Prof. Dr. J. Giesl M. Hark

Notes:

- Please solve these exercises in **groups of four**!
- The solutions must be handed in **directly before (very latest: at the beginning of)** the exercise course on Wednesday, 22.05.2019, 14:30, in lecture hall **AH I**. Alternatively you can drop your solutions into a box which is located right next to Prof. Giesl's office (until 30 minutes before the exercise course starts).
- Please write the **names** and **immatriculation numbers** of all students on your solution. Also please staple the individual sheets!

Exercise 1 (Semantics of Partial Functions): (1 + 1 + 1 = 3 points)

Consider the following function:

```
plus :: Int -> Int -> Int
plus 0 y = y
plus x 0 = x
plus x y = x + y
```

a) Define the semantics of `plus` as a function $f_{\texttt{plus}} : \mathbb{Z}_\perp \times \mathbb{Z}_\perp \to \mathbb{Z}_\perp$. Take the evaluation strategy of Haskell into account. In your definition, "+" may only be applied to arguments from $\mathbb{Z}$.

b) Prove or disprove that $f_{\texttt{plus}}$ is strict.

c) Prove that $f_{\texttt{plus}}$ is monotonic. Do not make use of the fact that any Haskell function is computable and hence monotonic.

Exercise 2 (Monotonicity): (1 + 3 = 4 points)

Consider the following total functions (with their usual interpretation):

a) $- : \mathbb{Z} \to \mathbb{Z}_\perp$

b) $*, \max : \mathbb{N} \times \mathbb{N} \to \mathbb{N}_\perp$

Here, "$-$" denotes the function that maps 2 to -2 or -3 to 3. Moreover, "max" denotes the "maximum" function.
For all $f \in \{-, \max, *\}$, give all monotonic functions f' such that f' is an extension of f, i.e., $f'(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n$ different from $\perp$. The functions f' should map between the following sets:

a) $-' : \mathbb{Z}_\perp \to \mathbb{Z}_\perp$

b) $*', \max' : \mathbb{N}_\perp \times \mathbb{N}_\perp \to \mathbb{N}_\perp$

Exercise 3 (Completeness): (4 + 3 = 7 points)

Consider Theorem 2.1.13 (c) from the lecture:
Let D_1, D_2 be domains. If $\sqsubseteq_{D_2}$ is complete on D_2, then $\sqsubseteq_{D_1 \to D_2}$ is complete on $D_1 \to D_2$.
The other direction of this theorem holds as well:
Let D_1, D_2 be arbitrary nonempty sets where D_2 is partially ordered with $\sqsubseteq_{D_2}$. If $\sqsubseteq_{D_1 \to D_2}$ is complete on $D_1 \to D_2$, then $\sqsubseteq_{D_2}$ is complete on D_2.
To prove this theorem, please prove the following two properties:

a) If $\sqsubseteq_{D_1 \to D_2}$ is complete on $D_1 \to D_2$, then $\perp_{D_2}$ exists.

Hints:

- You can **not** assume that $\perp_{D_1 \to D_2}(d) = \perp_{D_2}$ for each $d \in D_1$, as one does not know yet whether $\perp_{D_2}$ exists. Instead, first prove that $\perp_{D_1 \to D_2}$ is a constant function, i.e., that $\perp_{D_1 \to D_2}(d) = \perp_{D_1 \to D_2}(d')$ for each $d, d' \in D_1$.

b) If $\sqsubseteq_{D_1 \to D_2}$ is complete on $D_1 \to D_2$, then for all chains S on D_2, the least upper bound $\sqcup S$ of S exists in D_2.

Hints:

- You may use lemmas and theorems from the lecture before Thm. 2.1.13 for your proof.

Exercise 4 (Continuity): (1 + 2.5 + 2.5 + 3 = 9 points)

In this exercise we will compare topological–continuity from real analysis to continuity as defined in the lecture (Scott[1]–continuity).
Let $a, b \in \mathbb{R}$, then $[a, b] := \{r \mid a \leq r \leq b\}$. A function $f : [a, b] \to \mathbb{R}$ is topologically–continuous if for every converging sequence $(x_n)_{n \in \mathbb{N}}$ in $[a, b]$ we have $f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n)$. A function $f : [a, b] \to \mathbb{R}$ is Scott–continous if for any chain $S \subseteq [a, b]$ we have $f(\sup S) = \sup f(S)$, where $f(S) = \{f(s) \mid s \in S\}$. As usual, $\sup S$ denotes the least upper bound, i.e., the supremum of S.

a) Let $a, b \in \mathbb{R}$. Show that the standard ordering $\leq$ on the real numbers is complete on the closed interval $[a, b]$.

Hints:

- You may use that $\mathbb{R}$ is complete, i.e., every set of real numbers that is bounded from above has a least upper bound.

b) Let $a, b, c, d \in \mathbb{R}$. Show that **not** every topologically–continuous function $f : [a, b] \to [c, d]$ is Scott–continuous.

c) Let $a, b, c, d \in \mathbb{R}$. Prove that any monotonic and topologically–continuous function $f : [a, b] \to [c, d]$ is Scott–continuous. As usual, f is *monotonic* if $x \leq y$ implies $f(x) \leq f(y)$.

Hints:

- You can use the following lemma. Let $(x_n)_{n \in \mathbb{N}}$ be a monotonically increasing sequence of real numbers which is bounded from above. Then $(x_n)_{n \in \mathbb{N}}$ converges and $\lim_{n\to\infty} x_n = \sup\{x_n \mid n \in \mathbb{N}\}$.

d) Let $a, b, c, d \in \mathbb{R}$. Prove or disprove that any Scott–continuous function $f : [a, b] \to [c, d]$ is topologically–continuous.

[1] https://en.wikipedia.org/wiki/Dana_Scott