

Notes:

- Please solve these exercises in **groups of four!**
- The solutions must be handed in **directly before (very latest: at the beginning of)** the exercise course on Wednesday, 26.06.2019, 14:30, in lecture hall **AH I**. Alternatively you can drop your solutions into a box which is located right next to Prof. Giesl's office (until 30 minutes before the exercise course starts).
- Please write the **names** and **immatriculation numbers** of all students on your solution. Also please staple the individual sheets!

Exercise 1 (Transformation to Simple Haskell):

(7 points)

The data structure `List a` is defined by

```
data List a = Nil | Cons a (List a)
```

Transform the following HASKELL-expression to an equivalent simple HASKELL-expression using the transformation rules (1)-(9) from the lecture. Please give all intermediate expressions and indicate in each step which transformation rule was used. You may apply rules to several identical subterms simultaneously.

```
let length Nil = 0
    length (Cons x xs) = 1 + length xs
in length xs
```

Please simplify the intermediate expressions according to the following rules:

- if `(isan-tuple exp)` then `exp1` else `exp2` should be replaced by `exp1` for any $n \geq 0$.
- `(\var -> exp1) exp` should be replaced by `exp1'`, where `exp1'` results from `exp1` by replacing all (free) occurrences of `var` by `exp`.

Exercise 2 (Substitutions):

(2 + 2 + 3 = 7 points)

Identify the free variables of the following lambda terms t_1 , t_2 , and t_3 , and also apply the three substitutions $\sigma_1 = [x/\lambda x.x y]$, $\sigma_2 = [y/v y]$, and $\sigma_3 = [z/\lambda z.x z]$, respectively, to each of these terms (so that you obtain in total 9 possibly different lambda terms $t_i\sigma_j$ with $i, j \in \{1, 2, 3\}$):

- $t_1 = \lambda y.x (\lambda x.y x)$
- $t_2 = \lambda x.(\lambda y.x y) x y$
- $t_3 = \lambda y.(\lambda z.z x y) z y$

Exercise 3 (β -reduction):

(2 + 4 = 6 points)

- Give all reduction sequences with \rightarrow_β starting with the following term:

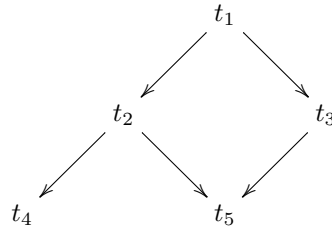
$$(\lambda x y.x y) \exp ((\lambda x z.\text{mult } x (\text{mult } z z)) 3 2)$$

- Give all reduction sequences with \rightarrow_β with at most 3 steps starting with the following term:

$$(\lambda f.(\lambda x. f (x x)) (\lambda x. f (x x))) (\lambda y.y)$$

Hints:

- You can save space by representing the reduction sequences as directed graphs. For example, $t_1 \rightarrow_\beta t_2 \rightarrow_\beta t_4$, $t_1 \rightarrow_\beta t_2 \rightarrow_\beta t_5$, and $t_1 \rightarrow_\beta t_3 \rightarrow_\beta t_5$ can be represented as:



- In a), you may abbreviate `exp` to `e` and `mult` to `m` to save space.
- In b), you may write A_f instead of $\lambda x.f(x\ x)$, B instead of $\lambda y.y$ and C instead of $\lambda x.x\ x$ to save space. The start term could then be represented as $(\lambda f.A_f\ A_f)\ B$. Furthermore, the term $\lambda x.(\lambda y.y)(x\ x)$ can be abbreviated to A_B .

Exercise 4 (Confluence):

(2 + 2 + 2 + 2 = 8 points)

- a) Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the natural numbers with the standard relation $>$.

Indicate whether the following properties hold. Explain your solution!

- For any $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that m is a normal form of n w.r.t. $>$.
- $>$ is confluent.

- b) Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the natural numbers with the following relation \succ : $m \succ n$ if there is some $0 < r \in \mathbb{N}$ with $m = r \cdot n$, i.e., n divides m .

Indicate whether the following properties hold. Explain your solution!

- For any $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that m is a normal form of n w.r.t. \succ .
- \succ is confluent.

- c) Let \mathbb{Z} be the integers with the standard relation $>$.

Indicate whether the following properties hold. Explain your solution!

- For any $z \in \mathbb{Z}$ there is a $q \in \mathbb{Z}$ such that q is a normal form of z w.r.t. $>$.
- $>$ is confluent.

- d) Let $\text{Pot}_{\neq \emptyset}(\mathbb{Z})$ be the set of nonempty subsets of the integers with the relation \subsetneq : $A \subsetneq B$ whenever $A \neq B$ and A is a subset of B .

Indicate whether the following properties hold. Explain your solution!

- For any $\emptyset \neq A \subseteq \mathbb{Z}$ there is a $B \subseteq \mathbb{Z}$ such that B is a normal form of A w.r.t. \subsetneq .
- \subsetneq is confluent.