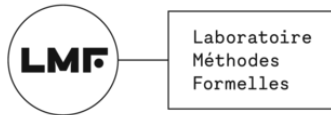


Star-Continuous Ésik Algebras

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Motivation



What is the **minimum amount of battery** required for the satellite to **always be able to send and receive messages**?

- The theory of **weighted automata** is very powerful
- Here: an application to **energy problems**

1 Semirings and Continuous Kleene Algebras

2 Semimodules and \acute{E} sik Algebras

3 Energy Problems

4 Conclusion

Semirings

A **semiring** is a structure $(S, \oplus, \otimes, 0, 1)$ such that

- $(S, \oplus, 0)$ is a commutative monoid,
 - ▶ $x \oplus y = y \oplus x$, $x \oplus (y \oplus z) = (x \oplus y) \oplus z$, $x \oplus 0 = x$
- $(S, \otimes, 1)$ is a monoid,
 - ▶ $x \otimes (y \otimes z) = (x \otimes y) \otimes z$, $x \otimes 1 = 1 \otimes x = x$
- and which satisfies distributive and annihilation laws:
 - ▶ $x(y \oplus z) = xy \oplus xz$, $(x \oplus y)z = xz \oplus yz$
 - ▶ $x \otimes 0 = 0 \otimes x = 0$

Examples:

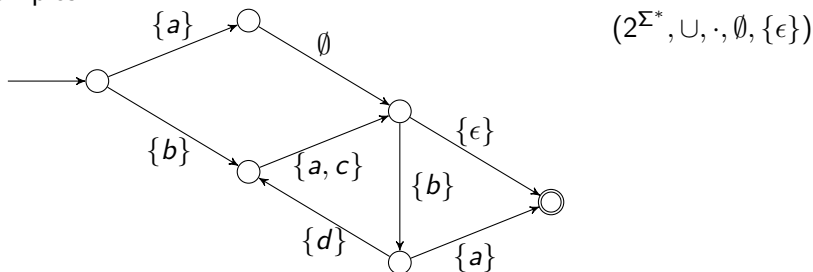
- natural numbers: $(\mathbb{N}, +, \cdot, 0, 1)$
- the boolean semiring: $(\{\mathbf{ff}, \mathbf{tt}\}, \vee, \wedge, \mathbf{ff}, \mathbf{tt})$
- max-plus algebra: $(\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$
- min-plus algebra: $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$
- languages over some alphabet Σ : $(2^{\Sigma^*}, \cup, \cdot, \emptyset, \{\epsilon\})$
- etc.

Weighted Automata

A **weighted automaton** (over a semiring S) is a structure (Q, I, K, T) :

- Q : finite set of states, $I, K \subseteq Q$ initial / accepting states
- $T \subseteq Q \times S \times Q$

Examples:



- **along** paths: \cdot
- choice **between** paths: \cup
- usual automata

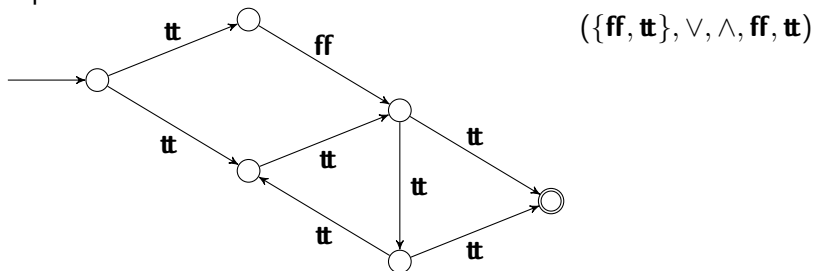
$$\underline{\underline{\{b\}\{a, c\} \cup \{b\}\{a, c\}\{b\}\{a\} \cup \dots}}$$

Weighted Automata

A **weighted automaton** (over a semiring S) is a structure (Q, I, K, T) :

- Q : finite set of states, $I, K \subseteq Q$ initial / accepting states
- $T \subseteq Q \times S \times Q$

Examples:



- **along** paths: \wedge
- choice **between** paths: \vee
- digraphs

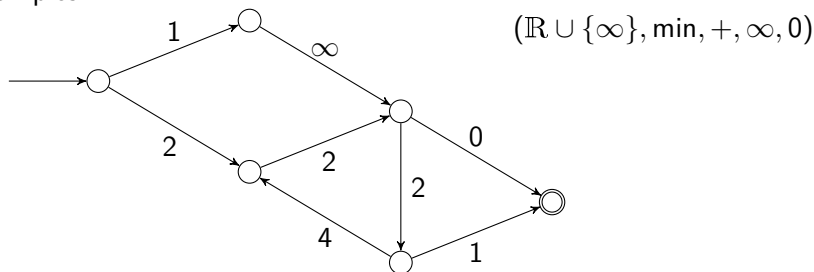
\odot is reachable

Weighted Automata

A **weighted automaton** (over a semiring S) is a structure (Q, I, K, T) :

- Q : finite set of states, $I, K \subseteq Q$ initial / accepting states
- $T \subseteq Q \times S \times Q$

Examples:



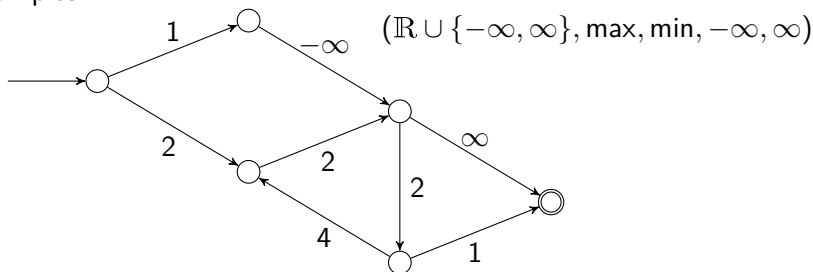
- **along** paths: $+$
- choice **between** paths: \min
- shortest path

Weighted Automata

A **weighted automaton** (over a semiring S) is a structure (Q, I, K, T) :

- Q : finite set of states, $I, K \subseteq Q$ initial / accepting states
- $T \subseteq Q \times S \times Q$

Examples:



- **along** paths: min
- choice **between** paths: max
- maximum flow

Reachability in Weighted Automata

Let $S = (S, \oplus, \otimes, 0, 1)$ be a semiring and $A = (Q, I, K, T)$ a weighted automaton over S .

- a **path** in A : $\pi = q_1 \xrightarrow{x_1} q_2 \xrightarrow{x_2} \cdots \xrightarrow{x_{n-1}} q_n$ with all $(q_i, x_i, q_{i+1}) \in T$
- the **value** of π : $|\pi| = x_1 \otimes x_2 \otimes \cdots \otimes x_{n-1}$
- π **accepting** if $q_1 \in I$ and $q_n \in K$

Definition

The **reachability value** of A is

$$|A| = \bigoplus \{ |\pi| \mid \pi \text{ accepting path in } A \}$$

- \otimes **along** paths; \oplus **between** paths
- needs some provision for infinite sums!

Complete Semirings

Definition (repeat)

The **reachability value** of A is

$$|A| = \bigoplus \{ |\pi| \mid \pi \text{ accepting path in } A \}$$

- needs some provision for infinite sums!

Definition

A semiring $(S, \oplus, \otimes, 0, 1)$ is **complete** if all infinite sums $\bigoplus X$ for $X \subseteq S$ exist.

- now the definition of $|A|$ makes sense
- but completeness is a rather restrictive condition
- we'll do something different

Continuous Kleene Algebras

From now on, restrict to **idempotent** semirings $(S, \oplus, \otimes, 0, 1)$.

- that is, $x \oplus x = x$ for all $x \in S$
- \mathbb{N} is not idempotent, but \mathbb{B} , max-plus, min-plus, max-min, 2^{Σ^*} are, as are most other important examples
- write $\vee = \oplus$ and $\perp = 0$ for emphasis

Definition

A **continuous Kleene algebra** is an idempotent semiring $(S, \vee, \otimes, \perp, 1)$ in which $\bigvee X$ exists for all $X \subseteq S$, and such that for all $Y \subseteq S$, $x, z \in S$, $x(\bigvee Y)z = \bigvee xYz$.

- a complete idempotent semiring in which multiplication distributes over infinite suprema
- again, too restrictive

Star-Continuous Kleene Algebras

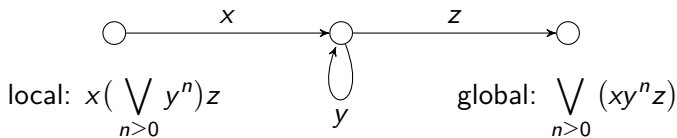
Definition (repeat)

A **continuous Kleene algebra** is an idempotent semiring $(S, \vee, \otimes, \perp, 1)$ in which $\bigvee X$ exists for all $X \subseteq S$, and such that for all $Y \subseteq S$, $x, z \in S$, $x(\bigvee Y)z = \bigvee xYz$.

Definition

A **star-continuous Kleene algebra** is an idempotent semiring $(S, \vee, \otimes, \perp, 1)$ in which $\bigvee \{x^n \mid n \geq 0\}$ exists for all $x \in S$, and such that for all $x, y, z \in S$, $x(\bigvee \{y^n \mid n \geq 0\})z = \bigvee x\{y^n \mid n \geq 0\}z$.

- **loop abstraction:**



Star-Continuous Kleene Algebras

For $x \in S$ in a star-continuous Kleene algebra S , define

$$x^* = \bigvee_{n \geq 0} x^n$$

- for languages, that's the **Kleene star**
- **poor man's inverse**: the equation

$$x^* = 1 \oplus x \oplus x^2 \oplus \dots = \frac{1}{1 - x}$$

does make surprisingly much sense!

Star-Continuous Kleene Algebras

- all continuous Kleene algebras are star-continuous, but not vice-versa
 - ▶ 2^{Σ^*} is a continuous Kleene algebra
 - ▶ the set of regular languages over Σ is star-continuous, but **not** continuous
- not all idempotent semirings are star-continuous Kleene algebras
 - ▶ counterexample is necessarily infinite

Matrix Semirings

Let S be a semiring and $n \geq 1$.

- $S^{n \times n}$: semiring of $n \times n$ matrices over S
- (with matrix addition and multiplication)
- If S is a star-continuous Kleene algebra, then so is $S^{n \times n}$
- with $M_{i,j}^* = \bigvee_{m \geq 0} \bigvee_{1 \leq k_1, \dots, k_m \leq n} M_{i,k_1} M_{k_1,k_2} \cdots M_{k_m,j}$

- and for $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ any partition,

$$M^* = \begin{bmatrix} (a \vee bd^*c)^* & (a \vee bd^*c)^* bd^* \\ (d \vee ca^*b)^* ca^* & (d \vee ca^*b)^* \end{bmatrix}$$

(recursively)

- “generalized Floyd-Warshall”

Reachability in Weighted Automata, II

Let $S = (S, \vee, \otimes, \perp, 1)$ be a star-continuous Kleene algebra and $A = (Q, I, K, T)$ a weighted automaton over S .

- transform A to **matrix form**:
 - ▶ recall $T : Q \times Q \rightarrow S$
 - ▶ write $Q = \{1, \dots, n\}$
 - ▶ then $I, K \subseteq Q$ become $\iota, \kappa \in \{\perp, 1\}^n$
 - ▶ and $T \in S^{n \times n}$: the **transition matrix**
- recall $|A| = \bigoplus \{|\pi| \mid \pi \text{ accepting path in } A\}$

Theorem

$$|A| = \iota T^* \kappa$$

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Motivation: Büchi Conditions in Weighted Automata

Let $A = (Q, I, K, T)$ be a weighted automaton over a semiring S

- an **infinite path** in A : $\pi = q_1 \xrightarrow{x_1} q_2 \xrightarrow{x_2} \dots$ with all $(q_i, x_i, q_{i+1}) \in T$
- π **Büchi** if $q_1 \in I$ and

$$\{q \in Q \mid \forall n \geq 0 : \exists i \geq n : q_i = q\} \cap K \neq \emptyset$$

Goal: make sense of the “definition”

$$\|A\| = \bigoplus \{\|\pi\| \mid \pi \text{ Büchi path in } A\}$$

- but what is the value $\|\pi\|$ of an infinite path? an **infinite product**?
- and, how to compute the sum \bigoplus ?

Semiring-Semimodule Pairs

- **semiring** $S = (S, \oplus, \otimes, 0, 1)$
- plus commutative **monoid** $V = (V, \oplus, 0)$
- **left S -action** $S \times V \rightarrow V, (s, v) \mapsto sv$
- such that for all $s, s' \in S, v \in V$:

$$(s \oplus s')v = sv \oplus s'v$$

$$(ss')v = s(s'v)$$

$$s0 = 0$$

$$s(v \oplus v') = sv \oplus sv'$$

$$0s = 0$$

$$1v = v$$

- (think of vector spaces over fields, or modules over rings)

Definition

An **Ésik algebra** is an idempotent semiring-semimodule pair (S, V) together with an **infinite product** $\prod : S^\omega \rightarrow V$, such that

- for all $x_0, x_1, \dots \in S$, $\prod x_n = x_0 \prod x_{n+1}$;
 - for any sequence $x_0, x_1, \dots \in S$ and any sequence $0 = n_0 \leq n_1 \leq \dots$ which increases without a bound, let $y_k = x_{n_k} \cdots x_{n_{k+1}-1}$ for all $k \geq 0$; then $\prod x_n = \prod y_k$.
-
- that is, \prod generalizes the finite product in S
 - a new name for an old notion (Zoltán Ésik died in 2016)
 - S for values of **finite** paths; V for values of **infinite** paths:

$$\| q_1 \xrightarrow{x_1} q_2 \xrightarrow{x_2} \cdots \| = \prod x_{n-1}$$

Continuous Ésik Algebras

Definition

An Ésik algebra (S, V, \prod) is **continuous** if

- S is a **continuous Kleene algebra** and V is a **complete lattice**,
- the S -action on V **preserves all suprema** in either argument, and
- for all $X_0, X_1, \dots \subseteq S$, $\prod(\bigvee X_n) = \bigvee \{ \prod x_n \mid x_n \in X_n, n \geq 0 \}$.

- Ésik-Kuich 2004
- “continuous” \implies **too restrictive**

Star-Continuous Ésik Algebras

Definition

An Ésik algebra (S, V, Π) is **star-continuous** if

- S is a **star-continuous Kleene algebra**,
- for all $x, y \in S, v \in V, xy^*v = \bigvee_{n \geq 0} xy^n v$,
- for all $x_0, x_1, \dots, y, z \in S, \prod (x_n(y \vee z)) = \bigvee_{x'_0, x'_1, \dots \in \{y, z\}} \prod x_n x'_n$,
- for all $x, y_0, y_1, \dots \in S, \prod x^* y_n = \bigvee_{k_0, k_1, \dots \geq 0} \prod x^{k_n} y_n$.

- Ésik-Fahrenberg-Legay-Quaas 2015

Matrix Semiring-Semimodule Pairs

Let (S, V) be a semiring-semimodule pair and $n \geq 1$.

- $(S^{n \times n}, V^n)$ is again a semiring-semimodule pair
- (the action is matrix-vector product)
- if (S, V) is a star-continuous Ésik algebra, then there is an operation ${}^\omega : S^{n \times n} \rightarrow V^n$ given by

$$M_i^\omega = \bigvee_{1 \leq k_1, k_2, \dots \leq n} M_{i, k_1} M_{k_1, k_2} \cdots$$

- ▶ (not a general infinite product)

- and for $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ any partition,

$$M^\omega = \begin{bmatrix} (a \vee bd^*c)^\omega \vee (a \vee bd^*c)^*bd^\omega \\ (d \vee ca^*b)^\omega \vee (d \vee ca^*b)^*ca^\omega \end{bmatrix}$$

(recursively)

Büchi Conditions in Weighted Automata, II

Let (S, V) be a star-continuous Ésik algebra and $A = (n, \iota, \kappa, T)$ a weighted automaton over S .

- reorder $Q = \{1, \dots, n\}$ so that $\kappa = (1, \dots, 1, \perp, \dots, \perp)$
 - ▶ that is, now the first $k \leq n$ states are accepting
- write $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $a \in S^{k \times k}$

Theorem

$$\|A\| = \iota \begin{bmatrix} (a + bd^*c)^\omega \\ d^*c(a + bd^*c)^\omega \end{bmatrix}$$

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Energy Problems



What is the **minimum amount of battery** required for the satellite to **always be able to send and receive messages**?

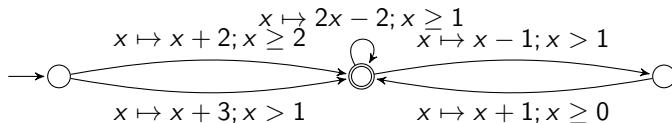
Energy Automata

Energy function:

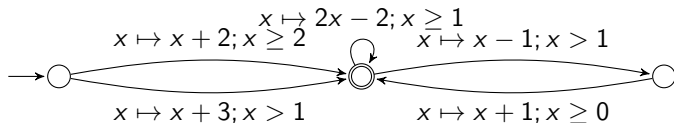
- partial function $f : \mathbb{R}_{\geq 0} \hookrightarrow \mathbb{R}_{\geq 0}$
- which is defined on some closed interval $[l_f, \infty[$ or on some open interval $]l_f, \infty[$,
- and such that for all $x \leq y$ for which f is defined,

$$f(y) - f(x) \geq y - x$$

Energy automaton: finite automaton labeled with energy functions



Energy Automata, Semantically



- start with **initial energy** x_0 and update at transitions according to label function
- if label function **undefined** on input, transition is **disabled**
- a discrete-time hybrid automaton (?)

Reachability: Given x_0 , does there exist an accepting (finite) run with initial energy x_0 ?

Büchi: Given x_0 , does there exist a Büchi (infinite) run with initial energy x_0 ?

Reachability in Energy Automata

- Let $L = [0, \infty]_{\perp}$: extended nonnegative real numbers plus \perp (for “undefined”)
 - ▶ (a complete lattice)
- **Extended energy function**: function $f : L \rightarrow L$
- with $f(\perp) = \perp$, and $f(\infty) = \infty$ unless $f(x) = \perp$ for all $x \in L$,
- and $f(y) - f(x) \geq y - x$ for all $x \leq y$.
- Set \mathcal{E} of such functions is an idempotent semiring with operations \vee (pointwise max) and \circ (composition)
- in fact, a star-continuous Kleene algebra
 - ▶ $f^*(x) = x$ if $f(x) \leq x$; $f^*(x) = \infty$ if $f(x) > x$
 - ▶ **not** a continuous Kleene algebra

Theorem (Reachability)

There exists an accepting run from initial energy x_0 iff $|A|(x_0) \neq \perp$.

Büchi Runs in Energy Automata

- Let $\mathbf{2} = \{\mathbf{ff}, \mathbf{tt}\}$: the Boolean lattice
- Let \mathcal{V} be the set of monotone and \top -continuous functions $L \rightarrow \mathbf{2}$
 - ▶ $f : L \rightarrow \mathbf{2}$ is called \top -continuous if $f(x) \equiv \mathbf{ff}$ or for all $X \subseteq L$ with $\bigvee X = \infty$, also $\bigvee f(X) = \mathbf{tt}$.
- $(\mathcal{E}, \mathcal{V})$ is an idempotent semiring-semimodule pair
- Define $\prod : \mathcal{E}^\omega \rightarrow \mathcal{V}$ by

$$(\prod f_n)(x) = \mathbf{tt} \text{ iff } \forall n \geq 0 : f_n(f_{n-1}(\cdots(x)\cdots)) \neq \perp$$

- Lemma: $\prod f_n$ is indeed \top -continuous for all $f_0, f_1, \dots \in \mathcal{E}$
- Theorem: $(\mathcal{V}, \mathcal{E})$ is a star-continuous Ésik algebra
 - ▶ **not** a continuous Ésik algebra

Theorem (Büchi)

There exists a Büchi run from initial energy x_0 iff $\|A\|(x_0) \neq \mathbf{ff}$.

Computability

Let $\mathcal{E}' \subseteq \mathcal{E}$, $\mathcal{V}' \subseteq \mathcal{V}$ such that

- \mathcal{E}' is closed under \vee , \circ , and $*$
- \mathcal{V}' is closed under \vee and contains all infinite products of elements of \mathcal{E}'
- all elements of \mathcal{E}' and \mathcal{V}' are **finitely representable**

Theorem

Reachability and Büchi acceptance are decidable for \mathcal{E}' -weighted energy automata.

The above holds for example for \mathcal{E}' all **piecewise linear** energy functions.






Conclusion

- semirings and weighted automata: a very versatile framework
 - ▶ barely touched applications here
 - ▶ see Droste, Kuich, Vogler (eds.): Handbook of Weighted Automata, Springer 2009
- star-continuous \mathcal{E} sik algebras: a useful generalization of continuous \mathcal{E} sik algebras
 - ▶ (like star-continuous Kleene algebras are a useful generalization of continuous Kleene algebras)
- can be used to solve general energy problems

Ongoing work:

- **real-time** energy problems (FORMATS 2008; FM 2018; LMCS 2019; FAC 2025)
- hybrid systems?
- non-idempotent case?

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