Automates, algèbre, applications - AAA

CM 3

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EPITA

S6 2022

Foreword

Program of the course

1. CM 1 : Weighted automata — 5 May
2. TD : Weighted automata — 12 May
3. CM 2 : LTL model checking — 12 May
4. CM 3 : ω-Automata — 19 May
5. DM (noté)
6. TP : ω-Automata — 2 June
7. CM 4 : Automata learning — 16 June

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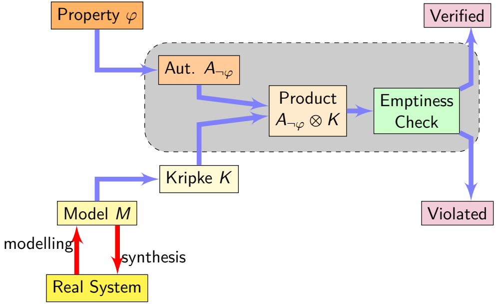
CM 3 : ω-Automata

Sources:

- Farwer, B, ω-automata. In: Automata, Logics, and Infinite Games (pp. 3-21), Springer, 2002.
- `https://spot.lrde.epita.fr/concepts.html`

Introduction

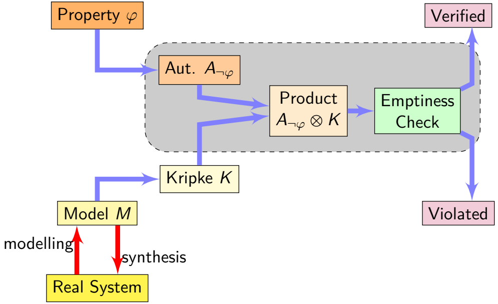
Model Checking



Recall

Model checking translates LTL to automata and checks for emptiness of intersection of model and specification.

Model Checking



Recall

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Goal of this lecture

Understand ω-automata

Ordinal Numbers

0

1

2

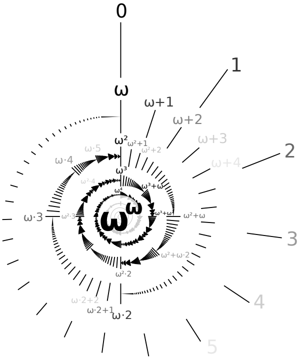
⋮

n

⋮

ω order type of $\mathbb{N}$

⋮



specify indices in ordered sequences

ω-Words

Σ: finite alphabet, $n \in \mathbb{N} \cup \{\omega\}$: ordinal number.

Definition

An *n-word* w over Σ is a sequence of length n,
i.e., a function $w : [0, n[\to \Sigma$.

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Notations:

Σ^n set of n-words,

Σ^* set of finite words $(n < \omega)$,

Σ^ω set of ω-words $(n = \omega)$,

$\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$ set of all words $(n \leq \omega)$,

ω-Rational Expressions

Definition (ω-Rational Expressions)

ω-languages are called ω-regular if they are of the form:

Above: L is a regular language (of finite words)
K, K_1, K_2 are ω-regular languages

ω-Rational Expressions

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ω-languages are called ω-regular if they are of the form:

- $K_1 \cup K_2$,
- $LK = \{vw \mid v \in L, w \in K\}$,
- $L^\omega = \{w_0 w_1 w_2 \cdots \mid w_i \in L\}$ for $\{\varepsilon\} \notin L$,

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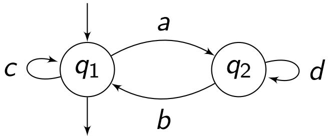
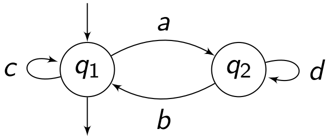
Example



Which words are accepted?

Example



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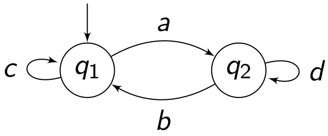

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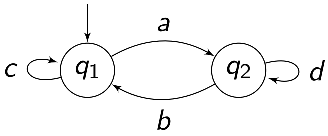

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$$(c^* a d^* b)^\omega + (c^* a d^* b)^* c^\omega + (c^* a d^* b)^* c^* a d^\omega$$

Büchi Automata

Infinite Sequences

Definition

$$\mathrm{Inf}((q_i)_{i\geq 0}) = \{q \mid q = q_i \text{ for infinitely many } i \geq 0\}$$

Example

$$\mathrm{Inf}(0,1,2,3,4,5,\ldots) = \emptyset$$
$$\mathrm{Inf}(0,1,2,3,3,3,\ldots) = \{3\}$$
$$\mathrm{Inf}(0,1,2,3,2,3,2,3,\ldots) = \{2,3\}$$

Büchi Automata: Definition

Definition (Büchi Automaton)

$\mathcal{A} = (Q, I, T, F)$ over alphabet Σ where

- Q: finite set of states,
- $I \subseteq Q$: initial states,
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Run of $\mathcal{A}$ on $w \in \Sigma^\omega$: $\rho = (q_i)_{i \geq 0}$ with

- $q_0 \in I$,
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i.e., $\mathrm{Inf}(\rho) \cap F \neq \emptyset$.

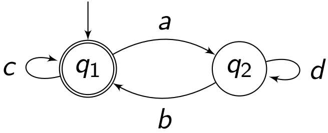
Büchi Automata: Definition 2

The language recognized by a Büchi Automaton $\mathcal{A}$:

$$\mathcal{L}(\mathcal{A}) = \{w \in \Sigma^\omega \mid \exists \text{ Büchi-accepted run of } \mathcal{A} \text{ on } w\}$$

A language recognized by Büchi Automata is called Büchi-recognizable.

Büchi Automata: Example



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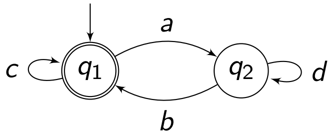

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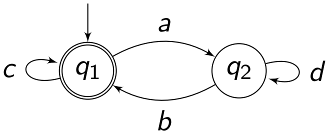

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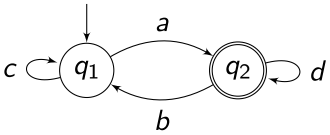

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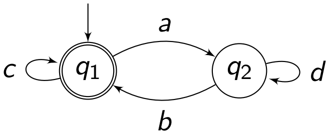

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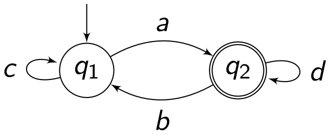

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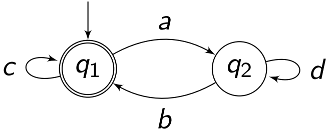

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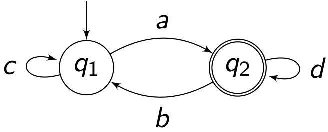

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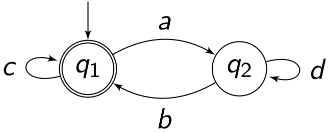

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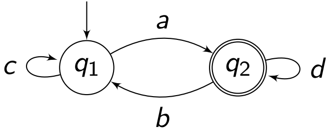

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$F = \{q_1, q_2\} \quad \Rightarrow$
$\mathcal{L}(\mathcal{A}) = (c^* ad^* b)^\omega + (c^* ad^* b)^* c^\omega + (c^* ad^* b)^* c^* ad^\omega$

Emptiness is Decidable

Büchi automaton: $\mathcal{A} = (Q, I, T, F)$

- Interpret $\mathcal{A}$ as graph (Q, T)
- Check reachability of states in F from states in I (Floyd-Warshall)
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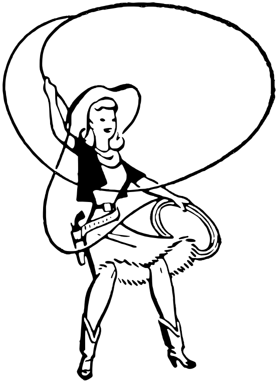
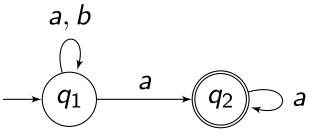


Deterministic Büchi Automata



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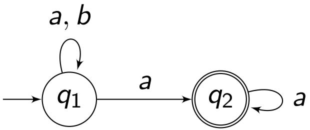

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Lemma

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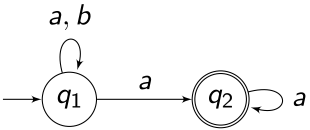

$$L := \mathcal{L}(\mathcal{A}) = (a+b)^* a^\omega$$

Lemma

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Proof.

Suppose: $\mathcal{B} = (Q, I, T, F)$ is a det. Büchi automaton recognizing L.
$\mathcal{B}$ accepts a^ω. After some finite prefix of a^ω, $\mathcal{B}$ will visit some state in F, say after i_0 letters.
$\mathcal{B}$ also accepts $a^{i_0} b a^\omega$. Therefore, for some i_1, after the prefix $a^{i_0} b a^{i_1}$, $\mathcal{B}$ will visit some state in F.
Continue: $a^{i_0} b a^{i_1} b a^{i_2} \cdots$ is recognized by $\mathcal{B}$ as states in F are visited infinitely often. But: $a^{i_0} b a^{i_1} b a^{i_2} \cdots \notin L$. $\lightning$

Büchi Automata: Properties

The class of languages accepted by Büchi automata is closed under:

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But: Büchi automata are not always determinisable.

Büchi vs. ω-Regular Expressions

Theorem

Büchi-recognizable languages are exactly the ω-regular languages

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prefix: $\rho_f = q_0 \to \ldots \to q_f$ for $q_0 \in I$ and $q_f \in F$

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All possible runs: $\bigcup_{f \in F} \rho_f \rho_f'^{\omega}$ is an ω-regular expression

Muller Automata

Muller Automata: Definition

Definition (Muller Automaton)

$\mathcal{A} = (Q, I, T, \mathcal{F})$ over alphabet Σ where

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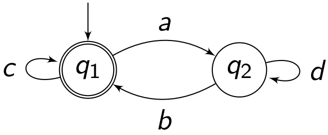

$$\mathcal{F} = \{\{q_1\}\}$$

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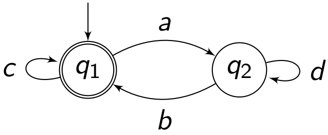

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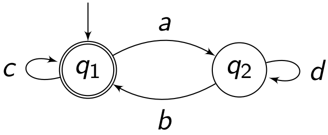

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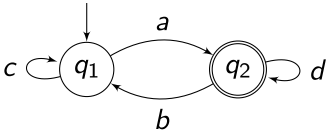

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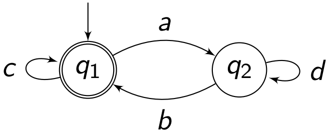

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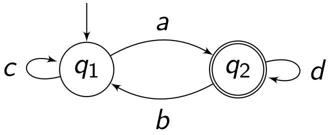

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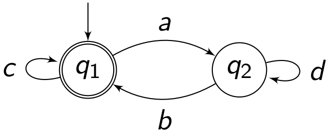

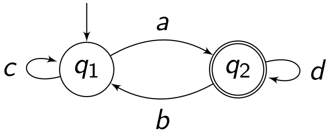

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$\mathcal{F} = \{\{q_1\}, \{q_2\}\} \quad \Rightarrow$

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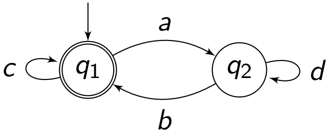

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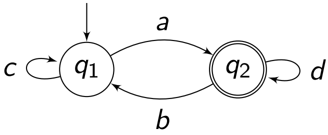


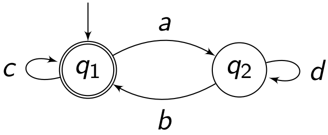
$$\mathcal{F} = \{\{q_2\}\}$$

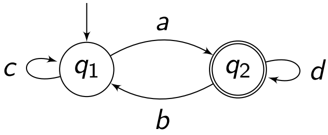
Which ω-words are accepted? $(\mathrm{Inf}(\rho) \in \mathcal{F})$

$$(c^* a d^* b)^* c^\omega \qquad (c^* a d^* b)^* c^* a d^\omega$$

$$\mathcal{F} = \{\{q_1\}, \{q_2\}\} \quad \Rightarrow \quad \mathcal{L}(\mathcal{A}) = (c^* a d^* b)^* c^\omega + (c^* a d^* b)^* c^* a d^\omega$$

Muller Automata: Example





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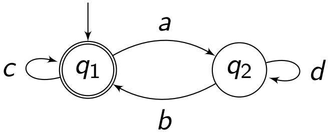

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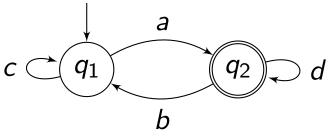

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Conversion Büchi to Muller

Easy direction:

Take Büchi automaton $\mathcal{B} = (Q, I, T, F)$ and transform it into Muller automaton $\mathcal{M} = (Q, I, T, \mathcal{F})$ with

$$\mathcal{F} = \{S \subseteq Q \mid S \cap F \neq \emptyset\}$$

Recall: Acceptance condition for Büchi: $\mathrm{Inf}(\rho) \cap F \neq \emptyset$

Conversion Muller to Büchi

Harder direction:

Take Muller automaton $\mathcal{M} = (Q, I, T, \mathcal{F})$.
Note that Büchi automata are closed under union.
Thus, assume $\mathcal{F} = \{F\}$ a singleton.

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Harder direction:

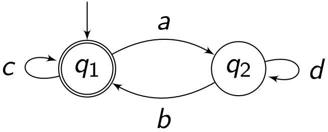
Take Muller automaton $\mathcal{M} = (Q, I, T, \mathcal{F})$.
Note that Büchi automata are closed under union.
Thus, assume $\mathcal{F} = \{F\}$ a singleton.

Transform $\mathcal{M}$ into Büchi automaton $\mathcal{B} = (Q', I, T', F')$ with:

$$
\begin{aligned}
Q' &= Q \cup (Q \times 2^F) \\
T' &= T \\
&\qquad \cup \{(q, \sigma, (q', \{q'\})) && \mid (q, \sigma, q') \in T, q' \in F\} \\
&\qquad \cup \{((q, S), \sigma, (q', S \cup \{q'\})) && \mid (q, \sigma, q') \in T, q, q' \in F, S \subsetneq F\} \\
&\qquad \cup \{((q, F), \sigma, (q', \{q'\})) && \mid (q, \sigma, q') \in T, q, q' \in F\} \\
F' &= F \times \{F\}
\end{aligned}
$$

Conversion Muller to Büchi: Example

$\mathcal{M}$:



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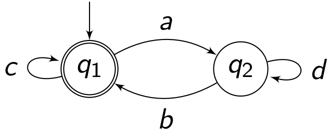

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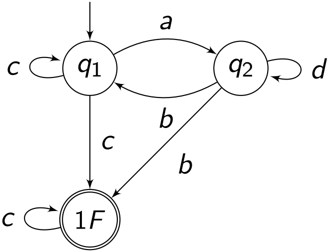
translates to

$\mathcal{B}$:



From Büchi to Spot

Transition-Based Generalized Büchi Automata

Definition (Transition-Based Generalized Büchi Automata (TGBA))

$\mathcal{A} = (Q, I, T, \mathcal{M})$ over alphabet Σ where

- Q: finite set of states,
- $I \subseteq Q$: initial states,
- $T \subseteq Q \times \Sigma \times 2^{\mathcal{M}} \times Q$: transitions,
- $\mathcal{M}$: finite set of acceptance conditions (colors).

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Run $\rho = q_0 \xrightarrow{w_0}_{M_0} q_1 \xrightarrow{w_1}_{M_1} q_2 \to \dots$
is TGBA-accepted if *all colors are seen infinitely often*
i.e., $\forall m \in \mathcal{M}. \forall i \in \mathbb{N}. \exists j > i. m \in M_j$.

Automata Over Propositions

The following can be done for any automaton:

Take a (finite) set of atomic propositions AP.
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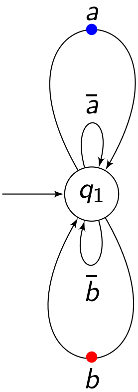
Note that we could interpret the set of minterms over AP as Σ.

Example

$$(a \wedge b \wedge \bar{c}); (a \wedge \bar{b} \wedge \bar{c}); (a \wedge \bar{b} \wedge c); \ldots$$

Transition-Based Generalized Büchi Automata: Example



Degeneralization

Lemma

TGBA can be degeneralized.

Construction is similar to Muller -> Büchi translation.

Product Construction

Let $A_1 = (Q_1, I_1, T_1, \mathcal{M}_1)$ and $A_2 = (Q_2, I_2, T_2, \mathcal{M}_2)$ be two TGBA over the same set of propositions Σ.

The product construction of A_1 and A_2 is $B = (Q, I, T, \mathcal{M})$ with

- $Q = Q_1 \times Q_2$
- $I = I_1 \times I_2$
- $T = \Big\{ ((q_1, q_2), p_1 \cap p_2, M_1 \cup M_2, (q_1', q_2')) \mid$ $(q_1, p_1, M_1, q_1') \in T_1, (q_2, p_2, M_2, q_2') \in T_2,\ p_1 \cap p_2 \neq \emptyset \Big\}$
- $\mathcal{M} = \mathcal{M}_1 \uplus \mathcal{M}_2$ (disjoint union)

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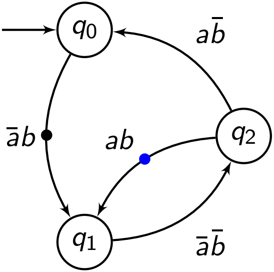
We can show: $\mathcal{L}(B) = \mathcal{L}(A_1) \cap \mathcal{L}(A_2)$.

Product Construction: Example



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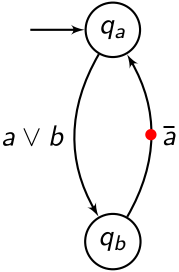

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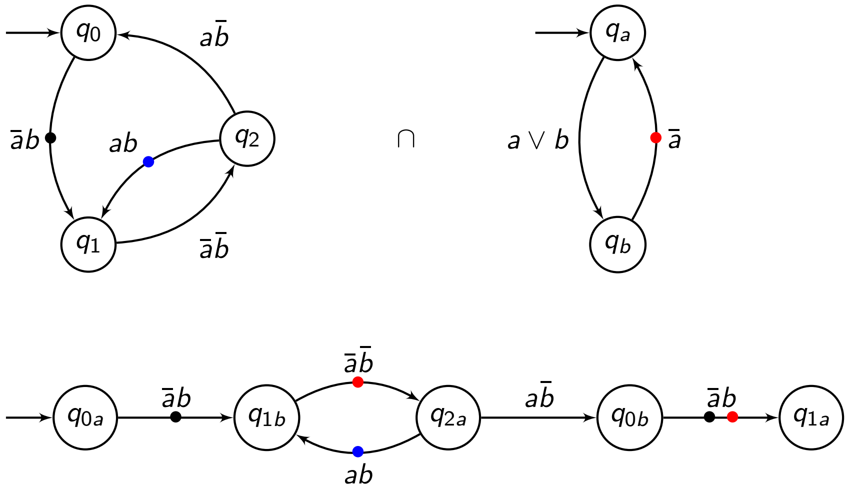

Emerson-Lei Automata

Description (Emerson-Lei Automaton)

- contains set of colors $\mathcal{M}$

  either states or transitions can be marked by colors
- acceptance condition: positive Boolean formula over the atoms
  - `t` or `f`: true and false
  - `Fin(m)`: color m occurs finitely many times
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Emerson-Lei implemented in Spot (see TP in two weeks!)

Example + Further Acceptance Conditions

none	`f`
all	`t`
Buchi	`Inf(0)`
gen.-Buchi	`Inf(0) & Inf(1) & ...`
co-Buchi	`Fin(0)`
gen.-co-Buchi	`Fin(0) | Fin(1) | ...`
Rabin	`(Fin(0)&Inf(1)) | (Fin(2)&Inf(3)) | ...`
Streett	`(Fin(0)|Inf(1)) & (Fin(2)|Inf(3)) & ...`
parity min odd 5	`Fin(0)&(Inf(1)|(Fin(2)&(Inf(3)|Fin(4))))`

Summary: Expressive Power

Overview: Expressive Power in Σ^ω

- ω-rational expressions $= \omega$-regular languages
- **Büchi automata**
- GBA
- **TGBA**
- Muller automata
- Emerson-Lei automata

- **LTL**

- Deterministic Büchi automata (incl. GBA, TGBA)


That's all Folks!

Backup

Notes

Generalized Büchi Automata

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Run ρ generalized-Büchi-accepted if
for all $F \in \mathcal{F}$, ρ is Büchi-accepted,
i.e., $\forall F \in \mathcal{F}. \mathrm{Inf}(\rho) \cap F \neq \emptyset$.

Degeneralization

Lemma

TGBA can be degeneralized.

Proof.

Let $\mathcal{A} = (Q, I, T, \mathcal{M})$ be a TGBA with $\mathcal{M} = \{m_1, \dots, m_k\}$.
We construct $\bar{\mathcal{A}} = (Q \times \{1, \dots, k\}, I \times \{1\}, \bar{T}, \{\bar{m}\})$ with

$$\begin{aligned} \bar{T} = {} & \{((q, i), \sigma, \emptyset, (q', i)) \mid (q, \sigma, M, q') \in T, m_i \notin M\} \\ & \cup \{((q, i), \sigma, \emptyset, (q', i+1)) \mid i \neq k, (q, \sigma, M, q') \in T, m_i \in M\} \\ & \cup \{((q, k), \sigma, \{m_a\}, (q', 1)) \mid (q, \sigma, M, q') \in T, m_k \in M\} \end{aligned}$$

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