Discrete and Continuous Models for Concurrent Systems

1. The Geometry of Concurrency

Uli Fahrenberg

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Introduction

¿ Discrete and Continuous Models for Concurrent Systems ?

Introduction

Discrete and Continuous Models for Concurrent Systems

Introduction

Discrete and Continuous Models for Concurrent Systems

- plane ticket reservation system
- fleet of robots
- your computer!

Introduction

Discrete and Continuous Models for Concurrent Systems

Introduction

Discrete and Continuous Models for Concurrent Systems

- networks of automata
- Petri nets
- event structures
- higher-dimensional automata

Introduction

Discrete and Continuous Models for Concurrent Systems

Introduction

Discrete and Continuous Models for Concurrent Systems

- Variants of directed topological spaces

Introduction

¿ Uli Fahrenberg ?

Introduction

Uli Fahrenberg

- 2005: PhD, *Aalborg* University, DK, Directed algebraic topology
- 2005-10: Aalborg University, Timed automata
- 2010-16: University of *Rennes*, FR, Specification theories
- 2016-21: École polytechnique, *Paris*, Geometry and concurrency

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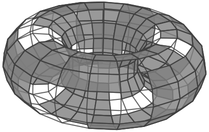
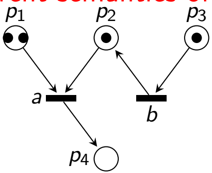
- 2016-21: École polytechnique, *Paris*, Geometry and concurrency
- 2021-: EPITA, *Rennes/Paris*, Automata and concurrency

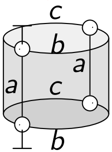
1. The geometry of concurrency



2. Concurrent semantics of Petri nets



3. Languages of higher-dimensional aut.



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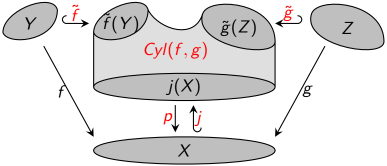

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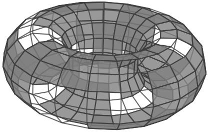

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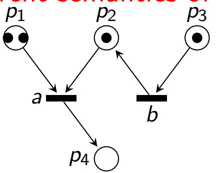

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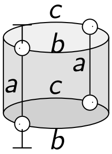

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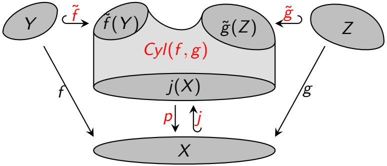


1 Introduction

2 Geometric Semantics

3 Directed Algebraic Topology

Algebraic View

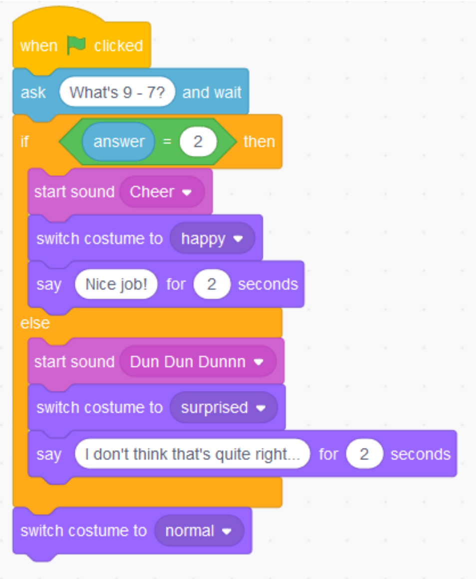
A program is

- a sequence of instructions
- plus branches
- and loops

Kleene algebra: set S with

- concatenation $\otimes$
- choice $\oplus$
- repetition *

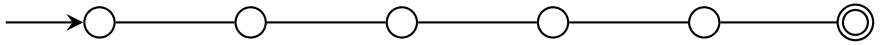
- idempotent semiring with unary * which computes fixed points



Geometric View

A program is a sequence of instructions

- ignoring branches and loops for now



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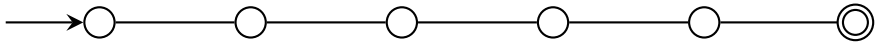

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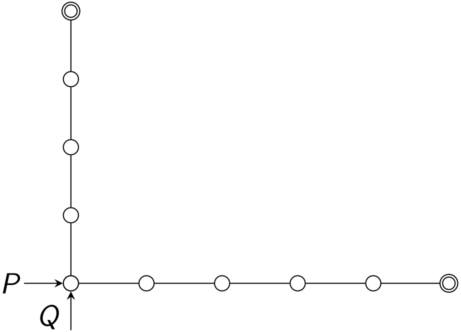

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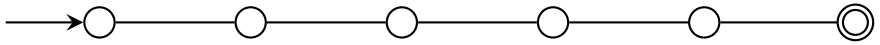

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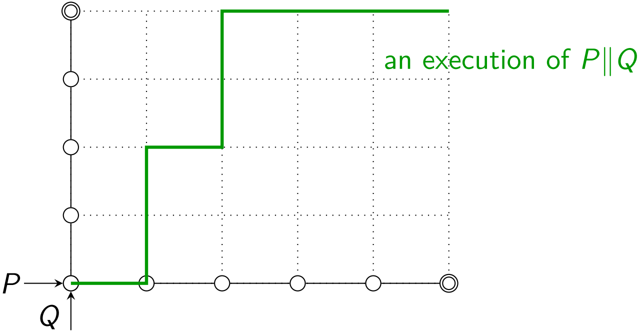

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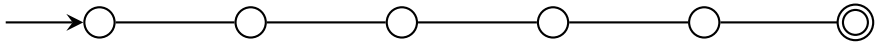

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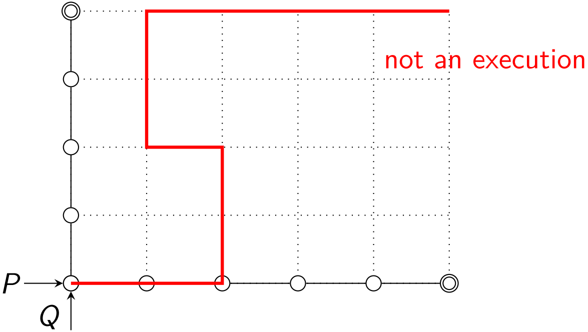

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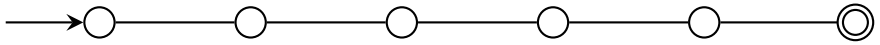

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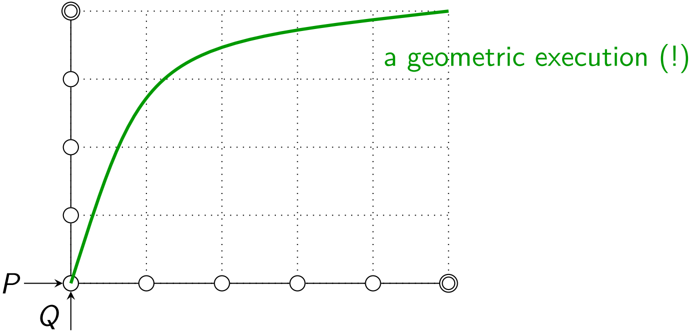

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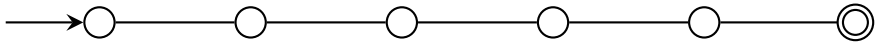

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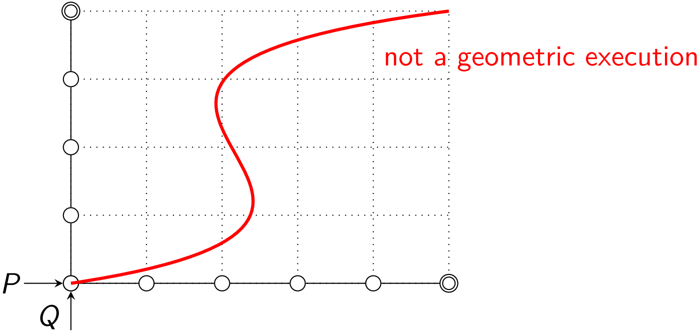
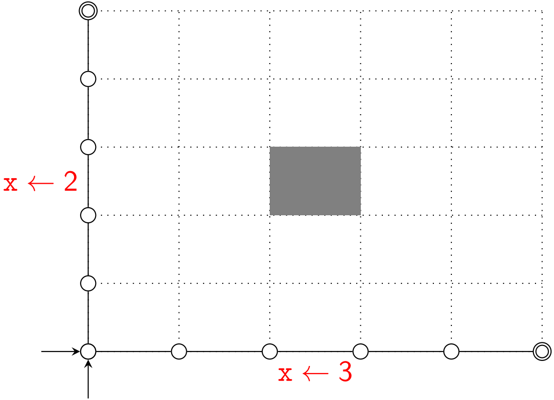


Holes

Adding mutual exclusion:



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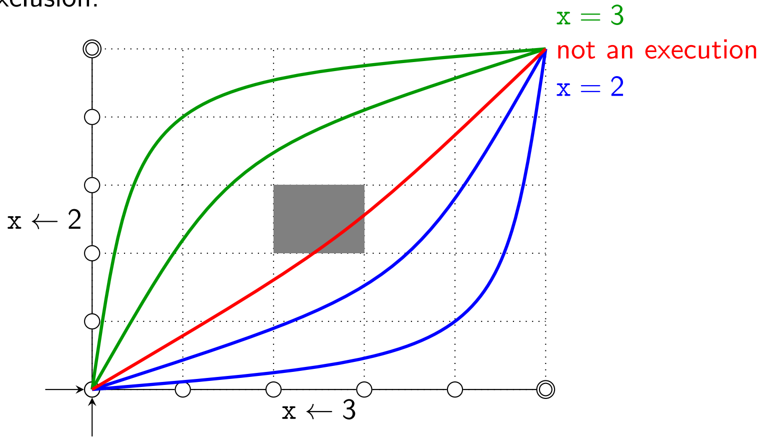

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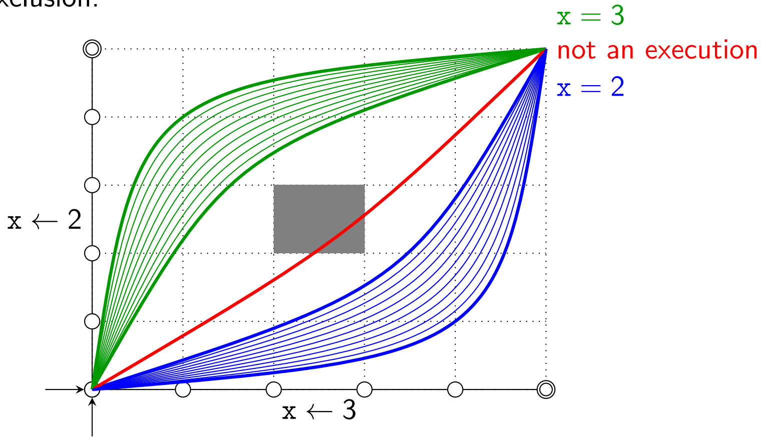
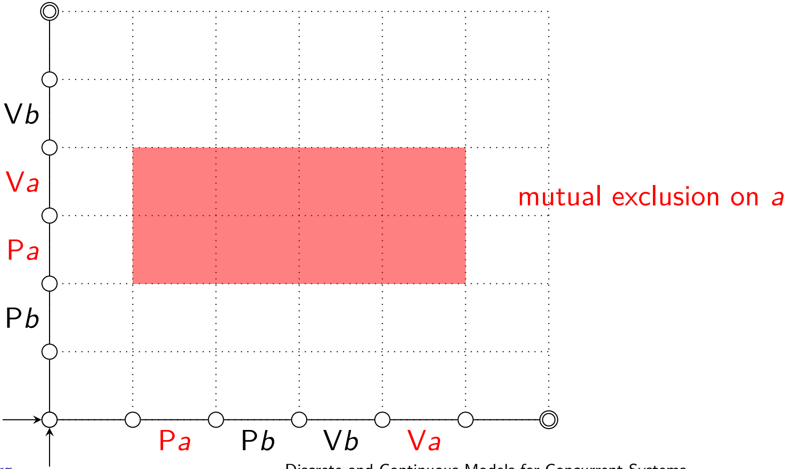


- homotopic paths ≙ equivalent executions

More Holes

Semaphores à la Dijkstra (P $\triangleq$ acquire; V $\triangleq$ release):



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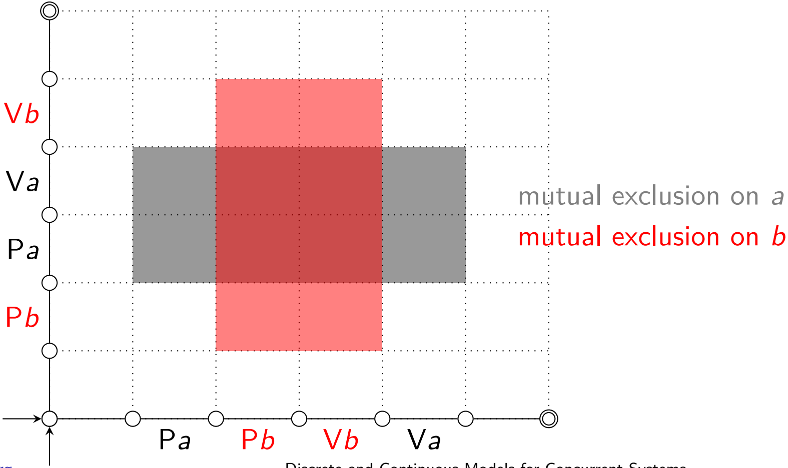

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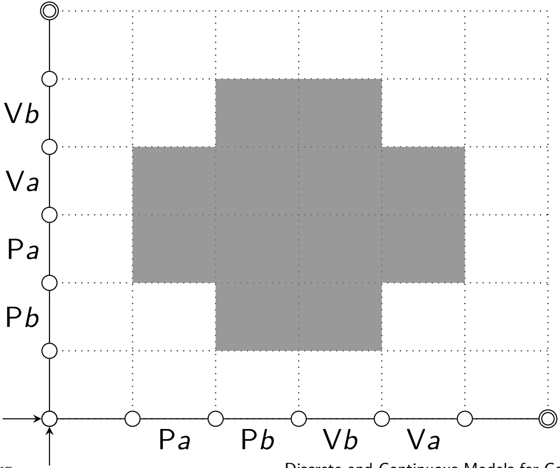

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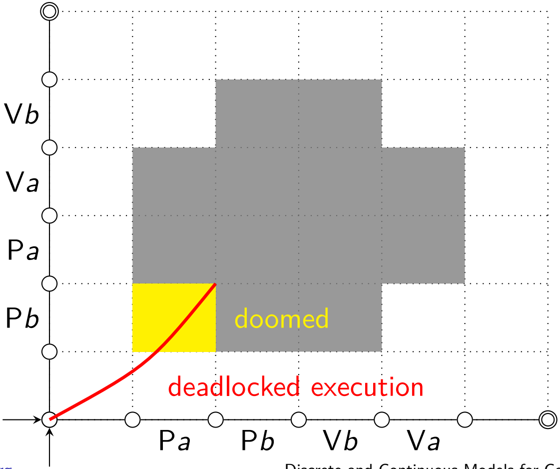

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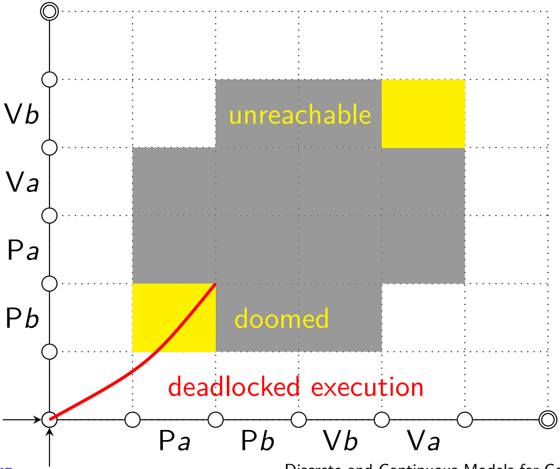
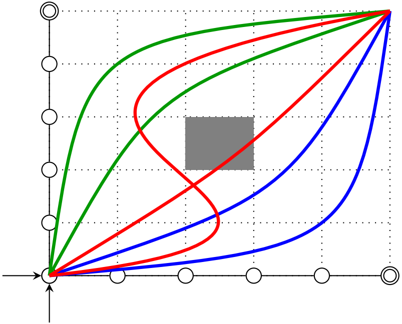
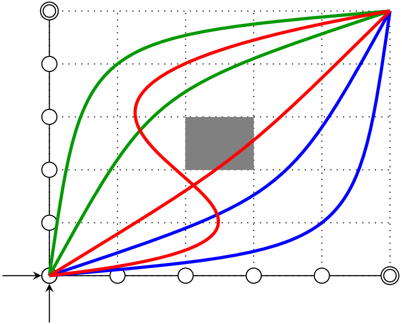


Summing Up



- A program is a topological space
- An execution is a path through said space
- Two executions are equivalent iff their paths are homotopic
- Deadlocks and unreachable states are concave corners

Summing Up



- A program is a directed topological space
- An execution is a directed path through said space
- Two executions are equivalent iff their dipaths are dihomotopic
- Deadlocks and unreachable states are concave corners

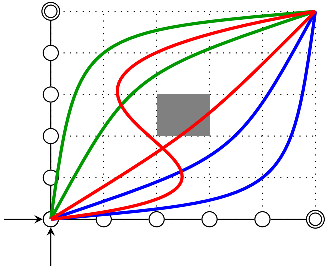
1. Introduction
2. Geometric Semantics
3. Directed Algebraic Topology

Directed Spaces

Definition (po-space)

A partially ordered space is a topological space X together with a partial order $\leq$ on X such that $\leq \subseteq X \times X$ is *closed* in the product topology.
A morphism of po-spaces is a $\leq$-preserving continuous function.

- a dipath in a po-space is a continuous & monotone path
  - a morphism $\vec{I} \to X$
- directed interval $\vec{I} = [0, 1]$ with usual order
- directed square $\vec{I} \times \vec{I}$, cube, etc.
- concatenation $\otimes$, branching $\oplus$
- no loops

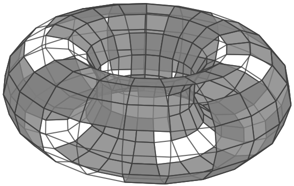


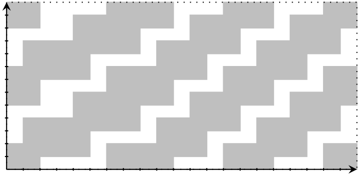
Directed Spaces

Definition (lpo-space)

A locally partially ordered space is a *Hausdorff* topological space X together with a relation $\leq$ on X in which any $x \in X$ has an open neighborhood $U \ni x$ such that the restriction of $\leq$ to U is a closed partial order.
A morphism of po-spaces is a continuous function which is *locally* $\leq$-preserving.

- $\leq$ may be taken globally reflexive and transitive





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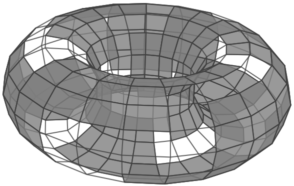
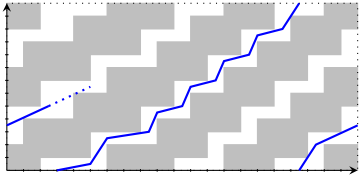
- $\leq$ may be taken globally reflexive and transitive
- a dipath is a morphism $\vec{I} \to X$





Directed Spaces

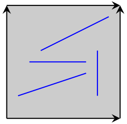
Definition (d-space)

A directed space is a topological space X together with a set $\vec{P}X$ of paths $I \to X$, called directed paths, such that

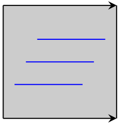
- all constant paths are directed,
- concatenations of directed paths are directed, and
- reparametrizations and restrictions of directed paths are directed.

A morphism of d-spaces is a continuous function which preserves directed paths.

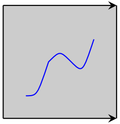
- a dipath is a morphism $p : \vec{I} \to X$, equivalently $p \in \vec{P}X$



$\vec{I} \times \vec{I}$



$\vec{I} \times I$



$\vec{I} \times \tilde{I}$

Directed Spaces

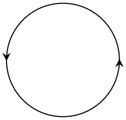
Definition (d-space; unpacked)

A directed space is a topological space X together with a set $\vec{P}X \subseteq \text{Top}(I, X)$ such that

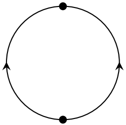
- $\forall p \in X : \lambda x.p \in \vec{P}X$ [const]
- $\forall \alpha, \beta \in \vec{P}X : \alpha(1) = \beta(0) \implies \alpha * \beta \in \vec{P}X$ [conc]
- $\forall \alpha \in \vec{P}X, \rho : \vec{I} \to \vec{I} : \alpha \circ \rho \in \vec{P}X$ [rep-rest]

A morphism of d-spaces X, Y is $f \in \text{Top}(X, Y)$ such that $\forall \alpha \in \vec{P}X : f \circ \alpha \in \vec{P}Y$.

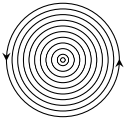
Resulting category dTop is complete and cocomplete (and dLCHaus is cartesian closed).



$\vec{S}_1$



$\vec{O}_1$



$C(\vec{S}_1)$

origin is a vortex

Directed (?) Intervals

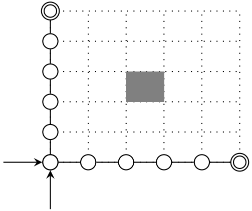
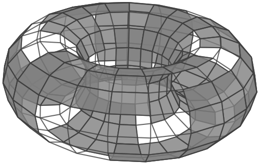
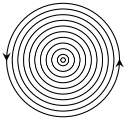
Three types of intervals:

- $\vec{I}$: $I = [0,1]$ with usual order
  - po-space; lpo-space
  - as d-space: $\vec{P}\vec{I}$: all monotone paths
- I: trivial order: $x \leq y$ iff $x = y$
  - po-space; lpo-space
  - as d-space: $\vec{P}I$: only constant paths
- $\tilde{I}$: chaotic d-space
  - d-structure: $\vec{P}\tilde{I}$: all paths
  - not an lpo-space (every point is a vortex!), neither a po-space

Directed Spaces, Summary

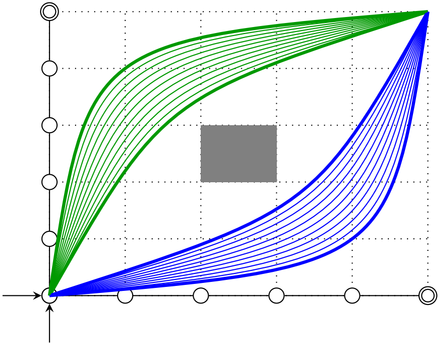
- po-spaces: partially ordered topological spaces: no loops, nice but restrictive
- lpo-spaces: locally partially ordered top. spaces: loops OK, difficult to work with
- d-spaces: top. spaces with distinguished paths: nice category, but include vortices
- (other models exist)

- po-spaces $\hookrightarrow$ lpo-spaces $\hookrightarrow$ d-spaces (not full as categories)

Directed Paths and Homotopies

- the directed interval $\vec{I}$: $([0,1], \leq)$ (usual order): po-space
- dipaths in X: morphisms $\vec{I} \to X$
  - for d-space $(X, \vec{P}X)$: dipaths $\triangleq \vec{P}X$
- a dihomotopy $H : \tilde{I} \times \vec{I} \to X$:
  - all $H(s, \cdot)$ dipaths
  - $H : I \times I \to X$ continuous
  - $H(\cdot, 0)$ and $H(\cdot, 1)$ constant
  - (some variants exist)



The Fundamental Category

- central object in algebraic topology: the fundamental group of a space X
- for $x \in X$, $\pi_1(X, x) = \{\alpha : I \to X \mid \alpha(0) = \alpha(1) = x\}$ modulo homotopy
- captures all information on homotopy of paths / 1-dimensional holes

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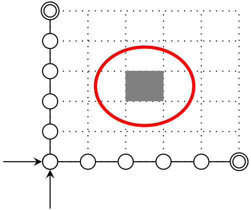
- in d-spaces, loops carry little information!



Definition (fundamental category)

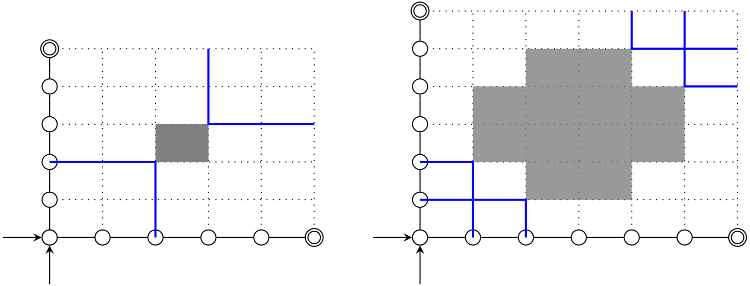
The fundamental category $\vec{\pi}_1(X)$ of a d-space X has as

- objects points $x \in X$,
- morphisms $\vec{\pi}_1(X)(x, y) = \{\alpha : \vec{I} \to X \mid \alpha(0) = x, \alpha(1) = y\}$ modulo dihomotopy

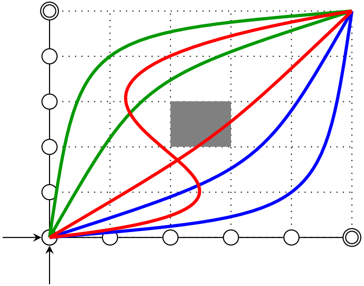
The Fundamental Category

The fundamental cat $\vec{\pi}_1(X)$ of a d-space X has as objects points $x \in X$ and as morphisms $\vec{\pi}_1(X)(x, y) = \{\alpha : \vec{I} \to X \mid \alpha(0) = x, \alpha(1) = y\}$ modulo dihomotopy.

- related: fundamental groupoid of topological spaces ("blow-up" of fundamental group)
- $\vec{\pi}_1(X)$ is huge $\implies$ identify components "where nothing happens"



Summing Up



- A program is a directed topological space X
  - po-space, lpo-space, d-space, etc.
- An execution is a directed path $\vec{I} \to X$
- Two executions are equivalent iff they are related by a dihomotopy $\tilde{I} \times \vec{I} \to X$
- The fundamental category: useful invariant, but too big

- Directed homotopy equivalence; directed coverings; directed homology; directed topological complexity; etc.

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