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Global exponential stability of recurrent neural networks with time-varying delays in the presence of strong external stimuli☆

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Abstract

This paper presents new theoretical results on the global exponential stability of recurrent neural networks with bounded activation functions and bounded time-varying delays in the presence of strong external stimuli. It is shown that the Cohen–Grossberg neural network is globally exponentially stable, if the absolute value of the input vector exceeds a criterion. As special cases, the Hopfield neural network and the cellular neural network are examined in detail. In addition, it is shown that criteria herein, if partially satisfied, can still be used in combination with existing stability conditions. Simulation results are also discussed in two illustrative examples.

Keywords: Recurrent neural networks; Time-varying; Exponential stability; Strong external stimuli

1. Introduction

The buds of some recurrent neural network models may be traced back to the nonlinear difference–differential equations in learning theory or prediction theory (Grossberg, 1967, 1968a, 1968b, 1969a, 1969b, 1969c, 1971, 1972). The global exponential stability for such systems was analyzed (Grossberg, 1968b). In particular, a general neural network, which is called the Cohen–Grossberg neural network (CGNN) and can function as stable associative memory, was developed and studied (Cohen & Grossberg, 1983; Grossberg, 1988).

As a special case of the Cohen–Grossberg neural network, the continuous-time Hopfield neural network (HNN) (Hopfield, 1984) was proposed and applied for optimization, associative memories, pattern classification, image processing, etc. In parallel, cellular neural networks (CNNs) (Chua & Yang, 1988a, 1988b) were developed and have attracted much attention due to their great perspective on applications. CNNs and delayed cellular neural networks (DCNNs) have been applied to signal processing, image processing, and pattern recognition (Chua & Yang, 1988a). For example, CNNs with opposite-sign templates have been successfully applied in connected component detection (CCD) in feature extraction (Roska, Wu, Balsi, & Chua, 1992; Roska, Wu, & Chua, 1993).

The stability of recurrent neural networks is a prerequisite for almost all neural network applications. The stability analysis is primarily concerned with the existence and uniqueness of equilibrium points and the global asymptotic stability, global exponential stability, and global robust stability of neural networks at equilibria. In recent years, the stability analysis of recurrent neural networks with delays received much attention; e.g., Arik (2002), Arik (2003), Cao (2001a, 2001b), Cao and Zhou (1998), Cao and Wang (2003), Chen, Cao, and Huang (2002), Civalleri, Gilli, and Pandolfi (1993), Dong (2002), Feng and Plamondon (2001), Gopalsamy and Sariyasa (2002), Hou and Qian (1998), Jiang, Li, and Teng (2003), Li, Huang, and Zhu (2003), Liao and Wang (2000), Liao, Chen, and Sanchez (2002), Liao and Wang (2003), Mohamad and Gopalsamy (2003), Peng, Qiao, and Xu (2002), Roska et al. (1992), Roska et al. (1993), Yi, Heng, and Leung (2001), Yu, Lu, Tsai, Su, and Liu (2003), Zeng and Wang (2006a, 2006b), Zeng, Wang, and Liao (2003, 2004), Zhang,

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Ma, and Xu (2001) and Zhang, Ma, Wang, and Xu (2003). In the existing stability analysis of neural networks, the common treatment is to translate its equilibrium to a zero solution, then analyze the stability of the resulting translated system around the origin. If the existence of the equilibrium of the neural network can be guaranteed, then stability of the neural network is equivalent to the stability of the zero solution of the translated system. However, the external stimuli are cancelled by the translation. As a result, almost all existing stability conditions are independent of external stimuli. Because the dynamic behavior of a neural network depends on external stimuli as well as other parameters, the stability conditions independent of external stimuli are over-conservative. Furthermore, as the locations of equilibria of recurrent neural networks depend also on the stimuli, ignoring the external stimuli in stability analysis will lose important information.

In this paper, it is shown that recurrent neural networks with bounded activation functions and bounded time-varying delays are globally exponentially stable, if the external stimuli are sufficiently strong, without the need for any other condition. In other words, the existing stability criteria can be relaxed or weakened for the recurrent neural networks with sufficiently strong biases.

The remainder of this paper is organized as follows. Section 2 describes some preliminaries. The main results on Cohen–Grossberg neural networks are stated in Section 3. Specific results on the delayed HNN and DCNN can be found in Sections 4 and 5, respectively. Simulation results of two illustrative examples are given in Section 6. Finally, concluding remarks are made in Section 7.

2. Preliminaries

Consider a general form of Cohen–Grossberg neural network model with time-varying delays: for $i = 1, 2, \dots, n$;

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = \alpha_i(x_i(t)) \left[-\beta_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t - \tau_{ij}(t))) + u_i \right], \tag{1}$$

where $x_i(t)$ is the state of the ith neuron, $\alpha_i(\cdot)$ and $\beta_i(\cdot)$ are continuous functions, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ are connection weight matrices that are not assumed to be symmetric, and $f(\cdot) = (f_1(\cdot), f_2(\cdot), \dots, f_n(\cdot))^{\mathrm{T}}$ and $g(\cdot) = (g_1(\cdot), g_2(\cdot), \dots, g_n(\cdot))^{\mathrm{T}}$ are vector-valued activation continuous functions, $\tau_{ij}(t)$ is the time-varying delay that satisfies $0 \le \tau_{ij}(t) \le \tau$ (τ is constant), and u_i is an external input (bias) to the ith neuron.

$\forall \xi \in \Re$, let $\Omega_\xi^+ = \{(p, q) \mid p > \xi; q > \xi,\ p, q \in \Re\}$; $\Omega_\xi^- = \{(p, q) \mid p < -\xi; q < -\xi,\ p, q \in \Re\}$. In this paper, we always assume that for $i = 1, 2, \dots, n$,

A_1: there exist constants $\bar{f}_i > 0$, $\bar{g}_i > 0$ such that $\forall r \in \Re$,

$$|f_i(r)| \le \bar{f}_i; \qquad |g_i(r)| \le \bar{g}_i;$$

A_2: $\forall (x_i, y_i) \in \Omega_\xi^+$ or $(x_i, y_i) \in \Omega_\xi^-$, $x_i \ne y_i$,

$$\lim_{\xi\to+\infty} (f_i(x_i) - f_i(y_i))/(x_i - y_i) = 0,$$
$$\lim_{\xi\to+\infty} (g_i(x_i) - g_i(y_i))/(x_i - y_i) = 0;$$

A_3: f_i, and g_i are continuous functions;

A_4: there exist constants $\bar{\alpha}_i > 0$, $\underline{\alpha}_i > 0$ such that $0 < \underline{\alpha}_i \le \alpha_i(x_i) \le \bar{\alpha}_i$, for all $x_i \in \Re$;

A_5: $\beta_i(0) = 0$, and there exist constants $\bar{c}_i > 0$, $\underline{c}_i > 0$ such that $0 < \underline{c}_i \le \frac{\beta_i(x_i)-\beta_i(y_i)}{x_i-y_i} \le \bar{c}_i$, for all $x_i, y_i \in \Re$, $x_i \ne y_i$.

Obviously, if f_i and g_i are differentiable, and $\lim_{|r|\to+\infty} \frac{\mathrm{d}f_i(r)}{\mathrm{d}r} = 0$, $\lim_{|r|\to+\infty} \frac{\mathrm{d}g_i(r)}{\mathrm{d}r} = 0$, then f_i and g_i satisfy the assumption A_2. Hence, the sigmoid activation function in HNN (Hopfield, 1984), and the linear saturation activation function in CNN (Chua & Yang, 1988a) satisfy the above assumptions A_1, A_2 and A_3.

Based on assumptions A_1–A_5, it is well known that the equilibrium points of the neural network (1) exist by Brouwer's fixed-point theorem. Let $x^* = (x_1^*, x_2^*, \dots, x_n^*)^{\mathrm{T}}$ be an equilibrium point of the neural network (1). Denote

$$z_i(t) = x_i(t) - x_i^*, \quad i = 1, 2, \dots, n,$$

then the neural network (1) can be rewritten as

$$\frac{\mathrm{d}z_i(t)}{\mathrm{d}t} = \alpha_i(z_i(t) + x_i^*) \left[-\beta_i^*(z_i(t)) + \sum_{j=1}^{n} a_{ij} f_j^*(z_j(t)) + \sum_{j=1}^{n} b_{ij} g_j^*(z_j(t - \tau_{ij}(t))) \right], \tag{2}$$

where for $i, j = 1, 2, \dots, n$,

$$\beta_j^*(z_j(t)) := \beta_j(z_j(t) + x_j^*) - \beta_j(x_j^*), \tag{3}$$
$$f_j^*(z_j(t)) := f_j(z_j(t) + x_j^*) - f_j(x_j^*), \tag{4}$$
$$g_j^*(z_j(t)) := g_j(z_j(t) + x_j^*) - g_j(x_j^*). \tag{5}$$

Denote C^0 as the set of continuous functions. For any $t_0 \ge 0$, the initial condition of the neural network (1) is assumed to be

$$\begin{cases} x_i(t) = \phi_i(t) & t_0 - \tau \le t \le t_0 \\ \phi_i \in C^0 & i = 1, 2, \dots, n. \end{cases} \tag{6}$$

The initial condition of the neural network (2) can be similarly defined.

Definition 1. The neural network (1) is said to be globally exponentially stable, if there exist positive constants β and a constant vector $(x_1^*, x_2^*, \dots, x_n^*)^{\mathrm{T}}$ such that the solution $x(t)$ of (1) with any initial condition (6) satisfies

$$|x_i(t) - x_i^*| \le \sup_{t_0-\tau\le s\le t_0} \{|x_i(s) - x_i^*|\} \exp\{-\beta(t - t_0)\}, \quad t \ge t_0 \ge 0.$$

Definition 2. A matrix is said to be a nonsingular M-matrix, if its off-diagonal entries are nonpositive and its every principal minor is positive.

3. Main Theorem

Lemma 1. *If assumption A_2 holds, then $\forall \ell_i, \mu_i > 0$ $(i = 1, 2, \ldots, n)$, there exists $\xi_i \geq 0$ such that when $x_i \neq y_i$, $(x_i, y_i) \in \Omega^+_{\xi_i}$ and $(x_i, y_i) \in \Omega^-_{\xi_i}$,*

$$-\ell_i \leq (f_i(x_i) - f_i(y_i))/(x_i - y_i) \leq \ell_i, \tag{7}$$

$$-\mu_i \leq (g_i(x_i) - g_i(y_i))/(x_i - y_i) \leq \mu_i. \tag{8}$$

Proof. From assumption A_2, for $x_i \neq y_i$, $(x_i, y_i) \in \Omega^+_\xi$ and $(x_i, y_i) \in \Omega^-_\xi$,

$$\lim_{\xi \to +\infty} (f_i(x_i) - f_i(y_i))/(x_i - y_i) = 0,$$
$$\lim_{\xi \to +\infty} (g_i(x_i) - g_i(y_i))/(x_i - y_i) = 0.$$

Hence for any given $\varepsilon > 0$, there exists $\xi_i \geq 0$ such that when $(x_i, y_i) \in \Omega^+_{\xi_i}$ and $(x_i, y_i) \in \Omega^-_{\xi_i}$, $(i = 1, 2, \ldots, n)$,

$$|(f_i(x_i) - f_i(y_i))/(x_i - y_i)| \leq \varepsilon,$$
$$|(g_i(x_i) - g_i(y_i))/(x_i - y_i)| \leq \varepsilon.$$

Especially, choose $\varepsilon = \min_{1 \leq i \leq n}\{\ell_i, \mu_i\}$, then Lemma 1 holds. $\square$

Next, we present the main theorem of the paper.

Main Theorem. *Suppose that there exist $\xi_i \geq 0, \gamma_i > 0$ $(i = 1, 2, \ldots, n)$ such that Eqs. (7) and (8) hold, where $\ell_i, \mu_i > 0$ $(i = 1, 2, \ldots, n)$ satisfy*

$$\gamma_i \underline{c}_i > \sum_{j=1}^{n} (|a_{ij}|\ell_j + |b_{ij}|\mu_j)\gamma_j. \tag{9}$$

The Cohen–Grossberg neural network (1) is globally exponentially stable, if for $i = 1, 2, \ldots, n$,

$$u_i > \bar{c}_i \xi_i + \sum_{j=1}^{n} (|a_{ij}|\bar{f}_j + |b_{ij}|\bar{g}_j) \quad \text{or}$$

$$u_i < -\underline{c}_i \xi_i - \sum_{j=1}^{n} (|a_{ij}|\bar{f}_j + |b_{ij}|\bar{g}_j). \tag{10}$$

Proof. Without loss of generality, let $u_i > \bar{c}_i \xi_i + \sum_{j=1}^{n} (|a_{ij}|\bar{f}_j + |b_{ij}|\bar{g}_j)$ $(i = 1, 2, \ldots, l), u_k < -\underline{c}_k \xi_k - \sum_{j=1}^{n} (|a_{kj}|\bar{f}_j + |b_{kj}|\bar{g}_j)$ $(k = l + 1, \ldots, n)$, where $l = 1, \ldots, n$. If $x_i(t) \leq \xi_i, i \in \{1, 2, \ldots, l\}$, then

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= \alpha_i(x_i(t)) \left[-\beta_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) \right. \\ &\quad \left. + \sum_{j=1}^{n} b_{ij} g_j(x_j(t - \tau_{ij}(t))) + u_i \right] \\ &\geq \alpha_i(x_i(t)) \left[-\bar{c}_i \xi_i - \sum_{j=1}^{n} (|a_{ij}|\bar{f}_j + |b_{ij}|\bar{g}_j) + u_i \right] > 0.\end{aligned} \tag{11}$$

Similarly, if $x_k(t) \geq -\xi_k, k \in \{l + 1, \ldots, n\}$, then

$$\begin{aligned}\frac{\mathrm{d}x_k(t)}{\mathrm{d}t} &= \alpha_k(x_k(t)) \left[-\beta_k(x_k(t)) + \sum_{j=1}^{n} a_{kj} f_j(x_j(t)) \right. \\ &\quad \left. + \sum_{j=1}^{n} b_{kj} g_j(x_j(t - \tau_{kj}(t))) + u_k \right] \\ &\leq \alpha_k(x_k(t)) \left[\underline{c}_k \xi_k + \sum_{j=1}^{n} (|a_{kj}|\bar{f}_j + |b_{kj}|\bar{g}_j) + u_k \right] \\ &< 0.\end{aligned} \tag{12}$$

From (11) and (12), there exists $T > 0$ that only depends on ε and the initial condition of the neural network (1), such that $\forall t > T, h = 1, 2, \ldots, n$,

$$|x_h(t)| > \xi_h.$$

Let $x^* = (x^*_1, x^*_2, \ldots, x^*_n)^{\mathrm{T}}$ be an equilibrium point of the neural network (1). Then $|x^*_h| > \xi_h$, $(h = 1, 2, \ldots, n)$. Let $z_h(t) = (x_h(t) - x^*_h)/\gamma_h$. From (7) and (8), $\forall t > T, h = 1, 2, \ldots, n, z_h(t) \neq 0$,

$$-\ell_h \leq (f_h(\gamma_h z_h(t) + x^*_h) - f_h(x^*_h))/(\gamma_h z_h(t)) \leq \ell_h;$$
$$-\mu_h \leq (g_h(\gamma_h z_h(t) + x^*_h) - g_h(x^*_h))/(\gamma_h z_h(t)) \leq \mu_h.$$

From (2)–(5), for $t > T + \tau, h = 1, 2, \ldots, n$,

$$\begin{aligned}D^+|z_h(t)| &= \frac{\mathrm{d}z_h(t)}{\mathrm{d}t} \mathrm{sign}(z_h(t)) \\ &\leq \alpha_h(z_h(t) + x^*_h) \left[-\underline{c}_h |z_h(t)| + \sum_{j=1}^{n} (|a_{hj}|\ell_j |z_j(t)| \right. \\ &\quad \left. + |b_{hj}|\mu_j |z_j(t - \tau_{hj}(t))|)\gamma_j/\gamma_h \right],\end{aligned} \tag{13}$$

where $D^+|z_h(t)|$ is the upper-right Dini-derivative of $|z_h(t)|$ with respect to t along the trajectory of the neural network (2). From (9) and $\tau_{ij}(t) \leq \tau$, there exists $\theta > 0$ such that for $\forall t > T + \tau, h = 1, 2, \ldots, n$,

$$\begin{aligned}&-\underline{c}_h + \theta/\underline{\alpha}_h + \sum_{j=1}^{n} (|a_{hj}|\ell_j + |b_{hj}|\mu_j \\ &\quad \times \exp\{\theta \tau_{hj}(t)\})\gamma_j/\gamma_h \leq 0.\end{aligned} \tag{14}$$

For $t > T + \tau, h = 1, 2, \ldots, n$, let $e_h(t) = |z_h(t)| - Z(T)\exp\{-\theta(t - \tau - T)\}$, where $Z(T) = \max\{\max_{t_0 - \tau \leq s \leq T + \tau} |z_i(s)|, i = 1, 2, \ldots, n\}$, then for all $i \in \{1, 2, \ldots, n\}, t \geq T + \tau, e_i(t) \leq 0$. Otherwise, for all $i \in \{1, 2, \ldots, n\}, \forall s \in [t_0 - \tau, T + \tau], e_i(s) \leq 0$, then there exist $k \in \{1, 2, \ldots, n\}, t_2 > t_1 \geq T + \tau$ and $\varepsilon_1 > 0$ sufficiently small such that

$$e_k(t_1) = 0, \qquad e_k(t_2) = \varepsilon_1, \tag{15}$$

$$D^+e_k(t_1) \geq 0, \qquad D^+e_k(t_2) > 0; \tag{16}$$

for $s \in (t_1, t_2], e_k(s) > 0$; and for $j = 1, 2, \ldots, n, t \in [t_0 - \tau, t_2], e_j(t) \leq \varepsilon_1$; i.e.,

$$|z_j(t)| \leq \varepsilon_1 + Z(T)\exp\{-\theta(t - \tau - T)\}. \tag{17}$$

On the other hand, by (13)–(15) and (17),

$$D^+e_k(t_2) \le \alpha_k(z_k(t)+x_k^*)\left[-\underline{c}_k|z_k(t_2)| + \sum_{j=1}^{n}(|a_{kj}|\ell_j|z_j(t_2)| + |b_{kj}|\mu_j|z_j(t_2-\tau_{kj}(t_2))|)\gamma_j/\gamma_k\right] + \theta Z(T)\exp\{-\theta(t_2-\tau-T)\}$$
$$\le \underline{\alpha}_k\left[-\underline{c}_k|z_k(t_2)| + \sum_{j=1}^{n}(|a_{kj}|\ell_j|z_j(t_2)| + |b_{kj}|\mu_j|z_j(t_2-\tau_{kj}(t_2))|)\gamma_j/\gamma_k\right] + \theta Z(T)\exp\{-\theta(t_2-\tau-T)\}$$
$$\le \underline{\alpha}_k\left(-c_k + \sum_{j=1}^{n}(|a_{kj}|\ell_j + |b_{kj}|\mu_j\exp\{\theta\tau_{kj}(t_2)\})\gamma_j/\gamma_k + \theta/\underline{\alpha}_k\right) \times Z(T)\exp\{-\theta(t_2-\tau-T)\}$$
$$+ \underline{\alpha}_k\left(-c_k + \sum_{j=1}^{n}(|a_{kj}|\ell_j + |b_{kj}|\mu_j)\gamma_j/\gamma_k\right)\varepsilon_1 \le 0.$$

This contradicts (16), thus $|z_h(t)| \le Z(T)\exp\{-\theta(t-\tau-T)\}$, for all $h \in \{1,2,\ldots,n\}, t \ge T+\tau$. Hence the Cohen–Grossberg neural network (1) is globally exponentially stable. □

Consider a recurrent neural network with time-varying delays:

$$\frac{dx_i(t)}{dt} = -c_ix_i(t) + \sum_{j=1}^{n}a_{ij}f_j(x_j(t)) + \sum_{j=1}^{n}b_{ij}g_j(x_j(t-\tau_{ij}(t))) + u_i, \tag{18}$$

where $i = 1,2,\ldots,n$, $x_i(t)$, a_{ij}, b_{ij}, $f_j(\cdot)$ and $g_j(\cdot)$ are defined the same as those in the neural network (1).

It is obvious that the neural network (18) is a special case of the Cohen–Grossberg neural network. Let $\alpha_i(x_i(t)) \equiv 1$; i.e., $\bar{\alpha}_i = \underline{\alpha}_i = 1$, $\bar{c}_i = \underline{c}_i = c_i > 0$ $(i = 1,2,\ldots,n)$, $C = \text{diag}\{c_1,c_2,\ldots,c_n\}$. According to the Main Theorem, we have the following Main Corollary

Main Corollary. *Suppose that there exist $\xi_i, \gamma_i > 0$ $(i = 1,2,\ldots,n)$ such that Eqs. (7) and (8) hold, where $\ell_i, \mu_i > 0$ $(i = 1,2,\ldots,n)$ satisfy*

$$\gamma_ic_i > \sum_{j=1}^{n}(|a_{ij}|\ell_j + |b_{ij}|\mu_j)\gamma_j. \tag{19}$$

The recurrent neural network (18) is globally exponentially stable, if for $i = 1,2,\ldots,n$,

$$|u_i| > c_i\xi_i + \sum_{j=1}^{n}(|a_{ij}|\bar{f}_j + |b_{ij}|\bar{g}_j). \tag{20}$$

Remark 1. Obviously, for any given $c_i > 0$ $(i = 1,2,\ldots,n)$, there exist small enough $\ell_i, \mu_i > 0$ $(i = 1,2,\ldots,n)$ such that (19) holds. If assumption A_2 holds, according to Lemma 1, for $\ell_i, \mu_i > 0$ $(i = 1,2,\ldots,n)$ satisfying (19), there exist ξ_i such that (7) and (8) hold. Hence, if assumption A_2 holds, then there exist $\ell_i > 0, \mu_i > 0, \xi_i$ $(i = 1,2,\ldots,n)$ such that (7), (8) and (19) always hold. Thus, according to the Main Corollary, if the external input is strong enough (i.e., $|u_i|$ satisfies (20) $i = 1,2,\ldots,n$), then the recurrent neural network (18) is globally exponentially stable. In other words, for any given parameter, sufficiently strong stimuli will make an RNN with bounded activation function to be globally exponentially stable.

If there exist ℓ_j, μ_j such that (7), (8) and (19) always hold, then ξ_i can be sufficiently negative such that $c_i\xi_i + \sum_{j=1}^{n}(|a_{ij}|\bar{f}_j + |b_{ij}|\bar{g}_j) < -1$. As a result, (20) always holds. According to the Main Corollary, it is unnecessary to restrict u_i. In other words, if there exist ℓ_j, μ_j such that (7), (8) and (19) hold, then the recurrent neural network (18) is globally exponentially stable, which is the main result of Zeng et al. (2003). Hence, the results in this paper includes a previous work in Zeng et al. (2003) as a special case.

When $\tau_{ij}(t) \equiv 0$, according to the Theorem in Cao and Zhou (1998) and Theorems 1 and 2 in Feng and Plamondon (2001),

$$\frac{dy_i(t)}{dt} = -c_iy_i(t) + \sum_{j=1}^{n}|a_{ij}|\gamma_jy_j(t) + \sum_{j=1}^{n}|b_{ij}|\beta_jy_j(t) \tag{21}$$

is global exponential stable at an equilibrium point. Since system (21) is linear, the real parts of all eigenvalues of the matrix $C - |\bar{A}| - |\bar{B}|$ are negative under the condition of the Theorem in Cao and Zhou (1998) and Theorems 1 and 2 in Feng and Plamondon (2001), where $C = \text{diag}\{c_1,c_2,\ldots,c_n\}$; $|\bar{A}| = (|a_{ij}|\ell_j)_{n\times n}$; $|\bar{B}| = (|b_{ij}|\mu_j)_{n\times n}$. Hence, $C - |\bar{A}| - |\bar{B}|$ is an M-matrix. Thus there exist $\gamma_1,\gamma_2,\ldots,\gamma_n > 0$ such that Eq. (19) holds. Namely, the results in this paper extend the work in Cao and Zhou (1998) and Feng and Plamondon (2001).

4. HNN with time-varying delay

When $f(r) = f_j(r) = g_j(r) = (\exp(r) - \exp(-r))/(\exp(r)+\exp(-r))$, $\tau_{ij}(t) \le \tau$ $(i,j = 1,2,\ldots,n)$, the neural network (18) is the continuous-time HNN with bounded time-varying delays or delayed HNN (DHNN)

$$\frac{dx_i(t)}{dt} = -c_ix_i(t) + \sum_{j=1}^{n}a_{ij}f(x_j(t)) + \sum_{j=1}^{n}b_{ij}f(x_j(t-\tau_{ij}(t))) + u_i. \tag{22}$$

Corollary 1. *DHNN (22) is globally exponentially stable, if there exist $\ell_i \leq 1$ $(i = 1, 2, \ldots, n)$ such that $C - \bar{A} - \bar{B}$ is a nonsingular M-matrix, and*

$$|u_i| > c_i \left(\ln 2 - \frac{1}{2} \ln \ell_i \right) + \sum_{j=1}^{n} (|a_{ij}| + |b_{ij}|), \tag{23}$$

where $\bar{A} = (\bar{a}_{ij})_{n\times n}$, $\bar{B} = (\bar{b}_{ij})_{n\times n}$, $\bar{a}_{ij} = |a_{ij}|\ell_j$, $\bar{b}_{ij} = |b_{ij}|\ell_j$.

Proof. Because $C - \bar{A} - \bar{B}$ is a nonsingular M-matrix, there exist positive numbers $\gamma_1, \gamma_2, \ldots, \gamma_n$ such that for $i = 1, 2, \ldots, n$,

$$c_i \gamma_i > \sum_{j=1}^{n} \gamma_j \ell_j (|a_{ij}| + |b_{ij}|). \tag{24}$$

Let $z_i(t) = x_i(t)/\gamma_i$, then from (22),

$$\frac{\mathrm{d}z_i(t)}{\mathrm{d}t} = -c_i z_i(t) + \frac{1}{\gamma_i} \left(\sum_{j=1}^{n} (a_{ij} f(\gamma_j z_j(t)) + b_{ij} f(\gamma_j z_j(t - \tau_{ij}(t)))) + u_i \right). \tag{25}$$

Since $\mathrm{d}f(r)/\mathrm{d}r = 4/(\exp(r) + \exp(-r))^2 \leq \min\{4/\exp(2r), 4/\exp(-2r)\}$, when $|\bar{x}_i| \geq \ln 2 - (\ln \ell_i)/2$ $(i = 1, 2, \ldots, n)$, $\mathrm{d}f(r)/\mathrm{d}r|_{r=\bar{x}_i} \leq \ell_i$. Also since $f(r + y) - f(r) = y\mathrm{d}f(x)/\mathrm{d}x|_{x=\eta}$, where $\eta \in (\min\{r, r + y\}, \max\{r, r + y\})$, when $x_i > \ln 2 - (\ln \ell_i)/2$, $x_i + \gamma_i z_i > \ln 2 - (\ln \ell_i)/2$ or $x_i < -\ln 2 + (\ln \ell_i)/2$, $x_i + \gamma_i z_i < -\ln 2 + (\ln \ell_i)/2$ $(i = 1, 2, \ldots, n)$,

$$-\ell_i \leq (f(x_i + \gamma_i z_i) - f(x_i))/(\gamma_i z_i) \leq \ell_i. \tag{26}$$

Choose $\xi_i = (\ln 2 - \frac{1}{2} \ln \ell_i)/\gamma_i$. From (24) and (26), according to the Main Corollary, (25) is globally exponentially stable; i.e., DHNN (22) is globally exponentially stable. □

Remark 2. Denote $|A| = (|a_{ij}|)_{n\times n}$, $|B| = (|b_{ij}|)_{n\times n}$. When $C - |A| - |B|$ is not a nonsingular M-matrix, the conditions in Cao and Wang (2003), Chen et al. (2002), Mohamad and Gopalsamy (2003), Zhang et al. (2001, 2003) and Zeng et al. (2003) cannot be used to ascertain the stability of DHNN (22). But $\forall c_i > 0$, when ℓ_i is small enough, $C - \bar{A} - \bar{B}$ is always a nonsingular M-matrix. Hence, according to Corollary 1, if input u_i is strong enough $(i = 1, \ldots, n)$, then DHNN (22) is globally exponentially stable.

Corollary 2. *DHNN (22) is globally exponentially stable, if*

$$|u_i| > c_i \left(\ln 2 - \frac{1}{2} \ln \frac{c_i}{\sum_{j=1}^{n} (|a_{ij}| + |b_{ij}|)} \right) + \sum_{j=1}^{n} (|a_{ij}| + |b_{ij}|).$$

Proof. Choose $\ell_i < c_i/(\sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|))$ in Corollary 1, then $C - \bar{A} - \bar{B}$ is a nonsingular M-matrix. According to Corollary 1, Corollary 2 holds. □

Remark 3. If $c_i > \sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)$ $(i = 1, 2, \ldots, n)$, then DHNN (22) is globally exponentially stable (Zeng et al., 2003). But when $c_i \leq \sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)$ $(i = 1, 2, \ldots, n)$, the stability of DHNN (22) is an interesting open problem. Corollaries 1 and 2 above partially respond this problem.

The ensuing corollary shows that the results herein can be used in conjunction with existing ones if the conditions hold partially only.

Corollary 3. *DHNN (22) is globally exponentially stable, if there exist $\ell_i \leq 1$ $(i \in N_1)$ such that $C - \tilde{A} - \tilde{B}$ is a nonsingular M-matrix, and $\forall i \in N_1$,*

$$|u_i| > c_i(\ln 2 - (\ln \ell_i)/2) + \sum_{j=1}^{n} (|a_{ij}| + |b_{ij}|), \tag{27}$$

where $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$, $\tilde{B} = (\tilde{b}_{ij})_{n\times n}$,

$$\tilde{a}_{ij} = \begin{cases} |a_{ij}|\ell_j, & j \in N_1 \\ |a_{ij}|, & j \in N_2; \end{cases} \qquad \tilde{b}_{ij} = \begin{cases} |b_{ij}|\ell_j, & j \in N_1 \\ |b_{ij}|, & j \in N_2, \end{cases}$$

$N_1 \cup N_2 = \{1, 2, \ldots, n\}$, $N_1 \cap N_2$ is empty.

Proof. Choose $\ell_j = 1$ for all $i \in N_2$. Then when $i \in N_2$, (7) and (8) are satisfied for any pair of (x_i, y_i). This means that ξ_i, $i \in N_2$ in (20) can take a sufficiently small negative value. From this fact and (20), we can conclude that there is no restriction on u_i, $\forall i \in N_2$. According to the Main Corollary, similar to the proof of Corollary 1, DHNN (22) is globally exponentially stable. □

Remark 4. According to Corollary 3, it is possible to obtain new theoretical results by synthesizing the main theorem in this paper and the results of the existing ones (e.g., Chen et al. (2002)); i.e., if a part separated from input vector is sufficiently strong, the constraints of the corresponding parameters can be relaxed or be weakened.

5. CNN with time-varying delays

When $f(r) = f_j(r) = g_j(r) = (|r + 1| - |r - 1|)/2$ $(j = 1, 2, \ldots, n)$, the neural network (18) is a CNN with time-varying delays or delayed CNN (DCNN)

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f(x_j(t)) + \sum_{j=1}^{n} b_{ij} f(x_j(t - \tau_{ij}(t))) + u_i. \tag{28}$$

Corollary 4. *DCNN (28) is globally exponentially stable, if*

$$|u_i| > c_i + \sum_{j=1}^{n} (|a_{ij}| + |b_{ij}|). \tag{29}$$

Proof. Since $(f(x_i + v_i) - f(x_i))/v_i = 0$ hold for any pair of $(x_i + v_i, x_i) \in \Omega_1^+$ and $(x_i + v_i, x_i) \in \Omega_1^-$. The assumption of the main theorem is fulfilled with $\ell_i = \mu_i = 0$ and $\xi_i = 1$.

Substituting $\xi_i = 1$ and $\bar{f}_i = \bar{g}_i = 1$ into (20), we have (29). Hence, according to the Main Corollary, DCNN (28) is globally exponentially stable. □

Remark 5. In Arik (2002), Cao (2001a, 2001b), Cao and Zhou (1998), Jiang et al. (2003), Li et al. (2003), Liao and Wang (2003), Mohamad and Gopalsamy (2003), Yu et al. (2003) and Zhang et al. (2001), the stability of neural networks is restricted by their parameters c_i, a_{ij}, b_{ij}. For example, in Cao (2001a, 2001b), Cao and Zhou (1998) and Mohamad and Gopalsamy (2003), the stability of the networks requires that $C-|A|-|B|$ is a nonsingular M-matrix. But Corollary 3 indicates that if input u_i is strong enough, any other constraints on the parameters c_i, a_{ij}, b_{ij} of the networks can be relaxed.

Corollary 5. *DCNN (28) is globally exponentially stable, if $C - \tilde{A} - \tilde{B}$ is a nonsingular M-matrix, and*

$$i \in N_1, |u_i| > c_i + \sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|), \tag{30}$$

where $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$, $\tilde{B} = (\tilde{b}_{ij})_{n\times n}$,

$$\tilde{a}_{ij} = \begin{cases} 0, & j \in N_1 \\ |a_{ij}|, & j \in N_2; \end{cases} \qquad \tilde{b}_{ij} = \begin{cases} 0, & j \in N_1 \\ |b_{ij}|, & j \in N_2, \end{cases}$$

$N_1 \cup N_2 = \{1, 2, \ldots, n\}$, $N_1 \cap N_2$ *is empty.*

Proof. Since $C - \tilde{A} - \tilde{B}$ is a nonsingular M-matrix, there exist positive numbers $\gamma_1, \gamma_2, \ldots, \gamma_n$ such that for $i = 1, 2, \ldots, n$,

$$c_i\gamma_i > \sum_{j=1}^{n}\gamma_j(|a_{ij}| + |b_{ij}|)\ell_j, \tag{31}$$

where $\ell_i = 0, i \in N_1$ and $\ell_j = 1, j \in N_2$.

Let $z_i(t) = x_i(t)/\gamma_i$, then from (28),

$$\frac{\mathrm{d}z_i(t)}{\mathrm{d}t} = -c_iz_i(t) + \frac{1}{\gamma_i}\left(\sum_{j=1}^{n}(a_{ij}f(\gamma_jz_j(t)) + b_{ij}f(\gamma_jz_j(t-\tau_{ij}(t)))) + u_i\right). \tag{32}$$

When $j \in N_2$, for any pair of $(x_j + v_j, x_j) \in \Omega_1^+$ and $(x_j + v_j, x_j) \in \Omega_1^-, v_j \neq 0$,

$$-\ell_j = -1 \leq (f(x_j + v_j) - f(x_j))/v_j \leq 1 = \ell_j.$$

Choose $\xi_i = 1, i \in N_1$ and $\xi_j, (j \in N_2)$ be a sufficiently small negative such that

$$j \in N_2, \quad |u_j| \geq 0 > c_j\xi_j + \sum_{k=1}^{n}(|a_{jk}| + |b_{jk}|). \tag{33}$$

According to the Main Corollary, (30), (31) and (33) imply that (32) is globally exponentially stable; i.e., DCNN (28) is globally exponentially stable. □

Corollary 6. *Let $\tau_{ij}(t) \equiv \tau_j(t)$ $(i, j = 1, 2, \ldots, n)$. DCNN (28) is globally exponentially stable, if*

$$\forall i \in N_1, \quad |u_i| > c_i + \sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|), \tag{34}$$

$$\text{and } \lambda_A + \lambda_B < c_{\min}, \tag{35}$$

where $c_{\min} := \min_{i\in N_2} c_i$, λ_A and λ_B are respectively the maximum eigenvalues of matrices

$$\begin{pmatrix} 0 & A_{N_2} \\ A_{N_2}^{\mathrm{T}} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & B_{N_2} \\ B_{N_2}^{\mathrm{T}} & 0 \end{pmatrix},$$

and A_{N_2} and B_{N_2} are square matrices respectively made up of i-row j-line of matrices A and B, respectively, $(i, j \in N_2)$.

Proof. Let $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^{\mathrm{T}}$ be an equilibrium point of DCNN (28), $z_i(t) = x_i(t) - x_i^*$, $f^*(z_i(t)) := f(z_i(t) + x_i^*) - f(x_i^*)$ $(i = 1, 2, \ldots, n)$, $z(t) = (z_1(t), z_2(t), \ldots, z_n(t))^{\mathrm{T}}$, $f^*(z(t)) = (f^*(z_1(t)), f^*(z_2(t)), \ldots, f^*(z_n(t)))^{\mathrm{T}}$. From (34), there exists T, which only depends on the initial condition of DCNN (28), such that for $t > T + \tau, i = 1, 2, \ldots, n$, DCNN (28) is equivalent to

$$\frac{\mathrm{d}z_i(t)}{\mathrm{d}t} = -c_iz_i(t) + \sum_{j\in N_2}a_{ij}f^*(z_j(t)) + \sum_{j\in N_2}b_{ij}f^*(z_j(t-\tau_j(t))). \tag{36}$$

Consider a subsystem of (36),

$$\frac{\mathrm{d}z_k(t)}{\mathrm{d}t} = -c_kz_k(t) + \sum_{j\in N_2}(a_{kj}f^*(z_j(t)) + b_{kj}f^*(z_j(t-\tau_j(t)))), \quad \forall k \in N_2. \tag{37}$$

Let $V(t) = V(z(t)) = \sum_{i\in N_2} z_i^2(t)$. Then

$$\begin{aligned}
\left.\frac{\mathrm{d}V(t)}{\mathrm{d}t}\right|_{(37)} &= 2\sum_{i\in N_2} z_i(t)\frac{\mathrm{d}z_i(t)}{\mathrm{d}t} \\
&= -2\sum_{i\in N_2} c_iz_i^2(t) + 2\sum_{i\in N_2} z_i(t) \times \sum_{j\in N_2}(a_{ij}f^*(z_j(t)) + b_{ij}f^*(z_j(t-\tau_j(t)))) \\
&= -2\sum_{i\in N_2} c_iz_i^2(t) + z(t)^{\mathrm{T}}\begin{pmatrix} 0 & A_{N_2} \\ A_{N_2}^{\mathrm{T}} & 0 \end{pmatrix} f^*(z(t)) + z(t)^{\mathrm{T}}\begin{pmatrix} 0 & B_{N_2} \\ B_{N_2}^{\mathrm{T}} & 0 \end{pmatrix} f^*(z(t-\tau(t))) \\
&\leq -2\sum_{i\in N_2} c_iz_i^2(t) + \lambda_A\left(\sum_{i\in N_2}(z_i^2(t) + f^*(z_i(t)))\right) + \lambda_B\left(\sum_{i\in N_2}(z_i^2(t) + f^*(z_i(t-\tau_i(t))))\right) \\
&\leq (-2c_{\min} + 2\lambda_A + \lambda_B)V(z(t)) + \lambda_BV(z(t-\tau(t))).
\end{aligned}$$

By (35), there exists $\theta > 0$ such that

$$-2c_{\min} + 2\lambda_A + \lambda_B + \theta + \lambda_B \exp\{\theta\tau\} \le 0.$$

Similar to the last part of the proof of the Main Theorem, $\sum_{i\in N_2} z_i^2(t) \le Z(T)\exp\{-\theta(t-T-\tau)\}$, where $z(T) = \max\{\max_{t_0-\tau\le s\le T+\tau} |z_i(s)|, i \in N_2\}$. It implies that system (37) is globally exponentially stable.

Now, consider a subsystem of (36),

$$\frac{\mathrm{d}z_i(t)}{\mathrm{d}t} = -c_i z_i(t) + \sum_{j\in N_2} (a_{ij} f^*(z_j(t)) + b_{ij} f^*(z_j(t-\tau_j(t)))), \quad i \in N_1. \tag{38}$$

Using the variations of constants formula, for $t > T + \tau$, the solution of (38) satisfies

$$z_i(t) = z_i(T+\tau)\exp\{-c_i(t-T-\tau)\} + \int_{T+\tau}^{t} \exp\{-c_i(t-s)\}\left\{\sum_{j\in N_2} (a_{ij} f^*(z_j(s)) + b_{ij} f^*(z_j(s-\tau_j(s))))\right\}\mathrm{d}s, \quad i \in N_1.$$

Since (37) is globally exponentially stable, there exist γ and $\beta > 0, \gamma \le \min\{c_i, i \in N_2\}$ such that for $t > T$,

$$|z_j(t)| \le \beta\exp\{-\gamma(t-T)\}, \quad j \in N_2.$$

From the definition of $f^*(\cdot)$,

$$|f^*(z_j(t))| \le |z_j(t)|.$$

Hence, for $t > T + \tau$,

$$\begin{aligned}
&\left|\sum_{j\in N_2} (a_{ij} f^*(z_j(t)) + b_{ij} f^*(z_j(t-\tau_j(t))))\right| \\
&\le \sum_{j\in N_2} (|a_{ij}||z_j(t)| + |b_{ij}||z_j(t-\tau_j(t))|) \\
&\le \beta \sum_{j\in N_2} (|a_{ij}| + |b_{ij}|\exp\{\gamma\tau\})\exp\{-\gamma(t-T)\} \\
&\le \Lambda\exp\{-\gamma(t-T)\}, \quad i \in N_1,
\end{aligned}$$

where $\Lambda = \max_{i\in N_1}\{\beta\sum_{j\in N_2}(|a_{ij}| + |b_{ij}|\exp\{\gamma\tau\})\}$. Thus for $i \in N_1, t > T + \tau$,

$$\begin{aligned}
&\left|\int_{T+\tau}^{t} \exp\{-c_i(t-s)\}\left\{\sum_{j\in N_2} (a_{ij} f^*(z_j(s)) + b_{ij} f^*(z_j(s-\tau_j(s))))\right\}\mathrm{d}s\right| \\
&\le \Lambda\int_{T+\tau}^{t} \exp\{-c_i(t-s)\}\exp\{-\gamma(s-T)\}\mathrm{d}s \\
&= \exp\{-\gamma(t-T)\}\int_{T+\tau}^{t} \Lambda\exp\{-(c_i-\gamma)(t-s)\}\mathrm{d}s.
\end{aligned}$$

It implies that the zero solution of (38) is globally exponentially stable. Hence DCNN (28) is globally exponentially stable. □

Remark 6. The method used in Corollaries 5 and 6 can be extended to obtain new stability results by synthesizing the main theorem in this paper and other existing results (e.g., Li et al. (2003)).

6. Illustrative examples

In this section, we give two numerical examples to illustrate the new results.

Example 1. Consider a DHNN in an n-dimensional array of neurons:

$$\begin{cases}
\dot{x}_1(t) = -x_1(t) + f(x_1(t-1)) \\
\quad + f(x_2(t-2)) + (3\ln 2)/2 + 2.1 \\
\dot{x}_2(t) = -x_2(t) + f(x_2(t-1)) \\
\quad + f(x_3(t-2)) + (3\ln 2)/2 + 2.1 \\
\vdots \\
\dot{x}_{n-1}(t) = -x_{n-1}(t) + f(x_{n-1}(t-1)) \\
\quad + f(x_n(t-1)) + (3\ln 2)/2 + 2.1 \\
\dot{x}_n(t) = -x_n(t) + f(x_n(t-1)) \\
\quad + f(x_1(t-2)) + (3\ln 2)/2 + 2.1,
\end{cases} \tag{39}$$

where $f(x(t)) = (\exp(x(t)) - \exp(-x(t)))/(\exp(x(t)) + \exp(-x(t)))$.

Since $C = I$, $A = 0$, and

$$B = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

which is a circular matrix, $C - |A| - |B|$ is not a nonsingular M-matrix, the conditions in Cao and Wang (2003), Chen et al. (2002) and Zeng et al. (2003) cannot be used to ascertain the stability of DHNN (39). But for $i = 1, 2, \ldots, n$,

$$\begin{aligned}
u_i &= (3\ln 2)/2 + 2.1 > (3\ln 2)/2 + 2 \\
&= c_i\left(\ln 2 - \frac{1}{2}\left(\ln c_i - \ln\left(\sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)\right)\right)\right) \\
&\quad + \sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|).
\end{aligned}$$

According to Corollary 2, DHNN (39) is globally exponentially stable. Specifically, when $n = 2$, simulation results with several random initial points are depicted in Figs. 1–3.

Example 2. Consider a DCNN in a $2q$-dimensional array of neurons:

$$\begin{cases}
\dot{x}_1(t) = -x_1(t) + g(x_1(t-1)) + g(x_2(t-2)) + 3.1 \\
\dot{x}_2(t) = -x_2(t) + g(x_2(t-1))/3 + g(x_3(t-2))/3 \\
\vdots \\
\dot{x}_{2q-1}(t) = -x_{2q-1}(t) + g(x_{2q-1}(t-1)) \\
\quad + g(x_{2q}(t-2)) + 3.1 \\
\dot{x}_{2q}(t) = -x_{2q}(t) + g(x_{2q}(t-1))/3 + g(x_1(t-2))/3,
\end{cases} \tag{40}$$

where $g(x(t)) = (|x(t)+1| - |x(t)-1|)/2$.

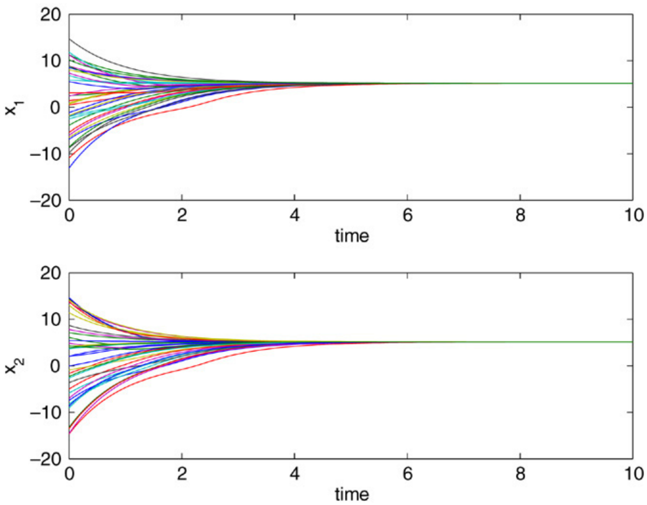

Fig. 1. Transient behavior of the DHNN in Example 1.

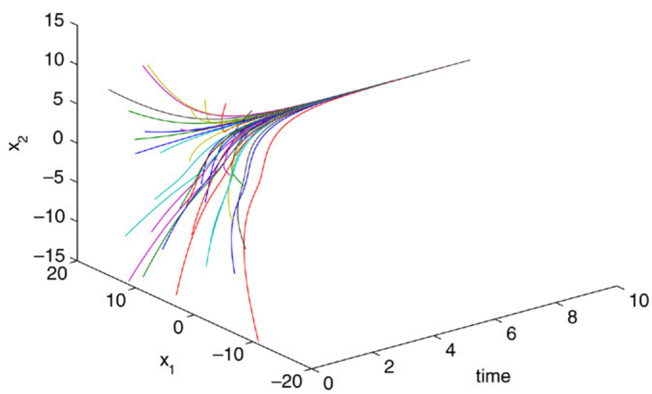

Fig. 2. Transient behavior (3D view) of the DHNN in Example 1.

Since $C = I$, $A = 0$, and

$$B = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1/3 & 1/3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 1/3 & 0 & 0 & \cdots & 0 & 1/3 \end{pmatrix},$$

$C - |A| - |B|$ is not a nonsingular M-matrix, the conditions in Cao and Wang (2003), Mohamad and Gopalsamy (2003) and Zhang et al. (2001) cannot be used to ascertain the stability of DCNN (40). But choose $N_1 = \{1, 3, \ldots, 2q-1\}$, $N_2 = \{2, 4, \ldots, 2q\}$, then for $\forall i \in N_1$,

$$u_i = 3.1 > 3 = c_i + \sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|).$$

Moreover, $C - \tilde{A} - \tilde{B}$ (i.e., $I - \tilde{B}$) is a nonsingular M-matrix, where $\tilde{A}$ and $\tilde{B}$ are defined in Corollary 5. According to Corollary 5, DCNN (40) is globally exponentially stable. Specifically, when $q = 1$, simulation results are depicted in Figs. 4–6. All the trajectories from several random initial points converge to the equilibrium point $(4.6, 0.5)^{\mathrm{T}}$.

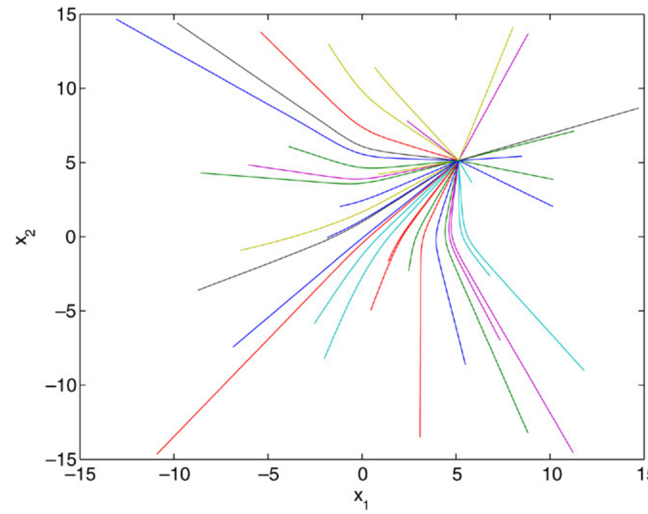

Fig. 3. State phase plot of the DHNN in Example 1.

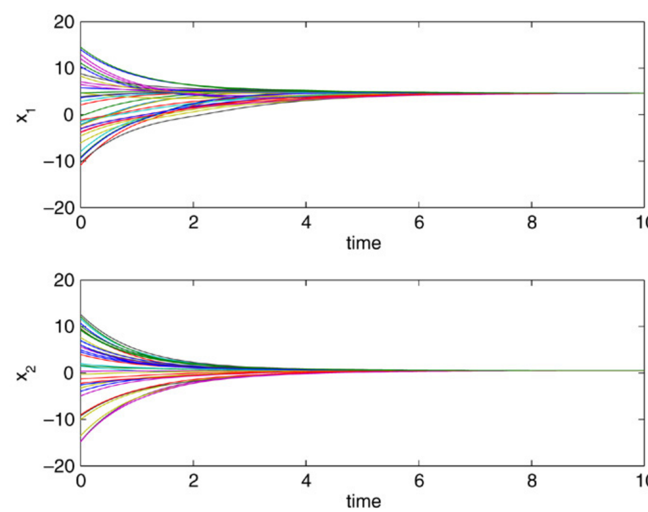

Fig. 4. Transient behavior of the DCNN with strong external input in Example 2.

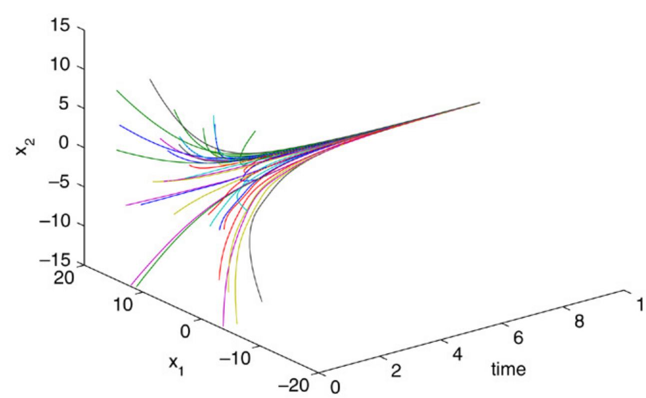

Fig. 5. Transient behavior (3D view) of the DCNN with strong external input in Example 2.

Now consider the DCNN without any external stimuli. There exist three equilibria $(0, 0, \ldots, 0)^{\mathrm{T}}$, $(3/2, 1/2, 3/2, 1/2,$

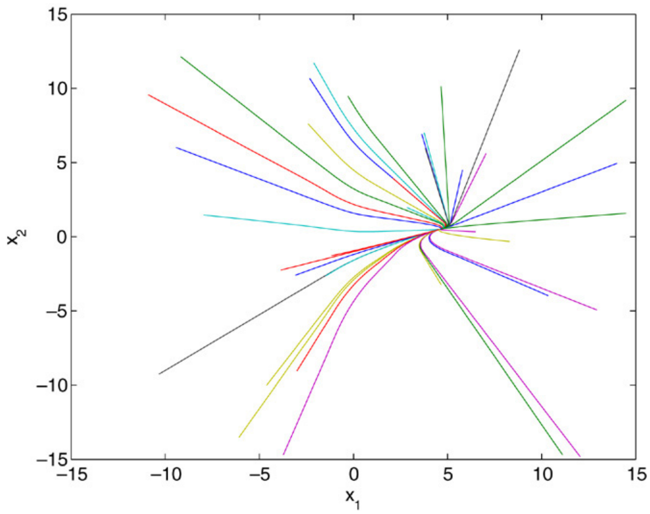

Fig. 6. State phase plot of the DCNN with strong external input in Example 2.

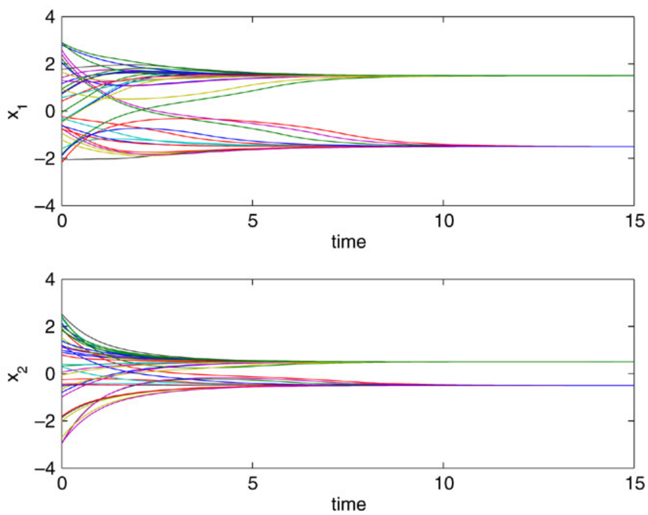

Fig. 7. Transient behavior of the DCNN without external input in Example 2.

$\ldots, 3/2, 1/2)^{\mathrm{T}}$ and $(-3/2, -1/2, -3/2, -1/2, \ldots, -3/2, -1/2)^{\mathrm{T}}$ for DCNN (40), where the origin is unstable. Hence, it is impossible for it to be globally stable. Specifically, when $q = 1$, simulation results are depicted in Figs. 7–9. The trajectories converge to the equilibrium points $(3/2, 1/2)^{\mathrm{T}}$ or $(-3/2, -1/2)^{\mathrm{T}}$ depending on initial points.

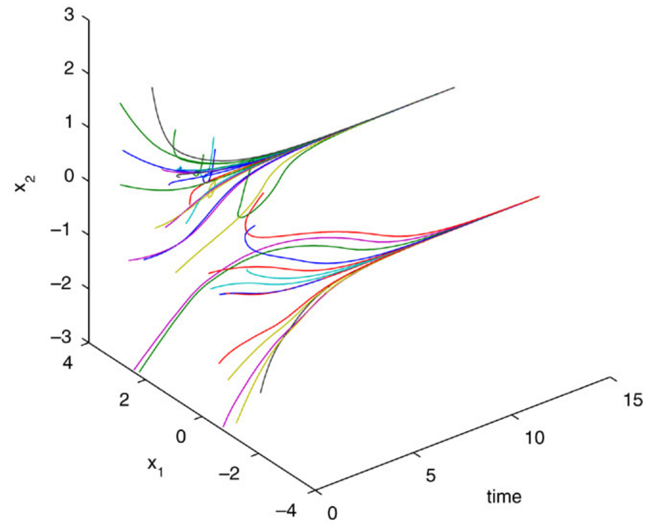

Fig. 8. Transient behavior (3D view) of the DCNN without external input in Example 2.

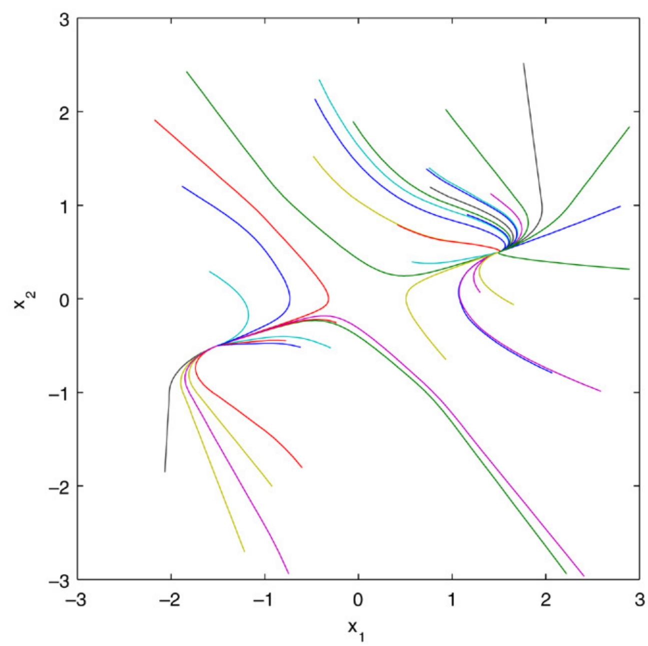

Fig. 9. State phase plot of the DCNN without external input in Example 2.

7. Concluding remarks

Stability analysis of neural networks is an important topic in neural network research. In this paper, we present new theoretical results on the global exponential stability of a general class of recurrent neural networks with bounded activation functions and bounded time-varying delays in the presence of strong external stimuli. It is shown that global exponential stability of such neural networks can be achieved when the external stimuli are sufficiently strong, without the need for other conditions. A sufficient condition on the bounds of stimuli is derived for global exponential stability of general recurrent neural networks. Specific criteria are deduced for the continuous-time Hopfield neural network with time-varying delays and cellular neural networks with time-varying delays. It is also shown that the present global exponential stability criteria can be used in conjunction with existing ones when they are partially satisfied. Simulation results illustrate the uses of the criteria to ascertain the global exponential stability of specific neural networks. The results in this paper are useful for both analysis and design of recurrent neural networks. From the quantitative analysis point of view, the input-dependent criteria for ascertaining global exponential stability are more reasonable as they contain more complete information about the neural networks. From the model design point of view, the stability conditions herein can serve as a guideline for system synthesis in specific applications.

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