TD AAA : Weighted Automata

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Bonus exercises are more difficult, and some of them might take a long time. Better keep them for the end...

1 Semirings

1.1 Exercise

Give a detailed proof that $(\mathbb{N}, +, \cdot, 0, 1)$ forms a semiring.

Solution

Trivial.

1.2 Exercise

Which of the following structures form semirings?

1. $(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$
2. $(\mathbb{N} \cup \{-\infty\}, +, \max, 0, -\infty)$
3. $(\mathbb{N} \cup \{\infty\}, \max, \min, 0, \infty)$
4. $(\mathbb{N} \cup \{\infty\}, \min, \max, \infty, 0)$

Solution

1. Yes
2. No: no distributivity
3. Yes: this is called the *max-min* semiring
4. Yes: the max-min semiring is self-dual

1.3 Exercise (bonus)

Let $\mathcal{F}$ be the set of functions $\mathbb{R}_+ \to \mathbb{R}_+$ (where $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$). Let max be pointwise maximum on $\mathcal{F}$, that is, $(\max(f, g))(x) = \max(f(x), g(x))$. Let $f \circ g$ denote usual function composition, that is, $(f \circ g)(x) = f(g(x))$.

1. What are the identity elements $\mathbb{0}$ and $\mathbb{1}$?
2. $(\mathcal{F}, \max, \circ, \mathbb{0}, \mathbb{1})$ does *not* form a semiring. Why?
3. There is an obvious subset $\mathcal{G} \subset \mathcal{F}$ such that $(\mathcal{G}, \max, \circ, \mathbb{0}, \mathbb{1})$ does form a semiring. Which is it?

Solution

1. $\mathbb{0} = \lambda x.0$, $\mathbb{1} = \lambda x.x$

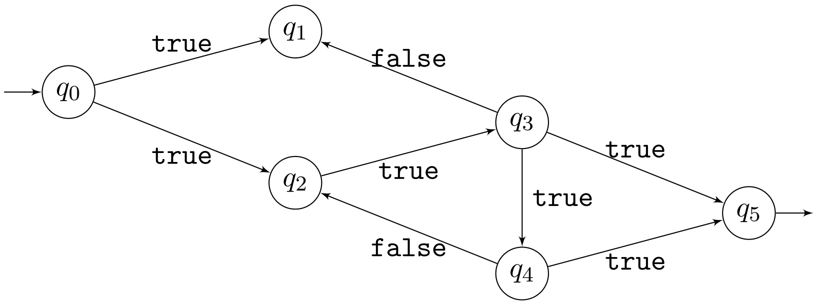
2. Left distributivity fails: we have $(\max(f \circ g, f \circ h))(x) = \max(f(g(x)), f(h(x)))$ and $(f \circ \max(g, h))(x) = f(\max(g(x), h(x)))$, and these two may not be equal if f is not increasing

3. The subset of increasing functions: $\mathcal{G} = \{f \in \mathcal{F} \mid x \leq y \Longrightarrow f(x) \leq f(y)\}$

2 Weighted automata

2.1 Exercise

Let A be the following automaton over the boolean semiring:



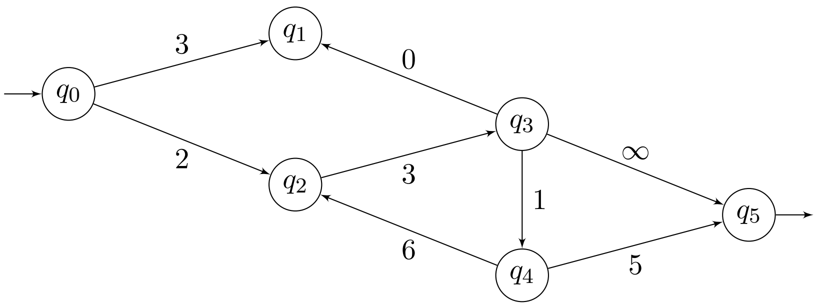
What is $|A|$?

Solution

`true`

2.2 Exercise

Let A be the following automaton over the semiring $(\mathbb{N} \cup \{\infty\}, \max, \min, 0, \infty)$:



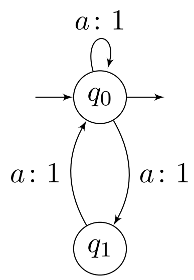
What is $|A|$?

Solution

2

2.3 Exercise

Let A be the following automaton with input over the semiring $(\mathbb{N}, +, \cdot, 0, 1)$:



What is $|A|(a^6)$?

Solution

For $F_0 = 0, F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$, we have $|A|(a^k) = F_{k+1}$ and $|A|(a^6) = 13$.

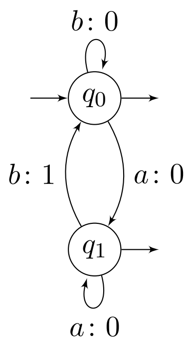
2.4 Exercise

Let $|w|_x$ denote the number of occurrences of substring x in word w and consider the semirings $S_1 = (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ and $S_2 = (\mathbb{N}, +, \cdot, 0, 1)$.

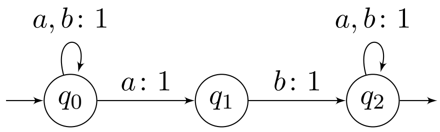
Find automata with input A_1, A_2, over the alphabet $\{a, b\}$ and over S_1, S_2, respectively and with $|A_i|(w) = |w|_{ab}$.

Solution

A_1:



A_2:



3 Small semirings

3.1 Exercise

Show that there are precisely two semirings S with two elements ($|S| = 2$). What are they? For each of the two S:

1. Describe weighted automata over S.
2. Describe $S\langle\langle\Sigma^*\rangle\rangle$.
3. Describe weighted automata over $S\langle\langle\Sigma^*\rangle\rangle$.

Hint: start by writing down the addition and multiplication tables for S, filling in all the cells which are given by the axioms.

Solution

S_1 is the boolean semiring, $S_2 = \mathbb{Z}/2\mathbb{Z}$. (The only difference is in the value of $1 + 1$)

For S_1:

1. See exo 2.1: if A is an S_1-weighted automaton, then $|A| = \texttt{true}$ iff a final state is reachable using only `true`-labeled transitions
2. $S_1\langle\langle\Sigma^*\rangle\rangle = \mathcal{P}(\Sigma^*)$: a function f from Σ^* to booleans is the same as the subset of Σ^* on which f has value `true`
3. ... hence $S_1\langle\langle\Sigma^*\rangle\rangle$-weighted automata are usual finite automata

For S_2:

1. If A is an S_2-weighted automaton, then $|A| = 1$ iff A admits an odd number of accepting paths using only 1-labeled transitions
2. the elements of $S_2\langle\langle\Sigma^*\rangle\rangle$ are "mod-2 multisets": like sets, with intersection $X \cap Y$ as usual; but for $X \cup Y$, elements which occur in both X and Y are thrown out. (In other words, $\cup$ is like `xor`)
3. If A is an $S_2\langle\langle\Sigma^*\rangle\rangle$-weighted automaton, then $|A|(w) = 1$ iff there is an odd number of accepting 1-labeled paths on w

3.2 Exercise (bonus)

Find all semirings with three elements. Are any of them non-commutative?

Solution

Let's write $S = \{0, 1, a\}$, then we have the following underspecified addition and multiplication tables:

$+$	0	1	a
0	0	1	a
1	1		
a	a		

$\cdot$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	

This already answers the second question: we see that multiplication in S is commutative: so no, all of them are commutative.

Now we need to decide the values of $1 + 1$, $1 + a$, $a + 1$, $a + a$, and aa. But $1 + a = a + 1$ and $a + a = a(1 + 1)$, so that leaves only three variables. Let's call them $x = 1 + 1$, $y = 1 + a$, and $z = aa$, and insert them into our tables:

$+$	0	1	a
0	0	1	a
1	1	x	y
a	a	y	ax

$\cdot$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	z

Now let's collect the constraints that the semiring axioms imply on our variables. We have already taken care of the identity and annihilation laws; $\cdot$ is already commutative; and associativity of $\cdot$ adds no new constraints. (Why?) That leaves associativity of $+$ and distributivity:

$$1 + 1 + a = x + a = 1 + y \qquad \Longrightarrow \qquad a + x = 1 + y \tag{1}$$

$$1 + a + a = y + a = 1 + ax \qquad \Longrightarrow \qquad a + y = 1 + ax \tag{2}$$

No other associativity constraints are interesting. (Why?)

For distributivity:

$$a(1+a) = ay = a + aa = a + z \qquad \Longrightarrow \qquad a + z = ay \tag{3}$$

$$a(a+a) = aax = aa + aa = z + z \qquad \Longrightarrow \qquad z + z = zx \tag{4}$$

(We have already used the expansion of $a(1+1)$.)

There are no other constraints. (Convince yourself that this is true.)

Now let's get to it.

1. Case $x = 0$. By (1) this implies $1+y = a$. But $1+0 = 1$ and $1+1 = x = 0$, so the only space left in the table is $1+a = y$, hence $y = a$. But then by (2), $0 = ax = a+a = a+y = 1+ax = 1+0 = 1$, a contradiction.

2. Case $x = 1$. By (1), $1 + y = a + 1 = y$, hence $y \neq 0$.

   i. Case $y = 1$. (2) is now trivially satisfied, and (3) gives $a + z = a$, forcing $z \neq 1$. (4) becomes $z + z = z$, which is satisfied for both $z = 0$ and $z = a$. Hence, two solutions: $(x, y, z) = (1, 1, 0)$ and $(x, y, z) = (1, 1, a)$.

   ii. Case $y = a$. Again (2) is trivially satisfied. (3) gives $a + z = aa = z$; but $a + 0 = a$ and $a + 1 = a$, so the only possibility is $z = a$, which also satisfies (4). One solution, $(x, y, z) = (1, a, a)$.

3. Case $x = a$. Now by (1), $1 + y = a + x = a + a = ax = aa = z$. Nothing else from the other equations.

   i. Case $y = 0$. Then $z = 1$, giving a contradiction to (4) which wants $1 + 1 = 1$.

   ii. Case $y = 1$. Then $z = 1 + 1 = a$, and the other equations hold. One solution: $(x, y, z) = (a, 1, a)$.

   iii. Case $y = a$. Then $z = 1 + a = y = a$, and the other equations hold. One solution: $(x, y, z) = (a, a, a)$.

Altogether we have found five solutions: five different semirings with three elements. For completeness, their addition and multiplication tables:

$+$	0	1	a
0	0	1	a
1	1	1	1
a	a	1	a

$\cdot$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	0

$+$	0	1	a
0	0	1	a
1	1	1	1
a	a	1	a

$\cdot$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	a

$+$	0	1	a
0	0	1	a
1	1	1	a
a	a	a	a

$\cdot$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	a

$+$	0	1	a
0	0	1	a
1	1	a	1
a	a	1	a

$\cdot$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	a

$+$	0	1	a
0	0	1	a
1	1	a	a
a	a	a	a

$\cdot$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	a

4 Star continuity

4.1 Exercise

Prove the lemma on p.70 of the slides of CM 1:

Lemma: If S is star-continuous, then $a^* = aa^* + 1 = a^*a + 1$ for all $a \in S$.

Solution

$aa^* + 1 = a\big(\bigoplus_{n\geq 0} a^n\big) + 1 = \bigoplus_{n\geq 0} aa^n + 1$ (by infinite distributivity), $= \bigoplus_{n\geq 1} a^n + 1 = \bigoplus_{n\geq 0} a^n = a^*$

4.2 Exercise (bonus)

Use the recursive algorithm to compute stars of matrices, to compute

$$\begin{bmatrix} \{a\} & \{a,b\} & \emptyset \\ \emptyset & \emptyset & \{a\} \\ \{a\} & \emptyset & \{b\} \end{bmatrix}^*$$

in the language semiring.

Solution

For example, splitting off the top-left corner: let

$$\begin{aligned} \alpha &= \{a\} \cup \begin{bmatrix} \{a,b\} & \emptyset \end{bmatrix} \begin{bmatrix} \emptyset & \{a\} \\ \emptyset & \{b\} \end{bmatrix}^* \begin{bmatrix} \emptyset \\ \{a\} \end{bmatrix} \\ \delta &= \begin{bmatrix} \emptyset & \{a\} \\ \emptyset & \{b\} \end{bmatrix} \cup \begin{bmatrix} \emptyset \\ \{a\} \end{bmatrix} \{a\}^* \begin{bmatrix} \{a,b\} & \emptyset \end{bmatrix} \end{aligned}$$

Then

$$\begin{bmatrix} \{a\} & \{a,b\} & \emptyset \\ \emptyset & \emptyset & \{a\} \\ \{a\} & \emptyset & \{b\} \end{bmatrix}^* = \begin{bmatrix} \alpha^* & \alpha^* \begin{bmatrix} \{a,b\} & \emptyset \end{bmatrix} \begin{bmatrix} \emptyset & \{a\} \\ \emptyset & \{b\} \end{bmatrix}^* \\ \delta^* \begin{bmatrix} \emptyset \\ \{a\} \end{bmatrix} \{a\}^* & \delta^* \end{bmatrix}.$$

Further

$$\begin{aligned} \alpha &= \{a\} \cup \begin{bmatrix} \{a,b\} & \emptyset \end{bmatrix} \begin{bmatrix} \{\epsilon\} & \{a\}\{b\}^* \\ \emptyset & \{b\}^* \end{bmatrix} \begin{bmatrix} \emptyset \\ \{a\} \end{bmatrix} \\ &= \{a\} \cup \begin{bmatrix} \{a,b\} & \emptyset \end{bmatrix} \begin{bmatrix} \{a\}\{b\}^*\{a\} \\ \{b\}^*\{a\} \end{bmatrix} \\ &= \{a\} \cup \{a,b\}\{a\}\{b\}^*\{a\}\,, \end{aligned}$$

and

$$\begin{aligned} \delta &= \begin{bmatrix} \emptyset & \{a\} \\ \emptyset & \{b\} \end{bmatrix} \cup \begin{bmatrix} \emptyset \\ \{a\} \end{bmatrix} \{a\}^* \begin{bmatrix} \{a,b\} & \emptyset \end{bmatrix} \\ &= \begin{bmatrix} \emptyset & \{a\} \\ \emptyset & \{b\} \end{bmatrix} \cup \begin{bmatrix} \emptyset & \emptyset \\ \{a\}^+\{a,b\} & \emptyset \end{bmatrix} \\ &= \begin{bmatrix} \emptyset & \{a\} \\ \{a\}^+\{a,b\} & \{b\} \end{bmatrix}. \end{aligned}$$

Now,

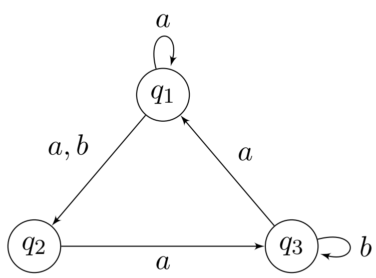
$$
\begin{aligned}
\delta^* &= \begin{bmatrix} \emptyset & \{a\} \\ \{a\}^+\{a,b\} & \{b\} \end{bmatrix}^* \\
&= \begin{bmatrix} (\{a\}\{b\}^*\{a\}^+\{a,b\})^* & (\{a\}\{b\}^*\{a\}^+\{a,b\})^*\{a\}\{b\}^* \\ (\{b\}+\{a\}^+\{a,b\}\{a\})^*\{a\}^+\{a,b\} & (\{b\}+\{a\}^+\{a,b\}\{a\})^* \end{bmatrix} \\
&= \begin{bmatrix} \beta^* & \beta^*\{a\}\{b\}^* \\ \gamma^*\{a\}^+\{a,b\} & \gamma^* \end{bmatrix},
\end{aligned}
$$

for $\beta = \{a\}\{b\}^*\{a\}^+\{a,b\}$ and $\gamma = \{b\} + \{a\}^+\{a,b\}\{a\}$.

In total, we have

$$
\begin{aligned}
&\begin{bmatrix} \{a\} & \{a,b\} & \emptyset \\ \emptyset & \emptyset & \{a\} \\ \{a\} & \emptyset & \{b\} \end{bmatrix}^* \\
&= \begin{bmatrix} \alpha^* & \alpha^* \begin{bmatrix} \{a,b\} & \emptyset \end{bmatrix} \begin{bmatrix} \emptyset & \{a\} \\ \emptyset & \{b\} \end{bmatrix}^* \\ \delta^* \begin{bmatrix} \emptyset \\ \{a\} \end{bmatrix} \{a\}^* & \delta^* \end{bmatrix} \\
&= \begin{bmatrix} \alpha^* & \alpha^* \begin{bmatrix} \{a,b\} & \emptyset \end{bmatrix} \begin{bmatrix} \{\epsilon\} & \{a\}\{b\}^* \\ \emptyset & \{b\}^* \end{bmatrix} \\ \delta^* \begin{bmatrix} \emptyset \\ \{a\}^+ \end{bmatrix} & \delta^* \end{bmatrix} \\
&= \begin{bmatrix} \alpha^* & \alpha^* \begin{bmatrix} \{a,b\} & \{a,b\}\{a\}\{b\}^* \end{bmatrix} \\ \delta^* \begin{bmatrix} \emptyset \\ \{a\}^+ \end{bmatrix} & \delta^* \end{bmatrix} \\
&= \begin{bmatrix} \alpha^* & \alpha^* \begin{bmatrix} \{a,b\} & \{a,b\}\{a\}\{b\}^* \end{bmatrix} \\ \begin{bmatrix} \beta^* & \beta^*\{a\}\{b\}^* \\ \gamma^*\{a\}^+\{a,b\} & \gamma^* \end{bmatrix} \begin{bmatrix} \emptyset \\ \{a\}^+ \end{bmatrix} & \begin{bmatrix} \beta^* & \beta^*\{a\} \\ \gamma^*\{a\}^+\{a,b\} & \gamma^* \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} \alpha^* & \alpha^*\{a,b\} & \alpha^*\{a,b\}\{a\}\{b\}^* \\ \beta^*\{a\}\{b\}^*\{a\}^+ & \beta^* & \beta^*\{a\} \\ \gamma^*\{a\}^+ & \gamma^*\{a\}^+\{a,b\} & \gamma^* \end{bmatrix} \\
&= \begin{bmatrix} (\{a\}\cup\{a,b\}\{ab^*a\})^* & (\{a\}\cup\{a,b\}\{ab^*a\})^*\{a,b\} & (\{a\}\cup\{a,b\}\{ab^*a\})^*\{a,b\}\{ab\}^* \\ (\{ab^*a^+\}\{a,b\})^*\{ab^*a^+\} & (\{ab^*a^+\}\{a,b\})^* & (\{ab^*a^+\}\{a,b\})^*\{a\} \\ (\{b\}+\{a\}^+\{a,b\}\{a\})^*\{a\}^+ & (\{b\}+\{a\}^+\{a,b\}\{a\})^*\{a\}^+\{a,b\} & (\{b\}+\{a\}^+\{a,b\}\{a\})^* \end{bmatrix}.
\end{aligned}
$$

Compare this to the following automaton:



4.3 Exercise

Intuitively, permuting the states in a weighted automaton $A = (\vec{i}, M, \vec{f})$ over a semiring S should not change the value $|A|$. Let's prove that this is true:

1. Let P be a permutation matrix in S. Show that P is invertible.
2. Permuting the states in A is done by changing M to PMP^{-1}. Show that $(PMP^{-1})^2 = PM^2P^{-1}$.
3. Show that $(PMP^{-1})^n = PM^nP^{-1}$ for any n and conclude that $(PMP^{-1})^* = PM^*P^{-1}$.
4. Conclude that if A' is the permuted automaton, i.e., $A' = (\vec{i}P^{-1}, PMP^{-1}, P\vec{f})$, then $|A'| = |A|$.

Solution

1. $P^{-1} = P^T$
2. $(PMP^T)^2 = PMP^TPMP^T = PMIMP^T = PM^2P^T$
3. Inductively, $(PMP^T)^{n+1} = (PMP^T)^nPMP^T = PM^nP^TPMP^T = PM^nIMP^T = PM^{n+1}P^T$
   Hence $(PMP^T)^* = \bigoplus_{n\geq 0}(PMP^T)^n = \bigoplus_{n\geq 0} PM^nP^T = P\big(\bigoplus_{n\geq 0} M^n\big)P^T = PM^*P^T$
4. $|A'| = \vec{i}P^T(PMP^T)^*P\vec{f} = \vec{i}P^TPM^*P^TP\vec{f} = \vec{i}IM^*I\vec{f} = |A|$

5 Idempotency

Definition: A semiring $(S, +, \cdot, 0, 1)$ is *idempotent* if $1 + 1 = 1$.

Idempotent semirings form an important subclass of semirings, mainly because many example semirings are idempotent.

5.1 Exercise

Show that $a + a = a$ for any $a \in S$ in an idempotent semiring S.

Solution

$a + a = a(1 + 1) = a \cdot 1 = a$.

5.2 Exercise

Which of the following semirings are idempotent?

1. $(\mathbb{N} \cup \{-\infty\}, \max, +, 0, -\infty)$
2. $(\mathbb{N}, +, \cdot, 0, 1)$
3. $(\mathbb{N} \cup \{\infty\}, \max, \min, 0, \infty)$
4. the language semiring

Solution

1. Yes: max is an idempotent operation
2. No: $1 + 1 = 2$
3. Yes
4. Yes: $\cup$ is an idempotent operation

5.3 Exercise (bonus)

Let S be an idempotent semiring. Let $\leq$ be the relation on S defined by $a \leq b$ iff $a + b = b$.

1. Show that $\leq$ is a partial order (i.e., it is reflexive, transitive, and antisymmetric)

$\leq$ is called the *natural order* on S.

Let now S also be star-continuous. Let $a \in S$. We know that $a^* = aa^* + 1$, that is, a^* is a fixed point for the mapping $f_a : S \to S$ given by $f_a(x) = ax + 1$.

2. Show that a^* is the *least* fixed point for f_a.

Solution

1. Trivial.
2. Let's first observe that our definition of $a \leq b$ iff $a + b = b$ is equivalent to $a \leq b$ iff $\exists x.\, a + x = b$:

   $\Rightarrow$ Assume that $a + b = b$. Take $x = b$, then we get $a + x = b$.
   $\Leftarrow$ Assume that $\exists x.\, a + x = b$. Then, we have $a + b = a + (a + x) = a + x = b$.

   Now let x be any fixed point of f_a; we need to show that $a^* \leq x$. We have $ax + 1 = x$, and by the above, this implies that
   $$ax \leq x \qquad \text{and} \qquad 1 \leq x\,.$$

   In the following, we want to argue about inequalities of products. Note therefore that from $a \leq b$ follows that $ac \leq bc$ because we know $a + b = b$ and can thus derive that $ac + bc = (a + b)c = bc$. (Equally, $ca \leq cb$ because $ca + cb = c(a + b) = cb$.)

   Applying the first inequality above multiple times, we get
   $$x \geq ax \geq aax \geq aaax \geq \dots\,,$$
   and combining it with the second, we have
   $$a = a1 \leq ax \leq x, \quad aa = aa1 \leq aax \leq x, \quad aaa \leq x, \quad \dots\,.$$
   Now,
   $$\begin{aligned} a^* &= \sum_{n=0}^{\infty} a^n \\ &= 1 + a + aa + aaa + aaaa + \dots \\ &\leq x + x + x \;\;\, + x \quad\; + x \qquad + \dots \\ &= x, \end{aligned}$$
   which proves that a^* is indeed the least fixed point of f_a.

6 Linear systems and Conway semirings

6.1 Exercise

Let S be any star-continuous semiring. Compute the solution of the following linear system (using variable y) over S:

$$y = (a + b)y + 1$$

Solution

As described on slide 35 of CM1, the solution of

$$\vec{L} = M\vec{L} + \vec{f}$$

is the the following fixed point:

$$\vec{L} = M^*\vec{f}$$

Here, we get

$$y = (a + b)^*1 = (a + b)^*$$

6.2 Exercise

Let S be any semiring. Compute the solution of the following linear system over S (by first translating the system into matrix notation):

$$\begin{aligned} y &= ay + bz + 1 \\ z &= y \end{aligned}$$

Solution

We have

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

By the observation above, we get

$$\begin{aligned} \begin{bmatrix} y \\ z \end{bmatrix} &= \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}^* \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} (a+b)^* & (a+b)^*b \\ (a^*b)^*a^* & (a^*b)^* \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} (a+b)^* \\ (a^*b)^*a^* \end{bmatrix} \end{aligned}$$

Note that $0^* = \sum_{n \geq 0} 0^n = 0^0 = 1$ because a^0 is the neutral element of multiplication.

6.3 Exercise

In exercise 6.2 above, the equality $z = y$ implies the equality of the two components of the solution. Argue why this equality holds in general.

This equality is called the *sum-star-identity* and is one of two axioms of *Conway semirings.*

Solution

$$\text{Sum-star-identity:} \qquad \forall a, b \in S.\, (a+b)^* = (a^*b)^*a^*$$

Conway himself explained in *Conway, J. H. (1971). Regular algebra and finite machines.*: "We observe that

$$(a+b)^* = 1 + a + b + aa + ab + ba + bb + aaa + \dots\,,$$

the sum of all products of a's and b's. On the other hand, the typical term of $(a^*b)^*a^*$ is

$$(a^i b)(a^j b)\dots(a^m b)a^n\,,$$

the general product of a's and b's partitioned by occurrences of b."

6.4 Exercise (bonus)

A *Conway semiring* is a semiring S together with a star operation $a \mapsto a^*$ in which the following hold for any $a, b \in S$:

$$(a+b)^* = (a^*b)^*a^* \qquad (ab)^* = 1 + a(ba)^*b$$

Derive the equalities

$$\begin{aligned} a^* &= 1 + aa^* = 1 + a^*a \\ a(ba)^* &= (ab)^*a \end{aligned}$$

from the axioms of Conway semirings.

Solution

The first is trivial. For the second:

$$\begin{aligned} a(ba)^* &= a(1 + b(ab)^*a) \\ &= a + ab(ab)^*a \\ &= (1 + ab(ab)^*)a \\ &= (ab)^*a \end{aligned}$$

6.5 Exercise (bonus)

Find (i.e., guess) two solutions of the (non-linear!) algebraic system

$$x = xx + a\,.$$

Compare your solutions to the language produced by the following grammar:

$$S \to SS \mid a$$

Argue why we are generally interested in *least* solutions to algebraic systems. (Compare with exercise 5.3.)

Solution

The algebraic system has two solutions: a^* and a^+.

a^+ is smaller because $a^+ \subsetneq a^*$.

When comparing the algebraic system above to the context-free grammar above (call it G), we can see that the language produced by this grammar can be found incrementally: Initially, $a \in \mathcal{L}(G)$, then, $aa \in \mathcal{L}(G)$, continuing, we get

$$\mathcal{L}(G) = \{a, aa, aaa, \ldots\}$$

Here, we approach the fixed point from below and will therefore reach the *least* fixed point. As we want weighted grammars to be a generalization of unweighted grammars, we are interested in least fixed points.