Timed Automata with Observers under Energy Constraints*

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ABSTRACT

In this paper, we study one-clock priced timed automata in which prices can grow linearly ($\dot{p} = k$) or exponentially ($\dot{p} = kp$), with discontinuous updates on edges. We propose EXPTIME algorithms to decide the existence of controllers that ensure existence of infinite runs or reachability of some goal location with non-negative observer value all along the run. These algorithms consist in computing the optimal delays that should be elapsed in each location along a run, so that the final observer value is maximized (and never goes below zero).

1. INTRODUCTION

Priced timed automata [5, 3] are emerging as a useful formalism for formulating and solving a broad range of real-time resource allocation problems of importance in application areas such as, *e.g.*, embedded systems. In [6] we began the study of a new class of resource scheduling problems, namely that of constructing infinite schedules or strategies subject to boundary constraints on the accumulation of resources.

More specifically, we proposed priced timed automata with *positive* as well as *negative* price-rates. This extension allows for the modelling of systems where resources are not only consumed but also occasionally produced or regained, *e.g.* would be useful in scheduling the behaviour of an autonomous robot which during operation occasionally may need to return to its base in order not to run out of energy. As an example

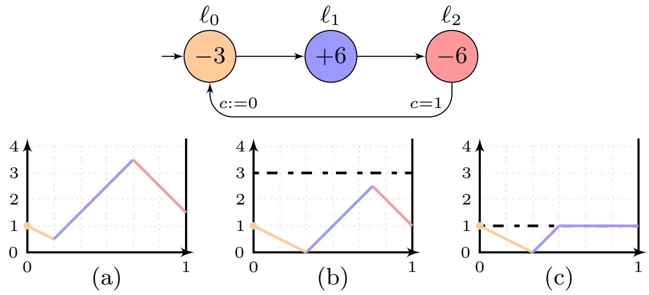

Figure 1: One-clock priced timed automaton and three types of infinite schedules: lower-bound (a), lower-upper-bound (b) and lower-weak-upper-bound (c).

consider the priced timed automaton in Figure 1 with infinite behaviours repeatedly delaying in ℓ_0, ℓ_1 and ℓ_2 for a total duration of one time unit. The negative weights (-3 and -6) in ℓ_0 and ℓ_2 indicate the rates by which energy will be consumed, and the positive rate ($+6$) in ℓ_1 indicates the rate by which energy will be gained. Thus, for a given iteration the effect on the energy remaining will highly depend on the distribution of the one time unit over the three locations.

In [6] three infinite scheduling problems for one-clock priced timed automata have been considered: the existence of an infinite schedule (run) during which the energy level never goes below zero (*lower-bound*), never goes below zero nor above an upper bound (*interval-bound*), and never goes below zero nor above a *weak* upper bound, which does not prevent energy-increasing behaviour from proceeding once the upper bound is reached but merely maintains the energy level at the upper bound (*lower-weak-upper-bound*). Figure 1 illustrates the three types of schedules given an initial energy level of one.

For one-clock priced timed automata both the lower-bound and the lower-weak-upper-bound problems are shown decidable (in polynomial time) [6], whereas the interval-bound problem is proved to be undecidable in a game setting. Decidability of the interval-bound problem for one-clock priced timed automata as well as decidability of all of the considered scheduling problems for priced timed automata with two or

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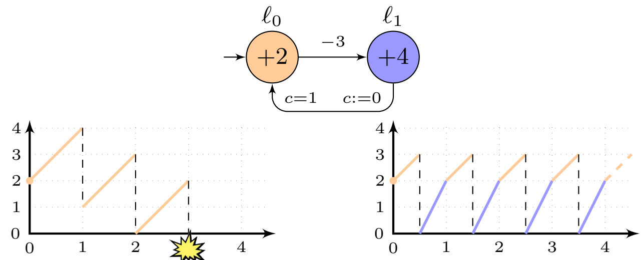
more clocks are still unsettled.

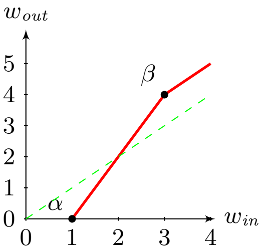


(a) (b)

Figure 2: One-clock priced timed automaton with discrete updates. Infeasibility of region-stable lower-bound schedule (a) and optimal lower-bound schedule (b).

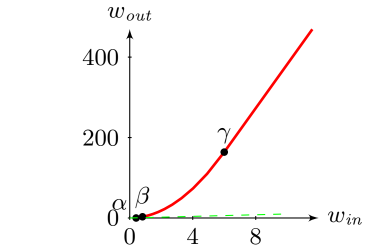
In this paper, we extend the decidability result of [6] for the lower-bound problem to "$1\frac{1}{2}$-clock" priced timed automata and with prices growing either linearly (*i.e.* $\dot{p} = k$) or exponentially (*i.e.* $\dot{p} = kp$). By "$1\frac{1}{2}$-clock" priced timed automata we refer to one-clock priced timed automata augmented with discontinuous (discrete) updates (*i.e.*, $p := p + c$) of the price on edges: discrete updates can easily be encoded using a second clock but do not provide the full expressive power of two clocks.

Surprisingly, the presence of discrete updates makes the lower-bound problem significantly more intricate. In particular, whereas region-stable strategies suffice in the search for infinite lower-bound schedules for one-clock priced timed automata, this is no longer the case when discrete updates are permitted. For the priced timed automaton in Figure 1 the infinite lower-bound schedule in Figure 1(a) could be replaced by the region-stable schedule in which the entire one time unit is spent in the location with the highest price-rate, *i.e.*, ℓ_1. In contrast, given initial energy-level of two, the (only possible) region-stable schedule for the "$1\frac{1}{2}$-clock" priced timed automaton of Figure 2 requires the one time unit to be spent in location ℓ_0 and will eventually lead to energy-level below zero as indicated in Figure 2(a). However, choosing to leave for location ℓ_1 after having spent 0.5 time units in ℓ_0—and thus having achieved an energy-level of 3 matching the subtracting update of the edge—provides an infinite lower-bound schedule (Figure 2(b)).



point	w_{in}	w_{out}
α	1	0
β	3	4

(a)



pt	w_{in}	w_{out}
α	≈ 0.406006	0
β	≈ 0.812012	3
γ	6	≈ 163.794

(b)

Figure 3: Energy functions for $\langle \ell_0, \ell_1 \rangle$ of the priced timed automaton of Figure 2 with linear rates (a) and exponential rates (b).

Not being able to rely on the classical region construction, the key to our decidability result is the notion of an *energy function* providing an abstraction of a path in the priced timed automaton. Given a path π, the energy function f_π maps an initial energy level w_{in} at the beginning of the path to the maximal energy w_{out} which may remain after the path. For the path $\pi = \langle \ell_0, \ell_1 \rangle$ of the priced timed automaton of Figure 2, we have already seen that $f_\pi(2) = 2$. We note that $f_\pi(1) = 0$ as the full one time-unit needs to be spent in location ℓ_0 in order to allow for a feasible run. Also $f_\pi(w_{in})$ is undefined for $w_{in} < 1$. On the other hand, $f_\pi(3) = 4$ as the full one time-unit may be spent in location ℓ_1; in fact $f_\pi(w_{in}) = w_{in} + 1$ whenever $w_{in} \geq 3$. Figure 3(a) shows the energy-function f_π. Also Figure 3(b) shows the energy function g_π for the same path π, but with exponential rates (*i.e.*, $\dot{p} = 2p$ in ℓ_0 and $\dot{p} = 4p$ in ℓ_1).

As we shall demonstrate in the remainder of the paper—for both linearly and exponentially priced timed automata—the energy function f_π for an arbitrary path π is a piecewise collection of rational power functions satisfying $f_\pi(x) - f_\pi(x') \geq x - x'$ whenever $x \geq x'$. The key for finding infinite lower-bound schedules now reduces to identifying (minimal) fixpoints $w = f_\pi(w)$, indicating that an initial energy level of w suffices for an infinite repetition of the path π. For the two energy functions f_π and g_π of Figure 2 the minimal fixpoints are $f_\pi(2) = 2$ and $g_\pi(\frac{3}{e^2-1}) = \frac{3}{e^2-1} \approx 0.47$, respectively, indicating the minimal initial energy level for infinite lower-bound schedules under the relevant linear or exponential interpretation.

Note that due to space restrictions, most of the proofs in the paper had to be deferred to an appendix.

2. TIMED AUTOMATA WITH OBSERVERS

The general formalism we introduce below, *timed automata with observers*, is intended to model control problems where resources may grow or decrease linearly or exponentially and with discontinuous updates. This includes oil tanks with pipes and drains which may be shut and opened using valves, electronic devices which may instantaneously lose energy when turned on or off, and bank accounts, or an investment portfolio, where the amount of money increases exponentially with time but where the transfer of money from one account to another typically has a fixed fee associated.

The formalism is quite general and unifies several concepts of timed automata with hybrid information found in the literature, *e.g.* in [9, 2]. In particular, it generalizes the notion of *priced*, or *weighted*, *timed automata* introduced in [4, 5]. In the definition, $\Phi(C)$ denotes the set of *clock constraints* on C given by the grammar $\varphi ::= x \bowtie k \mid \varphi_1 \wedge \varphi_2$ with $x \in X$, $k \in \mathbb{Z}$ and $\bowtie \in \{\leq, <, \geq, >, =\}$.

DEFINITION 1. *A* timed automaton with observers *is a tuple* $(L, C, I, urg, E, X, fl, upd)$ *consisting of a finite set* L *of locations, a finite set* C *of clocks, location invariants* $I\colon L \to \Phi(C)$*, an urgency mapping* $urg\colon L \to \{\top, \bot\}$*, a finite set* $E \subseteq L \times \Phi(C) \times 2^C \times L$ *of edges, a finite set* X *of variables, flow conditions* $fl\colon L \to (\mathbb{R}^X \to \mathbb{R}^X)$*, and update conditions* $upd\colon E \to (\mathbb{R}^X \to \mathbb{R}^X)$*.*

Note that a timed automaton with observers has indeed an underlying timed automaton (L, C, I, E). In the following we shall write $\ell \xrightarrow{g,r} \ell'$ instead of (ℓ, g, r, ℓ') for edges.

A timed automaton (with observers) is said to be *closed* if only non-strict inequalities $\leq$ and $\geq$ are used in guards and invariants. We shall later restrict development to closed timed automata: this case contains the important aspects of our algorithm and makes exposition easier.

Using a standard construction for timed automata, urgency of locations (for which we use the *urg* mapping above) can be encoded using an extra clock, hence is not strictly necessary in the above definition. However we shall later consider the special case of *one-clock* timed automata with observers, and for these, urgency indeed adds expressivity.

In the below definition, we use the standard reset and delay operators $v[r]$, $v+d$ on valuations given by $v[r](x)=0$ if $x\in r$, $v[r](x)=v(x)$ if $x\notin r$, and $(v+d)(x)=x+d$. Also, $\mathcal{D}([0,d],\mathbb{R}^X)$ denotes the set of continuous functions $[0,d]\to\mathbb{R}^X$ which are differentiable on the open interval $]0,d[$.

Definition 2. *The* semantics *of a timed automaton A with observers is given by the (infinite) transition system $[\![A]\!]=(S,T)$ with*

$$S=\big\{(\ell,v,w)\in L\times\mathbb{R}^C_{\geq 0}\times\mathbb{R}^X \;\big|\; v\models I(\ell)\big\}$$

$$\begin{aligned}T=\;&\big\{(\ell,v,w)\xrightarrow{e}(\ell',v',w') \;\big|\; \exists e=\ell\xrightarrow{g,r}\ell'\in E: v\models g,\\ &\qquad\qquad v'=v[r], w'=upd(e)(w)\big\}\\ \cup\;&\big\{(\ell,v,w)\xrightarrow{d}(\ell,v+d,w') \;\big|\; urg(\ell)=\top,\\ &\qquad d\in\mathbb{R}_{\geq 0},\exists f\in\mathcal{D}([0,d],\mathbb{R}^X):\\ &\qquad f(0)=w,\; f(d)=w',\; \textit{and}\;\forall t\in\,]0,d[\,:\\ &\qquad v+t\models I(\ell)\;\textit{and}\;\dot f(t)=fl(\ell)(f(t))\big\}\end{aligned}$$

A run of a timed automaton A with observers is a path in its semantics $[\![A]\!]$. Hence A admits both *discrete* behaviour, indicated by transitions $\xrightarrow{e}$, and *continuous* behaviour indicated by *delay* transitions $\xrightarrow{d}$. Note that whether or not a discrete or continuous transition is available does not depend on the value w of the observer variables; the semantics defined above is indeed just the semantics of the underlying timed automaton, augmented with observer values.

We shall henceforth mostly write $\dot p$ instead of the more cumbersome $fl(\ell)(w)(p)$, provided that the location ℓ is clear from the context, and similarly p' instead of $upd(e)(w)(p)$. We also write p instead of $w(p)$ when no ambiguity arises from such an abuse of notation.

Note also that timed automata with observers form a special class of hybrid automata [1] in which the clock variables c have the restricted flow $\dot c=1$ customary for timed automata.

In the sequel we shall consider two special classes of observers: *linear* and *exponential* ones. For a linear observer p, flow conditions are restricted to be of the form $\dot p=k$ for some constants k (possibly depending on the current location), hence linear observers admit a constant derivative (and linear growth) in locations. For an exponential observer p, flow conditions are restricted to be of the form $\dot p=kp$; that is, exponential observers have linear derivatives (hence exponential growth) in locations. We also restrict development to timed automata with *one* linear or exponential observer, with *additive* updates of the form $p'=p+c$, and with *one clock* only.

3. PROBLEMS AND RESULTS

The general problems with which we are concerned in this paper concern the existence of paths along which the observer value always remains positive:

Definition 3. *A run ρ in a timed automaton A with observers is* feasible *if the values of all the observers remain nonnegative all along ρ.*

Our problems can then be defined as follows:

Problem 1. (Reachability) Given a timed automaton A with observers X, an initial location ℓ_0, an initial valuation $w_0\colon X\to\mathbb{R}$, and a set of goal locations $L_G\subseteq L$, either exhibit a feasible run in A with initial location ℓ_0, initial clock values $v(c)=0$ for all $c\in C$, and initial valuation w_0, and visiting one of the locations in L_G, or establish that no such run exists.

Problem 2. (Infinite runs) Given a timed automaton A with observers X, an initial location ℓ_0, and an initial valuation $w_0\colon X\to\mathbb{R}$, either exhibit a feasible infinite run in A with initial location ℓ_0, initial clock values $v(c)=0$ for all $c\in C$, and initial valuation w_0, or establish that no such run exists.

In the case of linear observers, we also deal with a stronger notion of being feasible, in which the value of the observer must be larger than a given value m. The problem of *interval* bounds $m\leq w(p)\leq M$ appears to be much more difficult to handle however, see [6].

Below we give a precise definition of the classes of timed automata with observers which we shall consider in this paper and state the decidability results whose proof the rest of the paper is devoted to:

Definition 4. *A* one-clock timed automaton with one linear observer and additive updates *is a timed automaton with $C=\{c\}$, $X=\{p\}$, and for which there exist rate and weight functions $rate\colon L\to\mathbb{Z}$, $weight\colon E\to\mathbb{Z}$ such that $fl(\ell)(w)(p)=rate(\ell)$ and $upd(e)(w)(p)=w(p)+weight(e)$ for all $\ell\in L$ and $e\in E$.*

Definition 5. *A* one-clock timed automaton with one exponential observer and additive nonpositive updates *is a timed automaton with $C=\{c\}$, $X=\{p\}$, and for which there exist rate and weight functions $rate\colon L\to\mathbb{Z}$ and $weight\colon E\to\mathbb{Z}_{\leq 0}$ such that $fl(\ell)(w)(p)=rate(\ell)\,w(p)$ and $upd(e)(w)(p)=w(p)+weight(e)$ for all $\ell\in L$ and $e\in E$.*

Hence a linear observer indeed has $\dot p=rate(\ell)$ in all locations, and an exponential one has $\dot p=rate(\ell)\cdot p$. Notice that we require additive updates to be *nonpositive* for exponential observers; the general case poses additional difficulties (in particular, the value of the observer may not be monotone anymore).

Theorem 6. *Problems 1 and 2 are decidable in* EXPTIME *for closed one-clock timed automata with one linear observer and additive updates.*

Theorem 7. *Problems 1 and 2 are decidable in* EXPTIME *for closed one-clock timed automata with one exponential observer and additive* non-positive *updates.*

The rest of this paper is devoted to the proofs of these theorems, as follows: In Sections 4 to 6, we prepare the proofs by showing how to abstract observer values along *paths* in a timed automaton. In Section 7, we show how to use this abstraction to translate a timed automaton with one linear or exponential observer into a finite automaton *with energy functions* as defined in Section 8, for which the problems then can be decided.

For the sake of readability, we only present our proofs for the case of *closed* timed automata, *i.e.*, timed automata in which guards and invariants to not involve strict inequalities. This case already comprises the important ideas of our constructions.

4. OPTIMIZATIONS ALONG PATHS

Before attempting the general problem, we solve an optimization problem for paths without clock resets, both for a linear and for an exponential observer: We compute optimal delays in order to maximize the exit value along a path, as a function of the initial observer value. This is a special case of our problem (as a path is a "linear automaton"), and will be the keystone of our general algorithm.

More precisely, we assume we are given an *annotated unit path*, *i.e.*, a sequence

$$\pi: \quad \ell_0 \xrightarrow[\{c\}]{\varphi} \xrightarrow[\geq b_0]{p_0} \ell_1 \xrightarrow[\geq b_1]{p_1} \ell_2 \cdots \xrightarrow[\geq b_{n-1}]{p_{n-1}} \ell_n \xrightarrow[\{c\}]{c=1} \xrightarrow[\geq b_n]{p_n} \ell_{n+1}$$

along which all unspecified guards are $0 \leq c \leq 1$, clock c is only reset along the first and last edges, and there is a global invariant $c \leq 1$. We write r_i for the rate in location ℓ_i, and assume (w.l.o.g.) that $r_0 = r_{n+1} = 0$. Each edge carries its weight p_i and an *annotation* $\geq b_i$, which refers to observer value just before firing the transition: the transition is only fireable if observer value is larger than or equal to b_i. Notice that this kind of constraint can be encoded in our one-clock models thanks to urgency, by adding two transitions with weights $-b_i$ and $+b_i$ with an urgent location in-between. An annotated unit path can thus be seen as a special kind of one-clock timed automaton with one observer and additive updates.

Let π be an annotated unit path. A *run* along π with initial observer value w is a run

$$\rho: \quad (\ell_0, 0, w'_0) \xrightarrow{e_0} (\ell_1, v_1, w_1) \xrightarrow{t_1} (\ell_1, v'_1, w'_1) \xrightarrow{e_1} \cdots$$
$$\cdots (\ell_n, v_n, w_n) \xrightarrow{t_n} (\ell_n, v'_n, w'_n) \xrightarrow{e_n} (\ell_{n+1}, 0, w_{n+1})$$

in the corresponding timed automaton with observer with $w'_0 = w$. (Note that the precise values of w_i and w'_i depend on the type of observer we are considering.)

We write $\rho = (w, t_1, \ldots, t_n)_\pi$ to denote the run along π with initial observer value w and elapsing t_i time units in ℓ_i. Such a run ρ is a *feasible run* if it satisfies the additional constraint that $w'_i \geq b_i$ for every $0 \leq i \leq n$. Notice that these constraints are more general than our original aim of keeping observer value above 0: it suffices to let $b_i = \max(0, -p_i)$ to ensure that the value will remain nonnegative all along the run.

The *energy function* along an annotated unit path π is defined as

$$f_\pi(w) = \sup \left\{ w_{n+1} \;\middle|\; (w, t_1, \ldots, t_n)_\pi \text{ feasible run along } \pi \right\}$$

with $f_\pi(w)$ being undefined in case no feasible run along π with $w'_0 = w$ exists.

In the sequel, we explain how to compute f_π for an annotated unit path π, first in the linear and then in the exponential setting.

The first step, common to linear and exponential observers, is to remove urgent locations from our paths. Clearly enough, as no time elapses in urgent locations, the following two sequence of transitions are equivalent (w.r.t. time and observer values):

$$\ell_i \xrightarrow[\geq b_i]{p_i} \ell_{i+1}^{\mathrm{urg}} \xrightarrow[\geq b_{i+1}]{p_{i+1}} \ell_{i+2} \quad \rightsquigarrow \quad \ell_i \xrightarrow[\geq \max(b_i, b_{i+1} - p_i)]{p_i + p_{i+1}} \ell_{i+2}$$

Hence:

Lemma 8. *For any annotated unit path π, an annotated unit path $\overline{\pi}$ containing no urgent locations can be computed in polynomial time with $f_\pi = f_{\overline{\pi}}$.*

5. PATHS WITH LINEAR OBSERVER

In the following two sections, we show how to turn an annotated path into a *normal form* and how to compute f_π for normal-form paths. Both the notion of normal form, and how to compute energy functions for normal-form paths, depend on whether the observer is linear or exponential.

From now on, we can assume that π has the form

$$\pi: \quad \ell_0 \xrightarrow[\{c\}]{\varphi} \xrightarrow[\geq b_0]{p_0} \ell_1 \xrightarrow[\geq b_1]{p_1} \ell_2 \cdots \xrightarrow[\geq b_{n-1}]{p_{n-1}} \ell_n \xrightarrow[\{c\}]{c=1} \xrightarrow[\geq b_n]{p_n} \ell_{n+1}$$

with $n \geq 1$, and that it contains no urgent locations.

Normal form. An annotated unit path as above is said to be in *normal form* (for linear observers) if all locations are non-urgent, $n \geq 1$, and one of the following three conditions holds:

- $n = 1$ (*trivial normal form*);
- all rates are positive, and $r_i < r_{i+1}$ for $1 \leq i \leq n-1$, and for every $1 \leq i \leq n-1$, it holds that $b_{i-1} + p_{i-1} < b_i$ (*positive normal form*);
- all rates are negative, and $r_i > r_{i+1}$ for $1 \leq i \leq n-1$, and for every $2 \leq i \leq n$, it holds that $b_{i-1} + p_{i-1} > b_i$ (*negative normal form*).

The proof of the fact that any annotated path can be converted into normal form, and the kind of normal form one arrives at, depend on the path's maximal location rate $\max\{r_i \mid i = 1, \ldots, n\}$. There are three cases to consider:

Case $\max\{r_i \mid i = 1, \ldots, n\} = 0$**.** In this case, any run which maximizes observer value will delay in one of the locations with rate 0, hence all other locations can be removed from the path (and the corresponding edges contracted). As a matter of fact, one only needs to keep *one* of the locations with zero rate; all others can be removed as well. Hence one arrives at the *trivial normal form*; the details of the proof are in Appendix A.2:

Lemma 9. *For any annotated path π (without urgent locations) such that* $\max\{r_i \mid i = 1, \ldots, n\} = 0$*, an annotated path $\widetilde{\pi}$ in trivial normal form can be constructed in polynomial time with $f_\pi = f_{\widetilde{\pi}}$.*

Case $\max\{r_i \mid i = 1, \ldots, n\} > 0$**.** In this case, we can transform π into an equivalent path in positive (or trivial) normal form:

Lemma 10. *For any annotated path π (without urgent locations) such that* $\max\{r_i \mid i = 1, \ldots, n\} > 0$*, an annotated path $\widetilde{\pi}$ in positive or trivial normal form can be constructed in polynomial time with $f_\pi = f_{\widetilde{\pi}}$.*

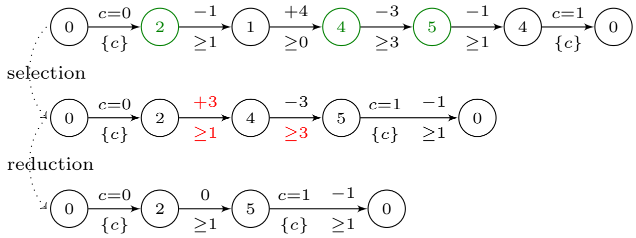


Figure 4: Conversion of annotated path into positive normal form

Proof sketch. The intuition is as follows: As before, the aim is to spend time in the most profitable location. However, due to annotations, we may have to delay some time in earlier locations, in order to have high enough observer value to fire transitions up to this optimal location.

We explain the construction here, and refer to Appendix A.3 for a detailed proof. First we construct a sequence $(n_j)_{j\geq 0}$ of *location indices with increasing rates* as follows:

- $n_0 = 0$
- assuming n_j has been computed for some $j \geq 0$, then
  - if $r_{n_j} = \max\{r_i \mid i = 1, \ldots, n\}$ is the maximal rate along π, then the sequence stops there;
  - otherwise, we let n_{j+1} be the least index $i > n_j$ for which $r_i > r_{n_j}$.

Let m be the index of the last item in $(n_j)_{j\geq 0}$. We add another last item $n_{m+1} = n + 1$, and define an intermediary annotated path $\bar{\pi}$, having $m + 2$ locations $\bar{\ell}_0$ to $\bar{\ell}_{m+1}$, with rates $\bar{r}_k = r_{n_k}$ when $0 \leq k \leq m + 1$. Notice that this sequence of locations satisfies the first part of the condition for being in *positive* normal form (or in trivial normal form if $m = 1$).

We now define the transitions of $\bar{\pi}$. For $0 \leq j \leq m$, the annotated edge $\bar{\ell}_j \xrightarrow[\geq \bar{b}_j]{\bar{p}_j} \bar{\ell}_{j+1}$ is defined by

$$\bar{p}_j = \sum_{k=n_j}^{n_{j+1}-1} p_k \qquad \bar{b}_j = \max\Big\{b_k - \sum_{l=n_j}^{k-1} p_l \mid n_j \leq k \leq n_{j+1}-1\Big\}$$

Hence $\bar{b}_j$ is the minimum observer value needed in $\bar{\ell}_j = \ell_{n_j}$ to complete the sub-path from ℓ_{n_j} to $\ell_{n_{j+1}}$ without delaying and under observance of the lower bounds b_k.

It remains to enforce the second condition ($\bar{b}_{i-1} + \bar{p}_{i-1} < \bar{b}_i$ for $1 \leq i \leq m - 2$) on $\bar{\pi}$. This is achieved by inductively replacing any offending pair of consecutive annotated edges $\bar{\ell}_{i-1} \xrightarrow[\geq \bar{b}_{i-1}]{\bar{p}_{i-1}} \bar{\ell}_i \xrightarrow[\geq \bar{b}_i]{\bar{p}_i} \bar{\ell}_{i+1}$ by a single annotated edge $\bar{\ell}_{i-1} \xrightarrow[\geq \bar{b}_{i-1}]{\bar{p}_{i-1}+\bar{p}_i} \bar{\ell}_{i+1}$. The resulting annotated path $\widetilde{\pi}$ is in normal form, and satisfies $f_\pi = f_{\widetilde{\pi}}$. □

Figure 4 shows an example of a path being converted into positive normal form.

Case $\max\{r_i \mid i = 1, \ldots, n\} < 0$. The case with only negative rates is dual to the above one and can be handled using similar techniques; see Appendix A.4 for details:

Lemma 11. *For any annotated path π such that $\max\{r_i \mid i = 1, \ldots, n\} < 0$, an annotated path $\bar{\pi}$ in negative (or trivial) normal form can be constructed in polynomial time with $f_\pi = f_{\bar{\pi}}$.*

Energy function. We now turn to the computation of the function mapping initial to final observer value along a unit path in normal form for linear observers. For the trivial normal form this is easy, as there is only one possible run along π. For the positive normal form we detail the computations below, and the negative normal form can be handled in an analogous manner.

Let

$$\pi:\quad \ell_0 \xrightarrow[\{c\}]{\varphi} \xrightarrow[\geq b_0]{p_0} \ell_1 \xrightarrow[\geq b_1]{p_1} \ell_2 \cdots \xrightarrow[\geq b_{n-1}]{p_{n-1}} \ell_n \xrightarrow[\{c\}]{c=1} \xrightarrow[\geq b_n]{p_n} \ell_{n+1}$$

be an annotated unit path in positive normal form, and define the n-tuple $t^{\mathrm{opt}} = (t_i^{\mathrm{opt}})_{1\leq i\leq n}$ by

$$t_i^{\mathrm{opt}} = \begin{cases} 0 & \text{if } i = m \text{ and } b_m \leq b_{m-1} + p_{m-1} \\ \frac{b_i-(b_{i-1}+p_{i-1})}{r_i} & \text{otherwise} \end{cases}$$

Since the rates are all positive and $b_i > b_{i-1} + p_{i-1}$ for all $1 \leq i \leq m - 1$, these values are well-defined and positive. An important equality to notice is the following:

$$b_{n-1} + p_{n-1} + r_n \cdot t_n^{\mathrm{opt}} = \max(b_{n-1} + p_{n-1}, b_n)$$

We prove in the sequel that those delays represent the "optimal" delays one should wait in each location, and correspond to the policy where each transition is fired as soon as the observer value satisfies the lower-bound constraint ($\geq b_i$ for the transition leaving ℓ_i).

As it may be the case that the optimal delays collected in t^{opt} do not sum up to 1 (which is the total time to be spent along π), we define another tuple $t^\star$ containing the delays which (as we shall show) have to be spent on an optimal run.

- In case $\sum_{i=1}^n t_i^{\mathrm{opt}} > 1$, we have to cut down on the time we delay in the locations. The more profitable locations are the ones with higher rates at the end of the path, hence this is where we shall spend the delays: Letting ι_π be the largest index for which $\sum_{i=\iota_\pi}^n t_i^{\mathrm{opt}} > 1$ (so that $\sum_{i=\iota_\pi+1}^n t_i^{\mathrm{opt}} \leq 1$), we set
$$t_i^\star = \begin{cases} 0 & \text{for } i < \iota_\pi \\ 1 - \sum_{i=\iota_\pi+1}^n t_i^{\mathrm{opt}} & \text{for } i = \iota_\pi \\ t_i^{\mathrm{opt}} & \text{for } i > \iota_\pi \end{cases}$$
- In case $\sum_{i=1}^n t_i^{\mathrm{opt}} \leq 1$, we may have to spend some extra time in one of the locations. The most profitable location for this delay is the last, hence we define $t_i^\star = t_i^{\mathrm{opt}}$ for $1 \leq i \leq n - 1$, and $t_n^\star = 1 - \sum_{i=1}^{n-1} t_i^{\mathrm{opt}}$. We also let $\iota_\pi = 0$ in this case.

Before we prove that those delays are indeed optimal, we first compute the initial observer value needed to traverse the whole path under this policy.

- in the first case ($\iota_\pi \geq 1$), the minimal initial observer value is
$$w_{\iota_\pi}^* = b_{\iota_\pi - 1} - \sum_{k=0}^{\iota_\pi - 2} p_k + (t_{\iota_\pi}^{\mathrm{opt}} - t_{\iota_\pi}^\star) \cdot r_{\iota_\pi},$$
and the final accumulated cost is $\omega_{\iota_\pi}^* = \max(b_n, b_{n-1} + p_{n-1}) + p_n$;
- if $\iota_\pi = 0$, the minimal initial observer value is $w_0^* = b_0$, and the final accumulated cost is $\omega_0^* = \max(b_n, b_{n-1} + p_{n-1}) + p_n + (t_n^\star - t_n^{\mathrm{opt}}) \cdot r_n$. These values actually equal w_1^* and ω_1^* defined below.

We generalize the previous construction by letting, for $\iota_\pi + 1 \le i \le n$:

$$w_i^* = b_{i-1} - \sum_{k=0}^{i-2} p_k$$

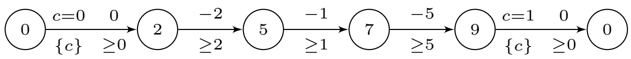
$$\omega_i^* = \max(b_n, b_{n-1} + p_{n-1}) + p_n + ((t_n^\star - t_n^{\mathrm{opt}}) + \sum_{j<i} t_j^\star) \cdot r_n$$

We claim that w_i^* is the minimal initial observer value for which it is possible to spend no delay in locations ℓ_0 to ℓ_{i-1} along a feasible run, and ω_i^* is the corresponding optimal observer value at the end of the run. This can be expressed as follows:

Proposition 12. *The function f_π is a piecewise affine function defined on the interval $[w^*_{\iota_\pi}, \infty[$, visiting points (w_i^*, ω_i^*), for all $\iota_\pi \le i \le n$, with constant slope $\dot{f}_\pi(x) \ge 1$ between two consecutive such points, and with slope $\dot{f}_\pi(x) = 1$ after (w_n^*, ω_n^*).*

A detailed proof of this result is presented in Appendix A.5; for the case of negative normal form see Appendix A.6.

Example 1. *We consider the following example, which is already in normal form. The corresponding function f_π then looks as depicted on Figure 5:*



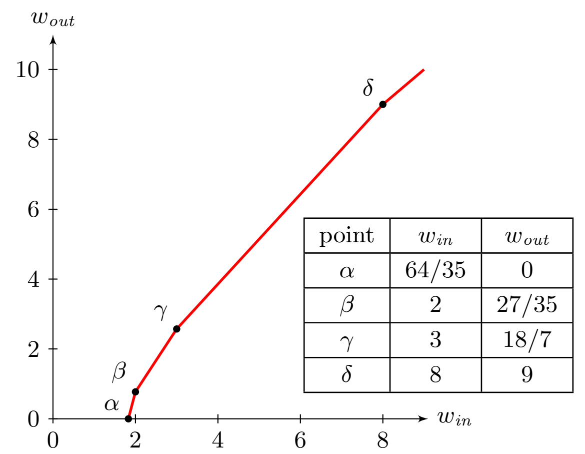


Figure 5: Function f_π for example with linear observer

For instance, if we enter the path with initial observer value 2, the optimal policy is to spend no time in the location with rate 2 (as we can leave it directly), then spend 1/5 time units in the next location (so that we have value 1 and can fire the outgoing transition), then spend 5/7 time units with rate 7, and the remaining 3/35 time units in the location with rate 9, ending with final observer value 27/35 (point β).

Remark 1. *We note that the above considerations easily can be adapted to paths (without resets) with a general guard $c = k$ on the last transition (instead of $c = 1$), hence it is straight-forward to handle these. Also the restriction to closed timed automata can easily be lifted: we showed above how to handle paths with non-strict guards only, and the general case is similar. In this case, the energy function f_π gives, for each input value w, the* supremum *$f_\pi(w)$ of the observer values obtainable as output, an whether or not this value is actually attained for an input can be decided by looking at the delays spent in each location.*

6. PATHS WITH EXPONENTIAL OBSERVER

Normal form. As for linear observers, we are interested in computing the energy function along a unit path, by first transforming it into a normal form and then computing the energy function for normal-form paths. In this case however, we have to restrict the kinds of paths we can handle:

- We assume that the edge weights p_i are nonpositive, and that at least one of the rates r_i is nonnegative;
- paths are not annotated, *i.e.* we do not impose "local" constraints of the form "$\ge b_i$" in this case, and only require that observer value always be nonnegative along the run.

These restrictions amount to only considering the *positive* normal form, without local observer constraints. As in the previous case, we could handle the case where all rates are negative in a similar way (with a suitable notion of *negative* normal form). The other restrictions are purely technical: Currently we do not know how to handle paths with mixed positive and negative updates, or with local constraints, but we expect our techniques to also extend to these settings.

For the sequel, we again fix a unit path

$$\pi: \quad \ell_0 \xrightarrow[\{c\}]{\varphi \quad p_0} \ell_1 \xrightarrow{p_1} \ell_2 \cdots \xrightarrow{p_{n-1}} \ell_n \xrightarrow[\{c\}]{c=1 \quad p_n} \ell_{n+1}$$

satisfying the above constraints, and with $r_0 = r_{n+1} = 0$. As in the previous section, our aim is to compute f_π for such a path (but now with exponential observer), mapping initial to maximum final observer value.

A path as above is said to be in *normal form* (for exponential observers) if all locations are non-urgent, $m \ge 1$, and one of the following two conditions holds:

- $m = 1$ (*trivial normal form*);
- all rates are positive, and $r_i < r_{i+1}$ for $1 \le i \le m-1$, and for every $2 \le i \le m-1$, it holds that $\frac{p_{i-1} r_{i-1} r_i}{r_{i-1} - r_i} < \frac{p_i r_i r_{i+1}}{r_i - r_{i+1}}$ (*positive normal form*);

The last condition for being in positive normal form is the counterpart, for exponential observers, to the condition "$b_i > b_{i-1} + p_{i-1}$" which we had in the case of linear observers.

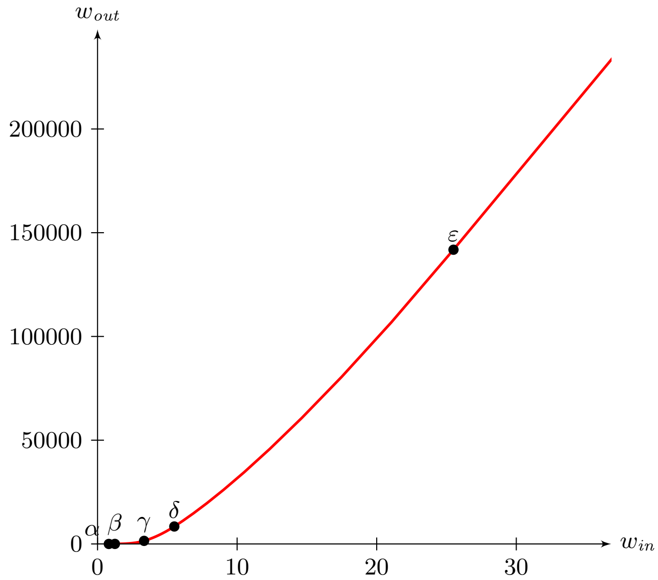
Such a normal form can be computed:

Proposition 13. *Assume π is a unit path with nonpositive edge weights and such that* $\max\{r_i \mid i = 1, \ldots, n\} \ge 0$. *Then we can construct in polynomial time a path $\widetilde{\pi}$ in normal form for exponential observers so that $f_\pi = f_{\widetilde{\pi}}$.*

The proof relies on arguments similar to the ones we used for the linear case. Details can be found in Appendix B.2.

Energy function. Along a path in positive normal form, we can decide whether a given initial observer value is sufficient to reach the last location:

Proposition 14. *Let π be a path in positive normal form (for exponential observers) and w an initial observer value. Then we can decide whether there is a feasible run along π with initial observer value w, and we can compute the value $f_\pi(w)$.*



pt	w_{in}	w_{out}
α	$e^{-2}*10/3*(21/8)^{2/5}*(2)^{2/7}$ ≈ 0.809003	0
β	$e^{-2}*10/3*(21/8)^{2/5}*(9)^{2/7}$ ≈ 1.24332	35/2
γ	10/3	$e^{9}*35/2*(8/21)^{9/5}*(1/9)^{9/7}$ ≈ 1480.38
δ	11/2	$e^{9}*35/2*(1/9)^{9/7}$ ≈ 8410.18
ε	51/2	$e^{9}*35/2$ ≈ 141804

interval	equation of the curve
$\alpha - \beta$	$w_{\text{out}} = \frac{45}{2} \cdot \left(\frac{w_{\text{in}}}{e^{-2}*10/3*(21/8)^{2/5}*(9)^{2/7}} \right)^{7/2} - 5$
$\beta - \gamma$	$w_{\text{out}} = \frac{35}{2} \cdot \left(\frac{w_{\text{in}}}{e^{-2}*10/3*(21/8)^{2/5}*(9)^{2/7}} \right)^{9/2}$
$\gamma - \delta$	$w_{\text{out}} = \frac{35}{2} \cdot \left(\frac{w_{\text{in}}-2}{e^{-5}*7/2*(9)^{5/7}} \right)^{9/5}$
$\delta - \varepsilon$	$w_{\text{out}} = \frac{35}{2} \cdot \left(\frac{w_{\text{in}}-3}{e^{-7}*45/2} \right)^{9/7}$
$\varepsilon - +\infty$	$w_{\text{out}} = (w_{\text{in}} - 8) \cdot e^{9}$

Figure 6: Function f_π for example with exponential observer

Notice that contrary to the linear case, it is not sufficient to fire a transition as soon as the observer value can afford paying the nonpositive update: Consider the two-state automaton of Figure 2. If the initial observer value is 3, it is allowed to immediately fire the transition to ℓ_1, but this would set the energy level to 0, and the exponential growth would be annihilated.

The proof of Proposition 14, which is presented in Appendix B.3, relies on the computation of optimal exit values for the observer: letting $w_i^{\text{opt}} = \frac{p_i \cdot r_{i+1}}{r_i - r_{i+1}}$, we prove that w_i^{opt} is the optimal value of the observer with which to exit location ℓ_i (as long as time permits). The optimal time to be spent in location ℓ_i is then

$$t_i^{\text{opt}} = \frac{1}{r_i} \ln \left(\frac{w_i^{\text{opt}}}{w_{i-1}^{\text{opt}} + p_{i-1}} \right),$$

with the convention that w_0^{opt} is the initial observer value. The technical condition for being in normal form implies that these values are positive (except possibly t_1^{opt}: having t_1^{opt} negative means that the initial observer value is sufficient to go with no delay to the next location).

This allows us to compute the exact value of f_π (see Appendix B.4 for a detailed proof):

Proposition 15. *The function f_π is defined on an interval $[w_0^*, \infty[$, and there is a sequence $w_0^* < w_1^* < \cdots < w_n^*$ of algebraic numbers $w_0^*, w_1^*, \ldots, w_n^* \in \mathbb{R}_{\geq 0}$ such that on each interval $[w_i^*, w_{i+1}^*]$ and on $[w_n^*, \infty[$, f_π can be obtained in closed form as*

$$f_\pi(w) = \alpha_i \cdot (w - \beta_i)^{r_i / r'_i} + \gamma_i$$

where r_i and r'_i are rates of π with $r_i \geq r'_i$, and α_i, β_i and γ_i are algebraic numbers which can be computed from the constants appearing in π. Moreover, f_π is continuous and has continuous derivative $\dot{f}_\pi(x) \geq 1$ on its domain (also at the points w_i^*).

Example 2. *We consider the same path as depicted in Example 1, but assuming an exponential observer with additive updates. This automaton satisfies the restrictions and is already in normal form for exponential observer. The resulting function f_π is depicted in Figure 6. For instance, if we enter the path with initial observer value 1, the optimal policy is to exit the location with rate 2 when the value reaches $w_1^{opt} = 10/3$ (which occurs after $\ln(5/3)/2$ time units), then leave the next location when the value reaches $w_2^{opt} = 7/2$, and then spend the remaining $1 - \ln(5/3)/2 - \ln(21/8)/5$ time units in the location with rate 7 (there is not enough time remaining to reach $w_3^{opt} = 45/2$). When we leave this location, observer value then equals $(7/2 - 1) \cdot \exp(7 \cdot (1 - \ln(5/3)/2 - \ln(21/8)/5)) \approx 21/2$, which makes it possible to fire the last transition directly and end up with final observer value around $11/2$.*

7. THE DISCRETE ABSTRACTION

In this section we show how a timed automaton with one clock and one observer with additive updates can be converted into a form in which the analysis of paths without resets from the preceding sections can be applied. After applying this, we arrive at a *finite automaton with energy functions* which we subsequently analyse in the next section.

Clock bounded above by 1. As a first step, we show that A can be converted to a timed automaton where the clock is bounded above by 1 and with only three different types of edges. A similar simplification technique for priced games was used in [8]; the proof of the lemma can be found in Appendix C.1.

Lemma 16. *Let A be a closed one-clock timed automaton with observers, ℓ_0 a location of A, and $L_G \subseteq L$ a set of goal locations. One can construct in exponential time another one-clock timed automaton A' with observers, together with a new initial location ℓ'_0 and a new set of goal locations L'_G, such that*

- *in any state ℓ' of A', the invariant is $c' \leq 1$;*
- *for any edge $\ell \xrightarrow{g,r} \ell'$ in A', either $r = \emptyset$ and g is the*

constraint $0 \le c' \le 1$, or $r = \{c'\}$ and g is an equality constraint $c' = 0$ or $c' = 1$,

and such that, for any w_0 and m, $\langle A, \ell_0, w_0, L_G, m\rangle$ is a positive instance of the reachability (resp. infinite-run) problem iff $\langle A', \ell'_0, w_0, L'_G, m\rangle$ is also a positive instance of that problem.

Note that the lemma applies to automata with an arbitrary number of observers, with arbitrary updates instead of only additive ones. For the following conversions however, we have to assume a single linear or exponential observer with additive updates.

Remark 2. *It is convenient for the sequel to have global invariant $c \le 1$ for the clock, but not strictly necessary. We already remarked on page that it is possible to handle paths with general invariant $c \le k$. A variant of the second property of the lemma can be ensured using the coarse clock regions of [10], and using this construction, one avoids exponential blowup.*

Eliminating cycles without resets. To be able to apply our analysis of paths without resets in Sections 5 and 6, we need to ensure that there are only finitely many such paths between any two locations. Using a partial unfolding of the timed automaton where we only unfold along edges without resets, and afterwards pruning infinite reset-free paths, we can construct a timed automaton without reset-free cycles. In order to be correct, this construction must be preceded by a detection of feasible Zeno runs in the original automaton, and this information has to be stored in the unfolding. For reachability, we also have to take into account positive Zeno cycles from which some final location is reachable. The precise construction, together with a proof of its correctness as stated in Lemma 17, can be found in Appendix C.2.

Lemma 17. *Let A be a closed one-clock timed automaton with one linear or exponential observer and additive updates, and with clock bound $c \le 1$. Let $\ell_0 \in L$ be a location of A, and $L_G \subseteq L$ be a set of goal locations. We can compute in exponential time*

- *two labelling functions $w_{Zeno}, w^{L_G}_{Zeno} \colon L \to \mathbb{R} \cup \{+\infty\}$,*
- *another such automaton A', with set of locations L', and a projection $\mathsf{lab} \colon L' \to L$,*
- *a location $\ell'_0 \in L'$*
- *a set of goal locations $L'_G \subseteq L'$,*

such that A' does not contain reset-free cycles, and for any initial observer value w_0, we have the following:

- *There is an infinite feasible run in A from $(\ell_0, c = 0)$ with initial observer value w_0 if and only if there is such a run in A' from $(\ell'_0, c = 0)$, or there is a feasible run in A' from $(\ell'_0, c = 0)$ with observer value w_0 to a configuration $(\ell, c = 0)$ with observer value w, and such that $w \ge w_{Zeno}(\mathsf{lab}(\ell))$.*
- *There is a feasible run in A from $(\ell_0, c = 0)$ with initial observer value w_0 to a location in L_G if and only if there is such a run in A' from $(\ell'_0, c = 0)$ to a location in L'_G, or there is a feasible run in A' from $(\ell'_0, c = 0)$ with observer value w_0 to a configuration $(\ell, c = 0)$ with observer value $w \ge w^{L_G}_{Zeno}(\mathsf{lab}(\ell))$.*

8. AUTOMATA WITH ENERGY FUNCTIONS

We are now left with a timed automaton A' in which all cycles have at least one resetting transition. We shall construct a discrete abstraction B of A' which will contain all the information we need for solving our problem. This abstraction will be a *finite automaton with energy functions*:

Definition 18. *A* finite automaton with energy functions *is a finite transition system (S, T) equipped with a function $f \colon T \to (\mathbb{R} \rightharpoonup \mathbb{R})$ decorating transitions with energy functions. The semantics $[\![B]\!]$ of such an automaton is given by an infinite transition system with states $(s, w) \in S \times \mathbb{R}$ and transitions $(s, w) \to (s', w')$ whenever there is $e = (s, s') \in T$ such that $f(e)$ is defined in w and $f(e)(w) = w'$.*

Discrete abstraction. For the discrete abstraction of A, we take as states of B the locations of A having at least one incoming resetting transition. It is intended that state ℓ of B represents configuration $(\ell, c = 0)$ of A. For each pair of states (ℓ, ℓ') in B, there is an edge from ℓ to ℓ' iff there is a reset-free path from ℓ to ℓ' in A. We label each edge (ℓ, ℓ') of B with a (partial) function $f_{\ell,\ell'} \colon \mathbb{R} \to \mathbb{R}$ computed as follows: $f_{\ell,\ell'}(w)$ is the maximal achievable observer value when entering ℓ', if starting from ℓ with observer value w and visiting only reset-free paths in A'. Since there are only exponentially many such paths, $f_{\ell,\ell'}$ can be computed in exponential time, using our procedures of Sections 5 or 6.

Finally, we label states of B with their values of w_{Zeno} and $w^{L_G}_{\mathrm{Zeno}}$ (as locations in A), obtained from Lemma 17: the values $w_{\mathrm{Zeno}}(\ell)$ (respectively $w^{L_G}_{\mathrm{Zeno}}(\ell)$) represent the minimal observer value needed in ℓ for which there exists a simple reset-free path from ℓ to a reset-free simple cycle with nonnegative accumulated update (respectively with positive accumulated update and from which L_G can be reached).

The following lemma directly follows from this construction and Lemma 17:

Lemma 19. *Let A be a closed one-clock timed automaton with one linear or exponential observer and additive updates, and with clock bound $c \le 1$. Let $\ell_0 \in L$ be a location of A and $L_G \subseteq L$ a set of goal locations, and assume w.l.o.g. that locations in L_G only have resetting incoming transitions and have no outgoing transitions. Let B be the finite automaton with energy functions as constructed above and $w_0 \in \mathbb{R}+$ an initial observer value. Then*

- *there is a feasible infinite run in A from $(\ell_0, c = 0)$ with initial credit w_0 iff either there is an infinite path in $[\![B]\!]$ from (ℓ_0, w_0), or there is a finite path in $[\![B]\!]$ from (ℓ_0, w_0) to a configuration (ℓ, w) such that $w \ge w_{Zeno}(\ell)$;*
- *there is a feasible run in A reaching a location in L_G from $(\ell_0, c = 0)$ with initial credit w_0 iff either there is a finite path in $[\![B]\!]$ from (ℓ_0, w_0) to (ℓ, w) for some $\ell \in L_G$ and some w, or there is a finite path in $[\![B]\!]$ from (ℓ_0, w_0) to (ℓ, w) for some ℓ such that $w \ge w^{L_G}_{Zeno}(\ell)$.*

Energy functions. We now take advantage of the special shape of the energy functions that we get in the case of linear and exponential cases.

Definition 20. *A* rational power function *is a function of the form $f \colon x \mapsto \alpha \cdot x^r + \beta$ where r is rational and α and β are algebraic numbers.*

As a consequence of our results of Sect. 5 and 6, we have:

Lemma 21. *Let π be an annotated path (with linear or exponential observer, under the corresponding restrictions* *of Sections 5 and 6). Then the energy function f_π has the following property:*

($\star$) *there exists an increasing sequence $x_1 < x_2 < \ldots < x_n$ of algebraic numbers such that*
- *the domain of f_π, written $\mathrm{dom}(f_\pi)$, is $[x_1, +\infty[$;*
- *for all i, the restrictions $(f_\pi)_{\restriction [x_i, x_{i+1}[}(x)$ and $(f_\pi)_{\restriction [x_n, \infty[}$ are rational power functions;*
- *for all $x \geq x' \geq x_1$, it holds that $f_\pi(x) - f_\pi(x') \geq x - x'$.*

Notice that the last condition follows from the fact that $\dot{f}_\pi(x) \geq 1$. Also, functions satisfying this condition are injective.

We shall need operations of (binary) maximum and composition on functions with property ($\star$) above; these are defined in the standard way: Given partial functions f and f' with right-infinite domain,
- $\max(f, f')$ is the function with domain $\mathrm{dom}(f) \cup \mathrm{dom}(f')$ defined by $x \mapsto \max\{g(x) \mid g \in \{f, f'\}, x \in \mathrm{dom}(g)\}$;
- $f' \circ f$ is the function with domain $\mathrm{dom}(f) \cap f^{-1}(\mathrm{dom}(f'))$ defined by $x \mapsto f'(f(x))$.

LEMMA 22. *If f and f' are partial functions satisfying Property ($\star$) above, then $\max(f, f')$ and $f' \circ f$ also satisfy ($\star$).*

PROOF. Let (x_i) and (x'_i) be the corresponding sequences of algebraic numbers as of Lemma 21. Let (y_j) be the increasing sequence of algebraic numbers given by $\{y_j\} = \{x_i\} \cup \{x'_i\}$. Then $\max(f, f')$ is defined on $[y_1, +\infty)$ and is clearly a piecewise rational power function. The proof of the third property is straightforward.

For composition $f' \circ f$, we let (y_j) be the increasing sequence of algebraic numbers for which $\{y_j\} = \{f^{-1}(x'_i) \mid x'_i \geq f(x_1)\} \cup \{x_i \mid f(x_i) \geq x'_1\}$. (Note that, indeed, these numbers are roots of polynomials and hence algebraic.) Then $f' \circ f$ is a piecewise rational power function on $[y_1, +\infty)$. The third property is straightforward. $\square$

We now study fixed points of those functions:

LEMMA 23. *Let f be a function satisfying Property ($\star$) of Lemma 21. Then*

1. *The set of fixed points of f is either empty or a left-closed interval.*
2. *Let $[x^*, x^\dagger\rangle$ be the set of fixed points of f (assuming it is not empty, and allowing $x^\dagger$ to equal $+\infty$). Then $f(x) < x$ for all $x < x^*$ and $f(x) > x$ for all $x > x^\dagger$.*
3. *If x^* exists, then for all $x \in \mathrm{dom}(f)$, there exists an infinite sequence $(f^n(x))_n$ of iterated values iff $x \geq x^*$.*

PROOF. For the first claim: From the fact that $f(x) - f(y) \geq x - y$ whenever $x \geq y$, we get that any point between two fixed points is a fixed point. Moreover, f is left-continuous, so that if $f(x) = x$ on a left-open interval, then also $f(x) = x$ at the left end-point.

For the second claim: Let $x < x^*$. Then $f(x^*) - f(x) \geq x^* - x$, which entails $f(x) \leq x$. Since x cannot be a fixed point, we must have $f(x) < x$. Similarly for the other claim.

Finally, for the third claim: For any $z \geq x^*$, we have $f(z) \geq z$. Hence if $x \geq x^*$, then for any k such that $f^k(x)$ is defined, we have $f^k(x) \geq x^*$, so that $f^{k+1}(x)$ is defined. On the other hand, assume that $x < x^*$, and that the infinite sequence $(f^n(x))_n$ is defined. Then $f(x) < x$ and, by induction, $f^{n+1}(x) < f^n(x)$ for all n. The sequence being decreasing and bounded by x_1, it converges. As f is left-continuous, the limit $\bar{x}$ satisfies $f(\bar{x}) = \bar{x}$, which contradicts the fact that x^* is the smallest fixed point. $\square$

Algorithm. We now gather everything together in order to solve Problems 1 and 2. We first explain how we detect feasible non-Zeno runs of A: Assume that such a run exists in A, and let ρ be the corresponding infinite run in B. Then some simple cycle σ in ρ must be repeated infinitely often. For all $i \geq 0$, write w_{2i} for the observer value when entering the $(i+1)$-st occurrence of σ, and w_{2i+1} for the value when exiting the $(i+1)$-st occurrence.

Assume that $w_{2i+1} < w_{2i}$ for all i, and that either $w_{2i+2} < w_{2i+1}$, or $w_{2i+2} = w_{2i+1}$ and the $(i+2)$-nd occurrence of σ directly follows the $(i+1)$-st one. Notice that it cannot be the case that we are in the latter situation for all i, as this would give an infinite iteration $f^n(w)$ for an energy function with $f(w) < w$, contradicting Lemma 23. Hence there are two occurrences of σ having a non-empty sub-path in-between, and this sub-path has negative effect on observer value. Dropping the earliest such path yields a feasible infinite non-Zeno run ρ'. This procedure can be repeated recursively, as long as the sequence (w_i) satisfies the condition above.

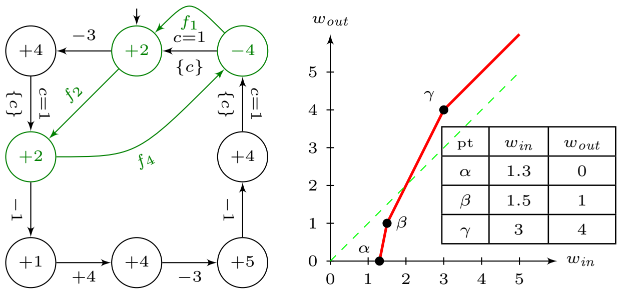
We first assume that the sequence $(w_i)_i$ is decreasing. This means that between the first and second occurrences of σ, observer value has decreased. Then this part between the first two occurrences of σ can be dropped, yielding a new feasible infinite non-Zeno run. Apply this procedure recursively, as long as the sequence $(w_i)_i$ in the resulting run is decreasing. This yields a sequence of feasible runs $(\rho_n)_n$ of the form $\rho_0 \cdot (\sigma)^n \cdot \pi_n$. There must exist an index at which the procedure cannot be repeated, since otherwise it would contradict Lemma 23. At that point, we end up with a run in which a simple cycle has a positive effect on observer value. Following Lemma 23, this cycle can be iterated from that point on, yielding a feasible lasso-shaped non-Zeno run.

Now, consider the non-periodic part of this run: if it contains a (simple) cycle with negative effect, then again we can drop this cycle while obtaining a feasible lasso-shaped infinite non-Zeno run. On the other hand, if the cycle has nonnegative effect, it can be iterated itself. In the end, we have proved the following lemma:

LEMMA 24. *If there is a feasible infinite non-Zeno run in A from $(\ell_0, c = 0)$ with observer value w_0, then there is a lasso-shaped one in which the initial part is acyclic and the periodic part is a simple cycle. In particular, both parts have linear size.*

Now consider the case of feasible Zeno runs in A. Such a run corresponds to a finite run ρ in B, ending in a location ℓ with observer value at least $w_{\text{Zeno}}(\ell)$. Using similar arguments as above, if there is a simple cycle in ρ with nonnegative effect, then we can deduce a feasible infinite run in B, hence in A. If the effect of the cycle is negative, then dropping the cycle yields another feasible Zeno run. In the end, we have

LEMMA 25. *If there is a feasible infinite Zeno run in A from $(\ell_0, c = 0)$ with observer value w_0, then either there is an acyclic path in B reaching a configuration (ℓ, w) with $w \geq$*



pt	w_{in}	w_{out}
α	1.3	0
β	1.5	1
γ	3	4

Figure 7: Solving the infinite-run problem for an example timed automaton

$w_{Zeno}(\ell)$*, or there is a lasso-shaped run in which the initial part is acyclic and the periodic part is a simple cycle.*

As B has size linear in A, we can enumerate in exponential time the possible witnesses given by the above two lemmas. Since constructing B is already in exponential time, our global procedure is in exponential time. For reachability, similar techniques give the following lemma:

LEMMA 26. *If there is a feasible run in A from $(\ell_0, c = 0)$ with observer value w_0 to a goal location in L_G, then there is an acyclic path in B reaching a configuration (ℓ, w) with $\ell \in L_G$ or $w \geq w^{L_G}_{Zeno}(\ell)$.*

EXAMPLE 3. *The simple linearly priced timed automaton in Figure 7 is a collection of some of the paths we have seen earlier. Specifically, we have taken the paths from Figures 2 and 4 and connected them using a trivial path with one location with rate -4. The figure also displays the discrete abstraction of the automaton, consisting of three states and three transitions labeled with energy functions f_2 for the path from Figure 2, f_4 for the path from Figure 4, and f_1 for the trivial path. Note that no Zeno runs are possible in the automaton, hence the w_{Zeno} annotation has been omitted.*

To compute the minimum observer value necessary for having an infinite run in the example automaton, we need to find the least fixed point of the energy function $f_1 \circ f_4 \circ f_2$ which is displayed in the right part of the figure. This point can be computed to be at $w_{in} = 2$, hence the automaton admits an infinite run if and only if initial observer value is at least 2.

9. CONCLUSION AND FUTURE WORK

We have shown that the infinite-runs and reachability problems are decidable for closed one-clock timed automata with one linear or exponential observer and additive updates, using a novel technique of energy functions. We expect this technique, and the general notion of finite automata with energy functions, to have applications in other areas as well.

The restriction to a one-clock setting is essential in our approach, and currently we do not know how to extend it to timed automata with more than one clock. However one-clock models are expressive enough to model a large class of interesting examples, and in the context of priced timed automata, it is well-known that models with more than one clock are very difficult to handle.

The restriction to closed timed automata is not essential. It was mainly adapted to ease exposition, and can be lifted by during the analysis carefully taking note of which values of energy functions can actually be attained. On another note, our technique also applies to the lower-soft-upper-bound problem mentioned in the introduction. We have concentrated on solving lower-bound problems here, but by adapting our analysis of paths, we can also solve the problem with soft upper bound. The interval-bound problem however is much more difficult.

Considering more general observers than only linear or exponential ones is also of interest. The present work is part of a general project concerning "hybridization" of timed-automata technology, and linear and exponential flow conditions combined cover the whole class of first-order differentiable observers. We can easily handle observers which can have either linear or exponential behaviour, but the general first-order differential case is more difficult, partly because our technique relies on flows being monotonous, either increasing or decreasing, in locations.

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APPENDIX

A. PROOFS OF SECTION 5

A.1 Proof of Lemma 8

LEMMA 8. *For any annotated unit path π, an annotated unit path $\bar{\pi}$ containing no urgent locations can be computed in polynomial time with $f_\pi = f_{\bar{\pi}}$.*

PROOF. In order to fix notation, let

$$\pi\colon \quad \ell_0 \xrightarrow[\{c\} \quad \geq b_0]{c=0/1 \quad p_0} \ell_1 \xrightarrow[\geq b_1]{p_1} \ell_2 \xrightarrow[\geq b_2]{p_2} \cdots \xrightarrow[\geq b_{n-1}]{p_{n-1}} \ell_n \xrightarrow[\{c\} \quad \geq b_n]{c=1 \quad p_n} \ell_{n+1}$$

as above, and assume that ℓ_k is urgent. Let $\bar{\pi}$ be the following path:

$$\bar{\pi}\colon \quad \ell_0 \xrightarrow[\{c\} \quad \geq b_0]{c=0/1 \quad p_0} \ell_1 \xrightarrow[\geq b_1]{p_1} \ell_2 \xrightarrow[\geq b_2]{p_2} \cdots \ell_{k-1} \xrightarrow[\geq \bar{b}_{k-1}]{\bar{p}_{k-1}} \ell_{k+1} \xrightarrow[\geq b_{k+1}]{p_{k+1}} \cdots \xrightarrow[\geq b_{n-1}]{p_{n-1}} \ell_n \xrightarrow[\{c\} \quad \geq b_n]{c=1 \quad p_n} \ell_{n+1}$$

In this path, location ℓ_k has been dropped, and the transitions entering and leaving this location have been merged: the transition entering ℓ_{k+1} now has

$$\bar{p}_{k-1} = p_{k-1} + p_k \qquad\qquad \bar{b}_{k-1} = \max\{b_{k-1}, b_k - p_{k-1}\}$$

As ℓ_k was urgent, it is easy to see that $f_\pi = f_{\bar{\pi}}$. □

A.2 Proof of Lemma 9

LEMMA 9. *For any annotated path π (without urgent locations) such that* $\max\{r_i \mid i = 1, \ldots, n\} = 0$*, an annotated path $\widetilde{\pi}$ in trivial normal form can be constructed in polynomial time with $f_\pi = f_{\widetilde{\pi}}$.*

PROOF. Let j_0 be such that $r_{j_0} = 0$, and

$$\widetilde{p}_0 = \sum_{k=0}^{j_0-1} p_k \qquad\qquad \widetilde{b}_0 = \max\Big\{b_k - \sum_{l=i_{m+1-j}}^{k-1} p_l \mid 0 \leq k \leq j_0 - 1\Big\}$$

$$\widetilde{p}_1 = \sum_{k=j_0}^{n} p_k \qquad\qquad \widetilde{b}_1 = \max\Big\{b_k - \sum_{l=i_1}^{k-1} p_l \mid j_0 \leq k \leq n\Big\}$$

We prove that the path

$$\widetilde{\pi}\colon \quad \widetilde{\ell}_0 \xrightarrow[\{c\} \quad \geq \widetilde{b}_0]{c=0 \quad \widetilde{p}_0} \ell_1 \xrightarrow[\{c\} \quad \geq \widetilde{b}_1]{c=1 \quad \widetilde{p}_1} \ell_2$$

is such that $f_\pi = f_{\widetilde{\pi}}$. Notice that it is in normal form, the second condition being satisfied vacuously.

Let w be a nonnegative real. We first assume that there exists a feasible run along $\widetilde{\pi}$ with initial credit w. In the special cases of a trivial normal form, one time unit elapses in $\widetilde{\ell}_1$, but this does not modify the accumulated cost as $r_{j_0} = 0$. Thus, such a feasible run exists if, and only if, it holds $w \geq \widetilde{b}_0$ and $w + \widetilde{p}_0 \geq \widetilde{b}_1$. By definition of $\widetilde{b}_0$ and $\widetilde{b}_1$, this is equivalent to $w + \sum_{l=0}^{k-1} p_l \geq b_k$ for all $0 \leq k \leq n$. Thus, the run along π which elapses one time unit in ℓ_{j_0} is feasible and has the same final accumulated cost, so that $f_{\widetilde{\pi}}(w) \leq f_\pi(w)$.

Conversely, let $\rho = (w, t_1, \ldots, t_n)_\pi$ be a feasible run along π with initial credit w. We assume that ρ is optimal, *i.e.*, that $w_{n+1} = f_\pi(w)$. Let $1 \leq j \leq n$ be the least index such that $t_j \neq 0$ and $j \neq j_0$. If no such j exists, then ρ spends one time unit in ℓ_{j_0}, and can thus be mimicked along $\widetilde{\pi}$, ending with the same accumulated credit. This would imply $f_{\widetilde{\pi}}(w) \geq f_\pi(w)$, and conclude the proof.

If j exists, consider the run $\bar{\rho} = (w, \bar{t}_1, \ldots, \bar{t}_n)_\pi$ obtained from ρ by transfering time t_j to ℓ_{j_0} (*i.e.*, by letting $\bar{t}_k = t_k$ except that $\bar{t}_j = 0$ and $\bar{t}_{j_0} = t_{j_0} + t_j$). It is easily proved, by induction, that $w_i \leq \bar{w}_i$ and $w'_i \leq \bar{w}'_i$ in all locations ℓ_i of π. Since ρ was assumed to be optimal, so is $\bar{\rho}$. This procedure can be applied recursively, yielding an optimal feasible run where time only elapses in ℓ_{j_0}, which, as seens above, entails that $f_{\widetilde{\pi}}(w) \geq f_\pi(w)$. □

A.3 Proof of Lemma 10

LEMMA 10. *For any annotated path π (without urgent locations) such that* $\max\{r_i \mid i = 1, \ldots, n\} > 0$*, an annotated path $\widetilde{\pi}$ in positive or trivial normal form can be constructed in polynomial time with $f_\pi = f_{\widetilde{\pi}}$.*

PROOF. Let $(n_j)_{j \geq 0}$ be the sequence of indices such that

- $n_0 = 0$
- assuming n_j has been computed for some $j \geq 0$, then
  - if $r_{n_j} = \max\{r_i \mid i = 1, \ldots, n\}$ is the maximal rate along π, then the sequence stops there;
  - otherwise, we let n_{j+1} be the least index $i > n_j$ for which $r_i > r_{n_j}$.

Let m be the index of last item in $(n_j)_{j\geq 0}$. We add a last item $n_{m+1} = n+1$, and define an intermediary annotated path $\bar{\pi}$, having $m+2$ locations $\bar{\ell}_0$ to $\bar{\ell}_{m+1}$, with rates $\bar{r}_k = r_{n_k}$ when $0 \leq k \leq m+1$. Notice that this sequence of locations satisfies the first part of the condition for being in *positive* normal form (or in trivial normal form if $m=1$).

We now define the transitions of $\bar{\pi}$. For $0 \leq j \leq m$, the annotated edge $\bar{\ell}_j \xrightarrow[\geq \bar{b}_j]{\bar{p}_j} \bar{\ell}_{j+1}$ is defined by

$$\begin{cases} \bar{p}_j &= \sum_{k=n_j}^{n_{j+1}-1} p_k \\ \bar{b}_j &= \max\left\{b_k - \sum_{l=n_j}^{k-1} p_l \mid n_j \leq k \leq n_{j+1}-1\right\} \end{cases}$$

Notice that the condition $b_i + p_i > b_{i+1}$ is not fulfilled yet, but it will be an easy task, which we delay after the proof that $f_{\bar{\pi}} = f_\pi$.

► *Proof that $f_{\bar{\pi}} \leq f_\pi$:* Pick a feasible run $\bar{\rho} = (w, \bar{t}_1, \ldots, \bar{t}_m)_{\bar{\pi}}$ along $\bar{\pi}$, for some initial credit w. Clearly enough, this run can be mimicked along π by elapsing each delay $\widetilde{t}_j$ of $\widetilde{\ell}_j$ in $\ell_{i_{m+1-j}}$, for $1 \leq j \leq m$, and elapsing zero time in the other locations. This yields a run $\rho = (w, t_1, \ldots, t_n)_\pi$ with $t_{n_i} = \bar{t}_i$ for all $1 \leq i \leq m$, and $t_j = 0$ otherwise. That ρ is a run along π follows from the definition of $\bar{b}_j$ and $\bar{p}_j$. Clearly enough, the accumulated cost along π and $\bar{\pi}$ are the same each time the runs enter (or leaves) corresponding locations (ℓ_{n_j} and $\bar{\ell}_j$). In particular, if $\bar{\rho}$ is chosen optimal, we get that $f_\pi(w) \geq f_{\bar{\pi}}(w)$ for any w where $f_{\bar{\pi}}$ is defined.

► *Proof that $f_{\bar{\pi}} \geq f_\pi$.* Pick a feasible run $\rho = (w, t_0, \ldots, t_n)_\pi$ along π with initial credit w, and define the run $\bar{\rho} = (w, \bar{t}_1, \ldots, \bar{t}_m)_{\bar{\pi}}$ with $\bar{t}_1 = \sum_{j=n_0}^{n_1-1} t_j + \sum_{j=n_1}^{n_2-1} t_j$ and $\bar{t}_i = \sum_{j=n_i}^{n_{i+1}-1} t_j$ when $2 \leq i \leq m$. We show that $\bar{\rho}$ is a feasible run along π', and that it achieves final accumulated cost as least as high as ρ does.

Let us first fix notations: $\bar{v}_j$ is the value of the clock when entering location $\bar{\ell}_j$, and $\bar{v}'_j$ is the valuation when leaving (so that $\bar{v}'_j = \bar{v}_j + \bar{t}_j$ for $1 \leq j \leq m$). Similarly, $\bar{w}_j$ and $\bar{w}'_j$ are the accumulated costs when entering and leaving ℓ_j, so that $\bar{w}'_j = \bar{w}_j + \bar{r}_j \cdot \bar{t}_j$ for $1 \leq j \leq m$.

Simple computations show that, along this run $\bar{\rho}$, it holds:

$$\begin{cases} \bar{v}'_i = \bar{v}_{i+1} = \sum_{j=0}^{n_{i+1}-1} t_j & \text{for all } 1 \leq i \leq m \\ \bar{w}'_i = \bar{w}_i + \bar{r}_i \cdot \bar{t}_i & \text{for all } 1 \leq i \leq m \\ \bar{w}_i = \bar{w}'_{i-1} + \bar{p}_{i-1} & \text{for all } 0 \leq i \leq m \end{cases}$$

Pick now some $1 \leq j < m$. Since ρ is feasible, we have, for all $n_j \leq k \leq n_{j+1}-1$,

$$w'_k = w'_{n_j} + \sum_{l=n_j}^{k-1} (p_l + t_{l+1} \cdot r_{l+1}) \geq b_k. \tag{1}$$

As $w'_{n_j} = w_{n_j} + t_{n_j} \cdot r_{n_j}$, we get that for every $n_j \leq k \leq n_{j+1}-1$:

$$w_{n_j} + \sum_{l=n_j}^{k} t_l \cdot r_l \geq b_k - \sum_{l=n_j}^{k-1} p_l \tag{2}$$

We now prove, by induction, that $\bar{w}_j \geq w_{n_j}$, for all $1 \leq j \leq m$. This holds for $j=1$, because $\bar{w}_1 = w + \bar{p}_0$, while $w_{n_1} = w + \sum_{l=0}^{n_1-1}(p_l + r_l t_l)$ and r_l are nonpositive for $1 \leq l < n_1$.

Assuming $\bar{w}_j \geq w_{n_j}$ for some $1 \leq j \leq m-1$, we have:

$$\begin{aligned} w_{n_{j+1}} &= w'_{n_{j+1}-1} + p_{n_{j+1}-1} \\ &= w'_{n_j} + \sum_{l=n_j}^{n_{j+1}-2} (p_l + t_{l+1} \cdot r_{l+1}) + p_{n_{j+1}-1} && \text{from (1)} \\ &= w_{n_j} + \sum_{l=n_j}^{n_{j+1}-1} r_l t_l + \sum_{l=n_j}^{n_{j+1}-1} p_l \\ &= w_{n_j} + \sum_{l=n_j}^{n_{j+1}-1} r_l t_l + \bar{p}_j \\ &\leq \bar{w}_j + \sum_{l=n_j}^{n_{j+1}-1} r_l t_l + \bar{p}_j \\ &\leq \bar{w}_j + \bar{t}_j \bar{r}_j + \bar{p}_j && \text{because } r_l < \bar{r}_j \text{ when } n_j \leq l \leq n_{j+1}-1, \text{ and } \textstyle\sum_{l=n_j}^{n_{j+1}-1} t_l \leq \bar{t}_j. \\ &= \bar{w}_{j+1} \end{aligned}$$

We conclude by proving that $\bar{\rho}$ is feasible, *i.e.*, that all its cost constraints are fulfilled. Indeed, for all $1 \le j \le m$ and for all $n_j \le k \le n_{j+1} - 1$, we have:

$$\begin{aligned}
\bar{w}'_j &\ge \bar{w}_j + \sum_{l=n_j}^{n_{j+1}-1} t_l \cdot \bar{r}_j \\
&\ge w_{n_j} + \sum_{l=n_j}^{n_{j+1}-1} t_l \cdot \bar{r}_j \\
&\ge w_{n_j} + \sum_{l=n_j}^{k} t_l \cdot \bar{r}_j && \text{because } \bar{r}_j \ge 0 \\
&\ge w_{n_j} + \sum_{l=n_j}^{k} t_l \cdot r_l && \text{as } \bar{r}_j \ge r_l \text{ for } l \le n_{j+1} - 1 \\
&\ge b_k - \sum_{l=n_j}^{k-1} p_l && \text{from (2).}
\end{aligned}$$

As this holds for any k between n_j and $n_{j+1} - 1$, we conclude that $\widetilde{w}'_j \ge \widetilde{b}_{j+1}$. We get that $\widetilde{\rho}$ is a feasible run along $\widetilde{\pi}$, and that the value of the observer variable at the end of $\widetilde{\rho}$ is larger than or equal to that at the end of ρ, hence that $f_{\widetilde{\pi}}(w) \ge f_\pi(w)$, for any w where $f_\pi(w)$ is defined.

It remains to enforce the second condition ($\bar{b}_{i-1} + \bar{p}_{i-1} < \bar{b}_i$ for $1 \le i \le m-2$) on $\bar{\pi}$. This is achieved in a similar way as for removing urgent locations: we inductively replace any offending pair of consecutive annotated edges $\bar{\ell}_{i-1} \xrightarrow[\ge \bar{b}_{i-1}]{\bar{p}_{i-1}} \bar{\ell}_i \xrightarrow[\ge \bar{b}_i]{\bar{p}_i} \bar{\ell}_{i+1}$ by a single annotated edge $\bar{\ell}_{i-1} \xrightarrow[\ge \bar{b}_{i-1}]{\bar{p}_{i-1}+\bar{p}_i} \bar{\ell}_{i+1}$. It is rather easy to see that this does not affect the energy function (a feasible run in the new path can be mimicked in the original path, and conversely, a feasible run along the original path is transformed into a feasible run along the new path by transfering the delay from $\bar{\ell}_i$ to $\bar{\ell}_{i+1}$, which is more profitable because $i \le m-1$). Recursively applying this transformation yields a path $\widetilde{\pi}$ in positive normal form, or in trivial normal form if there remains only one location with positive cost. □

A.4 Proof of Lemma 11

LEMMA 11. *For any annotated path π such that* $\max\{r_i \mid i = 1, \ldots, n\} < 0$*, an annotated path $\bar{\pi}$ in negative (or trivial) normal form can be constructed in polynomial time with $f_\pi = f_{\bar{\pi}}$.*

PROOF. The sequence of locations to be kept is defined as follows:

- $n_0 = 0$;
- if n_j has been constructed for some $j \ge 0$,
  - if $r_{n_j} = \min\{r_i \mid i = 1, \ldots, n\}$ is the minimal rate along π, then the sequence stops there;
  - otherwise, we let n_{j+1} be the greatest index $i > n_j$ for which $r_i = \max\{r_i \mid n_j + 1 \le i \le n\}$.

We let m be the index of the last item in this sequence, and add $n_{m+1} = n+1$. This defines an intermediary path $\bar{\pi}$ with $m+2$ locations $\bar{\ell}_0$ to $\bar{\ell}_{m+1}$, with rates $\bar{r}_i = r_{n_i}$. Notice that the selected sequence of locations satisfies the first part of the requirement for being in *negative* normal form (or in trivial normal form if $m = 1$).

The set of transitions is defined in the same way as in the previous proof. The proof that $f_{\bar{\pi}}$ equals f_π is also similar:

- $f_{\bar{\pi}} \ge f_\pi$ by transfering to $\ell_{n_{j+1}}$ the delays elapsed between ℓ_{n_j+1} and $\ell_{n_{j+1}-1}$. This preserves feasability of the run, and can only increase the final credit.
- $f_{\bar{\pi}} \le f_\pi$ by mimicking a feasible run of $\bar{\pi}$ along π.

Now, assume that $b_{i-1} + p_{i-1} > b_i$ for some i. Since r_i is negative, the accumulated cost in ℓ_i can only decrease, so that, when entering ℓ_i, the cost must already be larger than or equal to b_i. In other terms, the constraint "$\ge b_{i-1}$" can be strengthened into "$\ge b_i - p_{i-1}$". This way, we necessarily have $b_{i-1} + p_{i-1} \le b_i$. Assume that the equality holds for some i. In this case, it is more profitable to delay in ℓ_{i-1} than in ℓ_i (as long as $i > 1$), and location ℓ_i can be removed in the same way as for removing urgent locations. □

A.5 Proof of Proposition 12

We begin with recalling the setting: we consider an annotated path

$$\pi: \quad \ell_0 \xrightarrow[\{c\}]{c=0} \xrightarrow[\ge b_0]{p_0} \ell_1 \xrightarrow[\ge b_1]{p_1} \ell_2 \xrightarrow[\ge b_2]{p_2} \cdots \xrightarrow[\ge b_{n-1}]{p_{n-1}} \ell_n \xrightarrow[\{c\}]{c=1} \xrightarrow[\ge b_n]{p_n} \ell_{n+1}$$

in positive normal form, and define the n-tuple $t^{\text{opt}} = (t_i^{\text{opt}})_{1 \le i \le n}$ by

$$t_i^{\text{opt}} = \begin{cases} 0 & \text{if } i = m \text{ and } b_m \le b_{m-1} + p_{m-1} \\ \frac{b_i - (b_{i-1} + p_{i-1})}{r_i} & \end{cases}$$

We then consider two cases:
- if $\sum_{i=1}^{n} t_i^{\text{opt}} > 1$, let ι_π be the largest index for which $\sum_{i=\iota_\pi}^{n} \tau_i > 1$ (so that $\sum_{i=\iota_\pi-1}^{n} \tau_i \leq 1$). In this case, we let $t_i^\star = t_i^{\text{opt}}$ for all $\iota_\pi + 1 \leq i \leq n$, and $t_{\iota_\pi}^\star = 1 - \sum_{i=\iota_\pi+1}^{n} t_i^{\text{opt}}$. The other values, $t_i^\star$ with $i < \iota_\pi$, are set to zero.
- otherwise, we define $t_i^\star = t_i^{\text{opt}}$ for $1 \leq i \leq n-1$, and $t_n^\star = 1 - \sum_{i=1}^{n-1} t_i^{\text{opt}}$: the extra time will be spent in location ℓ_n, where the rate is maximal. We let $\iota_\pi = 0$ in this case.

Finally, we define w_i^* and ω_i^*, for $\iota_\pi + 1 \leq i \leq n$, as follows:

$$\begin{cases} w_i^* = b_{i-1} - \displaystyle\sum_{k=0}^{i-2} p_k \\ \omega_i^* = \max(b_n, b_{n-1} + p_{n-1}) + p_n + ((t_n^\star - t_n^{\text{opt}}) + \displaystyle\sum_{j<i} t_j^\star) \cdot r_n \end{cases}$$

Notice that we have the equality

$$\omega_i^* = \max(b_n, b_{n-1} + p_{n-1}) + p_n + (1 - \sum_{j=i}^{n} t_j^{\text{opt}}) \cdot r_n$$

because $(t_j^\star)_{j \leq n}$ sum up to one, and $t_j^\star = t_j^{\text{opt}}$ when $\iota_\pi < j < n$. However, the definition above will be a bit more convenient for the proof.

LEMMA 27. *For every $\iota_\pi \leq i \leq n$, we have that:*

1. *for every $0 \leq j < i$, it holds $w_i^* + \sum_{k=0}^{j-1} p_k \geq b_j$;*
2. *for every $i \leq j < n$, we have $w_i^* + \sum_{k=0}^{j-1} p_k + \sum_{k=i}^{j} t_k^\star \cdot r_k = b_j$;*
3. *$\omega_i^* = w_i^* + \sum_{j=0}^{n} p_j + \sum_{j=i}^{n-1} t_j^\star \cdot r_j + (t_n^\star + \sum_{j<i} t_j^\star) \cdot r_n$.*

PROOF. We begin with Property 1, proved by induction on j. For any $i \geq \iota_\pi$, we have

$$w_i \geq b_{i-1} - \sum_{k=0}^{i-2} p_k$$

(with equality when $i \neq \iota_\pi$). Hence the result when $j = i-1$. Now, if we have $w_i^* + \sum_{k=0}^{j-1} p_k \geq b_j$ for some $j > 0$, then $w_i^* + \sum_{k=0}^{j-2} p_k \geq b_j + p_{j-1} \geq b_{j-1}$, because π is in normal form.

We turn to the second property, again by induction on j. First, when $i = j = \iota_\pi$ (hence $\iota_\pi \neq n$), we have

$$w_i + \sum_{k=0}^{j-1} p_k + \sum_{k=i}^{j} t_k^\star \cdot r_k = b_{\iota_\pi - 1} + p_{\iota_\pi - 1} + t_{\iota_\pi}^{\text{opt}} \cdot r_{\iota_\pi}$$

which equald b_{ι_π} by definition of $t_{\iota_\pi}^{\text{opt}}$. If $i = j > \iota_\pi$, we have

$$w_i + \sum_{k=0}^{j-1} p_k + \sum_{k=i}^{j} t_k^\star \cdot r_k = b_{j-1} + p_{j-1} + t_j^\star \cdot r_j.$$

Since $\iota_\pi < j < n$, we have $t_j^\star = t_j^{\text{opt}}$, and the result follows from the definition of t_j^{opt}. The induction step is proved similarly: if the result holds for $i \geq \iota_\pi$ and $i \leq j < n-1$, we have

$$w_i + \sum_{k=0}^{j} p_k + \sum_{k=i}^{j+1} t_k^\star \cdot r_k = b_j + p_j + t_{j+1}^\star \cdot r_{j+1}.$$

As $\iota_\pi < j+1 < n$, we have $t_{j+1}^\star = t_{j+1}^{\text{opt}}$. By definition of t_{j+1}^{opt}, the result follows.

Applying the previous result for $\iota_\pi \leq i \leq j = n-1$, we get

$$w_i^* + \sum_{k=0}^{n-2} p_k + \sum_{k=i}^{n-1} t_k^\star \cdot r_k = b_{n-1}.$$

Hence

$$w_i^* + \sum_{j=0}^{n} p_j + \sum_{j=i}^{n-1} t_j^\star \cdot r_j + (t_n^\star + \sum_{j<i} t_j^\star) \cdot r_n = b_{n-1} + p_{n-1} + p_n + ((t_n^\star - t_n^{\text{opt}}) + \sum_{j<i} t_j^\star) \cdot r_n + t_n^{\text{opt}} \cdot r_n.$$

If $b_n > (b_{n-1} + p_{n-1})$, then $\omega_i^* = b_n + p_n + ((t_n^\star - t_n^{\text{opt}}) + \sum_{j<i} t_j^\star) \cdot r_n$ and $t_n^{\text{opt}} \cdot r_n = b_n - (b_{n-1} + p_{n-1})$, so that our result follows. Otherwise, $\omega_i^* = b_{n-1} + p_{n-1} + p_n + ((t_n^\star - t_n^{\text{opt}}) + \sum_{j<i} t_j^\star) \cdot r_n$ and $t^{\text{opt}} = 0$, which also entails our result. ◻

We now come to the main result of this section:

Proposition 12. *The function f_π is a piecewise affine function defined on the interval $[w^*_{\iota_\pi}, \infty[$, visiting points (w^*_i, ω^*_i), for all $\iota_\pi \le i \le n$, with constant slope $\dot{f}_\pi(x) \ge 1$ between two consecutive such points, and with slope $\dot{f}_\pi(x) = 1$ after (w^*_n, ω^*_n).*

Proof. Let $\iota_\pi \le i \le n$, and consider the following two runs:
- $\bar\rho = (w^*_i, \bar t_1, \bar t_2, \ldots, \bar t_n)_\pi$, with $\bar t_j = 0$ when $j < i$, $\bar t_j = t^\star_j$ when $i \le j \le n-1$, and $\bar t_n = t^\star_n + \sum_{j<i} t^\star_j$;
- $\rho = (w^*_i, t_1, \ldots, t_n)_\pi$, with $\sum_{j=1}^n t_j = 1$.

We write $\bar w_i$ and $\bar w'_i$ for the accumulated cost when entering and leaving ℓ_i along $\bar\rho$, and w_i and w'_i the the corresponding values along ρ.

First notice that $\bar\rho$ is feasible, as a direct consequence of Lemma 27. This entails $f_\pi(w) \ge \omega^*_i$. We now prove that the converse inequality also holds.

Assuming that ρ is feasible, we prove that it cannot achieve better final accumulated credit than $\bar w_{n+1}$. This will entail $f_\pi(w) = \omega^*_i$. To this aim, we prove, by induction on k, that the following properties hold:

Lemma 28. *For any $1 \le k \le n$,*
- *the time elapsed along ρ when entering location ℓ_k is larger than the time elapsed along $\bar\rho$ when entering ℓ_k:*

$$\sum_{j=1}^{k} t_j \ge \sum_{j=1}^{k} \bar t_j$$

- *the difference in the accumulated cost when leaving ℓ_k can be bounded as follows:*

$$w'_k - \bar w'_k \le r_k \cdot \Big(\sum_{j=1}^{k} t_j - \sum_{j=1}^{k} \bar t_j\Big).$$

Moreover, the inequality is an equality iff $t_j = \bar t_j$ when $1 \le j < k$ and $t_k \ge \bar t_k$.

Proof. First notice that, from Lemma 27, it holds $\bar w'_j = b_j$ for all $i \le j \le n-1$, so that it must be the case that $w'_j \ge \bar w'_j$ for all $i \le j \le n-1$. Moreover, since no time elapses up to ℓ_{i-1} along $\bar\rho$, and since rates are positive in π, we also have $w'_j \ge \bar w_j$ for all $j < i$.

Now, for any $1 \le k \le n$, we have

$$\begin{aligned} w'_k &= w^*_i + \sum_{j=0}^{k-1} (p_j + t_{j+1} \cdot r_{j+1}) \\ &= w^*_i + \sum_{j=0}^{k-1} p_j + \sum_{j=1}^{k} \bar t_j \cdot r_j + \sum_{j=1}^{k} (t_j - \bar t_j) \cdot r_j \\ &= \bar w'_k + \sum_{j=1}^{k} (t_j - \bar t_j) \cdot r_j && \text{(according to Lemma 27)} \end{aligned}$$

Hence for all k, $\sum_{j=1}^k (t_j - \bar t_j) \cdot r_j \ge 0$, and in particular $t_1 \ge \bar t_1$. Assume that the first assertion of the lemma holds for any $l \le k$, with $1 \le k \le n-1$, but fails to hold for $k+1$. Then we must have in particular $t_{k+1} \le \bar t_{k+1}$. As we just saw, we have

$$\sum_{j=1}^{k} r_j (t_j - \bar t_j) \ge r_{k+1}(\bar t_{k+1} - t_{k+1})$$

and we also have $r_{k+1}(\bar t_{k+1} - t_{k+1}) \ge r_k(\bar t_{k+1} - t_{k+1})$ because $r_k \le r_{k+1}$ and we have $\bar t_{k+1} - t_{k+1} \ge 0$.

On the other hand, because the first assertion holds up to k, we have

$$\sum_{j=1}^{k} r_j (t_j - \bar t_j) \le r_l (\sum_{j=1}^{l} t_j - \bar t_j) + \sum_{j=l+1}^{k} r_j (t_j - \bar t_j)$$

for all $1 \le l \le k$: indeed, this is obvious for $l = 1$, and if it holds for some index $l < k$, then

$$\begin{aligned}
\sum_{j=1}^{k} r_j(t_j - \bar{t}_j) &\le r_l(\sum_{j=1}^{l} t_j - \bar{t}_j) + \sum_{j=l+1}^{k} r_j(t_j - \bar{t}_j) && \text{(by ind. hyp. on } l\text{)} \\
&= r_l(\sum_{j=1}^{l} t_j - \bar{t}_j) + r_{l+1}(t_{l+1} - \bar{t}_{l+1}) + \sum_{j=l+2}^{k} r_j(t_j - \bar{t}_j) \\
&\le r_{l+1}(\sum_{j=1}^{l} t_j - \bar{t}_j) + r_{l+1}(t_{l+1} - \bar{t}_{l+1}) + \sum_{j=l+2}^{k} r_j(t_j - \bar{t}_j) && \text{(because } \sum_{j=1}^{l} t_j - \bar{t}_j \ge 0 \text{ by ind. hyp. on } k\text{)} \\
&= r_{l+1}(\sum_{j=1}^{l+1} t_j - \bar{t}_j) + \sum_{j=l+2}^{k} r_j(t_j - \bar{t}_j)
\end{aligned}$$

When $l = k - 1$, this entails

$$\sum_{j=1}^{k} r_j(t_j - \bar{t}_j) \le r_k(\sum_{j=1}^{k} t_j - \bar{t}_j)$$

and, gathering all together, we finally get

$$\bar{t}_{k+1} - t_{k+1} \le \sum_{j=1}^{k} t_j - \bar{t}_j$$

i.e., $\sum_{j=1}^{k+1} t_j \ge \sum_{j=1}^{k+1} \bar{t}_j$, which contradicts our initial assumption that the first property fails at $k + 1$. Hence this property holds.

Moreover, we just proved that

$$\sum_{j=1}^{k} r_j(t_j - \bar{t}_j) \le r_l(\sum_{j=1}^{l} t_j - \bar{t}_j) + \sum_{j=l+1}^{k} r_j(t_j - \bar{t}_j)$$

for all $1 \le l \le k$, hence

$$w'_k - \bar{w}'_k = \sum_{j=1}^{k}(t_j - \bar{t}_j) \cdot r_j \le \sum_{j=1}^{k}(t_j - \bar{t}_j) r_k. \tag{3}$$

It remains to prove that this is an equality if, and only if, $t_j = \bar{t}_j$ for all $j < k$. When $k = 1$, (3) is obviously an equality, and $t_j = \bar{t}_j$ for all $j < k$ holds vacuously.

Now, assume that the equivalence holds at k, and that

$$w'_{k+1} - \bar{w}'_{k+1} = \sum_{j=1}^{k+1}(t_j - \bar{t}_j) r_{k+1}.$$

We also have

$$w'_{k+1} - \bar{w}'_{k+1} = w'_k - \bar{w}'_k + r_{k+1}(t_{k+1} - \bar{t}_{k+1})$$

so that we must have

$$w'_k - \bar{w}'_k = \sum_{j=1}^{k}(t_j - \bar{t}_j) r_{k+1}$$

and, since the sum must be nonnegative and $r_{k+1} \ge r_k$, we get

$$w'_k - \bar{w}'_k \ge \sum_{j=1}^{k}(t_j - \bar{t}_j) r_k.$$

Since the converse inequality must also hold, this is an equality, which, from the induction hypothesis, entails that $t_j = \bar{t}_j$ for all $j < k$. Thus

$$w'_{k+1} - \bar{w}'_{k+1} = r_k(t_k - \bar{t}_k) + r_{k+1}(t_{k+1} - \bar{t}_{k+1}).$$

We want this quantity to equal $r_{k+1}(t_k + t_{k+1} - \bar{t}_k - \bar{t}_{k+1})$, which, since $r_k < r_{k+1}$, requires $t_k = \bar{t}_k$.

Finally, the converse implication is trivial. $\square$

Now, taking the second assertion of the Lemma when $k = n$ yields $w'_n - \bar{w}'_n \leq 0$, because the total time equals 1 along both ρ and $\bar{\rho}$. Hence the final accumulated credit along ρ is less than that along $\bar{\rho}$, for any feasible run ρ, which proves that $f_\pi(w^*_i) = \omega^*_i$. As a side result, this also proves that the optimal feasible run is unique along paths in normal form.

We continue the proof of Prop. 12 by showing that f_π is an affine function between two consecutive points. We first deal wth the case where $\iota_\pi < n$: fix $\iota_\pi \leq i < n$, and $w^*_i \leq w \leq w^*_{i+1}$. For any $j < i$, we have $w + \sum_{k=0}^{j-1} p_k \geq b_j$ (from Lemma 27, and because $w \geq w^*_i$). Moreover, since $w \leq w^*_{i+1}$, we have

$$w + \sum_{k=0}^{i-1} p_k \leq w^*_{i+1} + \sum_{k=0}^{i-1} p_k = b_i.$$

We let $t_i = (b_i - (w + \sum_{k=0}^{i-1} p_k))/r_i$. This value is nonnegative. Moreover, since $w \geq w^*_i$:

- if $i > \iota_\pi$, we have

$$b_i - (w + \sum_{k=0}^{i-1} p_k) \leq b_i - (b_{i-1} - \sum_{k=0}^{i-2} p_k + \sum_{k=0}^{i-1} p_k) = b_i - (b_{i-1} + p_{i-1})$$

  so that $t_i \leq t_i^{\text{opt}} \leq t_i^\star$.
- if $i = \iota_\pi$, we have

$$b_i - (w + \sum_{k=0}^{i-1} p_k) \leq b_i - (b_{i-1} + p_{i-1}) - (t_i^{\text{opt}} - t_i^\star) r_i = t_i^\star r_i$$

  and $t_i \leq t_i^\star$.

Let $\rho = (w, t_1, \ldots, t_n)$, with t_i defined as above, and $t_j = 0$ when $j < i$, and $t_j = t_j^{\text{opt}}$ when $i < j < n$, and $t_n = 1 - \sum_{j=1}^{n-1} t_j$. Since $t_j \leq t_j^\star$ for all $1 \leq j < n$, we get that $t_n \geq t_n^\star$, so that it is between 0 and 1. By definition of t_i, the accumulated cost along ρ when leaving ℓ_i is exactly b_i. By definition of t_j^{opt}, this entails that the accumulated cost along ρ when leaving locations ℓ_j, with $i \leq j < n$, is exactly b_j. Since $t_n \geq t_n^\star \geq t_n^{\text{opt}}$, the accumulated cost when leaving ℓ_n is larger than b_n, so that ρ is feasible. Finally, the delay elapsed in ℓ_n is

$$t_n = 1 - \sum_{j=i+1}^{n-1} t_j^{\text{opt}} - t_i = t_n^\star + \sum_{j=\iota_\pi}^{i} t_j^\star - t_i.$$

and the final accumulated cost along ρ is

$$\begin{aligned}
w_{n+1} &= b_{n-1} + p_{n-1} + r_n \cdot t_n + p_n \\
&= b_{n-1} + p_{n-1} + r_n(t_n^\star + \sum_{j=\iota_\pi}^{i} t_j^\star - t_i) + p_n \\
&= b_{n-1} + p_{n-1} + r_n t_n^{\text{opt}} + r_n(t_n^\star - t_n^{\text{opt}} + \sum_{j=\iota_\pi}^{i-1} t_j^\star + t_i^\star - t_i) + p_n \\
&= \max(b_{n-1} + p_{n-1}, b_n) + r_n(t_n^\star - t_n^{\text{opt}} + \sum_{j=\iota_\pi}^{i-1} t_j^\star + t_i^\star - t_i) + p_n \\
&= \omega^*_i + r_n(t_i^\star - t_i).
\end{aligned}$$

Notice that the last equality holds because

- either $i > \iota_\pi$, so that the equality follows from the definition of ω^*_i in this case. In this case, $t_i^\star = t_i^{\text{opt}}$, and

$$\begin{aligned}
w_{n+1} &= \omega^*_i + r_n(\frac{b_i - (b_{i-1} + p_{i-1})}{r_i} - \frac{b_i - (w + \sum_{k=0}^{i-1} p_k)}{r_i}) \\
&= \omega^*_i + \frac{r_n}{r_i}(w - w^*_i)
\end{aligned}$$

- or $i = \iota_\pi$, which entails that $\iota_\pi \neq 0$, and $t_n^\star = t_n^{\text{opt}}$. In this case, the penultimate line simplifies, and the equality holds by definition of $\omega^*_{\iota_\pi}$. Then

$$\begin{aligned}
w_{n+1} &= \omega^*_i + r_n(t^\star_i - t^{\text{opt}}_i + t^{\text{opt}}_i - t_i) \\
&= \omega^*_i + r_n(t^\star_i - t^{\text{opt}}_i) + \frac{r_n}{r_i}[(b_i - (b_{i-1} + p_{i-1})) - (b_i - (w + \sum_{j=0}^{i-1} p_j))] \\
&= \omega^*_i + \frac{r_n}{r_i}\big(w - (b_i - \sum_{j=0}^{i-2} p_j + r_i(t^{\text{opt}}_i - t^\star_i))\big) \\
&= \omega^*_i + \frac{r_n}{r_i}(w - w^*_{\iota_\pi})
\end{aligned}$$

Hence

$$f_\pi(w) \geq f_\pi(w^*_i) + \frac{r_n)}{rate(r_i)} \cdot (w - w^*_i)$$

when $w^*_i \leq w \leq w^*_{i+1}$. The converse inequality can be proven by following the same lines as for the proof that $f_\pi(w^*_i) = \omega^*_i$.

Finally, when $w \geq w^*_n = b_{n-1} - \sum_{j=0}^{n-2} p_j$, we can go straight to ℓ_n, and spend one time unit there. Since $r_n \geq r_i$ for all $i \leq n$, this run clearly maximizes the accumulated cost when leaving ℓ_n, which is then

$$w'_n = w + \sum_{j=0}^{n-1} p_j + r_n.$$

As $\iota_\pi < n$, then $t^{\text{opt}}_n \leq 1$, which entails that $r_n \geq b_n - (b_{n-1} + p_{n-1})$. This entails

$$w'_n \geq b_{n-1} - \sum_{j=0}^{n-2} p_j + \sum_{j=0}^{n-1} p_j + b_n - (b_{n-1} + p_{n-1})$$

which simplifies to $w'_n \geq b_n$. The run where we elapse one time unit in ℓ_n is thus feasible, and yields final accumulated cost

$$\begin{aligned}
w_{n+1} &= w + \sum_{j=0}^{n} p_j + r_n \\
&= w + \sum_{j=0}^{n} p_j + r_n(1 - t^{\text{opt}}_n) + r_n \cdot t^{\text{opt}}_n + b_{n-1} + p_{n-1} - b_{n-1} - p_{n-1} \\
&= w + \sum_{j=0}^{n-2} p_j + p_n - b_{n-1} + r_n(\sum_{j=0}^{n} t^\star_j - t^{\text{opt}}_n) + \max(b_n, b_{n-1} + p_{n-1}) \\
&= w - b_{n-1} + \sum_{j=0}^{n-2} p_j + p_n + r_n((t^\star_n - t^{\text{opt}}_n) + \sum_{j=0}^{n-1} t^\star_j) + \max(b_n, b_{n-1} + p_{n-1}) \\
&= \omega^*_n + (w - w^*_n).
\end{aligned}$$

As argued above, this is the optimal achievable final accumulated cost, so that we have

$$f_\pi(w) = \omega^*_n + (w - w^*_n)$$

when $w \geq w^*_n$.

To be exhaustive, we have to handle the case where $\iota_\pi = n$, *i.e.*, when $t^{\text{opt}}_n > 1$. This case is handled as the case where $w \geq w^*_n$ above: the optimal run goes straight to ℓ_n and let one time unit elapse there. Such a run is feasible thanks to Lemma 27 and because the accumulated cost when leaving ℓ_n is

$$\begin{aligned}
w'_n &= w + \sum_{j=0}^{n-1} p_j + r_n \\
&\geq= w^*_n + \sum_{j=0}^{n-1} p_j + r_n \\
&\geq b_{n-1} - \sum_{j=0}^{n-2} p_j + (t^{\text{opt}}_n - t^\star_n) r_n + \sum_{j=0}^{n-1} p_j + r_n \\
&\geq b_{n-1} + p_{n-1} + t^{\text{opt}}_n \cdot r_n && \text{(because } t^\star_n = 1 \text{ in this case)} \\
&\geq b_n.
\end{aligned}$$

This run is thus feasible. As argued above, it is optimal, so that

$$
\begin{aligned}
f_\pi(w) &= w + \sum_{j=0}^{n-1} p_j + r_n + p_n \\
&= (w - w_n^*) + w_n^* + \sum_{j=0}^{n-1} p_j + r_n + p_n \\
&= (w - w_n^*) + b_{n-1} - \sum_{j=0}^{n-2} p_j + (t_n^{\mathrm{opt}} - t_n^\star) r_n + \sum_{j=0}^{n-1} p_j + r_n + p_n \\
&= (w - w_n^*) + b_{n-1} + p_{n-1} + t_n^{\mathrm{opt}} \cdot r_n + p_n && \text{(because } t_n^\star = 1 \text{ in this case)} \\
&= (w - w_n^*) + \max(b_n, b_{n-1} + p_{n-1}) \\
&= \omega_n^* + (w - w_n^*).
\end{aligned}
$$

Finally, proving that f_π is undefined on the left of $(w^*_{\iota_\pi}, \omega^*_{\iota_\pi})$ is achieved using arguments similar to those used in the proof of Lemma 28. □

A.6 Computing f_π for negative-normal-form paths

The computation of f_π for negative-normal-form paths is slightly different to the case of positive-normal-form paths, but the proof techniques are similar. We thus only detail the construction here, and leave the proof to the dubious reader.

We begin with defining the n-tuple $t^{\mathrm{opt}} = (t_i^{\mathrm{opt}})_{1 \le i \le n}$ by

$$
t_i^{\mathrm{opt}} = \begin{cases} 0 & \text{if } i = 1 \text{ and } b_1 \ge b_0 + p_0 \\ \frac{b_i - (b_{i-1} + p_{i-1})}{r_i} & \end{cases}
$$

Notice that, in this case, we have the equality

$$
b_1 - r_1 \cdot t_1^{\mathrm{opt}} = \max(b_0 + p_0, b_1).
$$

We then define $t_i^\star$ as follows:

- if $\sum_{i=1}^n t_i^{\mathrm{opt}} > 1$, let ι_π be the least index for which $\sum_{i=1}^{\iota_\pi} \tau_i > 1$ (so that $\sum_{i=1}^{\iota_\pi - 1} \tau_i \le 1$). In this case, we let $t_i^\star = t_i^{\mathrm{opt}}$ for all $1 \le i \le \iota_\pi - 1$, and $t^\star_{\iota_\pi} = 1 - \sum_{i=1}^{\iota_\pi - 1} t_i^{\mathrm{opt}}$. The other values, $t_i^\star$ with $i > \iota_\pi$, are set to zero.
  In this case, we let $w^*_{\iota_\pi} = \max(b_0, b_1 - p_0)$, and

$$
\omega^*_{\iota_\pi} = b_{\iota_\pi} + r_{\iota_\pi}(t^\star_{\iota_\pi} - t^{\mathrm{opt}}_{\iota_\pi}) + \sum_{j=\iota_\pi}^{n} p_j.
$$

- otherwise, we define $t_i^\star = t_i^{\mathrm{opt}}$ for $2 \le i \le n$, and $t_1^\star = 1 - \sum_{i=2}^n t_i^{\mathrm{opt}}$: the extra time will be spent in location ℓ_1, where the rate is maximal. We let $\iota_\pi = n+1$ in this case, and $w^*_{\iota_\pi} = \max(b_0, b_1 - p_0) - r_1(t_1^\star - t_1^{\mathrm{opt}})$ and $\omega^*_{\iota_\pi} = b_n + p_n$.

The other important points on the curve are computed as follows: for each $1 \le i \le \iota_\pi - 1$,

$$
\begin{cases}
w_i^* = \max(b_0, b_1 - p_0) - r_1(1 - \sum_{j=1}^{i} t_j^{\mathrm{opt}}) \\
\omega_i^* = b_{i-1} + \sum_{j=i-1}^{n} p_j
\end{cases}
$$

PROPOSITION 29. *When π is an annotated unit path in negative normal form, the function f_π is a piecewise affine function visiting points (w_i^*, ω_i^*), for all $1 \le i \le \iota_\pi$, with constant slope between two consecutive such points, and with slope $+1$ after (w_1^*, ω_1^*).*

B. PROOFS OF SECTION 6

B.1 Preliminary results on paths with exponential observers

Before focusing on normal forms for paths with exponential observers, we prove some easy lemmas which will be very useful in the sequel.

Our first lemma takes advantage of the conditions we impose on π:

LEMMA 30. *Let ρ be a run along π with final credit w_n. Then ρ is feasible (i.e., the accumulated credit is always nonnegative along ρ) iff w_n is nonnegative.*

PROOF. If the accumulated cost is always nonnegative, then it is in particular at the end of the run. On the other hand, assume that the accumulated cost ever goes below zero. Then action transitions can only make the accumulated even worse, and delay transitions cannot make the sign change (because the cost function when delaying in a location has the form $w \cdot \exp(r \cdot t)$, whose sign cannot change). Thus, if the accumulated cost is negative at some point, it will remain negative all along the run, and be negative at the end. ☐

Next, we show that, despite the "derivative" of the observer depends on its value, we always have the motto "the higher, the better":

LEMMA 31. *Let $c \in [0,1]$ be a real (representing the value of the clock) and $0 \leq i \leq n$ be an integer (the index of the current location along π). Let $w \leq w'$ (resp. $w < w'$) be two nonnegative reals, and ρ be a feasible run visiting state (ℓ_i, c, w). Let w_n be the accumulated cost at the end of ρ. Then there is a feasible run ρ' visiting (ℓ_i, c, w') and ending with accumulated cost $w'_n \geq w_n$ (resp. $w'_n > w_n$).*

PROOF. The run ρ' is built by following the very same transitions as along ρ. It is easily proved (by induction) that the accumulated credit along ρ' always remains above the accumulated credit along ρ at the same date. ☐

B.2 Proof of Proposition 13

PROPOSITION 13. *Assume π is a unit path with nonpositive edge weights and such that $\max\{r_i \mid i = 1, \ldots, n\} \geq 0$. Then we can construct in polynomial time a path $\widetilde{\pi}$ in normal form for exponential observers so that $f_\pi = f_{\widetilde{\pi}}$.*

Removing urgent locations..
In the same way as we did for the linear case, we can assume that the path contains no urgent locations: if location ℓ_i is urgent, the sequence of transitions $\ell_{i-1} \xrightarrow{p_{i-1}} \ell_i \xrightarrow{p_i} \ell_{i+1}$ is replaced with $\ell_{i-1} \xrightarrow{p_{i-1}+p_i} \ell_{i+1}$. As with linear observers, this does not modify the global cost function f_π.

We first assume that $\max\{r_i \mid 1 \leq i \leq n\} = 0$..
Let us first rule out the easy case: if $\max\{r_i \mid 1 \leq i \leq n\} = 0$, then at least one of the rates, r_k, must be zero. The normal-form path will only have one location with rate 0:

LEMMA 32. *If π is such that $r_k = \max\{r_i \mid 1 \leq i \leq n\} = 0$, then the path*

$$\widetilde{\pi}: \quad \widetilde{\ell}_0 \xrightarrow[\{c\}]{c=0 \quad \widetilde{p}_0} \widetilde{\ell}_1 \xrightarrow[\{c\}]{c=1 \quad \widetilde{p}_1} \widetilde{\ell}_2$$

with $\widetilde{p}_0 = p_0 + \cdots + p_{k-1}$ and $\widetilde{p}_1 = p_k + \cdots + p_n$ and $\widetilde{r}_0 = \widetilde{r}_1 = \widetilde{r}_2 = 0$, is in trivial normal form and achieves $f_\pi = f_{\widetilde{\pi}}$.

PROOF. Clearly, $f_\pi(w) \geq f_{\widetilde{\pi}}(w)$ since any feasible run along π' can be mimicked along π. On the other hand, let

$$\rho: \quad (\ell_0, 0, w'_0) \xrightarrow{e_0} (\ell_1, v_1, w_1) \xrightarrow{t_1} (\ell_1, v'_1, w'_1) \cdots \xrightarrow{e_n} (\ell_{n+1}, 0, w_{n+1})$$

be a feasible run with $t_j > 0$ for $j \neq k$. Consider the run

$$\bar{\rho}: \quad (\ell_0, 0, \bar{w}'_0) \xrightarrow{e_0} (\ell_1, \bar{v}_1, \bar{w}_1) \xrightarrow{t'_1} (\ell_1, \bar{v}'_1, \bar{w}'_1) \cdots \xrightarrow{e_n} (\ell_{n+1}, 0, \bar{w}_{n+1})$$

with $t'_i = t_i$ except $t'_j = 0$ and $t'_k = t_k + t_j$. If $j < k$, then it holds $\bar{w}'_i = w'_i$ when $i < j$, and $\bar{w}'_j \geq w'_j$, and $\bar{w}'_i \geq w'_i$ when $j < i < k$. Thus, when leaving ℓ_k, both runs have the same clock value, but the credit along ρ' is higher. According Lemma 31, the credit at the end of ρ' is higher than that at the end of ρ.
On the other hand, if $j > k$, then the accumulated cost when arriving in ℓ_j is the same along both runs. Since the rate in ℓ_j is nonpositive, the credit when leaving ℓ_j is higher along ρ' than along ρ, with the same clock value. Again, Lemma 31 show that this is preserved at the end of the run. This proves that the "better" feasible runs along π are those that only delay in ℓ_k, and can thus be mimicked in π', hence proving that $f_{\pi'}(w) \geq f_\pi(w)$. ☐

In that case, the function f_π can be very easily computed:

$$f_\pi(w) = (w + p'_0) \cdot \exp(r_1) + p'_1$$

Hence the function f_π is linear with rate $\exp(r_1)$, and there is a feasible run along π if and only if $w \geq -p'_1 \cdot \exp(-r_1) - p'_0$.

We now assume that $\max\{r_i \mid 1 \leq i \leq n\} > 0$..
Assuming π contains at least one location with positive rate, we will compute a normal form for π in two steps:
- we first make the sequence of rates be increasing and positive, and drop transitions with zero cost;
- we then focus on the more technical condition $\frac{p_{i-1} r_{i-1} r_i}{r_{i-1} - r_i} < \frac{p_i r_i r_{i+1}}{r_i - r_{i+1}}$.

The first step corresponds to the following result:

PROPOSITION 33. *If π is such that $\max\{r_i \mid 1 \leq i \leq n\} > 0$ and discrete costs are nonpositive, we can construct in polynomial time a unit path π' such that all rates along π' are positive and increasing and discrete costs are negative (except possibly the costs on the first and last transitions), and $f_{\pi'} = f_\pi$.*

This result will be a consequence of the coming lemmas and corollaries. Those intermediary results basically show how to "transfer" delays to more profitable locations, achieving better final accumulated cost. We say that a run is *at least as good* as another one if the final accumulated cost along this run is at least as high.

LEMMA 34. *Assume that* $\max\{r_i \mid 1 \leq i \leq n\} > 0$*, and let* $\rho = (w, t_1, \ldots, t_n)_\pi$ *be a feasible run along* π*. Assume that* $1 \leq j, k \leq n$ *are such that* $r_j \leq 0$ *and* $t_j > 0$*, and* $r_k > 0$*. Then the run* $\rho' = (w, t'_1, \ldots, t'_n)_\pi$*, with* $t'_j = 0$*,* $t'_k = t_k + t_j$ *and* $t'_i = t_i$ *if* $i \neq j, k$*, is at least as good as* ρ*.*

PROOF. The cost at the end of ρ is

$$(\ldots(((w + p_0) \cdot \exp(r_1 t_1) + p_1) \cdot \exp(r_2 t_2) + p_2) \ldots) \cdot \exp(r_n t_n) + p_n$$

The expression for the cost at the end of the run ρ' is the same except that each t_i is replaced with the corresponding t'_i. Then, for all $i \neq k$, we have $r_i t_i \leq r_i t'_i$, and $r_k t_k < r_k t'_k$. It easily follows that ρ' is at least as good as ρ. □

COROLLARY 35. *If* π *is such that* $\max\{r_i \mid 1 \leq i \leq n\} > 0$*, then we can construct a unit path* π' *such that* $f_\pi = f_{\pi'}$ *and all rates along* π' *are positive.*

PROOF. Let ℓ_i be a location with $r_i \leq 0$ along π, and π' be the run obtained from π by replacing $\ell_{i-1} \xrightarrow{p_{i-1}} \ell_i \xrightarrow{p_i} \ell_{i+1}$ with $\ell_{i-1} \xrightarrow{p_{i-1}+p_i} \ell_{i+1}$. Clearly, $f_{\pi'}(w) \leq f_\pi(w)$ for any w, since any run along π' can be mimicked along π. Conversely, for any run ρ along π, Lemma 34 entails the existence of a run ρ' which is at least as good asρ and spends no time in ℓ_i. Thus ρ' is a run along π', and $f_{\pi'}(w) \geq f_\pi(w)$. Repeating this procedure as long as the path contains nonpositive rates provides us with the desired path. □

LEMMA 36. *Let* π *be a path such that all rates are positive and all discrete costs are nonpositive, and* $\rho = (w, t_1, \ldots, t_n)_\pi$ *be a feasible run along* π*. Assume that* $1 \leq j < k \leq n$ *are such that* $r_j \geq r_k$*,* $t_{j+l} = 0$ *for all* $1 \leq l \leq k - j - 1$*, and* $t_k > 0$*. Then the run* $\rho' = (w, t'_1, \ldots, t'_n)_\pi$*, with* $t'_j = t_j + t_k$*,* $t'_k = 0$ *and* $t'_i = t_i$ *if* $i \notin \{j, k\}$*, is at least as good as* ρ*.*

PROOF. The accumulated cost along ρ when leaving location ℓ_k is:

$$w'_k = (w_j \cdot \exp(r_j t_j) + p_j + \cdots + p_{k-1}) \cdot \exp(r_k t_k)$$

where w_j is the accumulated cost when entering ℓ_j (and is thus nonnegative). For $0 \leq \delta \leq t_k$, let $\rho^\delta = (w, \bar{t}_1, \ldots, \bar{t}_n)_\pi$ with $\bar{t}_j = t_j + \delta$, $\bar{t}_k = t_k - \delta$, and $\bar{t}_i = t_i$ when $i \notin \{j, k\}$. The accumulated cost along ρ^δ when exiting ℓ_k is

$$w'^\delta_k = (w^\delta_j \cdot \exp(r_j(t_j + \delta)) + p_j + \cdots + p_{k-1}) \cdot \exp(r_k(t_k - \delta)),$$

where w'^δ_j is the accumulated cost along ρ^δ when entering ℓ_j. Since the prefixes of ρ and ρ^δ up to ℓ_j are the same, it must be the case that $w_j = w^\delta_j$. It follows that $w'^0_k = w'_k$. We then compute the derivative of w'^δ_k against δ:

$$\frac{\partial w'^\delta_k}{\partial \delta} = r_j \cdot w_j \cdot \exp(r_j(t_j + \delta)) \cdot \exp(r_k(t_k - \delta)) - r_k \cdot (w_j \cdot \exp(r_j(t_j + \delta)) + p_j + \cdots + p_{k-1}) \cdot \exp(r_k(t_k - \delta))$$

The sign of this expression is the same as the sign of

$$(r_j - r_k) \cdot w_j \cdot \exp(r_j(t_j + \delta)) - r_k \cdot (p_j + \cdots + p_{k-1})$$

which is positive. Hence, $w^\delta_k \geq w_k$ for every $0 \leq \delta \leq t_k$. Lemma 31 concludes. □

COROLLARY 37. *If the rates along* π *are positive (except* $r_0 = r_{n+1} = 0$*) and discrete costs are nonpositive, then we can construct a unit path* π' *such that* $f_\pi = f_{\pi'}$ *and rates along* π' *(except* r_0 *and* r_{n+1}*) are positive and increasing.*

PROOF. The proof follows the same techniques as the proof of Corollary 35, namely removing offending locations. Lemma 36 ensures that the cost function f_π is preserved by this transformation. □

Finally we show we can assume that all discrete costs are negative (except possibly the first and last ones).

LEMMA 38. *Let* π *be a path such that* $0 < r_i < r_{i+1}$ *for all* $1 \leq i < n$*, and all discrete costs are nonpositive. Let* $\rho = (w, t_1, \ldots, t_n)_\pi$ *be a feasible run along* π*. Assume that* $p_j = 0$ *for some* $1 \leq j \leq n - 1$*. Then the run* $\bar{\rho} = (w, \bar{t}_1, \ldots, \bar{t}_n)_\pi$*, with* $\bar{t}_j = 0$*,* $\bar{t}_{j+1} = t_{j+1} + t_j$*, and* $\bar{t}_i = t_i$ *if* $i \notin j, j+1$*, is at least as good as* ρ*.*

PROOF. The accumulated cost along ρ when leaving ℓ_{j+1} is

$$w'_{j+1} = w_j \cdot \exp(r_j t_j) \cdot \exp(r_{j+1} t_{j+1})$$

whereas along $\bar{\rho}$, it is is

$$\bar{w}'_{j+1} = \bar{w}_j \cdot \exp(r_{j+1}(t_j + t_{j+1})),$$

where w_j and $\bar{w}_j$ are the accumulated cost when entering ℓ_j along ρ and $\bar{\rho}$, respectively. Since both runs are identical up to ℓ_j, it holds $w_j = \bar{w}_j$. By assumption $r_{j+1} > r_j$. Hence, $w'_{j+1} \geq w_{j+1}$. Lemma 31 concludes. □

Corollary 39. *If π is such that $0 < r_i < r_{i+1}$ for all $1 \leq i < n$, and all discrete costs are nonpositive, then we can construct a unit path π' such that rates along π' are positive and increasing, discrete costs along π' are negative (except possibly the discrete costs on the first and last transitions), and $f_\pi = f_{\pi'}$.*

Proof. This is again achieved by recursively dropping transitions with discrete cost zero, until no such transition exists. $\square$

The above results entail Proposition 33, which concludes the first step of the transformation of a path into normal form.

We now tackle the second part, assuming that the path π under study has positive increasing rates and negative discrete costs. We will prove the following result:

Proposition 40. *If π is such that $0 < r_i < r_{i+1}$ and $p_i < 0$ for all $1 \leq i \leq n-1$, we can construct in polynomial time a unit path π' in positive normal form such that $f_{\pi'} = f_\pi$.*

Again, this proposition is proven by transfering delays to more profitable locations. We assume the initial credit be w, and let $w_0^{\mathrm{opt}} = w$, and $w_i^{\mathrm{opt}} = \frac{p_i \cdot r_{i+1}}{r_i - r_{i+1}}$ for every $1 \leq i < n$. Notice that for all $1 \leq i < n$, w_i^{opt} is well-defined and positive. Each w_i^{opt} will actually be the optimal accumulated cost for leaving location ℓ_i, as we will prove later.

Lemma 41. *Let $\rho = (w, t_1, \ldots, t_n)_\pi$ be a feasible run along π. Pick $1 \leq j < n$, and write w'_j for the accumulated cost when leaving location ℓ_j along ρ. Then for every $0 < \delta \leq t_j$ with $w'_j \cdot \exp(-r_j\delta) \geq w_j^{opt}$, the run $\rho^\delta = (w, t_1, \ldots, t_{j-1}, t_j - \delta,$ $t_{j+1} + \delta, t_{j+1}, \ldots, t_n)_\pi$ is (strictly) better than ρ.*

Proof. The accumulated cost along ρ when leaving ℓ_{j+1} is:

$$w'_{j+1} = (w_j \cdot \exp(r_j t_j) + p_j) \cdot \exp(r_{j+1} t_{j+1})$$

with w_j it the accumulated cost when entering ℓ_j, and is positive (because $\alpha \cdot \exp(r_j t_j) + p_j \geq 0$, and $p_j < 0$ since $1 \leq j \leq n-1$). Along ρ_δ, it is:

$$w'^{\delta}_{j+1} = (w_j^\delta \cdot \exp(r_j(t_j - \delta)) + p_j) \cdot \exp(r_{j+1}(t_{j+1} + \delta))$$

with $w_j^\delta = w_j$ since ρ and ρ^δ coincide up to ℓ_j. Notice that $w'_j = w_j \cdot \exp(r_j t_j)$, so that $w_j \cdot \exp(r_j(t_j - \delta)) \geq w_j^{\mathrm{opt}}$.

Now, we have that ${w'_{j+1}}^0 = w'_{j+1}$, and the derivative of w'^{δ}_{j+1} against δ is:

$$\frac{\partial w'^{\delta}_{j+1}}{\partial \delta} = -r_j \cdot w_j \cdot \exp(r_j(t_j - \delta)) \cdot \exp(r_{j+1}(t_{j+1} + \delta)) + \ r_{j+1} \cdot (w_j \cdot \exp(r_j(t_j - \delta)) + p_j) \cdot \exp(r_{j+1}(t_{j+1} + \delta))$$

Its sign is the sign of $(r_{j+1} - r_j) \cdot w_j \cdot \exp(r_j(t_j - \delta)) + r_{j+1} p_j$, and

$$(r_{j+1} - r_j) \cdot w_j \cdot \exp(r_j(t_j - \delta)) + r_{j+1} p_j \geq (r_{j+1} - r_j) \cdot w_j^{\mathrm{opt}} + r_{j+1} p_j = 0$$

and the inequality is strict as soon as $w_j \cdot \exp(r_j(t_j - \delta)) \neq w_j^{\mathrm{opt}}$. Thus, for every δ satisfying the hypotheses, $w'^{\delta}_{j+1} > w'_{j+1}$. Lemma 31 concludes. $\square$

A run $\rho = (w, t_1, \ldots, t_n)_\pi$ is said *right-normalized in location ℓ_i* if either the accumulated cost w'_i when leaving ℓ_i is at most w_i^{opt}, or $t_i = 0$. It is *right-normalized* if it is right-normalized in all locations between ℓ_1 and ℓ_{n-1}.

Corollary 42. *Let ρ be a feasible run along π. Then we can construct a feasible run ρ' along π which is right-normalized and at least as good as ρ. Furthermore if ρ is not right-normalized, then ρ' is strictly better than ρ.*

Proof. Lemma 41 provides a way of right-normalizing a run in location ℓ_j. It suffices to apply it iteratively in the earliest location where ρ is not right-normalized in order to end up with a right-normalized run. $\square$

We now do the converse, and give conditions under which it is better to delay *earlier* in π.

Lemma 43. *Let $\rho = (w, t_1, \ldots, t_n)_\pi$ be a feasible run along π. Pick $1 \leq j < n$, and write w'_j for the accumulated cost when leaving location ℓ_j along ρ. Let k be the smallest index such that $k > j$ and $t_k > 0$, if any. Assume furthermore that $r_{l-1} \cdot w_{l-1}^{opt} \leq r_l \cdot w_l^{opt}$ for all $j+1 \leq l \leq k-1$. Then for any $0 < \delta \leq t_k$ such that $w'_j \cdot \exp(r_j\delta) \leq w_j^{opt}$, the run $\rho^\delta = (w, t_1, \ldots, t_{j-1}, t_j + \delta, 0, \ldots, 0, t_k - \delta, t_{k+1}, \ldots, t_n)_\pi$ is (strictly) better than ρ.*

Proof. The accumulated cost along ρ when leaving location ℓ_k is:

$$w'_k = (w_j \cdot \exp(r_j t_j) + p_j + \cdots + p_{k-1}) \cdot \exp(r_k t_k)$$

where w_j is the accumulated cost when entering ℓ_j, whereas that along ρ_δ is:

$$w'^{\delta}_k = (w_j^\delta \cdot \exp(r_j(t_j + \delta)) + p_j + \cdots + p_{k-1}) \cdot \exp(r_k(t_k - \delta))$$

where w_j^δ is the accumulated cost when entering ℓ_j along ρ^δ. Again, we have $w_j = w_j^\delta$ as both run coincide before ℓ_j. Hence $w_k^0 = w_k$. Also, $w'_j = w_j \cdot \exp(r_j(t_j + \delta)) \leq w_j^{\mathrm{opt}}$ from the hypotheses. We compute the derivative of w'^{δ}_k against δ:

$$\frac{\partial w'^{\delta}_k}{\partial \delta} = r_j \cdot w_j \cdot \exp(r_j(t_j + \delta)) \cdot \exp(r_k(t_k - \delta)) - r_k \cdot (w_j \cdot \exp(r_j(t_j + \delta)) + p_j + \cdots + p_{k-1}) \cdot \exp(r_k(t_k - \delta))$$

Its sign is the sign of $(r_j - r_k)\cdot\alpha\cdot\exp(r_j(t_j+\delta)) - r_k\cdot(p_j+\cdots+p_{k-1})$, and

$$(r_j - r_k)\cdot\alpha\cdot\exp(r_j(t_j+\delta)) - r_k\cdot(p_j+\cdots+p_{k-1}) \geq (r_j-r_k)\cdot w_j^{\mathrm{opt}} - r_k\cdot(p_j+\cdots+p_{k-1})$$

We now prove by descending induction on h, from $k-1$ downto j, that

$$(r_h - r_k)\cdot w_h^{\mathrm{opt}} - r_k\cdot(p_h+\cdots+p_{k-1}) \geq 0.$$

If $h = k-1$, this is obvious. We assume this holds for some $j < h < k$, and we prove this also holds for $h-1$. By induction hypothesis we have that

$$(r_h - r_k)\cdot w_h^{\mathrm{opt}} - r_k\cdot(p_h+\cdots+p_{k-1}) \geq 0$$

Since $r_h\cdot w_h^{\mathrm{opt}} \geq r_{h-1}\cdot w_{h-1}^{\mathrm{opt}}$, we get that:

$$(r_h - r_k)\cdot\frac{r_{h-1}}{r_h}\cdot w_{h-1}^{\mathrm{opt}} + r_k\cdot p_{h-1} - r_k\cdot(p_{h-1}+p_h+\cdots+p_{k-1}) \geq 0$$

It remains to compute:

$$\begin{aligned}
(r_h - r_k)\frac{r_{h-1}}{r_h}\cdot w_{h-1}^{\mathrm{opt}} + r_k\cdot p_{h-1} &= (r_h-r_k)\cdot\frac{r_{h-1}}{r_h}\cdot\frac{p_{h-1}r_h}{r_{h-1}-r_h} + r_k\cdot p_{h-1}\\
&= \frac{p_{h-1}\cdot(r_hr_{h-1}(r_h-r_k) + r_kr_h(r_{h-1}-r_h))}{r_h\cdot(r_{h-1}-r_h)}\\
&= \frac{p_{h-1}}{r_h\cdot(r_{h-1}-r_h)}\cdot\left(r_{h-1}r_h^2 - r_kr_h^2\right)\\
&= \frac{p_{h-1}r_h}{r_{h-1}-r_h}\cdot(r_{h-1}-r_k)\\
&= (r_{h-1}-r_k)\cdot w_{h-1}^{\mathrm{opt}}
\end{aligned}$$

Hence we get that

$$(r_{h-1} - r_k)\cdot w_{h-1}^{\mathrm{opt}} - r_k\cdot(p_{h-1}+p_h+\cdots+p_{k-1}) \geq 0$$

This concludes the induction step, and we get that the derivative is nonnegative, which implies that for every $0 < \delta \leq t_k$ such that $w_j\cdot\exp(r_j\delta) \leq w_j^{\mathrm{opt}}$, it holds $w_{j+1}^{\delta} \geq w_{j+1}$. Furthermore the derivative is equal to 0 only when $w_j\cdot\exp(r_j(t_j+\delta)) = w_j^{\mathrm{opt}}$, which implies that the run ρ_δ is strictly better than ρ. Again, Lemma 31 concludes the proof. ☐

We are now in a position to prove our main result, that a unit path can be turned into normal form.

Proof of Proposition 13. Thanks to Proposition 33, we can assume that π satisfies $0 < r_i < r_{i+1}$ and $p_i < 0$ for all $1 \leq i \leq n-1$. It remains to enforce the technical condition

$$\frac{p_{i-1}r_{i-1}r_i}{r_{i-1}-r_i} < \frac{p_ir_ir_{i+1}}{r_i-r_{i+1}} \tag{4}$$

for all $1 \leq i \leq n-2$, assuming $n \geq 3$. Let $w \in \mathsf{R}_+$ such that there exists a feasible run along π with initial credit w. In addition to the values $(w_i^{\mathrm{opt}})_{0\leq i\leq n-1}$ we have already defined, we define, for every $2 \leq i \leq n-1$:

$$t_i^{\mathrm{opt}} = \frac{1}{r_i}\cdot\ln\left(\frac{w_i^{\mathrm{opt}}}{w_{i-1}^{\mathrm{opt}}+p_{i-1}}\right) = \frac{1}{r_i}\cdot\ln\left(\frac{r_i\cdot w_i^{\mathrm{opt}}}{r_{i-1}\cdot w_{i-1}^{\mathrm{opt}}}\right)$$

which is well-defined because $w_j^{\mathrm{opt}} > 0$ for every $1 \leq j \leq n-1$. We also define

$$t_1^{\mathrm{opt}} = \frac{1}{r_1}\cdot\ln\left(\frac{w_1^{\mathrm{opt}}}{w_0^{\mathrm{opt}}+p_0}\right) = \frac{1}{r_1}\cdot\ln\left(\frac{w_1^{\mathrm{opt}}}{w+p_0}\right)$$

which is also well-defined, because there is a feasible run along π with initial credit w, so that it must be the case that $w + p_0 > 0$. As already mentionned, w_i^{opt} is the optimal accumulated cost with which to exit location ℓ_i, and t_i^{opt} is actually the delay to spend in ℓ_i to go from $w_{i-1}^{\mathrm{opt}} + p_{i-1}$ (the accumulated cost when entering ℓ_i) to w_i^{opt}. However, it might be the case that t_i^{opt}, as computed above, is larger than 1 or, worse, nonpositive. The latter happens precisely when the second condition for being in normal form is not satisfied.

Assume π does not satisfy condition (4). We construct a path π', shorter than π and still having positive increasing rates and negative discrete costs, and such that $f_\pi = f_{\pi'}$. Applying this procedure recursively, we get a path in positive (or trivial) normal form. Since the length of the path decreases at each step, this procedure terminates in a finite number of iterations. At the end, either the length of the path is 2 (so that condition (4) holds vacuously), or no locations violate condition (4) anymore.

Let $2 \leq k \leq n-1$ be the smallest index for which $t_k^{\mathrm{opt}} \leq 0$, and $\rho = (w, t_1, \ldots, t_n)_\pi$ be a feasible run along π. We assume w.l.o.g. that ρ is right-normalized (see Corollary 42). We construct a right-normalized feasible run ρ' along π which is at least as good as ρ, and which spends no time in location ℓ_k. This run will thus be a run along the path π' obtained from π by dropping location ℓ_k, which entails that $f_{\pi'} \geq f_\pi$. The other inequality is obvious, so that the equality holds.

The intuition for defining ρ' is as follows: we transfer the delay t_k of ℓ_k to other locations, so that ρ' is at least as good as ρ. As we proved above, this delay will be better spent in locations where the optimal cost is not reached yet, starting with the leftmost locations.

We thus define the *distorsion* of ρ as $k-h$, where $h = \max\{0 \leq j \leq k \mid \forall l \leq j.\ w'_l \geq w_l^{\text{opt}}\}$, where w'_l is the accumulated cost along ρ when leaving ℓ_l. This measure is well-defined as $w'_0 = w = w_0^{\text{opt}}$. We prove that either $t_k = 0$, or we can construct a right-normalized feasible run ρ' which is at least as good as ρ, and satisfies one of the following two conditions:

(a) it spends no time in ℓ_k;

(b) its distorsion is (strictly) smaller than that of ρ.

This will be achieved by transfering some of the delay spent in location ℓ_k along ρ to location ℓ_{h+1}.

We proceed by a distinction on the value of the distorsion of ρ:

- Assume the distorsion of ρ is 0. Then either $w'_k = w_k^{\text{opt}}$, or $t_k = 0$ (because ρ is right-normalized). In the former case:

$$\begin{aligned} w_k^{\text{opt}} = w'_k &= (w'_{k-1} + p_{k-1}) \cdot \exp(r_k t_k) \\ &\geq (w_{k-1}^{\text{opt}} + p_{k-1}) \cdot \exp(r_k t_k) && \text{because } w'_{k-1} \geq w_{k-1}^{\text{opt}} \\ &= \frac{r_{k-1}}{r_k} \cdot w_{k-1}^{\text{opt}} \cdot \exp(r_k t_k) \\ &\geq w_k^{\text{opt}} \cdot \exp(r_k t_k) && \text{because } t_k^{\text{opt}} \leq 0 \end{aligned}$$

  This is possible only if $t_k = 0$. In that case, we are done.

- Assume that the distorsion of ρ is 1. Then, $w'_{k-1} = w_{k-1}^{\text{opt}}$, and we can write:

$$\begin{aligned} w'_k &= (w'_{k-1} + p_{k-1}) \cdot \exp(r_k t_k) \\ &= (w_{k-1}^{\text{opt}} + p_{k-1}) \cdot \exp(r_k t_k) \\ &= w_{k-1}^{\text{opt}} \cdot \frac{r_{k-1}}{r_k} \cdot \exp(r_k t_k) \\ &\geq w_k^{\text{opt}} \cdot \exp(r_k t_k) && \text{because } t_k^{\text{opt}} \leq 0 \end{aligned}$$

  This is possible only if $t_k = 0$ (because either $w'_k \leq w_k^{\text{opt}}$ or $t_k = 0$).

- Assume now that the distorsion of ρ is (strictly) larger than 1, and that $t_k > 0$. It means that $h+1 < k$, and implies that $w'_k \leq w_k^{\text{opt}}$. By definition of k, we also have that

$$r_{h+1} \cdot w_{h+1}^{\text{opt}} < \cdots < r_{k-1} \cdot w_{k-1}^{\text{opt}}$$

  because for every $1 < j < k$, it holds $t_j^{\text{opt}} > 0$. Since $w'_{h+1} > 0$ (because ρ is feasible), we can apply Lemma 43: we transfer an amount of $\delta = \min\left(t_k, \frac{1}{r_{h+1}} \cdot \ln\left(\frac{w_{h+1}^{\text{opt}}}{w'_{h+1}}\right)\right)$ time units, which is positive, from location ℓ_k to location ℓ_{h+1}. This yields a feasible run ρ' which is (strictly) better than ρ. Furthermore ρ' is right-normalized, and if $\delta = t_k$, then the run ρ' spends no time in location ℓ_k. Otherwise, the value of the cost when leaving location ℓ_{h+1} along ρ' is precisely w_{h+1}^{opt}, and the distorsion of ρ' is strictly less than that of ρ.

By applying this reasoning several times we eventually construct a feasible run ρ' that is at least as good as ρ, and delays zero time unit in ℓ_k. □

B.3 Proof of Proposition 14

PROPOSITION 14. *Let π be a path in positive normal form (for exponential observers) and w an initial observer value. Then we can decide whether there is a feasible run along π with initial observer value w, and we can compute the value $f_\pi(w)$.*

PROOF. Let w be an initial credit. We claim that the following "algorithm" computes the optimal feasible run along π (if any): set

$$i = \min\left(\{1 \leq j \leq n-1 \mid w + p_0 + \cdots + p_{j-1} < w_j^{\text{opt}}\} \cup \{n\}\right)$$ [1]

Then set $t_j^* = 0$ for every $1 \leq j \leq i$.

1. If $i = n$, then we set $t_n^* = 1$, and we write

$$w^* = (w + p_0 + \cdots + p_{n-1}) \cdot \exp(r_n) + p_n$$

2. If $1 \leq i < n$, then we define

$$t_i^* = \min\left(1, \frac{1}{r_i} \cdot \ln\left(\frac{w_i^{\text{opt}}}{w + p_0 + \cdots + p_{i-1}}\right)\right)$$

[1] Note that due to the normal formal assumption, if $w + p_0 + \cdots + p_{j-1} < w_j^{\text{opt}}$ for some $1 \leq j \leq n-1$, then for every $j \leq k \leq n-1$, $w + p_0 + \cdots + p_{k-1} < w_k^{\text{opt}}$.

We also define

$$h = \min\left(\{i \le j \le n-1 \mid t_i^* + t_{i+1}^{\text{opt}} + \cdots + t_j^{\text{opt}} \ge 1\} \cup \{n\}\right)$$

and we then set:

$$\begin{cases} t_j^* = t_j^{\text{opt}} & \text{if } i < j < h \\ t_h^* = 1 - \sum_{j=i}^{h-1} t_j^* & \\ t_j^* = 0 & \text{if } h < j \le n \end{cases}$$

We write $w^* = (w_{h-1}^{\text{opt}} + p_{h-1}) \cdot \exp(r_h \cdot t_h^*) + p_h + \cdots + p_n$.

We claim that a feasible run exists if and only if $w^* \ge 0$, and that $f_\pi(w) = w^*$.

First note that the run $\rho^* = (w, t_1^*, \ldots, t_n^*)_\pi$ is a feasible run along π. In particular, $f_\pi(w) \ge w^*$.

To prove the converse we assume towards a contradiction that $f_\pi(w) < w^*$ and that there is a feasible run $\rho = (w, t_1, t_2, \ldots, t_n)_\pi$ along π such that its outgoing cost is $f_\pi(w)$. W.l.o.g. (see Corollary 42) we assume ρ is right-normalized: writing w_j for the cost when leaving location ℓ_j ($1 \le j \le n-1$), either $w_j \le w_j^{\text{opt}}$ or $t_j = 0$. In particular, for every $1 \le j < i$, $t_j = t_j^* = 0$. If $i = n$, we are done, and $t_n = 1 - \sum_{j=1}^{n-1} t_j = 1 = t_n^*$.

Assume that $i < n$, and take $k \ge i$ the smallest index such that $t_k \ne t_k^*$ (if this index does not exist, we have that $\rho = \rho^*$, which is not possible). We have that $k < n$ because $t_n = 1 - \sum_{j=1}^{n-1} t_j$ and $t_n^* = 1 - \sum_{j=1}^{n-1} t_j^*$. We distinguish between several cases:

- $t_i^* = 1$ and $k = i$, *i.e.* $t_i < 1$. In that case we have that the cost w_i when leaving location ℓ_i in ρ is equal to

$$(w + p_0 + \cdots + p_{i-1}) \cdot \exp(r_i t_i) < (w + p_0 + \cdots + p_{i-1}) \cdot \exp(r_i t_i^*) \le w_i^{\text{opt}}$$

  by definition of t_i^*. There must exist some location ℓ_l later (l minimal and $l > i$) so that $t_l > 0$, and we can apply Lemma 43 and transfer t_l time units from location ℓ_l to location ℓ_i (because $(w + p_0 + \cdots + p_{i-1}) \cdot \exp(r_i(t_i + t_l)) \le (w + p_0 + \cdots + p_{i-1}) \cdot \exp(r_i t_i^*) \le w_i^{\text{opt}}$) and get a better run. This contradicts the optimality of ρ.
- $t_i^* < 1$, in which case index h is properly defined:
  - either $k \le h$. This is similar to the previous case, we can transfer some delay from a location to the right to location ℓ_k and get a better run.
  - or $k > h$. This case is not possible because $\sum_{j=1}^{h} t_j = \sum_{j=1}^{h} t_j^* = 1$ (no delay remains for the rest of the run). □

B.4 Description of the function $w \mapsto f_\pi(w)$ when π is in positive normal form.

We have previously described an algorithm, that given w, computes $f_\pi(w)$ whenever there exists a feasible run along π with initial credit w.

The domain of f_π is the set

$$\mathfrak{W} = \{w \mid \text{there is a feasible run } \rho \text{ along } \pi \text{ with initial credit } w\}$$

We write $\mathfrak{w}$ for the lower bound of the set $\mathfrak{W}$: we have that $\mathfrak{W} = [\mathfrak{w}, +\infty)$.

For every $w \in \mathfrak{W}$ we define i_w the index i in the algorithm, and if defined, h_w the index h in the algorithm. We also write $t_j^*(w)$ for the various optimal time-points associated with initial credit w (as in the algorithm). We write $W_i = w_i^{\text{opt}} - (p_0 + \cdots + p_{i-1})$ for $1 \le i \le n-1$, $W_0 = 0$ and $W_n = +\infty$. We have that $0 \le \mathfrak{w} \le w \le w'$ implies $i_w \le i_{w'}$, and that for every $0 \le w \in \mathfrak{W}$, for every $1 \le i \le n$:[2]

$$i_w = i \iff W_{i-1} \le w < W_i$$

Furthermore for every $1 \le i < n$ such that $W_{i-1} \ge \mathfrak{w}$, $t_i^*(W_{i-1}) = \min(1, t_i^{\text{opt}})$.

We now analyze the form of the function f_π, depending on the position in the definition domain.

LEMMA 44. *The function f_π is a straight line:*

$$f_\pi(w) \quad = \quad (w + p_0 + \cdots + p_{n-1}) \cdot \exp(r_n) + p_n$$

on the interval $\{w \ge \mathfrak{w} \mid i_w = n\} = [W_{n-1}, +\infty) \cap \mathfrak{W}$.

PROOF. We assume that $i_w = n$. We have that $t_1^*(w) = 0, \ldots, t_{n-1}^*(w) = 0$, and $t_n^*(w) = 1$. We can thus compute:

$$f_\pi(w) \quad = \quad (w + p_0 + \cdots + p_{n-1}) \cdot \exp(r_n) + p_n$$

which is the expected result. □

Assume $0 \le i \le n-1$ and $i \le j \le n-1$. Define the constant W_i^j as follows:

$$W_i^j = w_i^{\text{opt}} \cdot \exp(-r_i(1 - \sum_{l=i+1}^{j} t_l^{\text{opt}})) - (p_0 + \cdots + p_{i-1})$$

[2] This is due to footnote 1.

Define for $1 \leq i \leq n-1$ the function

$$\begin{array}{rcl} H_i : [W_{i-1}, W_i) \cap \mathfrak{W} & \to & \{i, \dots, n\} \\ w & \mapsto & h_w \end{array}$$

and notice that it is nondecreasing (where defined).

LEMMA 45. *For every $1 \leq i \leq n-1$, for every $i \leq j \leq n-1$, and for every comparison operator $\sim$:*

$$\begin{cases} W_i \sim W_i^j & \textit{iff} \quad 1 \sim \displaystyle\sum_{l=i+1}^{j} t_l^{opt} \\ W_i^j \sim W_{i-1} & \textit{iff} \quad \displaystyle\sum_{l=i}^{j} t_l^{opt} \sim 1 \end{cases}$$

In particular, when defined:

- $W_i^{(\min H_i)-1} < W_{i-1} \leq W_i^{(\min H_i)}$
- $W_i^{(\max H_i)-1} < W_i \leq W_i^{(\max H_i)}$

Finally for every $0 \leq i \leq n-1$, for every $i \leq j \leq n-1$ such that $W_i^j \in [W_{i-1}, W_i] \cap \mathfrak{W}$,

$$W_i^j = \sup\{w \in [W_{i-1}, W_i) \cap \mathfrak{W} \mid H_i(w) = j\}$$

PROOF. The first result is simple arithmetic, taking into account that

$$\exp(r_i t_i^{\text{opt}}) = \frac{w_i^{\text{opt}}}{w_{i-1}^{\text{opt}} + p_{i-1}}$$

Then by monotonicity of H_i, $H_i(W_{i-1}) = \min H_i$. Thus, if $\min H_i \geq i+1$:

$$\begin{cases} t_i^*(W_{i-1}) + t_{i+1}^{\text{opt}} + \cdots + t_{(\min H_i)}^{\text{opt}} \geq 1 \\ t_i^*(W_{i-1}) + t_{i+1}^{\text{opt}} + \cdots + t_{(\min H_i)-1}^{\text{opt}} < 1 \end{cases}$$

In that case, $t_i^*(W_{i-1}) = \min(1, t_i^{\text{opt}}) = t_i^{\text{opt}}$, which implies:

$$\begin{cases} t_i^{\text{opt}} + t_{i+1}^{\text{opt}} + \cdots + t_{(\min H_i)}^{\text{opt}} \geq 1 \\ t_i^{\text{opt}} + t_{i+1}^{\text{opt}} + \cdots + t_{(\min H_i)-1}^{\text{opt}} < 1 \end{cases}$$

Hence the mentioned inequalities. If $\min H_i = i$, $t_i^*(W_{i-1}) \geq 1$, which implies $t_i^{\text{opt}} \geq 1$, and thus the expected result.

Similarly, $\max H_i = \lim_{w \to W_i^-} H_i(w)$, and for $w < W_i$ but close enough to W_i, it is the case that $H_i(w) = \max H_i$. In particular:

$$\begin{cases} t_i^*(w) + t_{i+1}^{\text{opt}} + \cdots + t_{(\max H_i)}^{\text{opt}} \geq 1 \\ t_i^*(w) + t_{i+1}^{\text{opt}} + \cdots + t_{(\max H_i)-1}^{\text{opt}} < 1 \end{cases}$$

If $\max H_i > i$, when w tends to W_i, we get the first equality, and for some value w such that $t_i^*(w) > 0$, we get the second equality:

$$\begin{cases} t_{i+1}^{\text{opt}} + \cdots + t_{(\max H_i)}^{\text{opt}} \geq 1 \\ t_{i+1}^{\text{opt}} + \cdots + t_{(\max H_i)-1}^{\text{opt}} \leq 1 \end{cases}$$

If $\max H_i = i$, then $\max H_i = \min H_i$ and we can apply the previous result.

Let γ_i^j be the minimum of the set. It is the unique solution to the equation:

$$t_i^*(\gamma_i^j) + t_{i+1}^{\text{opt}} + \cdots + t_j^{\text{opt}} = 1 \tag{5}$$

where $t_i^*(w) = \dfrac{1}{r_i} \cdot \ln\left(\dfrac{w_i^{\text{opt}}}{w + p_0 + \cdots + p_{i-1}}\right)$.

This is precisely the equation satisfied by W_i^j, hence the result. □

We will use the points W_i^j to refine the interval $[W_{i-1}, W_i)$.

REMARK 3. *If $\min H_i < \max H_i$, we have:*

$$W_{i-1} \leq W_i^{(\min H_i)} < \dots W_i^{(\max H_i)-1} < W_i \leq W_i^{(\max H_i)}$$

If $\min H_i = \max H_i$ (in which case $\min H_i > i$), we have:

$$W_{i-1} < W_i \leq W_i^{(\min H_i)}$$

Remark 4. *In the following we will have to manipulate these values. We thus give another formula that will be useful to manipulate W_i^j, and which is a direct consequence of equation (5):*

$$\left(\frac{w_i^{opt}}{W_i^j + p_0 + \dots p_{i-1}}\right)^{\frac{1}{r_i}} \cdot \left(\frac{w_{i+1}^{opt}}{w_i^{opt} + p_i}\right)^{\frac{1}{r_{i+1}}} \cdots \left(\frac{w_j^{opt}}{w_{j-1}^{opt} + p_{j-1}}\right)^{\frac{1}{r_j}} = e \tag{6}$$

Lemma 46. *We assume that $1 \leq i \leq n-1$ and $i \leq j < n$ with $\min H_i \leq j \leq \max H_i$. Then f_π is of the form*

$$f_\pi(w) = w_j^{opt} \cdot \left(\frac{w + p_0 + \cdots + p_{i-1}}{W_i^j + p_0 + \cdots + p_{i-1}}\right)^{\frac{r_j}{r_i}} + p_j + \cdots + p_n$$

on the interval $\{w \geq \mathfrak{w} \mid i_w = i \text{ and } H_i(w) = j\} = [W_{i-1}, W_i) \cap (W_i^{j-1}, W_i^j] \cap \mathfrak{W}$.

Proof. We first assume that $i < j$, and w is such that $i_w = i$ and $H_i(w) = j$. In that case, we know that $t_i^*(w) + t_{i+1}^{\text{opt}} + \cdots + t_j^{\text{opt}} \geq 1$ but $t_i^*(w) + t_{i+1}^{\text{opt}} + \cdots + t_{j-1}^{\text{opt}} < 1$. In particular, $t_i^*(w) < 1$, and thus:

$$t_i^*(w) = \frac{1}{r_i} \cdot \ln\left(\frac{w_i^{\text{opt}}}{w + p_0 + \cdots + p_{i-1}}\right)$$

which implies

$$\begin{aligned} t_j^*(w) &= 1 - \sum_{l=i}^{j-1} t_l^*(w) \\ &= 1 - \frac{1}{r_i} \cdot \ln\left(\frac{w_i^{\text{opt}}}{w + p_0 + \cdots + p_{i-1}}\right) - \sum_{l=i+1}^{j-1} t_l^{\text{opt}} \end{aligned}$$

We know the value of $f_\pi(w)$ in that case:

$$\begin{aligned} f_\pi(w) &= (w_{j-1}^{\text{opt}} + p_{j-1}) \cdot \exp\left(r_j \cdot t_j^*(w)\right) + p_j + \cdots + p_n \\ &= \psi_{i,j}(w) + p_j + \cdots + p_n \end{aligned}$$

We compute

$$\begin{aligned} \psi_{i,j}(w) &= (w_{j-1}^{\text{opt}} + p_{j-1}) \cdot \exp\left(r_j \cdot \left(1 - \frac{1}{r_i} \cdot \ln\left(\frac{w_i^{\text{opt}}}{w + p_0 + \cdots + p_{i-1}}\right) - \sum_{l=i+1}^{j-1} t_l^{\text{opt}}\right)\right) \\ &= (w_{j-1}^{\text{opt}} + p_{j-1}) \cdot \left(\frac{w_i^{\text{opt}}}{w + p_0 + \cdots + p_{i-1}}\right)^{-\frac{r_j}{r_i}} \cdot \exp\left(r_j \cdot \left(1 - \sum_{l=i+1}^{j-1} t_l^{\text{opt}}\right)\right) \\ &= w_j^{\text{opt}} \cdot \exp(-r_j t_j^{\text{opt}}) \cdot \left(\frac{w_i^{\text{opt}}}{w + p_0 + \cdots + p_{i-1}}\right)^{-\frac{r_j}{r_i}} \cdot \exp\left(r_j \cdot \left(1 - \sum_{l=i+1}^{j-1} t_l^{\text{opt}}\right)\right) \\ &\qquad \text{(because } t_j^{\text{opt}} \text{ is properly defined, due to } j < n) \\ &= w_j^{\text{opt}} \cdot \left(\frac{w_i^{\text{opt}}}{w + p_0 + \cdots + p_{i-1}}\right)^{-\frac{r_j}{r_i}} \cdot \exp\left(r_j \cdot \left(1 - \sum_{l=i+1}^{j} t_l^{\text{opt}}\right)\right) \end{aligned}$$

By definition of W_i^j, we have that:

$$\frac{W_i^j + p_0 + \cdots + p_{i-1}}{w_i^{\text{opt}}} = \exp\left(-r_i \cdot \left(1 - \sum_{l=i+1}^{j} t_l^{\text{opt}}\right)\right)$$

Thus:

$$\begin{aligned} \psi_{i,j}(w) &= w_j^{\text{opt}} \cdot \left(\frac{w_i^{\text{opt}}}{w + p_0 + \cdots + p_{i-1}}\right)^{-\frac{r_j}{r_i}} \cdot \left(\frac{W_i^j + p_0 + \cdots + p_{i-1}}{w_i^{\text{opt}}}\right)^{-\frac{r_j}{r_i}} \\ &= w_j^{\text{opt}} \cdot \left(\frac{w + p_0 + \cdots + p_{i-1}}{W_i^j + p_0 + \cdots + p_{i-1}}\right)^{\frac{r_j}{r_i}} \end{aligned}$$

which is the required expression.

We now assume $i = j$. This means that for such a w, $t_i^*(w) = 1$. In that case,

$$\begin{aligned} f_\pi(w) &= (w + p_0 + \cdots + p_{i-1}) \cdot \exp(r_i) + p_i + \cdots + p_n \\ &= (w + p_0 + \cdots + p_{i-1}) \cdot \frac{w_i^{\text{opt}}}{W_i^i + p_0 + \cdots + p_{i-1}} + p_i + \cdots + p_n \\ &= w_i^{\text{opt}} \cdot \left(\frac{w + p_0 + \cdots + p_{i-1}}{W_i^i + p_0 + \cdots + p_{i-1}} \right) + p_i + \cdots + p_n \end{aligned}$$

which is the expected form (as $i = j$, $\frac{r_j}{r_i} = 1$). □

It remains to handle the very last case, which we do now.

LEMMA 47. *We assume that $1 \leq i \leq n-1$ and $i < j = n$ with $\min H_i \leq j \leq \max H_i$. Then f_π is of the form*

$$f_\pi(w) = (w_{n-1}^{opt} + p_{n-1}) \cdot \left(\frac{w + p_0 + \cdots + p_{i-1}}{W_i^{n-1} + p_0 + \cdots + p_{i-1}} \right)^{\frac{r_j}{r_i}} + p_n$$

on the interval $\{w \geq \mathfrak{w} \mid i_w = i \text{ and } H_i(w) = n\} = [W_{i-1}, W_i) \cap (W_i^{n-1}, W_i^n] \cap \mathfrak{W}$.

PROOF. Assume that $i_w = i$ and $H_i(w) = j = n$. We have that $t_i^*(w) < 1$ and can be computed using the classical formula:

$$t_i^*(w) = \frac{1}{r_i} \cdot \left(\frac{w_i^{\text{opt}}}{w + p_0 + \cdots + p_{i-1}} \right)$$

We have that

$$t_n^*(w) = 1 - \frac{1}{r_i} \cdot \left(\frac{w_i^{\text{opt}}}{w + p_0 + \cdots + p_{i-1}} \right) - \sum_{l=i+1}^{n-1} t_l^{\text{opt}}$$

Thus, we can compute:

$$\begin{aligned} f_\pi(w) =& (w_{n-1}^{\text{opt}} + p_{n-1}) \cdot \exp\left(r_n \cdot \left(1 - \frac{1}{r_i} \cdot \left(\frac{w_i^{\text{opt}}}{w + p_0 + \cdots + p_{i-1}} \right) - \sum_{l=i+1}^{n-1} t_l^{\text{opt}} \right) \right) + p_n \\ =& (w_{n-1}^{\text{opt}} + p_{n-1}) \cdot \left(\frac{w + p_0 + \cdots + p_{i-1}}{w_i^{\text{opt}}} \right)^{\frac{r_n}{r_i}} \cdot \exp\left(r_n \cdot \left(1 - \sum_{l=i+1}^{n-1} t_l^{\text{opt}} \right) \right) + p_n \\ =& (w_{n-1}^{\text{opt}} + p_{n-1}) \cdot \left(\frac{w + p_0 + \cdots + p_{i-1}}{w_i^{\text{opt}}} \right)^{\frac{r_n}{r_i}} \cdot \exp(r_n \cdot \alpha) \cdot \exp\left(r_n \cdot \left(1 - \sum_{l=i+1}^{n-1} t_l^{\text{opt}} - \alpha \right) \right) + p_n \\ & \text{(for any } \alpha) \\ =& (w_{n-1}^{\text{opt}} + p_{n-1}) \cdot \left(\frac{w + p_0 + \cdots + p_{i-1}}{w_i^{\text{opt}}} \right)^{\frac{r_n}{r_i}} \cdot \exp(r_n \cdot \alpha) \cdot \left(\frac{W_i^n + p_0 + \cdots + p_{i-1}}{w_i^{\text{opt}}} \right)^{-\frac{r_n}{r_i}} + p_n \\ & \text{(where } W_i^n = w_i^{\text{opt}} \cdot \exp(-r_i(1 - \textstyle\sum + l = i + 1^j t_l^{\text{opt}} - \alpha)) - (p_0 + \cdots + p_{i-1})) \\ =& \beta \cdot \left(\frac{w + p_0 + \cdots + p_{i-1}}{w_i^{\text{opt}}} \right)^{\frac{r_n}{r_i}} \cdot \left(\frac{W_i^n + p_0 + \cdots + p_{i-1}}{w_i^{\text{opt}}} \right)^{-\frac{r_n}{r_i}} + p_n \\ & \text{(assuming } \beta = (w_{n-1}^{\text{opt}} + p_{n-1}) \cdot \exp(r_n \alpha)) \\ =& \beta \cdot \left(\frac{w + p_0 + \cdots + p_{i-1}}{W_i^n + p_0 + \cdots + p_{i-1}} \right)^{\frac{r_n}{r_i}} + p_n \\ =& (w_{n-1}^{\text{opt}} + p_{n-1}) \cdot \left(\frac{w + p_0 + \cdots + p_{i-1}}{W_i^{n-1} + p_0 + \cdots + p_{i-1}} \right)^{\frac{r_n}{r_i}} + p_n \qquad \text{(taking } \alpha = 0) \end{aligned}$$

This is the correct expression for f_π. □

PROPOSITION 48. *The application f_π is continuous on $\mathfrak{W}$.*

PROOF. Assume it is not the case. Then it means that it is not continuous in some of the points $\{W_i \mid 1 \leq i \leq n-1\} \cup \{W_i^j \mid W_i^j \in (W_{i-1}, W_i)\}$, because elsewhere it is defined by a continuous function.

- We assume it is not continuous in some W_{i-1} with $1 < i \leq n-1$. We have that $i_{W_{i-1}} = i$, $H_i(W_{i-1}) = \min H_i = j$, and that f_π is right-continuous. Furthermore, if $j < n$:

$$\begin{aligned}
f_\pi(W_{i-1}) &= w_j^{\text{opt}} \cdot \left(\frac{W_{i-1} + p_0 + \cdots + p_{i-1}}{W_i^j + p_0 + \cdots + p_{i-1}}\right)^{\frac{r_j}{r_i}} + p_j + \cdots + p_n \\
&= w_j^{\text{opt}} \cdot \left(\frac{w_{i-1}^{\text{opt}} + p_{i-1}}{w_i^{\text{opt}} \cdot \exp\left(-r_i\left(1 - \sum_{l=i+1}^{j} t_l^{\text{opt}}\right)\right)}\right)^{\frac{r_j}{r_i}} + p_j + \cdots + p_n \\
&= w_j^{\text{opt}} \cdot \left(\frac{w_i^{\text{opt}} \cdot \exp(-r_i t_i^{\text{opt}})}{w_i^{\text{opt}} \cdot \exp\left(-r_i\left(1 - \sum_{l=i+1}^{j} t_l^{\text{opt}}\right)\right)}\right)^{\frac{r_j}{r_i}} + p_j + \cdots + p_n \\
&= w_j^{\text{opt}} \cdot \left(\exp\left(r_i\left(1 - \sum_{l=i}^{j} t_l^{\text{opt}}\right)\right)\right)^{\frac{r_j}{r_i}} + p_j + \cdots + p_n \\
&= w_j^{\text{opt}} \cdot \exp\left(r_j\left(1 - \sum_{l=i}^{j} t_l^{\text{opt}}\right)\right) + p_j + \cdots + p_n
\end{aligned}$$

We notice also that if $i = j$, then $t_i^{\text{opt}} \geq 1$, and that if $i < j$, then $t_i^{\text{opt}} + \cdots + t_j^{\text{opt}} \geq 1$ whereas $t_i^{\text{opt}} + \cdots + t_{j-1}^{\text{opt}} < 1$. For $w < W_{i-1}$ but close to W_{i-1}, it is the case that $i_w = i-1$. Also for w close to W_{i-1}, $t_{i-1}^*(w)$ is small, say equal to some ε. In case $i < j$ we can choose it so that $\varepsilon + t_i^{\text{opt}} + \cdots + t_{j-1}^{\text{opt}} < 1$, and for that value, we also have $\varepsilon + t_i^{\text{opt}} + \cdots + t_j^{\text{opt}} \geq 1$. If $i = j$, any choice of $\varepsilon < 1$ is good, because then $\varepsilon + t_i^{\text{opt}} \geq 1$. In both cases, we get that if $w < W_{i-1}$ but close to W_{i-1}, $H_{i-1}(w) = j$. The value of f_π on the left of W_{i-1} is thus:

$$\begin{aligned}
f_\pi(w) &= w_j^{\text{opt}} \cdot \left(\frac{w + p_0 + \cdots + p_{i-2}}{W_{i-1}^j + p_0 + \cdots + p_{i-2}}\right)^{\frac{r_j}{r_{i-1}}} + p_j + \cdots + p_n \\
&= w_j^{\text{opt}} \cdot \left(\frac{w + p_0 + \cdots + p_{i-2}}{w_{i-1}^{\text{opt}} \cdot \exp(-r_{i-1}(1 - \sum_{l=i}^{j} t_l^{\text{opt}}))}\right)^{\frac{r_j}{r_{i-1}}} + p_j + \cdots + p_n \\
&= w_j^{\text{opt}} \cdot \left(\frac{w + p_0 + \cdots + p_{i-2}}{w_{i-1}^{\text{opt}}}\right)^{\frac{r_j}{r_{i-1}}} \cdot \exp\left(r_j\left(1 - \sum_{l=i}^{j} t_l^{\text{opt}}\right)\right) + p_j + \cdots + p_n
\end{aligned}$$

Thus,

$$\lim_{w \to W_{i-1}^-} f_\pi(w) = w_j^{\text{opt}} \cdot \exp\left(r_j\left(1 - \sum_{l=i}^{j} t_l^{\text{opt}}\right)\right) + p_j + \cdots + p_n$$

which expresses the continuity of f_π in W_{i-1}.

Assume now that $j = n$. Then:

$$\begin{aligned}
f_\pi(W_{i-1}) &= (w_{n-1}^{\text{opt}} + p_{n-1}) \cdot \left(\frac{W_{i-1} + p_0 + \cdots + p_{i-1}}{W_i^{n-1} + p_0 + \cdots + p_{i-1}}\right)^{\frac{r_n}{r_i}} + p_n \\
&= (w_{n-1}^{\text{opt}} + p_{n-1}) \cdot \left(\frac{w_{i-1}^{\text{opt}} + p_{i-1}}{w_i^{\text{opt}} \cdot \exp(-r_i(1 - \sum_{l=i+1}^{n-1} t_l^{\text{opt}}))}\right)^{\frac{r_n}{r_i}} + p_n \\
&= (w_{n-1}^{\text{opt}} + p_{n-1}) \cdot \left(\frac{w_{i-1}^{\text{opt}} + p_{i-1}}{(w_{i-1}^{\text{opt}} + p_{i-1}) \cdot \exp(r_i t_i^{\text{opt}}) \cdot \exp(-r_i(1 - \sum_{l=i+1}^{n-1} t_l^{\text{opt}}))}\right)^{\frac{r_n}{r_i}} + p_n \\
&= (w_{n-1}^{\text{opt}} + p_{n-1}) \cdot \left(\frac{1}{\exp(r_i t_i^{\text{opt}}) \cdot \exp(-r_i(1 - \sum_{l=i+1}^{n-1} t_l^{\text{opt}}))}\right)^{\frac{r_n}{r_i}} + p_n \\
&= (w_{n-1}^{\text{opt}} + p_{n-1}) \cdot \exp\left(r_n\left(1 - \sum_{l=i}^{n-1} t_l^{\text{opt}}\right)\right) + p_n
\end{aligned}$$

As in the previous case, we have that on the left of W_{i-1}, we have values $i-1$ and j for the indices. Hence, the expression for f_π is:

$$\begin{aligned} f_\pi(w) &= (w^{\text{opt}}_{n-1} + p_{n-1}) \cdot \left(\frac{w + p_0 + \cdots + p_{i-2}}{W^{n-1}_{i-1} + p_0 + \cdots + p_{i-2}} \right)^{\frac{r_n}{r_{i-1}}} + p_n \\ &= (w^{\text{opt}}_{n-1} + p_{n-1}) \cdot \left(\frac{w + p_0 + \cdots + p_{i-2}}{w^{\text{opt}}_{i-1} \cdot \exp(-r_{i-1}(1 - \sum_{l=i}^{n-1} t^{\text{opt}}_l))} \right)^{\frac{r_n}{r_{i-1}}} + p_n \\ &= (w^{\text{opt}}_{n-1} + p_{n-1}) \cdot \left(\frac{w + p_0 + \cdots + p_{i-2}}{w^{\text{opt}}_{i-1}} \right)^{\frac{r_n}{r_{i-1}}} \cdot \exp\left(r_n \left(1 - \sum_{l=i}^{n-1} t^{\text{opt}}_l \right) \right) + p_n \end{aligned}$$

Hence,

$$\lim_{w \to W^+_{i-1}} f_\pi(w) = (w^{\text{opt}}_{n-1} + p_{n-1}) \cdot \exp\left(r_n \left(1 - \sum_{l=i}^{n-1} t^{\text{opt}}_l \right) \right) + p_n$$

which yields the continuity of f_π in W_{i-1}.

- We assume it is not continuous in W_{n-1}. We have that f_π is right-continuous in W_{n-1}. Its value is moreover:

$$f_\pi(W_{n-1}) = (w^{\text{opt}}_{n-1} + p_{n-1}) \cdot \exp(r_n) + p_n$$

For $w < W_{n-1}$ but close to W_{n-1}, we have that $i_w = n - 1$. As previously we can show that $H_{n-1}(w) = n$. Hence:

$$\begin{aligned} f_\pi(w) &= (w^{\text{opt}}_{n-1} + p_{n-1}) \cdot \left(\frac{w + p_0 + \cdots + p_{n-2}}{W^{n-1}_{n-1} + p_0 + \cdots + p_{n-2}} \right)^{\frac{r_n}{r_{n-1}}} + p_n \\ &= (w^{\text{opt}}_{n-1} + p_{n-1}) \cdot \left(\frac{w + p_0 + \cdots + p_{n-2}}{w^{\text{opt}}_{n-1} \cdot \exp(-r_{n-1})} \right)^{\frac{r_n}{r_{n-1}}} + p_n \\ &= (w^{\text{opt}}_{n-1} + p_{n-1}) \cdot \left(\frac{w + p_0 + \cdots + p_{n-2}}{w^{\text{opt}}_{n-1}} \right)^{\frac{r_n}{r_{n-1}}} \cdot \exp(r_n) + p_n \end{aligned}$$

Thus, we deduce that

$$\lim_{w \to W^-_{n-1}} f_\pi(w) = (w^{\text{opt}}_{n-1} + p_{n-1}) \cdot \exp(r_n) + p_n$$

which coincides with the continuity of f_π in W_{n-1}.

- We assume f_π is not continuous in W^j_i with $W_{i-1} < W^j_i < W_i$ ($i \le j < n$). We have that f_π is left-continuous, and that:

$$f_\pi(W^j_i) = w^{\text{opt}}_j + p_j + \cdots + p_n$$

As $W^j_i < W_i$, on the right of W^j_i, we have that the indices are i and $j+1$, and (if $j + 1 < n$):

$$\begin{aligned} f_\pi(w) &= w^{\text{opt}}_{j+1} \cdot \left(\frac{w + p_0 + \cdots + p_{i-1}}{W^{j+1}_i + p_0 + \cdots + p_{i-1}} \right)^{\frac{r_{j+1}}{r_i}} + p_{j+1} + \cdots + p_n \\ &= (w^{\text{opt}}_j + p_j) \cdot \exp(r_{j+1} t^{\text{opt}}_{j+1}) \cdot \left(\frac{w + p_0 + \cdots + p_{i-1}}{w^{\text{opt}}_i \cdot \exp(-r_i(1 - \sum_{l=i+1}^{j+1} t^{\text{opt}}_l))} \right)^{\frac{r_{j+1}}{r_i}} + p_{j+1} + \cdots + p_n \\ &= (w^{\text{opt}}_j + p_j) \cdot \exp(r_{j+1} t^{\text{opt}}_{j+1}) \cdot \left(\frac{w + p_0 + \cdots + p_{i-1}}{w^{\text{opt}}_i} \right)^{\frac{r_{j+1}}{r_i}} \cdot \exp(r_{j+1}(1 - \sum_{l=i+1}^{j+1} t^{\text{opt}}_l)) + p_{j+1} + \cdots + p_n \\ &= (w^{\text{opt}}_j + p_j) \cdot \left(\frac{w + p_0 + \cdots + p_{i-1}}{w^{\text{opt}}_i} \right)^{\frac{r_{j+1}}{r_i}} \cdot \exp(r_{j+1}(1 - \sum_{l=i+1}^{j} t^{\text{opt}}_l)) + p_{j+1} + \cdots + p_n \end{aligned}$$

We have that

$$\begin{aligned} \lim_{w \to W^{j\,+}_i} f_\pi(w) &= (w^{\text{opt}}_j + p_j) \cdot \left(\frac{W^j_i + p_0 + \cdots + p_{i-1}}{w^{\text{opt}}_i} \right)^{\frac{r_{j+1}}{r_i}} \cdot \exp(r_{j+1}(1 - \sum_{l=i+1}^{j} t^{\text{opt}}_l)) + p_{j+1} + \cdots + p_n \\ &= (w^{\text{opt}}_j + p_j) \cdot \left(\exp(-r_i(1 - \sum_{l=i+1}^{j} t^{\text{opt}}_l)) \right)^{\frac{r_{j+1}}{r_i}} \cdot \exp(r_{j+1}(1 - \sum_{l=i+1}^{j} t^{\text{opt}}_l)) + p_{j+1} + \cdots + p_n \\ &= w^{\text{opt}}_j + p_j + \cdots + p_n \end{aligned}$$

which yields the continuity in W_i^j, when $j+1<n$.
Assume now that $j+1=n$. Then:

$$f_\pi(w) = (w_{n-1}^{\text{opt}} + p_{n-1}) \cdot \left(\frac{w + p_0 + \cdots + p_{i-1}}{W_i^{n-1} + p_0 + \cdots + p_{i-1}} \right)^{\frac{r_n}{r_i}} + p_n$$

We have that

$$\lim_{w \to W_i^{n-1}+} f_\pi(w) = w_{n-1}^{\text{opt}} + p_{n-1} + p_n$$

which is the continuity of f_π in W_i^j when $j = n-1$.
This concludes the proof of the continuity of f_π on $\mathfrak{W}$. $\square$

Proposition 49. *The function f_π is differentiable on $\mathfrak{W}$, and its derivative is continuous, for every $w \geq \mathfrak{w}$, $f'_\pi(w) > 0$, and for every $w > W_{n-1}$, $f'_\pi(w) > 1$.*

Proof. We compute the derivatives of f_π in all cases:

- if $1 \leq i \leq j < n$, on the set $\{w \mid i_w = i \text{ and } H_i(w) = j\}$, we have:

$$f'_\pi(w) = w_j^{\text{opt}} \cdot \frac{r_j}{r_i} \cdot \frac{(w + p_0 + \cdots + p_{i-1})^{\frac{r_j}{r_i}-1}}{\left(W_i^j + p_0 + \cdots + p_{i-1}\right)^{\frac{r_j}{r_i}}}$$

- if $i = j = n$, on the set $\{w \mid i_w = n\}$, we have:

$$f'_\pi(w) = \exp(r_n)$$

- if $1 \leq i < j = n$, on the set $\{w \mid i_w = i \text{ and } H_i(w) = n\}$, we have:

$$f'_\pi(w) = (w_{n-1}^{\text{opt}} + p_{n-1}) \cdot \frac{r_n}{r_i} \cdot \frac{(w + p_0 + \cdots + p_{i-1})^{\frac{r_n}{r_i}-1}}{\left(W_i^{n-1} + p_0 + \cdots + p_{i-1}\right)^{\frac{r_n}{r_i}}}$$

Thus, the function f_π is differentiable (in all points there is a derivative on the left and on the right of every point), it remains to show that the derivative is continuous. As for the continuity, we need to show that the derivative is continuous at all points in $\{W_i \mid 1 \leq i \leq n-1\} \cup \{W_i^j \mid W_{i-1} < W_i^j < W_i\}$.

- In W_{i-1} for $1 < i \leq n-1$. The function f'_π is right-continuous in W_{i-1}. We first assume $j = H_i(W_{i-1}) < n$. In that case:

$$\begin{aligned}
f'_\pi(W_{i-1}) &= w_j^{\text{opt}} \cdot \frac{r_j}{r_i} \cdot \frac{(W_{i-1} + p_0 + \cdots + p_{i-1})^{\frac{r_j}{r_i}-1}}{\left(W_i^j + p_0 + \cdots + p_{i-1}\right)^{\frac{r_j}{r_i}}} \\
&= w_j^{\text{opt}} \cdot \frac{r_j}{r_i} \cdot \frac{\left(w_{i-1}^{\text{opt}} + p_{i-1}\right)^{\frac{r_j}{r_i}-1}}{\left(W_i^j + p_0 + \cdots + p_{i-1}\right)^{\frac{r_j}{r_i}}} \\
&= w_j^{\text{opt}} \cdot \frac{r_j}{r_i} \cdot \frac{\left(w_i^{\text{opt}} \cdot \exp(-r_i t_i^{\text{opt}})\right)^{\frac{r_j}{r_i}-1}}{\left(w_i^{\text{opt}} \cdot \exp(-r_i(1 - \sum_{l=i+1}^{j} t_l^{\text{opt}}))\right)^{\frac{r_j}{r_i}}} \\
&= \frac{w_j^{\text{opt}}}{w_i^{\text{opt}}} \cdot \frac{r_j}{r_i} \cdot \frac{\left(\exp(-r_i t_i^{\text{opt}})\right)^{\frac{r_j}{r_i}-1}}{\left(\exp(-r_i(1 - \sum_{l=i+1}^{j} t_l^{\text{opt}}))\right)^{\frac{r_j}{r_i}}} \\
&= \frac{w_j^{\text{opt}}}{w_i^{\text{opt}}} \cdot \frac{r_j}{r_i} \cdot \frac{\exp(-(r_j - r_i) t_i^{\text{opt}})}{\exp(-r_j(1 - \sum_{l=i+1}^{j} t_l^{\text{opt}}))} \\
&= \frac{r_j \cdot w_j^{\text{opt}}}{r_i \cdot w_i^{\text{opt}}} \cdot \exp(r_i t_i^{\text{opt}}) \cdot \exp\left(r_j \left(1 - \sum_{l=i}^{j} t_l^{\text{opt}}\right)\right)
\end{aligned}$$

On the left of W_{i-1}, the indices are $i-1$ and j (see a justification in the proof of continuity). Thus:

$$\begin{aligned}
f'_\pi(w) &= w_j^{\text{opt}} \cdot \frac{r_j}{r_{i-1}} \cdot \frac{(w+p_0+\cdots+p_{i-2})^{\frac{r_j}{r_{i-1}}-1}}{\left(W_{i-1}^j+p_0+\cdots+p_{i-2}\right)^{\frac{r_j}{r_{i-1}}}} \\
&= w_j^{\text{opt}} \cdot \frac{r_j}{r_{i-1}} \cdot \frac{(w+p_0+\cdots+p_{i-2})^{\frac{r_j}{r_{i-1}}-1}}{\left(w_{i-1}^{\text{opt}} \cdot \exp(-r_{i-1}(1-\sum_{l=i}^j t_l^{\text{opt}}))\right)^{\frac{r_j}{r_{i-1}}}} \\
&= w_j^{\text{opt}} \cdot \frac{r_j}{r_{i-1}} \cdot \left(\frac{1}{w_{i-1}^{\text{opt}}}\right)^{\frac{r_j}{r_{i-1}}} \cdot (w+p_0+\cdots+p_{i-2})^{\frac{r_j}{r_{i-1}}-1} \cdot \exp\left(r_j\left(1-\sum_{l=i}^j t_l^{\text{opt}}\right)\right)
\end{aligned}$$

We have that:

$$\begin{aligned}
\lim_{w\to W_{i-1}^-} f_\pi(w) &= w_j^{\text{opt}} \cdot \frac{r_j}{r_{i-1}} \cdot \left(\frac{1}{w_{i-1}^{\text{opt}}}\right)^{\frac{r_j}{r_{i-1}}} \cdot (w_{i-1}^{\text{opt}})^{\frac{r_j}{r_{i-1}}-1} \cdot \exp\left(r_j\left(1-\sum_{l=i}^j t_l^{\text{opt}}\right)\right) \\
&= \frac{r_j \cdot w_j^{\text{opt}}}{r_{i-1} \cdot w_{i-1}^{\text{opt}}} \cdot \exp\left(r_j\left(1-\sum_{l=i}^j t_l^{\text{opt}}\right)\right)
\end{aligned}$$

Due to $t_i^{\text{opt}} = \frac{1}{r_i} \cdot \left(\frac{r_i \cdot w_i^{\text{opt}}}{r_{i-1} \cdot w_{i-1}^{\text{opt}}}\right)$, we get the continuity of f'_π in W_{i-1}.

We now assume that $H_i(W_{i-1}) = j = n$. We compute that (simple computation, similar to the previous case):

$$f'_\pi(W_{i-1}) = \frac{r_n \cdot (w_{n-1}^{\text{opt}}+p_{n-1})}{r_i \cdot w_i^{\text{opt}}} \cdot \exp(r_i t_i^{\text{opt}}) \cdot \exp\left(r_n\left(1-\sum_{l=i}^{n-1} t_l^{\text{opt}}\right)\right)$$

On the left of W_{i-1}, we get similarly that:

$$f'_\pi(w) = (w_{n-1}^{\text{opt}}+p_{n-1}) \cdot \frac{r_n}{r_{i-1}} \cdot \frac{1}{(w_{i-1}^{\text{opt}})^{\frac{r_n}{r_{i-1}}}} \cdot (w+p_0+\cdots+p_{i-2})^{\frac{r_n}{r_{i-1}}-1} \cdot \exp\left(r_n\left(1-\sum_{l=i}^{n-1} t_l^{\text{opt}}\right)\right)$$

Thus:

$$\begin{aligned}
\lim_{w\to W_{i-1}^-} f'_\pi(w) &= \frac{r_n \cdot (w_{n-1}^{\text{opt}}+p_{n-1})}{r_{i-1} \cdot w_{i-1}^{\text{opt}}} \cdot \exp\left(r_n\left(1-\sum_{l=i}^{n-1} t_l^{\text{opt}}\right)\right) \\
&= \frac{r_n \cdot (w_{n-1}^{\text{opt}}+p_{n-1})}{r_i \cdot w_i^{\text{opt}}} \cdot \exp(r_i t_i^{\text{opt}}) \cdot \exp\left(r_n\left(1-\sum_{l=i}^{n-1} t_l^{\text{opt}}\right)\right)
\end{aligned}$$

which concludes the continuity of f'_π in W_{i-1}.

- In W_{n-1}, we have that f'_π is right-continuous and that:

$$f'_\pi(W_{n-1}) = \exp(r_n)$$

On the left of W_{n-1}, we have that:

$$f'_\pi(w) = (w_{n-1}^{\text{opt}}+p_{n-1}) \cdot \frac{r_n}{r_{n-1}} \cdot \frac{1}{(w_{n-1}^{\text{opt}})^{\frac{r_n}{r_{n-1}}}} \cdot (w+p_0+\cdots+p_{n-2})^{\frac{r_n}{r_{n-1}}-1} \cdot \exp(r_n)$$

Thus,

$$\begin{aligned}
\lim_{w\to W_{n-1}^-} f'_\pi(w) &= (w_{n-1}^{\text{opt}}+p_{n-1}) \cdot \frac{r_n}{r_{n-1}} \cdot \frac{1}{(w_{n-1}^{\text{opt}})^{\frac{r_n}{r_{n-1}}}} \cdot (w_{n-1}^{\text{opt}})^{\frac{r_n}{r_{n-1}}-1} \cdot \exp\left(r_n\left(1-\sum_{l=n}^{n-1} t_l^{\text{opt}}\right)\right) \\
&= \frac{r_n \cdot (w_{n-1}^{\text{opt}}+p_{n-1})}{r_{n-1} \cdot w_{n-1}^{\text{opt}}} \cdot \exp(r_n) \\
&= \exp(r_n)
\end{aligned}$$

which yields the continuity of f'_π in W_{n-1}.

- In W_i^j with $W_{i-1} < W_i^j < W_i$ $(1 \le i \le j < n)$. The function f'_π is left-continuous, and

$$f'_\pi(W_i^j) = \frac{r_j \cdot w_j^{\text{opt}}}{r_i \cdot w_i^{\text{opt}}} \cdot \exp\left(r_i\left(1-\sum_{l=i+1}^j t_l^{\text{opt}}\right)\right)$$

On the right of W_i^j, the indices are i and $j+1$.
We first assume that $j+1 < n$. In that case, on the right of W_i^j, we have that

$$f'_\pi(w) = \frac{r_{j+1} \cdot w_{j+1}^{\text{opt}}}{r_i \cdot w_i^{\text{opt}}} \cdot \exp\left(r_i\left(1 - \sum_{l=i+1}^{j+1} t_l^{\text{opt}}\right)\right) \cdot \left(\frac{w + p_0 + \cdots + p_{i-1}}{W_i^{j+1} + p_0 + \cdots + p_{i-1}}\right)^{\frac{r_{j+1}}{r_i}-1}$$

Thus:

$$\begin{aligned}
\lim_{w \to W_i^{j+}} f'_\pi(w) &= \frac{r_{j+1} \cdot w_{j+1}^{\text{opt}}}{r_i \cdot w_i^{\text{opt}}} \cdot \exp\left(r_i\left(1 - \sum_{l=i+1}^{j+1} t_l^{\text{opt}}\right)\right) \cdot \left(\frac{W_i^j + p_0 + \cdots + p_{i-1}}{W_i^{j+1} + p_0 + \cdots + p_{i-1}}\right)^{\frac{r_{j+1}}{r_i}-1} \\
&= \frac{r_{j+1} \cdot w_{j+1}^{\text{opt}}}{r_i \cdot w_i^{\text{opt}}} \cdot \exp\left(r_i\left(1 - \sum_{l=i+1}^{j+1} t_l^{\text{opt}}\right)\right) \cdot \exp\left(-(r_{j+1} - r_i)\, t_{j+1}^{\text{opt}}\right) \\
&= \frac{r_{j+1} \cdot w_{j+1}^{\text{opt}}}{r_i \cdot w_i^{\text{opt}}} \cdot \exp\left(r_i\left(1 - \sum_{l=i+1}^{j} t_l^{\text{opt}}\right)\right) \cdot \exp\left(-r_{j+1} t_{j+1}^{\text{opt}}\right) \\
&= \frac{r_j \cdot w_j^{\text{opt}}}{r_i \cdot w_i^{\text{opt}}} \cdot \exp\left(r_i\left(1 - \sum_{l=i+1}^{j} t_l^{\text{opt}}\right)\right)
\end{aligned}$$

yielding the continuity of f'_π in W_i^j.
We now assume that $j+1 = n$. In that case, on the right of $W_i^j = W_i^{n-1}$, we have that:

$$f'_\pi(w) = (w_{n-1}^{\text{opt}} + p_{n-1}) \cdot \frac{r_n}{r_i} \cdot (w + p_0 + \cdots + p_{i-1})^{\frac{r_n}{r_i}-1} \cdot \frac{1}{(W_i^{n-1} + p_0 + \cdots + p_{i-1})^{\frac{r_n}{r_i}}}$$

Thus,

$$\begin{aligned}
\lim_{w \to W_i^{n-1+}} f'_\pi(w) &= (w_{n-1}^{\text{opt}} + p_{n-1}) \cdot \frac{r_n}{r_i} \cdot \left(W_i^{n-1} + p_0 + \cdots + p_{i-1}\right)^{\frac{r_n}{r_i}-1} \cdot \frac{1}{(W_i^{n-1} + p_0 + \cdots + p_{i-1})^{\frac{r_n}{r_i}}} \\
&= (w_{n-1}^{\text{opt}} + p_{n-1}) \cdot \frac{r_n}{r_i} \cdot \frac{1}{(W_i^{n-1} + p_0 + \cdots + p_{i-1})} \\
&= (w_{n-1}^{\text{opt}} + p_{n-1}) \cdot \frac{r_n}{r_i} \cdot \frac{1}{w_i^{\text{opt}} \cdot \exp(-r_i(1 - \sum_{l=i+1}^{n-1} t_l^{\text{opt}}))} \\
&= \frac{r_n \cdot (w_{n-1}^{\text{opt}} + p_{n-1})}{r_i \cdot w_i^{\text{opt}}} \cdot \exp\left(r_i\left(1 - \sum_{l=i+1}^{n-1} t_l^{\text{opt}}\right)\right)
\end{aligned}$$

yielding the continuity of f'_π in W_i^{n-1}.
This concludes the proof of the continuity of f'_π on $\mathfrak{W}$. The last part of the proposition is obvious, given the expressions for the function f'_π. □

We can depict the function f_π as in Figure 8.

C. PROOFS OF SECTION 7

C.1 Proof of Lemma 16

LEMMA 16. *Let A be a closed one-clock timed automaton with observers, ℓ_0 a location of A, and $L_G \subseteq L$ a set of goal locations. One can construct in exponential time another one-clock timed automaton A' with observers, together with a new initial location ℓ'_0 and a new set of goal locations L'_G, such that*

- *in any state ℓ' of A', the invariant is $c' \leq 1$;*
- *for any edge $\ell \xrightarrow{g,r} \ell'$ in A', either $r = \emptyset$ and g is the constraint $0 \leq c' \leq 1$, or $r = \{c'\}$ and g is an equality constraint $c' = 0$ or $c' = 1$,*

and such that, for any w_0 and m, $\langle A, \ell_0, w_0, L_G, m\rangle$ is a positive instance of the reachability (resp. infinite-run) problem iff $\langle A', \ell'_0, w_0, L'_G, m\rangle$ is also a positive instance of that problem.

PROOF. Recall that a timed automaton is bounded if clock values in all reachable states are bounded above. By [5, Thm. 2] we can assume A to be bounded by some constant $c_{\max}$ and then adapt a construction from [8, Prop. 2]: We let $L' = L \times \{0, \ldots, c_{\max} - 1\}$ and use the second component of a location (ℓ, α) for keeping track of the integer value of the original clock, so that the new clock is used for modeling the original clock's fractional value.

Due to the non-strict global invariant $c' \leq 1$, we have to modify the construction from [8] slightly so that an original configuration (ℓ, c) with c integer is mapped to the two configurations $((\ell, c-1), 1)$ and $((\ell, c), 0)$; hence edges have to be adapted accordingly. Figure 9 shows an example of the construction.

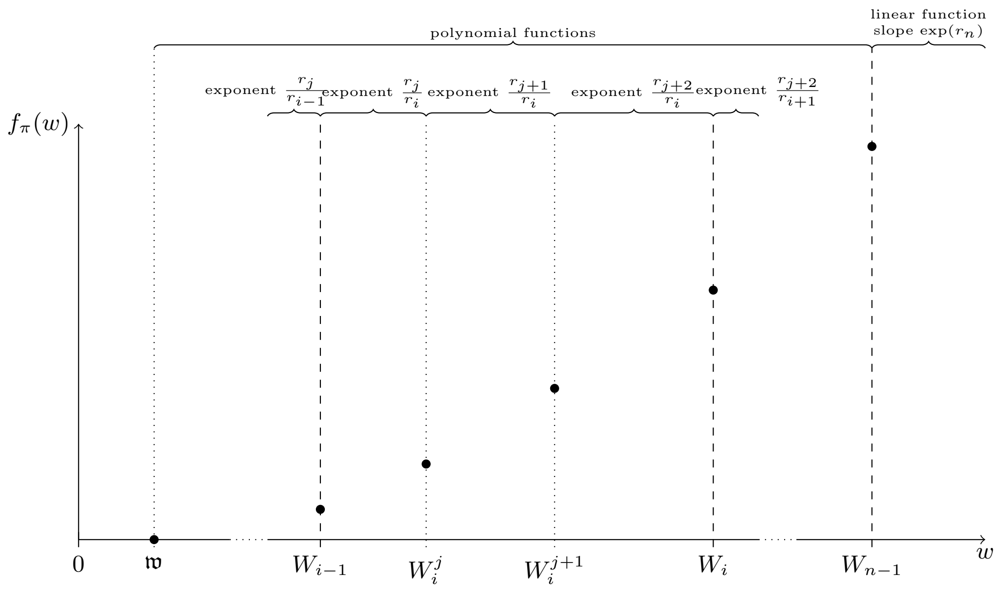

Figure 8: Energy function for exponential observer

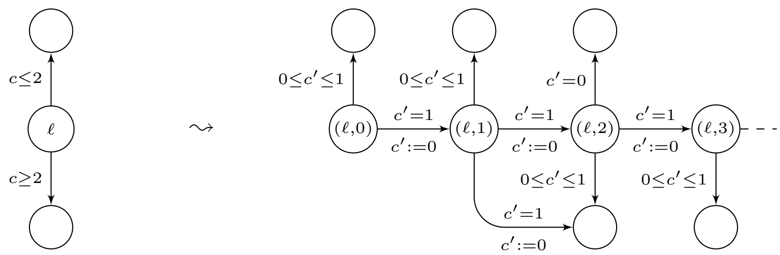

Figure 9: An example of the construction in Lemma 16

The observers of A' are the same as for A, and also their flow conditions and updates are inherited by A'. Then A and A' admit corresponding infinite runs.

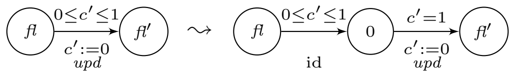

Figure 10: Reducing to three types of edges

For the second property, note that any offending edge $\ell \xrightarrow{g,r} \ell'$ has $g = (0 \leq c' \leq 1)$ and $r = \{c'\}$, and we can use the procedure depicted in Figure 10 to remove these. $\square$

C.2 Proof of Lemma 17

Lemma 17. *Let A be a closed one-clock timed automaton with one linear or exponential observer and additive updates, and with clock bound $c \leq 1$. Let $\ell_0 \in L$ be a location of A, and $L_G \subseteq L$ be a set of goal locations. We can compute in exponential time*
- *two labelling functions $w_{Zeno}, w_{Zeno}^{L_G} \colon L \to \mathbb{R} \cup \{+\infty\}$,*
- *another such automaton A', with set of locations L', and a projection $\mathsf{lab} \colon L' \to L$,*
- *a location $\ell'_0 \in L'$*
- *a set of goal locations $L'_G \subseteq L'$,*

such that A' does not contain reset-free cycles, and for any initial observer value w_0, we have the following:
- *There is an infinite feasible run in A from $(\ell_0, c = 0)$ with initial observer value w_0 if and only if there is such a run in A' from $(\ell'_0, c = 0)$, or there is a feasible run in A' from $(\ell'_0, c = 0)$ with observer value w_0 to a configuration $(\ell, c = 0)$ with observer value w, and such that $w \geq w_{Zeno}(\mathsf{lab}(\ell))$.*
- *There is a feasible run in A from $(\ell_0, c = 0)$ with initial observer value w_0 to a location in L_G if and only if there is such a run in A' from $(\ell'_0, c = 0)$ to a location in L'_G, or there is a feasible run in A' from $(\ell'_0, c = 0)$ with observer value w_0 to a configuration $(\ell, c = 0)$ with observer value $w \geq w_{Zeno}^{L_G}(\mathsf{lab}(\ell))$.*

The proof relies on the following lemmas:

Lemma 50. *Let A be a closed one-clock timed automaton with one linear or exponential observer and additive updates, with global invariant $c \leq 1$, and let ℓ_0 be a location of A. Assume that A has an infinite feasible run ρ from ℓ_0. Then there exists an infinite run ρ' such that*
- *either ρ' is Zeno,*
- *or ρ' has infinitely many resetting transitions, and between any two consecutive resetting transitions, any non-resetting simple cycle appears at most twice.*

Proof. Let ρ be an infinite feasible run in A. If ρ is Zeno, the result is immediate. Otherwise, ρ has infinitely many resetting transitions. Assume that, between a resetting transition and the subsequent one, some simple cycle is visited three times (or more). This situation can be depicted as in Figure 11.

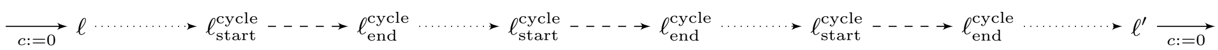

Figure 11: Three cycles between two consecutive resets.

From ρ, we define another feasible run as follows: in each occurrence of the cycle, for each location ℓ^{cycle} in that cycle, we transfer the delay spent in ℓ^{cycle} to the corresponding location
- in the first copy of the cycle in case the rate of ℓ^{cycle} is nonnegative;
- in the last copy of the cycle otherwise.

This way, no time is elapsed in any occurrence of the cycle, except the first and last ones. Moreover, the resulting run is feasible. Indeed, in both the linear and exponential cases, it can easily be shown by induction that the value of the observer when entering any location along ρ' is larger than or equal to its value when entering the corresponding location along ρ.

Now, if the sum of the additive updates is nonnegative along the cycle, then by repeating this cycle we get a infinite feasible Zeno run. If it is negative, then we drop the occurrences (except the first and last ones) from ρ', ending up with another feasible run. $\square$

Lemma 51. *Let A be a closed one-clock timed automaton with one linear or exponential observer and additive updates, with global invariant $c \leq 1$, and let ℓ be a location of A. We can compute in exponential time a value $w_{Zeno}(\ell) \in \mathbb{R} \cup \{+\infty\}$ such that there is a feasible infinite non-resetting run from ℓ with clock value 0 and observer value w iff $w \geq w_{Zeno}(\ell)$.*

Proof. Let ρ be a feasible non-resetting run from ℓ with initial clock value 0 and observer value w. Since it is non-resetting, ρ only contains guards $0 \leq c \leq 1$. Consider the run ρ' obtained from ρ by elapsing no time in locations with negative rate, and, for each location ℓ with nonnegative rate, elapsing in the first copy of ℓ along ρ' the sum of the delays elapsed in all occurrences of ℓ along ρ. In both the linear and exponential cases, it can be proved that ρ' is feasible. As a consequence, if there is a feasible non-resetting run in A, there is one along which, from some point on, no time elapses. In the part where no time elapses, simple cycles with negative accumulated updates can be removed, still resulting in a feasible run. This run must contain a simple cycle with nonnegative accumulated cost. The first such cycle can be iterated, giving a lasso-shaped non-resetting run with no time elapsing in the cyclic part.

This proves that there is a feasible non-resetting run from configuration $(\ell, c = 0, w)$ if, and only if, there is a lasso-shaped one from the same configuration with no time elapsing in the periodic part.

We now explain how to compute w_{Zeno}: the first part amounts to computing, for each location ℓ, the minimal observer value $w_0(\ell)$ for which there is a non-resetting, zero-delay feasible lasso from ℓ (this does not depend on the value of the clock, since guards on non-resetting transitions are $0 \le c \le 1$). This is a purely syntactic computation which can be achieved by a modified Bellman-Ford algorithm as in [7].

In a second phase, we have to compute, for each pair of locations (ℓ, ℓ') such that $w_0(\ell') \neq +\infty$, the minimal observer value needed from ℓ in order to be able to reach ℓ' with observer value $w_0(\ell')$. As we saw above, we can restrict to runs in which time elapses only during the first visit to each location, and containing no zero-delay cycles. The maximum length of such runs is $|S| \cdot (|S| + 1)$ (if there are more than $|S| + 1$ copies of a given location, then there must be two consecutive copies with no time elapsing inbetween, hence a zero-delay cycle). Hence it suffices to enumerate the candidate runs and compute the minimal observer value from ℓ to reach ℓ' with energy level $w_0(\ell')$. For each location ℓ, the smallest computed value is $w_{\text{Zeno}}(\ell)$.

The computation is in exponential time, as we have to enumerate exponentially many runs in the second part. □

Proof of Lemma 17. The procedure is as follows:

1. First label each location ℓ with $w_{\text{Zeno}}(\ell)$. This indicate the minimal observer value for which there is a non-resetting feasible run from $(\ell, c = 0)$.
2. Consider the (infinite) execution tree of A from ℓ_0, labelling nodes with the corresponding location in A and edges wit the corresponding transition. Then prune the subtrees rooted at nodes where a simple cycle has been visited three times without intermediary resetting transition. According to Lemma 50, there is an infinite feasible run in A iff there is one in this (infinite) tree, seen as an infinite timed automaton.
3. In the resulting tree, infinite branches necessarily contain infinitely many resetting transitions. For each resetting transition $(\ell, g, \{c\}, \ell')$, the target configuration is $(\ell', c = 0)$. We identify the corresponding nodes in the tree and turn this into a congruence by also identifying corresponding successors to nodes which have been identified.

   We end up with a finite timed automaton A' with observer (rates and updates are inherited from the original automaton) in which all cycles contain at least one resetting transition, and such that there exists an infinite feasible run in A from $(\ell_0, c = 0)$ with observer value w iff there is one in A' from $(\ell_0, c = 0)$ with observer value w, or a location ℓ of A' is reachable with observer value at least $w_{\text{Zeno}}(\ell)$ from $(\ell_0, c = 0)$ with observer value w.

The procedure runs in exponential time.

The second part of the claim can be proven along the same lines, by adapting Lemmas 50 and 51. Notice that here we also have to identify simple cycles with *strictly* positive accumulated energy and from which a goal location can be reached, which is achieved by the $w^{L_G}_{\text{Zeno}}$ mapping. □