Prof. Dr. J. Giesl — M. Hark

Notes:

- Please solve these exercises in **groups of four**!
- The solutions must be handed in **directly before (very latest: at the beginning of)** the exercise course on Wednesday, 26.06.2019, 14:30, in lecture hall **AH I**. Alternatively you can drop your solutions into a box which is located right next to Prof. Giesl's office (until 30 minutes before the exercise course starts).
- Please write the **names** and **immatriculation numbers** of all students on your solution. Also please staple the individual sheets!

Exercise 1 (Transformation to Simple Haskell): (7 points)

The data structure `List a` is defined by

```
data List a = Nil | Cons a (List a)
```

Transform the following Haskell-expression to an equivalent simple Haskell-expression using the transformation rules (1)-(9) from the lecture. Please give all intermediate expressions and indicate in each step which transformation rule was used. You may apply rules to several identical subterms simultaneously.

```
let length Nil = 0
    length (Cons x xs) = 1 + length xs
in length xs
```

Please simplify the intermediate expressions according to the following rules:

(i) `if (isa`$_{n-tuple}$ `exp) then exp1 else exp2` should be replaced by `exp1` for any $n \geq 0$.

(ii) `(\var->exp1) exp` should be replaced by `exp1'`, where `exp1'` results from `exp1` by replacing all (free) occurrences of `var` by `exp`.

Exercise 2 (Substitutions): (2 + 2 + 3 = 7 points)

Identify the free variables of the following lambda terms t_1, t_2, and t_3, and also apply the three substitutions $\sigma_1 = [x/\lambda x.x\ y]$, $\sigma_2 = [y/v\ y]$, and $\sigma_3 = [z/\lambda z.x\ z]$, respectively, to each of these terms (so that you obtain in total 9 possibly different lambda terms $t_i\sigma_j$ with $i, j \in \{1, 2, 3\}$):

a) $t_1 = \lambda y.x\ (\lambda x.y\ x)$

b) $t_2 = \lambda x.(\lambda y.x\ y)\ x\ y$

c) $t_3 = \lambda y.(\lambda z.z\ x\ y)\ z\ y$

Exercise 3 (β-reduction): (2 + 4 = 6 points)

a) Give all reduction sequences with $\rightarrow_\beta$ starting with the following term:

$$(\lambda x\ y.x\ y)\ \mathsf{exp}\ ((\lambda x\ z.\mathsf{mult}\ x\ (\mathsf{mult}\ z\ z))\ 3\ 2)$$

b) Give all reduction sequences with $\rightarrow_\beta$ with at most 3 steps starting with the following term:

$$(\lambda f.(\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ (\lambda y.y)$$

Hints:

- You can save space by representing the reduction sequences as directed graphs. For example, $t_1 \to_\beta t_2 \to_\beta t_4$, $t_1 \to_\beta t_2 \to_\beta t_5$, and $t_1 \to_\beta t_3 \to_\beta t_5$ can be represented as:

$$\begin{tikzcd}
 & & t_1 \arrow[dl] \arrow[dr] & \\
 & t_2 \arrow[dl] \arrow[dr] & & t_3 \arrow[dl] \\
t_4 & & t_5 &
\end{tikzcd}$$

- In a), you may abbreviate `exp` to `e` and `mult` to `m` to save space.
- In b), you may write A_f instead of $\lambda x.f\ (x\ x)$, B instead of $\lambda y.y$ and C instead of $\lambda x.x\ x$ to save space. The start term could then be represented as $(\lambda f.A_f\ A_f)\ B$. Furthermore, the term $\lambda x.(\lambda y.y)\ (x\ x)$ can be abbreviated to A_B.

Exercise 4 (Confluence): (2 + 2 + 2 + 2 = 8 points)

a) Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the natural numbers with the standard relation $>$.

Indicate whether the following properties hold. Explain your solution!

- For any $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that m is a normal form of n w.r.t. $>$.
- $>$ is confluent.

b) Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the natural numbers with the following relation $\succ$: $m \succ n$ if there is some $0 < r \in \mathbb{N}$ with $m = r \cdot n$, i.e., n divides m.

Indicate whether the following properties hold. Explain your solution!

- For any $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that m is a normal form of n w.r.t. $\succ$.
- $\succ$ is confluent.

c) Let $\mathbb{Z}$ be the integers with the standard relation $>$.

Indicate whether the following properties hold. Explain your solution!

- For any $z \in \mathbb{Z}$ there is a $q \in \mathbb{Z}$ such that q is a normal form of z w.r.t. $>$.
- $>$ is confluent.

d) Let $\mathrm{Pot}_{\neq\emptyset}(\mathbb{Z})$ be the set of nonempty subsets of the integers with the relation $\subsetneq$: $A \subsetneq B$ whenever $A \neq B$ and A is a subset of B.

Indicate whether the following properties hold. Explain your solution!

- For any $\emptyset \neq A \subseteq \mathbb{Z}$ there is a $B \subseteq \mathbb{Z}$ such that B is a normal form of A w.r.t. $\subsetneq$.
- $\subsetneq$ is confluent.