STOCKHOLM UNIVERSITY
Department of Statistics

Michael Carlson & Emma Pettersson
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Solutions to Plenary Exercises: Plenary Exercise 2

Basic Statistics for Economists, 15 ECTS, STE101

EXERCISE 1

Define the event "light bulb i stays lit for more than 1000 hours" with A_i , $i = 1,2,3$ and $P(A_i) = 0.9$.

Assume that the three events A_1, A_2, A_3 are independent of each other. This implies that $P(A_i \cap A_j) = P(A_i) \cdot P(A_j)$ for $i \neq j$, and also, that $P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2) \cdot P(A_3)$ and for any combination with complements we simply multiply the marginal probabilities, e.g. $P(\overline{A_1} \cap A_2 \cap \overline{A_3}) = P(\overline{A_1}) \cdot P(A_2) \cdot P(\overline{A_3})$ and so on.

As an alternative, you can define $\boldsymbol{X}$ *as the number of light bulbs (out of 3) that stay lit for more than 1000 hours. Then* $X \sim Bin(n = 3, p = 0.9)$.

To use of the binominal distribution table in the formula sheet, we have to define another random variable that is the complement of X. *Let* Y = *number of light bulbs that stay lit for* ***less*** *than 1000 hours,* $Y \sim Bin(n = 3, P^* = 0.1)$ *where:* $Y = n - X = 3 - X$.

a. None of the light bulbs stay lit; three different methods you can use:

$$P(\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}) = P(\overline{A_1}) \times P(\overline{A_2}) \times P(\overline{A_3}) = 0.1^3 = [1 - P(A_i)^3] = \mathbf{0.001}$$

Binomial formula: $P(X = 0) = \binom{3}{0} 0.9^0 (1 - 0.9)^{3-0} = 1 \times 1 \times 0.1^3 = 0.001$

Binomial Table 7: $P(X = 0) = P(Y = 3) = 1 - P(Y \leq 2) = 1 - 0.999 = 0.001$

b. Exactly one lamp stays lit; three methods you can use:

Three different configurations where exactly three are lit and one is not:

$$P\big((A_1 \cap \overline{A_2} \cap \overline{A_3}) \cup (\overline{A_1} \cap A_2 \cap \overline{A_3}) \cup (\overline{A_1} \cap \overline{A_2} \cap A_3)\big) = [\text{disjoint so add}]$$

$$= P(A_1)P(\overline{A_2})P(\overline{A_3}) + P(\overline{A_1})P(A_2)P(\overline{A_3}) + P(\overline{A_1})P(\overline{A_2})P(A_3)$$

$$= (0.9)(0.1)(0.1) + (0.1)(0.9)(0.1) + (0.1)(0.1)(0.9) = 3(0.1)(0.1)(0.9) = \mathbf{0.027}$$

Binomial formula: $P(X = 1) = \binom{3}{1} 0.9^1 (1 - 0.9)^{3-2} = 3 \times 0.9 \times 0.1^2 = \mathbf{0.027}$

Binomial Table 7: $P(\boldsymbol{X} = 1) = P(\boldsymbol{Y} = 2) = P(Y \leq 2) - P(Y \leq 1) = [\text{Table 7}] = 0.999 - 0.972 = \mathbf{0.027}$

c. At least two lamps stay lit; three methods you can use:

$$P(at\ least\ 2) = P(exactly\ 2\ \cup\ exactly\ 3) = 1 - P(exactly\ 0\ \cup exactly\ 1)$$
$$= 1 - [\ P(exactly\ 0) + P(exactly\ 1)] = 1 - [0.001 + 0.027] = \mathbf{0.972}$$

$$P(X \geq 2) = 1 - P(X \leq 1) = 1 - P(X = 0) - P(X = 1) = 1 - 0.001 - 0.027 = \mathbf{0.972}$$

$$P(X \geq 2) = P(Y < 2) = P(Y \leq 1) = [\text{Table 7}] = \mathbf{0.972}$$

EXERCISE 2

Assume that the demand for A and B are independent of each other so that we can use the multiplication rule. If we do not assume the demands for A and B are independent, we won't be able to solve the problem since we don't have enough information about the conditional or joint probabilities.

a. When no products are demanded, $X = 0$ **and** $Y = 0$.

$$P(X = 0 \cap Y = 0) = P(X = 0) \times P(Y = 0) = 0.05 \times 0.1 = \mathbf{0.005}$$

b. We are interested in finding the probability that $X \geq 3$ **and** that $Y \leq 4$.

$$P(X \geq 3 \cap Y \leq 4) = P(X \geq 3) \times P(Y \leq 4)$$

x	0	1	2	3	4	5	6
$P_x(x)$	0.05	0.07	0.20	0.25	0.30	0.10	0.03

$$P(X \geq 3) = P(X = 3) + \cdots + P(X = 6) = 0.25 + \cdots + 0.03 = \mathbf{0.68}$$

y	0	1	2	3	4	5	6
$P_y(y)$	0.10	0.12	0.15	0.20	0.25	0.10	0.08

$$P(Y \leq 4) = P(Y = 0) + \cdots + P(Y = 4) = 0.1 + \cdots + 0.25 = \mathbf{0.82}$$

$$P(X \geq 3 \cap Y \leq 4) = P(X \geq 3) \times P(Y \leq 4) = 0.68 \times 0.82 = \mathbf{0.5576}$$

c. Using the same method that we did in b. we want to find the probability that $X \leq 3$ and $Y \geq 3$.

$$P(X \leq 3 \cap Y \geq 3) = P(X \leq 3) \times P(Y \geq 3) = 0.57 \times 0.63 = \mathbf{0.3591}$$

d. We want to know the probability that exactly three items are demanded and that they are the same kind of item. This can occur when either $X = 3$ and $Y = 0$, **or** when $X = 0$ and $Y = 3$. Thus, we are interested in finding:

$$P(\ (X = 3 \cap Y = 0) \cup (X = 0 \cap Y = 3)\ )$$

Remember that the events $(X = 3 \cap Y = 0)$ and $(X = 0 \cap Y = 3)$ are disjoint. This means that:

$$P(\,(X = 3 \cap Y = 0) \cup (X = 0 \cap Y = 3)\,) = P(X = 3 \cap Y = 0) + P(X = 0 \cap Y = 3)$$

Because X and Y are independent we get that:

$$\begin{aligned}&P(\,(X = 3 \cap Y = 0) \cup (X = 0 \cap Y = 3)\,)\\&= (P(X = 3) \times P(Y = 0)) + (P(X = 0) \times P(Y = 3))\\&= (0.25)(0.1) + (0.05)(0.2) = \mathbf{0.035}\end{aligned}$$

e. We can find the mean μ_X and the variance σ_X^2, by setting a up a table and doing the following calculations.

x	0	1	2	3	4	5	6	Sum
$P_x(x)$	0.05	0.07	0.20	0.25	0.30	0.10	0.03	1
$xP_x(x)$	0	0.07	0.4	0.75	1.2	0.5	0.18	$\mu_X = 3.1$
$x - \mu$	−3.1	−2.1	−1.1	−0.1	0.9	1.9	2.9	
$(x - \mu)^2$	9.61	4.41	1.21	0.01	0.81	3.61	8.41	
$(x - \mu)^2 P_x(x)$	0.4805	0.3087	0.242	0.0025	0.243	0.361	0.2523	$\sigma_X^2 = 1.89$

We get the mean $\boldsymbol{\mu_X = 3.1}$, the variance $\boldsymbol{\sigma_X^2 = 1.89}$, and the standard deviation $\boldsymbol{\sigma_X = \sqrt{1.89} = 1.3738}$.

f. If the demand for A and B were independent of each other, the correlation between X and Y would be zero.
If they are dependent, the correlation can also be zero. The correlation coefficient measures the linear relationship between two variables, it is possible that a non-linear relationship between the variables exists despite the correlation being zero.

EXERCISE 3

Define the random variable X as the firms profit which has the sample space $S_X = \{-60,180\}$ thousand kronor.

$$P(-60) = 0.2 \quad ; \quad P(180) = 0.8$$

a. Mean: $\mu_X = E(X) = \sum_{x \in S_X} xP(x) = (-60)(0.2) + (180)(0.8) = 132$ thousand kronor.
Since the expected profit is positive, the company should take the job.

b. Now let $P(X=-60)=1-P$; $P(X=180)=P$

$$\mu_X = E(X) = \sum_{x\in S_X} xP(x) = (-60)(1-P)+(180)(P) = 240P-60$$

We set the above equation equal to zero since our expected profit should equal zero, then we solve for P.

$$E(X) = 240P-60 = 0 \Leftrightarrow 240P = 60 \Leftrightarrow \boldsymbol{P = 0.25}$$

EXERCISE 4

Define $X_1 =$ no. of correct answers on the first exam and $X_2 =$ no. of correct answers on the second exam. Each question has 4 alternatives of which only 1 is correct so by simply guessing there is a $P = 1/4 = 0.25$ probability of guessing the correct answer.

Guessing the right answer (i.e. choosing truly randomly), and assuming that the different questions will be independent of each other we have $X_1 \sim Bin(10, 0.25)$ and $X_2 \sim Bin(10, 0.25)$. Furthermore, it follows that X_1 and X_2 are independent of each other.

a. The probability of passing the first exam exam is

$$P(X_1 \geq 4) = 1 - P(X_1 \leq 3) = [\text{Table } 7, n = 10, p = 0.25] = 1 - 0.77588 = 0.22412$$

The probability of passing the second exam $P(X_2 \geq 4)$ is the same. The joint probability of passing both exams is then (due to independence):

$$P(\text{pass } 1^{\text{st}} \cap \text{pass } 2^{\text{nd}}) = P(X_1 \geq 4) \cdot P(X_2 \geq 4) = 0.22412 \cdot 0.22412 \approx 0.05023$$

However, to pass the course the total score has to be at least 10. By arranging the possible outcomes for X_1 and X_2 and the total score, we have the following table:

Total score					pass 2^{nd}						
	0	1	2	3	4	5	6	7	8	9	10
0	0	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10	11
2	2	3	4	5	6	7	8	9	10	11	12
3	3	4	5	6	7	8	9	10	11	12	13
pass 1^{st} 4	4	5	6	7	8	9	10	11	12	13	14
5	5	6	7	8	9	10	11	12	13	14	15
6	6	7	8	9	10	11	12	13	14	15	16
7	7	8	9	10	11	12	13	14	15	16	17
8	8	9	10	11	12	13	14	15	16	17	18
9	9	10	11	12	13	14	15	16	17	18	19
10	10	11	12	13	14	15	16	17	18	19	20

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We see that we have to subtract the probabilities of the three combined outcomes where the total is less than 10 (marked in orange).

$$P(\text{pass course})$$

$$= P(\text{pass } 1^{\text{st}} \cap \text{pass } 2^{\text{nd}}) - P(X_1 = 4 \cap X_2 = 4) - P(X_1 = 5 \cap X_2 = 4) - P(X_1 = 4 \cap X_2 = 5)$$

$$= [X_1 \text{ and } X_2 \text{ are independent}] =$$

$$= 0.05023 - [P(X_1 = 4) \cdot P(X_2 = 4)] - [P(X_1 = 5) \cdot P(X_2 = 4)] - [P(X_1 = 4) \cdot P(X_2 = 5)]$$

From Table 7 we have (for $k = 1$ or 2, and using $n = 10$ and $p = 0.25$)

$$P(X_k = 4) = P(X_k \leq 4) - P(X_k \leq 3) = 0.92187 - 0.77588 = 0.14599$$

$$P(X_k = 5) = P(X_k \leq 5) - P(X_k \leq 4) = 0.98027 - 0.92187 = 0.05840$$

Continuing,

$$P(pass\ course)$$

$$= 0.05023 - (0.14599 \cdot 0.14599) - (0.05840 \cdot 0.14599) - (0.14599 \cdot 0.05840) = \mathbf{0.01186}$$

b. First we realize that the event “pass course” is a subset of the event “pass exam 1”; a student cannot pass the course without passing the first exam. This means that the intersection of “pass course” and “pass exam 1” is equal to “pass course”:

$$\text{"pass course"} \cap \text{"pass exam 1"} = \text{"pass course"}$$

So,

$$P(\text{pass course}|\text{pass exam 1}) = \frac{P(\text{pass course} \cap \text{pass exam 1})}{P(\text{pass exam 1})}$$

$$= \frac{P(\text{pass course})}{P(\text{pass exam 1})} = \frac{P(\text{pass course})}{P(X_1 \geq 4)} = [\text{from a.}] = \frac{0.01186}{0.22412} = \mathbf{0.05294}$$

EXERCISE 6

Define the random variable X as the value of a house where $X \sim N(1\,400\,000; 800\,000^2)$.

$$P(X > 2\,000\,000) = [\text{standardize}] = P\left(Z > \frac{2\,000\,000 - 1\,400\,000}{800\,000}\right) = P(Z > 0.75)$$

$$= 1 - P(Z \leq 0.75) = [\text{Table 1}] = 1 - 0.77337 = 0.2263 \approx \mathbf{0.227}$$

We can also redefine X so we don’t need to deal with so many zero’s. Let the random variable Y represent the value of a house in millions of kronor.

$$Y = \frac{X}{1\,000\,000} \quad \Longrightarrow \quad \mu_Y = \frac{\mu_X}{1\,000\,000} = 1.4 \quad \Longrightarrow \quad \sigma_Y = \sqrt{\frac{\sigma_X^2}{1\,000\,000}} = 0.8$$

We get that $Y \sim N(1{,}4\ 0.8^2)$.

$$P(Y > 2) = P\left(Z > \frac{2 - 1.4}{0.8}\right) = P(Z > 0.75)$$

EXERCISE 7

Let the random variable X be the weight in grams and $X \sim N(\mu, 10^2)$.

We want to find a value of μ so that the probability that the weight is less than 300 is at most 5%.

$$P(X < 300) = [\text{standardize}] = P\left(Z < \frac{300 - \mu}{10}\right) \leq 0.05$$

Look at Table 2 in the formula sheet. There we see that $P(Z > 1.6449) = 0.05$, and by the symmetry of the normal distribution we also know that $P(Z < -1.6449) = 0.05$. Thus, we have that:

$$P\left(Z < \frac{300 - \mu}{10}\right) \leq P(Z < -1.6449) = 0.05$$

Now we solve for μ:

$$\frac{300 - \mu}{10} \leq -1.6449 \quad \Leftrightarrow \quad \mu \geq 300 + 16.449 \quad \Leftrightarrow \quad \boldsymbol{\mu \geq 316.449}$$