L3

Basic Statistics for Economists

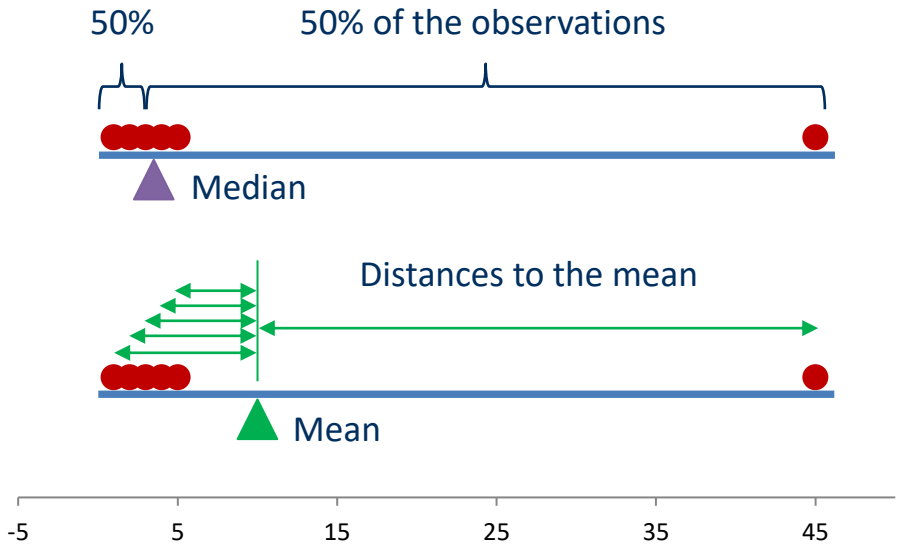
Spring 2020

Department of Statistics

Stockholm University

Short note on means and medians

- Dataset: $(\mathbf{1},\ \mathbf{2},\ \mathbf{3},\ \mathbf{4},\ \mathbf{5},\ \mathbf{45})$
- Median: $\boldsymbol{md} = \mathbf{3.5}$ Mean: $\bar{\boldsymbol{x}} = \mathbf{10}$



Equilibrium in **frequencies**

$50\% - 50\% = 0$

Equilibrium in **distances**

$\sum_{i=1}^{n}(x_i - \bar{x}) = 0$

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Today – two things

- **Descriptive statistics for two or more variables**
  - Tables, frequency- and magnitude tables
  - Graphical, diagrams
  - Covariation, measuring relationship between two variables
- **Introduction to Probability Theory**
  - Short intro about Models
  - Random experiments (sv. *slumpförsök*)
  - Sample space, outcome space (*utfallsrum*)
  - Events, outcomes (*händelser*)
  - A little **Set Theory** (*mängdlära*) and Venn diagrams
  - Definitions and interpretations of probability
  - Intro to **Combinatorics**

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More than one variable

Why study more than one variable at the same time?

- **Analyzing relationships**
  - Is there evidence in the data to support the idea that different variables/phenomena affect each other?
  - Can we utilize the relationship between variables to predict the value of one variable given the others?

- Study **cross tabulations** and different **graphs/diagrams**
- Also, numerical measures of relationship
  - **covariances** and **correlations**

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Frequency tables

- ***Cross tabulations***, ”crossing” two or more variables
- Absolute frequencies or relative frequencies (%)
- However, … now we can choose:

  - ***simultaneous*** or ***joint*** relative frequencies

  - ***conditional*** relative frequencies (sv. *betingade*)

- With conditional relative frequencies it is often easier to do comparisons between groups and discover relation-ships and dependencies, e.g. if two groups differ in some way

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Two variables in one table

Joint frequency distribution

No. of employees per income class and department		Department			Total
		A	B	C	
Income	0 – 500’	18	4	2	24
	500’ –	2	1	-	3
	Total	20	5	2	27

Bivariate distribution

The marginals show the univariate distributions

Joint relative frequency distribution

% employees per income class and department		Department			Total
		A	B	C	
Income	0 – 500’	67 %	15 %	7 %	89 %
	500’ –	7 %	4 %	-	11 %
	Total	74 %	19 %	7 %	100 %

Joint distribution – every combination dep./income

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Conditional distributions (sv. *betingade*)

Rows

Distribution over department per income class

		Department			Total
		A	B	C	
Income	0 – 500’	75 %	17 %	8 %	100 %
	500’ –	67 %	33 %	-	100 %
	Total	74 %	19 %	7 %	100 %

Conditioning on income class

Columns

Distribution over income class per department

		Department			Total
		A	B	C	
Income	0 – 500’	90 %	80 %	100 %	89 %
	500’ –	10 %	20 %	-	11 %
	Total	100 %	100 %	100 %	100 %

Conditioning on department

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Combining categorical and numerical

- **Magnitude tables** (SCB)
- Cross two (or more) categorical ***background variables***
- In each cell/combination a third numerical ***response variable***
- The response variable can be displayed as a mean, a total, proportion etc. …

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Example: Magnitude tables

Rows for categorical variable 1, columns for categorical variable 2 and an ***aggregate*** (e.g. sum) of a **third** numerical variable in each cell/combination

Net turnover, million SEK		NACE Code			Total
		1	2	3	
No. of employees	0 – 19	64 209	12 518	4 414	81 141
	20 – 49	3 071	11 679	2 257	17 007
	50 – 99	1 391	9 677	1 994	13 062
	100 – 249	4 967	14 022	2 941	21 930
	250 –	26 916	83 009	2 999	112 924
	Total	100 554	130 905	14 605	246 064

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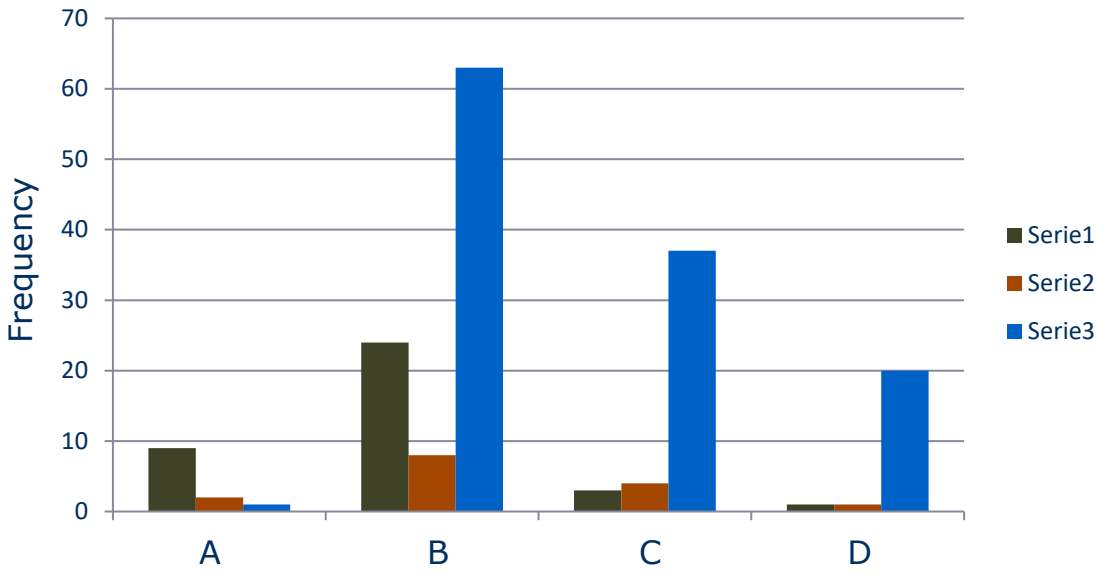
Graphical presentation multivariate data

- Grouped bars – absolute frequencies per group



Here, two categorical variables. Graph shows the frequency of each combination.

What about relative frequencies instead?

Categorical variable – nominal or ***ordinal***
Discrete numerical variable – ratio or ***interval***

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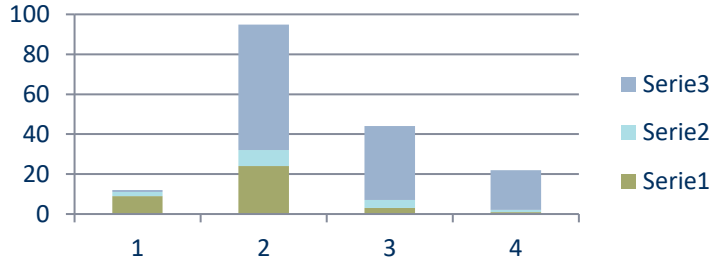
Multivariate data, cont.

- Stacked bars (categorical data or discrete numerical)

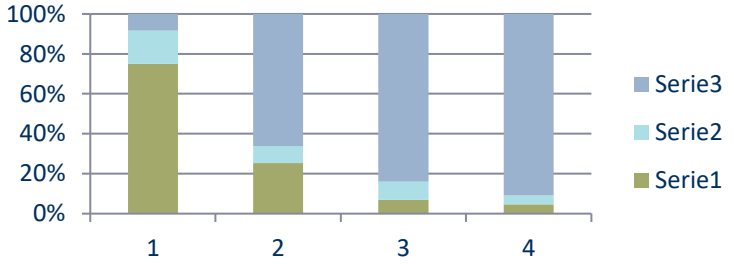
Absolute

frequencies per group (joint distribution)



Relative

frequencies per group **conditional** on groups



Categorical variable – nominal or ***ordinal***
Discrete numerical variable – ratio or ***interval***

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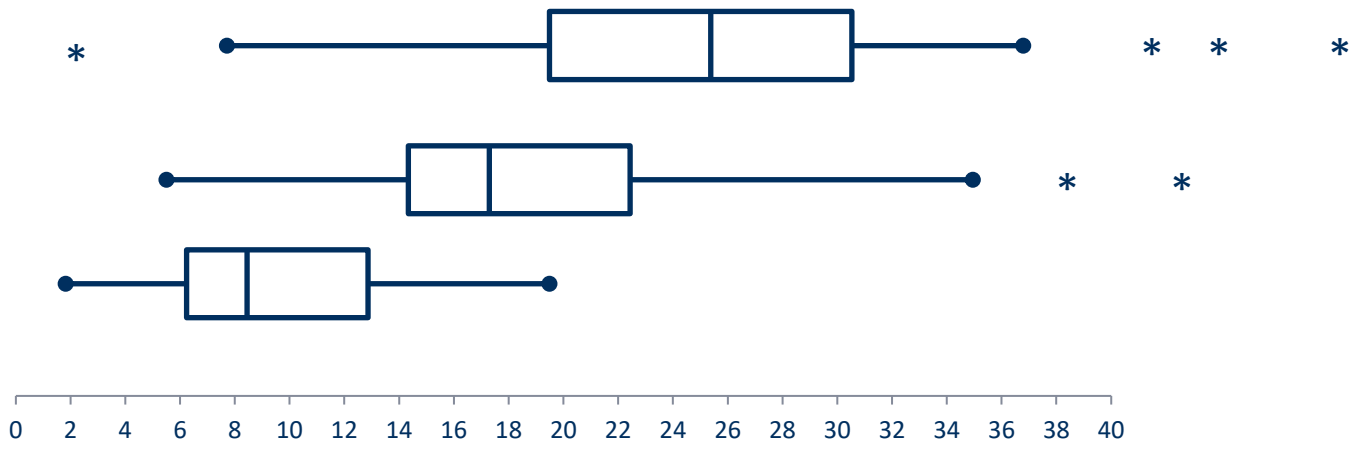
Numerical: comparing distributions

- Box plots again
- Three different groups



Continuous (or discrete) numerical variable – ratio or ***interval***

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Measuring relationship: Covariance

- Two numerical variables $\boldsymbol{x}$ and $\boldsymbol{y}$
- **Sample covariance** for $\boldsymbol{x}$ and $\boldsymbol{y}$:

Compare with the variance formulas

$$\boldsymbol{Cov(x,y)} = \boldsymbol{s_{xy}} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{n-1} = \frac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{n-1}$$

- Covariance = measure of covariation, ***linear*** relationships
- May be negative or positive or zero
- Not entirely easy to interpret the value one gets
- Q: How do we calculate the **population covariance** σ_{xy}?

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Correlation

- Measure of the ***linear*** relationship between $\boldsymbol{x}$ and $\boldsymbol{y}$
- Sample correlation coefficient for $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$\boldsymbol{Corr(x,y) = r_{xy} = \frac{S_{xy}}{S_x \cdot S_y} = \frac{S_{xy}}{\sqrt{S_x^2 \cdot S_y^2}}}$$

- Population correlation: $\boldsymbol{\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}}$ *("rho", or sv. "rhå")*
- Actually: *product moment correlation coefficient*
- **Important! Correlation is always in the interval [-1,1]**

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Covariances and correlations

- ***Covariance*** can in principal take on any value (-∞,∞)
  - Large absolute values suggest strong ***linear*** relationship
  - Scale dependent (e.g. different value if SEK, Euro or USD)
  - Difficult to compare covariances directly
- ***Correlation*** re-scales the covariance to **[-1,1]**
  - Scale independent, easier to compare values directly
  - Absolute values close to 1, i.e. -1 or +1, suggest strong ***linear*** relationships
  - Values close to zero suggest no linear relationship

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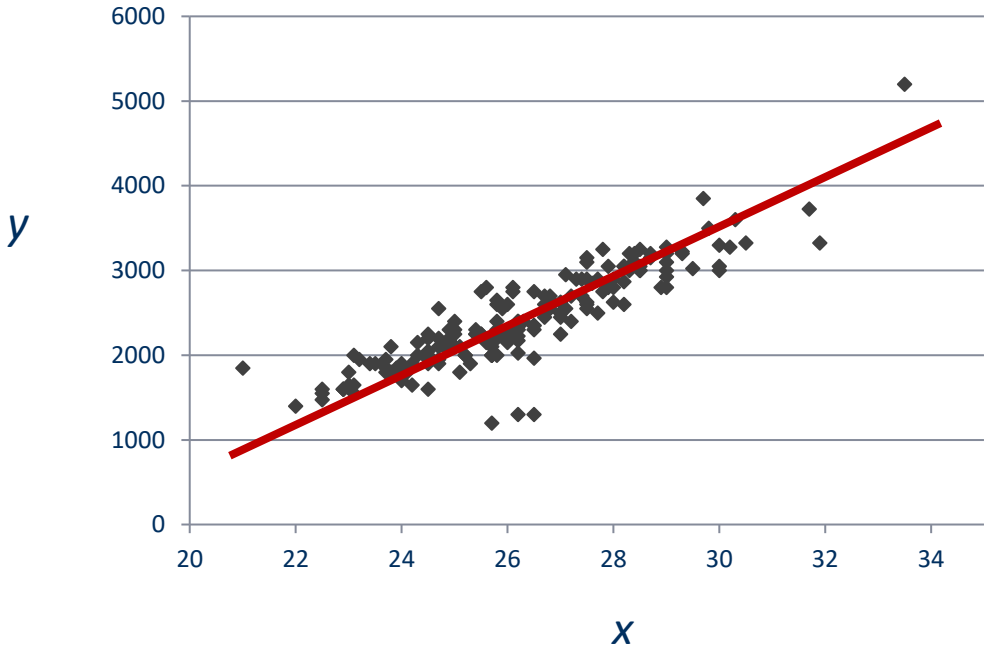
Scatter plots

- Sv. *punktplottar, spridningsdiagram*
- Two numerical continuous or discrete variables: $\boldsymbol{x}$ and $\boldsymbol{y}$
- Each pair $(\boldsymbol{x_i}, \boldsymbol{y_i})$, $i = 1, \ldots, n$, is represented by a dot:



Linear relationship if the imposed line is straight, not curved

If it's discrete numerical, data points (x_i, y_i) are likely to coincide with each other

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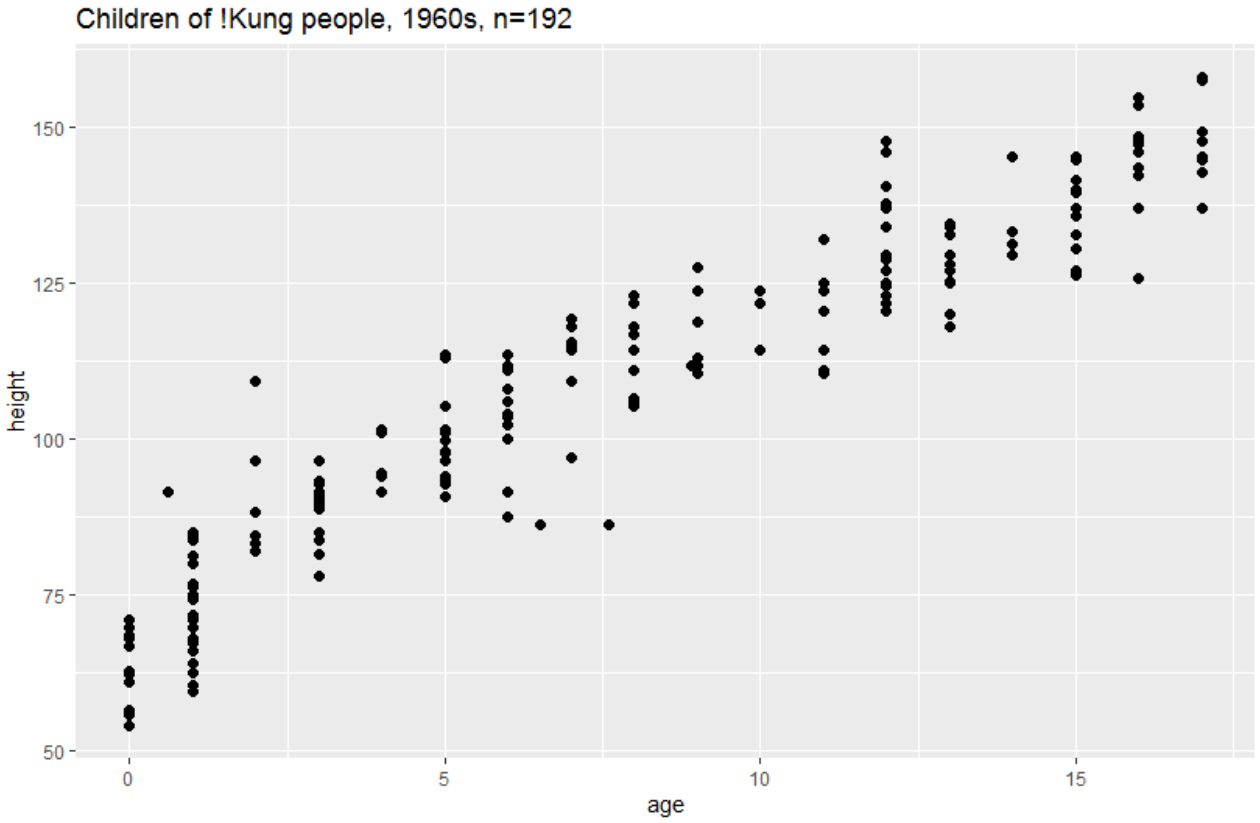
Example, height and age among children



$$s_{xy} = 130\ years \cdot cm \qquad r_{xy} = 0.94$$

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Scatter plots, cont.

What are we looking for?

- ***Strong*** or ***weak*** relationships
- ***Positive*** or ***negative*** relationships
- ***Linear*** or ***non-linear*** relationships
- Also, ***deviant*** and ***extreme*** values/observations

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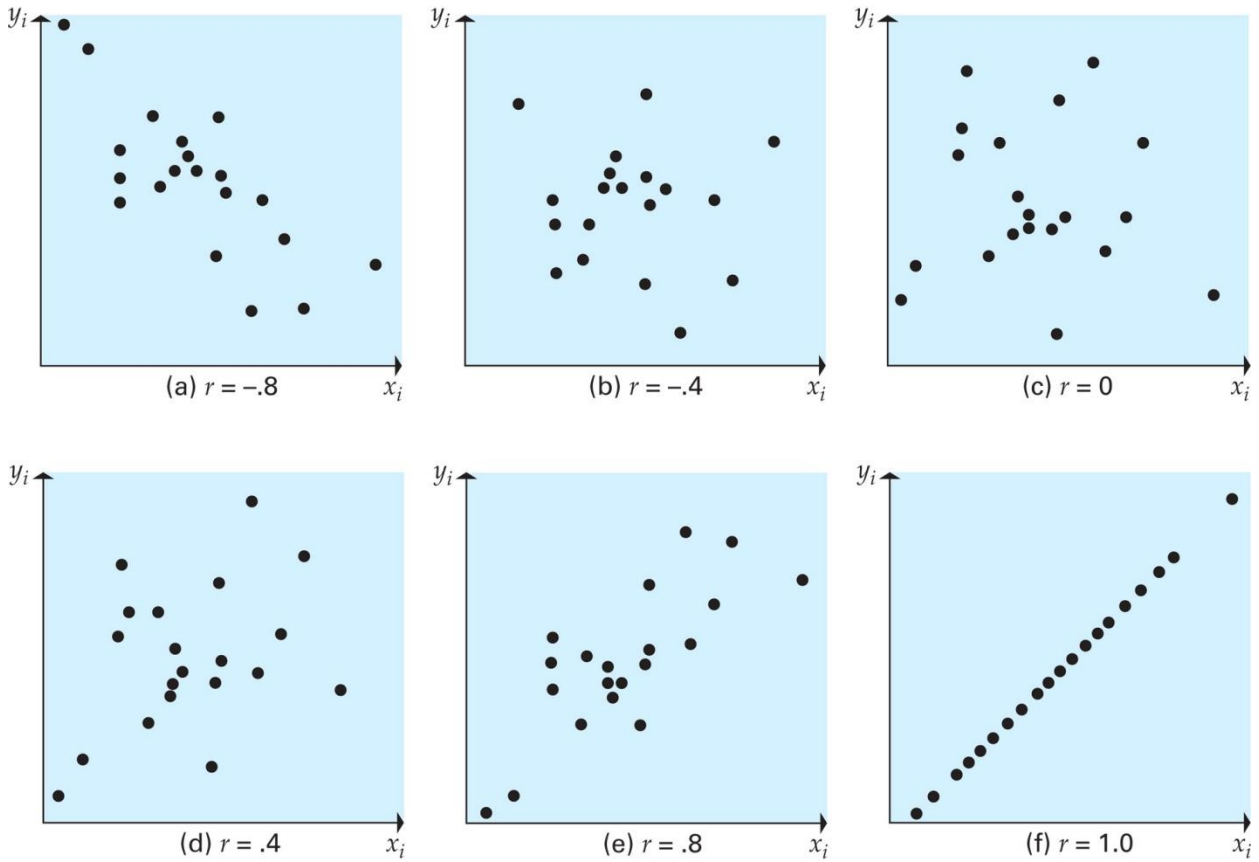
Strong/weak – positive/negative



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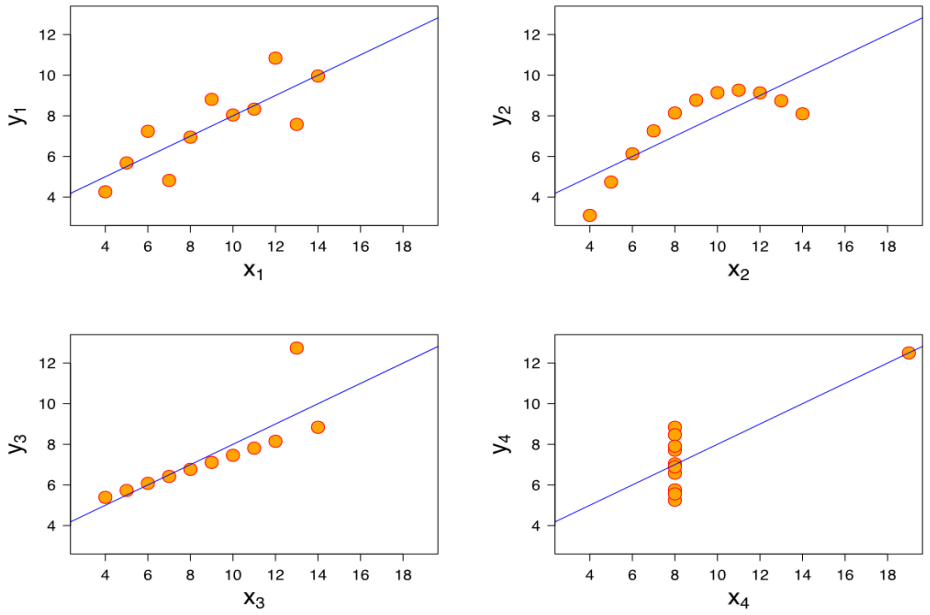
Linear/non-linear – deviant points

- Anscomb's data set



- Four pairs of variables all of which yield the same correlation: **0,816**
- Only one shows truly linear relationship (which?)
- One could be perfectly linear but for one extreme point (which?)

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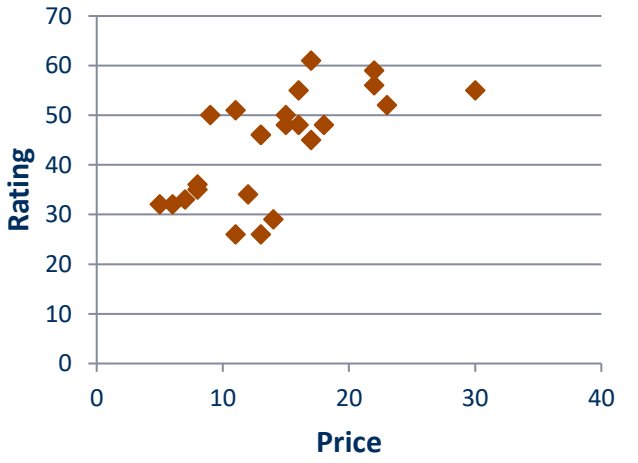
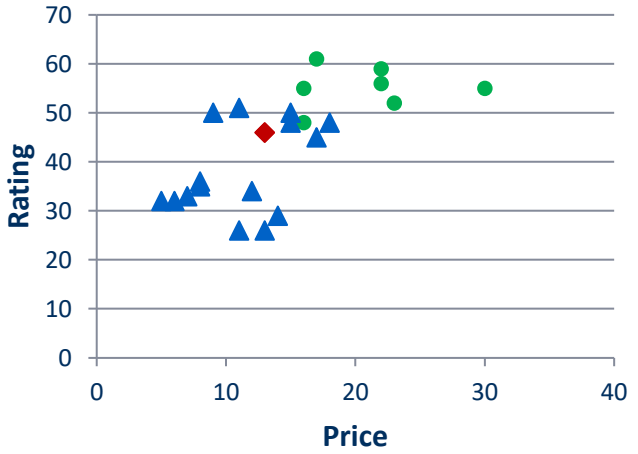
Example

ID	Type	Rating	PricePerLoad
1	Liquid	61	17
2	Liquid	59	22
3	Liquid	56	22
4	Liquid	55	16
5	Liquid	55	30
6	Liquid	52	23
7	Powder	51	11
8	Powder	50	15
9	Powder	50	9
10	Liquid	48	16
11	Powder	48	15
12	Powder	48	18
13	Gel	46	13
14	Gel	46	13
15	Powder	45	17
16	Powder	36	8
17	Powder	35	8
18	Powder	34	12
19	Powder	33	7
20	Powder	32	6
21	Powder	32	5
22	Powder	29	14
23	Powder	26	11
24	Powder	26	13



$s_{xy} = 43,2$

$r_{xy} = 0,67$



$r_{xy} = 0,42$

$r_{xy} = 0,07$

$r_{xy} =$ not def.

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Answers: $\bar{x} = 10$ $s_x^2 = 16$
$\bar{y} = 8$ $s_y^2 = 4$
$s_{xy} = 6$ $r_{xy} = 0{,}75$

DIY Exercise

- Calculate means and variances for *x* and *y* respectively
- Calculate the covariance and correlation between *x* and *y*
- Construct a scatter plot, displaying the data
- What is the relationship? How does it relate to the correlation?

i	1	2	3	4	5	6	7	8	9	10	Σ
x_i	10	6	14	10	12	16	8	14	6	4	
y_i	8	6	7	10	9	10	8	11	5	6	
x_i^2											
y_i^2											
$x_i y_i$											

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Summary

We have reviewed how to present empirical data, both one variable at a time and several together (**univariate** and **multivariate**)

- Tables
- Graphics, types of diagrams, charts
- Summarizing numerical measures
  - Location, variability and now linear correlation (relationship)

Now we will move towards ***models***, mathematical constructs that "explain" how the data was generated, how they relate and so on

- Specifically **probability models**, i.e. theoretical models that we trust will provide an approximate description of reality

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Models

- A model is a **simplified description** of something real based on **assumptions**
  - only relevant aspects are included in the model
  - to test the durability of a material, the color may not matter
  - we approximate (“almost right”)
- We denote the relevant aspects of reality with ***symbols***
  - we turn it into “mathematics”
  - e.g. $\boldsymbol{d = \frac{1}{2} \cdot g \cdot t^2}$
  - $\boldsymbol{d}$, $\boldsymbol{g}$ and $\boldsymbol{t}$ are symbols that represent the values of the different variables, i.e. phenomena in the real world

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Stochastic models – random models

In a **deterministic model**, there is no room for exemption, everything is precisely prescribed in the model (e.g. Newton) and all is perfectly predictable (if the model is "true").

Ex. Ohm's law (physics): $\boldsymbol{R = U/I}$

A **stochastic model** is characterized in that it contains a **random** component. We do not know exactly what the outcome will be but we might be able to say something about what could happen and how often.

Ex. $\boldsymbol{X}$ = "no. of deaths caused by horse kicks to the head"

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Random experiment

- A process or procedure that results in one of at least two possible outcomes (sv. *slumpförsök*)
- However, we cannot be certain which it will be

Sample space, outcome space

- An enumeration, listing or description of all possible **elementary outcomes** (**basic outcomes**) of a random experiment
  - numerical or categorical outcomes?
  - discrete or continuous?
  - limited or unlimited?
- Denoted in NCT by S (capital S)

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Events (sv. *händelser*)

- A ***subset*** of S
- May be a single basic outcome or a collection (set) of basic outcomes or even S itself
- Ex: Rolling a die (dice)
  - Sample space $S = \{1,2,3,4,5,6\}$ where $1,2,\ldots,6$ are basic outcomes
  - Some possible ***events***:
    - ”less than 3” $= \{1,2\}$
    - ”odd number” $= \{1,3,5\}$
    - ”equal to 4” $= \{4\}$
    - ”not 4” $= \{1,2,3,5,6\} = S - \{4\}$
    - ”larger than zero” $= \{1,2,3,4,5,6\} = S$

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Examples

Random experiment = ”toss two dice”

1. Define $\boldsymbol{Y}$ = “sum of the two outcomes”

   Describe the sample space

   Is every possible basic outcome equally probable?

2. Define $(\boldsymbol{X_1}, \boldsymbol{X_2})$ = ”dots on die no. 1 and die no. 2”

   Describe the sample space

   Is every possible basic outcome equally probable?

3. Define $\boldsymbol{Z}$ = “the age of a randomly chosen person in this room”

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Elementary Set Theory

- Let $\boldsymbol{O_1, O_2, \ldots}$ denote the ***basic elements of S (outcomes)***
- Let capital letters $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{S}$, … denote ***sets*** of elements
  - braces { } are used to list the elements in a set
  - ex. $A = \{1,2\}$, $B = \{4\}$, $C = \{O_1, \ldots, O_N\}$
- O_k is a **member of/lies in** A, denoted $\boldsymbol{O_k \in A}$
  - ex. $1 \in \{1,2\}$ but $3 \notin \{1,2\}$
- A is a ***subset*** of B, is denoted $\boldsymbol{A \subset B}$
  - Subset is denoted $\subseteq$, strict subset is denoted $\subset$
  - ex. $\{1,2\} \subset \{1,2,3,4,5,6\}$, $\{1,2\} \subseteq \{1,2,3,4,5,6\}$, $\{1,2\} \subseteq \{1,2\}$
  - ex. $\{5,6\} \not\subset \{1,2,3,4,5\}$

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Elementary Set Theory, cont.

Assume that $S = \{1,2,3,4,5,6\}$ and that $A = \{1,2\}$, $B = \{2,3,4\}$ and $C = \{3\}$

- ***Complement*** is all that is not a member of the set and is denoted $\overline{A}$, (alternative notations include A', A^c, A^*, $\complement A$, not A)
  - ex. $\bar{A} = \{3,4,5,6\}$

- ***Union***, denoted ∪: all members in A or B or both A and B
  - ex. $A \cup B = \{1,2,3,4\}$ $B \cup C = B$

- ***Intersection***, denoted ∩: all which are members in both *A* and *B*
  - ex. $A \cap B = \{2\}$ $B \cap C = C$

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Elementary Set Theory, cont.

- The ***empty set*** is the set containing nothing, denoted $\emptyset$.

- Two sets are ***disjoint*** or ***mutually exclusive*** (sv. *disjunkta, oförenliga*) if their intersection is empty, if they have nothing in common, they are ***non-overlapping***

  - ex. $A = \{1,2\}$ and $C = \{3\}$
    $A \cup C = \{1,2,3\}$
    $A \cap C = \emptyset$

- *What is the complement of S? What is $S \cup \emptyset$? $S \cap \emptyset$?*

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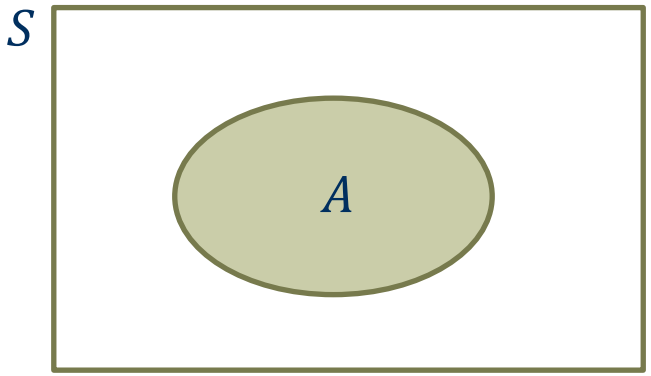
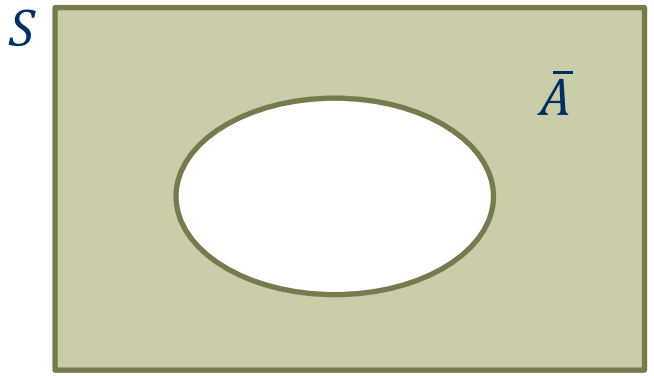
Venn diagrams

- A way to illustrate sets and their relationships to other sets
- The shaded areas denote the set in focus

$A \subseteq S$



$\bar{A} \subseteq S$



$A \cup \bar{A} = S$ $A \cap \bar{A} = \emptyset$

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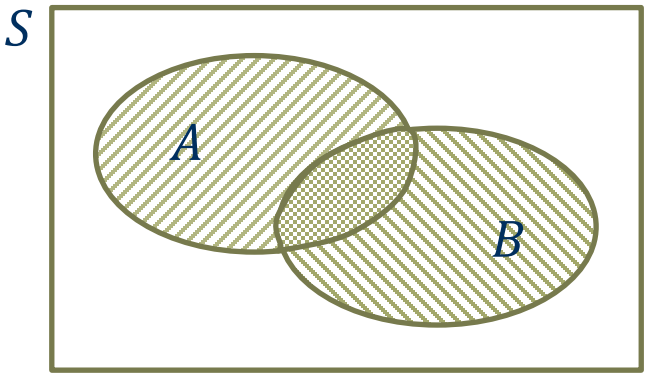
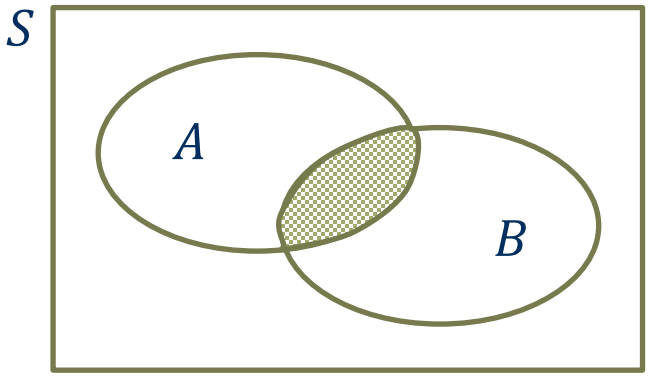
Venn diagrams, cont.

- Two sets, A and B
- The shaded areas denote the set in focus

$A \cup B$



$A \cap B$



- DIY: $A \cup \bar{B}$, $\bar{A} \cup B$, $\bar{A} \cup \bar{B}$, and $A \cap \bar{B}$, $\bar{A} \cap B$, $\bar{A} \cap \bar{B}$

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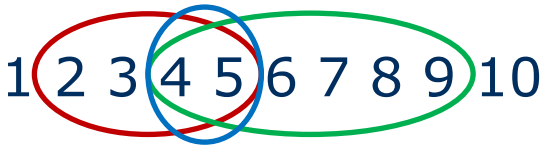
Or like this …

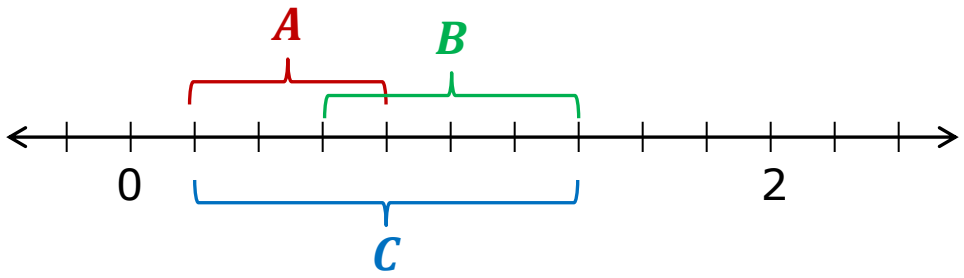
- List all set members and encircle the subsets (discrete sets):



$\boldsymbol{A} = \{2,3,4,5\}$ $\boldsymbol{B} = \{4,5,6,7,8,9\}$

$\boldsymbol{C} = A \cap B = \{4,5\}$

- Line of real numbers, mark subsets/intervals (continuous sets):



$\boldsymbol{A} = (0.2\,,0.8)$

$\boldsymbol{B} = (0.6\,,1{,}4)$

$\boldsymbol{C} = A \cup B = (0.2\,,1.4)$

$A \cap B = (0.6\,,0.8)$

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DIY Exercise – answers on Athena

Let $S = \{1,2,3,4,5,6\}$ and describe the relationship between them:

1. $A = \{1,2\}$ $B = \{3,4,5\}$ Disjoint, non-overlapping $A \cap B = \emptyset$
2. $A = \{1,2,3\}$ $B = \{3,4,5\}$ Overlapping, $A \cap B \neq \emptyset$
3. $A = \{1,2,3,4\}$ $B = \{1,2\}$ B is a strict subset of A, $B \subset A$

For each of the 3 cases above, what is the resulting set of

a) $A \cup B$ c) $\bar{A} \cup B$ e) $\bar{A} \cup \bar{B}$

b) $A \cap B$ d) $\bar{A} \cap B$ f) $\bar{A} \cap \bar{B}$

List the basic outcomes and encircle subsets!

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Probability

Once an ***experiment*** and the ***sample space*** $\boldsymbol{S}$ is defined we also need to know how probable any given event (subset) is.

- Let $\boldsymbol{O_k}$ denote a basic outcome (sv. *elementärt utfall*)
- $\boldsymbol{O_k}$ is in $\boldsymbol{S}$ and we write this as $\boldsymbol{O_k} \in \boldsymbol{S}$

- Let $\boldsymbol{P}(\boldsymbol{O_k})$ denote the ***probability*** that the outcome is $\boldsymbol{O_k}$
- The probability $\boldsymbol{P}(\boldsymbol{O_k})$ is a ***number,*** a numerical measure
- $\boldsymbol{P}$ for *probability*; $\boldsymbol{P}(\boldsymbol{O_k})$ can be viewed as a function

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Probability and event

- Let $\boldsymbol{A}$ be any event, i.e. $\boldsymbol{A} \subseteq \boldsymbol{S}$
  - ex. $A = \{1,2\} \subseteq \{1,2,3,4,5,6\} = S$

- The probability of $\boldsymbol{A}$: $\boldsymbol{P(A)}$ = **sum of** $\boldsymbol{P(O_k)}$ where $\boldsymbol{O_k} \in \boldsymbol{A}$
  - ex. $P(\text{"1 or 2"}) = P(\text{"1"}) + P(\text{"2"})$

Note! ”1” and ”2” are ***disjoint*** basic outcomes

An (almost) complete ***stochastic model*** is defined if

1. The sample space $\boldsymbol{S}$ is well-defined
2. For all $\boldsymbol{A} \subseteq \boldsymbol{S}$ we can provide $\boldsymbol{P(A)}$, including $\boldsymbol{P(S)}$ and $\boldsymbol{P(\emptyset)}$

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But what is a probability?

- Probabilities $P(A)$ are numbers
- Should for all practical reasons represent how "probable" an event is
- Our intuition tells us:
  - if the probability is zero, it should never happen?
  - if the probability is 100%, it should always happen?
- Interpretations of probabilities can vary:

  Classical, ***Frequentist*** or ***Subjectively***

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Classical interpretation

If possible, we can gauge the "size" of a subset $\boldsymbol{A}$ and compare it to the "size" of $\boldsymbol{S}$

- Assume that $\boldsymbol{S}$ can be described as set of basic outcomes $\boldsymbol{O_k}$, ***all of which are equally probable***
- Count the number of elements/members in $\boldsymbol{A}$ (size of $\boldsymbol{A}$)
- Compare to the total number of basic outcomes (size of $\boldsymbol{S}$)

$$\frac{\#(A)}{\#(S)} = \frac{\text{number of members in } A}{\text{number of members in } S} = P(A) \qquad \frac{size(A)}{size(S)} = P(A)$$

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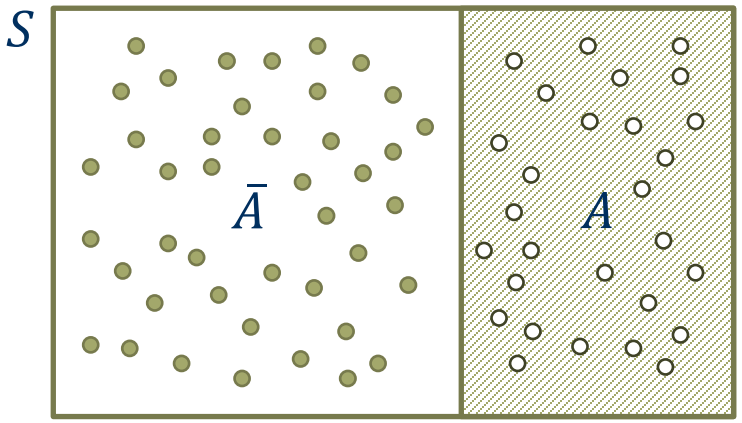
Classical interpretation – example

- 70 lottery tickets of which 28 are wins
- Draw one randomly
- Venn diagram

Every dot represents a ticket, dark are no-wins, white are wins



$$\#(A) = size(A) = 28$$

$$\#(S) = size(S) = 70$$

$$\boldsymbol{P(A) = 28/70 = 0,4}$$

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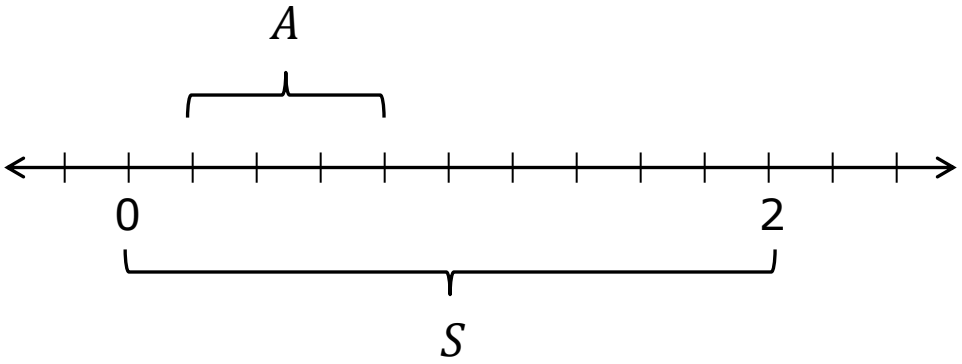
Variant with a continuous set

- Let $S = (0,2)$ i.e. an **interval**

  $S = \{x; 0 < x < 2\}$, all x such that $0 < x < 2$

- Assume that partial intervals that are equal in size, are equally probable
- Let $A = \{x; 0{,}2 < x < 0{,}8\}$
- Select a random point in the interval S



$size(A) = 0{,}6$

$size(S) = 2$

$\boldsymbol{P(A) = 0{,}6/2 = 0{,}3}$

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Combinatorics – counting the outcomes

Q. Menu with ***3 starters***, ***4 main courses*** and ***2 deserts***.

How many different 3-course dinners can you order?

A: $3 \cdot 4 \cdot 2 = \mathbf{24}$ Illustrate with a tree diagram!

In combinatorics, the number of possibilities often explode!

Ex. *Students are to be arranged in pairs. How many different configurations of 15 pairs are there in a group of 30 students?*

Answer: 6 190 283 353 629 375

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Principal of Multiplication

- An experiment has $\boldsymbol{m_1}$ possible outcomes
- Another experiment has $\boldsymbol{m_2}$ possible outcomes
- We carry out the first, then the second experiment
  - *the two experiments may be different or the same*

- In total there are

$$\boldsymbol{m_1 \times m_2}$$

possible combinations

Note! m_1 and m_2 may be different numbers or equal; depends on the experiments!

Note! Assumes that m_2 isn't affected by the outcome of experiment 1

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Combinatorics, cont.

Example

A bag with numbered balls $\mathbf{1}, \ldots, \boldsymbol{n}$

- Randomly draw one ball and write up the number

  How many possible outcomes?

- Randomly draw one ball again and write up the number

  How many possible outcomes?

- ***How many possible outcomes in total? First and second draw considered together?***

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Repeated draws, draw *x* times from *n*

- **Without replacement, *wor*** (sv. *utan återläggning)*
  - A ball that is drawn is put to the side and cannot be drawn again
  - How many possible outcomes, combinations?

    $n \cdot (n-1) \cdot \ldots \cdot (n-x+1)$ Multiplication principal

- **With replacement, *wr*** (sv. *med återläggning)*
  - A ball that is drawn is put back into the bag and may be drawn again, repeatedly
  - How many possible outcomes, combinations?

    $n \cdot n \cdot \ldots \cdot n = n^x$ Multiplication principal

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Combinatorics, cont.

Does the ***order*** in which the balls are drawn matter?

Assume that $x = 2$ letters are to be chosen (**wor**) from the set of $n = 4$ letters **A**, **B**, **C**, **D**

- Order matters:
  - the outcomes **A**,**B** och **B**,**A** are considered as **different**
  - the formulas on the preceding page – order matters
- Order doesn't matter:
  - the outcomes **A**,**B** och **B**,**A** are treated as **equal**
  - in many cases the ordering is not essential to the problem

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Factorial

- From the multiplication principal it should be clear that the number of ways we can arrange x different objects is

$$x \cdot (x-1) \cdot (x-2) \cdots 3 \cdot 2 \cdot 1 = x!$$

↑ 1st draw ↑ 2nd draw ↑ 3rd draw … ↑ last draw

- $x!$ pronounced ***x factorial*** (sv. *x fakultet*)

- Ex: 0! = 1 1! = 1 2! = 2 3! = 6 4! = 24

  10! = 3 628 800 20! = 2 432 902 008 176 640 000

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Summary

Understand the concepts, start with the Quiz F1-F4 today! :

- Random experiments – generating random data
- Sample space – all that is possible
- Events – a part of the sample space (basic outcomes)
- Set theory – structured way to describe
- Probability – what it is (really)
  - Classical interpretation ⇒ combinatorics, art of counting

Next time we will continue exploring combinatorics, set a foundation for probability theory and models and learn how to calculate probabilities.

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