TD AAA : Weighted Automata

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Bonus exercises are more difficult, and some of them might take a long time. Better keep them for the end...

1 Semirings

1.1 Exercise

Give a detailed proof that $(\mathbb{N}, +, \cdot, 0, 1)$ forms a semiring.

1.2 Exercise

Which of the following structures form semirings?

1. $(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$
2. $(\mathbb{N} \cup \{-\infty\}, +, \max, 0, -\infty)$
3. $(\mathbb{N} \cup \{\infty\}, \max, \min, 0, \infty)$
4. $(\mathbb{N} \cup \{\infty\}, \min, \max, \infty, 0)$

1.3 Exercise (bonus)

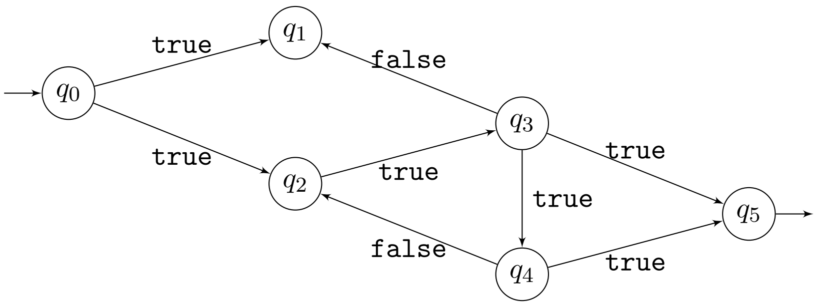
Let $\mathcal{F}$ be the set of functions $\mathbb{R}_+ \to \mathbb{R}_+$ (where $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$). Let max be pointwise maximum on $\mathcal{F}$, that is, $(\max(f, g))(x) = \max(f(x), g(x))$. Let $f \circ g$ denote usual function composition, that is, $(f \circ g)(x) = f(g(x))$.

1. What are the identity elements $\mathbb{0}$ and $\mathbb{1}$?
2. $(\mathcal{F}, \max, \circ, \mathbb{0}, \mathbb{1})$ does *not* form a semiring. Why?
3. There is an obvious subset $\mathcal{G} \subset \mathcal{F}$ such that $(\mathcal{G}, \max, \circ, \mathbb{0}, \mathbb{1})$ does form a semiring. Which is it?

2 Weighted automata

2.1 Exercise

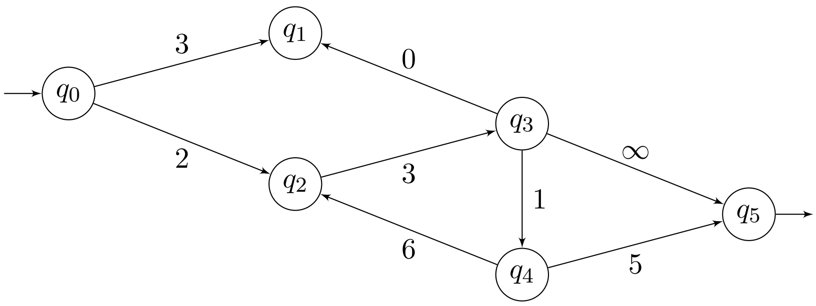
Let A be the following automaton over the boolean semiring:



What is $|A|$?

2.2 Exercise

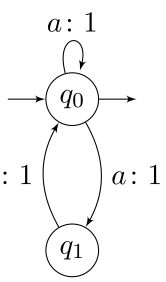
Let A be the following automaton over the semiring $(\mathbb{N} \cup \{\infty\}, \max, \min, 0, \infty)$:



What is $|A|$?

2.3 Exercise

Let A be the following automaton with input over the semiring $(\mathbb{N}, +, \cdot, 0, 1)$:



What is $|A|(a^6)$?

2.4 Exercise

Let $|w|_x$ denote the number of occurrences of substring x in word w and consider the semirings $S_1 = (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ and $S_2 = (\mathbb{N}, +, \cdot, 0, 1)$.

Find automata with input A_1, A_2, over the alphabet $\{a, b\}$ and over S_1, S_2, respectively and with $|A_i|(w) = |w|_{ab}$.

3 Small semirings

3.1 Exercise

Show that there are precisely two semirings S with two elements ($|S| = 2$). What are they? For each of the two S:

1. Describe weighted automata over S.
2. Describe $S\langle\langle\Sigma^*\rangle\rangle$.
3. Describe weighted automata over $S\langle\langle\Sigma^*\rangle\rangle$.

Hint: start by writing down the addition and multiplication tables for S, filling in all the cells which are given by the axioms.

3.2 Exercise (bonus)

Find all semirings with three elements. Are any of them non-commutative?

4 Star continuity

4.1 Exercise

Prove the lemma on p.70 of the slides of CM 1:

Lemma: If S is star-continuous, then $a^* = aa^* + 1 = a^*a + 1$ for all $a \in S$.

4.2 Exercise (bonus)

Use the recursive algorithm to compute stars of matrices, to compute

$$\begin{bmatrix} \{a\} & \{a,b\} & \emptyset \\ \emptyset & \emptyset & \{a\} \\ \{a\} & \emptyset & \{b\} \end{bmatrix}^*$$

in the language semiring.

4.3 Exercise

Intuitively, permuting the states in a weighted automaton $A = (\vec{i}, M, \vec{f})$ over a semiring S should not change the value $|A|$. Let's prove that this is true:

1. Let P be a permutation matrix in S. Show that P is invertible.
2. Permuting the states in A is done by changing M to PMP^{-1}. Show that $(PMP^{-1})^2 = PM^2P^{-1}$.
3. Show that $(PMP^{-1})^n = PM^nP^{-1}$ for any n and conclude that $(PMP^{-1})^* = PM^*P^{-1}$.
4. Conclude that if A' is the permuted automaton, i.e., $A' = (\vec{i}P^{-1}, PMP^{-1}, P\vec{f})$, then $|A'| = |A|$.

5 Idempotency

Definition: A semiring $(S, +, \cdot, 0, 1)$ is *idempotent* if $1 + 1 = 1$.

Idempotent semirings form an important subclass of semirings, mainly because many example semirings are idempotent.

5.1 Exercise

Show that $a + a = a$ for any $a \in S$ in an idempotent semiring S.

5.2 Exercise

Which of the following semirings are idempotent?

1. $(\mathbb{N} \cup \{\infty\}, \max, +, 0, \infty)$
2. $(\mathbb{N}, +, \cdot, 0, 1)$
3. $(\mathbb{N} \cup \{\infty\}, \max, \min, 0, \infty)$
4. the language semiring

5.3 Exercise (bonus)

Let S be an idempotent semiring. Let $\leq$ be the relation on S defined by $a \leq b$ iff $a + b = b$.

1. Show that $\leq$ is a partial order (i.e., it is reflexive, transitive, and antisymmetric)

$\leq$ is called the *natural order* on S.

Let now S also be star-continuous. Let $a \in S$. We know that $a^* = aa^* + 1$, that is, a^* is a fixed point for the mapping $f_a : S \to S$ given by $f_a(x) = ax + 1$.

2. Show that a^* is the *least* fixed point for f_a.

6 Linear systems and Conway semirings

6.1 Exercise

Let S be any semiring. Compute the solution of the following linear system over S:

$$y = (a + b)y + 1$$

6.2 Exercise

Let S be any semiring. Compute the solution of the following linear system over S:

$$\begin{aligned} y &= ay + bz + 1 \\ z &= y \end{aligned}$$

6.3 Exercise

In exercise 6.2 above, the equality $z = y$ implies the equality of the two components of the solution. Argue why this equality holds in general.

This equality is called the *sum-star-identity* and is one of two axioms of *Conway semirings*.

6.4 Exercise (bonus)

A *Conway semiring* is a semiring S together with a star operation $a \mapsto a^*$ in which the following hold for any $a, b \in S$:

$$(a+b)^* = (a^*b)^*a^* \qquad (ab)^* = 1 + a(ba)^*b$$

Derive the equalities

$$\begin{aligned} a^* &= 1 + aa^* = 1 + a^*a \\ a(ba)^* &= (ab)^*a \end{aligned}$$

from the axioms of Conway semirings.

6.5 Exercise (bonus)

Find (i.e., guess) two solutions of the (non-linear!) algebraic system

$$x = xx + a\,.$$

Compare your solutions to the language produced by the following grammar:

$$S \to SS \mid a$$

Argue why we are generally interested in *least* solutions to algebraic systems. (Compare with exercise 5.3.)