Basic Statistics for Economists

Spring 2020

Department of Statistics

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L4

From last lecture

- **Sample space** – all that is possible

$$S = \{1, 2, 3, 4, 5, 6\}$$

- **Events** – a subset of the sample space

$$A = \{\text{"odd number"}\} = \{1, 3, 5\} \subset S$$

- **Set theory** – formal theory for handling sets

$$A \cup B, \qquad A \cap B, \qquad A \cup \bar{A} = S \qquad A \cap \bar{A} = \emptyset$$

- **Venn diagram** – a method for visualizing sets and their relationships

- **Probability,** $P(A)$ – a function of A

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More from last lecture

- A **stochastic model** requires that
  1. The sample space S is well-defined
  2. For every event $A \subseteq S$, $P(A)$ is obtainable, incl. $P(S)$ and $P(\emptyset)$
- **Combinatorics – art of counting possibilities**
  - Multiplication principal: $\boldsymbol{m_1 \cdot m_2}$
  - Drawing without or with replacement

    $n \cdot (n-1) \cdot \dots \cdot (n-x-1)$ or $n \cdot n \cdot \dots \cdot n = n^x$
  - Factorial: $\boldsymbol{x! = x \cdot (x-1) \cdot \dots \cdot 2 \cdot 1}$

    $3! = 6 \qquad 6! = 720 \qquad 60! \approx 8{,}32 \cdot 10^{80}$
  - Order matters or not

An example of an infinite sample space

- Tossing a coin repeatedly
  - Sample space each individual toss: $\boldsymbol{S_i} = \{\mathbf{head}, \mathbf{tail}\}$
  - Count the number of tails until the first “head”
  - Sample space: $\boldsymbol{S} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots\}$
  - This is an often used model called “**geometric distribution**”

First Second Third

Outcome: “2 tails”

This lecture

- **Continuation of probability theory**
  - Some more combinatorics
  - Frequentist and subjective interpretations
  - Probability postulates
  - Calculating probabilities, probability rules
    - E.g. complement rule, addition rule, …
  - Conditional probabilities
    - multiplication rule
    - **statistical independence**
  - Bivariate probabilities – two events

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See L3 p. 44

Permutations

- Number of possible arrangements when $\boldsymbol{x}$ objects are drawn from a total of $\boldsymbol{n}$, without replacement, **order matters**:

$$\boldsymbol{P_x^n} = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-x+1) = \frac{\boldsymbol{n}!}{(\boldsymbol{n}-\boldsymbol{x})!}$$

- Note that $(\boldsymbol{n} - \boldsymbol{x})$ objects remain after drawing $\boldsymbol{x}$

- Based on the multiplication principal!

- Ex. $\boldsymbol{P_3^{10}} = \frac{\mathbf{10!}}{(\mathbf{10}-\mathbf{3})!} = \frac{\mathbf{10 \cdot 9 \cdot 8} \cdot \cancel{\mathbf{7 \cdot 6 \cdot \cdots \cdot 2 \cdot 1}}}{\cancel{\mathbf{7 \cdot 6 \cdot \cdots \cdot 2 \cdot 1}}} = \mathbf{10 \cdot 9 \cdot 8 = 720}$

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Example permutations

- Suppose that $\boldsymbol{x = 2}$ letters are selected (without replacement) from the set of $\boldsymbol{n = 4}$ letters **A**, **B**, **C**, **D**

- How many ***permutations*** are there?

$$P_2^4 = 4 \cdot 3 = 12$$

$$P_2^4 = \frac{4!}{(4-2)!} = \frac{4!}{2!} = \frac{24}{2} = 12$$

- The permutations are:

AB	**AC**	**AD**	**BC**	**BD**	**CD**
BA	**CA**	**DA**	**CB**	**DB**	**DC**

We consider different orderings as separate outcomes!

Combinations

- The number of outcomes when $\boldsymbol{x}$ objects are drawn from a total of $\boldsymbol{n}$, without replacement, and **order <u>doesn't</u> matter**:

$$\boldsymbol{C_x^n} = \frac{n!}{x!\,(n-x)!} = \binom{\boldsymbol{n}}{\boldsymbol{x}}$$

pronounced "n choose x"

- Note that $(n - x)$ objects are left "in the bag" after drawing $\boldsymbol{x}$
- Think about it and understand why

$$\binom{n}{x} = \binom{n}{n-x} \qquad \binom{n}{1} = \binom{n}{n-1} = n \qquad \binom{n}{0} = \binom{n}{n} = 1$$

- Ex. $\boldsymbol{C_3^{10} = \frac{10!}{3!\,7!} = \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7 \cdot 6 \cdot \dots \cdot 2 \cdot 1}}{3 \cdot 2 \cdot 1 \cdot \cancel{7 \cdot 6 \cdot \dots \cdot 2 \cdot 1}} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = \frac{720}{6} = 120}$

Example combinations

- Suppose that $\boldsymbol{x} = \mathbf{2}$ letters are selected (without replacement) from the set of $\boldsymbol{n} = \mathbf{4}$ letters **A**, **B**, **C**, **D**

- How many ***combinations*** are there??

$$C_2^4 = \binom{4}{2} = \frac{4!}{2!\,(4-2)!} = \frac{4!}{2! \cdot 2!} = \frac{24}{2 \cdot 2} = 6$$

- The combinations are:

AB	(same as BA)	**AC**	(same as CA)
AD	(same as DA)	**BC**	(same as CB)
BD	(same as DB)	**CD**	(same as DC)

We disregard the specific order in which they were drawn!

Select *x* from *n* – how many ways?

- Four cases:

	Order matters	Order doesn't matter
With replacement	n^x	C_x^{n+x-1} Skip this one!
Without replacement	$P_x^n = \dfrac{n!}{(n-x)!}$	$C_x^n = \binom{n}{x} = \dfrac{n!}{x!\,(n-x)!}$

- With replacement: objects may be drawn more than once
- Without replacement: objects can at most be drawn once
- Think **multiplication principal**!

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DIY Exercise

A musician has a very small piano with only 13 keys (*tangenter*), 8 white and 5 black (a full octave), where each key plays a unique tone.

a) If she plays a melody consisting of an ordered sequence of 3 optional tones, how many unique melodies can she play? A key may be played more than once. Does the order matter?

b) A basic chord usually consists of three unique tones/keys that are played together in one stroke. How many different chords can be played on this piano? Does the selection order matter? Selection with or without replacement?

Frequentistic interpretation

An intuitive interpretation/understanding of probability is to relate it to how often a given event $\boldsymbol{A}$ will occur.

- An experiment is repeated several times and we count the number of occurrences where the outcome is $\boldsymbol{A}$
- After $\boldsymbol{n}$ times we note how many times $\boldsymbol{A}$ has happened (successes); the number is $\boldsymbol{n_A}$
- The ratio $\boldsymbol{n_A/n}$ is the **relative frequency** of event $\boldsymbol{A}$
- The ratio tends to stabilize as $\boldsymbol{n}$ increases

$$\boldsymbol{n_A/n \to P(A)} \quad \text{when} \quad \boldsymbol{n \to \infty}$$

Note! Compare with L2 p. 25

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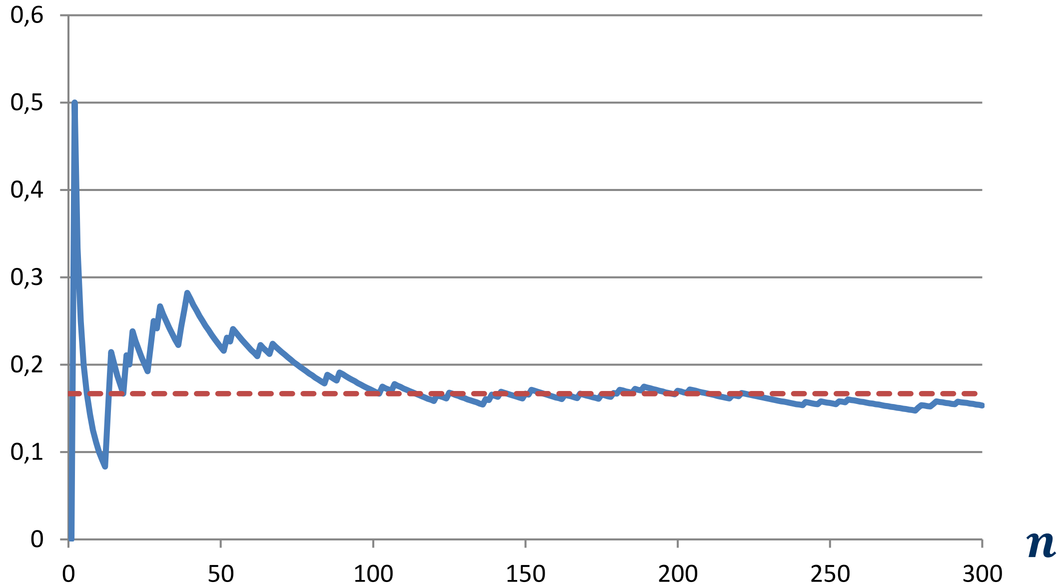
Frequentistic interpretation, cont.

Ex. Toss a die and count n_2 = the number of times we get a “2”:



$\frac{n_2}{n}$

Compare L2 p. 25!

After a sufficient number of tosses (n) the ratio tend to stabilize on a specific value.

What is this value?

(Not important that you fully understand this!)

Subjective probability

Probabilities represent a state of knowledge (or ignorance) before the experiment is conducted; it is a measure of personal belief.

We call these ***subjective probabilities***; you decide!

- Ex. The highest price **you** are willing to pay on a bet that $\boldsymbol{A}$ will occur where you gain 1 SEK if you win and 0 SEK if you lose
- You are willing to pay up to 0.80; what is your belief/probability that $\boldsymbol{A}$ will happen? Given that you are a rational person?
- Win: 0.20 in 8 of 10   Loss: 0.80 in 2 of 10 $P(A) = 0.80$

If many individual bets are placed, the total amount of bets on a win and against may be weighed together to a collective subjective probability.

Interpreting probabilities - summary

- ***Classical***

  $\#(A)/\#(S) = P(A)$

  $size(A)/size(S) = P(A)$

- ***Frequentist***

  $n_A/n \rightarrow P(A)$ when $n \rightarrow \infty$

- ***Subjective (personal)***

  rational personal judgement, expertise, fair bets

Really doesn't matter to us although the concepts are useful in determining how to calculate probabilities, especially the classical approach.

However, we have to define certain conditions in order to achieve mathematical consistency!

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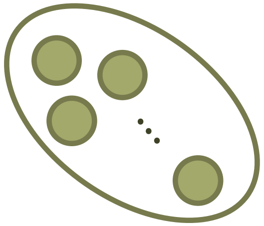
Probability postulates, NCT p. 107

1. If A is any event in S ($A \subseteq S$) then

   $$\mathbf{0 \leq P(A) \leq 1}$$ **Important!**

   i.e. all probabilities are numbers from 0 to 1

2. For all $A \subseteq S$ we have

   $A = O_1 \cup \cdots \cup O_K$

   $$\boldsymbol{P(A) = \sum_A P(O_k)}$$

   i.e. the sum of probabilities of each of the individual and **disjoint** basic outcomes O_k that are members of A

3. The probability for the entire sample space is

   $$\boldsymbol{P(S) = 1}$$ **Something has to happen!**

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An axiomatic theory – almost the same

Kolmogorov's axioms: A probability is a function $\boldsymbol{P}(\cdot)$ that assigns a number $\boldsymbol{P}(\boldsymbol{A})$ to every possible event $\boldsymbol{A} \subseteq \boldsymbol{S}$ in the sample space $\boldsymbol{S}$, such that the following conditions are met:

- $P(A) \geq 0$ **probabilities are never negative**
- $P(S) = 1$ **something must happen with probability 1**
- For all pairwise ***disjoint*** events A_1, ..., A_k in S, we have

$$P(A_1 \cup A_2 \cup \cdots \cup A_k) = P(A_1) + P(A_2) + \cdots + P(A_k)$$

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$$

Kolmogorov's system is often preferred

Conclusion

- **All** functions $\boldsymbol{P}$ that meet the probability postulates (Kolmogorov's axiom system) are **probability functions**

Example: ”Check if the following function $P(\cdot)$ is a valid probability function”. What do we do?

  – check that it always produces ***positive*** values for all $\boldsymbol{O_k} \in \boldsymbol{S}$
  – check that the ***sum*** of all probabilities is = 1 $\left(\sum_{\boldsymbol{S}} \boldsymbol{P}(\boldsymbol{O_k}) = \mathbf{1}\right)$

Example 1

- Experiment with four possible outcomes $S = \{\mathrm{C}, \mathrm{V}, \mathrm{R}, \mathrm{S}\}$

- ”Function”:

Outcome	C	V	R	S
$P(\cdot)$	0,35	0,15	0,25	0,25

- Is this a proper probability function / distribution?

1. All probabilities are $\geq \mathbf{0}$
2. $P(\mathrm{C}) + P(\mathrm{V}) + P(\mathrm{R}) + P(\mathrm{S}) = 0{,}35 + 0{,}15 + 0{,}25 + 0{,}25 \mathbf{= 1}$

Conclusion: Yes

Example 2

- An experiment med three possible outcomes: $S = \{1,2,3\}$

- Function: $P(x) = \dfrac{4-x}{a}$

- For what value on $\boldsymbol{a}$ is this a probability function?

1. $P(1) = \dfrac{4-1}{a} = \dfrac{\mathbf{3}}{\boldsymbol{a}} > \mathbf{0}$ $\quad P(2) = \dfrac{4-2}{a} = \dfrac{\mathbf{2}}{\boldsymbol{a}} > \mathbf{0}$ $\quad P(3) = \dfrac{4-3}{a} = \dfrac{\mathbf{1}}{\boldsymbol{a}} > \mathbf{0}$

2. $P(1) + P(2) + P(3) = \dfrac{3}{a} + \dfrac{2}{a} + \dfrac{1}{a} = \dfrac{\mathbf{6}}{\boldsymbol{a}} = \mathbf{1} \quad \Rightarrow \quad \boxed{\boldsymbol{a} = \mathbf{6}}$

Complement rule

- Let $\boldsymbol{A}$ be an event in $\boldsymbol{S}$ and $\overline{\boldsymbol{A}}$ its complement (opposite), then

$$\boldsymbol{P}(\overline{\boldsymbol{A}}) = \mathbf{1} - \boldsymbol{P}(\boldsymbol{A})$$

← Easy to understand and to remember!

Proof: according to the definition of $\overline{\boldsymbol{A}}$ we have

$$\left.\begin{array}{l} A \cup \bar{A} = S \\ A \cap \bar{A} = \emptyset \end{array}\right\} \Rightarrow P(A \cup \bar{A}) = P(A) + P(\bar{A}) = P(S) = 1$$

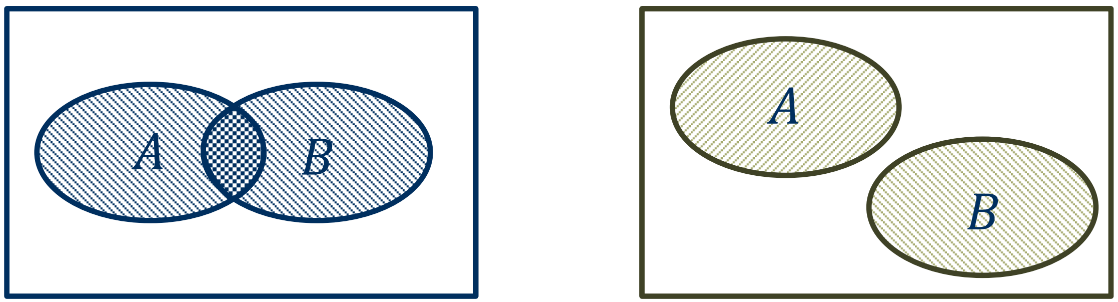
Compare with 2nd postulate or Kolmogorov's 3rd axiom

$\boldsymbol{A}$ and $\overline{\boldsymbol{A}}$ are ***collectively exhaustive*** (see NCT p. 98+109)
- together they cover the whole of $\boldsymbol{S}$

Addition rule

- The probability that A or B or both A and B occur:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$ **← Important!**



- If A and B are <u>disjoint</u> i.e. $A \cap B = \emptyset$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B)$$

(compare with 2nd postulate or Kolmogorov's 3rd axiom)

An axiomatic theory, cont.

- The frequentist, classical and subjective interpretations are all compatible with the postulates/Kolmogorov

- Formal definition of a probability model:
  - Probabilities $P(\cdot)$ are real-valued numbers
  - Samples space S is well-defined
  - For all $A \subseteq S$, $P(A)$ must be obtainable/calculable
  - The postulates/Kolmogorov's axioms apply

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Not the same definition of odds given by bookmakers, see e.g. https://en.wikipedia.org/wiki/Odds

Odds

- Alternative representation to probabilities
- Compare the probability of the event and it's complement:

$$Odds = \frac{P(A)}{P(\bar{A})} = \frac{P(A)}{1 - P(A)}$$

- Ex. Of the odds are say 1,5 = 3/2 the event will occur 3 out of 5 times and the complement 2 out of 5
- If you know the odds you know the probability :

$$P(A) = \frac{Odds}{1 + Odds}$$

Skip the section on "overinvolvement ratios"

Conditional probabilities

- (sv. *betingade sannolikheter*)

- The probability that A occurs given that B occurs or has occured
- Example tossing dice:
  - Probability for “6” **given** that it is an even number?
  - Probability for “6” **given** that it is an odd number?
  - Probability for “6” **given** that it is <u>not</u> a “5”?

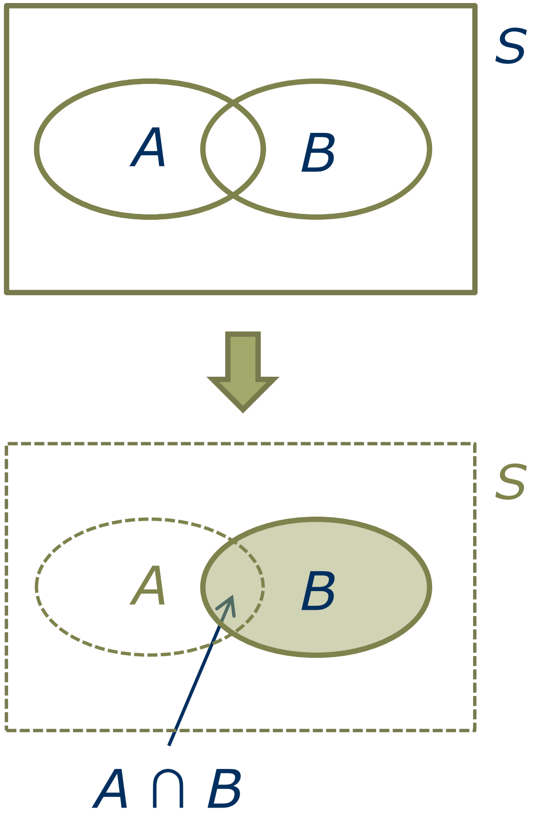
- ***What happens to the sample space when we condition?***

Conditional probabilities, cont.

- Since the event $\boldsymbol{B}$ has occurred we can in effect say that the available sample space S has been affected

- $\overline{\boldsymbol{B}}$ has <u>not</u> happened, i.e. we can disregard that part of $\boldsymbol{S}$

- The probability for $\boldsymbol{A}$ is obtained by comparing the part of $\boldsymbol{A}$ that intersects (overlaps) with $\boldsymbol{B}$, i.e. $\boldsymbol{P}(\boldsymbol{A} \cap \boldsymbol{B})$ relative to $\boldsymbol{P}(\boldsymbol{B})$



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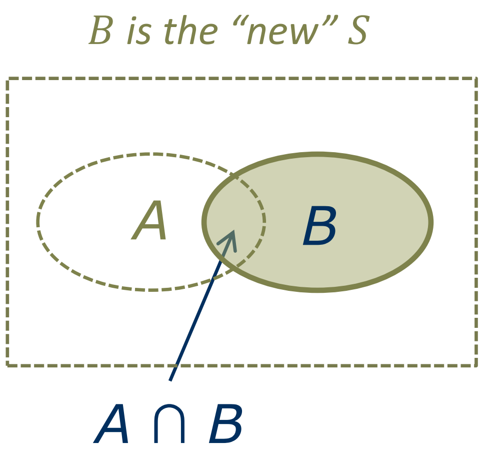
Conditioning ***(sv. betingning)***

- Instead of considering the ratio

$$size(A)/size(S) = P(A)$$

we consider the ratio

$$size(A \cap B)/size(B) = P(A|B)$$



$P(A|B)$ is read as "the probability of A **given** B"

and is calculated

$$\boldsymbol{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

Important!

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Conditioning, cont.

- We can re-arrange the formula:

$$P(A \cap B) = P(A|B) \cdot P(B)$$

- The ***reversed*** conditioning, $\boldsymbol{B}$ **given** $\boldsymbol{A}$:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \neq \frac{P(A \cap B)}{P(B)} = P(A|B)$$

Equal if and only if $P(A) = P(B)$

$$P(A \cap B) = P(B|A) \cdot P(A)$$

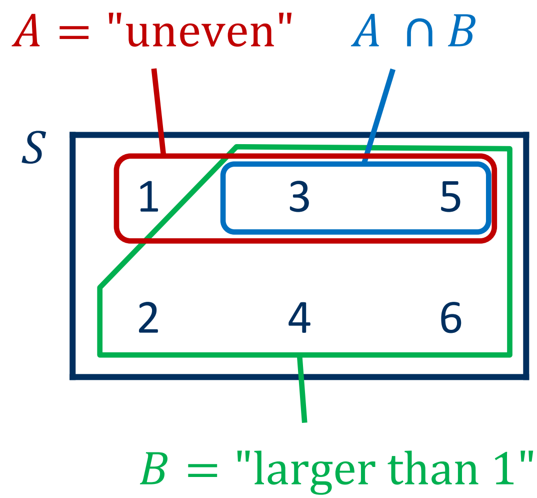
and thus

$$\boldsymbol{P(A|B) \cdot P(B) = P(B|A) \cdot P(A)}$$

Important!

Conditioning – example

Throw a die: $\boldsymbol{S} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}\}$



$P(\text{"larger than 1"}|\text{"uneven"}) = ?$

$$P(\text{"uneven"}) = P(1) + P(3) + P(5) = \frac{\mathbf{3}}{\mathbf{6}}$$

$$P(\text{"larger than 1"}) = 1 - \mathrm{P}(1) = \frac{\mathbf{5}}{\mathbf{6}}$$

$$P(\text{"larger than 1"} \cap \text{"uneven"}) = P(3) + P(5) = \frac{\mathbf{2}}{\mathbf{6}}$$

$$P(\text{"larger than 1"}|\text{"uneven"}) = \frac{P(\text{"larger than 1"} \cap \text{"uneven"})}{P(\text{"uneven"})} = \frac{2/6}{3/6} = \frac{\mathbf{2}}{\mathbf{3}}$$

NOTE: $\frac{2}{3} \neq P(\text{"larger than 1"})$ $\quad \frac{2}{3} \neq P(\text{"uneven"}|\text{"larger than 1"}) = \frac{2}{5}$

Multiplication rule

Let A and B be two events. The probability of the intersection $\boldsymbol{A \cap B}$ is obtainable from the conditional probabilities:

$$\boldsymbol{P(A \cap B) = P(A|B) \cdot P(B)}$$

and alternatively

Important!

$$\boldsymbol{P(A \cap B) = P(B|A) \cdot P(A)}$$

- What are the conditional probabilities if $\boldsymbol{A}$ and $\boldsymbol{B}$ are disjoint?

Statistical independence

- When an event A occurs or not without regard to the other event B

Example:

- You are in Stockholm and your friend is in Rome, you each toss a die:
  - What is the probability that you throw a six?
  - What is the probability that both of you throw sixes?
  - What is the probability that you throw a six **given** that your friend throws a six?

Definition of statistical independence

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two events. If

$$\boldsymbol{P}(\boldsymbol{A} \cap \boldsymbol{B}) = \boldsymbol{P}(\boldsymbol{A}) \cdot \boldsymbol{P}(\boldsymbol{B})$$

then $\boldsymbol{A}$ and $\boldsymbol{B}$ are **statistically independent**.

Important!

Equivalence: $\boldsymbol{A}$ and $\boldsymbol{B}$ independent $\Leftrightarrow$ $\boldsymbol{P}(\boldsymbol{A} \cap \boldsymbol{B}) = \boldsymbol{P}(\boldsymbol{A}) \cdot \boldsymbol{P}(\boldsymbol{B})$

If we know (assume) that they are independent, then we can calculate the probability of the intersection. If it turns out that the probability of the intersection equals the product $\boldsymbol{P}(\boldsymbol{A}) \cdot \boldsymbol{P}(\boldsymbol{B})$, then they are independent.

Consequences of independence

- If two events A and B are ***independent*** then

$$\boldsymbol{P(A|B) = P(A)}$$

$$\boldsymbol{P(B|A) = P(B)}$$

The probability that one will occur is not affected by what happens with the other.

- This is a direct consequence of the definitions since

$$\boldsymbol{P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \cdot P(B)}{P(B)} = P(A)}$$

Definition of *P(A|B)*

A and *B* are independent

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Consequences of independence, cont.

- If $\boldsymbol{A}$ and $\boldsymbol{B}$ are independent, then they are also both independent of each other's complement and the complements are also independent of each other.
- I.e. all four couples of events

$$(A, B)\ \ (\bar{A}, B)\ \ (A, \bar{B})\ \ (\bar{A}, \bar{B})$$

are statistically independent.

Ex. $\boldsymbol{P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B}) = [1 - P(A)] \cdot [1 - P(B)]}$

Example 1

Assume $P(A) = 0.4$, $P(B) = 0.3$ and $P(A|B) = 0$; $\boldsymbol{S}$ is discrete.

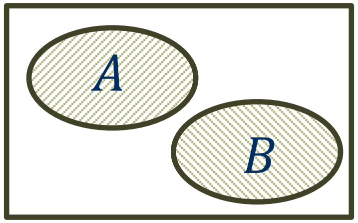
a) Are $\boldsymbol{A}$ and $\boldsymbol{B}$ **disjoint** (non-overlapping)?

Solution: check with the multiplication rule:

$$P(A \cap B) = P(A|B) \cdot P(B) = 0 \cdot 0.3 = 0 \quad \Rightarrow \quad A \cap B = \emptyset \Rightarrow$$

$\Rightarrow A$ and B are disjoint

b) Draw a Venn diagram:



c) Are $\boldsymbol{A}$ and $\boldsymbol{B}$ statistically **independent** or **dependent**?

$P(A \cap B) = 0$ from a) above but $P(A) \cdot P(B) = 0.12$

$\Rightarrow A$ and B are **dependent**

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Example 2

- Assume that we have the following probabilities:

$$P(A) = 0{,}8 \qquad P(B) = 0{,}2 \qquad P(A \cap B) = 0{,}16$$

a) Calculate the conditional probability of A given B

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0{,}16}{0{,}20} = 0{,}8$$

b) Are A and B independent events?

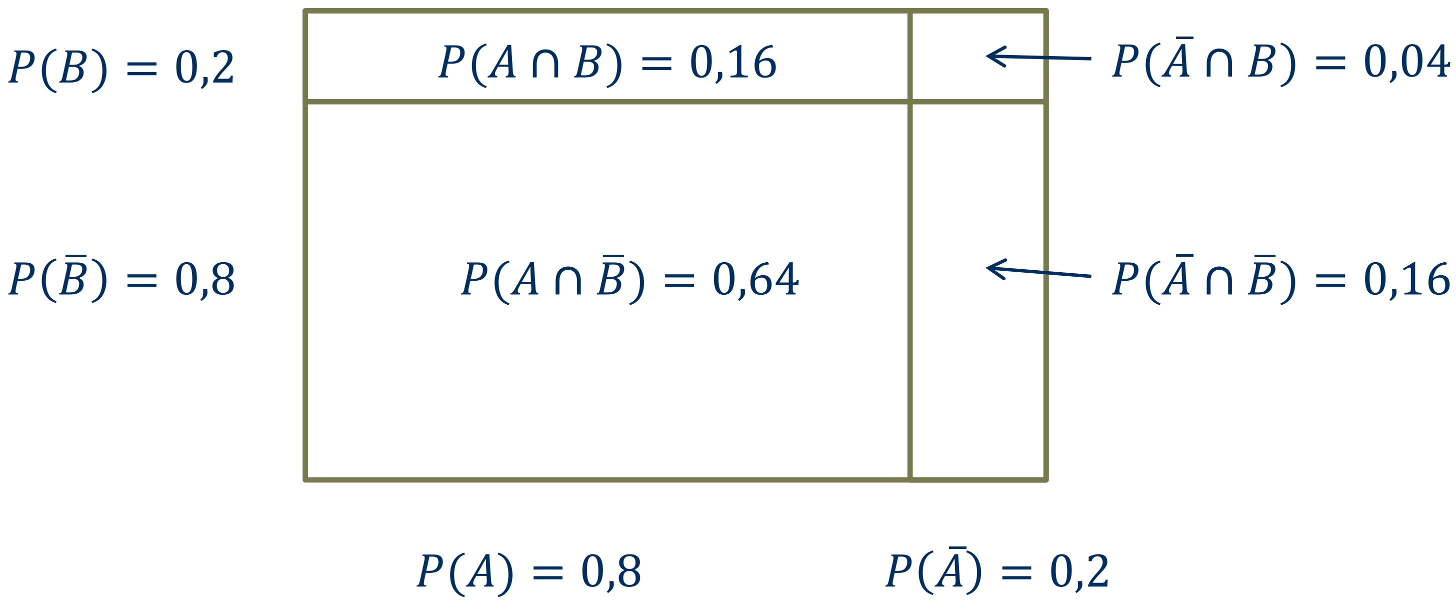
$$P(A|B) = P(A) = 0{,}8 \iff \text{yes, independent}$$

c) Calculate $P(\bar{A} \cap B)$

$$P(\bar{A} \cap B) = [1 - P(A)] \cdot P(B) = 0{,}2 \cdot 0{,}2 = 0{,}04$$

Example 2, cont.

- A Venn diagram:



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Example 2, cont.

- Or as a table of probabilities:

	A	$\bar{A}$	Sum
B	0,16	0,04	0,20
$\bar{B}$	0,64	0,16	0,80
Sum	0,80	0,20	1,00

The cell value is given as the product of the corresponding marginals:

$$\mathbf{0,2 \cdot 0,8 \ = \ 0,16}$$

$$\boldsymbol{P(A) \cdot P(B) = P(A \cap B)}$$

- This <u>only</u> holds when $\boldsymbol{A}$ and $\boldsymbol{B}$ are $\boxed{\textbf{independent}!}$

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Example 2, cont.

- Assume instead this table of probabilities:

	A	$\bar{A}$	Sum
B	0,10	0,10	0,20
$\bar{B}$	0,70	0,10	0,80
Sum	0,80	0,20	1,00

The product of the marginals:

$$\mathbf{0,2 \cdot 0,8 = 0,16 \neq 0,10}$$

$$\boldsymbol{P(A) \cdot P(B) \neq P(A \cap B)}$$

- Are $\boldsymbol{A}$ and $\boldsymbol{B}$ independent or dependent?

Joint probabilities, extending the idea

- Assume $A_1 \dots A_H$ are disjoint and collectively exhaustive
- Assume $B_1 \dots B_K$ are disjoint and collectively exhaustive

	A_1	…	A_H	Summa
B_1	$P(A_1 \cap B_1)$	…	$P(A_H \cap B_1)$	$P(B_1)$
B_2	$P(A_1 \cap B_2)$	…	$P(A_H \cap B_2)$	$P(B_2)$
$\vdots$	$\vdots$		$\vdots$	$\vdots$
B_K	$P(A_1 \cap B_K)$	…	$P(A_H \cap B_K)$	$P(B_K)$
Summa	$P(A_1)$	…	$P(A_H)$	$P(S) = 1$

Joint distribution for A and B

Marginal distribution for $B_1 \dots B_K$

Marginal distribution for $A_1 \dots A_H$

Joint probabilities, cont.

Column sum: $$P(A_1) = \sum_{k=1}^{K} P(A_1 \cap B_k) = \sum_{k=1}^{K} P(A_1|B_k)P(B_k)$$

	A_1	…	A_H	Sum
B_1	$P(A_1 \cap B_1)$	…	$P(A_H \cap B_1)$	$P(B_1)$
B_2	$P(A_1 \cap B_2)$	…	$P(A_H \cap B_2)$	$P(B_2)$
$\vdots$	$\vdots$		$\vdots$	$\vdots$
B_K	$P(A_1 \cap B_K)$	…	$P(A_H \cap B_K)$	$P(B_K)$
Sum	$P(A_1)$	…	$P(A_H)$	$P(S) = 1$

Row sum:

$$P(B_1) = \sum_{h=1}^{H} P(A_h \cap B_1)$$

$$= \sum_{h=1}^{H} P(B_1|A_h)P(A_h)$$

	1	2	3	4	5	6	sum
2	1/36	0	0	0	0	0	1/36
3	1/36	1/36	0	0	[illegible]	0	2/36
4	1/36	1/36	1/36	0	0	0	3/36
5	1/36	1/36	1/36	1/36	0	0	4/36
6	1/36	1/36	1/36	1/36	1/36	0	5/36
7	1/36	1/36	1/36	1/36	1/36	1/36	6/36
8	0	1/36	1/36	1/36	1/36	1/36	5/36
9	0	0	1/36	1/36	1/36	1/36	4/36
10	0	0	0	1/36	1/36	1/36	3/36
11	0	0	0	0	1/36	1/36	2/36
12	0	0	0	0	0	1/36	1/36
sum	1/6	1/6	1/6	1/6	1/6	1/6	1

Zeroes indicate dependency!

Marginal distribution for the sum of two dice

Marginal distribution for the first die

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Some more on conditioning

- Probabilities of the intersections are unknown but are obtainable through:

$$P(A \cap B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A)$$

see p. 28

- Hence, the **reversed conditioning is obtainable** through

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B) \cdot P(B)}{P(A)}$$

- We'll talk more about this next time (**Bayes' theorem**)