An Invitation to Higher-Dimensional Auomata Theory

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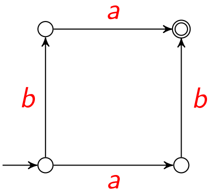
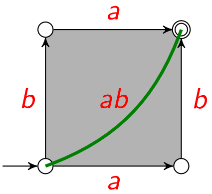
Higher-dimensional automata

a in parallel with b:



$a.b + b.a$



a and b are independent

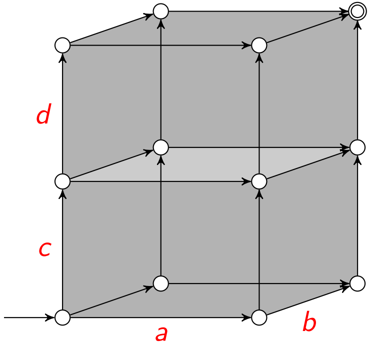
Higher-dimensional automata & concurrency

HDA as a model for concurrency:

- points: states
- edges: transitions
- squares, cubes etc.: independency relations (concurrently executing events)
- two-dimensional automata $\cong$ asynchronous transition systems [Bednarczyk]

[van Glabbeek 2006, TCS]: Up to history-preserving bisimilarity, HDA "generalize the main models of concurrency proposed in the literature" (notably, event structures and Petri nets)

A 3D example



- no cubes, all faces except middle horizontal
- *a* and *b* independent; *c* introduces conflict; *d* releases conflict

Precubical sets and higher dimensional automata

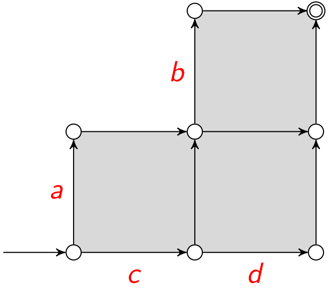
An loset is a finite, ordered and Σ-labelled set. (a list of events)

A precubical set X consists of:

- A set of cells X (cubes)
- Every cell $x \in X$ has an loset $\mathrm{ev}(x)$ (list of events active in x)
- We write $X[U] = \{x \in X \mid \mathrm{ev}(x) = U\}$ for an loset U (cells of type U)
- For every loset U and $A \subseteq U$ there are:
  upper face map $\delta^1_A : X[U] \to X[U - A]$ (terminating events A)
  lower face map $\delta^0_A : X[U] \to X[U - A]$ (unstarting events A)
- Precube identities: $\delta^\mu_A \delta^\nu_B = \delta^\nu_B \delta^\mu_A$ for $A \cap B = \emptyset$ and $\mu, \nu \in \{0, 1\}$

A higher dimensional automaton (HDA) is a precubical set X with start cells $X_\perp \subseteq X$ and accept cells $X^\top \subseteq X$ (not necessarily vertices)

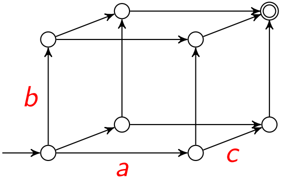
Example



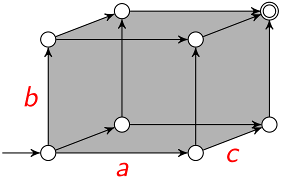
Languages of HDAs

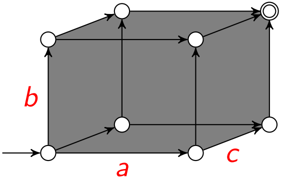
- Automata have languages
- HDA don't (hitherto)
- (focus has been on operational and *topological* aspects)

Languages of HDAs



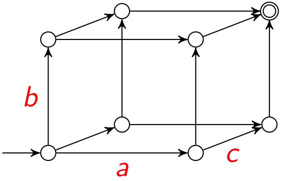
$$L_1 = \{abc, acb, bac, bca, cab, cba\}$$



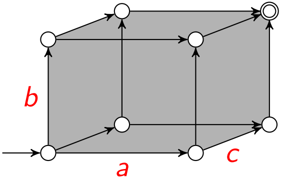


$$L_3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \dots \right\}$$

Languages of HDAs

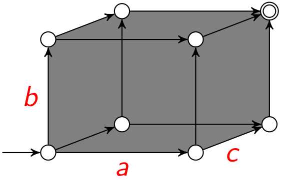


$$L_1 = \{abc, acb, bac, bca, cab, cba\}$$



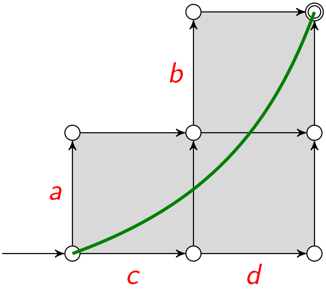
$$L_2 = \left\{ \begin{pmatrix} a \\ b \to c \end{pmatrix}, \begin{pmatrix} a \\ c \to b \end{pmatrix}, \begin{pmatrix} b \\ a \to c \end{pmatrix}, \begin{pmatrix} b \\ c \to a \end{pmatrix}, \begin{pmatrix} c \\ a \to b \end{pmatrix}, \begin{pmatrix} c \\ b \to a \end{pmatrix} \right\} \cup L_1$$

sets of pomsets



$$L_3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\} \cup L_2$$

N



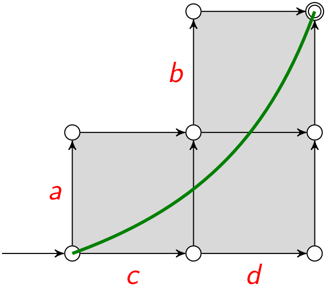
$$\left(\begin{tikzcd} a \arrow[r] & b \\ c \arrow[r] \arrow[ur] & d \end{tikzcd}\right)$$

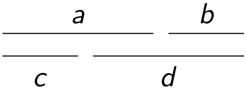
- not series-parallel!

Are all pomsets generated by HDA?

No, only (labeled) interval orders

- Poset $(P, \leq)$ is an interval order iff it has an interval representation:
  - a set $I = \{[l_i, r_i]\}$ of real intervals
  - with order $[l_i, r_i] \preceq [l_j, r_j]$ iff $r_i \leq l_j$
  - and an order isomorphism $(P, \leq) \leftrightarrow (I, \preceq)$
- [Fishburn 1970]





$$\left(\begin{tikzcd} a \arrow[r] & b \\ c \arrow[ur] \arrow[r] & d \end{tikzcd}\right)$$

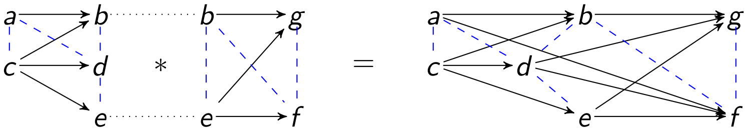
Pomsets with interfaces

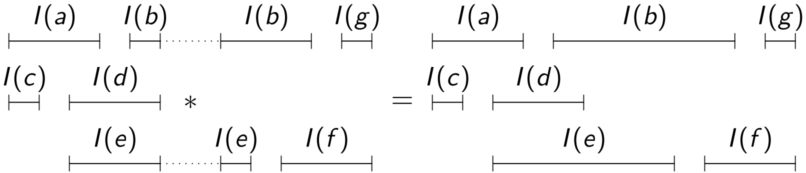
Definition (Ipomset)

A pomset with interfaces (and event order): $(P, <, \dashrightarrow, S, T, \lambda)$:

- finite set P;
- two partial orders $<$ (precedence order), $\dashrightarrow$ (event order)
  - s.t. $< \cup \dashrightarrow$ is a *total relation*;
- $S, T \subseteq P$ source and target interfaces
  - s.t. S is $<$-minimal, T is $<$-maximal.

Composition of ipomsets





- Gluing $P * Q$: P before Q, except for interfaces (which are identified)
- Parallel composition $P \parallel Q$: P above Q (disjoint union)

Languages of HDAs

- For an HDA X, $L(X)$ is a set of interval-order ipomsets
  - and closed under subsumption

MSCS

- For any interval order P, $\exists$ HDA $\square^P$ for which $L(\square^P) = \{P\}{\downarrow}$
  - and then for any HDA X, $P \in L(X)$ iff $\exists f : \square^P \to X$

Definition (Rational Languages over Σ)

- Generated by $\emptyset$, $\{\epsilon\}$, and all $\{[a]\}$, $\{[\bullet\, a]\}$, $\{[a\, \bullet]\}$, $\{[\bullet\, a\, \bullet]\}$ for $a \in \Sigma$
- under operations $\cup$, $*$, $\parallel$ and (Kleene plus) $^+$

Theorem (à la Kleene)

A language is rational iff it is recognized by an HDA.

CONCUR

Theorem (à la Myhill-Nerode)

A language is rational iff it has finite prefix quotient.

Myhill-Nerode

- $P \backslash L := \{Q \in \text{iiPoms} \mid PQ \in L\}$
- $\text{suff}(L) := \{P \backslash L \mid P \in \text{iiPoms}\}$

Theorem

L is rational iff $\textit{suff}(L)$ is finite.

Construction $L \rightsquigarrow M(L)$:

- $P \sim_L Q :\Leftrightarrow P \backslash L = Q \backslash L$
- $P \approx_L Q :\Leftrightarrow \forall A \subseteq T_P : (P - A) \backslash L = (Q - A) \backslash L$
- cells of $M(L)$ are $\approx_L$-equivalence classes

- $M(L)$ may be non-deterministic
- if L is determinizable, then $M(L)$ is deterministic (and minimal (?))
- but there exist non-determinizable languages
- in fact, there are infinitely ambiguous languages