplifier, motor and computer as shown in the figure below.



Note: The motor being tested is the large motor. The small motor will be driven and act as a generator. In this case we will refer to it as a tachometer.

3. Write a Labview program that will output an analog voltage to drive the motor amplifier. An analog input will be used to measure the motor speed from the tachometer.

4. Use a strobe light to find the relationship between the tachometer voltage and the angular speed.
5. Obtain velocity curves for the motor with different voltage step functions.
6. Use a fish scale and a lever arm to determine the torque when the motor is stalled with an input voltage.

• Post-lab:

1. Determine the values of K for the motor. Determine the J for the rotor, and calculate J values for different load masses added.
2. Use the values of R, J and K to compare theoretical to the actual motor response curves found in procedure step #5.
3. Use the values of R, J and K to determine what the stalled torque should be in procedure step #6. Compare this to the actual.
4. Find the time constant of the unloaded motor.

• Submit:

1. All work and results.

22.5 Lab 5 - Basic Control Systems

22.5.1 Prelab 6a - Servomotor Proportional Control Systems

Theory:

DC servomotors typically have a first-order (velocity) response as found in the previous lab.

$$\left(\frac{d}{dt}\right)\omega + \omega\left(\frac{K^2}{JR}\right) = V_s\left(\frac{K}{JR}\right)$$

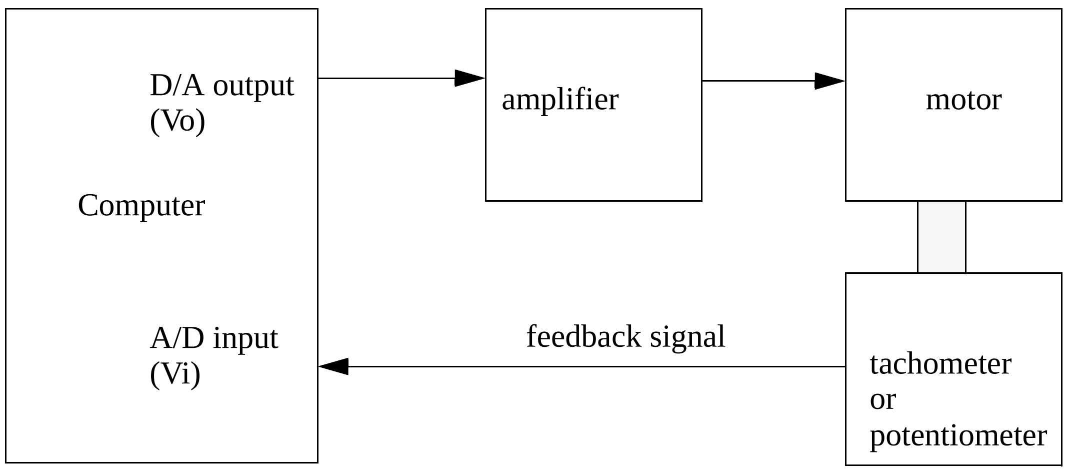
We can develop a simple control technique for control of the velocity using the equation below. For this form of control, we need to specify a desired velocity (or position) by setting a value ’Vd’. The difference between the desired speed and actual speed is calculated (Vd-Vi). This will give a voltage difference between the two values. This difference is multiplied by a constant ’K’. The value of ’K’ will determine how the system responds.

$$Vo = P(Vd - Vi)$$

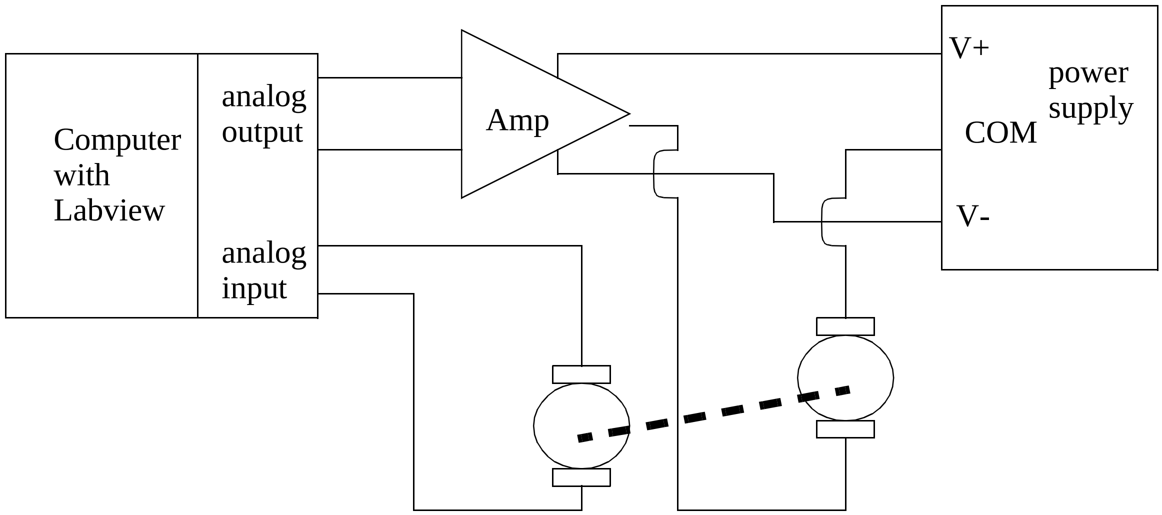
where,

Vo = Voltage to motor amplifier to control speed
Vi = Voltage from tachometer to measure speed or position
Vd = Desired motor speed voltage (user input)
P = Controller gain

The basic controller is set up as shown in the figure below. We can use a Labview program to implement the basic control equation described above.



The system below shows the components in the laboratory. The power supply may need to be constructed using two power supplies connected together. The analog output from the computer must be amplified (the computer can only output about 20mA maximum). The amplifier is constructed with an LM675 high current op-amp. The amplifier drives the large DC motor, which in turn drives a small DC motor being used as a tachometer. The voltage from the tachometer is input into the computer to determine the speed. A Labview program will subtract the tachometer voltage from the desired tachometer voltage, multiply the difference by a gain constant, and output the result to drive the motor.



Prelab:

1. Develop a Mathcad document that will model the velocity feedback controller given, motor parameters, desired velocity, an inertial load, and a gain constant. This is to be solved two different ways i) with Runge-Kutta integration, ii) integration of differential equations.
2. Test the controller model using a step function.

22.5.2 Experiment 5a - Servomotor Proportional Control Systems

Objective:

To investigate simple proportional servo motor control.

Procedure

1. Construct the equipment described in the
2. Apply a step function input and record the response.
3. For several values of proportional gain 'P', measure the response curves of the motor to a step function.

Post-lab:

1. Compare the theoretical and actual response curves on the same graphs.
2. Find and compare the time constants for experimental and theoretical results.

Submit:

1. All work and results.

22.6 Lab 6 - Basic System Components

22.6.1 Prelab 6a - Mechanical Components

• Theory:

Recall that for a rigid body we can sum forces. If the body is static (not moving), these forces and moments are equal to zero. If there is motion/acceleration, we use d'Alembert's equations for linear motion and rotation.

$$\sum F = Ma$$

$$\sum M = J\alpha$$

If we have a system that is comprised of a spring connected to a mass, it will oscillate. If the system also has a damper, it will tend to return to rest (static) as the damper dissipates energy. Recall that springs ideally follow Hooke's law. We can find the value of the spring constant by stretching the spring and measuring the forces at different points or we can apply forces and measure the displacements.

$$F = K_s \Delta x$$

$$\Delta x = x - x_0$$

$$F = K_s(x - x_0)$$

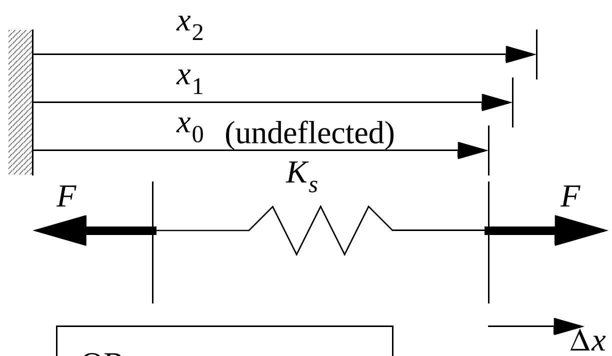
$$K_s = \frac{F_1}{x_1 - x_0} = \frac{F_2}{x_2 - x_0}$$

$$\frac{1}{K_s} = \frac{x_1 - x_0}{F_1} = \frac{x_2 - x_0}{F_2}$$

$$x_0 = x_1 - \frac{F_1}{K_s} = x_2 - \frac{F_2}{K_s}$$

$$\frac{1}{K_s}(F_2 - F_1) = x_2 - x_1$$

$$K_s = \frac{F_2 - F_1}{x_2 - x_1}$$



OR

$$F = K_s x$$

$$\Delta F = K_s \Delta x$$

$$K_s = \frac{\Delta F}{\Delta x}$$

$$K_s = \frac{F_2 - F_1}{x_2 - x_1}$$

In many cases we will get springs and devices that are preloaded. Both of the devices used in this lab have a preloaded spring. This means that when the spring has no force applied and appears to be undeflected, it is already under tension or compression, and we cannot use the unloaded length as the undeflected length. But, we can find the true undeflected length using the relationships from before.

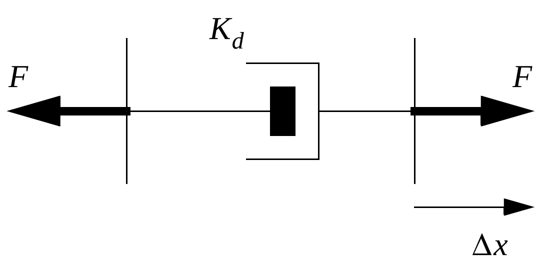
$$x_0 = x_1 - \frac{F_1}{K_s} = x_2 - \frac{F_2}{K_s}$$

Next, recall that the resistance force of a damper is proportional to velocity. Consider that when velocity is zero, the force is zero. As the speed increases, so does the force. We can measure this using the approximate derivatives as before.

$$F = K_d \frac{d}{dt} x$$

$$F = K_d\left(\frac{x(t + \Delta T) - x(t)}{\Delta T}\right)$$

$$K_d = \frac{F \Delta T}{x(t + \Delta T) - x(t)}$$



Now, consider the basic mass-spring combinations. If the applied forces are static, the mass and spring will remain still, but if some unbalanced force is applied, they will oscillate.

$$+\uparrow \quad \sum F_y = -F_s - F_g = M\left(\frac{d}{dt}\right)^2 y$$

$$-K_s y - Mg = M\left(\frac{d}{dt}\right)^2 y$$

$$M\left(\frac{d}{dt}\right)^2 y + K_s y = -Mg$$

$$y_h(t) = C_1 \cos\left(\sqrt{\frac{K_s}{M}}t + C_2\right)$$

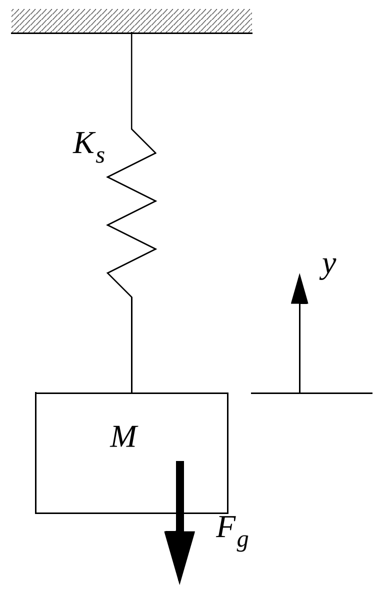
$$y_p(t) = A$$

$$\left(\frac{d}{dt}\right) y_p(t) = 0$$

$$\left(\left(\frac{d}{dt}\right)^2 y\right) y_p(t) = 0$$

$$M(0) + K_s(A) = -Mg \qquad \therefore A = \frac{-Mg}{K_s}$$

$$y(t) = y_h(t) + y_p(t) = C_1 \cos\left(\sqrt{\frac{K_s}{M}}t + C_2\right) + \frac{-Mg}{K_s}$$



Assume we start the mass at rest at the equilibrium height.

$$y(0) = 0 = C_1 \cos\left(\sqrt{\frac{K_s}{M}}(0) + C_2\right) + \frac{-Mg}{K_s}$$

$$\therefore C_1 = \frac{Mg}{K_s \cos(C_2)}$$

$$y'(0) = 0 = C_1 \sqrt{\frac{K_s}{M}} \sin\left(\sqrt{\frac{K_s}{M}}(0) + C_2\right)$$

$$\therefore C_2 = 0$$

$$y(t) = \left(\frac{Mg}{K_s}\right)\cos\left(\sqrt{\frac{K_s}{M}}t\right) + \frac{-Mg}{K_s}$$

The natural frequency is found by completing one time period,

$$2\pi = \sqrt{\frac{K_s}{M}}T = \sqrt{\frac{K_s}{M}}\frac{1}{f}$$

$$\therefore f = \frac{1}{2\pi}\sqrt{\frac{K_s}{M}}$$

In the lab an ultrasonic sensor will be used to measure the distances to the components as they move. The sensor used is an Allen Bradley 873C Ultrasonic Proximity Sensor. It emits sound pulses at 200KHz and waits for the echo from an object that is 30 to 100cm from it. It outputs an analog voltage that is proportional to distance. This sensor requires a 18-30 VDC supply to operate. The positive supply voltage is connected to the Brown wire, and the common is connected to the blue wire. The analog voltage output (for distance) is the black wire. The black wire and common can be connected to a computer with a DAQ card to read and record voltages. The sound from the sensor travels outwards in an 8 degree cone. A solid target will give the best reflection.

- Prelab:

1. Review the theory section.
2. Extend the theory by finding the response for a mass-spring, mass-damper, and mass-spring-damper system (assume values).
3. Set up a Mathcad sheet for the laboratory steps.

22.6.2 Experiment 6a - Mechanical Components

- Objective:

This lab will explore a simple translational system consisting of a spring mass and damper using instrumentation and Labview.

- Procedure

1. Use two masses to find the spring constant or stiffness of the spring. Use a measurement with a third mass to verify. If the spring is pretensioned determined the 'undeformed length'.
2. Hang a mass from the spring and determine the frequency of oscillation. Determine if the release height changes the frequency. Hint: count the cycles over a period of time.
3. Connect the computer to the ultrasonic sensor (an Allen Bradley Bulletin 873C Ultrasonic Range Sensor, see www.ab.com), and calibrate the voltages to the position of the target (DO NOT FORGET TO DO THIS). Write a Labview program to read the voltage values and save them to a data file. In the program set a time step for the voltage readings, or measure the relationship between the reading number and actual time for later calculation.
4. Attach a mass to the damper only and use Labview to collect position as a function of time as the mass drops. This can be used to find the damping coefficient.
5. Place the spring inside the damper and secure the damper. This will now be used as a combined spring damper. In this arrangement the spring will be precompressed. Make sure you know how much the spring has been compressed when the damper is in neutral position.
6. Use the spring-damper cylinder with an attached mass and measure the position of the mass as a function of time.
7. Use Working Model 2D to model the spring, damper and spring-damper responses.

- Post-lab:

1. Determine if the frequency of oscillation measured matches theory.
2. Compare the Labview data to the theoretical data for steps 2, 4 and 6.
3. Compare the Labview data to the working model simulations.

- Submit:

1. All results and calculations posted to a web page as a laboratory report.

22.7 Lab 7 - Oscillating Systems

• Many systems undergo periodic motion. For example, the pendulum of a clock.

22.7.1 Prelab 7a - Oscillating Systems

• Theory:

Suppose a large symmetric rotating mass has a rotational inertia J, and a twisting rod has a torsional spring coefficient K. Recall the basic torsional relationships.

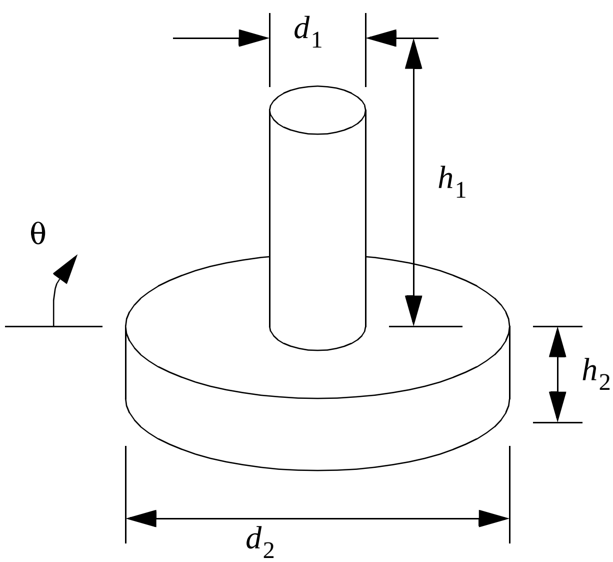
$$\sum T = -T = J\alpha = J\left(\frac{d}{dt}\right)^2\theta$$

$$T = K\Delta\theta = K(\theta - \theta_0)$$

We can calculate the torsional spring coefficient using the basic mechanics of materials

$$T = \frac{JG\theta}{L}$$

Finally, consider the rotating mass on the end of a torsional rod.



$$\sum T = -\left(\frac{J_1 G}{L}\right)\theta = J\left(\frac{d}{dt}\right)^2\theta$$

• Prelab:

1. Calculate the equation for the natural frequency for a rotating mass with a torsional spring.
2. Set up a Mathcad sheet that will
    - accept material properties and a diameter of a round shaft and determine the spring coefficient.
    - accept geometry for a rectangular mass and calculate the polar moment of inertia.
    - use the spring coefficient and polar moment of inertia to estimate the natural frequency.
    - use previous values to estimate the oscillations using Runge-Kutta.
    - plot the function derived using the homogeneous and particular solutions.

22.7.2 Experiment 7a - Oscillation of a Torsional Spring

• Objective:

To study torsional oscillation using Labview and computer data collection.

• Procedure

1. Calibrate the potentiometer so that the relationship between and output voltage and angle is known. Plot this on a graph and verify that it is linear before connecting it to the mass.
2. Set up the apparatus and connect the potentiometer to the mass. Apply a static torque and measure the deflected position.
3. Apply a torque to offset the mass, and release it so that it oscillates. Estimate the natural frequency by counting cycles over a long period of time.
4. Set up LabVIEW to measure the angular position of the large mass. The angular position of the mass will be measured with a potentiometer.

• Post-lab:

1. Compare the theoretical and experimental values.

• Submit:

1. All work and observations.
2. Post the laboratory report to a web page.

22.8 Lab 8 - Servo Control Systems

22.8.1 Prelab 5a - Research

• Theory:

This lab explores the use of advanced servo control systems. In particular Allen Bradley 5000 drives will be used. These drives are programmed using C. The drives can be interfaced to digital and analog IO, as well a handle serial commonucations.

• Prelab:

1. Visit the Allen Bradley web site (www.ab.com) and review the manuals for the Ultra5000 drives. In particular look at the installation manuals and programming manuals.
2. If necessary, review the principles of programming in C. Please note that the programming guide for the Ultra5000 drive contains a brief review of C pro-

gramming practices.

22.8.2 Experiment 5a - Tutorial and programming

• Objective:

To be able to develop programs for an AB Ultra5000 servo control system.

• Procedure

1. Follow the tutorial for the ULTRA5000 drives.
2. Develop a program that will use a voltage input (AIN1) from a potentiometer to position the drive. The drive will use forward and reverse limit switches on INPUT1 and INPUT2. When these switches are off the drive will be permitted to move in the given direction. INPUT3 should be used to stop the motor at any time, and stop the program. Ask the instructor to verify this program before continuing to the next stage.
3. Connect the servo drive to a linear slide. Attach two proximity sensors a few inches from each end of the slide to limit the stage motion. Use the ’Move’ function of the drive to verify that the forward and reverse limit switches are on the proper ends.
4. Verify that your program can be used to control the position of the slide using a potentiometer.

• Submit:

1. A lab report including the fully commented C program for the drive.

22.9 TUTORIAL - ULTRA 5000 DRIVES AND MOTORS

1. (If not installed, install the Ultraware software, version 1.1.) Connect the ULTRA 5000 drive to the serial port, set the drive to ’01’ on the front of the drive, and then apply power.
2. Run the ’Motor Configuration’ program and open the ’Motors’ file. Select the correct motor type and then select ’New Rotary....’. Notice the motor parameters. Close the program.
3. Run the ’Ultraware’ software. Select ’Create new file....’ and give it a unique name such as ’lab.udb’ (Note: you should put this file on a floppy disk, or zip disk so that you may use it again later.). It will scan the available drives to

locate the drive. When done, a ’5KDrive’ should appear under ’On-line drives’.

4. Right click on the ’%KDrive’ and select properties. For the ’Drive Type’ select the correct version indicated on the Ultra 5000 drive. Use ’Setup’ to update the drive parameters, then close the window.
5. Under the ’5KDrive’ select ’Motor’ and then select the motor type. Then select ’Setup’ to update the motor parameters. Notice the values and units. Close the window.
6. Turn the motor shaft by hand, it should turn easily. Enable the motor with ’Commands’ ’Enabled’ on the tool bar. The motor should now be controlled, and the shaft hard to turn.
7. For the motor select ’Motion’ then ’Jog’. In the window set the values below. After this hold the shaft and hit ’Jog forward’ then ’Jog reverse’ then ’Stop’. Notice how the values at the bottom of the screen change.

    Program Velocity 100,000
    Program Acceleration 100,000
    Program Deceleration 100,000

8. Now select ’Motion’ then ’Move’ and set the values below. When done, click ’Start’ to cause the motion. Change the value in ’Profile Distance’ and click ’Start’ again.

    Profile Distance 40,000
    Velocity 20,000
    Acceleration 20,000
    Deceleration 20,000

9. Create a project with ’Insert’ then ’Project’. Click on the ’Project’ that was created on the tree, then ’Insert Source File’. Type in the following file, and then click ’Build’. If there are any errors fix them before continuing to the next step.

```
#include <motion.h>

int main(void){
      InitMotionLibrary();   // Start the motor control functions

      AxisEnable();          // Enable control of the motor

      MoveSetAcc(2000);      // Set the maximum accelerations and velocity
      MoveSetVel(1000);
      MoveSetDec(2000);

      MoveDistance(10000);// Move in the positive direction
      while(MoveInProgress()){};// Wait until the move is complete
      Sleep(1000);           // Pause for 1 second
      MoveDistance(-10000);// Move in the negative direction
      while(MoveInProgress()){};
      Sleep(1000);

      AxisDisable();         // Release the motor drive control

      return 1;
}
```

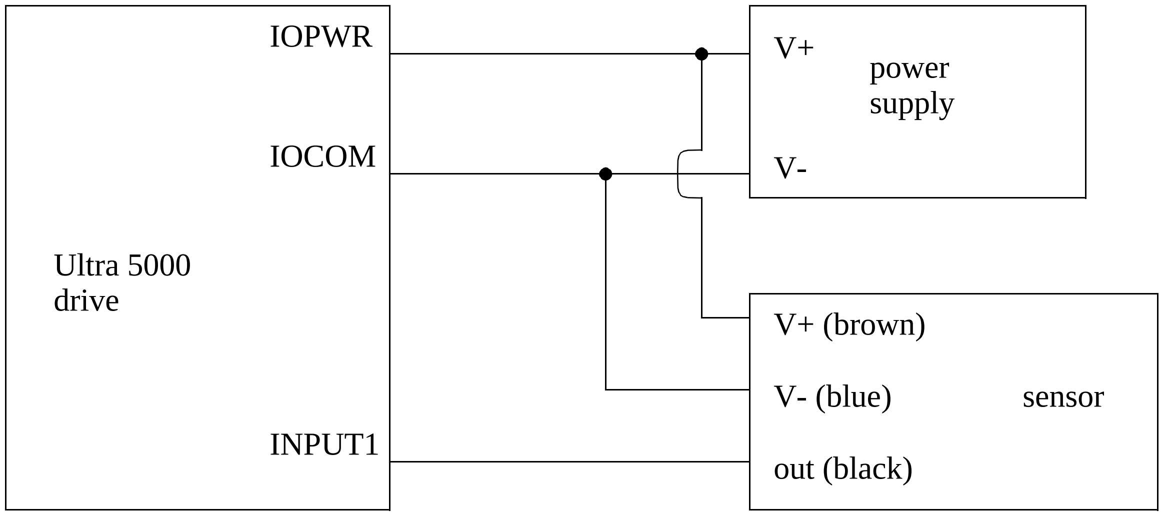
10. Drag the new file ’Project.exe’ to the ’Programs’ under the ’5KDrive’. Hold the shaft of the motor and right click on the new file just dropped and click ’Run’. The motor shaft should move forward, pause for 1 second, reverse, then stop.
11. Enter the following program and observe the results. It should behave the same as the previous program, but it has more structure. When developing control applications it is important to structure the programs so that they are easier to write and debug.

```
#include <motion.h>

void setup(){
	InitMotionLibrary();
	AxisEnable();
	MoveSetAcc(2000);
	MoveSetVel(1000);
	MoveSetDec(2000);
}

void shutdown(){
	AxisDisable();
}

void jog(int distance){
	MoveDistance(distance);
	while(MoveInProgress()){};
}

int main(void){
	setup();

	jog(10000);
	Sleep(1000);
	jog(-10000);
	Sleep(1000);

	shutdown();
	return 1;
}
```

12. Digital inputs and outputs for the drive are located on the upper connector, CN1A. The pin out for this connector is given below. Wires can be connected by pushing a thin slot screwdriver into the slot to release the pressure clamp, and then inserting the wire. The nominal voltages to activate an input are between 12V and 24V - do not exceed the maximum voltage as this may damage the drive.

Connector	Pin	Function
CN1A		
	1	INPUT 1 - Digital input 1
	2	INPUT 2 - Digital input 2
	3	INPUT 3 - Digital input 3
	4	INPUT 4 - Digital input 4
	5	INPUT 5 - Digital input 5
	6	INPUT 6 - Digital input 6
	7	INPUT 7 - Digital input 7
	8	INPUT 8 - Digital input 8
	9	OUTPUT 1 - Digital output 1
	10	OUTPUT 2 - Digital output 2
	11	OUTPUT 3 - Digital output 3
	12	OUTPUT 4 - Digital output 4
	13	SHIELD - for shielded cable termination
	14	IOPWR - V+ from external supply power for the IO
	15	INPUT 9 - Digital input 9
	16	INPUT 10 - Digital input 10
	17	INPUT 11 - Digital input 11
	18	INPUT 12 - Digital input 12
	19	INPUT 13 - Digital input 13
	20	INPUT 14 - Digital input 14
	21	INPUT 16 - Digital input 15
	22	INPUT 16 - Digital input 16
	23	OUTPUT 5 - Digital output 5
	24	OUTPUT 6 - Digital output 6
	25	OUTPUT 7 - Digital output 7
	26	OUTPUT 8+ - Relay output 8
	27	OUTPUT 8- - Relay output 8
	28	IOCOM - common from external IO power supply

13. Connect an external power supply to the digital IO and connect a proximity sensor to input 1, as shown in the diagram below. Verify that the sensor is connected properly using the Ultraware software. Also connect a multimeter to OUTPUT1 to read the output voltage.



14. Enter the program below and determine what the program does.

```
#include <motion.h>

void setup(){
    InitMotionLibrary();
    AxisEnable();
    MoveSetAcc(2000);
    MoveSetVel(1000);
    MoveSetDec(2000);
}

void shutdown(){
    AxisDisable();
}

void jog(int distance){
    MoveDistance(distance);
    while(MoveInProgress()){};
}

int main(void){
    int     count = 0,
            state_last = OFF,
            state_now;
    setup();

    while(count < 10){
            state_now = InputGetState(1); // get the state of INPUT1
            if( (state_now == ON) && (state_last == OFF) ){
                    jog(1000);
                    count++;
            }
            state_last = state_now;
            OutputSetState(1, state_now);// set OUTPUT1 to the INPUT1 value
    }

    shutdown();
    return 1;
}
```

15. Analog outputs (and connections for additional encoders) are available on connector CN1B. Connect a variable power supply to act as an analog input across AIN1 and +5VCOM (WARNING: TURN THE POWER SUPPLY OFF AND SET THE VOLTAGE AT THE MINIMUM BEFORE TURNING THE SUPPLY BACK ON). Connect a multimeter across AOUT1 and +5VCOM to read

an analog output voltage. Verify the connection using the Ultraware software.

Connector CN1B	Pin	Function
	1	+5V - internal power supply
	2	AX+ Encoder input/output A+
	3	AX- Encoder input/output A-
	4	BX+ Encoder input/output B+
	5	BX- Encoder input/output B-
	6	IX+ Encoder input/output I+
	7	IX- Encoder input/output I-
	8	+5VCOM internal power supply common
	9	AIN1 - analog input 1
	10	AIN2 - analog input 2
	11	+5VCOM internal power supply common
	12	AOUT1 - analog output 1
	13	AOUT2 - analog output 2
	14	SHIELD - termination point for shielded cables

16. Enter the program below and determine what it does.

```
#include <motion.h>

void setup(){
	InitMotionLibrary();
	AxisEnable();
	MoveSetAcc(2000);
	MoveSetVel(1000);
	MoveSetDec(2000);
}

void shutdown(){
	AxisDisable();
}

void locate(int position){
	MovePosition(position);
	while(MoveInProgress()){};
}

int main(void){
	float	a_in;
	int	new_location;
	setup();

	while(1 == 1){		// loop forever
		a_in = AnalogInputGetVoltage(1);// Get the voltage in from AIN1
		new_location = a_in * 1000;
		locate(new_location);
		AnalogOutputSetVoltage(1, a_in/2.0); // output half the input voltage
	}

	shutdown();
	return 1;
}
```

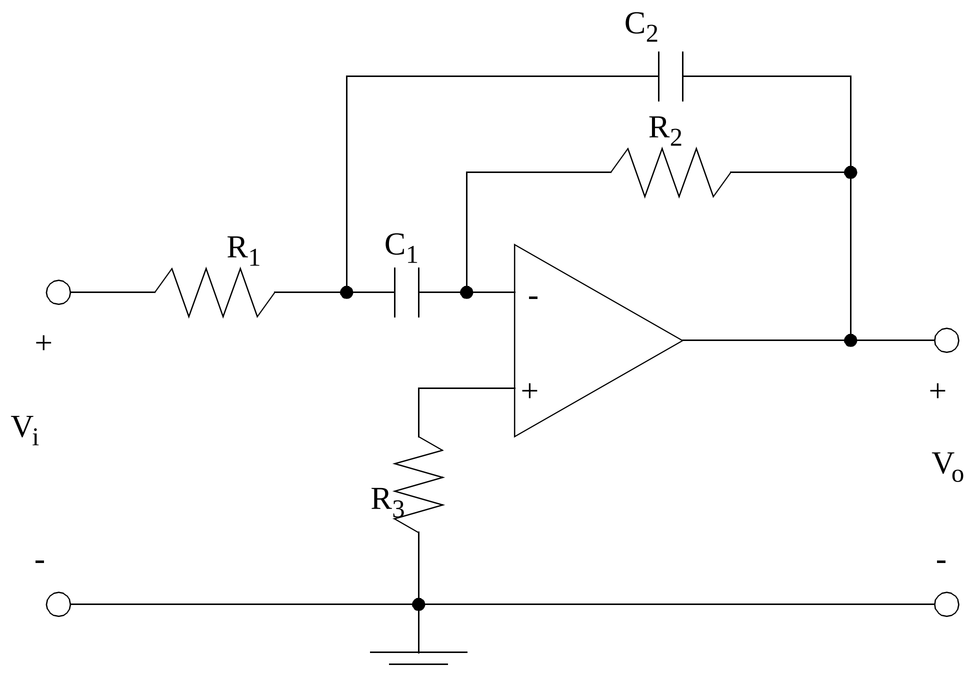
17. Communications????????????????????

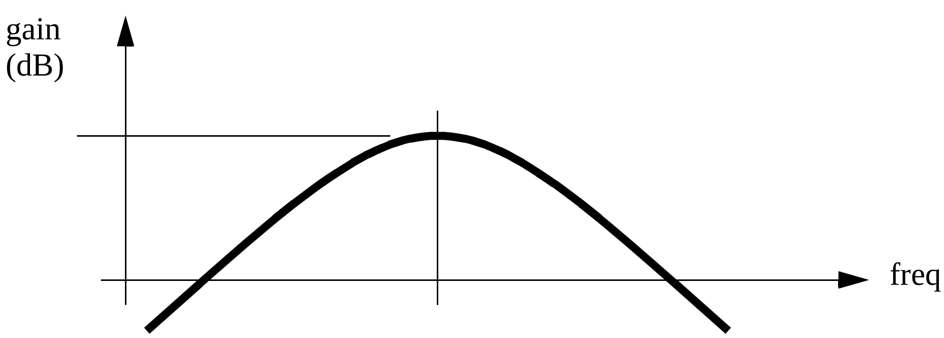
22.10 Lab 9 - Filters

22.10.1 Prelab 9 - Filtering of Audio Signals

Theory:

We can build simple filters using op-amps, and off the shelf components such as resistors and capacitors. The figure below shows a band pass filter. This filter will pass frequencies near a central frequency determined by the resistor and capacitor values. By changing the values we can change the overall gain of the amplifier, or the tuned frequency.

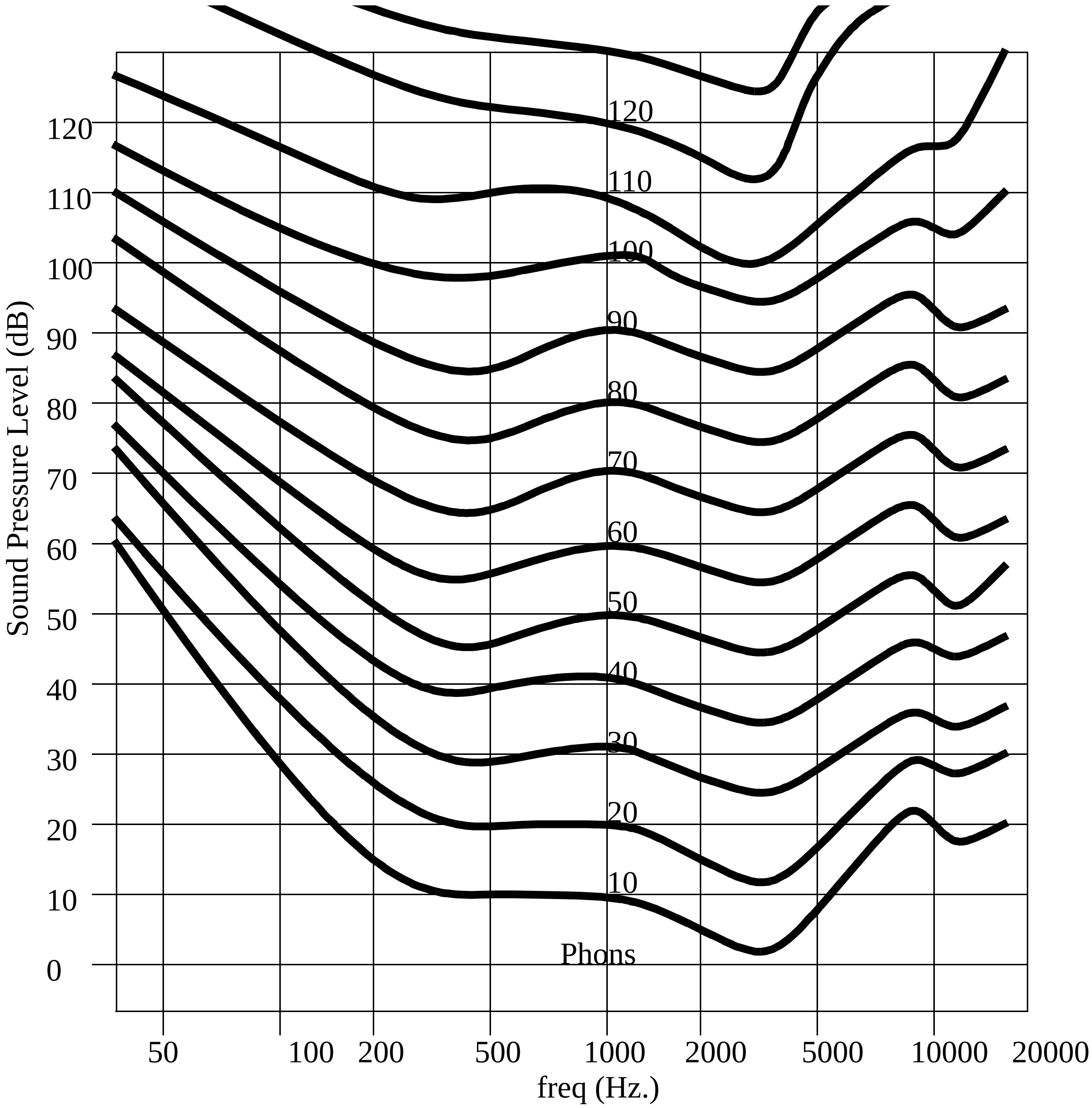




The equations for this filter can be derived with the node voltage method, and the final transfer function is shown below.

$$\frac{V_o}{V_i} = \left(-\frac{1}{R_1}\right)\left(\frac{DC_1R_1R_2}{1 + DC_2R_1 + DC_1R_1 + D^2C_1C_2R_1R_2}\right)$$

As dictated by the ear, audio signals have frequencies that are between 10Hz and 16KHz as illustrated in the graph below. This graph shows perceived sound level, with the units of 'phons'. For example, we can follow the 100 phons curve (this would be like a loud concert or very noisy factory requiring ear protection). At much lower and higher frequencies there would have to be more sound pressure for us to perceive the same loudness, or phon value. If the sound were at 50Hz and 113 dB it would sound as loud as 100dB sound at 1KHz.



You may appreciate that these curves are similar to transfer functions, but they are non-linear. For this lab it is important to know how the ear works because you will be using your ear as one of the experimental devices today.

Prelab:

1. Derive the transfer function given in the theory section.
2. Draw the Bode plot for the circuit given R1=R2=1000ohms and C1=C2=0.1uF. You are best advised to use Mathcad to do this.

22.10.2 Experiment 9 - Filtering of Audio Signals

Objective:

To build and test a filter for an audio system.

Procedure:

1. Set up the circuit shown in the theory section. Connect a small speaker to the output of the amplifier.
2. Apply a sinusoidal input from a variable frequency source. Use an oscilloscope to compare the amplitudes and the phase difference. Also record the relative volumes you notice.
3. Supply an audio signal, from a radio, CD player, etc. Record your observations about the sound.

Submit:

1. Bode plots for both theory and actual grain and phase angle.
2. A discussion of the sound levels you perceived.

22.11 Lab 10 - Stepper Motors

22.11.1 Prelab 10 - Stepper Motors

Theory:

Stepper motors move to positions, steps, but don't rotate continuously. To achieve continuous rotation the motor is moved, or stepped, continuosly in one direction. A full rotation of the motor is often divided into 200 steps (of 1.8 degrees each), although other resolutions are common. The motors we will use in the lab have 400 steps per revolution. When rotating the motor it can be stepped in the clockwise (CW) or counterclockwise (CCW) direction.

The equipment to be used in the lab is comprised of a stepper motor, a drive unit, a motion controller (indexer), and a computer running terminal software for programming. The motor is connected to the drive unit which will power the coils and step in the positive or negative directions, as indicated by pulses from the motion controller. The motion controller will accept motion commands, or run programs which generate the overall motion. The motion controller commands the motor drive to move by sending it pulses. The motor controller varies the period of the pulses. When motion is starting or ending a longer period corresponds to a slower angular velocity. At the midpoint of motion the pulse period is the shortest, at the maximum velocity.

The motion is normally described with a velocity profile. This can be given by defining the maximim acceleration/deceleration and maximum velocity. An alternative way to define these limits is to provide an acceleration and deceleration time, with a total motion time.

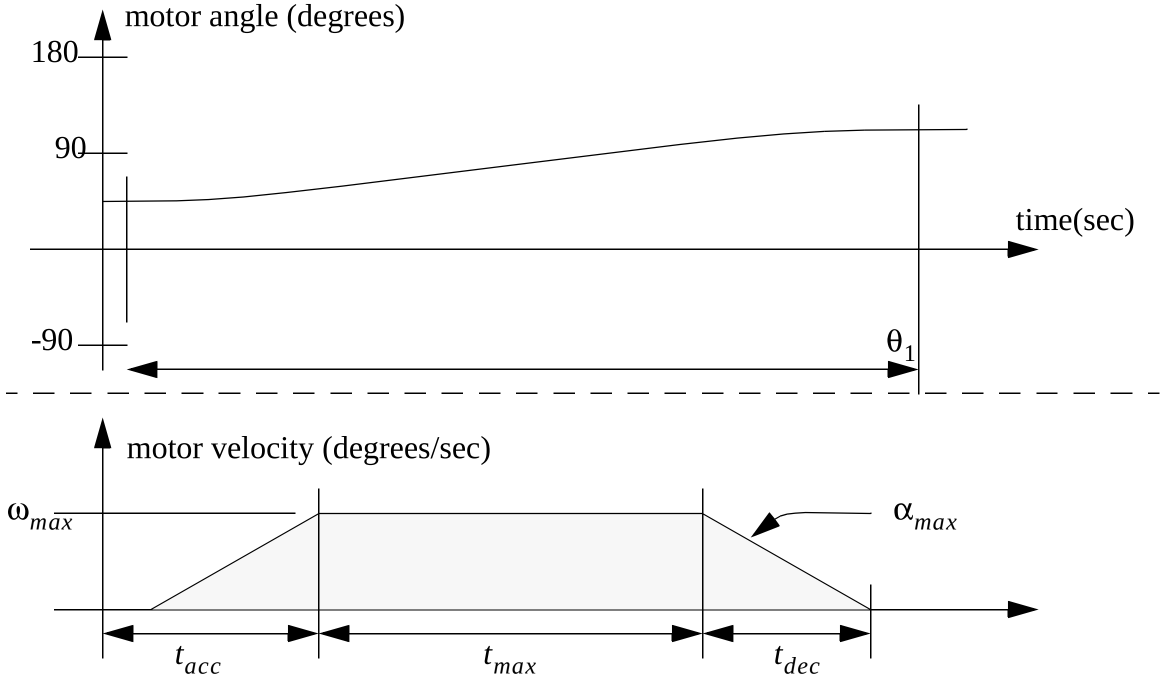


Figure 427 Velocity curves for motor control

Prelab:

1. In the textbook review the sections on stepper motors and motion control. Make sure you are familiar with the basic operation of a stepper motor and the structure of a velocity profile.
2. Review the manuals for the stepper motor on the course home page on claymore.

22.11.2 Experiment 9 - Stepper Motors

Objective:

To develop a stepper motor control program.

Procedure:

1. Follow the attached tutorial
2. Develop a program to accept an input to the motion controller and then move the

motor to different positions.

Submit:

1. The questions answered during the tutorial.
2. The program developed (with comments).

22.12 TUTORIAL - STEPPER MOTOR CONTROLLERS (by A. Blauch and H. Jack)

1. Examine the manuals for the stepper motor and controller.

2. Obtain and examine the stepper motor equipment. There are three main components: stepper motor, drive unit, and indexer. Identify each unit. Verify the components are connected together properly based on the diagram shown below. Note that the drive unit obtains power from an AC source (outlet) while the indexer receives power from a DC source (power supply). Once all of the connections have been verified, apply power to the system. If set up properly, the power LEDs on the drive unit and indexer will turn on and the stepper motor will have holding torque (the shaft won't turn freely).

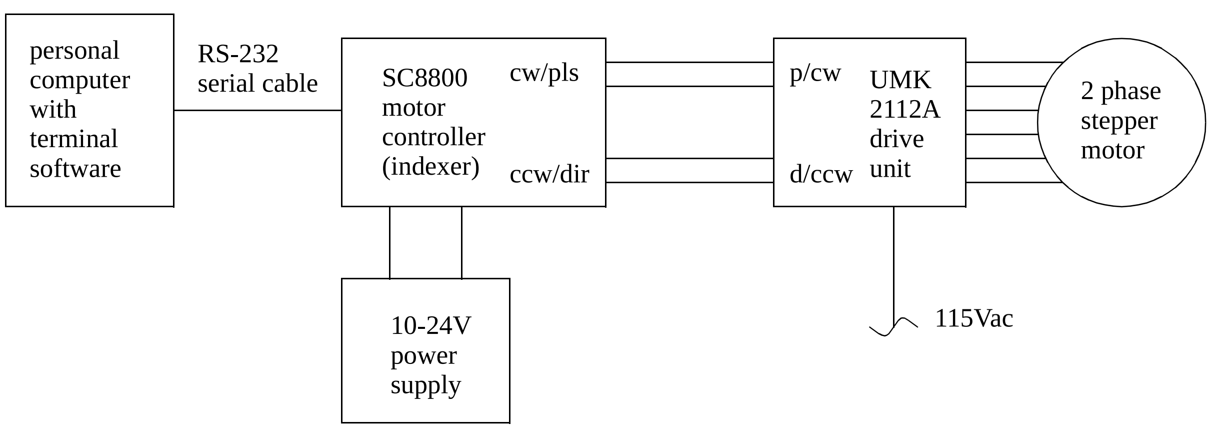


Figure 428 The stepper motor control system components

3. The indexer communicates with the PC via text transfers across a serial port. Connect a serial cable from the indexer to the COM port on the back of the PC. On the PC, run HyperTerminal (located under **`Programs->Accessories->Communications`**) or any other terminal emulation program. Set it to go "Direct to Com 1". Check to make sure the communication parameters are set to 9600 baud, 8 data bits, 1 stop bit, no parity, and no flow control/handshaking. Once connected the prompt **`0>`** will appear on the screen. If nothing appears, press enter several times. Cycling power on the indexer will cause a reset ban-

ner to appear. To display all of the available indexer commands, type **HELP**.

4. The indexer commands of interest are the ones used to generate and execute different velocity profiles. The list below highlights the commands used in this lab. Typing **HELP2, HELP3, HELP4, etc...** will display a more detailed lists of these and other profiling commands

Command	Description
H<+/->	Sets the direction of motion
T<xx.xx>	Sets the acceleration/deceleration time in seconds
VS<xxxxx>	Sets the starting velocity in pulses per second
V<xxxxx>	Sets the run velocity in pulses per second
D<xxxxx>	Sets the distance in pulses for the MI command
MC	Starts continuous motion
MI	Starts incremental motion
R	Displays the status of system parameters
RESET	Resets the system
S	Controlled stop of current motion
STOP	Immediate stop of current motion

5. To see what the current velocity profile parameters are, type the parameter name followed immediately by a question mark (i.e. **T?** will display the current acceleration/deceleration time).

6. Type in the following series of commands. Observe how the stepper motor responds and how the response relates to the parameter values entered.

```
VS0
V2000
T0.5
D10000
MI
```

7. Change the acceleration and decelleration times to be longer and shorter and observe the effects using the values below. As part of your observations, hold the shaft while the motion is in progress.

acc/dec	vel	distance
0.01	20000	50000
0.5	10	100
0.01	2000	6000

8. Generate and execute a velocity profile that will cause the motor to rotate at one revolution/second using the continuous motion command. Based on the run velocity parameter value, determine what angle one pulse (step) corresponds to. Compare your number to the resolution value on the stepper motor label.

9. Set the acceleration/deceleration parameter to several seconds. Using the run velocity parameter and the continuous motion command, make the stepper motor rotate at various velocities. Determine the maximum speed of rotation for the stepper motor (i.e. at what point does the motor lose synchronization with the input pulses).

10. Decrease the run velocity to about one half the maximum speed determined previously. Using the acceleration/deceleration parameter and the continuous motion command, make the stepper motor accelerate at various rates. Determine the maximum acceleration for the stepper motor (i.e. at what point does the motor lose synchronization with the input pulses). Make a note of any strange phenomena that may occur.

11. Determine the parameter values for a velocity profile with an acceleration of 18000 rpm^2, maximum velocity of 600 rpm, and a total run time of 10 seconds. Generate, execute, and verify the profile. For this profile you will need to use the incremental motion command.

12. Based on your experiences in lab with motors, discuss some advantages and disadvantages of using a stepper motor compared to a DC motor. Some issues to think about are torque, position control, velocity control, and interface.

13. Enter a program with the following commands. Notice that the editor has commands to alter ’Ax’ line ’x’, delete ’Dx’ line ’x’ or insert ’Ix’ a line ’x’. Each program will be assigned a number between 0 and 49. The program can be run with ’RUN TEST’.

```
EDIT TEST
PC0
TA0.5
TD0.5
VS2000
D6000
MI
Q
```

NOTE: After this point you should use the manual for a source of examples.

14. You can see a list of programs in the memory using ’LIST’. Programs can be deleted with ’DELETE’.

15. The motion controller has a limited set of variables available for user programs. Unlike normal programming variables, there are a fixed set availabe, as indicated below. The integers values are all 4 bytes and can range in value between +/-2,147,483,647. Write a simple program that sets a general variable value.

    W, X, Y, Z - general purpose integer
    V- velocity
    VS -
    LOOP -
    D - distance
    INx - input bit ’x’
    PC - position counter, the current motor position in steps

16. Outputs can be set using the ’OUTx’ command, where ’x’ is the output bit number. Connect a voltage supply to the inputs and a multimeter to the outputs and create a program to read an input and set an output as shown below. Refer to the manual for pinouts and electrical specifications.

17. Logical operations can be performed using ’IF’ and ’WHILE’ statements. The operations can include =, !=, <, <=, >, >=. The if statements can include ’ELSE’ conditions and should terminate with ’ENDIF’ statements. The while loops must end with ’ENDW’ statements. The equivalent to a for-loop is the ’LOOP x’ statement. It will loop the code ’x’ times to the ’ENDL’ statement. Write a simple program that will loop until an input becomes true.

18. It is possible to branch within a program using ’JMP’. To jump to another program (subroutine) the ’JMP_SEQ’ command can be used, the program can then return using the ’RET’ command. Write a program that has a subroutine.

22.13 Lab 11 - Variable Frequency Drives

22.13.1 Prelab 11 - Variable Frequency Drives

Theory: AC induction motors are designed with motor winding on the stator (outside) of the motor. The AC current in the stator coils sets up an alternating magnetic field. This field induces currents in the conductors (squirrel cage) in the rotor. This current creates a magnetic field that opposes the field from the stator. As a result a torque is created. In actuality the rotor must rotate somewhat slower than the field changes in the stator, this difference is called slip. For example a 3 phase motor (with two poles) that has a 60Hz power applied will with absolutly no rotational resistance rotate at 60 times per second. But in use it might rotate at 58 or 59Hz. As the number of poles in the motor rises, the speed of rotation decreases. For example a motor with four poles would rotate at half the speed of a two pole motor. The speed of the motor can be controlled by changing the frequency of the AC power supplied to the motor.The motor that we will use in the lab is a 3 phase AC motor made by Marathon Electric. It is a Black Max model number 8VF56H17T2001. The motor drives are Allen Bradley model 160 and 161 motor drives.

Prelab:

1. Visit the Marathon Electric and Allen Bradley web sites and review the manuals for the motor and controllers. Don't print these, but make a not of the web address so that you can find the manuals easily during the lab.

22.13.2 Experiment 11 - Variable Frequency Drives

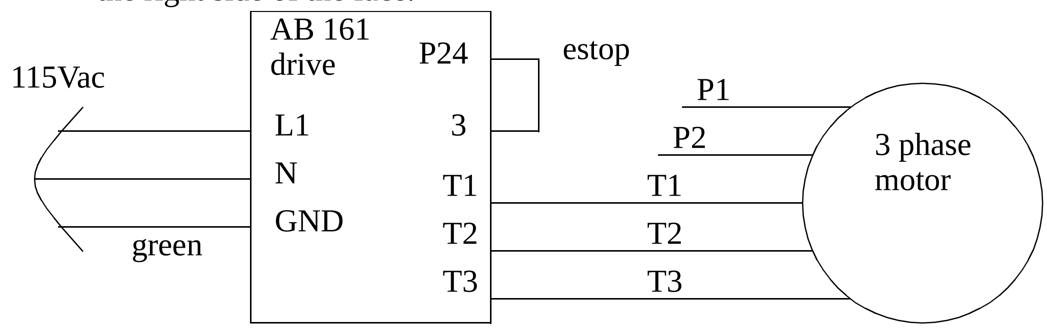
Objective: To learn the basic operation of Varaible Frequency Drives (VFD) and motors.

Procedure:

1. Follow the attached tutorial. (and make notes and observations.)
2. Connect the 3 phase motor to a tachometer for a velocity feedback. Using the manual determine how to change the PID parameters. Run a number of tests to see what it does.
3. Determine how to use an analog input to control the speed. Set it up and find the relationship between the input voltage and the motor speed.
4. Replace the 161 controller with the 160 controller, and determine how this controller operates.

22.14 TUTORIAL - ALLEN BRADLEY 161 VARIABLE FREQUENCY DRIVES

1. The motor should be connected the the drive as shown in the figure below. The three phase power lines are T1, T2 and T3. These can be connected to the drive by loosening the screw on the front face of the drive. (WARNING: make sure the power is off before opening the drive, and relseal the drive when done) The two other lines P1 and P2 from the drive are for a thermal overload relay, we will not use these. For general caution the ends of the wires should be covered with electrical tape to prevent accidental contact with other conductors. The wire from P24 (a 24V power source) and input 3 are for an emergency stop and must be connected for the drive to work. These terminals can be found under a flip up panel on the bottom front of the drive that can be opened by pulling on the right side of the face.

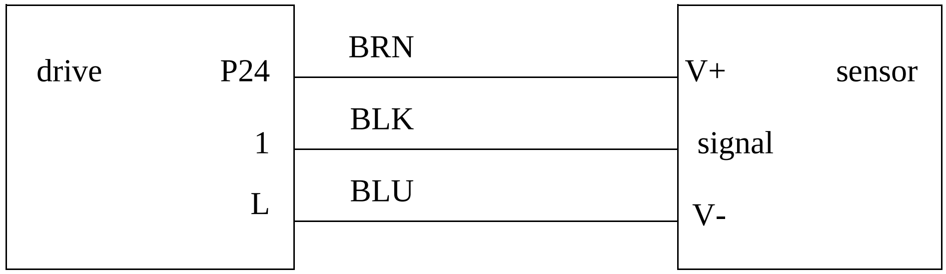


2. Notice the display and lights on the front face of the drive. When a program is running the ’RUN’ light is on. The program can be started using the green ’1’ button, and stopped with the red ’0’ button.
3. Program the unit using the buttons on the front panel, following the steps below (from page 17 in the manual). After the steps have been followed press the run button and turn the potentiometer to vary the speed. Try holding the shaft (cover the shaft to avoid cuts) at high and low speeds. What do you notice at very low speeds?

button(s)	result	description
sel	d01	move the the parameter number, from frequency
up or down	A--	move to the ’A’ parameters
sel	A01	select the ’A’ parameter group (use up or down if not A01)
up or down	00	select the velocity input source
sel	01	display the current setting
up or down	00	select the potentiometer (on the controller) as the input
enter	A01	accept the new value
up or down	A02	move the the start button selector
sel	01	move to parameter
up or down	02	select the value for start to the ’run’ button
enter	A02	accept the value

4. Connect a proximity to the controller as shown in the diagram below. Display

the input values using ’D05’.The bits on the screen should move up for an active input, and down for an inactive input. Input 3 will always be active because it is being used for the emergency stop (a factory default).



5. Various display parameters are listed below. Try these to see what information they show.
    - D02 - display the motor current (try holding the shaft while turning slowly)
    - D01 - display the frequency of rotation
    - D03 - the direction of rotation
    - D04 - PID parameters (when in use)
    - D05 - input status
    - D06 - output status
    - D16 - total drive run time in 10 hour blocks
6. Function parameters can be set with the ’F’ locations. Change the acceleration and decelleration times to 1 second using the ’F02’ and ’F03’ locations.
7. Restore the controller to the factory defaults using the sequence below from page 16 of the manual.

button(s)	result	description
up or down	b--	move to the ’B’ parameter groups
sel	b01	enter the
up or down	b84	move to the reset function
sel	01	select the function, make sure the value is ’01’
sel	b84	select the value
sel+up+down+0	b84	hold down the keys for 3 seconds
sel+up+down	00 then 0.0	release the ’0’ key and continue to hold the others until the display blinks.

22.15 TUTORIAL - ULTRA 100 DRIVES AND MOTORS

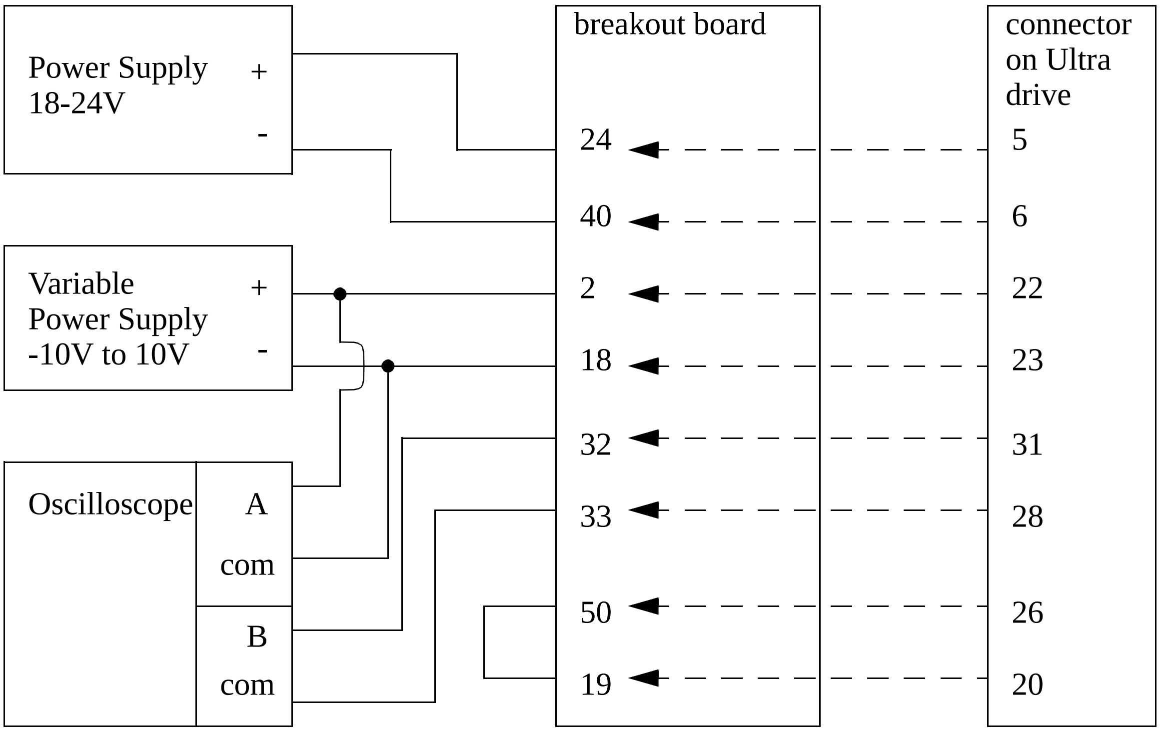
1. Obtain the motor and controller sets. This should include a brushless servo motor (Y-1003-2H), an Ultra 100 Controller and miscellaneous cables. The cables are described below. Locate and verify they are connected as indicated in the sequence given below.
   a) The motor is connected to the controller with two cables, one for the power to drive the motor, and the other a feedback from the encoder. The encoder is used to measure the position and velocity of the motor. If not connected already, connect these to the motor controller.
   b) The controller should also have a serial cable for connecting to a PC serial port. Connect this to a PC.
   c) A breakout board is provided with that will connect to the motor controller through a ribbon cable to J1. The pin mapping from the motor controller to the terminal numbers on the motor controller are given below.

controller pin	breakout pin	controller pin	breakout pin
1	9	26	50
2	25	27	17
3	41	28	33
4	8	29	49
5	24	30	16
6	40	31	32
7	7	32	48
8	23	33	15
9	39	34	31
10	6	35	47
11	22	36	14
12	38	37	30
13	5	38	46
14	21	39	13
15	37	40	29
16	4	41	45
17	20	42	12
18	36	43	28
19	3	44	44
20	19	45	11
21	35	46	27
22	2	47	43
23	18	48	10
24	34	49	26
25	1	50	42

   d) A power cord to be connected to a normal 120Vac power source. DO

NOT CONNECT THIS YET.

2. The controller needs to be connected as shown below. In this case it will require that the breakout board be connected to the motor controller by a ribbon cable. Note: the pins on the breakout board don't have the same numbers as on the motor controller. Connect the wires as shown below but do not turn on the devices yet.



3. Plug in the power cord for the Ultra 100 drive and look for a light on the front to indicate that it is working. WARNING: The motor might begin to turn after the next step --> Turn on the power supplies and the oscilloscope.
4. Run the 'Ultramaster' software on the PC. When prompted select 'drive 0'. At this point the software should find the drive and automatically load the controller parameters from the controller. When it is done a setup window will appear.
5. In the "drive setup" window ensure the following settings are made. When done close the setup window.
   - Motor Model: Y-1003-2-H - this is needed so that the proper electrical and mechanical properties of the motor will be used in the control of the motor (including rotor inertia).
   - Operation Mode Override: Analog Velocity Input - This will allow the voltage input (C+ and C-) to control motor velocity.
6. Select the 'IO Configuration' window and make the following settings.
   - Analog Output 1: Velocity - Motor Feedback - this will make the analog voltage output from the controller (R+ and R-) indicate the velocity of the motor, as measured by the encoder.
   - Scale: 500 RPM/VOLT - this will set the output voltage so that every

500RPM will produce an increase of 1V.

7. Open the "Oscilloscope" on the computer screen and make the following settings. When done position the oscilloscope to the side of the window so that it can be seen later.
   Channel A: Velocity - command
   Channel B: Velocity - motor feedback
   Time Base: 12.5ms
8. Open the control panel and change the speed with the slider, or by typing it in. Observe the corresponding changes on the oscilloscope on the screen, and the actual oscilloscope. Notice that the velocity change follows a ramp. The slope of the ramp is a function of the maximum acceleration.
9. Hold the motor shaft while it is turning slowly and notice the response on the oscilloscopes.
10. Open the tuning window and change the ’I’ value to 0. Change the motor speed again and notice the final error. It should be larger than in previous trials.
11. Leave the ’I’ value at ’0’ and change the ’P’ value to 50 and then change the speed again. This time the error should be larger, and the response to the change should be slower. Hold the shaft again, it should be easier to stop.
12. Try other motor parameters and see how the motor behaves. Note that very small or large values of the parameters may lead to controller faults - if this occurs set more reasonable ’P’ or ’I’ values and then reset the fault.
13. Notice that the left hand side of the tuning window has an autotune function. Select this to automatically pick the controller values. To perform this function click the ’stop’ then ’start’ buttons. The motor will move, and then reverse direction to determine the dynamic load. It will then pick parameters to leave the motor controller critically damped.
14. Set the parameters back to ’P’ = 200 and ’I’ = 66.
15. Stop the ’Control Panel’, this will allow the motor to be controlled by the external voltage supply. Change the voltage supply and notice how the motor responds. Reverse the voltage supply and notice that the motor turns in the other direction - remember to return it to normal when done.
16. Connect proximity sensors to digital inputs 1 and 2. The pins on the controllers are 32 and 33 respectively. The pins on the breakout board are 48 and 15 respectively. The wires for the proximity sensors are standardized as brown for V+, blue for the common and black for the signal.
17. Once the proximity sensors are attached test them with the "Display Digital IO" window. After this change the "IO Configuration window to set input 1 as "forward enable" and input 2 as the reverse enable". Test these to see that they actually stop the motor in both directions when not on.
18. Attach the motor to a ball screw slide and attach the proximity sensors. Change the motor controller to follow a position command. You should be able to control the position of the motor by changing the variable power supply voltage.

22.16 TUTORIAL - DVT CAMERAS (UPDATE FROM 450???)

(******To be done outside the lab without the hardware)

1. If the DVT software is not installed yet, install it now.
2. In either case put the DVT CDRom in the computer and follow the tutorials on the CD.
3. Run the ’Framework’ software. When it starts to run a window called ’PC Communications’ will appear. In this case we will only use the simulator, so select ’Emulator’ then ’Model 630’, then ’Connect’. The emulator will also start as another program.
4. At this point we can set up a simple set of vision tests for the camera. The camera is capable of supporting tests for more than one product. You can select which product the tests will be for by first selecting ’Products’ then ’Product Management...’. This will put up a window to define the product details. Select ’New’ and then enter ’part1’. Use ’OK’ to dismiss the windows and get back to the main programming window.
5. We need to now set up the emulator to feed images to the programming software. We can do this by loading a set of test images. To do this open the ’DVT SmartImage Sensor Emulator’ window. Then select ’Configure’ then ’Image sequence...’. Select ’Browse for Image’ and then look for images on the CD in the ’emulator\images\Measurement’ directory. Pick the file ’Measurement001.bmp’. The next window will ask for the last image so select ’Measurement010.bmp’. Select ’OK’ to get back to the main emulator window.
6. Go back to the Framework programming screen. The emulator can be started by selecting the right pointing black triangle near the red circle. You should see images appear in a sequence. Pause the images using the single vertical line.
7. Next we will add a test to the product. On the left hand side of the screen select ’Measurement’, select the small line with two red arrows and then move the mouse over the picture and draw a line over a single section of the part, for example the right hand most side by the large circle.
8. In the pop-up window enter the test name ’test1’. Apply the changes and dismiss the window and run the camera again. Notice that as the images flip by the result of the test is shown at the bottom of the screen. As the images are captured the part may move around. It is common to delete and recreate the sensor a number of times to find the right position for the line. Double click on the sensor and look at some of the options possible. Try changing these and see what happens.
9. Delete the previous sensor and create a sensor (rotational positioning: find edge in parallelogram), and a translation sensor (translational positioning: fudicial rectangle). For the translation sensor, under the ’Threshold’ select ’Dark Fudicial’. Run the software and note the rotation and translation data at the bottom of the screen as the images are processed.
10. Delete the sensors. Go to the emulator and load the two ’1dBarCode’ images in

the ’emulator\images\1DReader’. In Framework run the camera so that these appear on the screen, then pause the camera. On the left hand side of the screen select ’Readers’ then ’Barcode Sensor: Line for 1D codes’. Draw the sensor line over the middle of the barcode.

23. WRITING REPORTS

• As engineers, reports are the most common form of document we will write.

• Report writing is an art that we often overlook, but in many cases can make a dramatic impact on our progression.

• Your reports are most likely to find their way to a superiors desk than you are to meet the individual.

• Note: typically these documents are done as a long written document -this is done in point form for more mercenary reasons.

23.1 WHY WRITE REPORTS?

• Reports are written for a number of reasons,

- (lets forget about this one) as a student you must do them to get marks
- to let other engineers know the results of an experiment
- to leave a record of work done so that others may continue on
- as a record you may use yourself if you must do work again some time later
- they are required for legal reasons (contract or legislation)
- they bring closure to the project

23.2 THE TECHNICAL DEPTH OF THE REPORT

• This is the most common question for beginning report writers.

• Always follow the doctrine - “What happens if I am hit by a car; could another pick up my report and continue?” if the answer to this question is no the report is too short.

• Always ask the question - “If I was reading this before starting the project would I look at this section and say it is not needed?”. If the question we are probably best to condense the section or leave it out.

• Avoid more artistic sections. If you put these in, make it clear that they are an optional part of the report and can be skipped. It is somewhat arrogant to force the reader to

23.3 TYPES OF REPORTS

• We do different types of reports, including,

Laboratory - Theses ‘Lab Reports’ describe one or more experiments, the results, and the conclusions drawn from them.
Consulting - A summary of details, test results, observations, and a set of conclusions. Typically they will also contain a recommendation.
Project - A description of work done in a project to inform other engineers who may be asked to take up further work on the project.
Research - A summary of current advances in a topic. This should end with some comparison of alternatives.
Interim - A report to apprise supervisors and others as to the progress of a project or other major undertaking.
Executive - A brief summary of the report, and any implications for decision making at the management levels.

23.3.1 Laboratory

• Purpose: These reports should outline your procedure and results in detail. They should also contain the analysis and conclusions. The completeness of detail allows you (and others) to review these and verify the correctness of what has been done. These have been historically used for hundreds of years and are accepted as a form of scientific and legal evidence. It is completely unacceptable to make incorrect entries or leave out important steps or data.

• Standard Format:

1. Title, Author, Date - these make it clear what the labs contain, who did the work, and when it was done.
2. Purpose - a brief one line statement that allows a quick overview of what the experiment is about. This is best written in the form of a scientific goal using the scientific methods.
3. Theory - a review of applicable theory and calculations necessary. Any design work is done at this stage
4. Equipment - a list of the required equipment will help anybody trying to replicate the procedure. Specific identifying numbers should be listed when possible. If there are problems in the data, or an instrument is found to be out of

calibration, we can track the problems to specific sets of data and equipment.
5. Procedure - these are sequential operations that describe what was done during the experiment. The level of detail should be enough that somebody else could replicate the procedure. We want to use this as a scientific protocol.
6. Results (Note: sometimes procedure and results are mixed) - the results are recorded in tables, graphs, etc. as appropriate. It will also be very helpful to note other events that occur (e.g. power loss, high humidity, etc.)
7. Discussion - At this stage the results are reviewed for trends and other observations. At this point we want to consider the scientific method.
8. Conclusions - To conclude we will summarize the significant results, and make general statements either upholding or rejecting our purpose.

• Style: These are meant to be written AS the work is done. As a result the work should be past tense

• Laboratory reports should have one or more hypotheses that are to be tested. If testing designs these are the specifications. Examples might be,

- what is the thermal capacity of a material?
- what is the bandwidth of an amplifier?
- will the counter increment/decrement between 0 to 9?

• NOTE: These reports are much easier to write if you prepare all of the calculations, graphs, etc. before you start to write. If you sit down and decide to do things as you write it will take twice as long and get you half the marks...... believe me, I have written many in the past and I mark them now.

23.3.1.1 - An Example First Draft of a Report

Grand Valley State University

Padnos School of Engineering

EGR 345 Dynamics Systems Modeling and Control

Laboratory Exercise 7

Title: The Cooling of Coffee

Author: I. M. Wyred

Date: Dec., 23, 1998

Purpose: To derive a theoretical model of the rate at which coffee cools and experimentally verify the model and find coefficients.

Theory:

When coffee is heated kinetic energy is added, when coffee is cooled kinetic energy is removed. In a typical use, coffee cools as heat is lost through convection and conduction to the air and solids in contact. The factors involved in this convection/conduction can be difficult to measure directly, but we can approximate them with a simple thermal resistance. Consider the temperature difference between the coffee and the ambient temperature. The greater the temperature difference, the higher the rate of heat flow out of the coffee. This relationship can be seen formally in the equation below. We can also assume that the atmosphere is so large that the heat transfer will not change the temperature.

$$q = \frac{1}{R}(\theta_{coffee} - \theta_{air})$$

where,

q = heat flow rate from coffee to air (J/s)

R = thermal resistance between air and coffee

θ = temperatures in the coffee and air

We can also consider that coffee has a certain thermal capacity for the heat energy. As the amount of energy rises, there will be a corresponding temperature increase. This is known as the thermal capacitance, and this value is unique for every material. The basic relationships are given below. I will assume that the energy flow rate into the coffee is negligible.

$$\Delta\theta_{coffee} = \frac{1}{C_{coffee}}(q_{in} - q_{out}) = \frac{-1}{C_{coffee}}q \qquad \frac{d\theta_{coffee}}{dt} = \frac{-1}{C_{coffee}}q$$

where,

$$C_{coffee} = \text{thermal capacitance}$$
$$C_{coffee} = M_{coffee}\sigma_{coffee}$$

where,

$$M_{coffee} = \text{mass of thermal body}$$
$$\sigma_{coffee} = \text{specific heat of material in mass}$$

The temperatures can be found by consider that the energy flowing out of the cup, and into the atmosphere is governed by the resistance. And, the temperature in the coffee and air are governed by the two capacitances. We will make two assumptions, that the thermal capacitance of the atmosphere is infinite, and that there is no energy flowing into the coffee.

$$\frac{d\theta_{coffee}}{dt} = \frac{-1}{C_{coffee}}q$$

$$\therefore \frac{d\theta_{coffee}}{dt} = \frac{-1}{M_{coffee}\sigma_{coffee}}\left(\frac{1}{R}(\theta_{coffee} - \theta_{air})\right)$$

$$\therefore \frac{d\theta_{coffee}}{dt} + \left(\frac{1}{M_{coffee}\sigma_{coffee}R}\right)\theta_{coffee} = \left(\frac{1}{M_{coffee}\sigma_{coffee}R}\right)\theta_{air}$$

This differential equation can then be solved to find the temperature as a function of time.

Guess $\theta = A + Be^{Ct}$ $\frac{d}{dt}\theta = BCe^{Ct}$

$$\therefore BCe^{Ct} + \left(\frac{1}{M_{coffee}\sigma_{coffee}R}\right)(A + Be^{Ct}) = \left(\frac{1}{M_{coffee}\sigma_{coffee}R}\right)\theta_{air}$$

$$\therefore e^{Ct}\left(BC + \frac{B}{M_{coffee}\sigma_{coffee}R}\right) + \left(\frac{A}{M_{coffee}\sigma_{coffee}R} + \left(\frac{-1}{M_{coffee}\sigma_{coffee}R}\right)\theta_{air}\right) = 0$$

$$BC + \frac{B}{M_{coffee}\sigma_{coffee}R} = 0 \qquad C = \frac{-1}{M_{coffee}\sigma_{coffee}R}$$

$$\frac{A}{M_{coffee}\sigma_{coffee}R} + \left(\frac{-1}{M_{coffee}\sigma_{coffee}R}\right)\theta_{air} = 0 \qquad A = \theta_{air}$$

To find B, the initial temperature of the coffee should be used,

$$\theta_0 = A + Be^{C(0)} = \theta_{air} + B \qquad B = \theta_0 - \theta_{air}$$

The final equation is,

$$\theta = \theta_{air} + (\theta_0 - \theta_{air})e^{\frac{-t}{M_{coffee}\sigma_{coffee}R}}$$

The time constant of this problem can be taken from the differential equation above.

$$\tau = M_{coffee}\sigma_{coffee}R$$

Equipment:

1 ceramic coffee cup (14 oz.)
2 oz. ground coffee
1 coffee maker - Proctor Silex Model 1234A
1 thermocouple (gvsu #632357)
1 temperature meter (gvsu #234364)
1 thermometer

2 quarts of tap water
1 standard #2 coffee filter
1 clock with second hand
1 small scale (gvsu# 63424)

Procedure and Results:

1. The coffee pot was filled with water and this was put into the coffee maker. The coffee filter and grounds were put into the machine, and the machine was turned on. After five minutes approximately the coffee was done, and the pot was full.

2. The mass of the empty coffee cup was measured on the scale and found to be 214g.

3. The air temperature in the room was measured with the thermometer and found to be 24C. The temperature of the coffee in the pot was measured using the thermocouple and temperature meter and found to be 70C.

4. Coffee was poured into the cup and, after allowing 1 minute for the temperature to equalize, the temperature was measured again. The temperature was 65C. Readings of the coffee temperature were taken every 10 minutes for the next 60 minutes. These values were recorded in Table 1 below. During this period the cup was left on a table top and allowed to cool in the ambient air temperature. During this period the mass of the full coffee cup was measured and found to be 478g.

Table 1: Coffee temperatures at 10 minute intervals

time (min)	temperature (deg C)
0	65
10	53
20	43
30	35
40	30
50	28
60	26

Discussion:

The difference between the temperature of the coffee in the pot and in the cup was

5C. This indicates that some of the heat energy in the coffee was lost to heating the cup. This change is significant, but I will assume that the heating of the cup was complete within the first minute, and this will have no effect on the data collected afterwards.

The readings for temperature over time are graphed in Figure 1 below. These show the first-order response as expected, and from these we can graphically estimate the time constant at approximately 32 minutes.

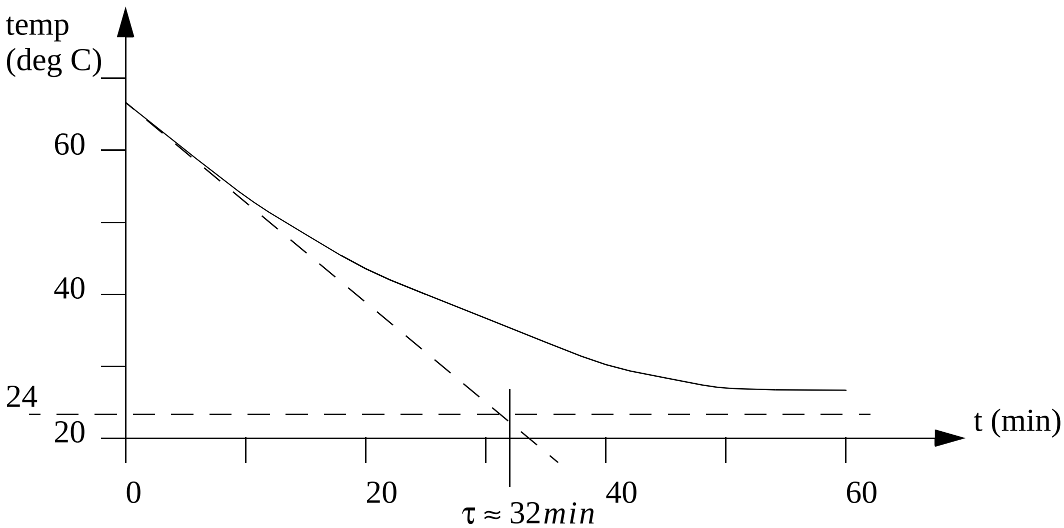


Figure 1 - A graph of coffee temperature measured at 10 minute intervals

We can compare the theoretical and experimental models by using plotting both on the same graph. The graph clearly shows that there is good agreement between the two curves, except for the point at 30 minutes, where there is a difference of 3.5 degrees C.

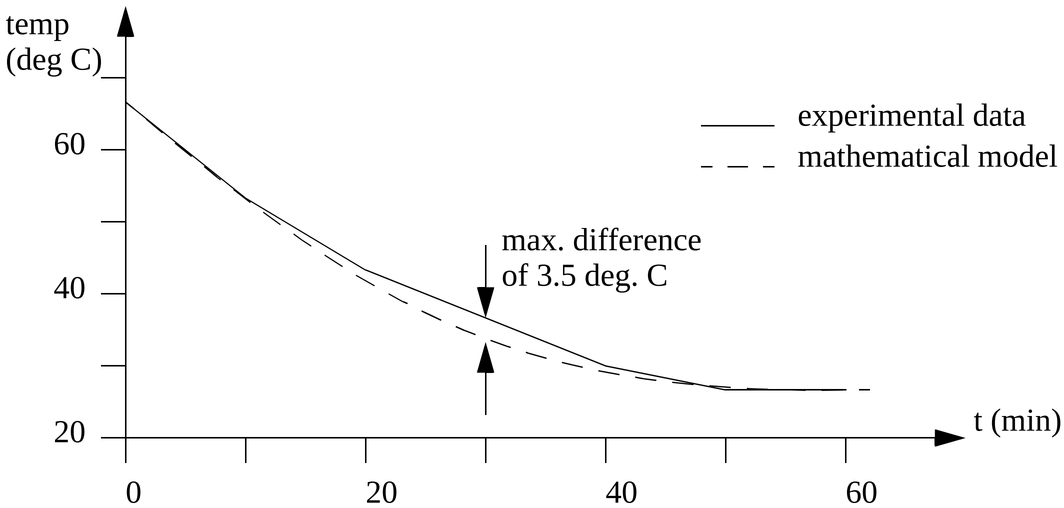


Figure 2 - Comparison of experimental and modelled curves

This gives an overall error of 8.5% between these two curves, compared to the total range of the data.

$$error = \frac{3.5}{65 - 24} 100 = 8.5\%$$

Finally, the results can be used to calculate a thermal resistance. If we know the mass of the coffee and assume that the coffee has the same specific heat as water, and have the time constant, the thermal resistance is found to be 1731sC/J.

$$\tau = M_{coffee} \sigma_{coffee} R$$

$$R = \frac{\tau}{M_{coffee} \sigma_{coffee}} = 1731 \frac{sC}{J}$$

$$M_{coffee} = 478g - 214g = 0.264Kg$$

$$\sigma_{coffee} = 4.2 \frac{C}{KgJ}$$

$$\tau = 32min = 1920s$$

Conclusion:

In general the models agreed well, except for a single data point. This error was relatively small, only being 8.5% of the entire data range. This error was most likely caused by a single measurement error. The error value is greater than the theoretical value, which suggests that the temperature might have been read at a "hot spot". In the procedure the temperature measuring location was not fixed, which probably resulted in a variation in measurement location.

23.3.1.2 - An Example Final Draft of a Report

• A final draft of the report is available on the course website in Mathcad format, and it will be distributed in the lab.

23.3.2 Research

• Purpose: After looking at a technical field we use these reports to condense the important details and differences. After reading a research report another reader should be able to discuss advanced topics in general terms.

• Strategy A:

1. Clearly define the objectives for the report
2. Outline what you know on a word processor in point form and find the ‘holes’
3. Do research to find the missing information
4. Incorporate the new and old information (still in point form)
5. Rearrange the points into a logical structure
6. Convert point form into full text
7. Proof read and edit

23.3.3 Project

• Purpose: These reports allow the developer or team to document all of the design decisions made during the course of the project. This report should also mention avenues not taken. Quite often the projects that we start will be handed off to others after a period of time. In many cases they will not have the opportunity to talk to us, or we may not have the time. These reports serve as a well known, central document that gathers all relevant information.

• Strategy A:

1. Define the goals for the project clearly in point form

2. Examine available options and also add these in point form
3. Start to examine engineering aspects of the options
4. Make engineering decisions, and add point form to the document
5. As work continues on the project add notes and figures
6. When the project is complete, convert the point form to full text.
7. Proof read and edit

23.3.4 Executive

• Purpose: These reports condense long topics into a very brief document, typically less than one page in length. Basically these save a manager from having to read a complete report to find the details that interest him/her.

23.3.5 Consulting

• Purpose: These reports are typically commissioned by an independent third party to review a difficult problem. The consultant will review the details of the problem, do tests as required, and summarize the results. The report typically ends with conclusions, suggestions or recommendations.

23.3.6 Interim

• Purpose: This report is normally a formal report to track the progress of a project. When a project is initially planned, it will be given a timeline to follow. The interim report will indicate progress relative to the initial timeline, as well as major achievements and problems.

23.4 ELEMENTS

• In reports we must back up our opinions with data, equations, drawings, etc. As a result we use a number of common items,

- figures
- tables
- equations
-

• When these elements are included, there MUST be a mention of them in the written text.

• These days it is common to cut and paste figures in software. Make sure

- the resolution is appropriate
- the colors print properly in the final form or print well as black and white
- the smallest features are visible
- scanned drawings are clean and cropped to size
- scanned photographs are clear and cropped to size
- digital photographs should be properly lit, and cropped to size
- screen captures are clipped to include only relevant data

23.4.1 Figures

• Figures include drawings, schematics, graphs, charts, etc.

• They should be labelled underneath sequentially and given a brief title to distinguish it from other graphs. For example “Figure 1 - Voltage and currents for 50 ohm resistor”

• In the body of the report the reference may be shortened to ‘Fig. 1’

• The figures do not need to immediately follow the reference, but they should be kept in sequence. We will often move figures to make the type setting work out better.

• If drawing graphs by computer,

- if fitting a line/curve to the points indicate the method used (e.g. linear regression)
- try not to use more than 5 curves on the same graph
- use legends that can be seen in black and white
- clearly label units and scales
- label axes with descriptive term. For example “Hardness (RHC)” instead of “RHC”
- scale the curve to make good use of the graph
- avoid overly busy graphs

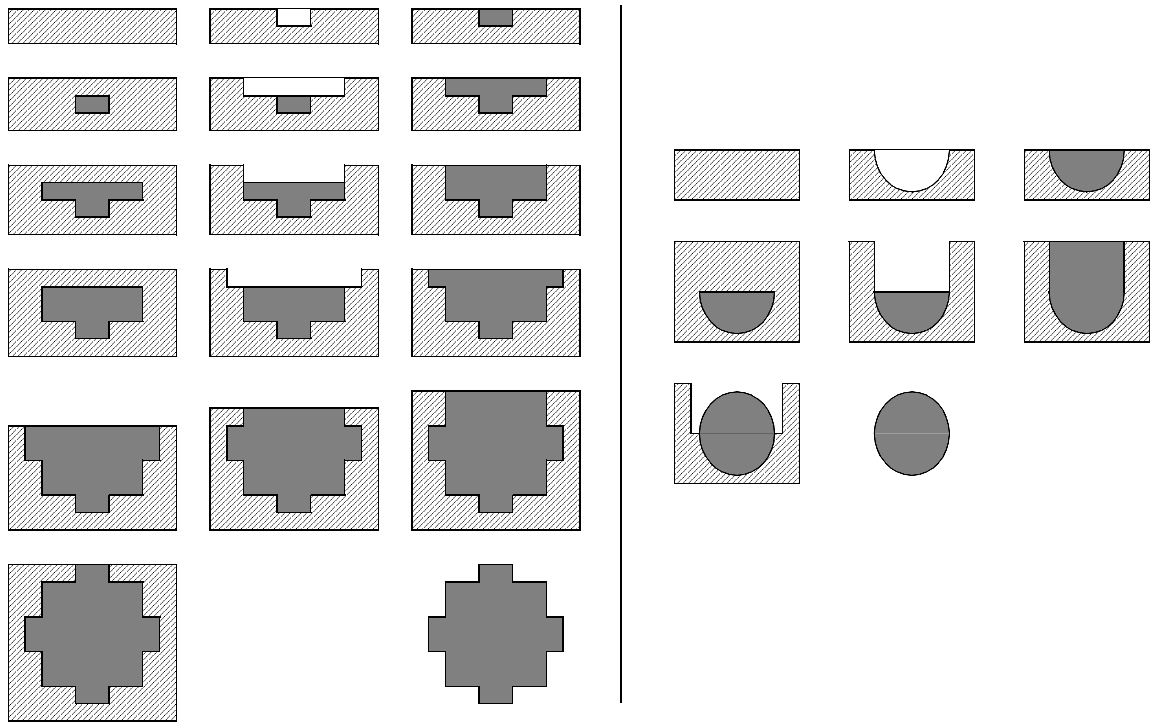


Figure 2 - Various Techniques for Making a Sphere with AMP

23.4.2 Tables

• Tables are often treated as figures.

• They allow dense information presentation, typically numerical in nature.

Table 2: A Comparison of Toy Vehicle Properties

Description	Number	Color	Shape	Material
car	3	red	rectangular	die cast
truck	6	blue	long	polyprop.
motorcycle	2	green	small	wood

• Legends can be added to tables to help condense size.

23.4.3 Equations

• When presenting equations, use a good equation editor, and watch to make sure subscripts, etc are visible.

• Number equations that are referred to in the text.

• Box in equations of great significance.

$$\overset{+}{\rightarrow}\sum F_x = -T_1 \sin 60° + F_R \sin\theta_R = 0 \qquad (1)$$

$$+\uparrow \sum F_y = -T_1 - T_1 \cos(60°) + F_R \cos\theta_R = 0 \qquad (2)$$

$$\therefore F_R = \frac{T_1 \sin 60°}{\sin\theta_R} = \frac{T_1 + T_1 \cos 60°}{\cos\theta_R} \qquad \text{sub (1) into (2)}$$

$$\therefore \frac{\sin 60°}{1 + \cos 60°} = \frac{\sin\theta_R}{\cos\theta_R} = \tan\theta_R$$

$$\therefore \tan\theta_R = \frac{0.866}{1 + 0.5}$$

$$\boxed{\therefore \theta_R = 30°}$$

$$98 \sin 60° = F_R \sin 30°$$

$$\boxed{\therefore F_R = 170N}$$

23.4.4 Experimental Data

• When analyzing the results from an experiment there are a few basic methods that may be used,

Percent difference -
Mean and standard deviation -
Point by point -
Matching functions -

etc......

XXXXXXXXXXXXXX Add more XXXXXXXXXXXXXXXXX

23.4.5 Result Summary

• It is very important to put a summary of results at the end of a report.

XXXXXXXXXXXXX

23.4.6 References

• References help provide direction to the sources of information when the information may be questioned, or the reader may want to get additional detail.

• Reference formats vary between publication sources. But, the best rule is be consistent.

• One popular method for references is to number them. The numbers are used in the body of the paper (eg, [14]), and the references are listed numerically at the end.

• Another method is to list the author name and year (eg, [Yackish, 1997]) and then list the references at the end of the report.

• Footnotes are not commonly used in engineering works.

23.4.7 Acknowledgments

• When others have contributed to the work but are not listed as authors we may choose to recognize them.

• Acknowledgments are brief statements that indicate who has contributed to a work.

23.4.8 Appendices

• When we have information that is needed to support a report, but is too bulky to include, one option is to add an appendix.

• Examples of appendices include,

- reviews of basic theory
- sample calculations
- long tables of materials data
- program listings
- long test results

23.5 GENERAL FORMATTING

• Some general formatting items are,

- number all pages sequentially,
  roman numerals starting from 'i)' on the first page
  arabic numerals starting from '1' on the
- or, number pages by section. This is very useful for multi part manuals
  for example '4-7' would be the 7th page in the 4th section
- if pages are blank label them 'this page left blank'
- number sections sequentially with roman or Arabic numerals
-

• For numbers,

- use engineering notation (move exponents 3 places) so that units are always micro, milli, kilo, mega, giga, etc.
- use significant figures to round the numbers
- units are required always

• General English usage,

- check spelling - note that you must read to double guess the smell checker.
- check grammar
- avoid informal phrases (e.g. "show me the money")
- define acronyms and jargon the first time you use them (e.g., IBM means "Ion Beam Manufacturing")

• General style rules,

- keep it simple (especially the introduction) - most authors trying to be eloquent end up sounding long winded and pretentious. For example, “Electronic computer based digital readings can provided a highly accurate data source to improve the quality of the ascertained data.” could be replaced with “Computer based data collection is more accurate.”
- get to the point and be concise. For example, “Readings of the pressure, as the probe was ascending up the chimney towards the top, were taken.” is better put “Pressure probe readings were taken as the probe was inserted”.
- it is fine to say ‘I’ or ‘we’, but don’t get carried away.
- don’t be afraid to reuse terms, phases or words if it is an exact description. For example, we could increase confusion by also describing translation as motion, movement, sliding, displacing, etc.

• General engineering rules are,

- all statements should be justified, avoid personal opinions or ‘gut feels’
- use exact engineering terms when needed, don’t try to get creative.

24. MATHEMATICAL TOOLS

***** This contains additions and sections by Dr. Andrew Sterian.

• We use math in almost every problem we solve. As a result the more relevant topics of mathematics are summarized here.

• This is not intended for learning, but for reference.

24.1 INTRODUCTION

• This section has been greatly enhanced, and tailored to meet our engineering requirements.

• The section outlined here is not intended to teach the elements of mathematics, but it is designed to be a quick reference guide to support the engineer required to use techniques that may not have been used recently.

• For those planning to write the first ABET Fundamentals of Engineering exam, the following topics are commonly on the exam.

- quadratic equation
- straight line equations - slop and perpendicular
- conics, circles, ellipses, etc.
- matrices, determinants, adjoint, inverse, cofactors, multiplication
- limits, L'Hospital's rule, small angle approximation
- integration of areas
- complex numbers, polar form, conjugate, addition of polar forms
- maxima, minima and inflection points
- first-order differential equations - guessing and separation
- second-order differential equation - linear, homogeneous, non-homogeneous, second-order
- triangles, sine, cosine, etc.
- integration - by parts and separation
- solving equations using inverse matrices, Cramer's rule, substitution
- eigenvalues, eigenvectors
- dot and cross products, areas of parallelograms, angles and triple product
- divergence and curl - solenoidal and conservative fields
- centroids
- integration of volumes

- integration using Laplace transforms
- probability - permutations and combinations
- mean, standard deviation, mode, etc.
- log properties
- taylor series
- partial fractions
- basic coordinate transformations - cartesian, cylindrical, spherical
- trig identities
- derivative - basics, natural log, small angles approx., chain rule, partial fractions

24.1.1 Constants and Other Stuff

• A good place to start a short list of mathematical relationships is with greek letters

lower case	upper case	name
α	Α	alpha
β	Β	beta
γ	Γ	gamma
δ	Δ	delta
ε	Ε	epsilon
ζ	Ζ	zeta
η	Η	eta
θ	Θ	theta
ι	Ι	iota
κ	Κ	kappa
λ	Λ	lambda
μ	Μ	mu
ν	Ν	nu
ξ	Ξ	xi
ο	Ο	omicron
π	Π	pi
ρ	Ρ	rho
σ	Σ	sigma
τ	Τ	tau
υ	Υ	upsilon
φ	Φ	phi
χ	Χ	chi
ψ	Ψ	psi
ω	Ω	omega

Figure 429 The greek alphabet

• The constants listed are amount some of the main ones, other values can be derived through calculation using modern calculators or computers. The values are typically given with more than 15 places of accuracy so that they can be used for double precision calculations.

$$e = 2.7182818 = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = \text{natural logarithm base}$$

$$\pi = 3.1415927 = \text{pi}$$

$$\gamma = 0.57721566 = \text{Eulers constant}$$

$$1radian = 57.29578°$$

Figure 430 Some universal constants

24.1.2 Basic Operations

• These operations are generally universal, and are described in sufficient detail for our use.

• Basic properties include,

commutative	$a + b = b + a$	
distributive	$a(b + c) = ab + ac$	
associative	$a(bc) = (ab)c$	$a + (b + c) = (a + b) + c$

Figure 431 Basic algebra properties

24.1.2.1 - Factorial

• A compact representation of a series of increasing multiples.

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n$$

$$0! = 1$$

Figure 432 The basic factorial operator

24.1.3 Exponents and Logarithms

• The basic properties of exponents are so important they demand some sort of mention

$$(x^n)(x^m) = x^{n+m} \qquad x^0 = 1 \text{ , if x is not 0} \qquad x^{\frac{1}{n}} = \sqrt[n]{x}$$

$$\frac{(x^n)}{(x^m)} = x^{n-m} \qquad x^{-p} = \frac{1}{x^p} \qquad x^{\frac{m}{n}} = \sqrt[n]{x^m}$$

$$(x^n)^m = x^{n \cdot m} \qquad (xy)^n = (x^n)(y^n) \qquad \sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

Figure 433 Properties of exponents

• Logarithms also have a few basic properties of use, The basic base 10 logarithm:

$$\log x = y \qquad\qquad x = 10^y$$

The basic base n logarithm:

$$\log_n x = y \qquad\qquad x = n^y$$

The basic natural logarithm (e is a constant with a value found near the start of this section:

$$\ln x = \log_e x = y \qquad\qquad x = e^y$$

Figure 434 Definitions of logarithms

• All logarithms observe a basic set of rules for their application,

$$\log_n(xy) = \log_n(x) + \log_n(y) \qquad\qquad \log_n(n) = 1$$

$$\log_n(1) = 0$$

$$\log_n\left(\frac{x}{y}\right) = \log_n(x) - \log_n(y)$$

$$\log_n(x^y) = y\log_n(x)$$

$$\log_n(x) = \frac{\log_m(x)}{\log_m(n)}$$

$$\ln(A\angle\theta) = \ln(A) + (\theta + 2\pi k)j \qquad\qquad k \in I$$

Figure 435 Properties of logarithms

24.1.4 Polynomial Expansions

• Binomial expansion for polynomials,

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \ldots + x^n$$

Figure 436 A general expansion of a polynomial

24.1.5 Practice Problems

1. Are the following expressions equivalent?

a) $A(5+B) - C = 5A + B - C$

b) $\frac{A+B}{C+D} = \frac{A}{C} + \frac{B}{D}$

c) $\log(ab) = \log(a) + \log(b)$

d) $5(5^4) = 5^5$

e) $3\log(4) = \log(16)$

f) $(x+6)(x-6) = x^2 + 36$

g) $10^{\log(5)} = \frac{10}{5}$

h) $\sqrt{\frac{(x+1)^6}{(x+1)^2}} = x^2 + 2x + 1$

2. Simplify the following expressions.

a) $x(x+2)^2 - 3x$

b) $\frac{(x+3)(x+1)x^2}{(x+1)^2 x}$

c) $\log(x^3)$

d) $\frac{64}{16}$

e) $\frac{15}{21} + \frac{3}{28}$

f) $(x^2 y^3)^4$

g) $\sqrt{4x^2 - 8y^4}$

h) $\frac{5}{3}\left(\frac{8}{9}\right)$

i) $\frac{\left(\frac{5}{4}\right)}{5}$

j) $(y+4)^3(y-2)$

k) $\sqrt{x^2 y}$

l) $\frac{x+1}{x+2} = 4$

3. Simplify the following expressions.

a) $\dfrac{A+B}{AB}$

b) $\dfrac{AB}{A+B}$

c) $\left(\dfrac{(x^4y^5)}{x^2}\right)^3$

d) $\log(x^5) + \log(x^3)$

(ans.

a) $\dfrac{A+B}{AB} = \dfrac{A}{AB} + \dfrac{B}{AB} = \dfrac{1}{B} + \dfrac{1}{A}$

b) $\dfrac{AB}{A+B}$ cannot be simplified

c) $\left(\dfrac{(x^4y^5)}{x^2}\right)^3 = (x^2y^5)^3 = x^6y^{15}$

d) $\log(x^5) + \log(x^3) = 5\log(x) + 3\log(x) = 8\log(x)$

4. Simplify the following expressions.

a) $n\log(x^2) + m\log(x^3) - \log(x^4)$

(ans.

a)

$$n\log(x^2) + m\log(x^3) - \log(x^4)$$

$$2n\log(x) + 3m\log(x) - 4\log(x)$$

$$(2n + 3m - 4)\log(x)$$

$$(2n + 3m - 4)\log(x)$$

5. Rearrange the following equation so that only 'y' is on the left hand side.

$$\frac{y + x}{y + z} = x + 2$$

(ans.

$$\frac{y + x}{y + z} = x + 2$$

$$y + x = (x + 2)(y + z)$$

$$y + x = xy + xz + 2y + 2z$$

$$y - xy - 2y = xz + 2z - x$$

$$y(-x - 1) = xz + 2z - x$$

$$y = \frac{xz + 2z - x}{-x - 1}$$

6. Find the limits below.

a) $$\lim_{t \to 0}\left(\frac{t^3 + 5}{5t^3 + 1}\right)$$

b) $$\lim_{t \to \infty}\left(\frac{t^3 + 5}{5t^3 + 1}\right)$$

(ans.

a) $$\lim_{t \to 0}\left(\frac{t^3 + 5}{5t^3 + 1}\right) = \frac{0^3 + 5}{5(0)^3 + 1} = 5$$

b) $$\lim_{t \to \infty}\left(\frac{t^3 + 5}{5t^3 + 1}\right) = \frac{\infty^3 + 5}{5(\infty)^3 + 1} = \frac{\infty^3}{5(\infty)^3} = 0.2$$

24.2 FUNCTIONS

24.2.1 Discrete and Continuous Probability Distributions

Binomial

$$P(m) = \sum_{t \le m} \binom{n}{t} p^t q^{n-t} \qquad q = 1 - p \qquad q, p \in [0, 1]$$

Poisson

$$P(m) = \sum_{t \le m} \frac{\lambda^t e^{-\lambda}}{t!} \qquad \lambda > 0$$

Hypergeometric

$$P(m) = \sum_{t \le m} \frac{\binom{r}{t}\binom{s}{n-t}}{\binom{r+s}{n}}$$

Normal

$$P(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2} dt$$

Figure 437 Distribution functions

24.2.2 Basic Polynomials

• The quadratic equation appears in almost every engineering discipline, therefore is of great importance.

$$ax^2 + bx + c = 0 = a(x - r_1)(x - r_2) \qquad r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Figure 438 Quadratic equation

• Cubic equations also appear on a regular basic, and as a result should also be considered.

$$x^3 + ax^2 + bx + c = 0 = (x - r_1)(x - r_2)(x - r_3)$$

First, calculate,

$$Q = \frac{3b - a^2}{9} \qquad R = \frac{9ab - 27c - 2a^3}{54} \qquad S = \sqrt[3]{R + \sqrt{Q^3 + R^2}} \qquad T = \sqrt[3]{R - \sqrt{Q^3 + R^2}}$$

Then the roots,

$$r_1 = S + T - \frac{a}{3} \qquad r_2 = \frac{S + T}{2} - \frac{a}{3} + \frac{j\sqrt{3}}{2}(S - T) \qquad r_3 = \frac{S + T}{2} - \frac{a}{3} - \frac{j\sqrt{3}}{2}(S - T)$$

Figure 439 Cubic equations

• On a few occasions a quartic equation will also have to be solved. This can be done by first reducing the equation to a quadratic,

$$x^4 + ax^3 + bx^2 + cx + d = 0 = (x - r_1)(x - r_2)(x - r_3)(x - r_4)$$

First, solve the equation below to get a real root (call it 'y'),

$$y^3 - by^2 + (ac - 4d)y + (4bd - c^2 - a^2 d) = 0$$

Next, find the roots of the 2 equations below,

$$r_1, r_2 = z^2 + \left(\frac{a + \sqrt{a^2 - 4b + 4y}}{2}\right)z + \left(\frac{y + \sqrt{y^2 - 4d}}{2}\right) = 0$$

$$r_3, r_4 = z^2 + \left(\frac{a - \sqrt{a^2 - 4b + 4y}}{2}\right)z + \left(\frac{y - \sqrt{y^2 - 4d}}{2}\right) = 0$$

Figure 440 Quartic equations

24.2.3 Partial Fractions

• The next is a flowchart for partial fraction expansions.

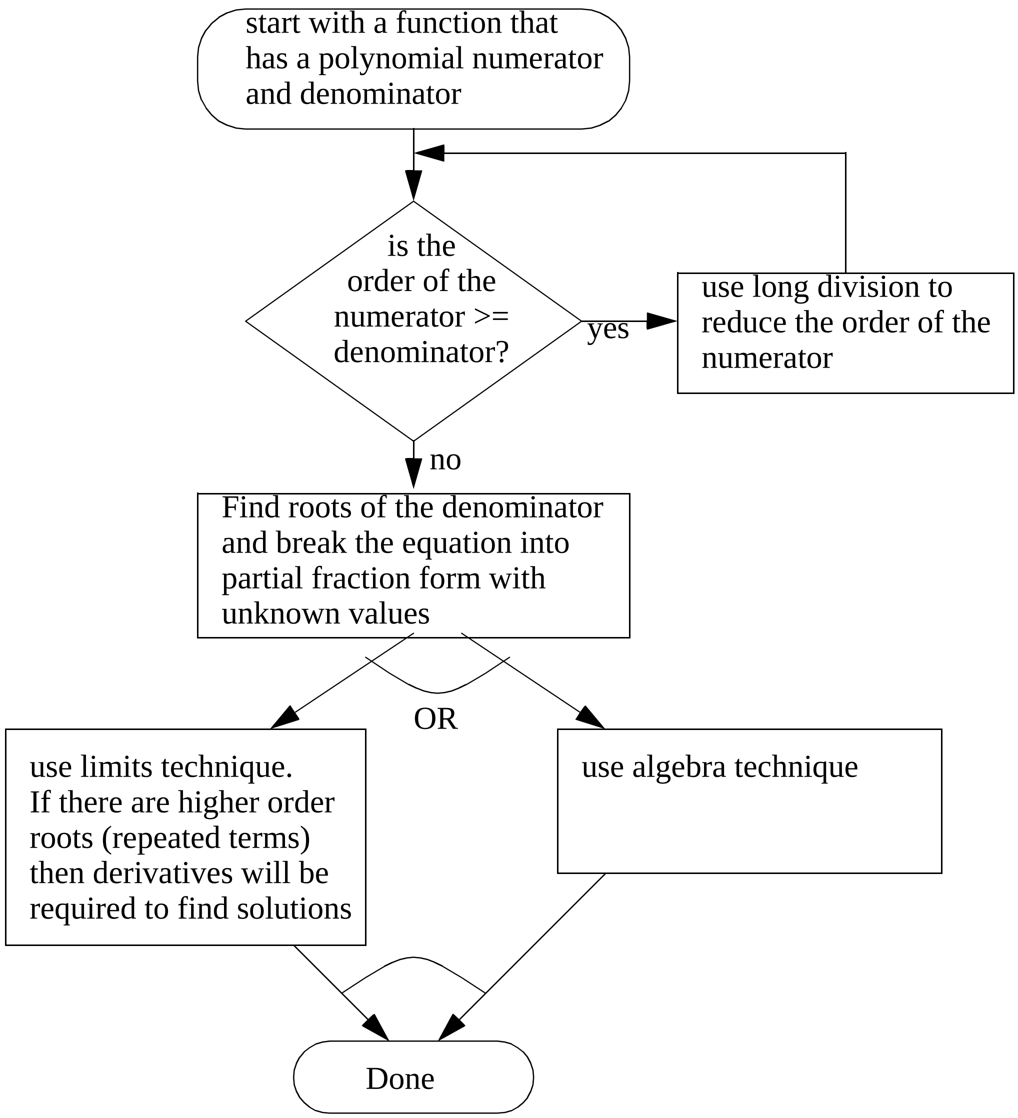


Figure 441 The methodolgy for solving partial fractions

• The partial fraction expansion for,

$$x(s) = \frac{1}{s^2(s+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}$$

$$C = \lim_{s \to -1}\left[(s+1)\left(\frac{1}{s^2(s+1)}\right)\right] = 1$$

$$A = \lim_{s \to 0}\left[s^2\left(\frac{1}{s^2(s+1)}\right)\right] = \lim_{s \to 0}\left[\frac{1}{s+1}\right] = 1$$

$$B = \lim_{s \to 0}\left[\frac{d}{ds}\left[s^2\left(\frac{1}{s^2(s+1)}\right)\right]\right] = \lim_{s \to 0}\left[\frac{d}{ds}\left(\frac{1}{s+1}\right)\right] = \lim_{s \to 0}[-(s+1)^{-2}] = -1$$

Figure 442 A partial fraction example

• Consider the example below where the order of the numerator is larger than the denominator.

$$x(s) = \frac{5s^3 + 3s^2 + 8s + 6}{s^2 + 4}$$

This cannot be solved using partial fractions because the numerator is 3rd order and the denominator is only 2nd order. Therefore long division can be used to reduce the order of the equation.

$$\begin{array}{r|l} & 5s + 3 \\ \hline s^2 + 4 & 5s^3 + 3s^2 + 8s + 6 \\ & \underline{5s^3 + 20s} \\ & \quad 3s^2 - 12s + 6 \\ & \quad \underline{3s^2 + 12} \\ & \qquad -12s - 6 \end{array}$$

This can now be used to write a new function that has a reduced portion that can be solved with partial fractions.

$$x(s) = 5s + 3 + \frac{-12s - 6}{s^2 + 4} \qquad \text{solve} \qquad \frac{-12s - 6}{s^2 + 4} = \frac{A}{s + 2j} + \frac{B}{s - 2j}$$

Figure 443 Solving partial fractions when the numerator order is greater than the denominator

• When the order of the denominator terms is greater than 1 it requires an expanded partial fraction form, as shown below.

$$F(s) = \frac{5}{s^2(s+1)^3}$$

$$\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}$$

Figure 444 Partial fractions with repeated roots

• We can solve the previous problem using the algebra technique.

$$\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}$$

$$= \frac{A(s+1)^3 + Bs(s+1)^3 + Cs^2 + Ds^2(s+1) + Es^2(s+1)^2}{s^2(s+1)^3}$$

$$= \frac{s^4(B+E) + s^3(A+3B+D+2E) + s^2(3A+3B+C+D+E) + s(3A+B) + (A)}{s^2(s+1)^3}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 & 2 \\ 3 & 3 & 1 & 1 & 1 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 & 2 \\ 3 & 3 & 1 & 1 & 1 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ -15 \\ 5 \\ 10 \\ 15 \end{bmatrix}$$

$$\frac{5}{s^2(s+1)^3} = \frac{5}{s^2} + \frac{-15}{s} + \frac{5}{(s+1)^3} + \frac{10}{(s+1)^2} + \frac{15}{(s+1)}$$

Figure 445 An algebra solution to partial fractions

24.2.4 Summation and Series

• The notation $\sum_{i=a}^{b} x_i$ is equivalent to $x_a + x_{a+1} + x_{a+2} + \ldots + x_b$ assuming a and b are integers and $b \geq a$. The index variable i is a placeholder whose name does not matter.

• Operations on summations:

$$\sum_{i=a}^{b} x_i = \sum_{i=b}^{a} x_i$$

$$\sum_{i=a}^{b} \alpha x_i = \alpha \sum_{i=a}^{b} x_i$$

$$\sum_{i=a}^{b} x_i + \sum_{j=a}^{b} y_j = \sum_{i=a}^{b} (x_i + y_i)$$

$$\sum_{i=a}^{b} x_i + \sum_{i=b+1}^{c} x_i = \sum_{i=a}^{c} x_i$$

$$\left(\sum_{i=a}^{b} x_i\right)\left(\sum_{j=c}^{d} y_j\right) = \sum_{i=a}^{b} \sum_{j=c}^{d} x_i y_j$$

• Some common summations:

$$\sum_{i=1}^{N} i = \frac{1}{2}N(N+1)$$

$$\sum_{i=0}^{N-1} \alpha^i = \begin{cases} \dfrac{1-\alpha^N}{1-\alpha}, & \alpha \neq 1 \\ N, & \alpha = 1 \end{cases}$$ for both real and complex α.

$$\sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha}, \ |\alpha| < 1$$ for both real and complex α. For $|\alpha| \geq 1$, the summation does not converge.

24.2.5 Practice Problems

1. Convert the following polynomials to multiplied terms as shown in the example.

e.g., $x^2 + 2x + 1 = (x + 1)(x + 1)$

a) $x^2 - 2x + 1$

b) $x^2 - 1$

c) $x^2 + 1$

d) $x^2 + x + 10$

e)

f)

2. Solve the following equation to find 'x'.

$$2x^2 + 8x = -8$$

(ans.

$$2x^2 + 8x = -8$$

$$x^2 + 4x + 4 = 0$$

$$(x + 2)^2 = 0$$

$$x = -2, -2$$

3. Reduce the following expression to partial fraction form.

$$\frac{x + 4}{x^2 + 8x}$$

(ans. $\frac{x + 4}{x^2 + 8x} = \frac{x + 4}{x(x + 8)} = \frac{A}{x} + \frac{B}{x + 8} = \frac{Ax + 8A + Bx}{x^2 + 8x}$

$$(1)x + 4 = (A + B)x + 8A$$

$8A = 4$ $\qquad$ $A = 0.5$

$1 = A + B$ $\qquad$ $B = 0.5$

$$\frac{x + 4}{x^2 + 8x} = \frac{0.5}{x} + \frac{0.5}{x + 8}$$

24.3 SPATIAL RELATIONSHIPS

24.3.1 Trigonometry

• The basic trigonometry functions are,

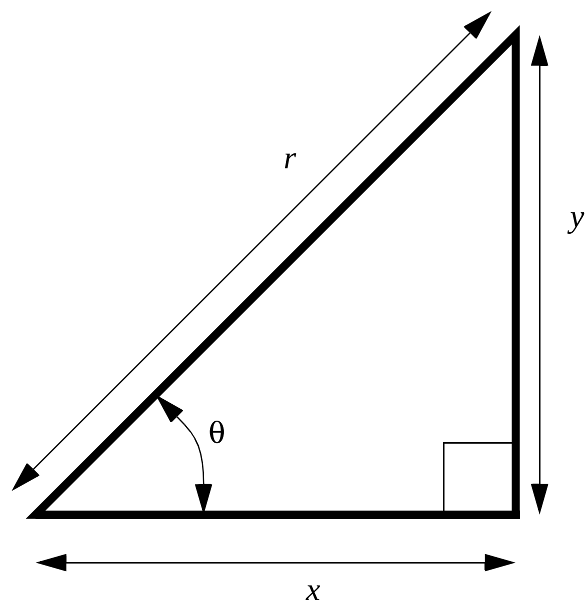
$$\sin\theta = \frac{y}{r} = \frac{1}{\csc\theta}$$

$$\cos\theta = \frac{x}{r} = \frac{1}{\sec\theta}$$

$$\tan\theta = \frac{y}{x} = \frac{1}{\cot\theta} = \frac{\sin\theta}{\cos\theta}$$

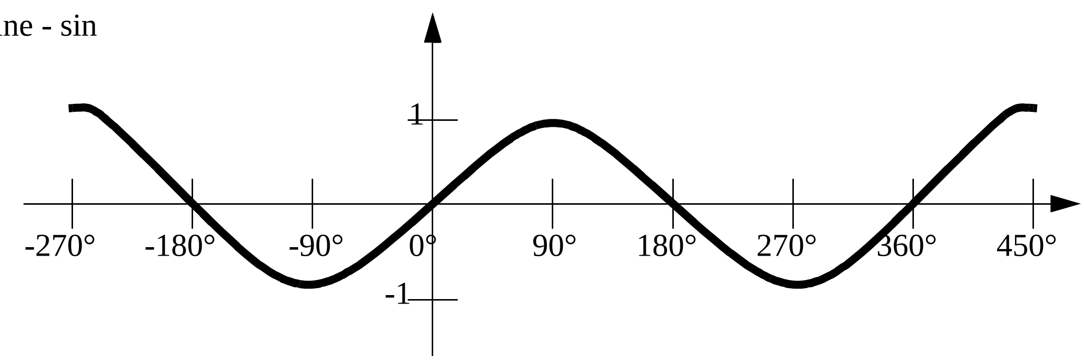
Pythagorean Formula:

$$r^2 = x^2 + y^2$$

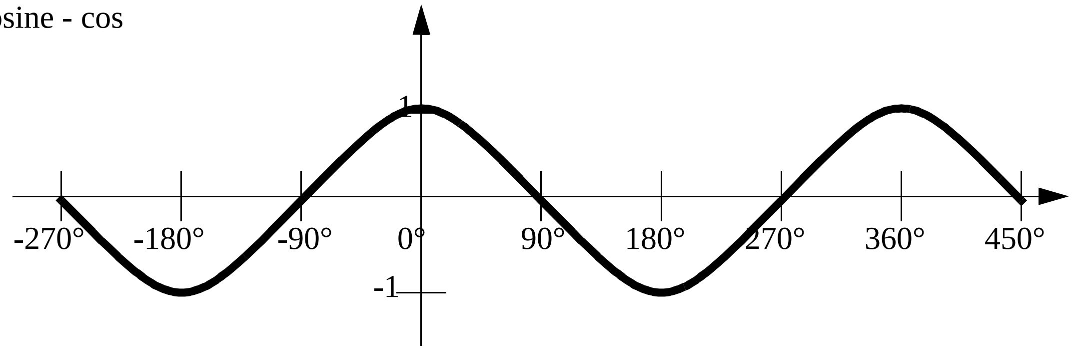


• Graphs of these functions are given below,

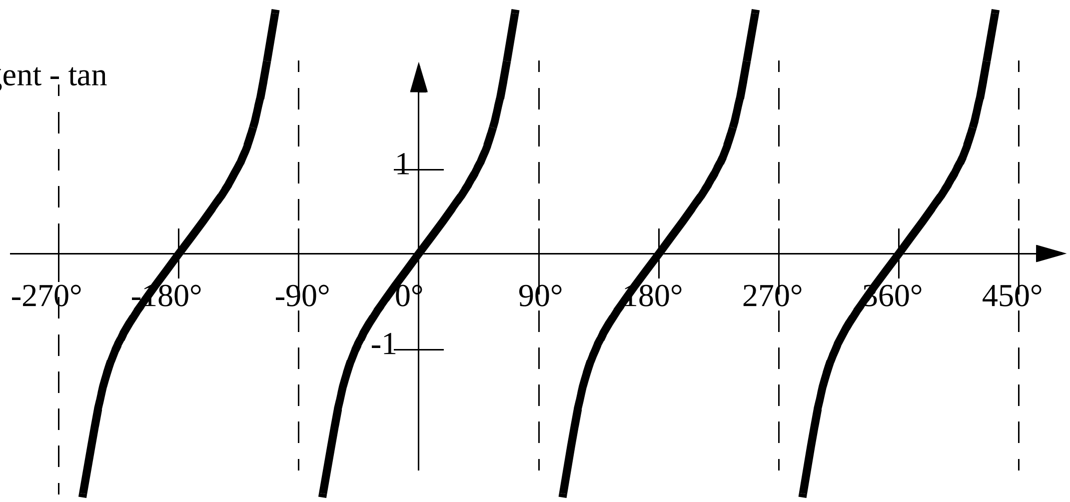
Sine - sin



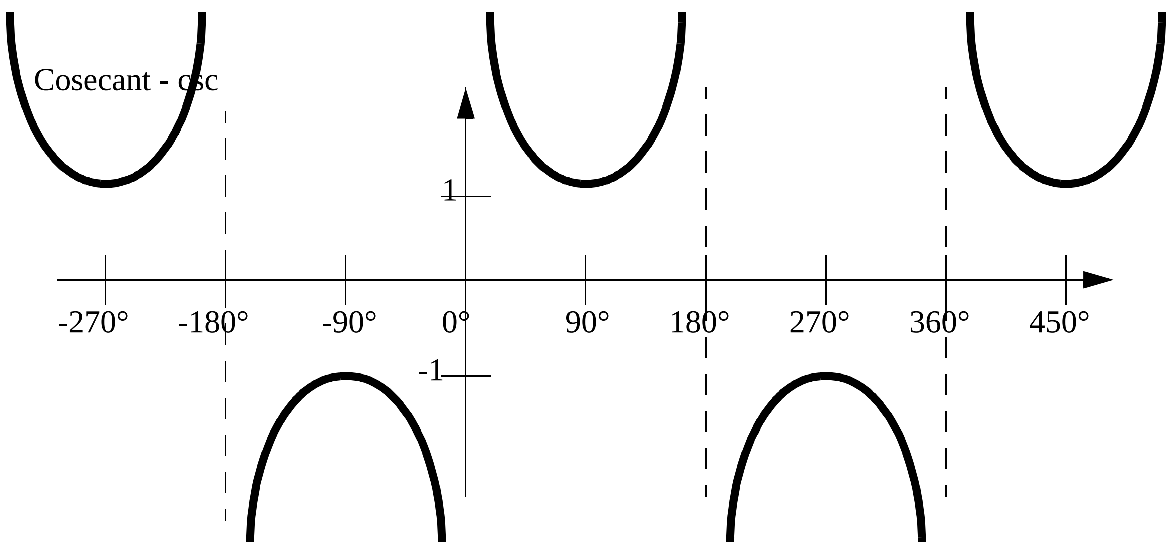
Cosine - cos

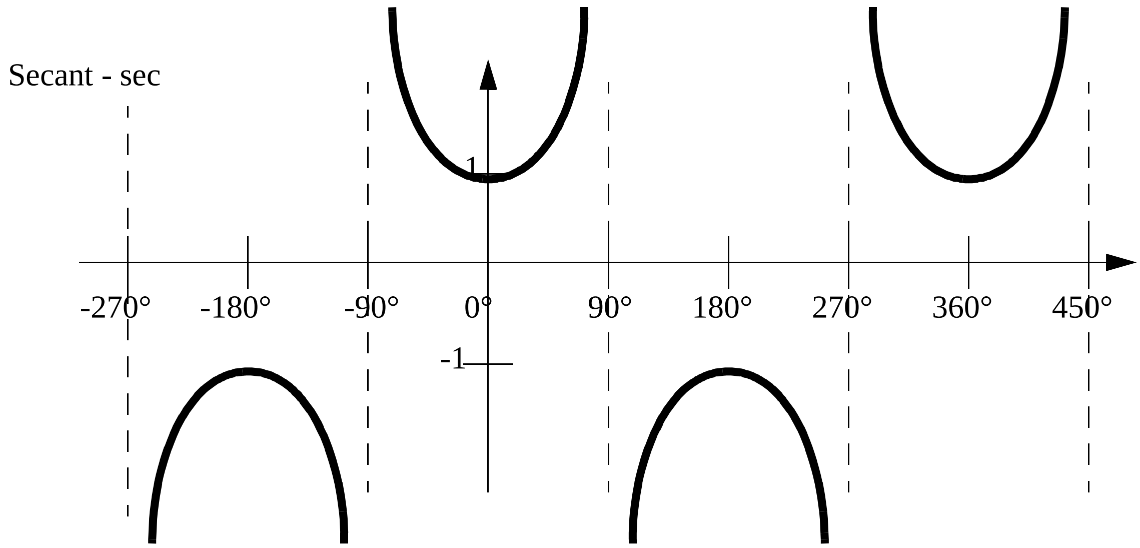


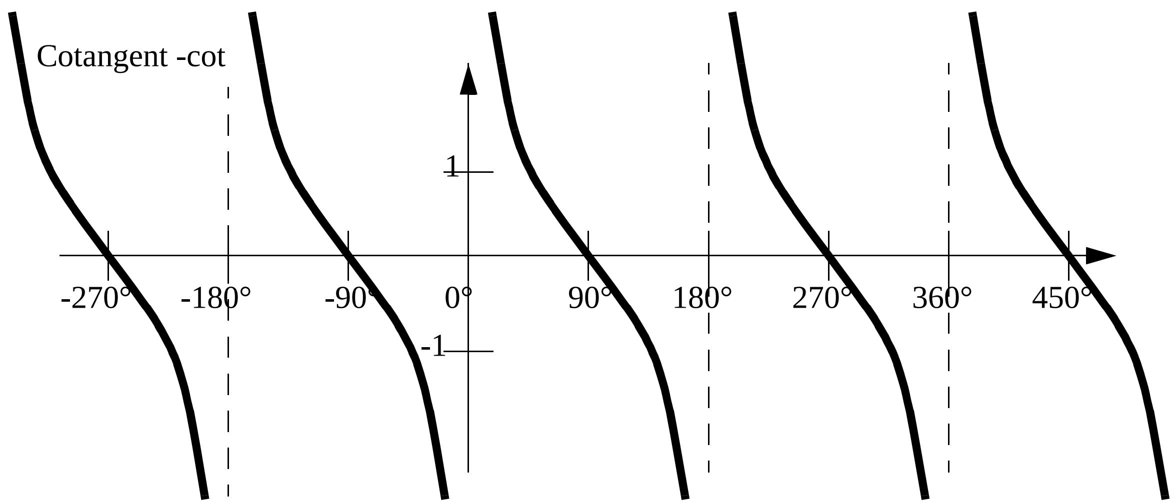
Tangent - tan



Cosecant - csc







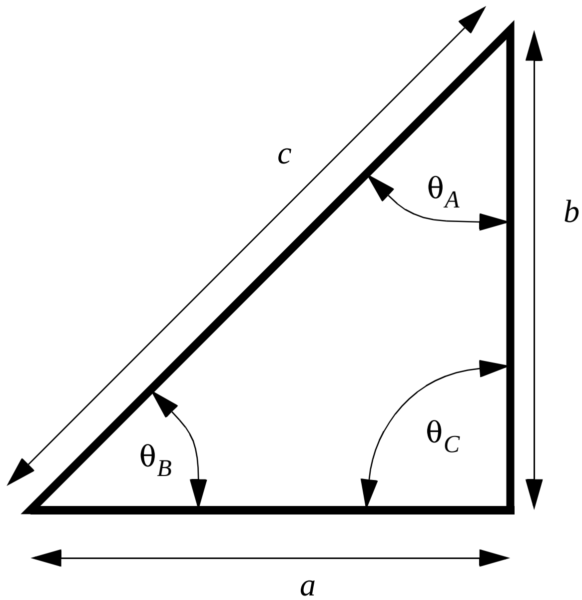
• NOTE: Keep in mind when finding these trig values, that any value that does not lie in the right hand quadrants of cartesian space, may need additions of ±90° or ±180°.

Cosine Law:

$$c^2 = a^2 + b^2 - 2ab\cos\theta_c$$

Sine Law:

$$\frac{a}{\sin\theta_A} = \frac{b}{\sin\theta_B} = \frac{c}{\sin\theta_C}$$



• Now a group of trigonometric relationships will be given. These are often best used when attempting to manipulate equations.

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

$$\sin\theta = \cos(\theta - 90°) = \cos(90° - \theta) = \text{etc.}$$

$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1\cos\theta_2 \pm \cos\theta_1\sin\theta_2 \qquad \text{OR} \qquad \sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2 \qquad \text{OR} \qquad \cos(2\theta) = (\cos\theta)^2 + (\sin\theta)^2$$

$$\tan(\theta_1 \pm \theta_2) = \frac{\tan\theta_1 \pm \tan\theta_2}{1 \mp \tan\theta_1\tan\theta_2}$$

$$\cot(\theta_1 \pm \theta_2) = \frac{\cot\theta_1\cot\theta_2 \mp 1}{\tan\theta_2 \pm \tan\theta_1}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

-ve if in left hand quadrants

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$$

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

• These can also be related to complex exponents,

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

24.3.2 Hyperbolic Functions

• The basic definitions are given below,

$$\sinh(x) = \frac{e^x - e^{-x}}{2} = \text{hyperbolic sine of x}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} = \text{hyperbolic cosine of x}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \text{hyperbolic tangent of x}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}} = \text{hyperbolic cosecant of x}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}} = \text{hyperbolic secant of x}$$

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \text{hyperbolic cotangent of x}$$

• some of the basic relationships are,

$$\sinh(-x) = -\sinh(x)$$

$$\cosh(-x) = \cosh(x)$$

$$\tanh(-x) = -\tanh(x)$$

$$\operatorname{csch}(-x) = -\operatorname{csch}(x)$$

$$\operatorname{sech}(-x) = \operatorname{sech}(x)$$

$$\coth(-x) = -\coth(x)$$

• Some of the more advanced relationships are,

$$(\cosh x)^2 - (\sinh x)^2 = (\operatorname{sech} x)^2 + (\tanh x)^2 = (\coth x)^2 - (\operatorname{csch} x)^2 = 1$$

$$\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \cosh(x)\sinh(y)$$

$$\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$$

$$\tanh(x \pm y) = \frac{\tanh(x) \pm \tanh(y)}{1 \pm \tanh(x)\tanh(y)}$$

• Some of the relationships between the hyperbolic, and normal trigonometry functions are,

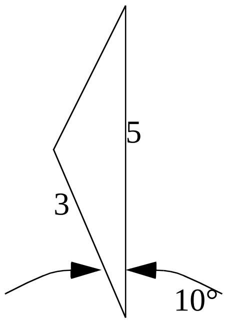
$$\sin(jx) = j\sinh(x) \qquad j\sin(x) = \sinh(jx)$$

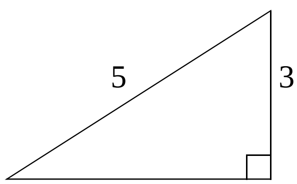
$$\cos(jx) = \cosh(x) \qquad \cos(x) = \cosh(jx)$$

$$\tan(jx) = j\tanh(x) \qquad j\tan(x) = \tanh(jx)$$

24.3.2.1 - Practice Problems

1. Find all of the missing side lengths and corner angles on the two triangles below.





2. Simplify the following expressions.

$$\cos\theta\cos\theta - \sin\theta\sin\theta =$$

$$(s + 3j)(s - 3j)(s + 2j)^2 =$$

(ans.

$$\cos\theta\cos\theta - \sin\theta\sin\theta = \cos(\theta + \theta) = \cos(2\theta)$$

$$(s + 3j)(s - 3j)(s + 2j)^2 = (s^2 - 9j^2)(s^2 + 4js + 4j^2)$$

$$s^4 + 4js^3 + 4j^2s^2 - 9j^2s^2 - 9j^2 4js - 9j^2 4j^2$$

$$s^4 + (4j)s^3 + (5)s^2 + (36j)s + (-36)$$

3. Solve the following partial fraction

$$\frac{4}{x^2 + 3x + 2} =$$

Note: there was a typo here, so an acceptable answer is also. $\frac{1}{x + 0.5}$

(ans.

$$\frac{4}{x^2 + 3x + 2} = \frac{A}{x + 1} + \frac{B}{x + 2} = \frac{Ax + 2A + Bx + B}{x^2 + 3x + 2} = \frac{(2A + B) + (A + B)x}{x^2 + 3x + 2}$$

$$A + B = 0 \qquad A = -B$$

$$2A + B = 4 = -2B + B = -B \qquad B = -4 \quad A = 4 \qquad \frac{4}{x + 1} + \frac{-4}{x + 2}$$

24.3.3 Geometry

• A set of the basic 2D and 3D geometric primitives are given, and the notation used is described below,

A = contained area

P = perimeter distance

V = contained volume

S = surface area

x, y, z = centre of mass

$\bar{x}, \bar{y}, \bar{z}$ = centroid

I_x, I_y, I_z = moment of inertia of area (or second moment of inertia)

AREA PROPERTIES:

$$I_x = \int_A y^2 dA = \text{ the second moment of inertia about the y-axis}$$

$$I_y = \int_A x^2 dA = \text{ the second moment of inertia about the x-axis}$$

$$I_{xy} = \int_A xy dA = \text{ the product of inertia}$$

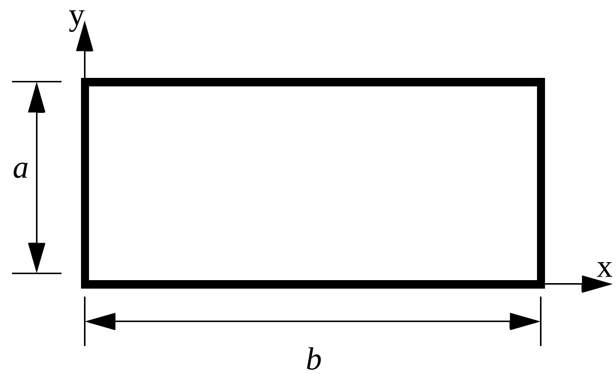
$$J_O = \int_A r^2 dA = \int_A (x^2 + y^2) dA = I_x + I_y = \text{ The polar moment of inertia}$$

$$\bar{x} = \frac{\int_A x dA}{\int_A dA} = \text{ centroid location along the x-axis}$$

$$\bar{y} = \frac{\int_A y dA}{\int_A dA} = \text{ centroid location along the y-axis}$$

Rectangle/Square:

$$A = ab$$
$$P = 2a + 2b$$

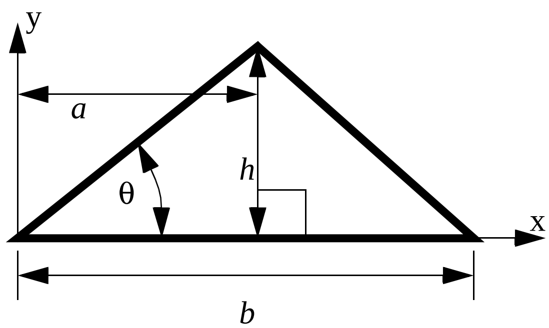


Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = \dfrac{b}{2}$	$\bar{I}_x = \dfrac{ba^3}{12}$	$I_x = \dfrac{ba^3}{3}$
$\bar{y} = \dfrac{a}{2}$	$\bar{I}_y = \dfrac{b^3a}{12}$	$I_y = \dfrac{b^3a}{3}$
	$\overline{I_{xy}} = 0$	$I_{xy} = \dfrac{b^2a^2}{4}$

Triangle:

$$A = \frac{bh}{2}$$

$$P =$$



Centroid:

$$\bar{x} = \frac{a+b}{3}$$

$$\bar{y} = \frac{h}{3}$$

Moment of Inertia
(about centroid axes):

$$\bar{I}_x = \frac{bh^3}{36}$$

$$\bar{I}_y = \frac{bh}{36}(a^2 + b^2 - ab)$$

$$\overline{I_{xy}} = \frac{bh^2}{72}(2a - b)$$

Moment of Inertia
(about origin axes):

$$I_x = \frac{bh^3}{12}$$

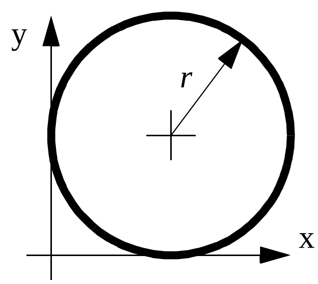
$$I_y = \frac{bh}{12}(a^2 + b^2 - ab)$$

$$I_{xy} = \frac{bh^2}{24}(2a - b)$$

Circle:

$$A = \pi r^2$$

$$P = 2\pi r$$



Centroid:

$$\bar{x} = r$$

$$\bar{y} = r$$

Moment of Inertia
(about centroid axes):

$$\bar{I}_x = \frac{\pi r^4}{4}$$

$$\bar{I}_y = \frac{\pi r^4}{4}$$

$$\overline{I_{xy}} = 0$$

Moment of Inertia
(about origin axes):

$$I_x =$$

$$I_y =$$

$$I_{xy} =$$

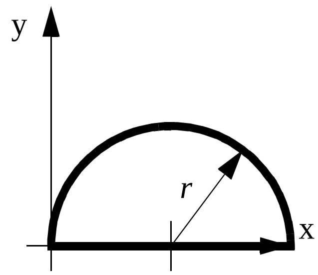
Mass Moment of Inertia
(about centroid):

$$J_z = \frac{Mr^2}{2}$$

Half Circle:

$$A = \frac{\pi r^2}{2}$$

$$P = \pi r + 2r$$



Centroid:

$$\bar{x} = r$$

$$\bar{y} = \frac{4r}{3\pi}$$

Moment of Inertia
(about centroid axes):

$$\bar{I}_x = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right)r^4$$

$$\bar{I}_y = \frac{\pi r^4}{8}$$

$$I_{xy} = 0$$

Moment of Inertia
(about origin axes):

$$I_x = \frac{\pi r^4}{8}$$

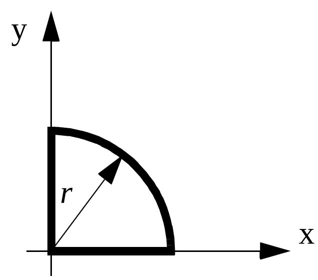
$$I_y = \frac{\pi r^4}{8}$$

$$I_{xy} = 0$$

Quarter Circle:

$$A = \frac{\pi r^2}{4}$$

$$P = \frac{\pi r}{2} + 2r$$



Centroid:

$$\bar{x} = \frac{4r}{3\pi}$$

$$\bar{y} = \frac{4r}{3\pi}$$

Moment of Inertia
(about centroid axes):

$$\bar{I}_x = 0.05488r^4$$

$$\bar{I}_y = 0.05488r^4$$

$$\overline{I_{xy}} = -0.01647r^4$$

Moment of Inertia
(about origin axes):

$$I_x = \frac{\pi r^4}{16}$$

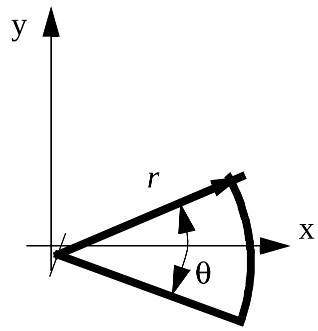
$$I_y = \frac{\pi r^4}{16}$$

$$I_{xy} = \frac{r^4}{8}$$

Circular Arc:

$$A = \frac{\theta r^2}{2}$$

$$P = \theta r + 2r$$



Centroid:

$$\bar{x} = \frac{2r\sin\frac{\theta}{2}}{3\theta}$$

$$\bar{y} = 0$$

Moment of Inertia (about centroid axes):

$$\bar{I}_x =$$

$$\bar{I}_y =$$

$$\overline{I_{xy}} =$$

Moment of Inertia (about origin axes):

$$I_x = \frac{r^4}{8}(\theta - \sin\theta)$$

$$I_y = \frac{r^4}{8}(\theta + \sin\theta)$$

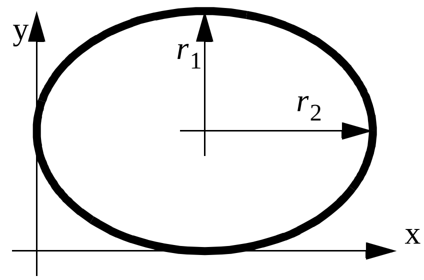
$$I_{xy} = 0$$

Ellipse:

$$A = \pi r_1 r_2$$

$$P = 4r_1 \int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{\sqrt{r_1^2 + r_2^2}}{a}(\sin\theta)^2}\, d\theta$$

$$P \approx 2\pi\sqrt{\frac{r_1^2 + r_2^2}{2}}$$



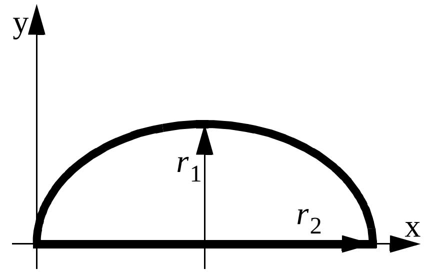
Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r_2$	$\bar{I}_x = \frac{\pi r_1^3 r_2}{4}$	$I_x =$
$\bar{y} = r_1$	$\bar{I}_y = \frac{\pi r_1 r_2^3}{4}$	$I_y =$
	$\overline{I_{xy}} =$	$I_{xy} =$

Half Ellipse:

$$A = \frac{\pi r_1 r_2}{2}$$

$$P = 2r_1\int_0^{\frac{\pi}{2}}\sqrt{1 - \frac{\sqrt{r_1^2 + r_2^2}}{a}(\sin\theta)^2}\,d\theta + 2r_2$$

$$P \approx \pi\sqrt{\frac{r_1^2 + r_2^2}{2}} + 2r_2$$



Centroid:

$$\bar{x} = r_2$$

$$\bar{y} = \frac{4r_1}{3\pi}$$

Moment of Inertia
(about centroid axes):

$$\bar{I}_x = 0.05488 r_2 r_1^3$$

$$\bar{I}_y = 0.05488 r_2^3 r_1$$

$$\bar{I}_{xy} = -0.01647 r_1^2 r_2^2$$

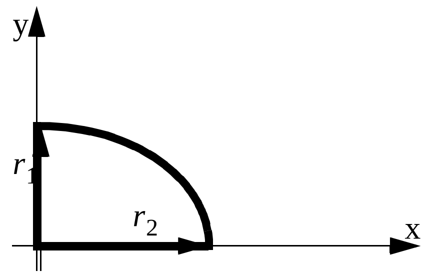
Moment of Inertia
(about origin axes):

$$I_x = \frac{\pi r_2 r_1^3}{16}$$

$$I_y = \frac{\pi r_2^3 r_1}{16}$$

$$I_{xy} = \frac{r_1^2 r_2^2}{8}$$

Quarter Ellipse:



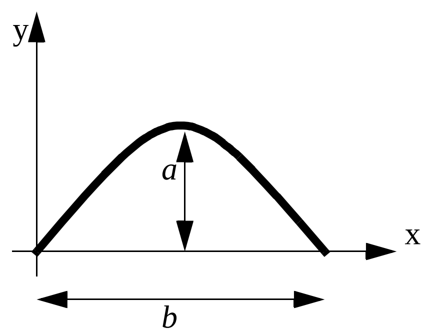
$$A = \frac{\pi r_1 r_2}{4}$$

$$P = r_1 \int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{\sqrt{r_1^2 + r_2^2}}{a} (\sin\theta)^2} d\theta + 2r_2$$

$$P \approx \frac{\pi}{2}\sqrt{\frac{r_1^2 + r_2^2}{2}} + 2r_2$$

Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = \frac{4r_2}{3\pi}$	$\overline{I_x} =$	$I_x = \pi r_2 r_1^3$
$\bar{y} = \frac{4r_1}{3\pi}$	$\overline{I_y} =$	$I_y = \pi r_2^3 r_1$
	$\overline{I_{xy}} =$	$I_{xy} = \frac{r_2^2 r_1^2}{8}$

Parabola:

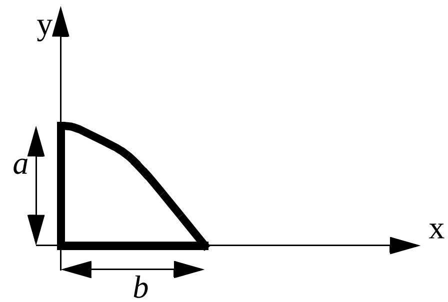


$$A = \frac{2}{3}ab$$

$$P = \frac{\sqrt{b^2 + 16a^2}}{2} + \frac{b^2}{8a}\ln\left(\frac{4a + \sqrt{b^2 + 16a^2}}{b}\right)$$

Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = \frac{b}{2}$	$\overline{I_x} =$	$I_x =$
$\bar{y} = \frac{2a}{5}$	$\overline{I_y} =$	$I_y =$
	$\overline{I_{xy}} =$	$I_{xy} =$

Half Parabola:



$$A = \frac{ab}{3}$$

$$P = \frac{\sqrt{b^2 + 16a^2}}{4} + \frac{b^2}{16a}\ln\left(\frac{4a + \sqrt{b^2 + 16a^2}}{b}\right)$$

Centroid:

$$\bar{x} = \frac{3b}{8}$$

$$\bar{y} = \frac{2a}{5}$$

Moment of Inertia (about centroid axes):

$$\bar{I}_x = \frac{8ba^3}{175}$$

$$\bar{I}_y = \frac{19b^3a}{480}$$

$$\overline{I_{xy}} = \frac{b^2a^2}{60}$$

Moment of Inertia (about origin axes):

$$I_x = \frac{2ba^3}{7}$$

$$I_y = \frac{2b^3a}{15}$$

$$I_{xy} = \frac{b^2a^2}{6}$$

• A general class of geometries are conics. This for is shown below, and can be used to represent many of the simple shapes represented by a polynomial.

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

Conditions		
	$A = B = C = 0$	straight line
	$B = 0, A = C$	circle
	$B^2 - AC < 0$	ellipse
	$B^2 - AC = 0$	parabola
	$B^2 - AC > 0$	hyperbola

VOLUME PROPERTIES:

$$I_x = \int_V r_x^2 dV = \text{ the moment of inertia about the x-axis}$$

$$I_y = \int_V r_y^2 dV = \text{ the moment of inertia about the y-axis}$$

$$I_z = \int_V r_z^2 dV = \text{ the moment of inertia about the z-axis}$$

$$\bar{x} = \frac{\int_V x dV}{\int_V dV} = \text{ centroid location along the x-axis}$$

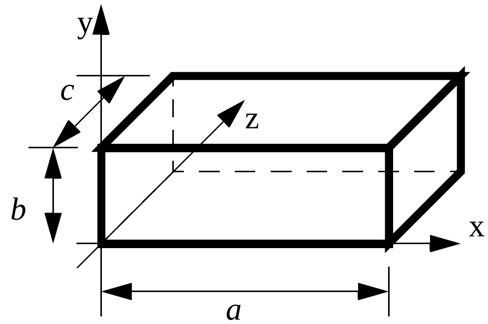
$$\bar{y} = \frac{\int_V y dV}{\int_V dV} = \text{ centroid location along the y-axis}$$

$$\bar{z} = \frac{\int_V z dV}{\int_V dV} = \text{ centroid location along the z-axis}$$

Parallelepiped (box):

$$V = abc$$

$$S = 2(ab + ac + bc)$$

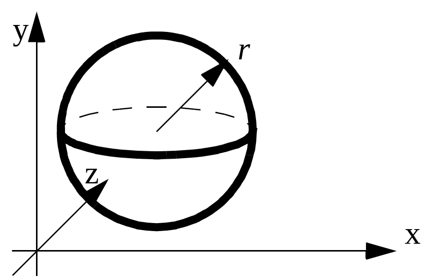


Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):	Mass Moment of Inertia (about centroid):
$\bar{x} = \frac{a}{2}$	$\bar{I}_x = \frac{M(a^2 + b^2)}{12}$	$I_x =$	$J_x =$
$\bar{y} = \frac{b}{2}$	$\bar{I}_y = \frac{M(a^2 + c^2)}{12}$	$I_y =$	$J_y =$
$\bar{z} = \frac{c}{2}$	$\bar{I}_z = \frac{M(b^2 + a^2)}{12}$	$I_z =$	$J_z =$

Sphere:

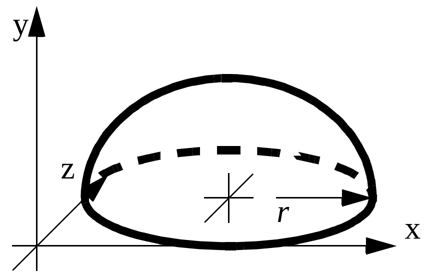
$$V = \frac{4}{3}\pi r^3$$

$$S = 4\pi r^2$$



Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):	Mass Moment of Inertia (about centroid):
$\bar{x} = r$	$\bar{I}_x = \frac{2Mr^2}{5}$	$I_x =$	$J_x = \frac{2Mr^2}{5}$
$\bar{y} = r$	$\bar{I}_y = \frac{2Mr^2}{5}$	$I_y =$	$J_y = \frac{2Mr^2}{5}$
$\bar{z} = r$	$\bar{I}_z = \frac{2Mr^2}{5}$	$I_z =$	$J_z = \frac{2Mr^2}{5}$

Hemisphere:

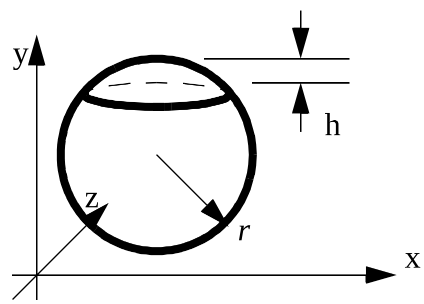


$$V = \frac{2}{3}\pi r^3$$

$$S =$$

Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r$	$\bar{I}_x = \frac{83}{320}Mr^2$	$I_x =$
$\bar{y} = \frac{3r}{8}$	$\bar{I}_y = \frac{2Mr^2}{5}$	$I_y =$
$\bar{z} = r$	$\bar{I}_z = \frac{83}{320}Mr^2$	$I_z =$

Cap of a Sphere:



$$V = \frac{1}{3}\pi h^2(3r - h)$$

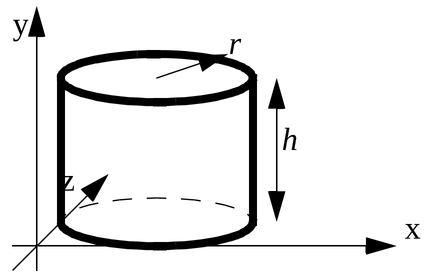
$$S = 2\pi r h$$

Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r$	$\bar{I}_x =$	$I_x =$
$\bar{y} =$	$\bar{I}_y =$	$I_y =$
$\bar{z} = r$	$\bar{I}_z =$	$I_z =$

Cylinder:

$$V = h\pi r^2$$

$$S = 2\pi rh + 2\pi r^2$$

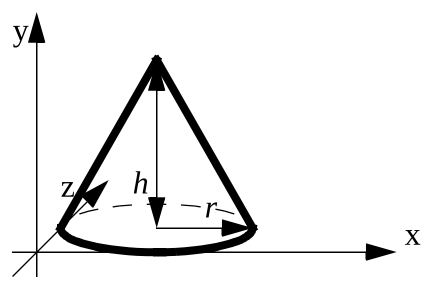


Centroid:	Moment of Inertia (about centroid axis):	Moment of Inertia (about origin axis):	Mass Moment of Inertia (about centroid):
$\bar{x} = r$	$\bar{I}_x = M\left(\frac{h^2}{12} + \frac{r^2}{4}\right)$	$I_x = M\left(\frac{h^2}{3} + \frac{r^2}{4}\right)$	$J_x = \frac{M(3r^2 + h^2)}{12}$
$\bar{y} = \frac{h}{2}$	$\bar{I}_y = \frac{Mr^2}{2}$	$I_y =$	$J_y = \frac{Mr^2}{2}$
$\bar{z} = r$	$\bar{I}_z = M\left(\frac{h^2}{12} + \frac{r^2}{4}\right)$	$I_z = M\left(\frac{h^2}{3} + \frac{r^2}{4}\right)$	$J_z = \frac{M(3r^2 + h^2)}{12}$

Cone:

$$V = \frac{1}{3}\pi r^2 h$$

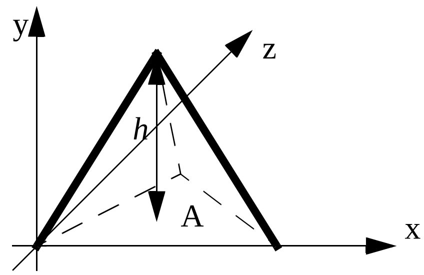
$$S = \pi r\sqrt{r^2 + h^2}$$



Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r$	$\bar{I}_x = M\left(\frac{3h^3}{80} + \frac{3r^2}{20}\right)$	$I_x =$
$\bar{y} = \frac{h}{4}$	$\bar{I}_y = \frac{3Mr^2}{10}$	$I_y =$
$\bar{z} = r$	$\bar{I}_z = M\left(\frac{3h^3}{80} + \frac{3r^2}{20}\right)$	$I_z =$

Tetrahedron:

$$V = \frac{1}{3}Ah$$

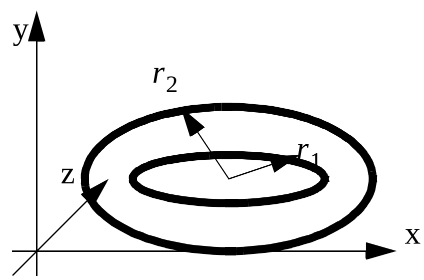


Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} =$	$\bar{I}_x =$	$I_x =$
$\bar{y} = \frac{h}{4}$	$\bar{I}_y =$	$I_y =$
$\bar{z} =$	$\bar{I}_z =$	$I_z =$

Torus:

$$V = \frac{1}{4}\pi^2(r_1 + r_2)(r_2 - r_1)^2$$

$$S = \pi^2(r_2^2 - r_1^2)$$

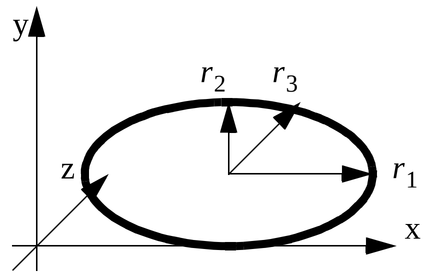


Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r_2$	$\bar{I}_x =$	$I_x =$
$\bar{y} = \left(\frac{r_2 - r_1}{2}\right)$	$\bar{I}_y =$	$I_y =$
$\bar{z} = r_2$	$\bar{I}_z =$	$I_z =$

Ellipsoid:

$$V = \frac{4}{3}\pi r_1 r_2 r_3$$

$$S =$$



Centroid:

$\bar{x} = r_1$

$\bar{y} = r_2$

$\bar{z} = r_3$

Moment of Inertia (about centroid axes):

$\bar{I}_x =$

$\bar{I}_y =$

$\bar{I}_z =$

Moment of Inertia (about origin axes):

$I_x =$

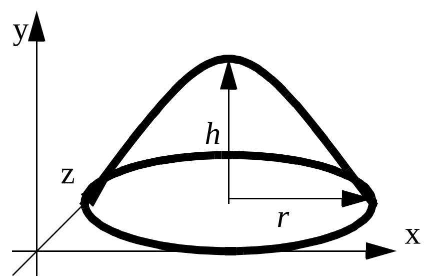
$I_y =$

$I_z =$

Paraboloid:

$$V = \frac{1}{2}\pi r^2 h$$

$$S =$$



Centroid:

$\bar{x} = r$

$\bar{y} =$

$\bar{z} = r$

Moment of Inertia (about centroid axes):

$\bar{I}_x =$

$\bar{I}_y =$

$\bar{I}_z =$

Moment of Inertia (about origin axes):

$I_x =$

$I_y =$

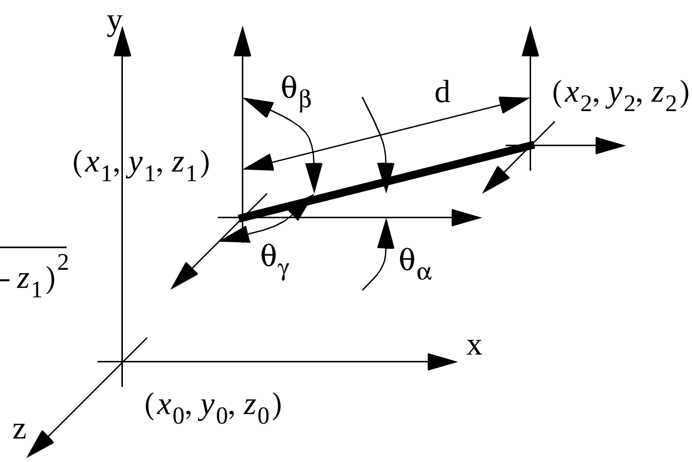
$I_z =$

24.3.4 Planes, Lines, etc.

• The most fundamental mathematical geometry is a line. The basic relationships are given below,

$y = mx + b$	defined with a slope and intercept
$m_{\text{perpendicular}} = \frac{1}{m}$	a slope perpendicular to a line
$m = \frac{y_2 - y_1}{x_2 - x_1}$	the slope using two points
$\frac{x}{a} + \frac{y}{b} = 1$	as defined by two intercepts

• If we assume a line is between two points in space, and that at one end we have a local reference frame, there are some basic relationships that can be derived.



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The direction cosines of the angles are,

$$\theta_\alpha = \operatorname{acos}\left(\frac{x_2 - x_1}{d}\right) \qquad \theta_\beta = \operatorname{acos}\left(\frac{y_2 - y_1}{d}\right) \qquad \theta_\gamma = \operatorname{acos}\left(\frac{z_2 - z_1}{d}\right)$$

$$(\cos\theta_\alpha)^2 + (\cos\theta_\beta)^2 + (\cos\theta_\gamma)^2 = 1$$

The equation of the line is,

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$ Explicit

$$(x, y, z) = (x_1, y_1, z_1) + t((x_2, y_2, z_2) - (x_1, y_1, z_1))$$ Parametric t=[0,1]

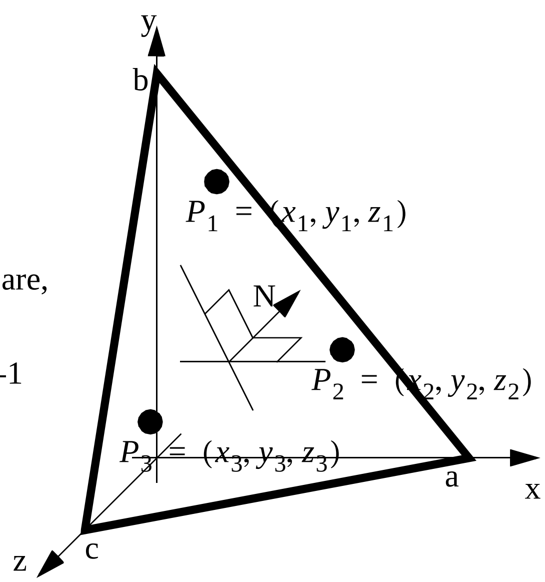
• The relationships for a plane are,

The explicit equation for a plane is,

$$Ax + By + Cz + D = 0$$

where the coefficients defined by the intercepts are,

$$A = \frac{1}{a} \qquad B = \frac{1}{b} \qquad C = \frac{1}{c} \qquad D = -1$$



The determinant can also be used,

$$det\begin{bmatrix} x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \\ x-x_3 & y-y_3 & z-z_3 \end{bmatrix} = 0$$

$$\therefore det\begin{bmatrix} y_2-y_1 & z_2-z_1 \\ y_3-y_1 & z_3-z_1 \end{bmatrix}(x-x_1) + det\begin{bmatrix} z_2-z_1 & x_2-x_1 \\ z_3-z_1 & x_3-x_1 \end{bmatrix}(y-y_1)$$

$$+ det\begin{bmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{bmatrix}(z-z_1) = 0$$

The normal to the plane (through the origin) is,

$$(x, y, z) = t(A, B, C)$$

24.3.5 Practice Problems

1. What is the circumferenece of a circle? What is the area? What is the ratio of the area to the circumference?

2. What is the equation of a line that passes through the points below?

a) (0, 0) to (2, 2)
b) (1, 0) to (0, 1)
c) (3, 4) to (2. 9)

3. Find a line that is perpendicular to the line through the points (2, 1) and (1, 2). The perpendicular line passes through (3, 5).

4. Manipulate the following equations to solve for ‘x’.

a) $x^2 + 3x = -2$
b) $\sin x = \cos x$

(ans.

a) $x^2 + 3x = -2$

$$x^2 + 3x + 2 = 0$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2(1)} = \frac{-3 \pm \sqrt{9-8}}{2} = \frac{-3 \pm 1}{2} = -1, -2$$

b) $\sin x = \cos x$

$$\frac{\sin x}{\cos x} = 1$$

$$\tan x = 1$$

$$x = \operatorname{atan} 1$$

$$x = \ldots, -135°, 45°, 225°, \ldots$$

5. Simplify the following expressions.

a) $$\sin 2\theta\left(\frac{(\cos 2\theta)^2}{\sin 2\theta} + \sin 2\theta\right)$$

(ans.

a) $$\sin 2\theta\left(\frac{(\cos 2\theta)^2}{\sin 2\theta} + \sin 2\theta\right) = (\cos 2\theta)^2 + (\sin 2\theta)^2 = 1$$

6. A line that passes through the point (1, 2) and has a slope of 2. Find the equation for the line, and for a line perpendicular to it.

(ans. given,

$$y = mx + b$$
$$m = 2$$
$$x = 1$$
$$y = 2$$

substituting

$$2 = 2(1) + b \qquad b = 0$$
$$y = 2x$$

perpendicular

$$m_{perp} = -\frac{1}{m} = -0.5$$
$$y = -0.5x$$

24.4 COORDINATE SYSTEMS

24.4.1 Complex Numbers

• In this section, as in all others, 'j' will be the preferred notation for the complex number, this is to help minimize confusion with the 'i' used for current in electrical engineering.

• The basic algebraic properties of these numbers are, The Complex Number:

$$j = \sqrt{-1} \qquad\qquad j^2 = -1$$

Complex Numbers:

$a + bj$ where, a and b are both real numbers

Complex Conjugates (denoted by adding an asterisk '*' the variable):

$$N = a + bj \qquad\qquad N^* = a - bj$$

Basic Properties:

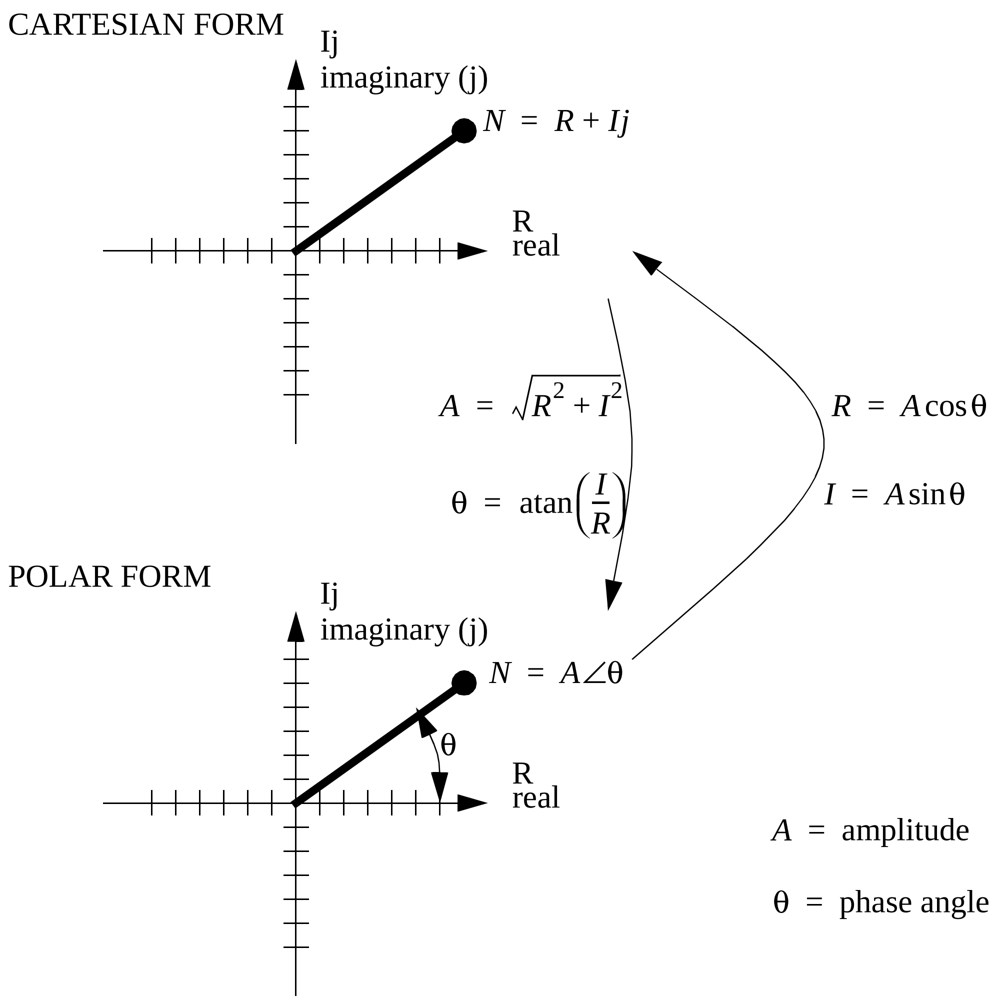
$$(a + bj) + (c + dj) = (a + c) + (b + d)j$$

$$(a + bj) - (c + dj) = (a - c) + (b - d)j$$

$$(a + bj) \cdot (c + dj) = (ac - bd) + (ad + bc)j$$

$$\frac{N}{M} = \frac{a + bj}{c + dj} = \frac{N}{M}\left(\frac{N^*}{N^*}\right) = \left(\frac{a + bj}{c + dj}\right)\left(\frac{c - dj}{c - dj}\right) = \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2}\right)j$$

• We can also show complex numbers graphically. These representations lead to alternative representations. If it in not obvious above, please consider the notation above uses a cartesian notation, but a polar notation can also be very useful when doing large calculations.



• We can also do calculations using polar notation (this is well suited to multiplication and division, whereas cartesian notation is easier for addition and subtraction),

$$A\angle\theta = A(\cos\theta + j\sin\theta) = Ae^{j\theta}$$

$$(A_1\angle\theta_1)(A_2\angle\theta_2) = (A_1A_2)\angle(\theta_1 + \theta_2)$$

$$\frac{(A_1\angle\theta_1)}{(A_2\angle\theta_2)} = \left(\frac{A_1}{A_2}\right)\angle(\theta_1 - \theta_2)$$

$$(A\angle\theta)^n = (A^n)\angle(n\theta) \qquad \text{(DeMoivre's theorem)}$$

• Note that DeMoivre's theorem can be used to find exponents (including roots) of complex numbers

• Euler's formula: $e^{j\theta} = \cos\theta + j\sin\theta$

• From the above, the following useful identities arise:

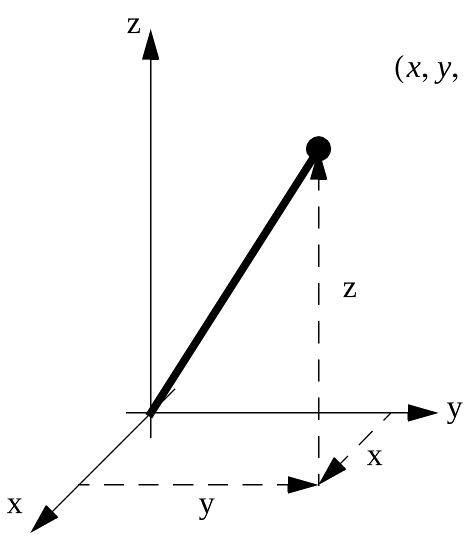
$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

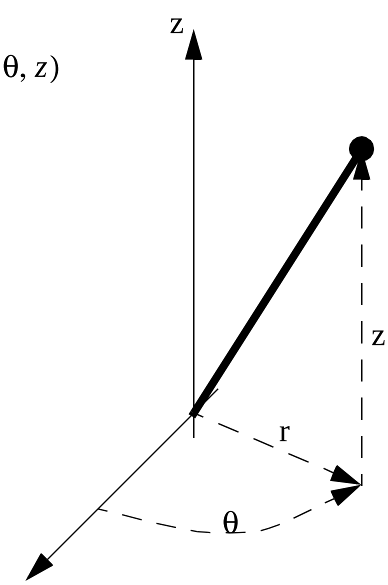
$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

24.4.2 Cylindrical Coordinates

• Basically, these coordinates appear as if the cartesian box has been replaced with a cylinder,

$$(x, y, z) \leftrightarrow (r, \theta, z)$$





$$x = r\cos\theta$$

$$y = r\sin\theta$$

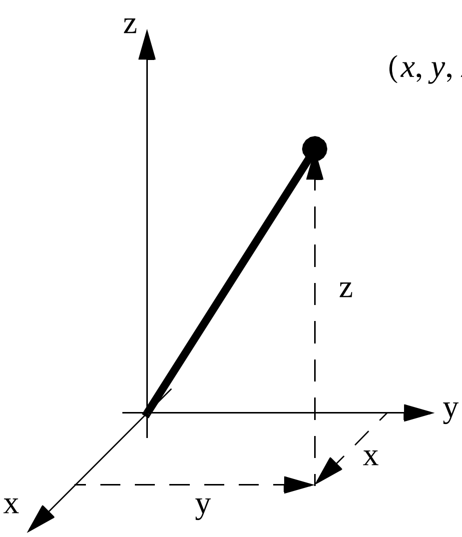
$$r = \sqrt{x^2 + y^2}$$

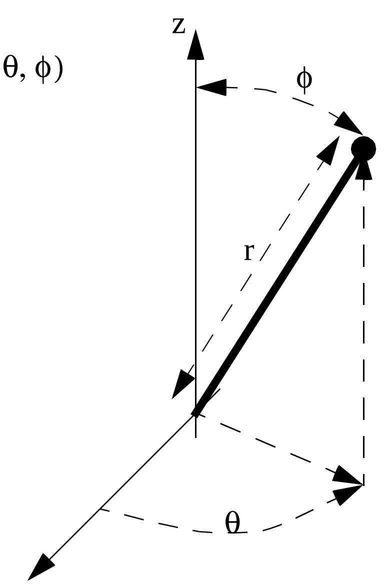
$$\theta = \operatorname{atan}\left(\frac{y}{x}\right)$$

24.4.3 Spherical Coordinates

• This system replaces the cartesian box with a sphere,

$$(x, y, z) \leftrightarrow (r, \theta, \phi)$$





$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \operatorname{atan}\left(\frac{y}{x}\right)$$

$$\phi = \operatorname{acos}\left(\frac{z}{r}\right)$$

24.4.4 Practice Problems

1. Simplify the following expressions.

a) $\dfrac{16}{(4j + 4)^2}$

b) $\dfrac{3j + 5}{(4j + 3)^2}$

c) $(3 + 5j)4j$ where, $j = \sqrt{-1}$

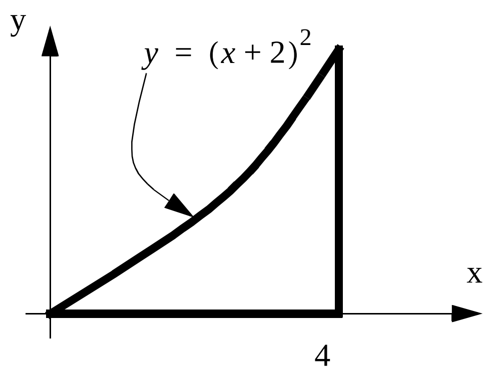
(ans. a) $$\frac{16}{(4j+4)^2} = \frac{16}{-16+32j+16} = \frac{16}{32j} = -0.5j$$

b) $$\frac{3j+5}{(4j+3)^2} = \frac{3j+5}{-16+24j+9} = \frac{3j+5}{-7+24j}\left(\frac{-7-24j}{-7-24j}\right) = \frac{-35-141j+72}{49+576} = \frac{37-141j}{625}$$

c) $$(3+5j)4j = 12j+20j^2 = 12j-20$$

2. For the shape defined below,

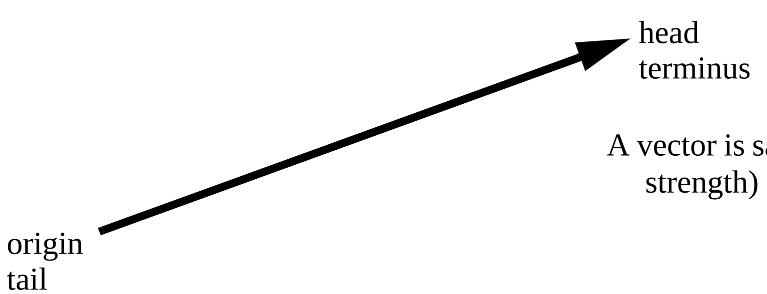
a) find the area of the shape.
b) find the centroid of the shape.
c) find the moment of inertia of the shape about the centroid.



24.5 MATRICES AND VECTORS

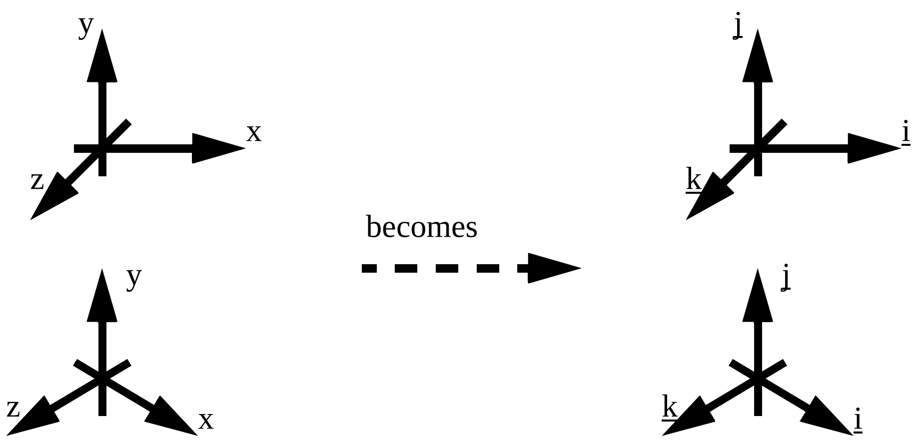
24.5.1 Vectors

• Vectors are often drawn with arrows, as shown below,



A vector is said to have magnitude (length or strength) and direction.

• Cartesian notation is also a common form of usage.

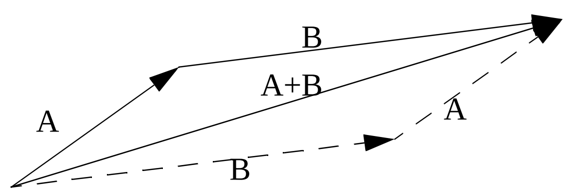


• Vectors can be added and subtracted, numerically and graphically,

$$A = (2, 3, 4) \qquad A + B = (2 + 7, 3 + 8, 4 + 9)$$

$$B = (7, 8, 9) \qquad A - B = (2 - 7, 3 - 8, 4 - 9)$$

Parallelogram Law



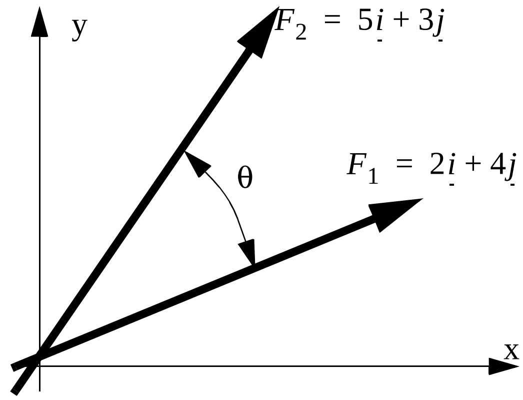
24.5.2 Dot (Scalar) Product

• We can use a dot product to find the angle between two vectors

$$\cos\theta = \frac{F_1 \bullet F_2}{|F_1||F_2|}$$

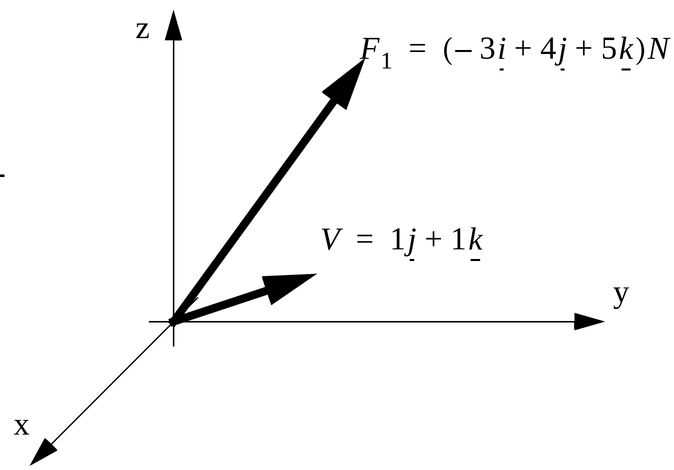
$$\therefore\theta = \operatorname{acos}\left(\frac{(2)(5) + (4)(3)}{\sqrt{2^2 + 4^2}\sqrt{5^2 + 3^2}}\right)$$

$$\therefore\theta = \operatorname{acos}\left(\frac{22}{(4.47)(6)}\right) = 32.5°$$



• We can use a dot product to project one vector onto another vector.

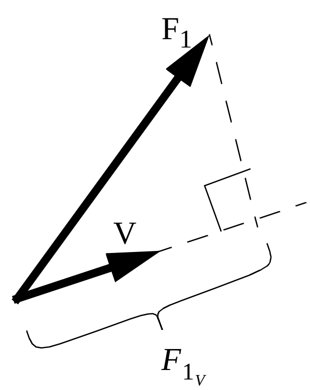
We want to find the component of force F_1 that projects onto the vector V. To do this we first convert V to a unit vector, if we do not, the component we find will be multiplied by the magnitude of V.



$$\lambda_V = \frac{V}{|V|} = \frac{1\underline{j} + 1\underline{k}}{\sqrt{1^2 + 1^2}} = 0.707\underline{j} + 0.707\underline{k}$$

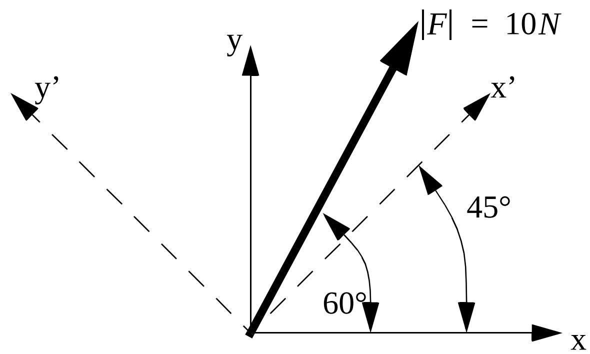
$$F_{1_V} = \lambda_V \bullet F_1 = (0.707\underline{j} + 0.707\underline{k}) \bullet (-3\underline{i} + 4\underline{j} + 5\underline{k})N$$

$$\therefore F_{1_V} = (0)(-3) + (0.707)(4) + (0.707)(5) = 6N$$



• We can consider the basic properties of the dot product and units vectors.

Unit vectors are useful when breaking up vector magnitudes and direction. As an example consider the vector, and the displaced x-y axes shown below as x'-y'.



We could write out 5 vectors here, relative to the x-y axis,

$$\text{x axis} = 2\underline{i}$$
$$\text{y axis} = 3\underline{j}$$
$$\text{x' axis} = 1\underline{i} + 1\underline{j}$$
$$\text{y' axis} = -1\underline{i} + 1\underline{j}$$
$$F = 10N\angle 60° = (10\cos 60°)\underline{i} + (10\sin 60°)\underline{j}$$

None of these vectors has a magnitude of 1, and hence they are not unit vectors. But, if we find the equivalent vectors with a magnitude of one we can simplify many tasks. In particular if we want to find the x and y components of F relative to the x-y axis we can use the dot product.

$$\lambda_x = 1\underline{i} + 0\underline{j} \quad \text{(unit vector for the x-axis)}$$

$$F_x = \lambda_x \bullet F = (1\underline{i} + 0\underline{j}) \bullet [(10\cos 60°)\underline{i} + (10\sin 60°)\underline{j}]$$
$$\therefore \quad = (1)(10\cos 60°) + (0)(10\sin 60°) = 10N\cos 60°$$

This result is obvious, but consider the other obvious case where we want to project a vector onto itself,

$$\lambda_F = \frac{F}{|F|} = \frac{10\cos 60° \underset{-}{i} + 10\sin 60° \underset{-}{j}}{10} = \cos 60° \underset{-}{i} + \sin 60° \underset{-}{j}$$

Incorrect - Not using a unit vector

$$F_F = F \bullet F$$
$$= ((10\cos 60°)\underset{-}{i} + (10\sin 60°)\underset{-}{j}) \bullet ((10\cos 60°)\underset{-}{i} + (10\sin 60°)\underset{-}{j})$$
$$= (10\cos 60°)(10\cos 60°) + (10\sin 60°)(10\sin 60°)$$
$$= 100((\cos 60°)^2 + (\sin 60°)^2) = \cancel{100}$$

Using a unit vector

$$F_F = F \bullet \lambda_F$$
$$= ((10\cos 60°)\underset{-}{i} + (10\sin 60°)\underset{-}{j}) \bullet ((\cos 60°)\underset{-}{i} + (\sin 60°)\underset{-}{j})$$
$$= (10\cos 60°)(\cos 60°) + (10\sin 60°)(\sin 60°)$$
$$= 10((\cos 60°)^2 + (\sin 60°)^2) = 10$$

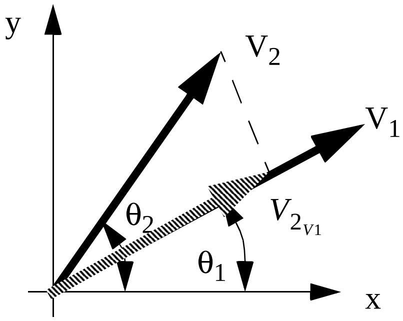
Correct

Now consider the case where we find the component of F in the x' direction. Again, this can be done using the dot product to project F onto a unit vector.

$$u_{x'} = \cos 45° \underset{-}{i} + \sin 45° \underset{-}{j}$$

$$F_{x'} = F \bullet \lambda_{x'} = ((10\cos 60°)\underset{-}{i} + (10\sin 60°)\underset{-}{j}) \bullet ((\cos 45°)\underset{-}{i} + (\sin 45°)\underset{-}{j})$$
$$= (10\cos 60°)(\cos 45°) + (10\sin 60°)(\sin 45°)$$
$$= 10(\cos 60° \cos 45° + \sin 60° \sin 45°) = 10(\cos(60° - 45°))$$

Here we see a few cases where the dot product has been applied to find the vector projected onto a unit vector. Now finally consider the more general case,



First, by inspection, we can see that the component of V_2 (projected) in the direction of V_1 will be,

$$\left|V_{2_{V1}}\right| = \left|V_2\right| \cos(\theta_2 - \theta_1)$$

Next, we can manipulate this expression into the dot product form,

$$= \left|V_2\right|(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)$$

$$= \left|V_2\right|[(\cos\theta_1 \underset{\sim}{i} + \sin\theta_1 \underset{\sim}{j}) \bullet (\cos\theta_2 \underset{\sim}{i} + \sin\theta_2 \underset{\sim}{j})]$$

$$= \left|V_2\right|\left[\frac{V_1}{\left|V_1\right|} \bullet \frac{V_2}{\left|V_2\right|}\right] = \left|V_2\right|\left[\frac{V_1 \bullet V_2}{\left|V_1\right|\left|V_2\right|}\right] = \frac{V_1 \bullet V_2}{\left|V_1\right|} = V_2 \bullet \lambda_{V_1}$$

Or more generally,

$$\left|V_{2_{V1}}\right| = \left|V_2\right| \cos(\theta_2 - \theta_1) = \left|V_2\right|\left[\frac{V_1 \bullet V_2}{\left|V_1\right|\left|V_2\right|}\right]$$

$$\therefore \left|V_2\right| \cos(\theta_2 - \theta_1) = \left|V_2\right|\left[\frac{V_1 \bullet V_2}{\left|V_1\right|\left|V_2\right|}\right]$$

$$\therefore \cos(\theta_2 - \theta_1) = \left[\frac{V_1 \bullet V_2}{\left|V_1\right|\left|V_2\right|}\right]$$

*Note that the dot product also works in 3D, and similar proofs are used.

24.5.3 Cross Product

- First, consider an example,

$$F = (-6.43\underline{i} + 7.66\underline{j} + 0\underline{k})N$$

$$d = (2\underline{i} + 0\underline{j} + 0\underline{k})m$$

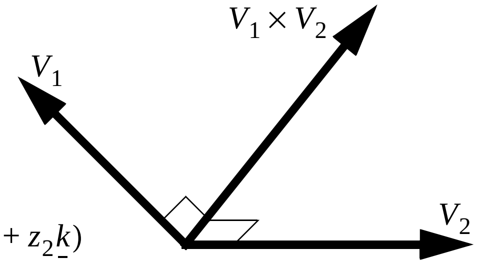
$$M = d \times F = \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2m & 0m & 0m \\ -6.43N & 7.66N & 0N \end{bmatrix}$$

NOTE: note that the cross product here is for the right hand rule coordinates. If the left handed coordinate system is used F and d should be reversed.

$$\therefore M = (0m0N - 0m(7.66N))\underline{i} - (2m0N - 0m(-6.43N))\underline{j} + (2m(7.66N) - 0m(-6.43N))\underline{k} = 15.3\underline{k}(\text{mN})$$

NOTE: there are two things to note about the solution. First, it is a vector. Here there is only a z component because this vector points out of the page, and a rotation about this vector would rotate on the plane of the page. Second, this result is positive, because the positive sense is defined by the vector system. In this right handed system find the positive rotation by pointing your right hand thumb towards the positive axis (the ‘k’ means that the vector is about the z-axis here), and curl your fingers, that is the positive direction.

- The basic properties of the cross product are, The cross (or vector) product of two vectors will yield a new vector perpendicular to both vectors, with a magnitude that is a product of the two magnitudes.



$$V_1 \times V_2 = (x_1\underline{i} + y_1\underline{j} + z_1\underline{k}) \times (x_2\underline{i} + y_2\underline{j} + z_2\underline{k})$$

$$V_1 \times V_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

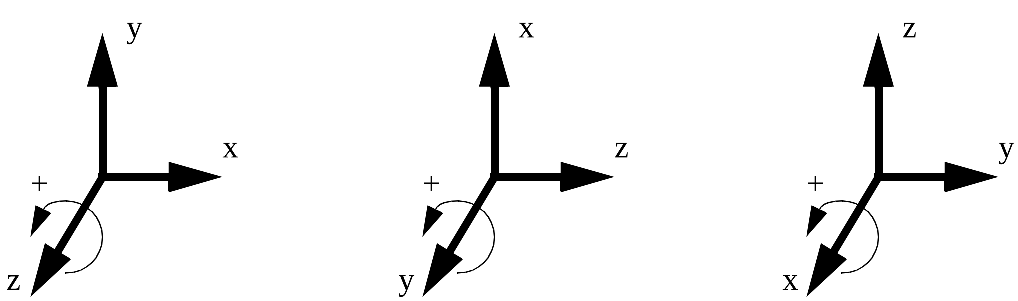
$$V_1 \times V_2 = (y_1z_2 - z_1y_2)\underline{i} + (z_1x_2 - x_1z_2)\underline{j} + (x_1y_2 - y_1x_2)\underline{k}$$

We can also find a unit vector normal 'n' to the vectors 'V1' and 'V2' using a cross product, divided by the magnitude.

$$\lambda_n = \frac{V_1 \times V_2}{|V_1 \times V_2|}$$

- When using a left/right handed coordinate system,

The positive orientation of angles and moments about an axis can be determined by pointing the thumb of the right hand along the axis of rotation. The fingers curl in the positive direction.



- The properties of the cross products are, The cross product is distributive, but not associative. This allows us to collect terms in a cross product operation, but we cannot change the order of the cross product.

$r_1 \times F + r_2 \times F = (r_1 + r_2) \times F$	DISTRIBUTIVE
$r \times F \neq F \times r$ but $r \times F = -(F \times r)$	NOT ASSOCIATIVE

24.5.4 Triple Product

When we want to do a cross product, followed by a dot product (called the mixed triple product), we can do both steps in one operation by finding the determinant of the following. An example of a problem that would use this shortcut is when a moment is found about one point on a pipe, and then the moment component twisting the pipe is found using the dot product.

$$(d \times F) \bullet u = \begin{vmatrix} u_x & u_y & u_z \\ d_x & d_y & d_z \\ F_x & F_y & F_z \end{vmatrix}$$

24.5.5 Matrices

• Matrices allow simple equations that drive a large number of repetitive calculations - as a result they are found in many computer applications.

• A matrix has the form seen below,

n columns

$$\begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{bmatrix}$$

m rows

If n=m then the matrix is said to be square.
Many applications require square matrices.
We may also represent a matrix as a 1-by-3 for a vector.

• Matrix operations are available for many of the basic algebraic expressions, examples are given below. There are also many restrictions - many of these are indicated.

$$A = 2 \qquad B = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 10 & 11 \end{bmatrix} \qquad C = \begin{bmatrix} 12 & 13 & 14 \\ 15 & 16 & 17 \\ 18 & 19 & 20 \end{bmatrix} \qquad D = \begin{bmatrix} 21 \\ 22 \\ 23 \end{bmatrix} \qquad E = \begin{bmatrix} 24 & 25 & 26 \end{bmatrix}$$

Addition/Subtraction

$$A + B = \begin{bmatrix} 3+2 & 4+2 & 5+2 \\ 6+2 & 7+2 & 8+2 \\ 9+2 & 10+2 & 11+2 \end{bmatrix} \qquad B + C = \begin{bmatrix} 3+12 & 4+13 & 5+14 \\ 6+15 & 7+16 & 8+17 \\ 9+18 & 10+19 & 11+20 \end{bmatrix}$$

$$B + D = \text{not valid}$$

$$B - A = \begin{bmatrix} 3-2 & 4-2 & 5-2 \\ 6-2 & 7-2 & 8-2 \\ 9-2 & 10-2 & 11-2 \end{bmatrix} \qquad B + C = \begin{bmatrix} 3-12 & 4-13 & 5-14 \\ 6-15 & 7-16 & 8-17 \\ 9-18 & 10-19 & 11-20 \end{bmatrix}$$

$$B - D = \text{not valid}$$

Multiplication/Division

$$A \cdot B = \begin{bmatrix} 3(2) & 4(2) & 5(2) \\ 6(2) & 7(2) & 8(2) \\ 9(2) & 10(2) & 11(2) \end{bmatrix} \qquad \frac{B}{A} = \begin{bmatrix} \frac{3}{2} & \frac{4}{2} & \frac{5}{2} \\ \frac{6}{2} & \frac{7}{2} & \frac{8}{2} \\ \frac{9}{2} & \frac{10}{2} & \frac{11}{2} \end{bmatrix}$$

$$B \cdot D = \begin{bmatrix} (3 \cdot 21 + 4 \cdot 22 + 5 \cdot 23) \\ (6 \cdot 21 + 7 \cdot 22 + 8 \cdot 23) \\ (9 \cdot 21 + 10 \cdot 22 + 11 \cdot 23) \end{bmatrix} \qquad D \cdot E = 21 \cdot 24 + 22 \cdot 25 + 23 \cdot 26$$

$$B \cdot C = \begin{bmatrix} (3 \cdot 12 + 4 \cdot 15 + 5 \cdot 18) & (3 \cdot 13 + 4 \cdot 16 + 5 \cdot 19) & (3 \cdot 14 + 4 \cdot 17 + 5 \cdot 20) \\ (6 \cdot 12 + 7 \cdot 15 + 8 \cdot 18) & (6 \cdot 13 + 7 \cdot 16 + 8 \cdot 19) & (6 \cdot 14 + 7 \cdot 17 + 8 \cdot 20) \\ (9 \cdot 12 + 10 \cdot 15 + 11 \cdot 18) & (9 \cdot 13 + 10 \cdot 16 + 11 \cdot 19) & (9 \cdot 14 + 10 \cdot 17 + 11 \cdot 20) \end{bmatrix}$$

$$\frac{B}{C}, \frac{B}{D}, \frac{D}{B}, \text{etc} = \text{not allowed (see inverse)}$$

Note: To multiply matrices, the first matrix must have the same number of columns as the second matrix has rows.

Determinant

$$|B| = 3 \cdot \begin{vmatrix} 7 & 8 \\ 10 & 11 \end{vmatrix} - 4 \cdot \begin{vmatrix} 6 & 8 \\ 9 & 11 \end{vmatrix} + 5 \cdot \begin{vmatrix} 6 & 7 \\ 9 & 10 \end{vmatrix} = 3 \cdot (-3) - 4 \cdot (-6) + 5 \cdot (-3) = 0$$

$$\begin{vmatrix} 7 & 8 \\ 10 & 11 \end{vmatrix} = (7 \cdot 11) - (8 \cdot 10) = -3$$

$$\begin{vmatrix} 6 & 8 \\ 9 & 11 \end{vmatrix} = (6 \cdot 11) - (8 \cdot 9) = -6$$

$$\begin{vmatrix} 6 & 7 \\ 9 & 10 \end{vmatrix} = (6 \cdot 10) - (7 \cdot 9) = -3$$

$|D|, |E|$ = not valid (matrices not square)

Transpose

$$B^T = \begin{bmatrix} 3 & 6 & 9 \\ 4 & 7 & 10 \\ 5 & 8 & 11 \end{bmatrix} \qquad D^T = \begin{bmatrix} 21 & 22 & 23 \end{bmatrix} \qquad E^T = \begin{bmatrix} 24 \\ 25 \\ 26 \end{bmatrix}$$

Adjoint

$$\|B\| = \begin{bmatrix} \begin{vmatrix} 7 & 8 \\ 10 & 11 \end{vmatrix} & -\begin{vmatrix} 6 & 8 \\ 10 & 11 \end{vmatrix} & \begin{vmatrix} 6 & 7 \\ 9 & 10 \end{vmatrix} \\ -\begin{vmatrix} 4 & 5 \\ 10 & 11 \end{vmatrix} & \begin{vmatrix} 3 & 5 \\ 9 & 11 \end{vmatrix} & -\begin{vmatrix} 3 & 4 \\ 9 & 10 \end{vmatrix} \\ \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} & -\begin{vmatrix} 3 & 5 \\ 6 & 8 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 6 & 7 \end{vmatrix} \end{bmatrix}^T$$

The matrix of determinant to the left is made up by getting rid of the row and column of the element, and then finding the determinant of what is left. Note the sign changes on alternating elements.

$$\|D\| = \text{invalid (must be square)}$$

Inverse

$$D = B \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

To solve this equation for x,y,z we need to move B to the left hand side. To do this we use the inverse.

$$B^{-1}D = B^{-1} \cdot B \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B^{-1}D = I \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B^{-1} = \frac{\|B\|}{|B|}$$

In this case B is singular, so the inverse is undetermined, and the matrix is indeterminate.

$$D^{-1} = \text{invalid (must be square)}$$

Identity Matrix

This is a square matrix that is the matrix equivalent to ‘1’.

$$B \cdot I = I \cdot B = B$$

$$D \cdot I = I \cdot D = D$$

$$B^{-1} \cdot B = I$$

$$\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{etc=I}$$

• The eigenvalue of a matrix is found using,

$$|A - \lambda I| = 0$$

24.5.6 Solving Linear Equations with Matrices

• We can solve systems of equations using the inverse matrix,

Given,

$$2 \cdot x + 4 \cdot y + 3 \cdot z = 5$$

$$9 \cdot x + 6 \cdot y + 8 \cdot z = 7$$

$$11 \cdot x + 13 \cdot y + 10 \cdot z = 12$$

Write down the matrix, then rearrange, and solve.

$$\begin{bmatrix} 2 & 4 & 3 \\ 9 & 6 & 8 \\ 11 & 13 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix} \qquad \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 4 & 3 \\ 9 & 6 & 8 \\ 11 & 13 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

• We can solve systems of equations using Cramer's rule (with determinants),

Given,

$$2 \cdot x + 4 \cdot y + 3 \cdot z = 5$$

$$9 \cdot x + 6 \cdot y + 8 \cdot z = 7$$

$$11 \cdot x + 13 \cdot y + 10 \cdot z = 12$$

Write down the coefficient and parameter matrices,

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 9 & 6 & 8 \\ 11 & 13 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix}$$

Calculate the determinant for A (this will be reused), and calculate the determinants for matrices below. Note: when trying to find the first parameter 'x' we replace the first column in A with B.

$$x = \frac{\begin{vmatrix} 5 & 4 & 3 \\ 7 & 6 & 8 \\ 12 & 13 & 10 \end{vmatrix}}{|A|} =$$

$$y = \frac{\begin{vmatrix} 2 & 5 & 3 \\ 9 & 7 & 8 \\ 11 & 12 & 10 \end{vmatrix}}{|A|} =$$

$$z = \frac{\begin{vmatrix} 2 & 4 & 5 \\ 9 & 6 & 7 \\ 11 & 13 & 12 \end{vmatrix}}{|A|} =$$

24.5.7 Practice Problems

1. Perform the matrix operations below.

Multiply

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} =$$

Determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 9 \end{vmatrix} =$$

Inverse

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 9 \end{bmatrix}^{-1} =$$

ANS.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} = \begin{bmatrix} 68 \\ 167 \\ 266 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 36$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 9 \end{bmatrix}^{-1} = \begin{bmatrix} -0.833 & 0.167 & 0.167 \\ 0.167 & -0.333 & 0.167 \\ 0.5 & 0.167 & -0.167 \end{bmatrix}$$

2. Perform the vector operations below,

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

Cross Product $A \times B =$

Dot Product $A \bullet B =$

ANS.

$A \times B = (-4, 17, -10)$

$A \bullet B = 13$

4. Solve the following equations using any technique,

$$5x - 2y + 4z = -1$$
$$6x + 7y + 5z = -2$$
$$2x - 3y + 6z = -3$$

ANS.

x= 0.273
y= -0.072
z= -0.627

5. Solve the following set of equations using a) Cramer's rule and b) an inverse matrix.

$$2x + 3y = 4$$

$$5x + 1y = 0$$

(ans. $\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

a) $$x = \frac{\left|\begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix}\right|}{\left|\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\right|} = \frac{4}{-13} \qquad y = \frac{\left|\begin{bmatrix} 2 & 4 \\ 5 & 0 \end{bmatrix}\right|}{\left|\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\right|} = \frac{-20}{-13} = \frac{20}{13}$$

b) $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \frac{\begin{bmatrix} 1 & -(5) \\ -(3) & 2 \end{bmatrix}^T}{\left|\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\right|}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \frac{\begin{bmatrix} 1 & -3 \\ -5 & 2 \end{bmatrix}}{-13}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \left(-\frac{1}{13}\right)\begin{bmatrix} 4 \\ -20 \end{bmatrix} = \begin{bmatrix} -\frac{4}{13} \\ \frac{20}{13} \end{bmatrix}$$

6. Perform the following matrix calculation. Show all work.

$$\left[\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} L \\ M \\ N \end{bmatrix}\right]^T$$

(ans.
$$\left[\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} L \\ M \\ N \end{bmatrix}\right]^T = \left[\begin{bmatrix} AX+BY+CZ \\ DX+EY+FZ \\ GX+YH+IZ \end{bmatrix} + \begin{bmatrix} L \\ M \\ N \end{bmatrix}\right]^T = \begin{bmatrix} AX+BY+CZ+L \\ DX+EY+FZ+M \\ GX+YH+IZ+N \end{bmatrix}^T$$
$$= \begin{bmatrix} AX+BY+CZ+L & DX+EY+FZ+M & GX+YH+IZ+N \end{bmatrix}$$

7. Perform the matrix calculations given below.

a) $\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} =$

b) $\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}\begin{bmatrix} X & Y & Z \end{bmatrix} =$

8. Find the dot product, and the cross product, of the vectors A and B below.

$$A = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

(ans.

$$A \bullet B = xp + yq + zr$$

$$A \times B = \begin{vmatrix} i & j & k \\ x & y & z \\ p & q & r \end{vmatrix} = i(yr - zq) + j(zp - xr) + k(xq - yp) = \begin{bmatrix} yr - zq \\ zp - xr \\ xq - yp \end{bmatrix}$$

9. Perform the following matrix calculations.

a) $\begin{bmatrix} a \\ b \\ c \end{bmatrix}^T \begin{bmatrix} d & e & f \\ g & h & k \\ m & n & p \end{bmatrix}$ b) $\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right|$ c) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$

(ans.

a)
$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}^T \begin{bmatrix} d & e & f \\ g & h & k \\ m & n & p \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} d & e & f \\ g & h & k \\ m & n & p \end{bmatrix} = \begin{bmatrix} (ad + bg + cm) & (ae + bh + cn) & (af + bk + cp) \end{bmatrix}$$

b)
$$\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = ad - bc$$

c)
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{adj}}{\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right|} = \frac{\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}}{ad - bc} = \begin{bmatrix} \frac{d}{ad - bc} & \frac{-c}{ad - bc} \\ \frac{-b}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix}$$

10. Find the value of 'x' for the following system of equations.

$$x + 2y + 3z = 5$$

$$x + 4y + 8z = 0$$

$$4x + 2y + z = 1$$

(ans. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 8 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$

$$x = \frac{\begin{bmatrix} 5 & 2 & 3 \\ 0 & 4 & 8 \\ 1 & 2 & 1 \end{bmatrix}}{\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 8 \\ 4 & 2 & 1 \end{bmatrix}} = \frac{5(4-16) + 2(8-0) + 3(0-4)}{1(4-16) + 2(32-1) + 3(2-16)} = \frac{-60 + 16 - 12}{-12 + 62 - 42} = \frac{-56}{8} = -7$$

11. Perform the matrix calculations given below.

a) $\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} =$

b) $\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X & Y & Z \end{bmatrix} =$

c) $det \begin{bmatrix} A & B \\ C & D \end{bmatrix} =$

d) $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^A =$

e) $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} =$

12. Solve the following set of equations with the specified methods.

$$3x + 5y = 7$$
$$4x - 6y = 2$$

a) Inverse matrix

b) Cramer’s rule

c) Gauss-Jordan row reduction

d) Substitution

24.6 CALCULUS

• NOTE: Calculus is very useful when looking at real systems. Many students are turned off by the topic because they "don’t get it". But, the secret to calculus is to remember that there is no single "truth" - it is more a loose collection of tricks and techniques. Each one has to be learned separately, and when needed you must remember it, or know where to look.

24.6.1 Single Variable Functions

24.6.1.1 - Differentiation

• The basic principles of differentiation are, Both u, v and w are functions of x, but this is not shown for brevity.

Also note that C is used as a constant, and all angles are in radians.

$$\frac{d}{dx}(C) = 0$$

$$\frac{d}{dx}(Cu) = (C)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(u + v + \dots) = \frac{d}{dx}(u) + \frac{d}{dx}(v) + \dots$$

$$\frac{d}{dx}(u^n) = (nu^{n-1})\frac{d}{dx}(u)$$

$$\frac{d}{dx}(uv) = (u)\frac{d}{dx}(v) + (v)\frac{d}{dx}(u)$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \left(\frac{v}{v^2}\right)\frac{d}{dx}(u) - \left(\frac{u}{v^2}\right)\frac{d}{dx}(v)$$

$$\frac{d}{dx}(uvw) = (uv)\frac{d}{dx}(w) + (uw)\frac{d}{dx}(v) + (vw)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(y) = \frac{d}{du}(y)\frac{d}{dx}(u) = \text{chain rule}$$

$$\frac{d}{dx}(u) = \frac{1}{\frac{d}{du}(x)}$$

$$\frac{d}{dx}(y) = \frac{\frac{d}{du}(y)}{\frac{d}{du}(x)}$$

- Differentiation rules specific to basic trigonometry and logarithm functions

$$\frac{d}{dx}(\sin u) = (\cos u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\cos u) = (-\sin u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\tan u) = \left(\frac{1}{\cos u}\right)^2\frac{d}{dx}(u)$$

$$\frac{d}{dx}(e^u) = (e^u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\cot u) = (-\csc u)^2\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\sec u) = (\tan u \sec u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\csc u) = (-\csc u \cot u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\sinh u) = (\cosh u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\cosh u) = (\sinh u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\tanh u) = (\operatorname{sech} u)^2\frac{d}{dx}(u)$$

• L'Hospital's rule can be used when evaluating limits that go to infinity.

$$\lim_{x \to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x \to a}\left(\frac{\left(\frac{d}{dt}\right)f(x)}{\left(\frac{d}{dt}\right)g(x)}\right) = \lim_{x \to a}\left(\frac{\left(\frac{d}{dt}\right)^2 f(x)}{\left(\frac{d}{dt}\right)^2 g(x)}\right) = \dots$$

• Some techniques used for finding derivatives are, Leibnitz's Rule, (notice the form is similar to the binomial equation) can be used for finding the derivatives of multiplied functions.

$$\left(\frac{d}{dx}\right)^n(uv) = \left(\frac{d}{dx}\right)^0(u)\left(\frac{d}{dx}\right)^n(v) + \binom{n}{1}\left(\frac{d}{dx}\right)^1(u)\left(\frac{d}{dx}\right)^{n-1}(v)$$

$$\binom{n}{2}\left(\frac{d}{dx}\right)^2(u)\left(\frac{d}{dx}\right)^{n-2}(v) + \ldots + \binom{n}{n}\left(\frac{d}{dx}\right)^n(u)\left(\frac{d}{dx}\right)^0(v)$$

24.6.1.2 - Integration

• Some basic properties of integrals include,

In the following expressions, u, v, and w are functions of x. in addition to this, C is a constant. and all angles are radians.

$$\int C dx = ax + C$$

$$\int Cf(x)dx = C\int f(x)dx$$

$$\int (u + v + w + \ldots)dx = \int u dx + \int v dx + \int w dx + \ldots$$

$$\int u dv = uv - \int v du = \text{integration by parts}$$

$$\int f(Cx)dx = \frac{1}{C}\int f(u)du \qquad u = Cx$$

$$\int F(f(x))dx = \int F(u)\frac{d}{du}(x)du = \int \frac{F(u)}{f'(x)}du \qquad u = f(x)$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad \int \frac{1}{x}dx = \ln|x| + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C \qquad \int e^x dx = e^x + C$$

• Some of the trigonometric integrals are,

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int (\sin x)^2 dx = -\frac{\sin x \cos x + x}{2} + C$$

$$\int (\cos x)^2 dx = \frac{\sin x \cos x + x}{2} + C$$

$$\int (\sin x)^3 dx = -\frac{\cos x((\sin x)^2 + 2)}{3} + C$$

$$\int (\cos x)^3 dx = \frac{\sin x((\cos x)^2 + 2)}{3} + C$$

$$\int (\cos x)^4 dx = \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$$

$$\int \cos x(\sin x)^n dx = \frac{(\sin x)^{n+1}}{n+1} + C$$

$$\int \sinh x dx = \cosh x + C$$

$$\int \cosh x dx = \sinh x + C$$

$$\int \tanh x dx = \ln(\cosh x) + C$$

$$\int x \cos(ax) dx = \frac{\cos(ax)}{a^2} + \frac{x}{a}\sin(ax) + C$$

$$\int x^2 \cos(ax) dx = \frac{2x\cos(ax)}{a^2} + \frac{a^2x^2 - 2}{a^3}\sin(ax) + C$$

• Some other integrals of use that are basically functions of x are,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int (a+bx)^{-1} dx = \frac{\ln|a+bx|}{b} + C$$

$$\int (a+bx^2)^{-1} dx = \frac{1}{2\sqrt{(-b)a}} \ln\left(\frac{\sqrt{a} + 2\sqrt{-b}}{\sqrt{a} - x\sqrt{-b}}\right) + C,\ a > 0,\ b < 0$$

$$\int x(a+bx^2)^{-1} dx = \frac{\ln(bx^2+a)}{2b} + C$$

$$\int x^2(a+bx^2)^{-1} dx = \frac{x}{b} - \frac{a}{b\sqrt{ab}} \operatorname{atan}\left(\frac{x\sqrt{ab}}{a}\right) + C$$

$$\int (a^2 - x^2)^{-1} dx = \frac{1}{2a} \ln\left(\frac{a+x}{a-x}\right) + C,\ a^2 > x^2$$

$$\int (a+bx)^{-1} dx = \frac{2\sqrt{a+bx}}{b} + C$$

CORRECT??

$$\int x(x^2 \pm a^2)^{-\frac{1}{2}} dx = \sqrt{x^2 \pm a^2} + C$$

$$\int (a+bx+cx^2)^{-1} dx = \frac{1}{\sqrt{c}} \ln\left[\sqrt{a+bx+cx^2} + x\sqrt{c} + \frac{b}{2\sqrt{c}}\right] + C,\ c > 0$$

$$\int (a+bx+cx^2)^{-1} dx = \frac{1}{\sqrt{-c}} \operatorname{asin}\left[\frac{-2cx - b}{\sqrt{b^2 - 4ac}}\right] + C,\ c < 0$$

$$\int (a+bx)^{\frac{1}{2}}dx = \frac{2}{3b}(a+bx)^{\frac{3}{2}}$$

$$\int (a+bx)^{\frac{1}{2}}dx = \frac{2}{3b}(a+bx)^{\frac{3}{2}}$$

$$\int x(a+bx)^{\frac{1}{2}}dx = -\frac{2(2a-3bx)(a+bx)^{\frac{3}{2}}}{15b^2}$$

$$\int (1+a^2x^2)^{\frac{1}{2}}dx = \frac{x(1+a^2x^2)^{\frac{1}{2}} + \frac{\ln\left(x+\left(\frac{1}{a^2}+x^2\right)^{\frac{1}{2}}\right)}{a}}{2}$$

$$\int x(1+a^2x^2)^{\frac{1}{2}}dx = \frac{a\left(\frac{1}{a^2}+x^2\right)^{\frac{3}{2}}}{3}$$

$$\int x^2(1+a^2x^2)^{\frac{1}{2}}dx = \frac{ax}{4}\left(\frac{1}{a^2}+x^2\right)^{\frac{3}{2}} - \frac{8}{8a^2}x(1+a^2x^2)^{\frac{1}{2}} - \frac{\ln\left(x+\left(\frac{1}{a^2}+x^2\right)^{\frac{1}{2}}\right)}{8a^3}$$

$$\int (1-a^2x^2)^{\frac{1}{2}}dx = \frac{1}{2}\left[x(1-a^2x^2)^{\frac{1}{2}} + \frac{\operatorname{asin}(ax)}{a}\right]$$

$$\int x(1-a^2x^2)^{\frac{1}{2}}dx = -\frac{a}{3}\left(\frac{1}{a^2}-x^2\right)^{\frac{3}{2}}$$

$$\int x^2(a^2-x^2)^{\frac{1}{2}}dx = -\frac{x}{4}(a^2-x^2)^{\frac{3}{2}} + \frac{1}{8}\left[x(a^2-x^2)^{\frac{1}{2}} + a^2\operatorname{asin}\left(\frac{x}{a}\right)\right]$$

$$\int (1+a^2x^2)^{-\frac{1}{2}}dx = \frac{1}{a}\ln\left[x+\left(\frac{1}{a^2}+x^2\right)^{\frac{1}{2}}\right]$$

$$\int (1-a^2x^2)^{-\frac{1}{2}}dx = \frac{1}{a}\operatorname{asin}(ax) = -\frac{1}{a}\operatorname{acos}(ax)$$

• Integrals using the natural logarithm base ‘e’,

$$\int e^{ax}dx = \frac{e^{ax}}{a} + C$$

$$\int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax - 1) + C$$

24.6.2 Vector Calculus

• When dealing with large and/or time varying objects or phenomenon we must be able to describe the state at locations, and as a whole. To do this vectors are a very useful tool.

• Consider a basic function and how it may be represented with partial derivatives.

$$y = f(x, y, z)$$

We can write this in differential form, but the right hand side must contain partial derivatives. If we separate the operators from the function, we get a simpler form. We can then look at them as the result of a dot product, and divide it into two vectors.

$$(d)y = \left(\left(\frac{\partial}{\partial x}\right)f(x, y, z)\right)dx + \left(\left(\frac{\partial}{\partial y}\right)f(x, y, z)\right)dy + \left(\left(\frac{\partial}{\partial z}\right)f(x, y, z)\right)dz$$

$$(d)y = \left[\left(\frac{\partial}{\partial x}\right)dx + \left(\frac{\partial}{\partial y}\right)dy + \left(\frac{\partial}{\partial z}\right)dz\right]f(x, y, z)$$

$$(d)y = \left[\left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}\right) \bullet (dx\underline{i} + dy\underline{j} + dz\underline{k})\right]f(x, y, z)$$

We then replace these vectors with the operators below. In this form we can manipulate the equation easily (whereas the previous form was very awkward).

$$(d)y = [\nabla \bullet dX]f(x, y, z)$$

$$(d)y = \nabla f(x, y, z) \bullet dX$$

$$(d)y = |\nabla f(x, y, z)||dX| \cos\theta$$

In summary,

$$\nabla = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}$$

$$F = F_x\underline{i} + F_y\underline{j} + F_z\underline{k}$$

$\nabla \bullet F$ = the divergence of function F

$\nabla \times F$ = the curl of function F

• Gauss's or Green's or divergence theorem is given below. Both sides give the flux across a surface, or out of a volume. This is very useful for dealing with magnetic fields.

$$\int_V (\nabla \bullet F)dV = \oint_A F dA$$

where,

V, A = a volume V enclosed by a surface area A

F = a field or vector value over a volume

• Stoke's theorem is given below. Both sides give the flux across a surface, or out of a volume. This is very useful for dealing with magnetic fields.

$$\int_A (\nabla \times F) dA = \oint_L F dL$$

where,

A, L = A surface area A, with a bounding parimeter of length L

F = a field or vector value over a volume

24.6.3 Differential Equations

• Solving differential equations is not very challenging, but there are a number of forms that need to be remembered.

• Another complication that often occurs is that the solution of the equations may vary depending upon boundary or initial conditions. An example of this is a mass spring combination. If they are initially at rest then they will stay at rest, but if there is some disturbance, then they will oscillate indefinitely.

• We can judge the order of these equations by the highest order derivative in the equation.

• Note: These equations are typically shown with derivatives only, when integrals occur they are typically eliminated by taking derivatives of the entire equation.

• Some of the terms used when describing differential equations are,

ordinary differential equations - if all the derivatives are of a single variable. In the example below 'x' is the variable with derivatives.

e.g., $$\left(\frac{d}{dt}\right)^2 x + \left(\frac{d}{dt}\right) x = y$$

first-order differential equations - have only first-order derivatives,

e.g., $$\left(\frac{d}{dt}\right) x + \left(\frac{d}{dt}\right) y = 2$$

second-order differential equations - have at least on second derivative,

e.g., $$\left(\frac{d}{dt}\right)^2 x + \left(\frac{d}{dt}\right)y = 2$$

higher order differential equations - have at least one derivative that is higher than second-order.
partial differential equations - these equations have partial derivatives

• Note: when solving these equations it is common to hit blocks. In these cases backtrack and try another approach.

• linearity of a differential equation is determined by looking at the dependant variables in the equation. The equation is linear if they appear with an exponent other than 1.

eg.

$y'' + y' + 2 = 5x$	linear
$(y'')^2 + y' + 2 = 5x$	non-linear
$y'' + (y')^3 + 2 = 5x$	non-linear
$y'' + \sin(y') + 2 = 5x$	non-linear

24.6.3.1 - First-order Differential Equations

• These systems tend to have a relaxed or passive nature in real applications.

• Examples of these equations are given below,

$$y' + 2xy^2 - 4x^3 = 0$$
$$y' - 2y = 0$$

• Typical methods for solving these equations include,

guessing then testing
separation

homogeneous

24.6.3.1.1 - Guessing

• In this technique we guess at a function that will satisfy the equation, and test it to see if it works.

$$y' + y = 0 \qquad \text{the given equation}$$

$$y = Ce^{-t} \qquad \text{the guess}$$

now try to substitute into the equation

$$y' = -Ce^{-t}$$

$$y' + y = -Ce^{-t} + Ce^{-t} = 0 \qquad \text{therefore the guess worked - it is correct}$$

$$\boxed{y = Ce^{-t}}$$

• The previous example showed a general solution (i.e., the value of 'C' was not found). We can also find a particular solution.

$$y = Ce^{-t} \qquad \text{a general solution}$$

$$y = 5e^{-t} \qquad \text{a particular solution}$$

24.6.3.1.2 - Separable Equations

• In a separable equation the differential can be split so that it is on both sides of the equation. We then integrate to get the solution. This typically means there is only a single derivative term.

e.g., $$\frac{dx}{dy} + y^2 + 2y + 3 = 0$$

$$\therefore dx = (-y^2 - 2y - 3)dy$$

$$\therefore x = \frac{-y^3}{3} - y^2 - 3y + C$$

e.g., $$\frac{dx}{dy} + x = 0$$

$$\therefore \left(-\frac{1}{x}\right)dx = dy$$

$$\therefore \ln(-x) = y$$

24.6.3.1.3 - Homogeneous Equations and Substitution

• These techniques depend upon finding some combination of the variables in the equation that can be replaced with another variable to simplify the equation. This technique requires a bit of guessing about what to substitute for, and when it is to be applied.

e.g., $\frac{dy}{dx} = \frac{y}{x} - 1$ the equation given

$u = \frac{y}{x}$ the substitution chosen

Put the substitution in and solve the differential equation,

$$\frac{dy}{dx} = u - 1$$

$$\therefore u + x\frac{du}{dx} = u - 1$$

$$\therefore \frac{du}{dx} = \frac{-1}{x}$$

$$\therefore -\frac{du}{dx} = \frac{1}{x}$$

$$\therefore -u = \ln(x) + C$$

Substitute the results back into the original substitution equation to get rid of 'u',

$$-\frac{y}{x} = \ln(x) + C$$

$$\therefore y = -x\ln(x) - Cx$$

24.6.3.2 - Second-order Differential Equations

• These equations have at least one second-order derivative.

• In engineering we will encounter a number of forms,

- homogeneous
- nonhomogeneous

24.6.3.2.1 - Linear Homogeneous

• These equations will have a standard form,

$$\left(\frac{d}{dt}\right)^2 y + A\left(\frac{d}{dt}\right)y + By = 0$$

• An example of a solution is,

e.g.,
$$\left(\frac{d}{dt}\right)^2 y + 6\left(\frac{d}{dt}\right)y + 3y = 0$$

Guess,

$$y = e^{Bt}$$

$$\left(\frac{d}{dt}\right)y = Be^{Bt}$$

$$\left(\frac{d}{dt}\right)^2 y = B^2 e^{Bt}$$

substitute and solve for B,

$$B^2 e^{Bt} + 6Be^{Bt} + 3e^{Bt} = 0$$

$$B^2 + 6B + 3 = 0$$

$$B = -3 + 2.449j, -3 - 2.449j$$

substitute and solve for B,

$$y = e^{(-3 + 2.449j)t}$$

$$y = e^{-3t} e^{2.449jt}$$

$$y = e^{-3t}(\cos(2.449t) + j\sin(2.449t))$$

Note: if both the roots are the same,

$$y = C_1 e^{Bt} + C_2 t e^{Bt}$$

24.6.3.2.2 - Nonhomogeneous Linear Equations

• These equations have the general form,

$$\left(\frac{d}{dt}\right)^2 y + A\left(\frac{d}{dt}\right)y + By = Cx$$

• to solve these equations we need to find the homogeneous and particular solutions and then add the two solutions.

$$y = y_h + y_p$$

to find yh solve,

$$\left(\frac{d}{dt}\right)^2 y + A\left(\frac{d}{dt}\right)y + B = 0$$

to find yp guess at a value of y and then test for validity, A good table of guesses is,

Cx form	Guess
A	C
$Ax + B$	$Cx + D$
e^{Ax}	Ce^{Ax} Cxe^{Ax}
$B\sin(Ax)$ or $B\cos(Ax)$	$C\sin(Ax) + D\cos(Ax)$
	or $Cx\sin(Ax) + xD\cos(Ax)$

• Consider the example below,

$$\left(\frac{d}{dt}\right)^2 y + \left(\frac{d}{dt}\right) y - 6y = e^{-2x}$$

First solve for the homogeneous part,

$$\left(\frac{d}{dt}\right)^2 y + \left(\frac{d}{dt}\right) y - 6y = 0 \qquad \text{try} \qquad y = e^{Bx}$$

$$\left(\frac{d}{dt}\right) y = Be^{Bx}$$

$$\left(\frac{d}{dt}\right)^2 y = B^2 e^{Bx}$$

$$B^2 e^{Bx} + Be^{Bx} - 6e^{Bx} = 0$$

$$B^2 + B - 6 = 0$$

$$B = -3, 2$$

$$y_h = e^{-3x} + e^{2x}$$

Next, solve for the particular part. We will guess the function below.

$$y = Ce^{-2x}$$

$$\left(\frac{d}{dt}\right) y = -2Ce^{-2x}$$

$$\left(\frac{d}{dt}\right)^2 y = 4Ce^{-2x}$$

$$4Ce^{-2x} + -2Ce^{-2x} - 6Ce^{-2x} = e^{-2x}$$

$$4C - 2C - 6C = 1$$

$$C = 0.25$$

$$y_p = 0.25e^{-2x}$$

Finally,

$$y = e^{-3x} + e^{2x} + 0.25e^{-2x}$$

24.6.3.3 - Higher Order Differential Equations

24.6.3.4 - Partial Differential Equations

• Partial difference equations become critical with many engineering applications involving flows, etc.

24.6.4 Other Calculus Stuff

• The Taylor series expansion can be used to find polynomial approximations of functions.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \ldots + \frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!}$$

24.6.5 Practice Problems

1. Find the derivative of the function below with respect to time.

$$\frac{3t}{(2+t)^2} + e^{2t}$$

(ans. $\left(\frac{d}{dt}\right)\left(\frac{3t}{(2+t)^2} + e^{2t}\right) = \left(\frac{d}{dt}\right)(3t(2+t)^{-2}) + \left(\frac{d}{dt}\right)(e^{2t}) = 3(2+t)^{-2} - 6t(2+t)^{-3} + 2e^{2t}$

2. Solve the following differential equation, given the initial conditions at t=0s.

$$x'' + 4x' + 4x = 5t \qquad x_0 = 0 \qquad x'_0 = 0$$

(ans. homogeneous solution:

$$x'' + 4x' + 4x = 0$$

guess: $x_h = e^{At} \quad x_h' = Ae^{At} \quad x_h'' = A^2 e^{At}$

$$A^2 e^{At} + 4Ae^{At} + 4e^{At} = 0 = A^2 + 4A + 4 = (A+2)(A+2)$$

$$x_h = C_1 e^{-2t} + C_2 te^{-2t}$$

particular solution:

guess: $x_p = At + B \quad x_p' = A \quad x_p'' = 0$

$$(0) + 4(A) + 4(At + B) = 5t$$

$$(4A + 4B) + (4A)t = (0) + (5)t$$

$$A = \frac{5}{4} = 1.25 \qquad 4(1.25) + 4B = 0 \qquad B = -1.25$$

$$x_p = 1.25t - 1.25$$

combine and solve for constants:

$$x(t) = x_h + x_p = C_1 e^{-2t} + C_2 te^{-2t} + 1.25t - 1.25$$

for x(0) = 0 $\quad 0 = C_1 e^{-2(0)} + C_2(0)e^{-2(0)} + 1.25(0) - 1.25 = C_1 - 1.25$

$$C_1 = 1.25$$

for (d/dt)x(0) = 0 $\quad 0 = (-2)C_1 e^{-2(0)} + (-2)C_2(0)e^{-2(0)} + C_2 e^{-2(0)} + 1.25$

$$-2C_1 + C_2 + 1.25 = 0 = -2(1.25) + C_2 + 1.25 \qquad C_2 = 1.25$$

$$x(t) = 1.25e^{2t} + 1.25te^{2t} + 1.25t - 1.25$$

3. Find the following derivatives.

a) $\frac{d}{dt}(\sin t + \cos t)$

b) $\frac{d}{dt}((t+2)^{-2})$

c) $\frac{d}{dt}(5te^{8t})$

d) $\frac{d}{dt}(5\ln t)$

(ans.

a) $\frac{d}{dt}(\sin t + \cos t) = \frac{d}{dt}(\sin t) + \frac{d}{dt}(\cos t) = \cos t - \sin t$

b) $\frac{d}{dt}((t+2)^{-2}) = -2(t+2)^{-3}$

c) $\frac{d}{dt}(5te^{8t}) = 5e^{8t} + 40te^{8t}$

d) $\frac{d}{dt}(5\ln t) = \frac{5}{t}$

4. Find the following integrals

a) $\int 6t^2 dt$

b) $\int 14e^{7t} dt$

c) $\int \sin(0.5t) dt$

d) $\int \frac{5}{x} dx$

(ans.

a) $\int 6t^2 dt = 6\left(\frac{t^3}{3}\right) = 2t^3 + C$

b) $\int 14e^{7t} dt = 14\left(\frac{e^{7t}}{7}\right) + C = 2e^{7t} + C$

c) $\int \sin(0.5t) dt = \frac{-\cos(0.5t)}{0.5} + C = -2\cos(0.5t) + C$

d) $\int \frac{5}{x} dx = 5\ln(x) + C$

5. Find the following derivative.

$$\frac{d}{dt}(5te^{4t} + (t+4)^{-3})$$

6. Find the following derivatives.

a) $\frac{d}{dx}\left(\frac{1}{x+1}\right)$

b) $\frac{d}{dt}(e^{-t}\sin(2t-4))$

(ans.

a) $\frac{d}{dx}\left(\frac{1}{x+1}\right) = \frac{-1}{(x+1)^2}$

b) $\frac{d}{dt}(e^{-t}\sin(2t-4)) = -e^{-t}\sin(2t-4) + 2e^{-t}\cos(2t-4)$

7. Solve the following integrals.

a) $\int e^{2t}dt$

b) $\int(\sin\theta + \cos 3\theta)d\theta$

(ans.

a) $\int e^{2t}dt = 0.5e^{2t} + C$

b) $\int(\sin\theta + \cos 3\theta)d\theta = -\cos\theta + \frac{1}{3}\sin 3\theta + C$

8. Solve the following differential equation.

$$x'' + 5x' + 3x = 3 \qquad x(0) = 1$$
$$x'(0) = 1$$

(ans. $x'' + 5x' + 3x = 3 \qquad x(0) = 1 \qquad x'(0) = 1$

Homogeneous:

$$A^2 + 5A + 4 = 0$$

$$A = \frac{-5 \pm \sqrt{25-16}}{2} = \frac{-5 \pm 3}{2} = -4, -1$$

$$x_h = C_1 e^{-4t} + C_2 e^{-t}$$

Particular:

$$x_p = A \qquad x'_p = 0 \qquad x''_p = 0$$

$$0 + 5(0) + 3A = 3 \qquad A = 1$$

$$x_p = 1$$

Initial values:

$$x = x_h + x_p = C_1 e^{-4t} + C_2 e^{-t} + 1$$

$$1 = C_1(1) + C_2(1) + 1 \qquad C_1 = -C_2$$

$$x' = -4C_1 e^{-4t} - C_2 e^{-t}$$

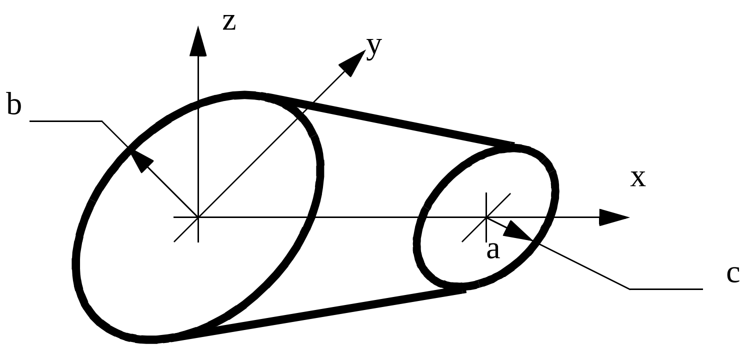
$$1 = -4C_1(1) - C_2(1)$$

$$4(-C_2) + C_2 = -1 \qquad C_2 = \frac{1}{3}$$

$$C_1 = -\frac{1}{3}$$

$$x = -\frac{1}{3}e^{-4t} + \frac{1}{3}e^{-t} + 1$$

9. Set up an integral and solve it to find the volume inside the shape below. The shape is basically a cone with the top cut off.



(ans.

$$V = \int dV = \int_0^a A dx$$

$$r = b + \left(\frac{c-b}{a}\right)x$$

$$A = \pi r^2 = \pi\left(b + \left(\frac{c-b}{a}\right)x\right)^2 = \pi b^2 + 2\pi\left(\frac{c-b}{a}\right)x + \pi\left(\frac{c-b}{a}\right)^2 x^2$$

$$V = \int_0^a \left(\pi b^2 + 2\pi\left(\frac{c-b}{a}\right)x + \pi\left(\frac{c-b}{a}\right)^2 x^2\right)dx$$

$$V = \pi b^2 x + \pi\left(\frac{c-b}{a}\right)x^2 + \frac{\pi}{3}\left(\frac{c-b}{a}\right)^2 x^3 \Big|_0^a$$

$$V = \pi b^2 a + \pi\left(\frac{c-b}{a}\right)a^2 + \frac{\pi}{3}\left(\frac{c-b}{a}\right)^2 a^3$$

$$V = \pi b^2 a + \pi(c-b)a + \frac{\pi}{3}(c-b)^2 a^2$$

10. Solve the first order non-homogeneous differential equation below. Assume the system starts at rest.

$$2x' + 4x = 5\sin 4t$$

11. Solve the second order non-homogeneous differential equation below.

$$2x'' + 4x' + 2x = 5 \qquad \text{where,} \quad x(0) = 2$$
$$x'(0) = 0$$

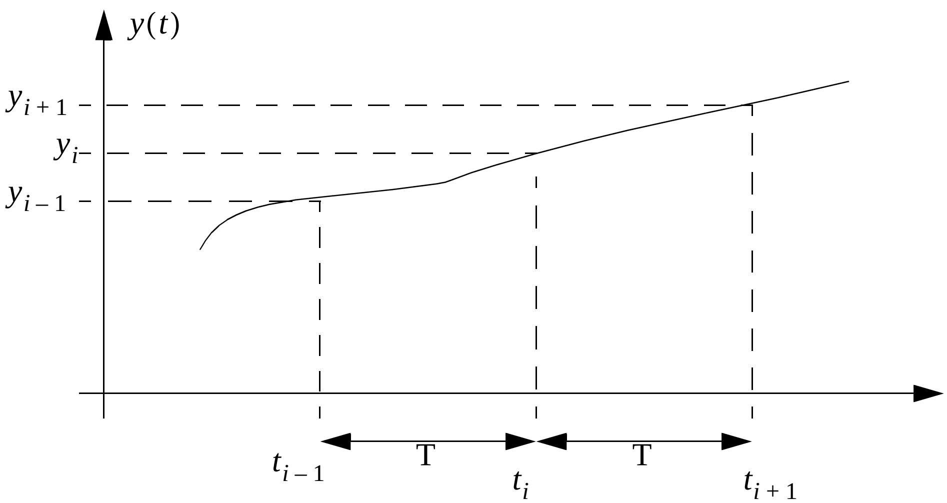
24.7 NUMERICAL METHODS

• These techniques approximate system responses without doing integrations, etc.

24.7.1 Approximation of Integrals and Derivatives from Sampled Data

• This form of integration is done numerically - this means by doing repeated calculations to solve the equation. Numerical techniques are not as elegant as solving differential equations, and will result in small errors. But these techniques make it possible to solve complex problems much faster.

• This method uses forward/backward differences to estimate derivatives or integrals from measured data.



$$\int_{t_{i-1}}^{t_i} y(t_i) \approx \left(\frac{y_i + y_{i-1}}{2}\right)(t_i - t_{i-1}) = \frac{T}{2}(y_i + y_{i-1})$$

$$\frac{d}{dt}y(t_i) \approx \left(\frac{y_i - y_{i-1}}{t_i - t_{i-1}}\right) = \left(\frac{y_{i+1} - y_i}{t_{i+1} - t_i}\right) = \frac{1}{T}(y_i - y_{i-1}) = \frac{1}{T}(y_{i+1} - y_i)$$

$$\left(\frac{d}{dt}\right)^2 y(t_i) \approx \frac{\frac{1}{T}(y_{i+1} - y_i) - \frac{1}{T}(y_i - y_{i-1})}{T} = \frac{-2y_i + y_{i-1} + y_{i+1}}{T^2}$$

24.7.2 Euler First-order Integration

• We can also estimate the change resulting from a derivative using Euler's equation for a first-order difference equation.

$$y(t+h) \approx y(t) + h\frac{d}{dt}y(t)$$

24.7.3 Taylor Series Integration

• Recall the basic Taylor series,

$$x(t+h) = x(t) + h\left(\frac{d}{dt}\right)x(t) + \frac{1}{2!}h^2\left(\frac{d}{dt}\right)^2 x(t) + \frac{1}{3!}h^3\left(\frac{d}{dt}\right)^3 x(t) + \frac{1}{4!}h^4\left(\frac{d}{dt}\right)^4 x(t) + \ldots$$

• When h=0 this is called a MacLaurin series.

• We can integrate a function by,

$$\left(\frac{d}{dt}\right)x = 1 + x^2 + t^3$$

$$\left(\frac{d}{dt}\right)x_0 = 0$$

$$x_0 = 0$$

$$h = 0.1$$

t (s)	x(t)	d/dt x(t)
0	0	0
0.1		
0.2		
0.3		
0.4		
0.5		
0.6		
0.7		
0.8		
0.9		

24.7.4 Runge-Kutta Integration

• The equations below are for calculating a fourth order Runge-Kutta integration.

$$x(t+h) = x(t) + \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4)$$

$$F_1 = hf(t, x)$$

$$F_2 = hf\left(t + \frac{h}{2}, x + \frac{F_1}{2}\right)$$

$$F_3 = hf\left(t + \frac{h}{2}, x + \frac{F_2}{2}\right)$$

$$F_4 = hf(t + h, x + F_3)$$

where,

x = the state variables
f = the differential function
t = current point in time
h = the time step to the next integration point

24.7.5 Newton-Raphson to Find Roots

• When given an equation where an algebraic solution is not feasible, a numerical solution may be required. One simple technique uses an instantaneous slope of the function, and takes iterative steps towards a solution.

$$x_{i+1} = x_i - \frac{f(x_i)}{\left(\frac{d}{dx}f(x_i)\right)}$$

• The function f(x) is supplied by the user.

• This method can become divergent if the function has an inflection point near the root.

• The technique is also sensitive to the initial guess.

• This calculation should be repeated until the final solution is found.

24.8 LAPLACE TRANSFORMS

• The Laplace transform allows us to reverse time. And, as you recall from before the inverse of time is frequency. Because we are normally concerned with response, the Laplace transform is much more useful in system analysis.

• The basic Laplace transform equations is shown below,

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

where,

$f(t)$ = the function in terms of time t

$F(s)$ = the function in terms of the Laplace s

24.8.1 Laplace Transform Tables

• Basic Laplace Transforms for operational transformations are given below,

TIME DOMAIN	FREQUENCY DOMAIN
$Kf(t)$	$Kf(s)$
$f_1(t) + f_2(t) - f_3(t) + \ldots$	$f_1(s) + f_2(s) - f_3(s) + \ldots$
$\frac{df(t)}{dt}$	$sf(s) - f(0^-)$
$\frac{d^2f(t)}{dt^2}$	$s^2f(s) - sf(0^-) - \frac{df(0^-)}{dt}$
$\frac{d^nf(t)}{dt^n}$	$s^nf(s) - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt} - \ldots - \frac{d^nf(0^-)}{dt^n}$
$\int_0^t f(t)dt$	$\frac{f(s)}{s}$
$f(t-a)u(t-a), a > 0$	$e^{-as}f(s)$
$e^{-at}f(t)$	$f(s-a)$
$f(at), a > 0$	$\frac{1}{a}f\left(\frac{s}{a}\right)$
$tf(t)$	$\frac{-df(s)}{ds}$
$t^nf(t)$	$(-1)^n\frac{d^nf(s)}{ds^n}$
$\frac{f(t)}{t}$	$\int_s^\infty f(u)du$

• A set of useful functional Laplace transforms are given below,

TIME DOMAIN	FREQUENCY DOMAIN
A	$\frac{A}{s}$
t	$\frac{1}{s^2}$
e^{-at}	$\frac{1}{s+a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
te^{-at}	$\frac{1}{(s+a)^2}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
Ae^{-at}	$\frac{A}{s-a}$
Ate^{-at}	$\frac{A}{(s-a)^2}$
$2\lvert A\rvert e^{-\alpha t}\cos(\beta t+\theta)$	$\frac{A}{s+\alpha-\beta j}+\frac{A^{\text{complex conjugate}}}{s+\alpha+\beta j}$
$2t\lvert A\rvert e^{-\alpha t}\cos(\beta t+\theta)$	$\frac{A}{(s+\alpha-\beta j)^2}+\frac{A^{\text{complex conjugate}}}{(s+\alpha+\beta j)^2}$

- Laplace transforms can be used to solve differential equations.

24.9 z-TRANSFORMS

• For a discrete-time signal $x[n]$, the two-sided z-transform is defined by $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$. The one-sided z-transform is defined by $X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$. In both cases, the z-transform is a polynomial in the complex variable z.

• The inverse z-transform is obtained by contour integration in the complex plane $x[n] = \frac{1}{j2\pi}\oint X(z)z^{n-1}dz$. This is usually avoided by partial fraction inversion techniques, similar to the Laplace transform.

• Along with a z-transform we associate its region of convergence (or ROC). These are the values of z for which $X(z)$ is bounded (i.e., of finite magnitude).

• Some common z-transforms are shown below.

Table 3: Common z-transforms

Signal $x[n]$	z-Transform $X(z)$	ROC
$\delta[n]$	1	All z
$u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$nu[n]$	$\dfrac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert > 1$
$n^2u[n]$	$\dfrac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$	$\lvert z\rvert > 1$
$a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$(-a^n)u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$(-na^n)u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\cos(\omega_0 n)u[n]$	$\dfrac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
$\sin(\omega_0 n)u[n]$	$\dfrac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
$a^n\cos(\omega_0 n)u[n]$	$\dfrac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Table 3: Common z-transforms

Signal $x[n]$	z-Transform $X(z)$	ROC
$a^n \sin(\omega_0 n) u[n]$	$\dfrac{az^{-1}\sin\omega_0}{1 - 2az^{-1}\cos\omega_0 + a^2 z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$\dfrac{n!}{k!(n-k)!}u[n]$	$\dfrac{z^{-k}}{(1-z^{-1})^{k+1}}$	$\lvert z\rvert > 1$

• The z-transform also has various properties that are useful. The table below lists properties for the two-sided z-transform. The one-sided z-transform properties can be derived from the ones below by considering the signal $x[n]u[n]$ instead of simply $x[n]$.

Table 4: Two-sided z-Transform Properties

Property	Time Domain	z-Domain	ROC
Notation	$x[n]$ $x_1[n]$ $x_2[n]$	$X(z)$ $X_1(z)$ $X_2(z)$	$r_2 < \lvert z\rvert < r_1$ ROC_1 ROC_2
Linearity	$\alpha x_1[n] + \beta x_2[n]$	$\alpha X_1(z) + \beta X_2(z)$	At least the intersection of ROC_1 and ROC_2
Time Shifting	$x[n-k]$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
z-Domain Scaling	$a^n x[n]$	$X(a^{-1}z)$	$\lvert a\rvert r_2 < \lvert z\rvert < \lvert a\rvert r_1$
Time Reversal	$x[-n]$	$X(z^{-1})$	$\dfrac{1}{r_1} < \lvert z\rvert < \dfrac{1}{r_2}$
z-Domain Differentiation	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	$r_2 < \lvert z\rvert < r_1$

Table 4: Two-sided z-Transform Properties

Property	Time Domain	z-Domain	ROC
Convolution	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of ROC_1 and ROC_2
Multiplication	$x_1[n]x_2[n]$	$\frac{1}{j2\pi}\oint X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv$	At least $r_{1l}r_{2l} < \|z\| < r_{1u}r_{2u}$
Initial value theorem	$x[n]$ causal	$x[0] = \lim_{z \to \infty} X(z)$	

24.10 FOURIER SERIES

• These series describe functions by their frequency spectrum content. For example a square wave can be approximated with a sum of a series of sine waves with varying magnitudes.

• The basic definition of the Fourier series is given below.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

24.11 TOPICS NOT COVERED (YET)

• To ensure that the omissions are obvious, I provide a list of topics not covered below. Some of these may be added later if their need becomes obvious.

• Frequency domain - Fourier, Bessel

24.12 REFERENCES/BIBLIOGRAPHY

Spiegel, M. R., Mathematical Handbook of Formulas and Tables, Schaum's Outline Series, McGraw-Hill Book Company, 1968.

25. A BASIC INTRODUCTION TO ‘C’

25.1 WHY USE ‘C’?

• ‘C’ is commonly used to produce operating systems and commercial software. Some examples of these are UNIX, Lotus-123, dBase, and some ‘C’ compilers.

• *Machine Portable*, which means that it requires only small changes to run on other computers.

• *Very Fast*, almost as fast as assembler.

• Emphasizes *structured programming*, by focusing on functions and subroutines.

• You may easily customize ’C’ to your own needs.

• Suited to Large and Complex Programs.

• Very Flexible, allows you to create your own functions.

25.2 BACKGROUND

• Developed at Bell Laboratories in the Early 70’s, and commercial compilers became available in the Late 70’s.Recnetly has become more popular because of its ties to UNIX. Over 90% of UNIX is written in ‘C’. AT&T originally developed ‘C’ with the intention of making it an in-house standard.

25.3 PROGRAM PARTS

• /* is the start of a comment.

• */ is the end of comment.

• The main program is treated like a function, and thus it has the name *main().*

• lower/UPPER case is crucial, and can never be ignored.

• Statements are separated by semi-colons ‘;’

• Statements consist of one operation, or a set of statements between curly brackets *{, }*

• There are no line numbers.

```
Program to Add two Numbers:
/* A simple program to add two numbers and print the results */

main()
{
int x, y = 2, z; /* define three variables and give one a value */
x = 3; /* give another variable a value */
z = x + y; /* add the two variables */
printf("%d + %d = %d\n", x, y, z); /*print the results */
}

Results (output):
3 + 2 = 5
```

• lines may be of any length.

• A very common function in 'C' is *printf()*. This function will do a formatted print. The format is the first thing which appears between the brackets. In this case the format says print an integer *%d* followed by a space then a '+' then another space, another integer, another space, '=', and another space, another integer, then a line feed '*\n*'. All variables that follow the format statement are those to be printed. *x, y*, and *z* are the three integers to be printed, in their respective orders.

• Major Data Types for variables and functions are (for IBM PC):

int (2 byte integer),
short (1 byte integer),
long (4 byte integer),
char (1 byte integer),
float (4 byte IEEE floating point standard),
double (8 byte IEEE floating point standard).

•*int, short, long, char* can be modified by the addition of *unsigned*, and *register*. An *unsigned integer* will not use 1 bit for number sign. A *register* variable will use a data register in the microprocessor, if possible, and it will speed things up (this is only available for integers).

```
Example of Defining Different Data Types:

main()
{
unsigned int i;
register j;
short k;
char l;
double m;
etc
```

• A *function* consists of a sub-routine or program, which has been assigned a name. This function is capable of accepting an argument list, and returning a single value. The function must be defined before it is called from within the program. (e.g. *sin()* and *read()*).

```
Program to add numbers with a function:
/* A simple program to add two numbers and print the results */

int add(); /* Declare a integer function called 'add' */

main()
{
int x = 3, y = 2, z; /* define three variables and give values */
z = add(x, y); /* pass the two values to 'add' and get the sum*/
printf("%d + %d = %d\n", x, y, z); /*print the results */
}

int add(a, b) /* define function and variable list */
int a, b; /* describe types of variable lists */
{
int c; /* define a work integer */
c = a + b; /* add the numbers */
return(c); /* Return the number to the calling program */
}
```

• Every variable has a *scope*. This determines which functions are able to use that variable. If a variable is *global*, then it may be used by any function. These can be modified by the addition of *static*, *extern* and *auto*. If a variable is defined in a function, then it will be local to that function, and is not used by any other function. If the variable needs to be initialized every time the subroutine is called, this is an *auto* type. *static* variables can be used for a variable that must keep the value it had the last time the function was called. Using *extern* will allow the variable types from other parts of the program to be used in a function.

```
Program example using global variables:
/* A simple program to add two numbers and print the results */

int x = 3, /* Define global x and y values */
y = 2,
add(); /* Declare an integer function called ‘add’ */

main()
{
printf(“%d + %d = %d\n”, x, y, add()); /*print the results */
}

int add() /* define function */
{
return(x + y); /* Return the sumto the calling program */
}
```

- Other variable types of variables are *union, enum, struct*, etc.

- Some basic control flow statements are *while(), do-while(), for(), switch()*, and *if()*. A couple of example programs are given below which demonstrate all the ’C’ flow statements.

```
Program example with a for loop:
/* A simple program toprint numbers from 1 to 5*/

main()
{
int i;
for(i = 1; i <= 5; i = i + 1){
printf(“number %d \n”, i); /*print the number */
}
}
```

or example with a while loop:

```
main()
{
int i = 1;
while(i <= 5){
printf("number %d \n", i);
i = i + 1;
}
}
```

or example with a do while loop

```
main()
{
int i = 1;
do{
printf("number %d \n", i);
i = i + 1;
}while(i <= 5)
}
```

or example with a do until loop:

```
main()
{
int i = 1;
do{
printf("number %d \n", i);
i = i + 1;
}until(i > 5)
}
```

Example Program with an if else:

```
main()
{
int x = 2, y = 3;
if(x > y){
printf(“Maximum is %d \n”, x);
} else if(y > x){
printf(“Maximum is %d \n”, y);
} else {
printf(“Both values are %d \n”, x);
}
}
```

Example Program using switch-case:

```
main()
{
int x = 3; /* Number of People in Family */
switch(x){ /* choose the numerical switch */
case 0: /* Nobody */
printf(“There is no family \n”);
break;
case 1: /* Only one person, but a start */
printf(“There is one parent\n”);
break;
case 2: /* You need two to start something */
printf(“There are two parents\n”);
break;
default: /* critical mass */
printf(“There are two parents and %d kids\n”, x-2);
break;
}
}
```

• *#include <filename.h>* will insert the file named *filename.h* into the program. The **.h* extension is used to indicate a header file which contains ‘C’ code to define functions and constants. This almost always includes “*stdio.h*”. As we saw before, a function must be defined (as with the ‘add’ function). We did not define *printf()* before we used it, this is normally done by using *#include <stdio.h>* at the top of your programs. “*stdio.h*” contains a line which says ‘*int printf();*’. If we needed to use a math function like *y = sin(x)* we would have to also use

#include <math.h>, or else the compiler would not know what type of value that *sin()* is supposed to return.

• *#define CONSTANT TEXT* will do a direct replacement of *CONSTANT* in the program with *TEXT*, before compilation. *#undef CONSTANT* will undefine the *CONSTANT*.

```
A Sample Program to Print Some sin() values
(using defined constatnts)

#include “stdio.h”
#include “math.h”
#define TWO_PI 6.283185307
#define STEPS 5

main()
{
double x; /* Current x value*/

for(x = 0.0; x <= TWO_PI; x = x + (TWO_PI / STEPS)){
printf(“%f = sin(%f) \n”, sin(x), x);
}
}
```

• *#ifdef, #ifndef, #if, #else* and *#else* can be used to conditionally include parts of a program. This is use for including and eliminating debugging lines in a program.

• *#define, #include, #ifdef, #ifndef, #if, #else, /** and **/* are all handled by the Preprocessor, before the compiler touches the program.

• Matrices are defined as shown in the example. In ‘C’ there are no limits to the matrix size, or dimensions. Arrays may be any data type. Strings are stored as arrays of characters.

• *i++* is the same as *i = i + 1.*

A Sample Program to Get a String
Then Print its ASCII Values (with matrix)

```
#include "stdio.h"
#define STRING_LENGTH 5

main()
{
int i;
char string[STRING_LENGTH]; /* character array */
gets(string); /* Input string from keyboard */
for(i = 0; i < STRING_LENGTH; i++){
printf("pos %d, char %c, ASCII %d \n", i, string[i], string[i]);
}
}
```

```
INPUT:
HUGH<return>

OUTPUT:
pos 0, char H, ASCII 72
pos 0, char U, ASCII 85
pos 0, char G, ASCII 71
pos 0, char H, ASCII 72
pos 0, char , ASCII 0
```

- Pointers are a very unique feature of 'C'. First recall that each variable uses a real location in memory. The computer remembers where the location of that variable is, this memory of location is called a pointer. This pointer is always hidden from the programmer, and uses it only in the background. In 'C', the pointer to a variable may be used. We may use some of the operations of 'C' to get the variable that the pointer, points to. This allows us to deal with variables in a very powerful way.

A Sample Program to Get a String
Then Print its ASCII Values (with pointers):

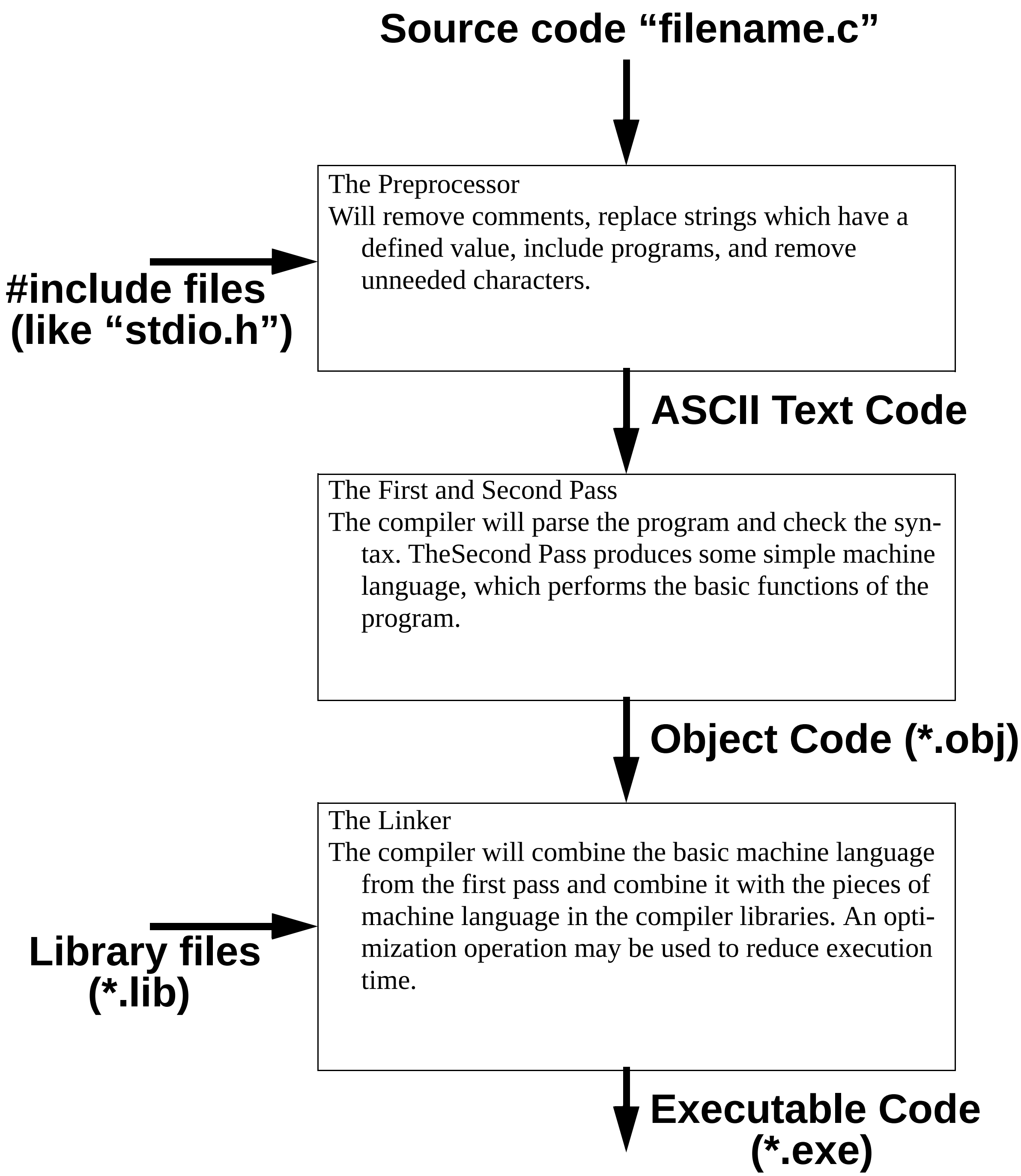
```
#include “stdio.h”

main()
{
int i;
char *string; /* character pointer */
gets(string); /* Input string from keyboard */
for(i = 0; string[i] != 0; i++){
printf(“ pos %d, char %c, ASCII %d \n”, i, string[i], string[i]);
}
}
```

INPUT:
HUGH<return>

OUTPUT:
pos 0, char H, ASCII 72
pos 0, char U, ASCII 85
pos 0, char G, ASCII 71
pos 0, char H, ASCII 72

25.4 HOW A ‘C’ COMPILER WORKS

- A ‘C’ compiler has three basic components: Preprocessor, First and Second Pass Compiler, and Linker.



25.5 STRUCTURED ‘C’ CODE

• A key to well designed and understandable programs.

• Use indents, spaces and blank lines, to make the program look less cluttered, and give it a block style.

• Comments are essential to clarify various program parts.

• Descriptive variable names, and defined constants make the purpose of the variable obvious.

• All declarations for the program should be made at the top of the program listing.

A Sample of a Bad Program Structure:

```
main(){int i;for(;i<10;i++)printf("age:%d\n",i);}
```

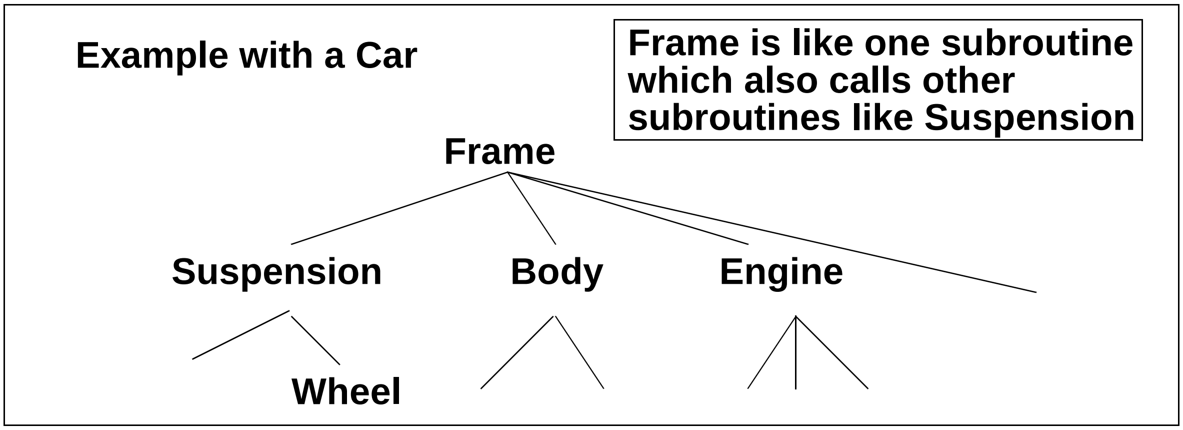
A Good Example of the same Program:

```
#include <stdio.h>
#define COUNT 10 /* Number of counts in loop */

main()
{
int i; /* counter */
        for(i = 0; i < COUNT; i++){ /* loop to print numbers */
        printf("age:%d\n", i);
        }
        exit(0);
}
```

25.6 ARCHITECTURE OF ‘C’ PROGRAMS (TOP-DOWN)

25.6.1 How?

• A program should be broken into fundamental parts (using functions for each part) and then assembled using functions. Each function consists of programs written using the previous simpler functions.



• A Clear division should be maintained between program levels.

• Never use goto's, they are a major source of logic errors. Functions are much easier to use, once written.

• Try to isolate machine specific commands (like graphics) into a few functions.

25.6.2 Why?

• A top-down design allows modules to be tested as they are completed. It is much easier to find an error in a few lines of code, than in a complete program.

• When programs are complete, errors tend to be associated with modules, and are thus much easier to locate.

• Updates to programs are much easier, when we only need to change one function.

• It is just as easy to change the overall flow of a program, as it is to change a function.
Application of 'C' to a CAD Program

25.7 CREATING TOP DOWN PROGRAMS

1. *Define Objectives* - Make a written description of what the program is expected to do.

2. *Define Problem* - Write out the relevant theory. This description should include variables, calculations and figures, which are necessary for a complete solution to the problem. From this we make a list of required data (inputs) and necessary results (output).

3. *Design User Interface* - The layout of the screen(s) must be done on paper. The method of data entry must also be considered. User options and help are also considered here. *(There are numerous factors to be considered at this stage, as outlined in the course notes.)*

4. *Write Flow Program* - This is the main code that decides when general operations occur. This is the most abstract part of the program, and is written calling dummy ‘program stubs’.

5. *Expand Program* - The dummy ‘stubs’ are now individually written as functions. These functions will call another set of dummy ‘program stubs’. This continues until all of the stubs are completed. *After the completion of any new function, the program is compiled, tested and debugged.*

6. *Testing and Debugging*- The program operation is tested, and checked to make sure that it meets the objectives. If any bugs are encountered, then the program is revised, and then retested.

7. *Document* - At this stage, the operation of the program is formally described. For Programmers, a top-down diagram can be drawn, and a written description of functions should also be given.

Golden Rule: If you are unsure how to proceed when writing a program, then work out the problem on paper, before you commit yourself to your programmed solution.

Note: Always consider the basic elements of Software Engineering, as outlined in the ES488 course notes.

25.8 HOW THE BEAMCAD PROGRAM WAS DESIGNED

25.8.1 Objectives:

• The program is expected to aid the design of beams by taking basic information about beam geometry and material, and then providing immediate feedback. The beam will be simply supported, and be under a single point load. The program should also provide a printed report on the beam.

25.8.2 Problem Definition:

• The basic theory for beam design is available in any good mechanical design textbook. In this example it will not be given.

• The inputs were determined to be few in number: Beam Type, Beam Material, Beam Thickness, Beam Width, Beam Height, Beam Length, Load Position, Load Force.

• The possible outputs are Cross Section Area, Weight, Axial Stiffness, Bending Stiffness, and Beam Deflection, a visual display of Beam Geometry, a display of Beam Deflection.

25.8.3 User Interface:

25.8.3.1 - Screen Layout (also see figure):

• The small number of inputs and outputs could all be displayed, and updated, on a single screen.

• The left side of the screen was for inputs, the right side for outputs.

• The screen is divided into regions for input(2), input display and prompts(1), Beam Cross section(3), Numerical Results(4), and Beam Deflection(5).

25.8.3.2 - Input:

• Current Inputs were indicated by placing a box around the item on the display(1).

• In a separate Prompt Box(2), this input could be made.

• The cursor keys could be used to cursor the input selector up or down.

• Single keystroke operation.

• Keys required: UP/DOWN Cursors, F1, F2, F4, numbers from ‘0’ to ‘9’, ‘.’, ‘-’, and <RETURN>. In the spirit of robustness it was decided to screen all other keys.

25.8.3.3 - Output:

• Equations, calculations, material types, and other relevant information were obtained from a text.

• Proper textual descriptions were used to ensure clarity for the user.

• For a printed report, screen information would be printed to a printer, with the prompt area replaced with the date and time.

25.8.3.4 - Help:

• A special set of help information was needed. It was decided to ensure that the screen always displays all information necessary(2).

25.8.3.5 - Error Checking:

• Reject any input which violates the input limits.

• A default design was given, which the user could modify.

• An error checking program was created, which gives error messages.

25.8.3.6 - Miscellaneous:

• The screen was expressed in normalized coordinates by most sub-routines.

• Colors were used to draw attention, and highlight areas.

25.8.4 Flow Program:

```
main()
/*
 * EXECUTIVE CONTROL LEVEL
 *
 * This is the main terminal point between the
 * various stages of setup, input, revision
 * and termination.
 *
 * January 29th, 1989.
 */
{
        static int error;

        if((error = setup()) != ERROR) {
                        screen(NEW);
                        screen(UPDATE);
                        while((error = input()) != DONE) {
                                if(error == REVISED) {
screen(NEW);
screen(UPDATE);
                                }
                        }
                        error = NO_ERROR;
        }
        kill();
        if(error == ERROR) {
                        printf("EGA Graphics Driver Not Installed");
        }
}
```

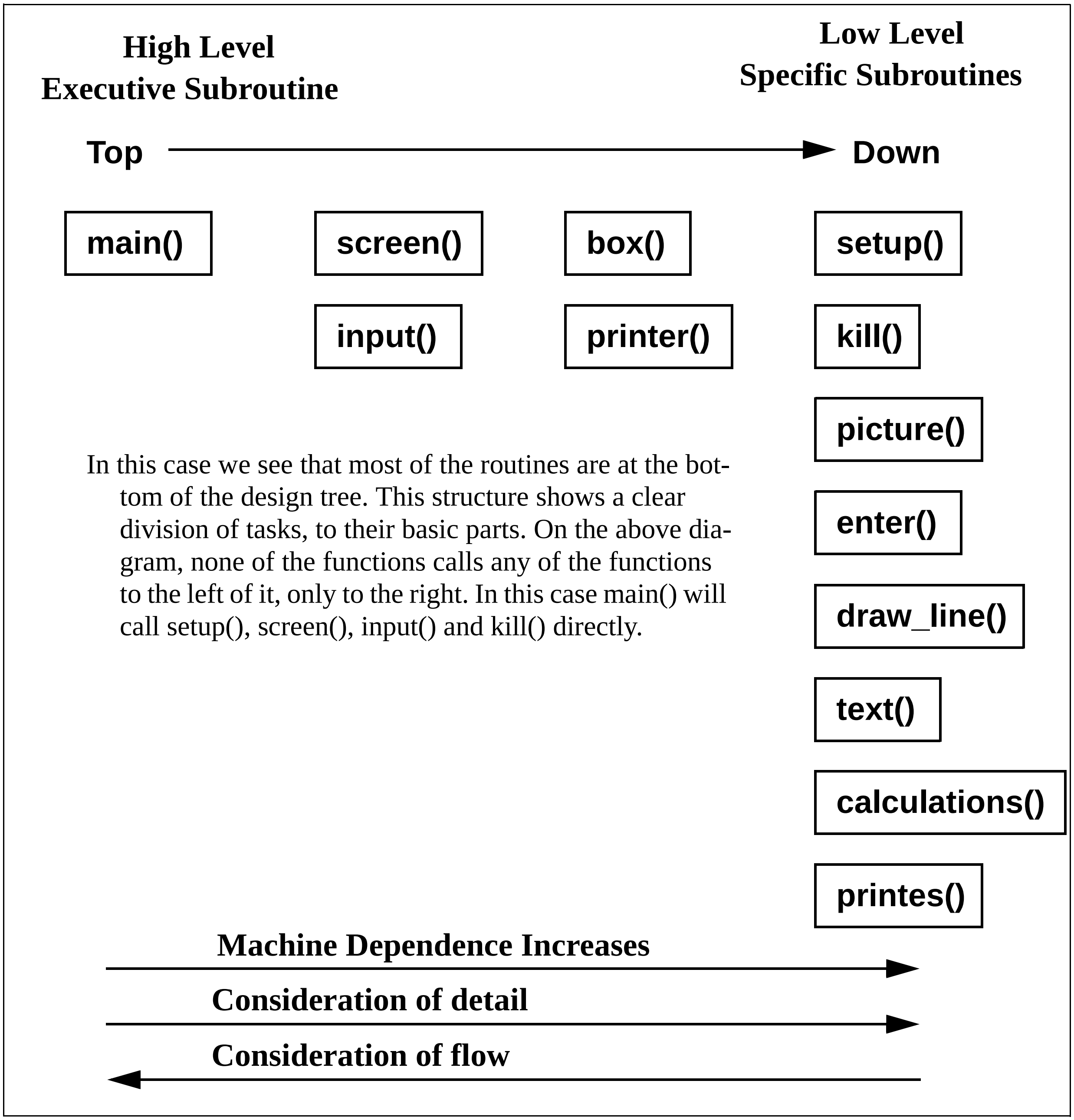
25.8.5 Expand Program:

• The routines were written in a top down fashion, in a time of about 30 hours. These routines are listed below.

Routines Used In Package:

- **main()** - to be used as the main program junction.
- **setup()** - to set up graphics mode and printer.
- **screen()** - A function to draw, or refresh part of the screen. In the interest of program speed, this function uses some low level commands.
- **calculations()** - perform the calculations of outputs from the inputs
- **picture()** - draws the beam cross section and deflection of beam. For the sake of speed, this section will use low level commands.
- **input()** - A function which controls the main input loop for numbers, controls, error screening, and any calls to desired routines. Input uses both higher and lower level commands for the sake of speed.
- **printes()** - A function to print the EGA screen.
- **printer()** - A function to remove help messages from the screen, and then dumps the screen to the printer.
- **enter()** - checks for entry error, against a set of limits for any input.
- **text()** - A function to print text on the screen at normalized coords.
- **draw_line()** - A Function to draw a line in normalized coords.
- **box()** - A Function to draw a double lined box anywhere on screen.
- **kill()** - deinitialize anything initialized in **setup()**.

- Condition and error flags were used to skip unnecessary operations, and thus speed up response. A response of more than 0.5 seconds will result in loss of attention by the user.



In this case we see that most of the routines are at the bottom of the design tree. This structure shows a clear division of tasks, to their basic parts. On the above diagram, none of the functions calls any of the functions to the left of it, only to the right. In this case main() will call setup(), screen(), input() and kill() directly.

25.8.6 Testing and Debugging:

• The testing and debugging was very fast, with only realignment of graphics being required. This took a couple of hours.

25.8.7 Documentation

25.8.7.1 - Users Manual:

• The documentation included an Executive Summary of what the Program does.

• The Objectives of the program were described.

• The theory for beam design was given for the reference of any program user, who wanted to verify the theory, and possible use it.

• A manual was given which described key layouts, screen layout, basic sequence of operations, inputs and outputs.

• Program Specifications were also given.

• A walk through manual was given. This allowed the user to follow an example which displayed all aspects of the program.

25.8.7.2 - Programmers Manual:

• Design Strategy was outlined and given.

• A complete program listing was given (with complete comments).

• Complete production of this Documentation took about 6 hours.

25.8.8 Listing of BeamCAD Program.

• Written for turbo 'C'

25.9 PRACTICE PROBLEMS

1. What are the basic components of a 'C' compiler, and what do they do?

2. You have been asked to design a CAD program which will choose a bolt and a nut to hold two pieces of sheet metal together. Each piece of sheet metal will have a hole drilled in it that is the size of the screw. You are required to consider that the two pieces are experiencing a single force. State your assumptions about the problem, then describe how you would produce this program with a Top Down design.

3. What are some reasons for using 'C' as a programming language?

4. Describe some of the reasons for Using Top-Down Design, and how to do it.

26. UNITS AND CONVERSIONS

• Units are essential when describing real things.

• Good engineering practice demands that each number should always be accompanied with a unit.

26.1 HOW TO USE UNITS

• This table does not give an exhaustive list of conversion factors, but instead a minimal (but fairly complete) set is given. From the values below any conversion value can be derived. If you are not sure about this, ask the instructor to show you how.

• A simple example of unit conversion is given below,

**a simple unit conversion example:

Given,

$$d_x = 10m \qquad d_y = 5ft$$

Find the distance 'd',

keep the units in the equation

$$d = \sqrt{d_x^2 + d_y^2}$$

$$\therefore d = \sqrt{(10m)^2 + (5ft)^2}$$

multiply by 1

$$\therefore d = \sqrt{100m^2 + 25ft^2}$$

$$\therefore d = \sqrt{100m^2 + 25ft^2\left(\frac{0.3048m}{1ft}\right)^2}$$

$$\therefore d = \sqrt{100m^2 + 25ft^2(0.092903)\frac{m^2}{ft^2}}$$

cancel out units

$$\therefore d = \sqrt{100m^2 + 25(0.092903)m^2}$$

$$\therefore d = \sqrt{102.32m^2} = 10.12m$$

From the tables

$$1ft = 0.3048m$$

$$\therefore 1 = \frac{0.3048m}{1ft}$$

26.2 HOW TO USE SI UNITS

1. Beware upper/lower case letter in many cases they can change meanings.

e.g. m = milli, mega

2. Try to move prefixes out of the denominator of the units.

e.g., N/cm or KN/m

3. Use a slash or exponents.

e.g., (kg•m/s^2) or (kg•m•s^{-2})

4. Use a dot in compound units.

e.g., N•m

5. Use spaces to divide digits when there are more than 5 figures, commas are avoided because their use is equivalent to decimal points in some cultures.

• In some cases units are non-standard. There are two major variations US units are marked with ‘US’ and Imperial units are marked with ‘IMP’.

26.3 THE TABLE

Major Division

Distance

1 ft. (feet) = 12 in. (inches) = 0.3048 m (meter)
1 mile = 1760 yards = 5280 ft = 1.609km
1 in.(inch) = 2.540 cm
1 yd (yard) = 3 ft.
1 nautical mile = 6080 ft. = 1852 m = 1.150782 mi
1 micron = 10^{-6} m
1 angstrom = 10^{-10} m
1 mil = 10^{-6} m
1 acre = 43,560 ft. = 0.4047 hectares

1 furlong = 660 ft
1 lightyear = 9.460528e15 m
1 parsec = 3.085678e16 m

Area

1 acre = 43,559.66 ft^2
1 Hectare (ha) = 10,000 m^2
1 Hectare (ha) = 10,000 m^2
1 Hectare (ha) = 10,000 m^2
1 Hectare (ha) = 10,000 m^2

Velocity

1 mph = 0.8689762 knot

Angle

1 rev = 2PI radians = 360 degrees = 400 gradians
1 degree = 60 minutes
1 minute = 60 seconds

Volume

1 US gallon = 231 in^3
1 CC = 1 cm^3
1 IMP gallon = 277.274 in^3
1 barrel = 31 IMP gal. = 31.5 US gal.
1 US gal. = 3.785 l = 4 quarts = 8 pints = 16 cups
1 liter (l) = 0.001 m^3 = 2.1 pints (pt) = 1.06 quarts (qt) = 0.26 gallons (gal)
1 qt (quart) = 0.9464 l
1 cup (c) = 0.2365882 l = 8 USoz
1 US oz = 1 dram = 456.0129 drops = 480 US minim = 1.040842 IMP oz
= 2 tablespoons = 6 teaspoons
1 IMP gal. = 1.201 U.S. gal.
1 US pint = 16 US oz
1 IMP pint = 20 IMP oz
1tablespoon = 0.5 oz.
1 bushel = 32 quarts
1 peck = 8 quarts

Force/Mass

1 N (newton) = 1 kg•m/s^2 = 100,000 dyne
1 dyne = $2.248*10^{-6}$ lb. (pound)
1 kg = 9.81 N (on earth surface) = 2.2046 lb
1lb = 16 oz. (ounce) = 4.448N
1 oz. = 28.35 g (gram) = 0.2780N

1 lb = 0.03108 slug
1 kip = 1000 lb.
1 slug = 14.59 kg
1 imperial ton = 2000 lb = 907.2 kg
1 metric tonne = 1000 kg
1 troy oz = 480 grain (gr)
1 g = 5 carat
1 pennyweight = 24 grain
1 stone = 14 lb
1 long ton = 2240 lb
1 short ton = 2000 lb

Pressure

1 Pascal (Pa) = 1 N/m^2 = 6.895 kPa
1 atm (metric atmos.) =760 mmHg at 0°C=14.223 lb/in^2=1.0132*10^5 N/m^2
1 psi = 2.0355 in. Hg at 32F = 2.0416 in. Hg at 62F
1 microbar = 0.1 N/m^2

Scale/Magnitude

atto (a) = 10^{-18}
femto (f) = 10^{-15}
pico (p) = 10^{-12}
nano (n) = 10^{-9}
micro (μ) = 10^{-6}
milli (m) = 10^{-3}
centi (c) 10^{-2}
deci (d) = 10^{-1}
deka (da) = 10
hecto (H) = 10^{2}
kilo (K) = 10^{3}
mega (M) = 10^{6}
giga (G) = 10^{9}
tera (T) = 10^{12}
peta (P) = 10^{15}
exa (E) = 10^{18}

Power

1 h.p. (horsepower) = 745.7 W (watts) = 2.545 BTU/hr. = 550 ft.lb./sec.
1 ft•lb/s = 1.356 W
1 J (joule) = 1 N•m = 10^7 ergs = 0.2389 cal.
1 W = 1 J/s

1 ev = $1.60219*10^{-19}$ J
1 erg = 10^{-7} J

Temperature

°F = [(°C*9)/5]+32, °C = Celsius (Centigrade), F = Fahrenheit
K = Kelvin
Rankine (R) = F - 459.666
0.252 calories = 1 BTU (British Thermal Unit)
-273.2 °C = -459.7 °F = 0 K = 0 R = absolute zero
0 °C = 32 °F = 273.3 K = 491.7 R = Water Freezes
100°C = 212°F = 373.3 K = 671.7 R = Water Boils (1 atm. pressure)
1 therm = 100,000 BTU

Mathematical

π radians = 3.1416 radians = 180 degrees = 0.5 cycles
1 Hz = 1 cycle/sec.
1 rpm (revolutions per minute) = 60 RPS (Revolutions per second) = 60Hz
1 fps (foot per second) = 1 ft/sec
1 mph (miles per hour) = 1 mi./hr.
1 cfm (cubic foot per minute) = 1 ft^3/min.
e = 2.718

Time

1 Hz (hertz) = 1 s^{-1}
1 year = 365 days = 52 weeks = 12 months
1 leap year = 366 days
1 day = 24 hours
1 fortnight = 14 days
1 hour = 60 min.
1 min = 60 seconds
1 millenium = 1000 years
1 century = 100 years
1 decade = 10 years

Physical Constants

R = 1.987 cal/mole K = ideal gas law constant
K = Boltzmann's constant = $1.3x10^{-16}$ erg/K = $1.3x10^{-23}$ J/K
h = Planck's constant = $6.62x10^{-27}$ erg-sec = $6.62x10^{-34}$ J.sec
Avagadro's number = $6.02x10^{23}$ atoms/atomic weight
density of water = 1 g/cm^3
electron charge = $1.60x10^{-19}$ coul.
electron rest mass = 9.11*10**-31 kg
proton rest mass = 1.67*10**-27 kg

speed of light (c) = 3.00x10^{10} cm/sec
speed of sound in dry air 25 C = 331 m/s
gravitational constant = 6.67*10**-11 Nm**2/kg**2
permittivity of free space = 8.85*10**-12 farad/m
permeability of free space = 1.26*10**-6 henry/m
mean radius of earth = 6370 Km
mass of earth = 5.98*10**24 kg

Electromagnetic

magnetic flux = weber (We) = 10**8 maxwell
inductance = henry
magnetic flux density = tesla (T) = 10**4 gauss
magnetic intensity = ampere/m = 0.004*PI oersted
electric flux density = coulomb/m**2
capacitance = farad
permeability = henry/m
electric field strength = V/m
luminous flux = lumen
luminance = candela/m**2
1 flame = 4 foot candles = 43.05564 lux = 43.05564 meter-candles
illumination = lux
resistance = ohm

26.4 ASCII, HEX, BINARY CONVERSION

• The table below will allow conversions between decimal, binary, hexadecimal, and ASCII values. The values shown only go up to 127. ASCII values above this are not commonly used in robust applications.

decimal	hexadecimal	binary	ASCII	decimal	hexadecimal	binary	ASCII
0	0	00000000	NUL	32	20	00100000	space
1	1	00000001	SOH	33	21	00100001	!
2	2	00000010	STX	34	22	00100010	“
3	3	00000011	ETX	35	23	00100011	#
4	4	00000100	EOT	36	24	00100100	$
5	5	00000101	ENQ	37	25	00100101	%
6	6	00000110	ACK	38	26	00100110	&
7	7	00000111	BEL	39	27	00100111	‘
8	8	00001000	BS	40	28	00101000	(
9	9	00001001	HT	41	29	00101001	)
10	A	00001010	LF	42	2A	00101010	*
11	B	00001011	VT	43	2B	00101011	+
12	C	00001100	FF	44	2C	00101100	,
13	D	00001101	CR	45	2D	00101101	-
14	E	00001110	S0	46	2E	00101110	.
15	F	00001111	S1	47	2F	00101111	/
16	10	00010000	DLE	48	30	00110000	0
17	11	00010001	DC1	49	31	00110001	1
18	12	00010010	DC2	50	32	00110010	2
19	13	00010011	DC3	51	33	00110011	3
20	14	00010100	DC4	52	34	00110100	4
21	15	00010101	NAK	53	35	00110101	5
22	16	00010110	SYN	54	36	00110110	6
23	17	00010111	ETB	55	37	00110111	7
24	18	00011000	CAN	56	38	00111000	8
25	19	00011001	EM	57	39	00111001	9
26	1A	00011010	SUB	58	3A	00111010	:
27	1B	00011011	ESC	59	3B	00111011	;
28	1C	00011100	FS	60	3C	00111100	<
29	1D	00011101	GS	61	3D	00111101	=
30	1E	00011110	RS	62	3E	00111110	>
31	1F	00011111	US	63	3F	00111111	?

decimal	hexadecimal	binary	ASCII	decimal	hexadecimal	binary	ASCII
64	40	01000000	@	96	60	01100000	‘
65	41	01000001	A	97	61	01100001	a
66	42	01000010	B	98	62	01100010	b
67	43	01000011	C	99	63	01100011	c
68	44	01000100	D	100	64	01100100	d
69	45	01000101	E	101	65	01100101	e
70	46	01000110	F	102	66	01100110	f
71	47	01000111	G	103	67	01100111	g
72	48	01001000	H	104	68	01101000	h
73	49	01001001	I	105	69	01101001	i
74	4A	01001010	J	106	6A	01101010	j
75	4B	01001011	K	107	6B	01101011	k
76	4C	01001100	L	108	6C	01101100	l
77	4D	01001101	M	109	6D	01101101	m
78	4E	01001110	N	110	6E	01101110	n
79	4F	01001111	O	111	6F	01101111	o
80	50	01010000	P	112	70	01110000	p
81	51	01010001	Q	113	71	01110001	q
82	52	01010010	R	114	72	01110010	r
83	53	01010011	S	115	73	01110011	s
84	54	01010100	T	116	74	01110100	t
85	55	01010101	U	117	75	01110101	u
86	56	01010110	V	118	76	01110110	v
87	57	01010111	W	119	77	01110111	w
88	58	01011000	X	120	78	01111000	x
89	59	01011001	Y	121	79	01111001	y
90	5A	01011010	Z	122	7A	01111010	z
91	5B	01011011	[	123	7B	01111011	{
92	5C	01011100	yen	124	7C	01111100	|
93	5D	01011101	]	125	7D	01111101	}
94	5E	01011110	^	126	7E	01111110	r arr.
95	5F	01011111	_	127	7F	01111111	l arr.

26.5 G-CODES

- A basic list of ‘G’ operation codes is given below. These direct motion of the tool.

G00 - Rapid move (not cutting)
G01 - Linear move
G02 - Clockwise circular motion
G03 - Counterclockwise circular motion
G04 - Dwell
G05 - Pause (for operator intervention)
G08 - Acceleration
G09 - Deceleration
G17 - x-y plane for circular interpolation
G18 - z-x plane for circular interpolation
G19 - y-z plane for circular interpolation
G20 - turning cycle or inch data specification
G21 - thread cutting cycle or metric data specification
G24 - face turning cycle
G25 - wait for input #1 to go low (Prolight Mill)
G26 - wait for input #1 to go high (Prolight Mill)
G28 - return to reference point
G29 - return from reference point
G31 - Stop on input (INROB1 is high) (Prolight Mill)
G33-35 - thread cutting functions (Emco Lathe)
G35 - wait for input #2 to go low (Prolight Mill)
G36 - wait for input #2 to go high (Prolight Mill)
G40 - cutter compensation cancel
G41 - cutter compensation to the left
G42 - cutter compensation to the right
G43 - tool length compensation, positive
G44 - tool length compensation, negative
G50 - Preset position
G70 - set inch based units or finishing cycle
G71 - set metric units or stock removal
G72 - indicate finishing cycle (EMCO Lathe)
G72 - 3D circular interpolation clockwise (Prolight Mill)
G73 - turning cycle contour (EMCO Lathe)
G73 - 3D circular interpolation counter clockwise (Prolight Mill)
G74 - facing cycle contour (Emco Lathe)
G74.1 - disable 360 deg arcs (Prolight Mill)
G75 - pattern repeating (Emco Lathe)
G75.1 - enable 360 degree arcs (Prolight Mill)
G76 - deep hole drilling, cut cycle in z-axis
G77 - cut-in cycle in x-axis
G78 - multiple threading cycle
G80 - fixed cycle cancel
G81-89 - fixed cycles specified by machine tool manufacturers
G81 - drilling cycle (Prolight Mill)
G82 - straight drilling cycle with dwell (Prolight Mill)
G83 - drilling cycle (EMCO Lathe)

G83 - peck drilling cycle (Prolight Mill)
G84 - taping cycle (EMCO Lathe)
G85 - reaming cycle (EMCO Lathe)
G85 - boring cycle (Prolight mill)
G86 - boring with spindle off and dwell cycle (Prolight Mill)
G89 - boring cycle with dwell (Prolight Mill)
G90 - absolute dimension program
G91 - incremental dimensions
G92 - Spindle speed limit
G93 - Coordinate system setting
G94 - Feed rate in ipm (EMCO Lathe)
G95 - Feed rate in ipr (EMCO Lathe)
G96 - Surface cutting speed (EMCO Lathe)
G97 - Rotational speed rpm (EMCO Lathe)
G98 - withdraw the tool to the starting point or feed per minute
G99 - withdraw the tool to a safe plane or feed per revolution
G101 - Spline interpolation (Prolight Mill)

• M-Codes control machine functions and these include,

M00 - program stop
M01 - optional stop using stop button
M02 - end of program
M03 - spindle on CW
M04 - spindle on CCW
M05 - spindle off
M06 - tool change
M07 - flood with coolant
M08 - mist with coolant
M08 - turn on accessory #1 (120VAC outlet) (Prolight Mill)
M09 - coolant off
M09 - turn off accessory #1 (120VAC outlet) (Prolight Mill)
M10 - turn on accessory #2 (120VAC outlet) (Prolight Mill)
M11 - turn off accessory #2 (120VAC outlet) (Prolight Mill) or tool change
M17 - subroutine end
M20 - tailstock back (EMCO Lathe)
M20 - Chain to next program (Prolight Mill)
M21 - tailstock forward (EMCO Lathe)
M22 - Write current position to data file (Prolight Mill)
M25 - open chuck (EMCO Lathe)
M25 - set output #1 off (Prolight Mill)
M26 - close chuck (EMCO Lathe)
M26 - set output #1 on (Prolight Mill)
M30 - end of tape (rewind)
M35 - set output #2 off (Prolight Mill)

M36 - set output #2 on (Prolight Mill)
M38 - put stepper motors on low power standby (Prolight Mill)
M47 - restart a program continuously, or a fixed number of times (Prolight Mill)
M71 - puff blowing on (EMCO Lathe)
M72 - puff blowing off (EMCO Lathe)
M96 - compensate for rounded external curves
M97 - compensate for sharp external curves
M98 - subprogram call
M99 - return from subprogram, jump instruction
M101 - move x-axis home (Prolight Mill)
M102 - move y-axis home (Prolight Mill)
M103 - move z-axis home (Prolight Mill)

• Other codes and keywords include,

Annn - an orientation, or second x-axis spline control point
Bnnn - an orientation, or second y-axis spline control point
Cnnn - an orientation, or second z-axis spline control point, or chamfer
Fnnn - a feed value (in ipm or m/s, not ipr), or thread pitch
Innn - x-axis center for circular interpolation, or first x-axis spline control point
Jnnn - y-axis center for circular interpolation, or first y-axis spline control point
Knnn - z-axis center for circular interpolation, or first z-axis spline control point
Lnnn - arc angle, loop counter and program cycle counter
Nnnn - a sequence/line number
Onnn - subprogram block number
Pnnn - subprogram reference number
Rnnn - a clearance plane for tool movement, or arc radius, or taper value
Qnnn - peck depth for pecking cycle
Snnn - cutting speed (rpm), spindle speed
Tnnn - a tool number
Unnn - relative motion in x
Vnnn - relative motion in y
Wnnn - relative motion in z
Xnnn - an x-axis value
Ynnn - a y-axis value
Znnn - a z-axis value
; - starts a comment (proLight Mill), or end of block (EMCO Lathe)

27. ATOMIC MATERIAL DATA

Element	Atomic #	Atomic weight	density g/cm^3	Valances	H_w
copper	29	63.57	8.96	1/2	
iron	26	55.85	7.86	2/3	
aluminum	13	26.97	2.67	3	
tungsten	74	183.85	19.3	6/8	
zinc	30	65.37	7.13	2	
silicon	14	28.09	2.33	4	
carbon	6	12.01	2.1	2/4	
titanium	22	47.9	4.51	3/4	

F = 96,500 coulombs

28. MECHANICAL MATERIAL PROPERTIES

Table 5:

Material	Type	Density	E	G	poisson	ten - yield stress	ten - ult. stress	ten - yield strain	yield stress - comp.	ult. stress - comp.	ult. stress - shear	thermal expansion
		(Mg/m3)	(GPa)	(GPa)		(MPa)		(%)	(MPa)			10-6/deg C
aluminum	typical	2.6-2.8	70-79	26-30	0.33	35-500	100-550	1-45				23
	1100-h14											
	2024-T6											
	5456-h116											
	2014-T6	2.8	73	36-41	0.33	410	480	13				
	6061-T6	2.8	70	26	0.33	270	310	17				
	7075-T6	2.7	72	26	0.33	480	550	11				

Table 5:

Material	Type	Density	E	G	poisson	ten - yield stress	ten - ult. stress	ten - yield strain	yield stress - comp.	ult. stress - comp.	ult. stress - shear	thermal expansion
		(Mg/m3)	(GPa)	(GPa)		(MPa)		(%)	(MPa)			10-6/deg C
Brass	typical	8.4-8.6	96-110	36-41	0.34	70-550	200-620	4-60				19.1-21.2
Yellow Brass												
	cold rolled											
	annealed											
Red Brass												
	cold rolled											
	annealed											
Bronze	typical	8.2-8.8	96-120	36-44	0.34	82-690	200-830	5-60				18-21
Cast Iron	typical	7.0-7.4	83-170	32-69	0.2-0.3	120	69-480	0-1		340-1400		9.9-12
Concrete	typical	2.3	17-31	na	0.1-0.2					10-70		7-14
	reinforced	2.4	17-31	na	0.1-0.2							
	lightweight	1.1-1.8	17-31	na	0.1-0.2							
Copper	typical	8.9	110-120	40-47	0.33-0.36	55-760	230-830	4-50				16.6-17.6
	annealed											
	hard drawn											
Glass	typical	2.4-2.8	48-83	19-35	0.17-0.27		30-1000	0				5-11
	Plate						70					
	Fibers						7000-20000					
Magnesium	typ. alloys	1.76-1.83	41-45	15-17	0.35	80-280	140-340	2-20				26.1-28.8
Monel	67%Ni,30% Cu	8.8	170	66	0.32	170-1100	450-1200	2-50				14
Nickel		8.8	210	80	0.31	100-620	310-760	2-50				13

Table 5:

Material	Type	Density	E	G	poisson	ten - yield stress	ten - ult. stress	ten - yield strain	yield stress - comp.	ult. stress - comp.	ult. stress - shear	thermal expansion
		(Mg/m3)	(GPa)	(GPa)		(MPa)		(%)	(MPa)			10-6/deg C
Plastics												
	Nylon	.88-1.1	2.1-3.4	na	0.4		40-80	20-100				70-140
	Polyethylene	.96-1.4	0.7-1.4	na	0.4		7-28	15-300				140-290
Rock												5-9
	Granite	2.6-2.9	40-100	na	0.2-0.3					50-280		
	Marble	2.6-2.9	40-100	na	0.2-0.3					50-280		
	Quartz	2.6-2.9	40-100	na	0.2-0.3					50-280		
	Limestone	2.0-2.9	20-70	na	0.2-0.3					20-200		
	Sandstone	2.0-2.9	20-70	na	0.2-0.3					20-200		
Rubber	typical	.96-1.3	0.0007-0.004	0.0002-0.001	0.45-0.50	1-7	7-20	100-800				130-200
Sand,soil,gravel		1.2-2.2										
Steel		7.85	190-210	75-80	0.27-0.30							10-18
	cold rolled											
	annealed											
	high strength					340-1000	550-1200	5-25				14
	machine					340-700	550-860	5-25				
	spring					400-1600	700-1900	3-15				
	stainless					280-700	400-1000	5-40				
	tool					520	900	8				
	structural					200-700	340-830	10-40				12
	astm-a48											

Table 5:

Material	Type	Density	E	G	poisson	ten - yield stress	ten - ult. stress	ten - yield strain	yield stress - comp.	ult. stress - comp.	ult. stress - shear	thermal expansion
		(Mg/m3)	(GPa)	(GPa)		(MPa)		(%)	(MPa)			10-6/deg C
	astm-a47											
	astm-a36					250	400	30				12
	astm-a5242											
	astm-a441											
	astm-a572					340	500	20				12
	astm-a514					700	830	15				12
	wire					280-1000	550-1400	5-40				
Stainless Steel	aisi 302											17
Titanium	typ. alloys	4.5	100-120	39-44	0.33	760-1000	900-1200	10				8.1-11
Tungsten		1.9	340-380	140-160	0.2		1400-4000	0-4				4.3
Water	fresh	1.0										
	sea	1.02										
Wood (dry)												
	Douglas fir	.48-.56	11-13			30-50	50-80		30-50	40-70		
	Oak	.64-.72	11-12			40-60	50-100		30-40	30-50		
	Southern pine	.56-.64	11-14			40-60	50-100		30-50	40-70		

28.1 FORMULA SHEET

- A collection of the essential mechanics of materials formulas are given below

Axial/Normal Stress/Strain

$$\sigma = \varepsilon E = \frac{P}{A} \qquad \delta = \varepsilon L = \frac{PL}{AE}$$

Shear Stress/Strain

$$\tau = \gamma G \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

Poisson's ratio

$$\nu = -\frac{\varepsilon_{lateral}}{\varepsilon_{longitudinal}} \qquad E = 2G(1 + \nu)$$

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

Torsion

$$\tau_{max} = \frac{Tc}{J} \qquad \gamma_{max} = \frac{\tau_{max}}{G}$$

$$\phi = \frac{TL}{JG} \qquad J = \frac{\pi r^4}{2}$$ (cylinder)

$$\sigma_{ten} = -\sigma_{comp} = \tau \qquad J = \frac{\pi}{2}(r_o^4 - r_i^4)$$ (hollow tube)

$$\varepsilon = \frac{\gamma}{2} \qquad P = 2\pi f T$$

Beams

$$\sigma = -\frac{My}{I} \qquad I = \frac{bh^3}{12} \text{ (for rectangle)}$$

$$\tau = \frac{VQ}{Ib} \qquad I = \frac{\pi r^4}{2} \text{ (for circle)}$$

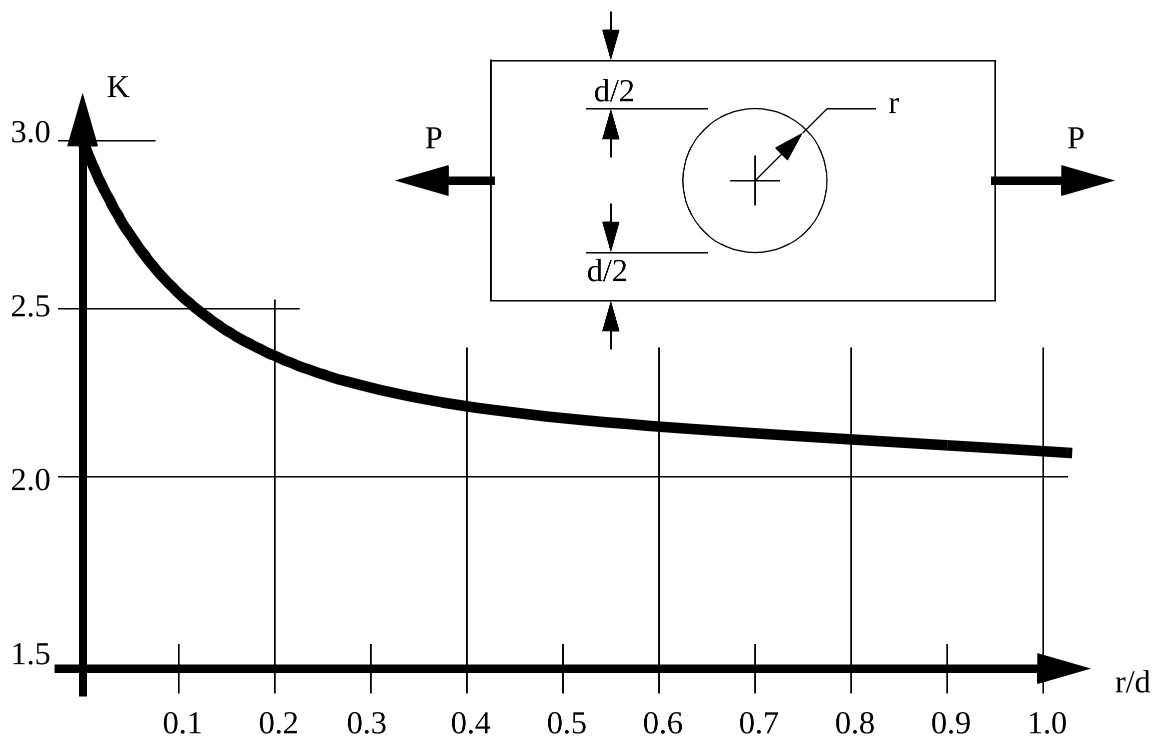
$$L = \rho\theta \qquad \varepsilon_{max} = \frac{c}{\rho}$$

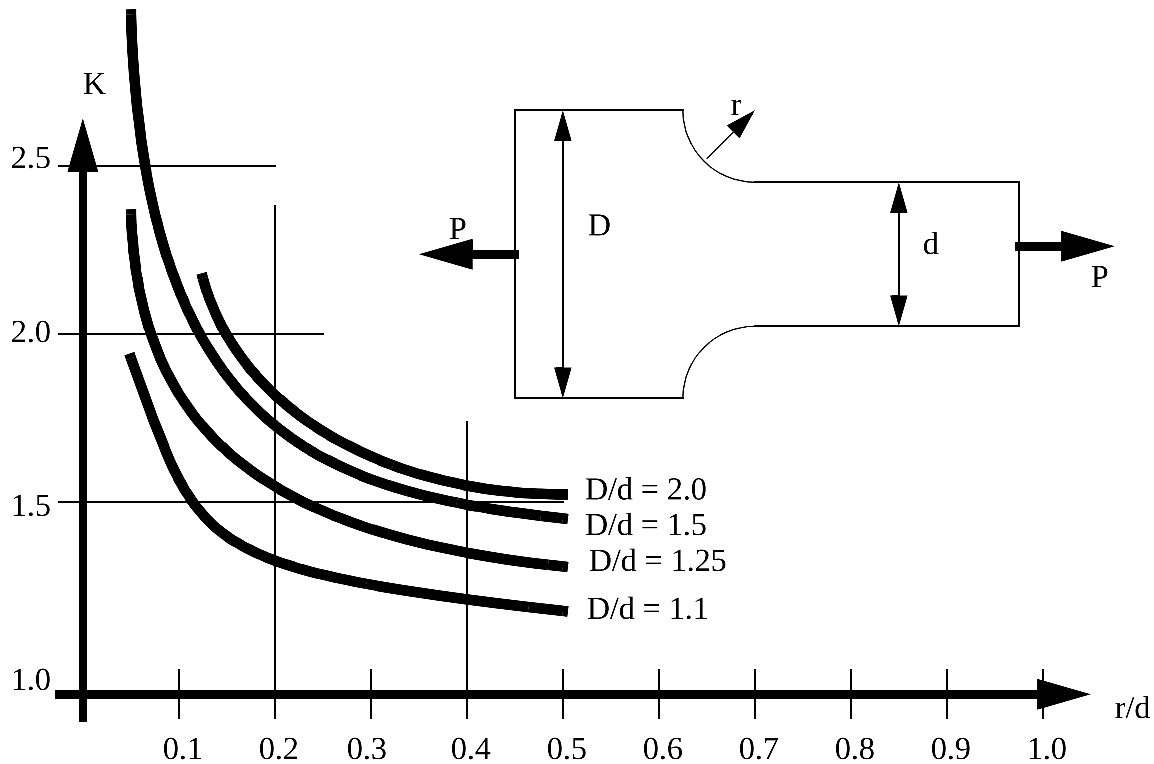
$$\sigma = -\frac{My}{I} \qquad \rho = \frac{EI}{M}$$

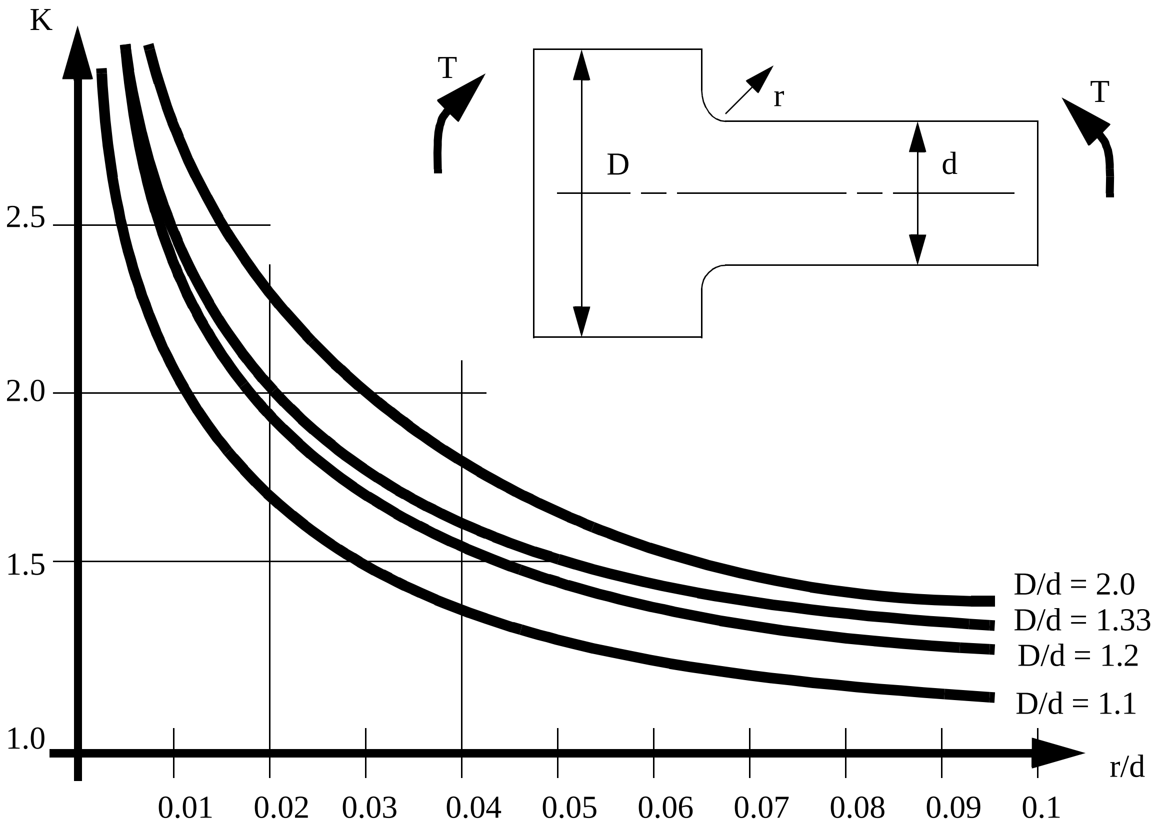
Buckling

$$P < \frac{\pi^2 EI}{k^2 L^2}$$

k=0.5 (both ends fixed)
k=0.7 (one end fixed, one pinned)
k=1.0 (both ends pinned)
k=2.0 (one end pinned, one free)




K
2.5
2.0
1.5
1.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
r/d
D/d = 2.0
D/d = 1.5
D/d = 1.25
D/d = 1.1
P
D
r
d
P


K
2.5
2.0
1.5
1.0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
r/d
D/d = 2.0
D/d = 1.33
D/d = 1.2
D/d = 1.1
T
T
D
d
r