L13

Basic Statistics for Economists

Spring 2020

Department of Statistics

Stockholm University

Summary of last time

- We want to estimate a parameter
  - e.g. μ or P
  - Two opposite claims regarding the parameter: H_0 and H_1
- Start by thinking about what is given and what must be assumed
  - e.g. normal distribution, known variance, sample size, indep.
- Identify the correct test variable, Z eller t_{n-1}
- Decide significance level α
- Find critical values
  - E.g. z_α or $-z_\alpha$ (one-sided) or $\pm z_{\alpha/2}$ (double-sided)
- Compute and make decision: reject or not reject. **Interpret!**

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Exercise – review

- A school class conducts a statistics experiment. They decide to collect and measure the volume of milk in one liter packages from the local dairy. They suspect that the packages might contain less than 100 cl (= 1 liter) on average. The pupils choose $\boldsymbol{n = 10}$ packages and calculate the following sums: $\boldsymbol{\sum x_i = 980}$ and $\boldsymbol{\sum x_i^2 = 96085}$.
- Test using a suitable hypothesis test whether the average volume of milk is less than 100 cl. Use significance level $\boldsymbol{\alpha = 0,05}$. At least two assumptions must be true in order for the test to be fully correct. Which assumptions?

Solution

- Assume: **independence** and **normal distribution** $\boldsymbol{X_i}$ **iid** $\boldsymbol{N(\mu, \sigma^2)}$.
  Normal distribution is necessary since $\boldsymbol{n = 10}$ is too few to use CGS.
- Note that we do not know the population variance $\boldsymbol{\sigma^2}$, it has be estimated.
- Hypotheses: $\boldsymbol{H_0: \mu = 100\ \text{cl}}$ (the claim that we want to disprove)
  $\boldsymbol{H_1: \mu < 100\ \text{cl}}$ (our belief that we want to support)
- Test variable (assumption of normal distribution ⇒ t-distribution):

$$\boldsymbol{t_9 = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t(9)} \qquad \boldsymbol{n-1} \text{ degrees of freedom}$$

- Decision: reject $\boldsymbol{H_0}$ if $\boldsymbol{t_{obs} < -t_{n-1;\alpha} = -t_{9;0,05} = -1,833}$

NOTE! ”Less than” in $\boldsymbol{H_1}$, use negative $\boldsymbol{t}$-value!

Solution, cont.

- Calculations:

$$\boldsymbol{\bar{x}} = \frac{\sum x_i}{n} = \frac{\mathbf{980}}{10} = \mathbf{98} \qquad \boldsymbol{s_x^2} = \frac{\sum x_i^2 - n\bar{x}^2}{n-1} = \frac{\mathbf{96085} - 10 \cdot 98^2}{9} = \mathbf{5}$$

$$\boldsymbol{t_{obs}} = \frac{\boldsymbol{\bar{x}_{obs}} - \boldsymbol{\mu_0}}{\boldsymbol{s_{obs}/\sqrt{n}}} = \frac{\mathbf{98-100}}{\sqrt{\mathbf{5/10}}} = \boxed{\mathbf{-2{,}82843 < -1{,}833}}$$

- Conclusion/interpretation:
  - $\boldsymbol{H_0}$ **rejected** on 5% significance level;
  - The average volume of milk is **significantly less than** 100 cl;
  - The result is **statistically significant** at the 5% level.

Today

NCT ch. 9.2 + 10.1-3 + 10.5 = cont. **Hypothesis testing**

- $\boldsymbol{p}$**-value** – instead of determining $\boldsymbol{\alpha}$ ahead of time
- The connection between **hypothesis test** and **KI**
- Test the **difference** between means, $\boldsymbol{\mu_X}$ and $\boldsymbol{\mu_Y}$
  - One case with two **dependent samples** – pairwise dependence
  - Some cases with two **independent samples**
- Test the **difference** between proportions, $\boldsymbol{P_X}$ and $\boldsymbol{P_Y}$
- Some discussion of *power* – probability of rejecting a false H_0

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Significance and power, error types

Consequence & probability	Actual situation	
decision	H_0 true	H_0 false
accept H_0	correct decision $(1-\alpha)$	Type II Error (β)
reject H_0	Type I Error (α)	correct decision $(1-\beta)$

- the probability of type I error = **significance level** (*p-value*)
- the probability $1-\beta$ = **power**

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NCT p. 354-361

p-values

- Suppose that we have a null hypothesis and an alternative hypothesis, a point estimate and a test variable.
- But we have not decided on a significance level α.
- Set the **critical value = the observed test variable**.
- **What significance level α would this correspond to?**
- How significant is our particular result?

Example

Assume:

- Normal distribution with known variance $\boldsymbol{\sigma^2 = 4}$, iid, size $\boldsymbol{n = 16}$
- Hypothesis (one-sided test) and test variable:

$$\boldsymbol{H_0\!: \mu = 40} \quad \text{vs.} \quad \boxed{\boldsymbol{H_1\!:\ \mu > 40}} \qquad \boldsymbol{z_{obs}} = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{\bar{x} - 40}{2/4} = \boldsymbol{2(\bar{x} - 40)}$$

- Decision rule: reject if $\boxed{\boldsymbol{z_{obs} > z_{krit}}}$

NOTE! significance level and critical value are not decided!

- **Under** $\boldsymbol{H_0}$, what is the probability to observe $\boxed{\boldsymbol{\bar{X} > \bar{x}_{obs}}\ ?}$
- Say that we observe $\boldsymbol{\bar{x}_{obs}} =$ (a) $\boldsymbol{41{,}1}$ (b) $\boldsymbol{39{,}5}$ (c) $\boldsymbol{40{,}82}$

Example, cont.

a) $\bar{\boldsymbol{x}} = \mathbf{41,1}$: $z_{obs} = 2(41{,}1 - \mathbf{40}) = \mathbf{2,2} \Rightarrow \boldsymbol{P}(\boldsymbol{Z} > \mathbf{2,2}) = 1 - P(Z \leq 2{,}2)$

$= 1 - 0{,}98610 = \mathbf{0,01390}$

critical value $\boldsymbol{z_\alpha} = \boldsymbol{z_{obs}} = \mathbf{2,2}$ corresponds to $\boldsymbol{\alpha} = \mathbf{1,39\%}$

b) $\bar{\boldsymbol{x}} = \mathbf{39,5}$: $z_{obs} = 2(39{,}5 - \mathbf{40}) = \mathbf{-1,0} \Rightarrow P(Z > -1{,}0) = P(Z \leq 1{,}0)$

$= \mathbf{0,84134}$

critical value $\boldsymbol{z_\alpha} = \boldsymbol{z_{obs}} = \mathbf{1,0}$ corresponds to $\boldsymbol{\alpha} = \mathbf{84,13\%}$

c) $\bar{\boldsymbol{x}} = \mathbf{40,82}$: $z_{obs} = 2(40{,}82 - \mathbf{40}) = \mathbf{1,64} \Rightarrow P(Z > 1{,}64) = 1 - P(Z \leq 1{,}64)$

$= 1 - 0{,}94950 = \mathbf{0,0505}$

critical value $\boldsymbol{z_\alpha} = \boldsymbol{z_{obs}} = \mathbf{1,64}$ corresponds to $\boldsymbol{\alpha} = \mathbf{5,05\%}$

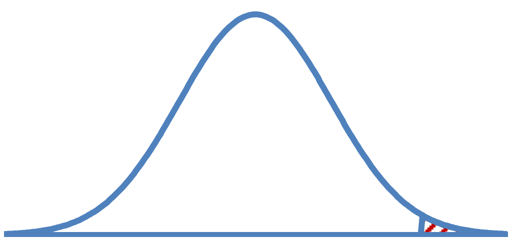
Example, cont.

- ***p-value*** = **value** α that would make the **critical value = observed value**, i.e. if $\boldsymbol{z_{krit} = z_{obs} = z_\alpha}$
- *p*-value $= \boldsymbol{p = P(Z > z_{obs} \mid H_0 \text{ true})}$

$\bar{x}_{obs} = 41{,}1$

$z_{obs} = 2{,}20$

$\boldsymbol{p = 1{,}39\ \%}$

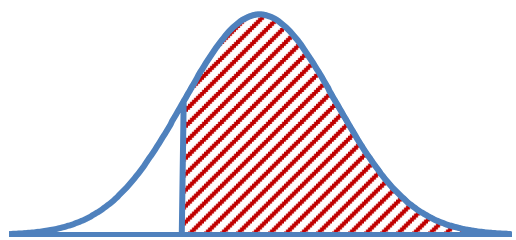


$\boldsymbol{z_{krit} = 2{,}2}$

$\bar{x}_{obs} = 39{,}5$

$z_{obs} = -1{,}00$

$\boldsymbol{p = 84{,}13\ \%}$

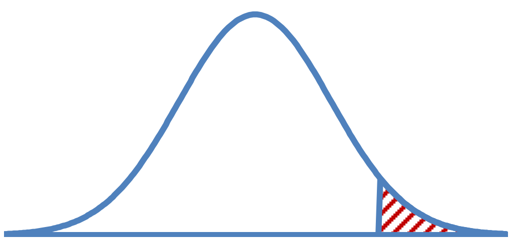


$\boldsymbol{z_{krit} = -1}$

$\bar{x}_{obs} = 40{,}82$

$z_{obs} = 1{,}64$

$\boldsymbol{p = 5{,}05\ \%}$



$\boldsymbol{z_{krit} = 1{,}64}$

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p-value and double-sided test

- Double sided test = **double** the probability
- $\boldsymbol{H_0: \mu = 40}$ vs. $\boldsymbol{H_1: \mu \neq 40}$
- Suspose that we observe

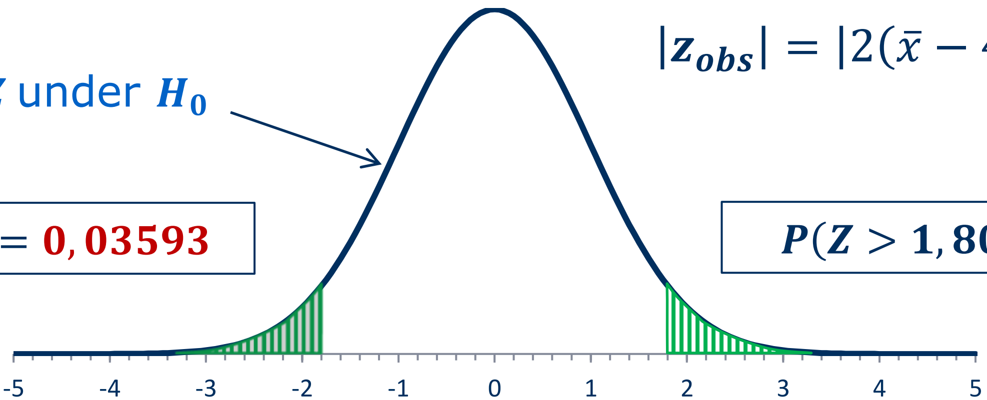
$$\boldsymbol{\bar{x}_{obs} = 40,9}$$

$$|z_{obs}| = |2(\bar{x} - 40)| = \mathbf{1,80}$$

Distribution of $\boldsymbol{Z}$ under $\boldsymbol{H_0}$

$\boldsymbol{P(Z < -1,80) = 0,03593}$

$\boldsymbol{P(Z > 1,80) = 0,03593}$



- The rule "reject if $|z_{obs}| > 1{,}80$" corresponds to

$$\boldsymbol{\alpha = 2 \cdot 3,593\,\% = 7,186\,\% = p\text{-value}}$$

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Using *p*-values

If the ***p*-value** is **low**, e.g. < 5%

⇒ corresponds to **rejected** on 5% significance level

If the ***p*-value** is **big**, e.g. > 5%

⇒ the null hypothesis **cannot be rejected** on the 5% level

Significance level

- "**statistically significant**" (Sv. "**statistiskt säkerställt**"), is synonymous with a **low *p*-value**, typically less than 5%.

Hypothesis test and confidence interval

- A **95 % CI** for $\boldsymbol{\mu}$ with the assumption of an iid sample of normally distributed observations with known variance:

$$\bar{x} + 1{,}96\frac{\sigma}{\sqrt{n}} = \text{[for example]} = 25 \pm 5$$

- A **double-sided hypothesis test** on the **5% level** with the same assumption as above:

$$H_0\colon \mu = \boldsymbol{\mu_0} \quad \text{vs.} \quad H_1\colon \mu \neq \boldsymbol{\mu_0} \qquad\qquad z_{obs} = \left|\frac{\bar{x} - \boldsymbol{\mu_0}}{\sigma/\sqrt{n}}\right| \leq 1{,}96$$

- **What is the connection between the confidence interval and the test?**

Hypothesis test and confidence intervall, cont.

- Test variable and (reversed) decision:
- **Do not** reject $\boldsymbol{H_0}$ if

$$\boldsymbol{z_{obs} = \left|\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}\right| \leq 1,96 \quad \Leftrightarrow \quad -1,96 \leq \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \leq 1,96}$$

$$\Leftrightarrow \quad -1,96\frac{\sigma}{\sqrt{n}} \leq (\bar{x} - \mu_0) \leq 1,96\frac{\sigma}{\sqrt{n}}$$

Rearrange a little ...

$$\Leftrightarrow \quad -\bar{x} - 1,96\frac{\sigma}{\sqrt{n}} \leq -\mu_0 \leq -\bar{x} + 1,96\frac{\sigma}{\sqrt{n}}$$

... a little more...

$$\Leftrightarrow \quad \boldsymbol{\bar{x} - 1,96\frac{\sigma}{\sqrt{n}} \leq \mu_0 \leq \bar{x} + 1,96\frac{\sigma}{\sqrt{n}}}$$

$\boxed{\textbf{95\% CI for } \boldsymbol{\mu}}$

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Compare μ_0 against a confidence interval

- Suppose you are going to test a null hypothesis against a **double-sided** alternative at significance level $\boldsymbol{\alpha = 0,05}$

$$H_0 : \mu = \mu_0 \quad \text{vs.} \quad H_1 : \mu \neq \mu_0$$

- Your friend has already calculated a $\mathbf{100(1-\alpha) = 95\%}$ confidence interval for $\boldsymbol{\mu}$ based on the same data that you are going use.

If your μ_0 **lies outside the interval**

$\Rightarrow$ **reject** the null hypothesis

If your μ_0 **lies within the interval**

$\Rightarrow$ **do not reject** null hypothesis

Comparing groups

- Suppose that you want to compare two means μ_X and μ_Y of group X and group Y, respectively.

- A typical null hypothesis assumes no difference between groups:

$$H_0\colon \mu_X = \mu_Y \iff \mu_X - \mu_Y = \mathbf{0} \quad \iff \mu_Y - \mu_X = \mathbf{0}$$

It is possible to test differences other than 0; in general $\mu_X - \mu_Y = D_0$ where D_0 is some number.

- The alternative can be one-sided or double:

$$H_1\colon \mu_X > \mu_Y \iff \mu_X - \mu_Y > 0 \iff \mu_Y - \mu_X < 0$$

$$H_1\colon \mu_X < \mu_Y \iff \mu_X - \mu_Y < 0 \iff \mu_Y - \mu_X > 0$$

$$H_1\colon \mu_X \neq \mu_Y \iff \mu_X - \mu_Y \neq 0 \iff \mu_Y - \mu_X \neq 0$$

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Comparing groups, cont.

- Draw one sample from group $\boldsymbol{X}$ and one sample from group $\boldsymbol{Y}$.
- The two **samples are independent** of each other.
- **Within** each sample the observations are **iid**
- If not normally distributed observations ⇒ CLT (large $\boldsymbol{n}$)
- This implies $\bar{X} \sim N\left(\mu_X, \frac{\sigma_X^2}{n_X}\right)$ $\bar{Y} \sim N\left(\mu_Y, \frac{\sigma_Y^2}{n_Y}\right)$

KNOWN SINCE EARLIER LECTURES

and $E(\bar{X} - \bar{Y}) = \mu_X - \mu_Y$ $Var(\bar{X} - \bar{Y}) = \frac{\sigma_X^2}{n_X} + \frac{\sigma_Y^2}{n_Y}$

LINEAR COMBINATION OF TWO INDEP. R.V.

and $(\bar{X} - \bar{Y}) \sim N\left(\mu_X - \mu_Y; \frac{\sigma_X^2}{n_X} + \frac{\sigma_Y^2}{n_Y}\right)$

WE USE Z OR t AS TEST VARIABLE

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Compare to L10 p. 9

Different cases, CI for $\mu_x - \mu_y$

- When should you use z and when should you use t?

Sample sizes n_x, n_y	Distribution	Variances σ_X^2 and σ_Y^2		
		known	unknown s_x^2, s_y^2	
large	normal	Z	**approx.** Z	
	not normal	**CLT** $\Rightarrow Z$	**approx CLT** $\Rightarrow Z$	
small	normal	Z	$\sigma_X^2 \neq \sigma_Y^2$: **Not incl.**	$\sigma_X^2 = \sigma_Y^2$: t
	not normal	**Not included**	**Not included**	

- In the cases marker blue above, where the sample is large and the variance unknown, one could argue that t should be used instead of z. This gives more conservative results, i.e. slightly smaller risk α; using t adjusts for the slightly larger uncertainty that follows from the estimation of the variance.

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exercise/example 1

- $\boldsymbol{X_i}$ **iid** $\boldsymbol{N(\mu_X, \sigma_X^2)}$ and $\boldsymbol{Y_i}$ **iid** $\boldsymbol{N(\mu_Y, \sigma_Y^2)}$ where $\boldsymbol{\sigma_X^2 = 5^2}$ and $\boldsymbol{\sigma_Y^2 = 3^2}$
- $\boldsymbol{\bar{X}}$ and $\boldsymbol{\bar{Y}}$ are independent; two small samples: $\boldsymbol{n_X = 20}$ and $\boldsymbol{n_Y = 12}$;
- Hypothesis: $\boldsymbol{H_0: \mu_X - \mu_Y = 0}$ vs. $\boldsymbol{H_1: \mu_X - \mu_Y > 0}$ (one-sided)
- significance level, set to $\boldsymbol{\alpha = 5\ \%}$
- Test variable:

$$\boldsymbol{Z = \frac{\bar{X} - \bar{Y} - 0}{\sqrt{\frac{\sigma_X^2}{n_X} + \frac{\sigma_Y^2}{n_Y}}} = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{25}{20} + \frac{9}{12}}} = \frac{\bar{X} - \bar{Y}}{\sqrt{2}} \sim N(0, 1)}$$

- critical value and decision rule:
  - reject $\boldsymbol{H_0}$ if $\boldsymbol{z_{obs} > z_{0,05} = 1,6449}$ (Tabell 2)

exercise/example 1, cont.

- We observe $\bar{\boldsymbol{x}}_{\boldsymbol{obs}} = \mathbf{50}$ and $\bar{\boldsymbol{y}}_{\boldsymbol{obs}} = \mathbf{48}$, substitution

$$\boldsymbol{z}_{\boldsymbol{obs}} = \frac{\mathbf{50-48}}{\sqrt{\mathbf{2}}} = \frac{\mathbf{2}}{\sqrt{\mathbf{2}}} = \sqrt{\mathbf{2}} = \mathbf{1{,}41421} < \mathbf{1{,}6449}$$

- Conclusion and interpretation:
  - $\boldsymbol{H}_{\mathbf{0}}: \boldsymbol{\mu}_{\boldsymbol{X}} = \boldsymbol{\mu}_{\boldsymbol{Y}}$ cannot be rejected at significance level $\alpha = 0{,}05$
  - $\bar{\boldsymbol{x}}_{\boldsymbol{obs}} = \mathbf{50}$ is not significantly greater than $\bar{\boldsymbol{y}}_{\boldsymbol{obs}} = \mathbf{48}$
  - The difference is not significant on the $5\,\%$ level

exercise/example 2

- Suppose $\boldsymbol{X_i}$ **iid** $\boldsymbol{N(\mu_X, \sigma_X^2)}$ and $\boldsymbol{Y_i}$ **iid** $\boldsymbol{N(\mu_Y, \sigma_Y^2)}$ where $\boldsymbol{\sigma_X^2}$ and $\boldsymbol{\sigma_Y^2}$ are **unknown**;
- $\boldsymbol{\bar{X}}$ and $\boldsymbol{\bar{Y}}$ are independent; large samples: $\boldsymbol{n_X = 400}$ and $\boldsymbol{n_Y = 400}$;
- Hypotheses: $\boldsymbol{H_0: \mu_X - \mu_Y = 0}$ vs. $\boldsymbol{H_1: \mu_X - \mu_Y < 0}$ $\boldsymbol{(\mu_Y - \mu_X > 0)}$
- significance level set $\boldsymbol{\alpha = 5\ \%}$
- Test variable:

$$\boldsymbol{Z = \frac{\bar{X} - \bar{Y} - 0}{\sqrt{\frac{S_X^2}{n_X} + \frac{S_Y^2}{n_Y}}} \sim N(0, 1)}$$

APPROXIMATE DUE TO THE FACT THAT WE ESTIMATE THE VARIANCES, NOT BECAUSE OF CLT

- critical value and decision rule:
  - reject $\boldsymbol{H_0}$ if $\boldsymbol{z_{obs} < -z_{0,05} = -1,6449}$ (Table 2)

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exercise/example 2, cont.

- We observe $\boldsymbol{\bar{x}_{obs} = 49,\ s_X^2 = 6^2}$ and $\boldsymbol{\bar{y}_{obs} = 50,\ s_Y^2 = 8^2}$:

$$\boldsymbol{z_{obs}} = \frac{49 - 50}{\sqrt{\frac{36}{400} + \frac{64}{400}}} = \frac{-1}{\sqrt{1/4}} = \boldsymbol{-2 < 1,6449}$$

- Conclusion and interpretation:
  - $\boldsymbol{H_0: \mu_X = \mu_Y}$ is **rejected** at significance level $\alpha = 0{,}05$
  - $\boldsymbol{\bar{x}_{obs} = 49}$ is significantly less than $\boldsymbol{\bar{y}_{obs} = 50}$
  - The claim $\boldsymbol{\mu_X}$ is less than $\boldsymbol{\mu_Y}$ is **statistically significant** at the $5\ \%$ level

exercise/example 3a

- Suppose that $\boldsymbol{X_i}$ **iid** $\boldsymbol{N(\mu_X, \sigma_X^2)}$ and $\boldsymbol{Y_i}$ **iid** $\boldsymbol{N(\mu_Y, \sigma_Y^2)}$ where $\boldsymbol{\sigma_X^2}$ and $\boldsymbol{\sigma_Y^2}$ are **unknown**
- $\bar{X}$ and $\bar{Y}$ are independent; two **small** samples: $\boldsymbol{n_X = 20}$ and $\boldsymbol{n_Y = 12}$;
- Hypotheses: $\boldsymbol{H_0: \mu_X - \mu_Y = 0}$ vs. $\boldsymbol{H_1: \mu_X - \mu_Y \neq 0}$
- significance level set to $\boldsymbol{\alpha = 5\,\%}$ and $\boldsymbol{\alpha/2 = 0{,}025\,\%}$
- Test variabl4: $$\boldsymbol{Z = \frac{\bar{X} - \bar{Y} - 0}{\sqrt{\frac{S_X^2}{n_X} + \frac{S_Y^2}{n_Y}}} \sim t(\nu)}$$

NOT PART OF THE COURSE

More difficult to calculate the degrees of freedom $\boldsymbol{\nu}$ (see NCT p. 397) and Table 3 is typically useless.

exercise/example 3b

- Same as 3a except that the **variances** are assumed to be of **equal size**, e.g. $\boxed{\sigma_X^2 = \sigma_Y^2}$ ← **Important assumption!**

- Test variable:

$$t = \frac{\bar{X} - \bar{Y} - 0}{\sqrt{\frac{S_p^2}{n_X} + \frac{S_p^2}{n_Y}}} \sim t(n_X + n_Y - 2)$$

$20 + 12 - 2 = 30$
DEGREES OF FREEDOM

where $S_p^2 = \frac{(n_X - 1)S_X^2 + (n_Y - 1)S_Y^2}{n_X + n_Y - 2}$

POOLED VARIANCE ESTIMATE

- Decision rule: reject H_0 if $|t_{obs}| > t_{30;0,025} = 2,042$

exercise/example 3b, cont.

- You observe $\bar{x}_{obs} = 47.5,\ s_X^2 = 5^2$ and $\bar{y}_{obs} = 51,\ s_Y^2 = 4^2$:

$$s_p^2 = \frac{(20-1)\cdot 25 + (12-1)\cdot 16}{20+12-2} = 21,7 \qquad s_p = 4,6583$$

$$|t_{obs}| = \left|\frac{47,5-51}{\sqrt{\frac{21,7}{20}+\frac{21,7}{12}}}\right| = \left|\frac{-3,5}{1,7010}\right| = 2,0576 > 2,042$$

- Conclusion and interpretation:
  - $H_0: \mu_X = \mu_Y$ **rejected** at significance level $\alpha = 0{,}05$
  - And so on…

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exercise/example 4 – pairwise differences

Same starting point as that of **L10 s. 16-20**:

- Samples of **pairs** of observations of the same object: $(\boldsymbol{X_k}, \boldsymbol{Y_k})$, the pairs are iid, small sample $\boldsymbol{n} = \mathbf{8}$
- Differences $\boldsymbol{D_k} = (\boldsymbol{X_k} - \boldsymbol{Y_k})$ assumed iid $\boldsymbol{N}(\boldsymbol{\mu_D}, \boldsymbol{\sigma_D^2})$; $\boldsymbol{\sigma_D^2}$ uknown
- Solution corresponding to **L12 p. 23 exercise 3**: $\boxed{\textbf{\textit{t}-test}}$
- Hypotheses: $\boldsymbol{H_0}: \boldsymbol{\mu_D} = \boldsymbol{\mu_0}$ vs. $\boldsymbol{H_1}: \boldsymbol{\mu_D} < \boldsymbol{\mu_0}$ $\quad \boldsymbol{\alpha} = \mathbf{5}\ \%$
- Test variable for $\boldsymbol{\mu_D}$:

$$\boldsymbol{t} = \frac{\overline{\boldsymbol{D}} - \boldsymbol{\mu_0}}{\boldsymbol{S_d}/\sqrt{\boldsymbol{n}}} \sim \boldsymbol{t}(\boldsymbol{n} - \mathbf{1}) = \boldsymbol{t}(\mathbf{7})$$

exercise/example 4, cont.

- Decision rule and critical value:
  - reject $\boldsymbol{H_0}$ if $\boldsymbol{t_{obs} < -t_{7\,;\,0,05} = -1,895}$ (Table 3)

- Start from $\boldsymbol{\mu_0 = 0}$ i $\boldsymbol{H_0}$, observered $\boldsymbol{\bar{d}_{obs} = 0,375}$ and $\boldsymbol{s_d = 1,408}$:

$$\boldsymbol{t_{obs}} = \frac{-0{,}375 - 0}{1{,}408/\sqrt{8}} = \boldsymbol{-0,7533 > -1,895}$$

- Conclusion and interpretation:
  - We **fail to reject** $\boldsymbol{H_0}\colon \boldsymbol{\mu_D = 0}$ at significance level $\alpha = 0{,}05$
  - The difference is not statistically significant.

Exercise/example 5 – proportion

- Assume that $X \sim Bin(n_X; P_X)$ and $Y \sim Bin(n_Y; P_Y)$, where X and Y are the number of observations with some particular property from groups X and Y respectively
- Two large samples: $n_X = 200$ and $n_Y = 200$

Note! This only works if the assumed difference is 0!

- The estimates of the proportions: $\hat{p}_X = X/n_x$, $\hat{p}_Y = Y/n_Y$
- Hypotheses: $H_0: p_x - p_Y = 0$ vs. $H_1: p_Y - p_Y \neq 0$ $\alpha = 10\ \%$
- Test variable:

$$Z = \frac{\hat{p}_X - \hat{p}_Y - 0}{\sqrt{\hat{p}_0(1 - \hat{p}_0)\left(\frac{1}{n_X} + \frac{1}{n_Y}\right)}} \sim N(0,1)$$

APPROXIMATION! CLT

where $\hat{p}_0 = \frac{n_X\hat{p}_X + n_Y\hat{p}_Y}{n_X + n_Y}$

COMPARE TO POOLED s_p^2 P. 24

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exercise/example 5, cont.

- critical value and decision rule:
  - reject $\boldsymbol{H_0}$ if $|\boldsymbol{z_{obs}}| > \boldsymbol{z_{\alpha/2}} = \boldsymbol{z_{0,05}} = \mathbf{1,6449}$

- We observe $\boldsymbol{\hat{p}_X} = \mathbf{0,30}$ and $\boldsymbol{\hat{p}_Y} = \mathbf{0,40}$

$$\boldsymbol{\hat{p}_0} = \frac{0{,}3 + 0{,}4}{2} = \mathbf{0,35} \qquad |\boldsymbol{z_{obs}}| = \left| \frac{0{,}30 - 0{,}40 - 0}{\sqrt{0{,}35 \cdot 0{,}65 \left(\frac{1}{200} + \frac{1}{200}\right)}} \right| = \mathbf{1,4825}$$

- Conclusion and interpretation:
  - Null hypothesis $\boldsymbol{P_x} = \boldsymbol{P_Y}$ cannot be rejected at $\alpha = 0{,}05$
  - The proportions are not significantly different

p-values

Red = reject H_0
Blue = fail to reject

exercise/example 1-5 (draw!):

1. $p = P(\boldsymbol{Z} > \mathbf{1,4142} \mid H_0 \text{ true}) \approx 1 - P(Z \leq 1{,}41) = \mathbf{0,07927}$

2. $p = P(\boldsymbol{Z} < \mathbf{-2,000} \mid H_0 \text{ true}) = 1 - P(Z \leq 2) = \mathbf{0,02275}$

3b. $p = P(|\boldsymbol{t}| > \mathbf{2,0576} \mid H_0 \text{ true}) = 2 \cdot P(t > 2{,}0576) = \mathbf{0,04841}$

4. $p = P(\boldsymbol{t} < \mathbf{-0,7533} \mid H_0 \text{ true}) = 1 - P(t \leq 0{,}7533) = \mathbf{0,2379}$

5. $p = P(|\boldsymbol{Z}| > \mathbf{1,4825} \mid H_0 \text{ true}) \approx 2 \cdot P(Z > 1{,}48) = \mathbf{0,1389}$

NOTE! Example 3b and 4 are ***t*-tests**. For these, we need a computer to calculate *p*-values. Tabell 3 cannot be used.

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Formulas for CI and hypothesis testing

- ***Do I need to remember all the formulas?***
  - No, use the formulary
- ***There are different formulas depending on the situation***
  - What you need to learn is whether to use ***z*** or ***t*** and how to read, interpret, and apply the formulas
- ***You are required to understand how assumptions and facts determine the solution***
- If you assume that the case is not secure (H_0) you can then choose the risk of Type I error, i.e. set α at an appropriate level.
  - Normal distribution, independence, large/small sample, known/unknown variance, easy estimates or comparisons between groups
  - suitable estimate and its distribution and CI
  - Test variable and its distribution, critical levels (hypothesis test)

power – using an example

- Given: normal distributed, independent observations, known variance $\boldsymbol{\sigma^2 = 16,\ n = 100,\ \ \sigma/\sqrt{n} = 0,4}$
- Hypothesis testing (one-sided):

$$\boldsymbol{H_0\!: \mu = 0} \quad \text{vs. } \boldsymbol{H_1\!: \mu > 0, \quad \alpha = 0,05}$$

- Decision rule: reject $\boldsymbol{H_0}$ if $\boldsymbol{z_{obs} > 1,6449} \ \Leftrightarrow\ \boldsymbol{\overline{x}_{obs} > 0,6580}$

$$\boldsymbol{z_{krit}} = \frac{\bar{x} - 0}{4/10} = \boldsymbol{1,6449} \ \Leftrightarrow\ \boldsymbol{\overline{x}_{krit}} = 0{,}4 \cdot 1{,}6449 = \boldsymbol{0,6580}$$

$$\textbf{power} = P(\text{reject } H_0 | H_0 \text{ not true}) \quad = \boldsymbol{P(\overline{X}_{obs} > \overline{x}_{krit} |\, \mu = \mu_1 \neq \mu_0)}$$

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power – using an example, cont.

- Some cases: $\boldsymbol{\mu = 0,5\,;1\,;2\,;5\,;0\,;-0,5\,;\,-1}$
- the probability of rejecting $\boldsymbol{H_0} = \boldsymbol{P(\bar{X} > 0,658)}$ for these μ:

$$\boldsymbol{\mu = 0,5}:\ P(\bar{X} > 0{,}658) = P\left(Z > \frac{0{,}658 - \mathbf{0{,}5}}{0{,}4}\right) = P(Z > 0{,}395) = \mathbf{0,3464}$$

$$\boldsymbol{\mu = 1}:\ P(\bar{X} > 0{,}658) = P\left(Z > \frac{0{,}658 - \mathbf{1}}{0{,}4}\right) = P(Z > -0{,}855) = \mathbf{0,8037}$$

$$\boldsymbol{\mu = 2}:\ P(\bar{X} > 0{,}658) = P\left(Z > \frac{0{,}658 - \mathbf{2}}{0{,}4}\right) = P(Z > -3{,}355) = \mathbf{0,9996}$$

$$\boldsymbol{\mu = 5}:\ P(\bar{X} > 0{,}658) = P\left(Z > \frac{0{,}658 - \mathbf{5}}{0{,}4}\right) = P(Z > -10{,}855) \approx \mathbf{1}$$

$$\boldsymbol{\mu = 0}:\ P(\bar{X} > 0{,}658) = P\left(Z > \frac{0{,}658 - \mathbf{0}}{0{,}4}\right) = P(Z > -1{,}6449) = \mathbf{0,050}$$

$= \boldsymbol{\alpha}$. Here $\boldsymbol{H_0}$ is true!

$$\boldsymbol{\mu = -0,5}:\ P(\bar{X} > 0{,}658) = P\left(Z > \frac{0{,}658 - (\mathbf{-0{,}5})}{0{,}4}\right) = P(Z > 2{,}895) = \mathbf{0,0019}$$

Note! The decision rule is always the same, it always starts from H_0

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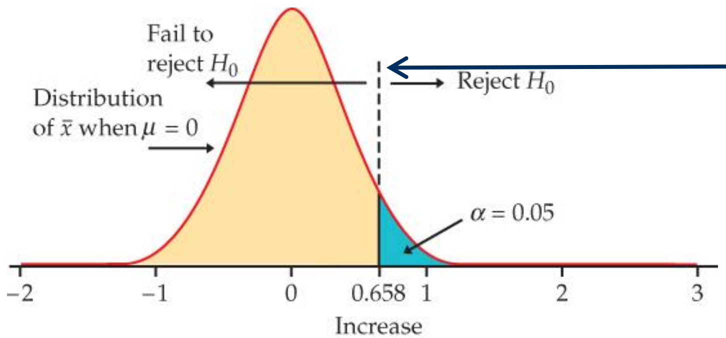
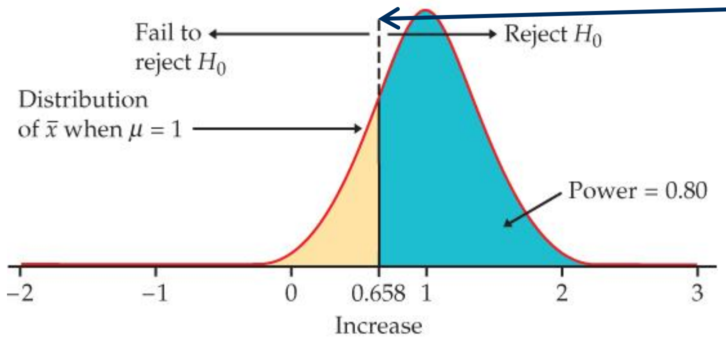
power, forts

When H_0 is true:
$\mu = \mu_0 = 0$



Distribution and critical level under H_0 and $\mu = \mu_0 = 0$

When H_0 is <u>not</u> true: e.g. $\mu = 1$



Same critical value as above, except $\mu = 1$ and $\mu_0 = 0$

Bild: Moore et al. (2012) *Introduction to the Practice of Statistics*. New York: Freeman

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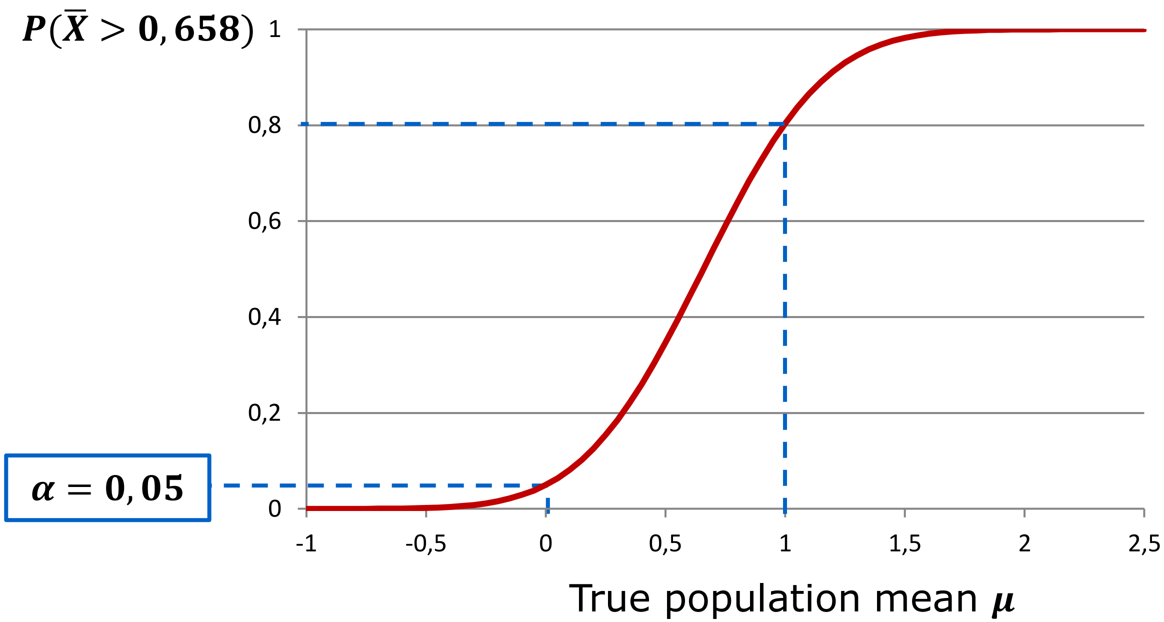
Power function of a test

- the probability to reject H_0 i.e. $P(\bar{X} > 0,658)$ as a function of μ
  (if $\mu =$ some value, what is the probability that we reject H_0?)



We want a test with **high power**, the test should be good at **identifying false null hypotheses**.

You want the **curve** to be **steep**!

If we **increase** n, the curve gets **steeper** .

What did you conclude?

The example of the lion cage.

What should be the null hypothesis?

- ”The cage is secure”

or

- ”The cage is not secure”