Higher-Dimensional Automata from a Topological Viewpoint

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Preface

This dissertation presents the results of the author's Ph.D. studies at the Department of Mathematical Sciences at Aalborg University, Denmark, conducted from August 2002 to September 2005 within the Mathematics and Physics Ph.D. programme headed by Prof. Jesper Møller, under supervision of Dr. Martin Raussen.

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In no particular order:

- Lisbeth Fajstrup and Martin Raussen, for teaching me quite a few things.
- Eric Goubault, for letting me visit him in Paris, twice, and teaching me quite a few things.
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- John Leth, for always being in the smoking room when needed.
- Lisbeth Grubbe, my favourite secretary.

Højere-dimensionelle automater fra en topologisk synsvinkel: Dansk resumé

Denne afhandlings tema er anvendelser af *directed topology* på højere-dimensionelle automater. Højere-dimensionelle automater er en formalisme for samtidige systemer, som har den egenskab at de kan udtrykke alle højere-ordens afhængigheder mellem processerne i et samtidigt system.

Beregninger i en højere-dimensionel automat er sammenhængende stier af kuber, som via geometrisk realisering kan oversættes til kontinuerte stier i et *directed* topologisk rum. Ækvivalente beregninger oversættes herved til homotope stier, og de beregningsmæssige egenskaber af en højere-dimensionel automat kan således undersøges ved metoder fra algebraisk topologi.

En simulering af højere-dimensionelle automater A og B er en afbildning $f : A \to B$ som bevarer beregningerne: Enhver beregning i A afbilledes ved f i en tilsvarende beregning i B. Sådan en simulering kaldes en bisimulering hvis enhver beregning i B også kan modsvares af en tilsvarende beregning i A.

I denne afhandling introduceres to forskellige bisimulerings-begreber for højere-dimensionelle automater, og det vises at disse bisimuleringer er karakteriseret ved at deres geometriske realiseringer løfter d-stier.

Contents

1 Introduction

Higher-dimensional automata: why they are useful, and how topology comes into the game.

This thesis is concerned with some applications of *directed topology* to the theory of *higher-dimensional automata*. Higher-dimensional automata are a formalism for concurrent systems, introduced by Vaughan Pratt [Pra91] in 1991. They have the specific feature that they can express all higher-order dependencies between the processes in a concurrent system, a capacity which most other formalisms are lacking.

Rob van Glabbeek [Gla06] has recently shown that, measured in terms of expressivity and up to a certain notion of *history-preserving bisimilarity*, higher-dimensional automata are the most general of the main formalisms for concurrent systems. This strongly suggests that

1. higher-dimensional automata are a useful formalism for modeling concurrent systems, especially if one needs to take into account dependency relations between processes, and that

2. investigating properties of the higher-dimensional automata formalism is a prudent research project, from a computer science point of view.

There is another, equally important, reason for dedicating one's research time to higher-dimensional automata: they are interesting from a *topological* point of view. A higher-dimensional automaton is basically a *precubical set*, and as such it has a *geometric realisation* as a topological space. Properties of higher-dimensional automata, and of mappings between them, translate thus to properties of topological spaces and continuous functions.

This translation allows one to use the huge and well-developed apparatus of *algebraic topology* to investigate properties of higher-dimensional automata, and of other concurrent systems. (Note that precubical sets *per se*, without being realised as topological spaces, also are proper objects of study in algebraic topology.) This line of research was founded by Eric Goubault [GJ92, Gou95] in 1992, when he was studying *homological* properties of higher-dimensional automata.

There is however a problem with the translation from higher-dimensional automata to topological spaces: one loses all information concerning *precedence* of events. In a higher-dimensional automaton, there are *causal relationships* between states or transitions, but a topological space is symmetric, so these non-symmetric relations are lost in the translation.

This and other problems are addressed in *directed topology*, where the objects of study are topological spaces equipped with precedence information. There is quite a zoo of categories of directed topological spaces, each of them useful in its own right; here are the ones we are aware of:

- The *po-spaces* and *local po-spaces* introduced by Lisbeth Fajstrup, Eric Goubault, and Martin Raussen [FGR99, FGR05] in 1999
- The *d-spaces* introduced by Marco Grandis [Gra03, Gra02] in 2001, and his *spaces with distinguished cubes* [Gra05] from 2003
- The *flows* [Gau03] introduced by Philippe Gaucher in 2002
- The *streams* of Sanjeevi Krishnan [Kri05], introduced in 2005

Using one of these frameworks, one can thus translate properties of higher-dimensional automata into properties of directed topological spaces. One cannot, however, apply the tools of algebraic topology directly anymore; also algebraic topology has to be expanded with notions of precedence. This *breaks the symmetries*[1] of algebraic topology; instead of fundamental groups (or groupoids) one has *fundamental categories* [Rau03, FRGH04], instead of homology groups one has homology *monoids* [Gra05] or some other structures [Fah04]. Other notions are still waiting to be carried over to directed topology, one particular being the theory of fibrations and cofibrations (we shall have more to say on this issue later).

[1] A slogan of Marco Grandis [Gra05].

Let us go a bit more into detail now: Computations in a higher-dimensional automaton, or in some other concurrent system, can be viewed as (discrete) *paths*. Through geometric realisation, these execution paths are translated to (continuous) *directed paths* in a directed space. Also, we might want to treat certain computations in our concurrent system as being *equivalent*, for example because they result in the same value for some variable (or, in general, *all* variables). Through geometric realisation, equivalence of computations translates to *directed homotopy* of directed paths. Hence the computational properties of a given concurrent system can be investigated using techniques from directed homotopy and homology theory.

Likewise, one is often interested in relations between concurrent systems, and especially in *equivalences* of concurrent systems. These come in two flavours: *observational* and *behavioural*, and from a topological viewpoint, the latter is the more interesting. Informally, two concurrent systems are behaviourally equivalent, or *bisimilar*, if any computation in the first system can be matched by the other, and vice versa. It is customary to express bisimilarity in terms of spans of simulations with a certain *open map* property, cf. [JNW96], hence it is natural to investigate the *topological* properties of such open maps.

In this thesis, we define two notions of bisimilarity for higher-dimensional automata, and we show that the open maps associated with each of them are described by certain lifting properties of their geometric realisations.

1.1 Motivation

An introduction to the basics of transition systems, and a demonstration of the usefulness of the higher-dimensional automata formalism.

Perhaps the most common formalism for modeling concurrent systems is the one of *transition systems*. A transition system is a directed graph with a specified *initial state* and labels on the edges.

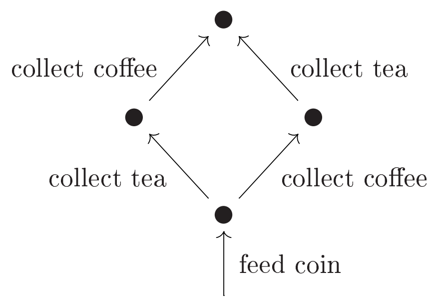

Figure 1.1: The transition system for V_1.

Consider a politically correct vending machine V_1, which when fed a coin will provide both a cup of tea *and* a cup of coffee, but which only has one vending slot. As a transition system, we can model the interaction of a customer with this vending machine as in Figure 1.1: First, a coin is inserted, and then tea and coffee are collected in an order specified by the vending machine.

Let V_2 be an enhanced version of the machine V_1; it still furnishes both a cup of tea and a cup of coffee when fed a coin, but it does so through *two* vending slots. Hence V_2 has the property that it can provide tea and coffee *concurrently*, whereas V_1 only can supply *one after the other*. Assume further that the customer can only collect one cup at a time; maybe she has a book in the other hand. This does not change the transition system for V_1, and the transition system for her interaction with V_2 is now the same as for V_1.

So we have two different vending machines which give raise to the same transition system. This is problematic for several reasons, one being that V_1 and V_2 *behave* differently; after all it only takes V_2 half the time to serve coffee and tea compared to V_1. Another reason is the following:

Suppose the vending machines run out of cups, so the customer now has to provide the cups herself. The action "collect tea" is thus replaced by "provide cup" and "take tea," and similarly for "collect coffee;" both are *refined*. This makes our transition systems look like in Figure 1.2, where we have omitted the verbs in the actions and the initial "feed coin" arrow:

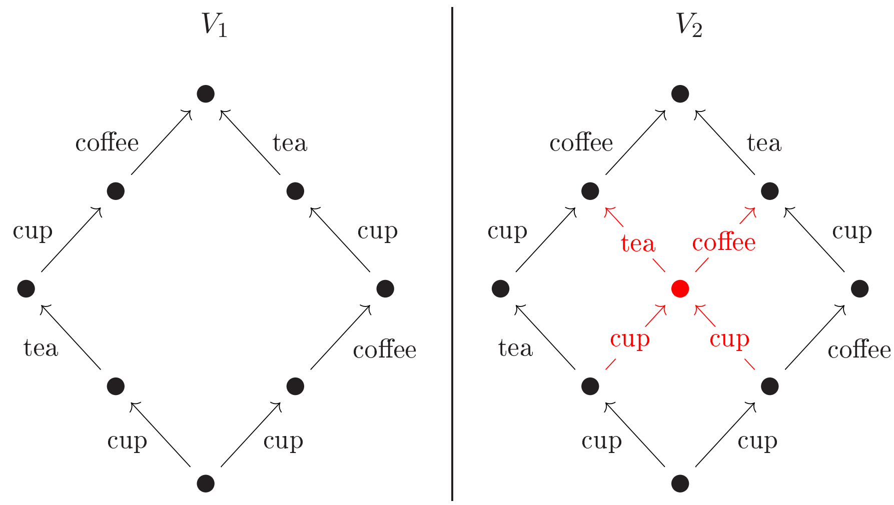

Figure 1.2: The transition systems for V_1 and V_2 after action refinement.

In vending machine V_1, the customer places a cup in the vending slot, waits for her tea, or coffee, and then takes her beverage and places another cup, afterwards collecting the other drink. Vending machine V_2 on the other hand has *two* slots, so here the customer can choose to place both her cups first, and then collect both her beverages.

So the original transition systems for V_1 and V_2 are the same, but after refinement of actions they are not; our model is not stable under action refinement.

This problem is trivially avoided when one disallows action refinement, declaring transition systems to be a formalism which is not "truly concurrent." If refinement has to be permitted, one can enrich the transition system formalism with notions of *independence of actions*. In our examples, the actions "collect tea" and "collect coffee" would be independent in the transition system V_2, but not in V_1, thus distinguishing the two transition systems from another.

Transition systems enriched with independence notions exist in a number of flavours; some notable examples are the *asynchronous transition*

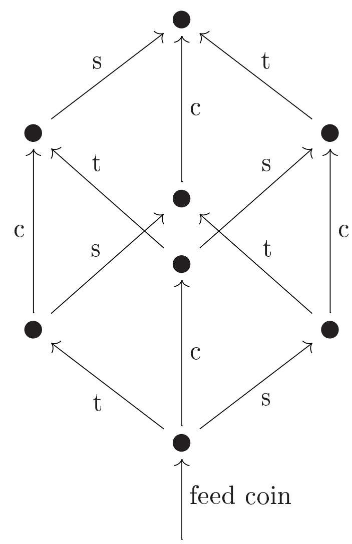

Figure 1.3: The transition system for V_3 and V_4.

systems of [Shi85, Bed87], the *concurrent transition systems* of [Sta89], and the *transition systems with independence* in [NSW94]. The objects in these categories are transition systems equipped with a binary relation of independence, either on the actions or on the transitions, and the morphisms are required to respect this relation.

We can refine our examples to show that *binary* independence relations are not sufficient for distinguishing *concurrent behaviour* (V_2) from *interleaving behaviour* (V_1): Let V_3 be a vending machine which when fed a coin, will furnish *three* drinks; *t*ea, *c*offee, and *s*ake, but through only *two* vending slots, and let V_4 be the same as V_3, except for having *three* vending slots. The transition systems for both machines look as in Figure 1.3, where we have abbreviated the actions by their first letters t, c, and s.

Now assume that we only encode *pairwise* independence of actions in our transition systems, and refine the actions as we did before. Then we fail to add the state and transitions in V_4 which are depicted in red in

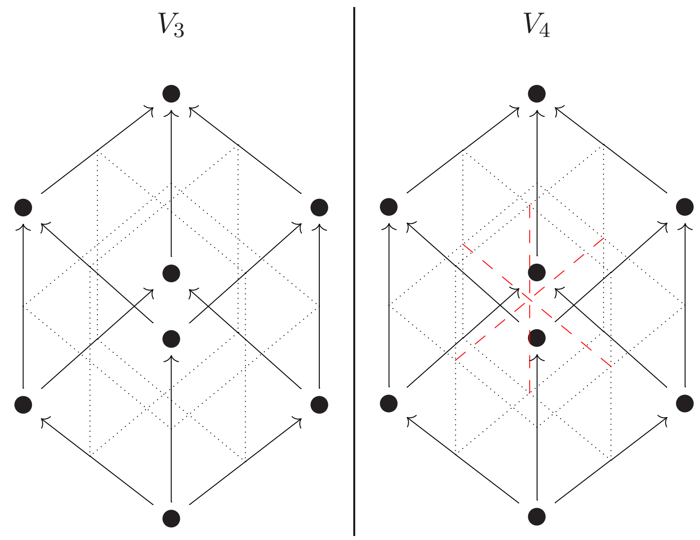

Figure 1.4: The transition systems for V_1 and V_2 after refinement.

Figure 1.4.

That is, to have a "truly concurrent" model, also *ternary* independence relations have to be considered, and iterating the examples, we end up with a formalism where there are n-ary independence relations for all $n \in \mathbb{N}$.

A generalisation of this approach, expressing independence of *transitions* rather than *actions*, is provided by the higher-dimensional automata formalism. In a higher-dimensional automaton, a square formed by independent transitions is *filled in*, whereas if the transitions are not independent, it is *empty*. Hence when refining actions, or rather *transitions*, the filled-in square is subdivided, whereas the empty square is left unchanged.

Similarly, *three* independent transitions give rise to a filled-in *cube*, whereas three *pairwise* independent transitions form an empty cube with all *faces* filled in. Thus n-ary independence relations are expressed by n-cubes in a precubical set.

1.2 Transition Systems

A more technical introduction to transition systems and their simulations, useful for later generalisation to higher-dimensional automata.

A transition system is a directed graph with a specified *initial state*. For us, directed graphs are permitted to contain both multiple edges and loops (so they are what graph theorists would refer to as *pseudodigraphs*, cf. [CL00]). We shall henceforth call them *predigraphs* rather than digraphs;[1] to be precise, a predigraph consists of two sets A_0 and A_1, which are glued together with two *structure maps* $\delta^0, \delta^1 : A_1 \to A_0$, specifying for each edge its start respectively end vertex.

A *labeled transition system* over some finite set Σ (an *alphabet*) is a transition system $\{A_0, A_1\}$ enriched with a *labeling map* $\lambda : A_1 \to \Sigma$. We can identify the finite set Σ with the predigraph (Σ_0, Σ_1), with $\Sigma_0 = \{*\}$ a one-element set, $\Sigma_1 = \Sigma$, and $\delta_0(p) = \delta_1(p) = *$ for all $p \in \Sigma$; we shall make this more precise in Lemma 5.6. With this identification, the labeling map λ is a *predigraph morphism*.

We can express the initial state of $A = \{A_0, A_1\}$ as a predigraph morphism $i : * \to A$, where $*$ is the *one-point predigraph* with just one vertex and no edges. Hence a labeled transition system is a diagram

$$* \xrightarrow{i} A \xrightarrow{\lambda} \Sigma \tag{1}$$

of predigraphs and predigraph morphisms.

A *simulation* of two transition systems $\langle * \overset{i}{\to} A \overset{\lambda}{\to} \Sigma \rangle$, $\langle * \overset{j}{\to} B \overset{\mu}{\to} \Xi \rangle$ consists of two predigraph morphisms $f : A \to B$, $\sigma : \Sigma \to \Xi$ for which

[1] In fact, we shall soon use the term *digraph* for another kind of structure.

the diagram

$$\begin{tikzcd} * \arrow[r, equal] \arrow[d, "i"'] & * \arrow[d, "j"] \\ A \arrow[r, "f"'] \arrow[d, "\lambda"'] & B \arrow[d, "\mu"] \\ \Sigma \arrow[r, "\sigma"'] & \Xi \end{tikzcd}$$

commutes, that is, $f \circ i = j$ and $\sigma \circ \lambda = \mu \circ f$. These conditions ensure that the initial state of A is mapped to the initial state of B, and that the label of any transition is mapped to the label of the image transition. A simulation $(f, \lambda) : \langle * \to A \to \Sigma \rangle \to \langle * \to B \to \Xi \rangle$ thus expresses the property that any transition a in A can be matched by a transition $f(a)$ whose label is the image, under σ, of the label of a.

Note that a predigraph morphism $\sigma : \Sigma \to \Xi$ is the same as a set function $\Sigma \to \Xi$. For applications, one wants to include *partial set functions* $\Sigma \to \Xi$ in the formalism; a common trick for doing this is to equip the sets with an extra element $\perp$, so $\Sigma_\perp = \Sigma \sqcup \{\perp\}$ and similarly for Ξ,[1] and then replacing a partial set function $\sigma : \Sigma \nrightarrow \Xi$ by a (total) set function $\sigma_\perp : \Sigma_\perp \to \Xi_\perp$ for which $\sigma_\perp(p) = \perp$ whenever $\sigma(p)$ is undefined.

For predigraph morphisms, one can achieve the same effect by equipping the predigraphs with *trivial loops* at all vertices, *i.e.* by passing from predigraphs to the *free digraphs* generated by the predigraphs. Specifically, a *digraph* consists of sets A_0, A_1 and functions

$$A_1 \underset{\delta^0}{\overset{\delta^1}{\underset{\xleftarrow{\varepsilon}}{\rightrightarrows}}} A_0$$

for which $\delta^0 \circ \varepsilon = \delta^1 \circ \varepsilon = \mathsf{id} : A_0 \to A_0$, and the free digraph on a predigraph $A = \{A_0, A_1\}$ is the digraph FA given by $(FA)_0 = A_0$ and $(FA)_1 = A_1 \sqcup \{\varepsilon a \mid a \in A_0\}$, with structure maps $\delta^0(\varepsilon a) = \delta^1(\varepsilon a) = a$. A partial set function $\sigma : \Sigma \nrightarrow \Xi$ is then the same as a *digraph morphism* $\sigma : F\Sigma \to F\Xi$.

[1] $\sqcup$ denotes disjoint union.

We can thus modify the above definition of simulation to also include partial functions as follows: A *simulation* of labeled transition systems is a commuting diagram

$$\begin{tikzcd} * \arrow[r, equal] \arrow[d, "i"'] & * \arrow[d, "j"] \\ A \arrow[r, "f"', red] \arrow[d, "\lambda"'] & B \arrow[d, "\mu"] \\ \Sigma \arrow[r, "\sigma"', red] & \Xi \end{tikzcd}$$

where the black arrows indicate *predigraph* morphisms, and the red arrows indicate *digraph* morphisms. Hence if $\mathsf{Cub}^{(1)}$ denotes the category of digraphs, then the *category of labeled transition systems* is the full subcategory of the *pointed arrow category* $\left\langle * \to \mathsf{Cub}^{(1)} \to \mathsf{Cub}^{(1)} \right\rangle$ spanned by those objects which are freely generated by predigraph diagrams $* \to A \to \Sigma$ as in (1).

Notes

The standard reference on transition systems (and other formalisms) is [WN95]. Though our exposition is somewhat different from the one in [WN95], the resulting categories of transition systems are isomorphic, except for one discrepancy: The transition systems of [WN95] are *extensional*; for any two states a, b and any label p, there can at most be *one* transition from a to b with label p.

1.3 Organisation, Main Results

The main body of this work consists of Chapters 4 and 5. Before this, we have included two chapters on some of the categories of directed topology and on (pre)cubical sets, and how these are connected through the adjunction between geometric realisation and the singular cubical set functor.

Chapter 4 is concerned with cube paths in precubical sets, and how these provide a combinatorial counterpart to directed paths (*d-paths*) in the geometric realisation, via *carrier sequences*. The analogy passes to *homotopies* of cube paths vs. *d-homotopies* of d-paths, and the main result is the following:

Theorem A. *Given d-paths $p, q : \vec{I} \to |A|$ in the geometric realisation of a geometric precubical set A, then the carrier sequences of p and q are homotopic if and only if p and q are d-homotopic rel carriers of their endpoints.*

We also introduce a notion of universal covering of a precubical set, which is closely related to the universal directed covering of its geometric realisation. Parts of Chapter 4 are based on results by Fajstrup [Faj05, Faj06].

In Chapter 5, we introduce higher-dimensional automata and their morphisms. In fact we define *two* categories of (labeled) higher-dimensional automata; one with (exact) simulations and bisimulations, the other with simulations and bisimulations *up to homotopy*.

The main result for the exact category is as follows:

Theorem B. *For a non-contracting simulation $(f, \mathsf{id}) : \langle * \xrightarrow{i} A \to !\Sigma\rangle \to \langle * \to B \to !\Sigma\rangle$ to be a bisimulation map it is necessary and sufficient that for any $x \in \uparrow[i*]$ and any d-path $q : \vec{I} \to |B|$ such that $q(0) = |f|(x)$, there exists a d-path $p : \vec{I} \to |A|$ such that $p(0) = x$ and $q = |f| \circ p$.*

The notation $x \in \uparrow y$ expresses that there exists a d-path in $|A|$ from y to x. Hence if all of A is reachable, (f, id) is a bisimulation map if and only if $|f|$ *lifts d-paths* in the diagram

$$\begin{tikzcd} 0 \arrow[r, "x"] \arrow[d, hook] & {|A|} \arrow[d, "|f|"] \\ \vec{I} \arrow[r, "q"'] \arrow[ur, dotted, "p"] & {|B|} \end{tikzcd}$$

For the category of homotopy simulations, we have the following modification of the above theorem, where $\tilde{A}$ denotes the universal covering of A with respect to the cube $i*$.

Theorem C. *A non-contracting homotopy simulation*

$$(f, \mathsf{id}) : \langle * \xrightarrow{i} A \to !\Sigma \rangle \to \langle * \xrightarrow{j} B \to !\Sigma \rangle$$

is a homotopy bisimulation map if and only if one of the following equivalent properties holds:

- *For any d-path $r : \vec{I} \to |A|$ with $r(0) = [i*]$ and any d-path $q : \vec{I} \to |B|$ with $q(0) = |f|(r(1))$ and $\mathsf{carr}\, q(1) \in B_0$, there exists a d-path $p : \vec{I} \to |A|$ such that $p(0) = r(1)$ and $|f| \circ (r * p) \sim (|f| \circ r) * q \text{ rel } \partial I$.*
- *For any $x \in |\tilde{A}|$ and any $y \in \uparrow|\tilde{f}|(x) \subseteq |\tilde{B}|$ with $\mathsf{carr}\, y \in \tilde{B}_0$, there is $z \in \uparrow x$ such that $y = |\tilde{f}|(z)$.*

Our results in the exact category, specifically Sections 5.1 through 5.6, have been published before in [Fah05b, Fah05a]. They are based on work by Goubault in [Gou95, Gou02].

2 Directed Topological Spaces

A short introduction to three categories of directed topological spaces; po-spaces, local po-spaces, and d-spaces, and the forgetful functors between them.

Notation

- The category of sets and mappings is denoted Set. The set $\mathbb{N}$ of natural numbers includes 0, the set $\mathbb{N}_+$ of positive natural numbers does not.
- Given a relation $R \subseteq S \times S$ on a set S and $T \subseteq S$, the *restriction* $R|_T$ of R to T is the relation $R \cap T \times T \subseteq T \times T$.
- A *partial order* is a relation which is reflexive, transitive, and antisymmetric. The category of partially ordered sets and (weakly) increasing mappings is denoted poSet.
- The category of topological spaces is denoted Top. The category of Hausdorff topological spaces is denoted Haus.

2.1 Po-Spaces

Po-spaces are perhaps the simplest category of directed topological spaces. They have a global sense of direction and thus rarely occur in applications.

2.1 Definition. A relation R on a topological space X is *closed* if $R \subseteq X \times X$ is a closed subset in the product topology.

2.2 Definition [FGR05] A *po-space* is a topological space X together with a closed partial order $\leq$ on X.

2.3 Lemma [Nac65] *A po-space is Hausdorff.*

2.4 Remark. This is a simple application of the well-known fact that a topological space X is Hausdorff if its diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in the product topology. The proof in [Nac65] makes use only of reflexivity and antisymmetry of $\leq$; transitivity is not used.

2.5 Examples.

- The standard unit interval $I = [0, 1]$ with its standard total order is a po-space denoted $\vec{I}$.

- The unit cubes I^n with the product partial order

$$(x_1, \dots, x_n) \leq (y_1, \dots, y_n) \quad \iff \quad x_i \leq y_i \text{ for all } i$$

  are po-spaces denoted $\vec{I}^n$.

2.6 Definition [FGR05] A continuous, increasing mapping $f : X \to Y$ of po-spaces X, Y is called a *dimap*, and the category of po-spaces and dimaps is denoted poTop. Isomorphisms in poTop are called *dihomeomorphisms*.

2.7 Proposition [Hau05] *The forgetful functor $U_1 : \mathsf{poTop} \to \mathsf{Haus}$ has a left adjoint F_1 assigning to a Hausdorff space the po-space with the discrete order $x \leq y$ iff $x = y$. The forgetful functor $U_2 : \mathsf{poTop} \to \mathsf{poSet}$ has a left adjoint F_2 assigning to a po-set X the po-space with the discrete topology $\tau = 2^X$.*

Proof: Any continuous mapping $F_1X \to Y$ is monotone, thus we have a bijection $\mathsf{poTop}(F_1X, Y) \approx \mathsf{Haus}(X, U_1Y)$. Similarly any monotone mapping $F_2X \to Y$ is continuous, hence we have a bijection $\mathsf{poTop}(F_2X, Y) \approx \mathsf{poSet}(X, U_2Y)$. Naturality is clear in both cases. $\square$

2.8 Proposition [Hau05] *The category poTop is complete and cocomplete.*

2.2 Local Po-Spaces

Local po-spaces are a generalisation of po-spaces, with a local sense of direction. This makes them well-suited for applications, however their categorical properties are rather bad.

2.9 Definition (cf. [FGR05]**)** A *local po-space* is a Hausdorff topological space X together with a reflexive relation $\leq$ on X in which any $x \in X$ has an open neighbourhood $U \ni x$ such that the restriction $\leq|_U$ is a closed partial order on U.

2.10 Remarks.

- An open neighbourhood as in the definition is called a *po-neighbourhood*.
- Our definition is different from the one in [FGR05] in that we have a "global" relation $\leq$ on X, whereas [FGR05] has "local" relations $\leq_U$ on each U. Otherwise, the definitions are equivalent.
- Contrary to po-spaces, the Hausdorff property of local po-spaces does not follow from the definition; hence we demand it explicitly.
- In a more restrictive variant of the above definition, one demands $\leq$ to be (globally) antisymmetric and closed. Most interesting examples of local po-spaces satisfy these stronger constraints, and the Hausdorff property of this restrictive variant of local po-spaces follows from $\leq$ being reflexive, antisymmetric, and closed, cf. Remark 2.4.

2.11 Definition (cf. [FGR05]**)** A continuous mapping $f : X \to Y$ of local po-spaces is called a *local dimap* provided that for any $x \in X$, there exists an open neighbourhood $U \ni x$ such that

$$\forall\, y, z \in U : y \leq z \Longrightarrow f(y) \leq f(z)$$

The category of local po-spaces and local dimaps is denoted lpoTop. Isomorphisms in lpoTop are called *local dihomeomorphisms*.

2.12 Definition (cf. [FGR05]**)** Two local po-structures $(X, \leq_1)$, $(X, \leq_2)$ on the same underlying topological space are *equivalent* if the identity map on X is a local dihomeomorphism, that is if any $x \in X$ has an open neighbourhood $U \ni x$ such that

$$\leq_1 \cap U \times U = \leq_2 \cap U \times U$$

2.13 Remark. Again, these two definitions are easily seen to be equivalent to the ones found in [FGR05].

2.14 Proposition. *The forgetful functor* $U : \mathsf{lpoTop} \to \mathsf{Haus}$ *has a left adjoint* $J \circ F_1$, *where* F_1 *is the functor* $\mathsf{Haus} \to \mathsf{poTop}$ *from Proposition 2.7 and* $J : \mathsf{poTop} \hookrightarrow \mathsf{lpoTop}$ *is the inclusion functor.*

Proof: As in the proof of Proposition 2.7, the bijection

$$\mathsf{lpoTop}(JF_1X, Y) \approx \mathsf{Haus}(X, UY)$$

is due to the fact that any continuous mapping $JF_1X \to Y$ is a local (global, in fact) dimap, and naturality is clear. $\square$

2.15 Proposition [Hau05] *The category* lpoTop *is finitely complete and admits all coproducts.*

2.16 Remark. We shall later show that lpoTop does not admit all coequalisers, see Example 4.11, hence that it is not cocomplete.

2.17 Example [Sok02] The inclusion functor $J : \mathsf{poTop} \hookrightarrow \mathsf{lpoTop}$ is not full: Let $X \in \mathsf{Top}$ be the boundary of the unit square, *i.e.*

$$X = \{(0, y), (1, y), (x, 0), (x, 1) \mid 0 \leq x, y \leq 1\} \subseteq I \times I$$

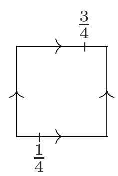

Figure 2.1: An example po-space.

see also Figure 2.1. Define two partial orders on X by

$$\begin{array}{rll} (x_1,y_1) \leq_1 (x_2,y_2) & \text{if} & x_1 \leq y_1, x_2 \leq y_2 \\ (x_1,y_1) \leq_2 (x_2,y_2) & \text{if} & x_1 = x_2, y_1 \leq y_2 \\ & \text{or} & y_1 = y_2, x_1 \leq x_2 \\ & \text{or} & x_1 = 0, y_2 = 1 \\ & \text{or} & y_1 = 0, x_2 = 1 \end{array}$$

Then the mapping $f = \mathsf{id} : (X, \leq_1) \to (X, \leq_2)$ is a local dimap, but as $\left(\frac{1}{4}, 0\right) \leq_1 \left(\frac{3}{4}, 1\right)$ and $\left(\frac{1}{4}, 0\right) \not\leq_2 \left(\frac{3}{4}, 1\right)$, $f \notin \mathsf{poTop}\big((X, \leq_1), (X, \leq_2)\big)$.

2.18 Definition. A local dimap $f : (X, \leq_1) \to (Y, \leq_2) \in \mathsf{lpoTop}$ is called a *global dimap* if f is in the image of the inclusion functor $J : \mathsf{poTop} \hookrightarrow \mathsf{lpoTop}$, *i.e.* if $x \leq_1 y$ implies $f(x) \leq_2 f(y)$ for all $x, y \in X$.

2.19 Lemma (cf. [Sok02]**)** *Let X be a po-space and $f \in \mathsf{lpoTop}(\vec{I}, JX)$. Then f is a global dimap.*

Proof: Let $\mathcal{U}$ be an open cover of I such that whenever $s, t \in U$ for some $U \in \mathcal{U}$, $f(s) \leq f(t)$ in JX. By compactness of I, we have a finite subfamily $\mathcal{V} = \{J_1, \ldots, J_n\} \subseteq \mathcal{U}$.

Order the J_i such that $\inf J_i \leq \inf J_{i+1}$, and write $J_1 = [0, t_1[$, $J_n =]s_n, 1]$, and $J_i =]s_i, t_i[$ for $i = 2, \ldots, n-1$. Note that for all $i = 1, \ldots, n-1$, $s_i \leq s_{i+1}$ and $s_{i+1} \in J_i$, hence $f(s_i) \leq f(s_{i+1}$ in JX.

Let $s \leq t \in I$, and let $k \leq \ell \in \{1, \ldots, n\}$ such that $s \in J_k$, $s \notin J_{k+1}$, and $t \in J_\ell$. Then $s \leq s_{k+1}$ in J_k, thus $f(s) \leq f(s_{k+1})$ in X. Also, $f(s_\ell) \leq f(t)$ in X, hence $f(s) \leq f(t)$. $\square$

2.3 D-Spaces

In d-spaces, the concept of directed path is not a derived notion as in (local) po-spaces, but rather part of the definition. D-spaces have good categorical properties.

2.20 Definition [Gra03] A *d-space* is a topological space X together with a set $\vec{P}X$ of continuous paths $I \to X$ such that

- $\vec{P}X$ contains all constant paths,
- for all $p \in \vec{P}X$ and $\varphi \in \mathsf{poTop}(\vec{I}, \vec{I})$, $p \circ \varphi \in \vec{P}X$, and
- for all $p, q \in \vec{P}X$ such that $p(1) = q(0)$, $p * q \in \vec{P}X$.

Here $p * q$ denotes the concatenation of p and q defined by

$$p * q(t) = \begin{cases} p(2t) & \text{for } t \leq \frac{1}{2}, \\ q(2t-1) & \text{for } t \geq \frac{1}{2}. \end{cases}$$

2.21 Terminology. Elements of $\vec{P}X$ are called *d-paths*. Paths $p \in \vec{P}X$ such that $p(0) = p(1)$ are called *d-loops*.

2.22 Definition. A d-space $(X, \vec{P}X)$ is *saturated* provided that whenever $p \circ \varphi \in \vec{P}X$ for a path $p : I \to X$ and $\varphi \in \mathsf{poTop}(\vec{I}, \vec{I})$ surjective, then also $p \in \vec{P}X$.

2.23 Definition. The *future* respectively *past* of a point $x \in X$ in a d-space X are the sets

$$\uparrow x = \{y \in X \mid \exists\, p \in \vec{P}X : p(0) = x, p(1) = y\}$$
$$\downarrow x = \{y \in X \mid \exists\, p \in \vec{P}X : p(0) = y, p(1) = x\}$$

2.24 Definition [Gra03] A continuous mapping $f : X \to Y$ of d-spaces $(X, \vec{P}X)$, $(Y, \vec{P}Y)$ satisfying $f \circ p \in \vec{P}Y$ for all $p \in \vec{P}X$ is called a *d-map*. The category of d-spaces and d-maps is denoted dTop. Isomorphisms in dTop are called *d-homeomorphisms*.

2.25 Proposition [Gra03] *The category* dTop *is complete and cocomplete.*

2.26 Definitions [Gra03] Let $(X, \vec{P}X)$ be a d-space.

- The *restriction* of X to a subset $A \subseteq X$ is the d-space $(A, \vec{P}A)$, where

$$\vec{P}A = \{p \in \vec{P}X \mid p(t) \in A \text{ for all } t \in I\}$$

- The *quotient* of X under an equivalence relation $\sim \subseteq X \times X$ is the d-space $(Y, \vec{P}Y)$, where $Y = X/\sim$ is the topological quotient, and

$$\vec{P}Y = \{(q \circ p_1) * \cdots * (q \circ p_n) \mid p_1, \ldots, p_n \in \vec{P}X\}$$

  with $q : X \to Y$ the quotient map.

2.4 D-Spaces without Small Loops

What characterises local po-spaces among d-spaces is that they do not have small loops.

2.27 Definition. A d-space $(X, \vec{P}X)$ is *loop-free* if all d-loops in $\vec{P}X$ are trivial. X is *without small loops* if any $x \in X$ admits a loop-free open neighbourhood.

2.28 Remark. In the terminology of [Gra03], a d-space is without small loops if and only if it contains no point-like vortices, and all its reversible d-paths are trivial.

2.29 Definition [Gra03] The functor $F : \mathsf{lpoTop} \to \mathsf{dTop}$ is defined on objects by

$$(X, \leq) \mapsto \big(X, \mathsf{lpoTop}(\vec{I}, X)\big)$$

and is the identity on morphisms. Composing with the inclusion functor $J : \mathsf{poTop} \hookrightarrow \mathsf{lpoTop}$, we get a functor $G = F \circ J : \mathsf{poTop} \to \mathsf{dTop}$.

2.30 Remark. It is straight-forward to check that F is indeed a functor. Also, the definition in [Gra03] is via *bitopological* spaces, but otherwise agrees with ours.

2.31 Corollary. *The functor* $F : \mathsf{lpoTop} \to \mathsf{dTop}$ *is faithful, and if* $(X, \leq_1)$, $(X, \leq_2)$ *are equivalent local po-structures, then* $F(X, \leq_1) = F(X, \leq_2)$.

2.32 Proposition. *If* $(X, \leq)$ *is a local po-space, then* $F(X, \leq)$ *is a saturated Hausdorff d-space without small loops. If* $(Y, \leq)$ *is a po-space, then* $G(Y, \leq)$ *is a saturated loop-free Hausdorff d-space.*

Proof: The saturation and Hausdorff properties are clear. To show that $G(Y, \leq)$ is loop-free, let $p \in \vec{P}Y = \mathsf{lpoTop}(\vec{I}, JY)$ be a d-loop, then p is a global dimap by Lemma 2.19. Let $s \in I$, then $p(0) \leq p(s) \leq p(1) = p(0)$ and hence $p(s) = p(0)$, *i.e.* p is trivial.

The d-space $F(X, \leq)$ contains no small loops because by the above, any po-neighbourhood in X is loop-free. □

2.5 Directed Homotopies of Directed Paths

D-spaces and (local) po-spaces have different "natural" notions of directed homotopy; d-homotopy respectively dihomotopy. We shall see later that for geometric realisations of precubical sets, these notions agree.

2.33 Definition (cf. [FGR05]) Given a d-space $(X, \vec{P}X)$ and subsets $U, V \subseteq X$, then a continuous mapping $H : I^2 \to X$ is called a *dihomotopy rel* (U, V) provided that

- $H(s, \cdot) \in \vec{P}X$ for all $s \in I$, and
- $H(s, 0) \in U$ and $H(s, 1) \in V$ for all $s \in I$.

If $H : I^2 \to X$ is a dihomotopy rel (U, V), then the d-paths $H(0, \cdot)$ and $H(1, \cdot)$ are said to be *dihomotopic rel* (U, V), denoted $H(0, \cdot) \sim H(1, \cdot)$ rel (U, V).

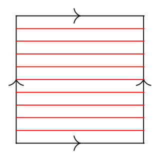

Figure 2.2: D-paths in the po-space $(I^2, \preceq)$.

2.34 Remarks.

- Letting U, V be the whole space X respectively one-point subsets of X, this includes both the notions of "free" dihomotopy and of dihomotopy with fixed endpoints from [FGR05].
- Dihomotopy of *d-maps* $f, g : Y \to X$ can be defined in a similar manner, but we will not need this here.

2.35 Definition (cf. [Gra03]**)** Given a d-space $(X, \vec{P}X)$ and subsets $U, V \subseteq X$, then a d-map $H : \vec{I}^2 \to X$ is called an *elementary d-homotopy rel* (U, V) provided that $H(s, 0) \in U$ and $H(s, 1) \in V$ for all $s \in I$. If $H : \vec{I}^2 \to X$ is an elementary d-homotopy, we write $H(0, \cdot) \dot{\prec} H(1, \cdot)$ rel (U, V). The relation $\dot{\sim}$ of *d-homotopy rel* (U, V) is the equivalence relation generated by $\dot{\prec}$.

2.36 Remarks.

- This includes again both the notions of "free" d-homotopy and of d-homotopy with fixed endpoints as defined in [Gra03].
- D-homotopy implies dihomotopy, but the reverse implication does not hold, cf. the example below. An important category of d-spaces where the notions of d-homotopy and dihomotopy agree is given by the geometric realisations of precubical sets, cf. Corollary 4.45.

2.37 Example. Define a partial order on I^2 by $(s_1, t_1) \preceq (s_2, t_2)$ iff $s_1 = s_2$ and $t_1 \leq t_2$. Any two d-paths in the po-space $(I^2, \preceq)$ are *dihomotopic*, whereas *d-homotopy* is the trivial relation, cf. Figure 2.2.

3 Precubical and Cubical Sets

For combinatorial counterparts to directed topological spaces, cubical sets are better suited than simplicial sets, due to the natural notion of direction inherent in the cubes in a cubical set.

3.1 Basic Definition

3.1 Definition [GM03, Cra95] The category $\square_{d,e,s,g}$ of *elementary cubes* has as objects the sets $\{0,1\}^n$, $n \in \mathbb{N}$, and the morphisms are generated by the mappings

$$\begin{aligned}
d_i^\nu : \{0,1\}^n &\to \{0,1\}^{n+1} \qquad (i = 1,\ldots,n+1; \nu = 0,1) \\
(t_1,\ldots,t_n) &\mapsto (t_1,\ldots,t_{i-1},\nu,t_i,\ldots,t_n) \\
e_i : \{0,1\}^n &\to \{0,1\}^{n-1} \qquad (i = 1,\ldots,n) \\
(t_1,\ldots,t_n) &\mapsto (t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n) \\
s_i : \{0,1\}^n &\to \{0,1\}^n \qquad (n \geq 2; i = 1,\ldots,n-1) \\
(t_1,\ldots,t_n) &\mapsto (t_1,\ldots,t_{i-1},t_{i+1},t_i,t_{i+2},\ldots,t_n) \\
g_i^0 : \{0,1\}^n &\to \{0,1\}^{n-1} \qquad (n \geq 1; i = 1,\ldots,n; \nu = 0,1) \\
(t_1,\ldots,t_n) &\mapsto (t_1,\ldots,t_{i-1},\max(t_i,t_{i+1}),t_{i+2},\ldots,t_n) \\
g_i^1 : \{0,1\}^n &\to \{0,1\}^{n-1} \qquad (n \geq 1; i = 1,\ldots,n; \nu = 0,1) \\
(t_1,\ldots,t_n) &\mapsto (t_1,\ldots,t_{i-1},\min(t_i,t_{i+1}),t_{i+2},\ldots,t_n)
\end{aligned}$$

It contains subcategories $\square_d$, $\square_{d,e}$, and $\square_{d,e,s}$ generated by the respective subsets of the above mappings.

3.2 Definitions.

- The category pCub of *precubical sets* is the presheaf $\mathsf{Set}^{\square_d^{\mathrm{op}}}$.

- The category Cub of *cubical sets* is the presheaf $\mathsf{Set}^{\square_{d,e}^{\mathrm{op}}}$.

- The category Cub_σ of *symmetric cubical sets* is the presheaf $\mathsf{Set}^{\square_{d,e,s}^{\mathrm{op}}}$.

- The category $\mathsf{Cub}_{\sigma,\gamma}$ of *symmetric cubical sets with connections* is the presheaf $\mathsf{Set}^{\square_{d,e,s,g}^{\mathrm{op}}}$.

3.3 Notation.

- The structure maps induced by the generating morphisms d_i^ν, e_i, s_i, g_i^ν will be denoted δ_i^ν, ε_i, σ_i, and γ_i^ν. They are called *face maps*, *degeneracies*, *symmetries*, respectively *connections*.

- The representables in the above categories will be denoted

$$\square^n = \mathsf{Hom}(\cdot, \{0,1\}^n)$$

3.4 Corollary. *The categories* pCub, Cub, Cub_σ, *and* $\mathsf{Cub}_{\sigma,\gamma}$ *are all complete and cocomplete. Their objects are graded sets* $A = \{A_n\}_{n\in\mathbb{N}}$ *with structure maps which satisfy the relevant subsets of the following equalities:*

$$\delta_i^\nu \delta_j^\mu = \delta_{j-1}^\mu \delta_i^\nu \quad (i<j) \qquad\qquad \varepsilon_i\varepsilon_j = \varepsilon_{j+1}\varepsilon_i \quad (i \le j)$$

$$\delta_i^\nu \varepsilon_j = \begin{cases} \varepsilon_{j-1}\delta_i^\nu & (i<j) \\ \varepsilon_j \delta_{i-1}^\nu & (i>j) \\ \mathsf{id} & (i=j) \end{cases} \qquad \begin{aligned} \sigma_i\sigma_i &= \mathsf{id} \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_j &= \sigma_j\sigma_i \quad (i \ne j-1, j, j+1) \end{aligned}$$

$$\delta_i^\nu \sigma_j = \begin{cases} \sigma_{j-1}\delta_i^\nu & (i<j) \\ \sigma_j\delta_i^\nu & (i>j+1) \\ \delta_{i+1}^\nu & (i=j) \\ \delta_{i-1}^\nu & (i=j+1) \end{cases} \qquad \sigma_i\varepsilon_j = \begin{cases} \varepsilon_{i+1} & (j=i) \\ \varepsilon_i & (j=i+1) \\ \varepsilon_j\sigma_i & (j \ne i, i+1) \end{cases}$$

$$\gamma_i^\nu \varepsilon_j = \begin{cases} \varepsilon_{j+1}\gamma_i^\nu & (i < j) \\ \varepsilon_j \gamma_{i-1}^\nu & (i > j) \\ \varepsilon_i \varepsilon_i & (i = j) \end{cases} \qquad \delta_i^\nu \gamma_j^\mu = \begin{cases} \gamma_{j-1}^\mu \delta_i^\nu & (i < j) \\ \gamma_j^\mu \delta_{i-1}^\nu & (i > j+1) \\ \mathsf{id} & (i = j, j+1;\ \nu = \mu) \\ \varepsilon_j \delta_j^\nu & (i = j, j+1;\ \nu \neq \mu) \end{cases}$$

$$\gamma_i^\nu \gamma_j^\mu = \begin{cases} \gamma_{j+1}^\mu \gamma_i^\nu & (i < j) \\ \gamma_{i+1}^\nu \gamma_i^\nu & (i = j, \nu = \mu) \end{cases} \qquad \begin{array}{l} \sigma_i \gamma_i^\nu = \gamma_i^\nu \\ \sigma_i \gamma_j^\nu = \gamma_j^\nu \sigma_i \quad (i \neq j-1, j, j+1) \end{array}$$

Limits and colimits are given pointwise; specifically, if $A = \{A_n\}$, $B = \{B_n\}$, then $(A \times B)_n = A_n \times B_n$, with structure maps $\ell(a,b) = (\ell a, \ell b)$.

3.5 Corollary. *The forgetful functors*

$$\mathsf{Cub}_{\sigma,\gamma} \longrightarrow \mathsf{Cub}_\sigma \longrightarrow \mathsf{Cub} \longrightarrow \mathsf{pCub}$$

all have left and right adjoints.

Proof: Indeed, these functors are induced by the inclusion functors

$$\Box_{d,e,s,g} \longleftarrow \Box_{d,e,s} \longleftarrow \Box_{d,e} \longleftarrow \Box_d$$

hence they have both left and right adjoints, cf. [MLM92, Thm. VII.2.2].□

3.6 Lemma. *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

be a pullback square in $\mathsf{Set}^{\Box^{\mathrm{op}}}$, for $\Box$ one of the categories of Definition 3.1, and let $b \in B$, $c \in C$ such that $h(b) = k(c)$. Then there exists a unique $a \in A$ fulfilling $f(a) = b$, $g(a) = c$.

Proof: Assume that $h(b) = k(c) \in D_n$, then $b \in B_n$, $c \in C_n$. Consider the diagram

$$\begin{tikzcd}
\square^n \arrow[dr, dotted, "\varphi"] \arrow[drr, bend left, "\hat{b}"] \arrow[ddr, bend right, "\hat{c}"'] & & \\
& A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\
& C \arrow[r, "k"'] & D
\end{tikzcd}$$

where $\hat{b} : \square^n \to B$, $\hat{c} : \square^n \to C$ are the morphisms induced by $\mathsf{id} \in \square^n(\{0,1\})^n \mapsto b$ respectively $\mathsf{id} \in \square^n(\{0,1\})^n \mapsto c$, and φ is defined by the universal property of the pullback. Let $a = \varphi(\mathsf{id})$, then by commutativity, $f(a) = b$ and $g(a) = c$. Uniqueness of a follows from the uniqueness of φ. $\square$

3.2 Truncation Functors, Coskeleta, Shells

We shall generally use $\square$ to mean any of the categories $\square_d$, $\square_{d,e}$, $\square_{d,e,s}$, and $\square_{d,e,s,g}$ of Definition 3.1.

3.7 Definitions. Let $n \in \mathbb{N}$.

- The category $\square(n)$ is the full subcategory of $\square$ generated by the objects $\{0,1\}^k$ for $k = 0, \dots, n$.
- Let $i_n : \square(n) \hookrightarrow \square(n+1)$, $j_n : \square(n) \hookrightarrow \square$ be the obvious inclusion functors. The *truncation functors* are

$$\mathrm{tr}_n = \mathsf{Set}^{i_n^{\mathrm{op}}} : \mathsf{Set}^{\square(n+1)^{\mathrm{op}}} \to \mathsf{Set}^{\square(n)^{\mathrm{op}}}$$
$$\mathrm{tr}^n = \mathsf{Set}^{j_n^{\mathrm{op}}} : \mathsf{Set}^{\square^{\mathrm{op}}} \to \mathsf{Set}^{\square(n)^{\mathrm{op}}}$$

3.8 Remark. The truncation functors are given by post-composing with the inclusions; $\mathrm{tr}_n A = A \circ i_n$, $\mathrm{tr}^n A = A \circ j_n$, and similarly for maps $f : A \to B$. By [MLM92, Thm. VII.2.2], they have both left and right adjoints, which are called *skeleton* and *coskeleton* functors, respectively, following the terminology established for simplicial sets in [Dus75].

Being left adjoints, the skeleton functors are "free" functors in the sense that they enrich the presheaf in question with higher-dimensional degeneracies. The coskeleton functors are more interesting, and we describe them below.

3.9 Definition. Let $A \in \mathsf{Set}^{\square^{\mathrm{op}}}$ and $n \in \mathbb{N}$. An *n-shell* is a $(2n+2)$-tuple $(s_1^0, s_1^1, \dots, s_{n+1}^0, s_{n+1}^1) \in A_n^{2n+2}$ such that $\delta_i^\nu s_j^\mu = \delta_{j-1}^\mu s_i^\nu$ for all $1 \le i < j \le n+1$, $\nu, \mu = 0, 1$. The set of such n-shells is denoted $\square A_n$.

3.10 Proposition. *Let $n \in \mathbb{N}$. The right adjoints to the truncation functors* tr_n, tr^n *are the* coskeleton *functors* $\mathrm{cosk}_n : \mathsf{Set}^{\square(n)^{\mathrm{op}}} \to \mathsf{Set}^{\square(n+1)^{\mathrm{op}}}$, $\mathrm{cosk}^n : \mathsf{Set}^{\square(n)^{\mathrm{op}}} \to \mathsf{Set}^{\square^{\mathrm{op}}}$ *defined as follows:*

If $A = \{A_0, \dots, A_n\} \in \mathsf{Set}^{\square(n)^{\mathrm{op}}}$, then $\mathrm{cosk}^n A = \{A_i\}_{i \in \mathbb{N}}$, with $A_{i+1} = \square A_i$ for $i \ge n$, and the relevant subset of the following structure maps:

$$
\begin{aligned}
\delta_i^\nu\big(s_1^0, \dots, s_{n+1}^1\big) &= s_i^\nu \\
\sigma_i\big(s_1^0, \dots, s_{n+1}^1\big) &= \big(s_1^0, \dots, s_{i-1}^1, s_{i+1}^0, s_{i+1}^1, s_i^0, s_i^1, s_{i+2}^0, \dots, s_{n+1}^1\big) \\
\varepsilon_i a &= \big(\varepsilon_{i-1}\delta_1^0 a, \dots, \varepsilon_{i-1}\delta_{i-1}^1 a, a, a, \varepsilon_i\delta_i^0 a, \dots, \varepsilon_i\delta_n^1 a\big) \\
\gamma_i^0 a &= \big(\gamma_{i-1}^0\delta_1^0 a, \dots, \gamma_{i-1}^0\delta_{i-1}^1 a, a, \varepsilon_i\delta_i^1 a, \\
&\qquad\qquad a, \varepsilon_i\delta_i^1 a, \gamma_i^0\delta_{i+1}^0 a, \dots, \gamma_i^0\delta_n^1 a\big) \\
\gamma_i^1 a &= \big(\gamma_{i-1}^1\delta_1^0 a, \dots, \gamma_{i-1}^1\delta_{i-1}^1 a, \varepsilon_i\delta_i^0 a, a, \\
&\qquad\qquad \varepsilon_i\delta_i^0 a, a, \gamma_i^0\delta_{i+1}^0 a, \dots, \gamma_i^0\delta_n^1 a\big)
\end{aligned}
$$

If $f = \{f_0, \dots, f_n\} : A \to B \in \mathsf{Set}^{\square(n)^{\mathrm{op}}}$, then $\mathrm{cosk}^n f : \mathrm{cosk}^n A \to \mathrm{cosk}^n B \in \mathsf{Set}^{\square^{\mathrm{op}}}$ is given by $(\mathrm{cosk}^n f)_i = f_i$ for $i = 0, \dots, n$ and

$$(\mathrm{cosk}^n f)_{i+1}\big(s_1^0, \dots, s_{i+1}^1\big) = \big((\mathrm{cosk}^n f)_i(s_1^0), \dots, (\mathrm{cosk}^n f)_i(s_{i+1}^1)\big)$$

for $i \ge n$. Furthermore, $\mathrm{cosk}_n = \mathrm{tr}^{n+1} \circ \mathrm{cosk}^n$.

3.11 Remark. A shell is uniquely determined by its faces;

$$\boldsymbol{s} = (s_1^0, \dots, s_{n+1}^1) = (\delta_1^0 \boldsymbol{s}, \dots, \delta_{n+1}^1 \boldsymbol{s})$$

The formulas for $\varepsilon_i a$ and $\gamma_i^\nu a$ above are obtained from the equations for $\delta_i^\nu \varepsilon_j$ respectively $\delta_i^\nu \gamma_j^\mu$ in Corollary 3.4.

3.12 Corollary. *The functors* cosk^n *are full and faithful for all* $n \in \mathbb{N}$.

3.13 Notation. In accordance with Definition 3.2, we let $\mathsf{pCub}^{(n)} = \mathsf{Set}^{\square(n)_d^{\mathrm{op}}}$, $\mathsf{Cub}^{(n)} = \mathsf{Set}^{\square(n)_{d,e}^{\mathrm{op}}}$, $\mathsf{Cub}_\sigma^{(n)} = \mathsf{Set}^{\square(n)_{d,e,s}^{\mathrm{op}}}$, and $\mathsf{Cub}_{\sigma,\gamma}^{(n)} = \mathsf{Set}^{\square(n)_{d,e,s,g}^{\mathrm{op}}}$.

3.3 Geometric Realisation and the Singular Cubical Set

Analogously to the situation in "usual" algebraic topology, there is a pair of adjoint functors connecting cubical sets and d-spaces. We shall see later that the geometric realisation of a precubical set is a local po-space.

3.14 Definition. The *elementary realisation* functor $\mathbb{I}_{d,e,s,g} : \square_{d,e,s,g} \to \mathsf{dTop}$ is defined by the assignments

$$\{0,1\}^n \mapsto \vec{I}^n \qquad\qquad \ell \mapsto \bar{\ell}$$

where ℓ stands for any of the generating morphisms d_i^ν, e_i, s_i, g_i^ν in $\square_{d,e,s,g}$, and the corresponding morphisms $\bar{d}_i^\nu$, $\bar{e}_i$, $\bar{s}_i$, $\bar{g}_i^\nu$ are given by

$$\begin{aligned}
\bar{d}_i^\nu : \vec{I}^n &\to \vec{I}^{n+1} \qquad (i = 1, \dots, n+1; \nu = 0,1)\\
(t_1, \dots, t_n) &\mapsto (t_1, \dots, t_{i-1}, \nu, t_i, \dots, t_n)\\
\bar{s}_i : \vec{I}^n &\to \vec{I}^n \qquad (n \geq 2; i = 1, \dots, n-1)\\
(t_1, \dots, t_n) &\mapsto (t_1, \dots, t_{i-1}, t_{i+1}, t_i, t_{i+2}, \dots, t_n)\\
\bar{e}_i : \vec{I}^n &\to \vec{I}^{n-1} \qquad (i = 1, \dots, n)\\
(t_1, \dots, t_n) &\mapsto (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n)\\
\bar{g}_i^0 : \vec{I}^n &\to \vec{I}^{n-1} \qquad (n \geq 1; i = 1, \dots, n; \nu = 0,1)\\
(t_1, \dots, t_n) &\mapsto (t_1, \dots, t_{i-1}, \max(t_i, t_{i+1}), t_{i+2}, \dots, t_n)\\
\bar{g}_i^1 : \vec{I}^n &\to \vec{I}^{n-1} \qquad (n \geq 1; i = 1, \dots, n; \nu = 0,1)\\
(t_1, \dots, t_n) &\mapsto (t_1, \dots, t_{i-1}, \min(t_i, t_{i+1}), t_{i+2}, \dots, t_n)
\end{aligned}$$

By restriction, this also defines functors $\mathbb{I}_d : \square_d \to \mathsf{dTop}$, $\mathbb{I}_{d,e} : \square_{d,e} \to \mathsf{dTop}$, and $\mathbb{I}_{d,e,s} : \square_{d,e,s} \to \mathsf{dTop}$.

3.15 Definition. Let $\mathbb{I} : \square \to \mathsf{dTop}$ be any of the functors of Definition 3.14. The *singular* $\mathbb{I}$*-set* functor $\vec{P}_{\mathbb{I}} : \mathsf{dTop} \to \mathsf{Set}^{\square^{\mathrm{op}}}$ is given by associating to $X \in \mathsf{dTop}$ the presheaf $\vec{P}_{\mathbb{I}}X : \square^{\mathrm{op}} \to \mathsf{Set}$ by the assignment

$$\begin{aligned} \vec{I}^n &\mapsto \mathsf{dTop}(\vec{I}^n, X) \\ \ell : \vec{I}^n \to \vec{I}^m &\mapsto \ \ell^{\square} : \mathsf{dTop}(\vec{I}^m, X) \to \mathsf{dTop}(\vec{I}^n, X) \\ &\qquad\qquad\qquad x \mapsto x \circ \ell \end{aligned}$$

For a d-map $f : X \to Y$, $\vec{P}_{\mathbb{I}}f : \vec{P}_{\mathbb{I}}X \stackrel{.}{\to} \vec{P}_{\mathbb{I}}Y$ is the natural transformation given by the assignment

$$\begin{aligned} \vec{I}^n \mapsto \ f_n : \mathsf{dTop}(\vec{I}^n, X) &\to \mathsf{dTop}(\vec{I}^n, Y) \\ \varphi &\mapsto f \circ \varphi \end{aligned}$$

3.16 Proposition (cf. [MLM92, Thm. I.5.2]**)** *For any of the functors* $\mathbb{I} : \square \to \mathsf{dTop}$ *of Definition 3.14, the functor* $\vec{P}_{\mathbb{I}} : \mathsf{dTop} \to \mathsf{Set}^{\square^{\mathrm{op}}}$ *has a left adjoint* $|\cdot|_{\mathbb{I}} : \mathsf{Set}^{\square^{\mathrm{op}}} \to \mathsf{dTop}$ *given as follows: For* $A \in \mathsf{Set}^{\square^{\mathrm{op}}}$,

$$|A|_{\mathbb{I}} = \Big(\coprod_{n \in \mathbb{N}} A_n \times \vec{I}^n \Big) / \equiv$$

where $\equiv$ *is the equivalence relation generated by*

$$(a \circ \ell; \boldsymbol{t}) \equiv (a; \bar{\ell}(\boldsymbol{t}))$$

for any morphism $\ell : \{0,1\}^n \to \{0,1\}^m \in \square$ *and all* $a \in A_m$, $\boldsymbol{t} \in \vec{I}^n$. *For* $f : A \to B \in \mathsf{Set}^{\square^{\mathrm{op}}}$,

$$|f|_{\mathbb{I}}(a; \boldsymbol{t}) = (f(a); \boldsymbol{t})$$

3.17 Remark. The functors $|\cdot|_{\mathbb{I}} : \mathsf{Set}^{\square^{\mathrm{op}}} \to \mathsf{dTop}$ are called *geometric realisation functors.* Being left adjoints, they preserve colimits.

3.4 Tensor Product

The product of cubical sets is not well-behaved with respect to geometric realisation, cf. [Kan55]. Hence tensor product is introduced.

3.18 Definition. The *tensor product* $C = A \otimes B$ of $A, B \in \mathsf{pCub}$ is defined by

$$C_n = \bigsqcup_{p+q=n} A_p \times B_q$$

with face maps

$$\delta_i^\nu(a,b) = \begin{cases} (\delta_i^\nu a, b) & \text{for } i \leq p \\ (a, \delta_{i-p}^\nu b) & \text{for } i \geq p+1 \end{cases} \qquad (a,b) \in A_p \times B_q$$

3.19 Definition [Kan55] The *tensor product* $C = A \otimes B$ of $A, B \in \mathsf{Cub}$ is defined by

$$C_n = \Big(\bigsqcup_{p+q=n} A_p \times B_q \Big) \Big/ {\sim_n}$$

where $\sim_n$ is the equivalence relation generated by, for all $(a,b) \in A_r \times B_s$, $r+s = n-1$, letting $(\varepsilon_{r+1} a, b) \sim_n (a, \varepsilon_1 b)$. If $a \otimes b$ denotes the equivalence class of $(a,b) \in A_p \times B_q$ under $\sim_n$, the face maps and degeneracies of C are given by

$$\delta_i^\nu(a \otimes b) = \begin{cases} \delta_i^\nu a \otimes b & (i \leq p) \\ a \otimes \delta_{i-p}^\nu b & (i \geq p+1) \end{cases}$$
$$\varepsilon_i(a \otimes b) = \begin{cases} \varepsilon_i a \otimes b & (i \leq p+1) \\ a \otimes \varepsilon_{i-p} b & (i \geq p+1) \end{cases} \tag{1}$$

3.20 Remark. This equips the categories pCub and Cub with symmetric monoidal structures.

3.21 Lemma. *If $F : \mathsf{pCub} \to \mathsf{Cub}$ denotes the free functor and $A, B \in \mathsf{pCub}$, then $F(A \otimes B)$ is isomorphic to $FA \otimes FB$.*

Proof: Using equation (1) above as a "rewriting rule," any degenerate cube $\varepsilon_{i_1}\cdots\varepsilon_{i_\ell}(a\otimes b)\in F(A\otimes B)$ can be transformed into a cube in $FA\otimes FB$. This defines a cubical morphism $F(A\otimes B)\to FA\otimes FB$, which has an inverse obtained by using the inverse rewriting rule. $\square$

3.22 Lemma. *For all $A,B\in\mathsf{pCub}$ there is a d-homeomorphism $\Theta_{A,B}:|A\otimes B|_{\mathbb{I}_d}\to|A|_{\mathbb{I}_d}\times|B|_{\mathbb{I}_d}$. Also, if $f:A\to C\in\mathsf{pCub}$, $g:B\to D\in\mathsf{pCub}$, then the diagram*

$$\begin{tikzcd}
|A\otimes B|_{\mathbb{I}_d} \arrow[r, "\Theta_{A,B}"'] \arrow[d, "|f\otimes g|"'] & |A|_{\mathbb{I}_d}\times|B|_{\mathbb{I}_d} \arrow[d, "|f|\times|g|"] \\
|C\otimes D|_{\mathbb{I}_d} \arrow[r, "\Theta_{C,D}"'] & |C|_{\mathbb{I}_d}\times|D|_{\mathbb{I}_d}
\end{tikzcd}$$

commutes. Similar statements hold with pCub replaced by Cub.

Proof: If one introduces tensor products as colimits, the lemma follows easily from the fact that the geometric realisation functor preserves colimits, cf. [Jar06]. In more elementary terms, we can write

$$\begin{aligned}
|A\otimes B|_{\mathbb{I}_d} &= \Big(\coprod_{p,q\in\mathbb{N}} A_p\times B_q\times\vec{I}^{p+q}\Big)/\equiv\\
|A|_{\mathbb{I}_d}\times|B|_{\mathbb{I}_d} &= \Big(\coprod_{p\in\mathbb{N}} A_p\times\vec{I}^{p}\Big)/\equiv\times\Big(\coprod_{q\in\mathbb{N}} B_q\times\vec{I}^{q}\Big)/\equiv\\
&= \Big(\coprod_{p,q\in\mathbb{N}} A_p\times\vec{I}^{p}\times B_q\times\vec{I}^{q}\Big)/\equiv
\end{aligned}$$

and then Θ is induced by the mapping

$$\big((a,b);t_1,\dots,t_{p+q}\big)\mapsto\big((a;t_1,\dots,t_p),(b;t_{p+1},\dots,t_{p+q})\big)$$

Commutativity of the diagram is clear. $\square$

4 Cube Paths in Precubical Sets

Cube paths in a precubical set correspond to d-paths in the geometric realisation in a certain precise way. This correspondence also lifts to homotopy of cube paths vs. d-homotopy of d-paths.

We shall not pay much attention to other cubical categories but pCub, so we will denote the geometric realisation functor simply by $|\cdot| : \mathsf{pCub} \to \mathsf{dTop}$. Also, we shall assume that for any precubical set $A = \{A_n\}$, all the A_n are disjoint.

4.1 Geometric Realisation of Geometric Precubical Sets

4.1 Definition. The *dimension* $\dim a$ of a cube $a \in A$ in a precubical set is the unique $n \in \mathbb{N}$ such that $a \in A_n$.

4.2 Definitions.

- Given $a, b \in A$ in a precubical set A such that $a = \delta_i^\nu b$ for some $i \in \mathbb{N}_+$, $\nu \in \{0, 1\}$, then a is a *direct face* of b, denoted $a \lhd b$. If $\nu = 0$, then a is a *direct lower face* of b, denoted $a \lhd_- b$. If $\nu = 1$, then a is a *direct upper face* of b, denoted $a \lhd_+ b$.

- The reflexive, transitive closures of the relations $\lhd$, $\lhd_-$, and $\lhd_+$ are denoted $\lhd^*$, $\lhd_-^*$, respectively $\lhd_+^*$. If $a \lhd^* b$ ($a \lhd_-^* b$, $a \lhd_+^* b$) for some $a, b \in A$ in a precubical set A, then a is a *face* (*lower face*, respectively *upper face*) of b.

4.3 Definitions.

- The *star* of a cube $a \in A$ in a precubical set is $\mathsf{St}\, a = \{b \in A \mid a \lhd^* b\}$. The *face set* of a is $\mathsf{Fs}\, a = \{b \in A \mid b \lhd^* a\}$.

- The *image* of a cube $a \in A_n$ in the geometric realisation of a precubical set A is the closed set
$$[a] = \{(a, t_1, \ldots, t_n) \mid 0 \leq t_j \leq 1 \text{ for all } j\} \subseteq |A|$$
The *interior image* of a is the set
$$]a[= \{(a, t_1, \ldots, t_n) \mid 0 < t_j < 1 \text{ for all } j\} \subseteq |A|$$

- The *carrier* $\mathsf{carr}\, x$ of a point $x \in |A|$ in the geometric realisation of a precubical set A is the unique cube $a \in A$ such that $x \in]a[$.

4.4 Remark. The set $]a[$ is open if and only if $\mathsf{St}\, a = \{a\}$. For $a \in A_0$, $]a[= [a]$.

4.5 Lemma. *The geometric realisation of a precubical set is Hausdorff.*

Proof: It is indeed a CW complex and hence Hausdorff. □

4.6 Lemma. *If $a \lhd^* b$ in a precubical set A, then $a = \delta_{j_1}^{\nu_1} \cdots \delta_{j_\ell}^{\nu_\ell} b$ for a sequence $(\nu_1, \ldots, \nu_\ell)$ and an* increasing *sequence $(j_1, \ldots, j_\ell)$.*

Proof: Use the precubical relation $\delta_i^\nu \delta_j^\mu = \delta_{j-1}^\mu \delta_i^\nu$ $(i < j)$ as a "rewriting rule." □

4.7 Definitions [FGR05]

- A precubical set A is *non-selflinked* if whenever $a \lhd^* b$ in A, then $a = \delta_{j_1}^{\nu_1} \cdots \delta_{j_\ell}^{\nu_\ell} b$ for a *unique* sequence $(\nu_1, \ldots, \nu_\ell)$ and a *unique* increasing sequence $(j_1, \ldots, j_\ell)$.

- A non-selflinked precubical set A is *geometric* provided that for any $a, b \in A$ such that $\mathsf{Fs}\, a \cap \mathsf{Fs}\, b \neq \emptyset$, there exists a unique $c \in \mathsf{Fs}\, a \cap \mathsf{Fs}\, b$ such that $d \lhd^* c$ for any $d \in \mathsf{Fs}\, a \cap \mathsf{Fs}\, b$.

4.8 Remark. In a *non-selflinked* precubical set A, all the natural morphisms $\square^n \to A$ are *embeddings*. In particular, any loop in a non-selflinked precubical set consists of at least two different cubes.

In the geometric realisation of a *geometric* precubical set, the intersection of two cubes, if nonempty, is a face in both cubes. Indeed, a non-selflinked precubical set A is geometric if and only if the property holds that for any $a, b \in A$ such that $[a] \cap [b] \neq \emptyset$, there exists $c \in A$ such that $[a] \cap [b] = [c]$. This is analogous to a condition one frequently sees required of *simplicial complexes*, cf. [Bre93, Def. IV.21.1].

4.9 Lemma [FGR05] *If $a = \delta_{j_1}^{\nu_1} \cdots \delta_{j_\ell}^{\nu_\ell} b$ for a sequence $(\nu_1, \dots, \nu_\ell)$ and an increasing sequence $(j_1, \dots, j_\ell)$ in a non-selflinked precubical set A, and if $J = \{j_1, \dots, j_\ell\}$, then*

$$[a] = \left\{(b; t_1, \dots, t_k) \in [b] \;\middle|\; t_j = \nu_j \text{ for all } j \in J\right\}$$

4.10 Proposition [FGR05] *For any non-selflinked precubical set A there exists a relation $\leq$ on $|A|$ such that $(|A|, \leq)$ is a local po-space and $\vec{P}|A| = \mathsf{lpoTop}(\vec{I}, |A|)$.*

4.11 Example (cf. [Gra03]**)** The following example demonstrates that the geometric realisation of a *cubical* set is not necessarily a local po-space, and also that lpoTop is not cocomplete, cf. Remark 2.16. Figure 4.1 shows a coequaliser diagram in Cub, which as geometric realisation preserves colimits, translates to a coequaliser diagram in dTop. The coequaliser space $|A|_{\mathbb{I}_{d,e}}$ contains small loops around x (some of them depicted in red), hence cannot be a local po-space. So the geometric realisation of the cubical set A is not a local po-space, and the category lpoTop does not have all coequalisers.

4.2 Cube Paths

4.12 Lemma. *Let $f : A \to B$ be a morphism of non-selflinked precubical sets and $a_1, a_2 \in A$ such that $a_1 \lhd^* a_2$ or $a_2 \lhd^* a_1$. If $f(a_1) = \delta_{i_1}^{\nu_1} \cdots \delta_{i_\ell}^{\nu_\ell} f(a_2)$, then also $a_1 = \delta_{i_1}^{\nu_1} \cdots \delta_{i_\ell}^{\nu_\ell} a_2$.*

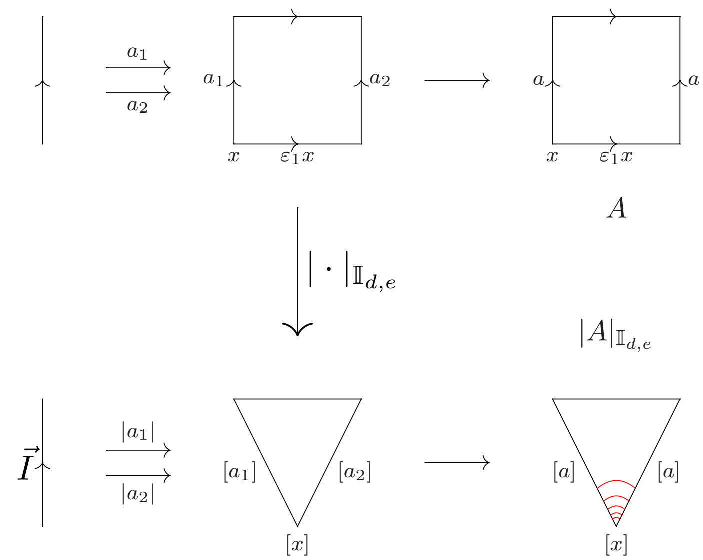

Figure 4.1: A coequaliser in Cub and its geometric realisation.

Proof: If $a_2 \lhd^* a_1$, then $f(a_2) \lhd^* f(a_1) \lhd^* f(a_2)$, hence $f(a_1) = f(a_2)$, and as B is non-selflinked, also $a_1 = a_2$. If $a_1 \lhd^* a_2$, *i.e.* $a_1 = \delta_{j_1}^{\mu_1} \cdots \delta_{j_m}^{\mu_m} a_2$, then also $f(a_1) = \delta_{j_1}^{\mu_1} \cdots \delta_{j_m}^{\mu_m} f(a_2)$, and as B is non-selflinked, $(j_1, \ldots, j_m) = (i_1, \ldots, i_\ell)$ and $(\mu_1, \ldots, \mu_m) = (\nu_1, \ldots, \nu_\ell)$. □

4.13 Definition. A *cube path* in a precubical set A is a sequence $(a_1, \ldots, a_n)$ of cubes in A such that $a_i \neq a_{i+1}$, and $a_i \lhd_-^* a_{i+1}$ or $a_{i+1} \lhd_+^* a_i$ for all $i = 1, \ldots, n-1$. The cube path is *full* if $a_i \lhd_- a_{i+1}$ or $a_{i+1} \lhd_+ a_i$ for all $i = 1, \ldots, n-1$. It is *sparse* if $a_i \lhd_-^* a_{i+1}$ if and only if $a_{i+2} \lhd_+^* a_{i+1}$ for all $i = 1, \ldots, n-2$.

4.14 Example. Consider the precubical set depicted in Figure 4.2, where a_1 is the leftmost 2-cube, and a_2, a_3 are the two 3-cubes.

- An example of a *sparse* cube path is $(a_1, \delta_1^1 a_1, a_2, \delta_1^1 \delta_1^1 a_2, a_3)$, with the directions indicated in the right diagram.
- A *full* cube path is given by $(a_1, \delta_1^1 a_1, \delta_2^0 a_2, a_2, \delta_1^1 a_2, \delta_1^1 \delta_1^1 a_2, \delta_1^0 a_3, a_3)$.

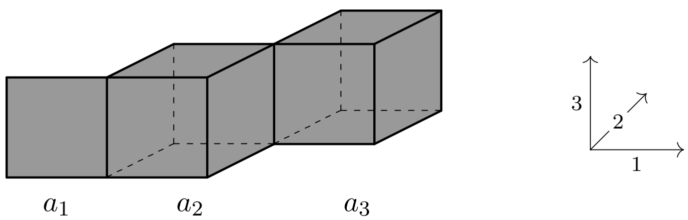

Figure 4.2: A cube path.

- An example of a general cube path, neither sparse nor full, is given by $(a_1, \delta_1^1 a_1, a_2, \delta_1^1 \delta_1^1 a_2, \delta_2^0 a_3, a_3)$.

4.15 Remarks.

- There is a different notion of cube path in [NR98], of which ours is a generalisation. For us, a cube path in the sense of [NR98] is a sparse cube path $(a_1, \ldots, a_n)$ in which $a_i \lhd_-^* a_{i+1}$ or $a_i \lhd_+^* a_{i-1}$ imply that $a_i \in A_0$.

- Cube paths in *cubical* sets can be defined analogously, and we note that if a cubical set is freely generated by a precubical set, then any of its cube paths can be turned into a *precubical* cube path by just deleting the degenerate cubes.

4.16 Definition. Given a precubical set A and $a, b \in A$, then b is *reachable from* a if there exists a cube path $(a_1, \ldots, a_n)$ in A with $a_1 = a$ and $a_n = b$.

4.17 Lemma [Faj05] *Given a d-path $p : \vec{I} \to |A|$ in the geometric realisation of a geometric precubical set A, there exists a partition of the unit interval $0 = s_1 \leq \cdots \leq s_{n+1} = 1$ and a unique cube path $(a_1, \ldots, a_n)$ in A such that $\mathsf{carr}\, p(s_i) \in \{a_{i-1}, a_i\}$, and $\mathsf{carr}\, p(s) = a_i$ for $s_i < s < s_{i+1}$.*

4.18 Definition. The *carrier sequence* $\mathsf{carrs}\, p$ of a d-path $p : \vec{I} \to |A|$ in the geometric realisation of a geometric precubical set A is the unique cube path given by Lemma 4.17.

4.19 Proposition. *Given a cube path $(a_1, \dots, a_n)$ in a geometric precubical set A and $x_1, \dots, x_n \in |A|$ such that $\mathsf{carr}\, x_i = a_i$ for all $i = 1, \dots, n$ and $x_i \le x_{i+1}$ for all $i = 1, \dots, n-1$, then there is a d-path $p : \vec{I} \to |A|$ and $0 = s_1 < \dots < s_n = 1$ such that $\mathsf{carrs}\, p = (a_1, \dots, a_n)$, and $p(s_i) = x_i$ for all $i = 1, \dots, n$.*

Proof: We shall construct d-paths p_i from x_i to x_{i+1} such that $\mathsf{carrs}\, p_i = (a_i, a_{i+1})$; then $p = p_1 * \dots * p_{n-1}$ will be as required.

Let $i \in \{1, \dots, n-1\}$, and assume first that $a_i \lhd_-^* a_{i+1}$, *i.e.* $a_i = \delta_{j_1}^0 \cdots \delta_{j_\ell}^0 a_{i+1}$ for a unique sequence $j_1 < \dots < j_\ell$. Let $J = \{j_1, \dots, j_\ell\}$, and write $x_i = (a_{i+1}; t_1, \dots, t_k)$ and $x_{i+1} = (a_{i+1}; t'_1, \dots, t'_k)$ for some $(t_1, \dots, t_k)$, $(t'_1, \dots, t'_k) \in \vec{I}^k$ and some $k \in \mathbb{N}_+$. By $x_{i+1} \in]a_{i+1}[$ we have $0 < t'_j < 1$ for all j, and by Lemma 4.9 and $\mathsf{carr}\, x_i = a_i$, $t_j = 0$ for $j \in J$ and $0 < t_j < 1$ for $j \notin J$.

Now $x_i \le x_{i+1}$ implies that $t_j \le t'_j$ for all j, hence we can define the d-path p_i by

$$p_i(s) = (a_{i+1}; (1-s)t_1 + st'_1, \dots, (1-s)t_k + st'_k)$$

Then $\mathsf{carr}\, p_i(0) = a_i$, and for $s > 0$, $0 < (1-s)t_j + st'_j < 1$ for all j, hence $\mathsf{carr}\, p_i(s) = a_{i+1}$ for $s > 0$.

In case $a_{i+1} \lhd_+^* a_i$, we can again apply Lemma 4.9 to get a sequence $j_1 < \dots < j_\ell$ such that $a_{i+1} = \delta_{j_1}^1 \cdots \delta_{j_\ell}^1 a_i$ and let $J = \{j_1, \dots, j_\ell\}$. Then $x_i = (a_i; t_1, \dots, t_k)$ and $x_{i+1} = (a_i; t'_1, \dots, t'_k)$, with $0 < t_j < 1$ for all j, and $t'_j = 1$ for $j \in J$, $0 < t'_j < 1$ for $j \notin J$. We can define p_i by

$$p_i(s) = (a_i; (1-s)t_1 + st'_1, \dots, (1-s)t_k + st'_k)$$

and then $\mathsf{carr}\, p_i(s) = a_i$ for $s < 1$ and $\mathsf{carr}\, p_i(1) = a_{i+1}$. □

4.20 Corollary. *Given a cube path $(a_1, \dots, a_n)$ in a geometric precubical set A and $x \in |A|$ such that $\mathsf{carr}\, x = a_k$ for some $k \in \{1, \dots, n\}$, then there is a d-path $p : \vec{I} \to |A|$ for which $\mathsf{carrs}\, p = (a_1, \dots, a_n)$ and $p(s) = x$ for some $0 \le s \le 1$.*

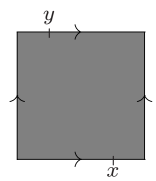

Figure 4.3: $y \notin \uparrow x$.

Proof: Starting with $x_k = x$, we shall inductively construct a sequence $x_1, \dots, x_n$ such that $\mathsf{carr}\, x_i = a_i$ for all $i = 1, \dots, n$ and $x_i \leq x_{i+1}$ for all $i = 1, \dots, n-1$. An application of Proposition 4.19 then yields the desired d-path.

Let $i \in \{1, \dots, n-1\}$ such that x_i has been constructed, but x_{i+1} has not. If $a_i \lhd_-^* a_{i+1}$, then by Lemma 4.9, $x_i = (a_{i+1}; t_1, \dots, t_k)$, with $t_j = 0$ if and only if $j \in J$ for some non-empty set $J \subseteq \{1, \dots, k\}$. Let $t'_j = t_j$ for $j \notin J$, $t'_j = \frac{1}{2}$ for $j \in J$, and let $x_{i+1} = (a_{i+1}; t'_1, \dots, t'_k)$.

If $a_{i+1} \lhd_+^* a_i$, we again apply Lemma 4.9 to get $J = \{j_1, \dots, j_\ell\}$ such that $a_{i+1} = \delta^1_{j_1} \cdots \delta^1_{j_\ell} a_i$. Write $x_i = (a_i; t_1, \dots, t_k)$, then we can let $x_{i+1} = (a_i; t'_1, \dots, t'_k)$, with $t'_j = t_j$ for $j \notin J$ and $t'_j = 1$ for $j \in J$.

The case in which x_i has been constructed, but x_{i-1} has not, is similar. □

4.21 Lemma. *Given $x, y \in |A|$ in the geometric realisation of a geometric precubical set A such that $\mathsf{carr}\, x \in A_0$ or $\mathsf{carr}\, y \in A_0$, then $y \in \uparrow x$ if and only if $\mathsf{carr}\, y$ is reachable from $\mathsf{carr}\, x$.*

4.22 Remark. The condition that $\mathsf{carr}\, x \in A_0$ or $\mathsf{carr}\, y \in A_0$ is necessary: In the precubical set in Figure 4.3, $\mathsf{carr}\, y$ is reachable from $\mathsf{carr}\, x$, but $y \notin \uparrow x$.

Proof of lemma. If $y \in \uparrow x$, and $p : \vec{I} \to |A|$ is a d-path such that $p(0) = x$ and $p(1) = y$, then $\mathsf{carrs}\, p$ is a cube path from $\mathsf{carr}\, x$ to $\mathsf{carr}\, y$. For the converse, let $\mathsf{carr}\, y$ be reachable from $\mathsf{carr}\, x$, and let $(a_1, \dots, a_n)$ be a cube path such that $a_1 = \mathsf{carr}\, x$ and $a_n = \mathsf{carr}\, y$.

If $\mathsf{carr}\, x \in A_0$, we can apply Corollary 4.20 to get a d-path $p : \vec{I} \to X$ with $\mathsf{carrs}\, p = (a_2, \dots, a_n)$ and $p(1) = y$. Hence $y \in \uparrow p(0)$, and as x is the lower corner of $a_2 = \mathsf{carr}\, p(0)$, also $p(0) \in {\uparrow}x$. Similarly, if $\mathsf{carr}\, y \in A_0$, Corollary 4.20 yields a d-path $p : \vec{I} \to X$ with $\mathsf{carrs}\, p = (a_1, \dots, a_{n-1})$ and $p(0) = x$, showing that $p(1) \in {\uparrow}x$, and also $y \in {\uparrow}p(1)$. $\square$

4.23 Lemma. *Given a morphism $f : A \to B$ of precubical sets and $x \in |A|$, then $\mathsf{carr}(|f|(x)) = f(\mathsf{carr}\, x)$.*

Proof: Let $\mathsf{carr}\, x = a \in A_n$, then $x = (a; t_1, \dots, t_n)$ for some $0 < t_1, \dots, t_n < 1$, hence $|f|(x) = (f(a); t_1, \dots, t_n)$ and $|f|(x) \in \,]f(a)[$. $\square$

4.24 Corollary. *Given a morphism $f : A \to B$ of geometric precubical sets and a d-path $p : \vec{I} \to |A|$, then $\mathsf{carrs}(|f| \circ p) = f(\mathsf{carrs}\, p)$.*

Proof: Let $\mathsf{carrs}\, p = (a_1, \dots, a_n)$, and let $0 = s_1 \le \cdots \le s_{n+1} = 1$ be a partition of the unit interval such that $\mathsf{carr}\, p(s_i) \in \{a_{i-1}, a_i\}$ and $\mathsf{carr}\, p(s) = a_i$ for $s_i < s < s_{i+1}$. By Lemma 4.23, $\mathsf{carr}(|f|(p(s_i))) \in \{f(a_{i-1}), f(a_i)\}$ and $\mathsf{carr}(|f|(p(s))) = f(a_i)$ for $s_i < s < s_{i+1}$, hence by uniqueness, $\mathsf{carrs}(|f| \circ p) = (f(a_1), \dots, f(a_n))$. $\square$

4.25 Lemma. *Given a morphism $f : A \to B$ of precubical sets, $y \in |B|$, and $a \in A$ such that $\mathsf{carr}\, y = f(a)$, then there is a unique $x \in |A|$ such that $\mathsf{carr}\, x = a$ and $y = |f|(x)$.*

Proof: If we write $y = (f(a); t_1, \dots, t_n)$, then $x = (a; t_1, \dots, t_n)$. $\square$

4.26 Lemma. *Given a morphism $f : A \to B$ of geometric precubical sets, a d-path $q : \vec{I} \to |B|$, and a cube path $(a_1, \dots, a_n)$ in A such that $\mathsf{carrs}\, q = (f(a_1), \dots, f(a_n))$, then there exists a unique d-path $p : \vec{I} \to |A|$ such that $\mathsf{carrs}\, p = (a_1, \dots, a_n)$ and $q = |f| \circ p$.*

Proof: Let $0 = s_1 \le \cdots \le s_{n+1} = 1$ be a partition of the unit interval such that $\mathsf{carr}\, q(s_i) \in \{f(a_{i-1}), f(a_i)\}$ and $\mathsf{carr}\, q(s) = f(a_i)$ for $s_i < s < s_{i+1}$. Then $q(s) \in [f(a_i)]$ for $s_i \le s \le s_{i+1}$, hence we have d-paths $q_i : [s_i, s_{i+1}] \to \vec{I}^{\dim a_i}$ such that

$$q\,|_{[s_i, s_{i+1}]}(s) = \big(f(a_i); q_i(s)\big)$$

Define $p : \vec{I} \to |A|$ by

$$p\,|_{[s_i, s_{i+1}]}(s) = \big(a_i; q_i(s)\big) \tag{1}$$

then p is continuous because q is, $q = |f| \circ p$, and $\mathsf{carrs}\, p = (a_1, \dots, a_n)$. Uniqueness of p follows from (1). $\square$

4.27 Lemma. *If*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

is a pullback square of geometric precubical sets, and $q : \vec{I} \to |B|$, $r : \vec{I} \to |C|$ are d-paths making the below diagram commute, then there exists a unique d-path $p : \vec{I} \to |A|$ filling in the diagram

$$\begin{tikzcd} \vec{I} \arrow[dr, dotted, "p"] \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "r"'] & & \\ & {|A|} \arrow[r, "|f|"'] \arrow[d, "|g|"] & {|B|} \arrow[d, "|h|"] \\ & {|C|} \arrow[r, "|k|"'] & {|D|} \end{tikzcd}$$

Moreover, if $x \in |A|$ is such that $q(0) = |f|(x)$ and $r(0) = |g|(x)$, then $p(0) = x$.

Proof: Let $\mathsf{carrs}(|h| \circ q) = \mathsf{carrs}(|k| \circ r) = (d_1, \dots, d_n)$. By Corollary 4.24, $\mathsf{carrs}\, q = (b_1, \dots, b_n)$ and $\mathsf{carrs}\, r = (c_1, \dots, c_n)$, with $h(b_i) = k(c_i) = d_i$ for all $i = 1, \dots, n$, hence Lemma 3.6 provides us with cubes $a_i \in A$ such that $f(a_i) = b_i$ and $g(a_i) = c_i$ for $i = 1, \dots, n$.

Let $i, j \in \{1, \dots, n\}$ such that $i = j \pm 1$ and $d_i = \delta^\nu_{j_1} \cdots \delta^\nu_{j_\ell} d_j$, for $\nu = 0$ or $\nu = 1$. By Lemma 4.12, also $b_i = \delta^\nu_{j_1} \cdots \delta^\nu_{j_\ell} b_j$, $c_i = \delta^\nu_{j_1} \cdots \delta^\nu_{j_\ell} c_j$, thus $f(a_i) = f(\delta^\nu_{j_1} \cdots \delta^\nu_{j_\ell} a_j)$ and $g(a_i) = g(\delta^\nu_{j_1} \cdots \delta^\nu_{j_\ell} a_j)$, and by uniqueness, $a_i = \delta^\nu_{j_1} \cdots \delta^\nu_{j_\ell} a_j$.

Hence $(a_1, \ldots, a_n)$ is a cube path in A, and we can apply Lemma 4.26 to get $p : \vec{I} \to |A|$ making the above diagram commute. Uniqueness of p follows from uniqueness of $(a_1, \ldots, a_n)$.

As for the last claim, we have

$$h\big(f(\mathsf{carr}\, x)\big) = h\big(f(\mathsf{carr}\, p(0))\big) = k\big(g(\mathsf{carr}\, p(0))\big) = k\big(g(\mathsf{carr}\, x)\big)$$

hence by the uniqueness part of Lemma 3.6, $\mathsf{carr}\, p(0) = \mathsf{carr}\, x$, and as $|f|(p(0)) = |f|(x)$, Lemma 4.25 implies $p(0) = x$ □

4.3 Homotopy of Cube Paths

4.28 Definition. A cube path $(b_1, \ldots, b_m)$ in a precubical set A is a *filler* of another cube path $(a_1, \ldots, a_n)$, denoted $(a_1, \ldots, a_n) \preceq (b_1, \ldots, b_m)$ provided that

- the sequence $(a_1, \ldots, a_n)$ is a subsequence of $(b_1, \ldots, b_m)$,
- $a_1 = b_1$ and $a_n = b_m$, and
- for each b_j there are a_i and a_k such that $a_i \lhd^* b_j \lhd^* a_k$.

4.29 Remark. The relation $\preceq$ is a partial order on the cube paths in A. Any cube path admits a (non-unique) filler which is a full cube path; these are the maximal elements with respect to $\preceq$. The minimal elements are the sparse cube paths; for any cube path there is a unique sparse cube path of which it is a filler.

4.30 Definition [Gla91] Two full cube paths $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_m)$ in a precubical set A are *adjacent* if $n = m$, $a_1 = b_1$, $a_n = b_m$, and there is at most one $i \in \{1, \ldots, n\}$ for which $a_i \neq b_i$. *Homotopy* of full cube paths in A, denoted by the symbol $\sim$, is the transitive closure of the adjacency relation.

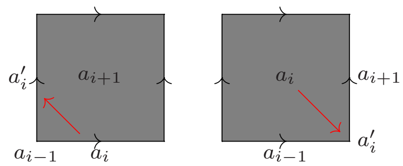

Figure 4.4: A flip and a throb.

4.31 Remarks.

- We define homotopy of *general* cube paths below.
- Adjacency of full cube paths comes in two flavours, see Figure 4.4. If $a_{i-1} \lhd_- a_i \lhd_- a_{i+1}$ for some i (or, equivalently, $a_{i+1} \lhd_+ a_i \lhd_+ a_{i-1}$), we can apply the precubical relation $\delta_i^\nu \delta_j^\mu = \delta_{j-1}^\mu \delta_i^\nu$ to replace a_i by another cube a'_i of the same dimension and get an adjacent cube path. We call this operation a *flip*.

  If $a_{i-1} \lhd_- a_i$ and $a_{i+1} \lhd_+ a_i$, and a_{i-1} and a_{i+1} have a common face a'_i, we can replace a_i by that cube a'_i (whose dimension is 2 lower than the one of a_i) and get an adjacent cube path. We call this operation, and its inverse, a *throb*.

4.32 Lemma. *Given a cube path* $(a_1, \dots, a_n)$ *in a precubical set* A *and two full cube paths* $(b_1, \dots, b_m) \succeq (a_1, \dots, a_n) \preceq (c_1, \dots, c_k)$, *then* $(b_1, \dots, b_m) \sim (c_1, \dots, c_k)$.

Proof: All full cube paths which are fillers of $(a_1, \dots, a_n)$ are connected through a sequence of flips. □

4.33 Definition. Two cube paths $(a_1, \dots, a_n)$, $(b_1, \dots, b_m)$ in a precubical set A are *homotopic*, denoted $(a_1, \dots, a_n) \sim (b_1, \dots, b_m)$, if there are full cube paths $(c_1, \dots, c_k) \succeq (a_1, \dots, a_n)$, $(d_1, \dots, d_k) \succeq (b_1, \dots, b_m)$ such that $(c_1, \dots, c_k) \sim (d_1, \dots, d_k)$. The homotopy class of a cube path $(a_1, \dots, a_n)$ is denoted $[a_1, \dots, a_n]$.

4.34 Remark. For applications, we also need homotopy of cube paths in *cubical* sets which are freely generated by precubical sets. By Remark 4.15,

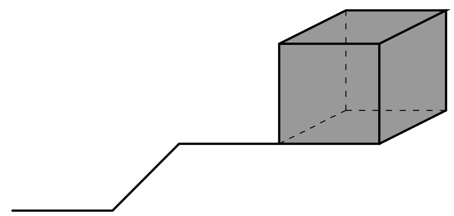

Figure 4.5: A fan-shaped cube path

these can be turned into precubical cube paths by deleting degenerate cubes, and we say that two such cube paths are homotopic if their precubical components are.

4.35 Definition. The *total dimension* of a cube path $(a_1, \ldots, a_n)$ in a precubical set A is $\mathsf{tdim}(a_1, \ldots, a_n) = \sum_{i=1}^{n} \dim a_i$.

4.36 Lemma. *If* $(a_1, \ldots, a_n) \sim (b_1, \ldots, b_n)$ *are full cube paths in a precubical set* A*, then* $\mathsf{tdim}(a_1, \ldots, a_n) \equiv \mathsf{tdim}(b_1, \ldots, b_n) \bmod 2$*.*

Proof: If two full cube paths are adjacent through a flip, their total dimensions are equal. If the adjacency is a throb, their total dimensions differ by 2. □

4.37 Definitions. Let $(a_1, \ldots, a_n)$ be a cube path in a precubical set A.

- $(a_1, \ldots, a_n)$ is *linear* if n is odd and

$$\dim a_i = \begin{cases} 0 & \text{for } i \text{ odd} \\ 1 & \text{for } i \text{ even} \end{cases}$$

- $(a_1, \ldots, a_n)$ is *fan-shaped* if either $(a_1, \ldots, a_n)$ or $(a_1, \ldots, a_{n-1})$ is linear.

See Figure 4.5 for an example of a fan-shaped cube path.

4.38 Lemma. *Given a cube path* $(a_1, \ldots, a_n)$ *in a precubical set* A *such that* $\dim a_1 = 0$*, there exists a fan-shaped cube path* $(b_1, \ldots, b_\ell) \sim (a_1, \ldots, a_n)$*.*

4.39 Remark. If also $\dim a_n = 0$, the cube path $(b_1, \dots, b_\ell)$ will be linear.

Proof of lemma. Let $(a'_1, \dots, a'_m) \succeq (a_1, \dots, a_n)$ be a full cube path. If $(a_1, \dots, a_n)$ already is fan-shaped, then

$$\dim a'_i = \begin{cases} 0 & \text{for } i \le m - \dim a_n \text{ odd} \\ 1 & \text{for } i \le m - \dim a_n \text{ even} \\ \dim a_n + i - m & \text{for } i > m - \dim a_n \end{cases}$$

hence $m - \dim a_n$ is odd, and

$$\begin{aligned} \mathsf{tdim}(a'_1, \dots, a'_m) &= \sum_{\substack{i=1 \\ i \text{ even}}}^{m-\dim a_n} 1 \quad + \sum_{i=m-\dim a_n+1}^{m} (\dim a_n + i - m) \\ &= \frac{(\dim a_n)^2 + m - 1}{2} \end{aligned}$$

Let $D_{\min} = \frac{(\dim a_n)^2+m-1}{2}$, then any full cube path $(b'_1, \dots, b'_m) \sim (a'_1, \dots, a'_m)$ has $\mathsf{tdim}(b'_1, \dots, b'_m) \ge D_{\min}$, and $\mathsf{tdim}(b'_1, \dots, b'_m) = D_{\min}$ if and only if the cube path $(b'_1, \dots, b'_{m-\dim a_n}, b'_m)$ is fan-shaped.

So if $\mathsf{tdim}(a'_1, \dots, a'_m) = D_{\min}$, we are done, and if $\mathsf{tdim}(a'_1, \dots, a'_m) > D_{\min}$, we shall construct a full cube path homotopic to $(a'_1, \dots, a'_m)$ for which the total dimension is $D - 2$. As $\mathsf{tdim}(a'_1, \dots, a'_m) \equiv D_{\min} \bmod 2$, one can iterate this construction and eventually arrive at a cube path with total dimension $D_{\min}$.

So assume that $\mathsf{tdim}(a'_1, \dots, a'_m) > D_{\min}$, then there is $k \in \{3, \dots, m-1\}$ such that $\dim a'_k \ge 2$, $a'_{k-1} \lhd_- a'_k$, and $a'_{k+1} \lhd_+ a'_k$. Taking the *minimal* such k, we can also assume that $a'_{k-2} \lhd_- a'_{k-1}$. Then $a'_{k-2} = \delta^0_{j_1} a'_{k-1}$, $a'_{k-1} = \delta^0_{j_2} a'_k$, and $a'_{k+1} = \delta^1_{j_3} a'_k$ for some indices j_1, j_2, j_3.

If $j_2 < j_3$, then

$$\delta^0_{j_2} a'_{k+1} = \delta^0_{j_2} \delta^1_{j_3} a'_k = \delta^1_{j_3-1} \delta^0_{j_2} a'_k = \delta^1_{j_3-1} a'_{k-1}$$

hence the cube path $(a'_1, \dots, a'_{k-1}, \delta^0_{j_2} a'_{k+1}, a'_{k+1}, \dots, a'_m)$ is adjacent to $(a'_1, \dots, a'_m)$, and

$$\mathsf{tdim}(a'_1, \dots, a'_{k-1}, \delta^0_{j_2} a'_{k+1}, a'_{k+1}, \dots, a'_m) = D - 2$$

If $j_2 > j_3$, then

$$\delta^1_{j_3} a'_{k-1} = \delta^1_{j_3}\delta^0_{j_2} a'_k = \delta^0_{j_2-1}\delta^1_{j_3} a'_k = \delta^0_{j_2-1} a'_{k+1}$$

so $(a'_1, \ldots, a'_{k-1}, \delta^1_{j_3} a'_{k-1}, a'_{k+1}, \ldots, a'_m)$ is adjacent to $(a'_1, \ldots, a'_m)$, and

$$\mathsf{tdim}(a'_1, \ldots, a'_{k-1}, \delta^1_{j_3} a'_{k-1}, a'_{k+1}, \ldots, a'_m) = D - 2$$

For the remaining case $j_2 = j_3$, we have to take a look at $a'_{k-2} = \delta^0_{j_1}\delta^0_{j_2} a'_k$. If $j_1 < j_2$, then $a'_{k-2} = \delta^0_{j_2-1}\delta^0_{j_1} a'_k$, so the cube path $(a'_1, \ldots, a'_m)$ is adjacent to

$$(a'_1, \ldots, a'_{k-2}, \delta^0_{j_1} a'_k, a'_k, \ldots, a'_m)$$

If $j_1 \geq j_2$, then $a'_{k-2} = \delta^0_{j_2}\delta^0_{j_1+1} a'_k$, hence $(a'_1, \ldots, a'_m)$ is adjacent to

$$(a'_1, \ldots, a'_{k-2}, \delta^0_{j_1+1} a'_k, a'_k, \ldots, a'_m)$$

In both cases, the total dimension of the cube path remains the same, and the index replacing j_2 (j_1 resp. $j_1 + 1$) is inequal to $j_2 = j_3$, hence we can go back to the case $j_2 \neq j_3$ and finish our argument. $\square$

4.40 Lemma. *Given d-paths $p, q : \vec{I} \to |\square^n|$ in the geometric realisation of the n-cube $\square^n$ for some $n \in \mathbb{N}$ and $x, y \in |\square^n|$ such that $p(0) = q(0) = x$ and $p(1) = q(1) = y$, then $p \sim q$ rel $(\{x\}, \{y\})$.*

Proof: Trivial. $\square$

4.41 Lemma. *Given d-paths $p, q : \vec{I} \to |A|$ in the geometric realisation of a geometric precubical set A such that $\mathsf{carrs}\, p = \mathsf{carrs}\, q$, then $p \sim q$ rel $\big([\mathsf{carr}\, p(0)], [\mathsf{carr}\, p(1)]\big)$.*

Proof: Let $\mathsf{carrs}\, p = \mathsf{carrs}\, q = (a_1, \ldots, a_n)$, and reparametrize p and q such that $\mathsf{carr}\, p(s) = \mathsf{carr}\, q(s)$ for all $s \in I$. Then there is a partition of the unit interval $0 = s_1 \leq \cdots \leq s_{n+1} = 1$ such that $p(s), q(s) \in [a_i]$ for $s_i \leq s \leq s_{i+1}$.

Hence we have d-paths $p_i, q_i : [s_i, s_{i+1}] \to \vec{I}^{\dim a_i}$ such that

$$p\,|_{[s_i,s_{i+1}]}(s) = \big(a_i; p_i(s)\big) \qquad\qquad q\,|_{[s_i,s_{i+1}]}(s) = \big(a_i; q_i(s)\big)$$

Define $H_i : I \times [s_i, s_{i+1}] \to |A|$ by

$$H_i(s,t) = \big(a_i; (1-s)p_i(t) + sq_i(t)\big)$$

then the horizontal concatenation $H = H_1 * \cdots * H_n$ is a dihomotopy $p \sim q$ rel $\big([\mathsf{carr}\, p(0)], [\mathsf{carr}\, p(1)]\big)$. □

4.42 Lemma. *Given d-paths $p, q : \vec{I} \to |A|$ in the geometric realisation of a geometric precubical set A such that $\mathsf{carrs}\, p \preceq \mathsf{carrs}\, q$, then $p \sim q$ rel $\big([\mathsf{carr}\, p(0)], [\mathsf{carr}\, p(1)]\big)$.*

Proof: We can assume that the carrier sequences differ at only *one* element, thus $\mathsf{carrs}\, q = (a_1, \ldots, a_n)$ and $\mathsf{carrs}\, p = (a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n)$. Assume without loss of generality that $a_{k-1} \lhd^- a_k \lhd^- a_{k+1}$. Let $0 \le s_1 < s_2 \le 1$ such that $\mathsf{carr}\, p(s_1) = a_{k-1}$ and $\mathsf{carr}\, p(s_2) = a_{k+1}$. Let $p_1 = p\,|_{[0,s_1]}$, $p_2 = p\,|_{[s_1,s_2]}$, $p_3 = p\,|_{[s_2,1]}$, then $p = p_1 * p_2 * p_3$.

Write $p(s_1) = (a_{k+1}; t_1^1, \ldots, t_1^m)$, $p(s_2) = (a_{k+1}; t_2^1, \ldots, t_2^m)$, then $0 < t_2^j < 1$ and $0 \le t_1^j \le t_2^j$ for all $j = 1, \ldots, m$. Use Lemma 4.9 to get $J = \{j_1, \ldots, j_\ell\}$ such that $a_k = \delta^0_{j_1} \cdots \delta^0_{j_\ell} a_{k+1}$ and $j_1 < \cdots < j_\ell$, and let

$$t_3^j = \begin{cases} 0 & \text{if } j \in J \\ t_2^j & \text{if } j \notin J \end{cases}$$

then the point $(a_{k+1}; t_3^1, \ldots, t_3^m) \in \,]a_k[$, and $t_1^j \le t_3^j \le t_2^j$ for all $j = 1, \ldots, m$.

Define d-paths $p_2', p_2'' : \vec{I} \to |A|$ by

$$p_2'(s) = \big(a_{k+1}; (1-s)t_1^1 + st_3^1, \ldots, (1-s)t_1^m + st_3^m\big)$$
$$p_2''(s) = \big(a_{k+1}; (1-s)t_3^1 + st_2^1, \ldots, (1-s)t_3^m + st_2^m\big)$$

then $p_2' * p_2'' \sim p_2$ rel $(p(s_1), p(s_2))$ by Lemma 4.40, and $\mathsf{carrs}(p_2' * p_2'') = (a_{k-1}, a_k, a_{k+1})$. Let $p' = p_1 * p_2' * p_2'' * p_3$, then $p \sim p'$ rel $(p(0), p(1))$, and $\mathsf{carrs}\, p' = \mathsf{carrs}\, q$, hence $p' \sim q$ rel $\big([\mathsf{carr}\, p(0)], [\mathsf{carr}\, p(1)]\big)$. □

4.43 Corollary. *Given a cube path $(a_1, \ldots, a_n)$ in a geometric precubical set A and $p, q : \vec{I} \to |A|$ such that $\mathsf{carrs}\, p \preceq (a_1, \ldots, a_n) \succeq \mathsf{carrs}\, q$, then $p \sim q$ rel $([a_1], [a_n])$.*

Proof: Let $r : \vec{I} \to |A|$ be a d-path such that $\mathsf{carrs}\, r = (a_1, \ldots, a_n)$, then by Lemma 4.42, $p \sim r$ rel $([a_1], [a_n])$ and $q \sim r$ rel $([a_1], [a_n])$. □

4.44 Lemma [Faj05] *Let $p, q : \vec{I} \to |A|$ be d-paths in the geometric realisation of a geometric precubical set A such that $\mathsf{carrs}\, p$, $\mathsf{carrs}\, q$ are* linear *cube paths and $p \sim q$ rel $(p(0), p(1))$. Then $\mathsf{carrs}\, p \sim \mathsf{carrs}\, q$.*

4.45 Corollary [Faj05] *Given d-paths $p, q : \vec{I} \to |A|$ in the geometric realisation of a geometric precubical set A such that $\mathsf{carr}\, p(0), \mathsf{carr}\, p(1) \in A_0$, then $p \sim q$ rel $(p(0), p(1))$ if and only if $p \dot{\sim} q$ rel $(p(0), p(1))$.*

Theorem A. Given d-paths $p, q : \vec{I} \to |A|$ in the geometric realisation of a geometric precubical set A, then $\mathsf{carrs}\, p \sim \mathsf{carrs}\, q$ if and only if $p \sim q$ rel $\big([\mathsf{carr}\, p(0)], [\mathsf{carr}\, p(1)]\big)$.

Proof: For the "only if" part, assume that $\mathsf{carrs}\, p \sim \mathsf{carrs}\, q$, and let $(a_1, \ldots, a_n) \succeq \mathsf{carrs}\, p$, $(b_1, \ldots, b_n) \succeq \mathsf{carrs}\, q$ be full cube paths such that $(a_1, \ldots, a_n) \sim (b_1, \ldots, b_n)$. Without loss of generality we can assume that $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are adjacent.

If the adjacency is a flip at a_i, then $(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ is again a cube path, and $(a_1, \ldots, a_n) \succeq (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \preceq (b_1, \ldots, b_n)$. Letting $r : \vec{I} \to |A|$ be a d-path with $\mathsf{carrs}\, r = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$, we can apply Corollary 4.43 twice to get $p \sim r$ rel $([a_1], [a_n])$ and $q \sim r$ rel $([a_1], [a_n])$.

If the adjacency is a throb at a_i, we can without loss of generality assume that $a_{i-1} = \delta^0_{j_1} a_i$, $a_{i+1} = \delta^1_{j_2} a_i$, $b_i = \delta^1_{j_3} a_{i-1}$, and $b_i = \delta^0_{j_4} a_{i+1}$ for some $j_1, \ldots, j_4$. Then $b_i \lhd^* a_i$. Let $p', q' : \vec{I} \to |A|$ be such that $\mathsf{carrs}\, p' = (a_1, \ldots, a_n)$ and $\mathsf{carrs}\, q' = (b_1, \ldots, b_n)$, then by Corollary 4.43, $p \sim p'$ rel $([a_1], [a_n])$ and $q \sim q'$ rel $([a_1], [a_n])$. We shall show that also $p' \sim q'$ rel $([a_1], [a_n])$.

Let $s_1 < s_2 \in I$ such that $q'(s_1) \in]a_{i-1}[$ and $q'(s_2) \in]a_{i+1}[$. Write $q'(s_1) = (a_i; t_1, \dots, t_k)$, $q'(s_2) = (a_i; t'_1, \dots, t'_k)$, then $0 < t_j < 1$ for $j \neq j_1$ and $0 < t'_j < 1$ for $j \neq j_2$, and $t_{j_1} = 0$, $t'_{j_2} = 1$. Also, $q'(s_1) \leq q'(s_2)$ implies that $t_j \leq t'_j$ for all j, hence we can define a d-path $r : \vec{I} \to |A|$ by

$$r(s) = (a_i; (1-s)t_1 + st'_1, \dots, (1-s)t_k + st'_k)$$

Then $\mathsf{carrs}\, r = (a_{i-1}, a_i, a_{i+1})$, and also $r \sim q\,|_{[s_1,s_2]}$ rel $(q(s_1), q(s_2))$ by Lemma 4.40.

Let q'' be the concatenation $q'' = q\,|_{[0,s_1]} * r * q\,|_{[s_2,1]}$, then $q' \sim q''$ rel $([a_1], [a_n])$, and as $\mathsf{carrs}\, q'' = \mathsf{carrs}\, p'$, Lemma 4.41 implies $q'' \sim p'$ rel $([a_1], [a_n])$.

To prove the "if" part of the theorem, assume that we have $p \sim q$ rel $\big([\mathsf{carr}\, p(0)], [\mathsf{carr}\, p(1)]\big)$, and let $c_0 = \delta_1^0 \cdots \delta_1^0\, \mathsf{carr}\, p(0) \in A_0$, $c_1 = \delta_1^1 \cdots \delta_1^1\, \mathsf{carr}\, p(1) \in A_0$ be the bottom left corner of $\mathsf{carr}\, p(0)$ respectively the top right corner of $\mathsf{carr}\, p(1)$.

Let $H : I \times \vec{I} \to |A|$ be a dihomotopy between p and q, *i.e.* $H(0,\cdot) = p$, $H(1,\cdot) = q$, and $H(s,0) \in [\mathsf{carr}\, p(0)]$, $H(s,1) \in [\mathsf{carr}\, p(1)]$ for all $s \in I$. Let $h_0 : I \to \vec{I}^{\dim \mathsf{carr}\, p(0)}$, $h_1 : I \to \vec{I}^{\dim \mathsf{carr}\, p(1)}$ such that

$$H(s,0) = \big(\mathsf{carr}\, p(0); h_0(s)\big) \qquad H(s,1) = \big(\mathsf{carr}\, p(1); h_1(s)\big)$$

and define $H_0, H_1 : I \times \vec{I} \to |A|$ by

$$\begin{aligned} H_0(s,t) &= \big(\mathsf{carr}\, p(0); t h_0(s)\big) \\ H_1(s,t) &= \big(\mathsf{carr}\, p(1); (1-t) h_1(s) + t(1,\dots,1)\big) \end{aligned}$$

Let $\tilde{H} = H_0 * H * H_1$ be the horizontal concatenation, and let $p_1 = \tilde{H}(0,\cdot)$, $q_1 = \tilde{H}(1,\cdot)$. Then $p_1 \sim q_1$ rel $([c_0], [c_1])$, and $\mathsf{carrs}\, p \preceq \mathsf{carrs}\, p_1$, $\mathsf{carrs}\, q \preceq \mathsf{carrs}\, q_1$.

Let $(a_1, \dots, a_n)$, $(b_1, \dots, b_m)$ be linear cube paths such that $\mathsf{carrs}\, p_1 \sim (a_1, \dots, a_n)$ and $\mathsf{carrs}\, q_1 \sim (b_1, \dots, b_m)$, cf. Lemma 4.38. Let $p_2, q_2 : \vec{I} \to |A|$ such that $\mathsf{carrs}\, p_2 = (a_1, \dots, a_n)$, $\mathsf{carrs}\, q_2 = (b_1, \dots, b_m)$. Then $\mathsf{carrs}\, p_1 \sim \mathsf{carrs}\, p_2$ and $\mathsf{carrs}\, q_1 \sim \mathsf{carrs}\, q_2$, hence $p_1 \sim p_2$ rel $([c_0], [c_1])$ and $q_1 \sim q_2$ rel $([c_0], [c_1])$ by the first part of our theorem. This implies that $p_2 \sim q_2$ rel $([c_0], [c_1])$, thus $\mathsf{carrs}\, p_2 \sim \mathsf{carrs}\, q_2$ by Lemma 4.44, whence $\mathsf{carrs}\, p \sim \mathsf{carrs}\, q$. $\square$

4.4 Universal Coverings of Precubical Sets

The universal covering of a precubical set is the combinatorial counterpart to the universal directed covering space of a local po-space.

We let $*$ denote the precubical set consisting of a single 0-cube.

4.46 Definitions.

- A *pointed cube path* in a pointed precubical set $i : * \to A$ is a cube path $(a_1, \dots, a_n)$ in A for which $a_1 = i*$.
- A cube $a \in A$ is *reachable* if it is reachable from $i*$.

4.47 Definition (cf. [Gla91]) The *universal covering* of a pointed precubical set $i : * \to A$ consists of a pointed precubical set $\tilde{i} : * \to \tilde{A}$ and a pointed morphism $\pi_A : \tilde{A} \to A$ defined as follows:

$$\begin{aligned} \tilde{A}_n &= \big\{[a_1, \dots, a_k] \bigm| (a_1, \dots, a_k) \text{ pointed cube path in } A, a_k \in A_n\big\} \\ \tilde{\delta}_i^1[a_1, \dots, a_k] &= [a_1, \dots, a_k, \delta_i^1 a_k] \\ \tilde{\delta}_i^0[a_1, \dots, a_k] &= \{(b_1, \dots, b_\ell) \mid b_\ell = \delta_i^0 a_k \\ &\qquad\qquad \text{and } (b_1, \dots, b_\ell, a_k) \sim (a_1, \dots, a_k)\} \\ \pi_A[a_1, \dots, a_k] &= a_k \end{aligned}$$

The morphism $\pi_A : \tilde{A} \to A$ is called the *covering map*.

4.48 Lemma. *The mappings $\tilde{\delta}_i^\nu$ are well-defined, and $\tilde{A}$ is indeed a precubical set.*

Proof: Let $(a_1, \dots, a_k) \sim (c_1, \dots, c_m)$ be cube paths in A. Then $a_k = c_m$ and hence also $(a_1, \dots, a_k, \delta_i^1 a_k) \sim (c_1, \dots, c_m, \delta_i^1 c_m)$, so the mappings $\tilde{\delta}_i^1$ are well-defined. For showing that the $\tilde{\delta}_i^0$ are well-defined, note first that $\tilde{\delta}_i^\nu[a_1, \dots, a_k] = \tilde{\delta}_i^\nu[c_1, \dots, c_m]$: For any cube path $(b_1, \dots, b_\ell)$ in A, $(b_1, \dots, b_\ell) \in \tilde{\delta}_i^\nu[a_1, \dots, a_k]$ if and only if $b_\ell = \delta_i^0 a_k = \delta_i^0 c_m$ and $(b_1, \dots, b_\ell, c_m) = (b_1, \dots, b_\ell, a_k) \sim (a_1, \dots, a_k) \sim (c_1, \dots, c_m)$, if and only if $(b_1, \dots, b_\ell) \in \tilde{\delta}_i^\nu[c_1, \dots, a_m]$.

We are left with showing that the sets $\tilde\delta_i^\nu[a_1,\dots,a_k]$ are non-empty. Let $(c_1,\dots,c_m)\sim(a_1,\dots,a_k)$ be fan-shaped; in case $\dim a_k=1$ we have $(c_1,\dots,c_{m-1})\in\tilde\delta_i^\nu[a_1,\dots,a_k]$, and $(c_1,\dots,c_{m-1},\delta_i^\nu c_m)\in\tilde\delta_i^\nu[a_1,\dots,a_k]$ for $\dim a_k>1$.

The precubical identity $\tilde\delta_i^\nu\tilde\delta_j^\mu=\tilde\delta_{j-1}^\mu\tilde\delta_i^\nu$ (for $i<j$) is clear for $\nu=\mu=1$. For $\nu=\mu=0$, let $(c_1,\dots,c_m)$ be a fan-shaped cube path, then

$$\tilde\delta_i^0\tilde\delta_j^0[c_1,\dots,c_m]=\begin{cases}[c_1,\dots,c_{m-1}] & \text{if } \dim c_m=2\\ [c_1,\dots,c_{m-1},\delta_i^0\delta_j^0c_m] & \text{if } \dim c_m>2\end{cases}$$

For $\nu=0$ and $\mu=1$, let again $(c_1,\dots,c_m)$ be a fan-shaped cube path, then

$$\tilde\delta_{j-1}^1\tilde\delta_i^0[c_1,\dots,c_m]=[c_1,\dots,c_{m-1},\delta_i^0c_m,\delta_i^0\delta_j^1c_m]$$

It will be enough to show that

$$(c_1,\dots,c_{m-1},\delta_i^0c_m,\delta_i^0\delta_j^1c_m)\in\tilde\delta_i^0\tilde\delta_j^1[c_1,\dots,c_m]=\tilde\delta_i^0[c_1,\dots,c_m,\delta_j^1c_m]$$

But

$$\begin{aligned}(c_1,\dots,c_m,\delta_j^1c_m)&\preceq(c_1,\dots,c_{m-1},\delta_i^0c_m,c_m,\delta_j^1c_m)\\&\sim(c_1,\dots,c_{m-1},\delta_i^0c_m,\delta_i^0\delta_j^1c_m,\delta_j^1c_m)\end{aligned}$$

so this is clear.

For $\nu=1$ and $\mu=0$, the argument is similar;

$$\tilde\delta_i^1\tilde\delta_j^0[c_1,\dots,c_m]=[c_1,\dots,c_{m-1},\delta_j^0c_m,\delta_i^1\delta_j^0c_m]$$

and

$$(c_1,\dots,c_{m-1},\delta_j^0c_m,\delta_i^1\delta_j^0c_m)\in\tilde\delta_{j-1}^0\tilde\delta_i^1[c_1,\dots,c_m]$$

because $(c_1,\dots,c_m,\delta_i^1c_m)\sim(c_1,\dots,c_{m-1},\delta_j^0c_m,\delta_i^1\delta_j^0c_m,\delta_i^1c_m)$. □

4.49 Remarks.

- As any cube path is equivalent to a fan-shaped one, it is enough to consider homotopy classes of *fan-shaped* cube paths for defining the universal covering of $i : * \to A$. This is indeed the approach taken in [Faj06], and in this paper it is also shown that the geometric realisation $|\tilde{A}|$, for a *geometric* precubical set A, is d-homeomorphic to the universal directed covering space of $|A|$ with respect to $[i*]$, cf. [Faj03]. Even though it has been a major source of inspiration for what follows, we shall make no direct use of this fact here.

- A pointed precubical morphism $f : A \to B$ induces a pointed morphism $\tilde{f} : \tilde{A} \to \tilde{B}$, given by $\tilde{f}[a_1, \dots, a_n] = [f(a_1), \dots, f(a_n)]$.

4.50 Proposition. *For any pointed precubical set $i : * \to A$, the covering map $\pi_A : \tilde{A} \to A$ is an isomorphism if and only if*

- *any $b \in A$ is reachable, and*
- *whenever $(a_1, \dots, a_n)$, $(b_1, \dots, b_m)$ are pointed cube paths in A such that $a_n = b_m$, then $(a_1, \dots, a_n) \sim (b_1, \dots, b_m)$.*

Proof: For the "if" part of the proposition, an inverse to π_A is provided by the precubical map $\psi : A \to \tilde{A}$ given by mapping $b \in A$ to $[a_1, \dots, a_n] \in \tilde{A}$, where $(a_1, \dots, a_n)$ is any pointed cube path in A with $a_n = b$.

For the other implication, assume π_A to be an isomorphism, and let $\psi : A \to \tilde{A}$ be its inverse.

Let $b \in A$, and let $\psi(b) = [b_1, \dots, b_n]$. Then $b_n = \pi_A[b_1, \dots, b_n] = b$ and $b_1 = i*$, hence b is reachable by the cube path $(b_1, \dots, b_n)$. This shows the first of the above properties.

For showing the second property, let $(a_1, \dots, a_n)$, $(b_1, \dots, b_m)$ be full pointed cube paths in A such that $a_n = b_m$, then $(\psi(a_1), \dots, \psi(a_n))$ and $(\psi(b_1), \dots, \psi(b_m))$ are cube paths in $\tilde{A}$. As $\psi(a_1) = \psi(b_1) = [i*]$, we can apply Lemma 4.52, hence $(\psi(a_1), \dots, \psi(a_n)) \sim (\psi(b_1), \dots, \psi(b_m))$. Any sequence of adjacencies from $(\psi(a_1), \dots, \psi(a_n))$ to $(\psi(b_1), \dots, \psi(b_m))$ maps via π_A to a sequence of adjacencies from $(a_1, \dots, a_n)$ to $(b_1, \dots, b_m)$, thus $(a_1, \dots, a_n) \sim (b_1, \dots, b_m)$. □

4.51 Lemma. *Let $i : * \to A$ be a pointed precubical set and $\tilde{b}, \tilde{c} \in \tilde{A}$. For $\tilde{c}$ to be reachable from $\tilde{b}$ it is necessary and sufficient that there exist $(c_1, \dots, c_m) \in \tilde{c}$ and $\ell \le m$ such that $(c_1, \dots, c_\ell) \in \tilde{b}$.*

Proof: As to sufficiency, the required cube path from $\tilde{b}$ to $\tilde{c}$ is

$$\big([c_1, \ldots, c_\ell], [c_1, \ldots, c_{\ell+1}], \ldots, [c_1, \ldots, c_m]\big)$$

For necessity, we first note that it is enough to show the claim for the case that $\tilde{c}$ is reachable from $\tilde{b}$ by a *one-step* cube path, *i.e.* we can assume that $\tilde{c} \lhd_+^* \tilde{b}$ or $\tilde{b} \lhd_-^* \tilde{c}$. Let $(b_1, \ldots, b_n) \in \tilde{b}$, and assume first that $\tilde{c} \lhd_+^* \tilde{b}$. By induction it suffices to consider the case $\tilde{c} \lhd_+ \tilde{b}$, but then $\tilde{c} = [b_1, \ldots, b_n, \delta_j^1 b_n]$ for some $j \in \mathbb{N}$.

In case $\tilde{b} \lhd_-^* \tilde{c}$, it is again sufficient to show the claim for $\tilde{b} \lhd_- \tilde{c}$. Then $(b_1, \ldots, b_n) \in \tilde{\delta}_j^0 \tilde{c}$ for some $j \in \mathbb{N}$, and thus $(b_1, \ldots, b_n, \pi_A(\tilde{c})) \in \tilde{c}$, where $\pi_A : \tilde{A} \to A$ is the covering map. □

4.52 Lemma. *Let $i : * \to A$ be a pointed precubical set, and let $(\tilde{a}_1, \ldots, \tilde{a}_n)$ and $(\tilde{b}_1, \ldots, \tilde{b}_m)$ be pointed cube paths in $\tilde{A}$ such that $\tilde{a}_n = \tilde{b}_m$. Then $(\tilde{a}_1, \ldots, \tilde{a}_n) \sim (\tilde{b}_1, \ldots, \tilde{b}_m)$.*

4.53 Remark. In light of Remark 4.49, this follows (for geometric precubical sets) from the fact that $|\pi_A| : |\tilde{A}| \to |A|$ lifts d-homotopies of d-paths, cf. [Faj03, Prop. 3.11], together with our Theorem A. We give a direct proof below.

Proof of lemma. By Lemma 4.51, there exist cube paths $(a_1^i, \ldots, a_{k_i}^i) \in \tilde{a}_i$, $(b_1^i, \ldots, b_{\ell_i}^i) \in \tilde{b}_i$ such that

$$\tilde{a}_{i+1} = [a_1^i, \ldots, a_{q_i}^i, a_{p_{i+1}}^{i+1}, \ldots, a_{k_{i+1}}^{i+1}]$$
$$\tilde{b}_{i+1} = [b_1^i, \ldots, b_{s_i}^i, b_{r_{i+1}}^{i+1}, \ldots, b_{\ell_{i+1}}^{i+1}]$$

for indices $p_i \leq q_i \leq k_i$, $r_i \leq s_i \leq \ell_i$, and $q_1 = s_1 = 1$. Then $\tilde{a}_n = [\alpha]$, $\tilde{b}_n = [\beta]$, where α, β are the cube paths

$$\alpha = (a_1^1, a_{p_2}^2, \ldots, a_{q_2}^2, a_{p_3}^3, \ldots, a_{q_3}^3, \ldots, a_{p_n}^n, \ldots, a_{k_n}^n)$$
$$\beta = (b_1^1, b_{r_2}^2, \ldots, b_{s_2}^2, b_{r_3}^3, \ldots, b_{s_3}^3, \ldots, b_{r_m}^m, \ldots, b_{\ell_m}^m)$$

Hence $\alpha \sim \beta$, so let $\alpha \preceq \gamma_1 \sim \cdots \sim \gamma_v \succeq \beta$ be a sequence of adjacencies of full cube paths. Write $\gamma_i = (c_1^i, \ldots, c_{t_i}^i)$, and let $\tilde{c}_j^i = [c_1^i, \ldots, c_j^i]$, for

$j \leq t_i$, be the homotopy classes of the prefixes of γ_i. This defines cube paths $\tilde{\gamma}_i = (\tilde{c}^i_1, \dots \tilde{c}^i_{t_i})$ in $\tilde{A}$, with the property that $\tilde{\gamma}_1 \succeq (\tilde{a}_1, \dots, \tilde{a}_n)$ and $\tilde{\gamma}_v \succeq (\tilde{b}_1, \dots, \tilde{b}_m)$.

We show that $\tilde{\gamma}_i$ is adjacent to $\tilde{\gamma}_{i+1}$ for all $i = 1, \dots, v-1$; this will finish the proof: Let $i \in \{1, \dots, v-1\}$, and let ρ be the index such that $c^i_\rho \neq c^{i+1}_\rho$ and $c^i_j = c^{i+1}_j$ for $j \neq \rho$, given by adjacency of γ_i and γ_{i+1}. Then $(c^i_1, \dots, c^i_j) \sim (c^{i+1}_1, \dots, c^{i+1}_j)$ for $j \neq \rho$ and thus $\tilde{c}^i_j = \tilde{c}^{i+1}_j$ for $j \neq \rho$, and also $\tilde{c}^i_\rho \neq \tilde{c}^{i+1}_\rho$, hence $\tilde{\gamma}_i$ and $\tilde{\gamma}_{i+1}$ are adjacent. □

4.54 Corollary (cf. [Gla91]**)** *For any pointed precubical set $i : * \to A$, the (double) covering map*

$$\pi_{\tilde{A}} = \tilde{\pi}_A : \tilde{\tilde{A}} \to \tilde{A}$$

is an isomorphism.

Proof: Indeed, any $[a_1, \dots, a_n] \in \tilde{A}$ is reachable by the pointed cube path $\big([a_1], [a_1, a_2], \dots, [a_1, \dots, a_n]\big)$, so the corollary follows from Lemma 4.52 and Proposition 4.50. □

4.55 Corollary. *Universal covering is an idempotent functor from pointed precubical sets to pointed precubical sets.*

5 Bisimulations for Higher-Dimensional Automata

5.1 One-Point Precubical Sets and ω-Tori

Some preliminary considerations, necessary for being able to label higher-dimensional automata.

5.1 Definitions.

- $A \in \mathsf{pCub}$ is called a *one-point* precubical set if A_0 is a one-element set.
- The *one-point union* of two one-point precubical sets $A, B \in \mathsf{pCub}$ is the precubical set

$$A \vee B = (A \sqcup B)/_{A_0 \sim B_0}$$

5.2 Remark. One-point precubical sets have a natural embedding into the comma category $\langle * \downarrow \mathsf{pCub}\rangle$ of pointed precubical sets. In that category, $A \vee B$ is the coproduct of A and B.

5.3 Definition. Let $A \in \mathsf{pCub}^{(1)}$ be a one-dimensional one-point precubical set, and denote by $F : \mathsf{pCub} \to \mathsf{Cub}_\sigma$ the free functor. The *free ω-torus* on A is the symmetric cubical set $\circledcirc A \subseteq \operatorname{cosk}^1 FA$ given by

$$(\circledcirc A)_n = \left\{ a \in (\operatorname{cosk}^1 FA)_n \;\middle|\; \delta_i^0 a = \delta_i^1 a \text{ for all } i = 1, \dots, n \right\}$$

5.4 Lemma. *For any one-point precubical set $A \in \mathsf{pCub}^{(1)}$, $\circledcirc A$ is freely generated by a precubical set.*

Proof (cf. [Gou02]**)** Let $\leq$ be a total order on A_1 and define $B \in \mathsf{pCub}$ by

$$\begin{aligned} B_n &= \big\{(b_1, \ldots, b_n) \in A_1^n \;\big|\; b_i \leq b_j \text{ for } 1 \leq i \leq j \leq n\big\} \\ \delta_i^\nu(b_1, \ldots, b_n) &= (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n) \end{aligned}$$

Note that B is isomorphic to a precubical subset of $\operatorname{cosk}^n A$.

If C is the symmetric cubical set freely generated by B and $\perp \notin A_1$ denotes a special symbol, then $C_n = (A_1 \cup \{\perp\})^n$ with structure maps given by

$$\begin{aligned} \delta_i^\nu(c_1, \ldots, c_n) &= (c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n) \\ \sigma_i(c_1, \ldots, c_n) &= (c_1, \ldots, c_{i-1}, c_{i+1}, c_i, c_{i+2}, \ldots, c_n) \\ \varepsilon_i(c_1, \ldots, c_n) &= (c_1, \ldots, c_{i-1}, \perp, c_i, \ldots, c_n) \end{aligned}$$

hence C is isomorphic to $\circledcirc A$. □

5.5 Remark. We shall denote the *cubical* set generating $\circledcirc A$ by $!_\leq A$, following the notation introduced in [Gou95]. This is given by

$$\begin{aligned} (!_\leq A)_n = \big\{&(b_1, \ldots, b_n) \in (A_1 \cup \{\perp\})^n \;\big| \\ &\quad b_i \leq b_j \text{ whenever } 1 \leq i \leq j \leq n \text{ and } b_i, b_j \in A_1\big\} \\ \delta_i^\nu(b_1, \ldots, b_n) &= (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n) \end{aligned}$$

As the $!_\leq A$ are all isomorphic, we can forget about the ordering $\leq$ and simply write $!A$.

5.6 Lemma. *The full subcategory of $\mathsf{Cub}^{(1)}$ spanned by one-dimensional cubical sets freely generated by one-dimensional precubical sets is isomorphic to the category of finite sets and partial functions.*

Proof: The isomorphism is provided by the functor Φ which forgets the 0-dimensional structure. That is, given $A = \{A_0, A_1\} \in \mathsf{pCub}^{(1)}$, let $\Phi A = A_1$, and for $f : FA \to FB \in \mathsf{Cub}^{(1)}$, $\Phi f : A_1 \to B_1$ is the partial function defined by $\Phi f(a) = f_1(a)$ if $f_1(a)$ is non-degenerate. □

5.7 Corollary. *The full subcategory of* Cub *spanned by the cubical sets* $!A$ *for all one-point precubical sets* $A \in \mathsf{pCub}^{(1)}$ *is isomorphic to the category of finite sets and partial functions.*

Proof: This follows from Lemma 5.6 and Corollary 3.12. □

5.8 Proposition. *If* $A, B \in \mathsf{pCub}^{(1)}$ *are one-point precubical sets, then*

$$!A \otimes !B \cong !(A \vee B)$$

Proof: We use the representation from Remark 5.5. Let $\leq_A$, $\leq_B$ be total orders on A_1 respectively B_1, and let $\leq$ be the total order on $A \sqcup B$ generated by

$$\big\{ \leq_A \cup \leq_B \cup (a,b) \;\big|\; a \in A_1, b \in B_1 \big\}$$

Then $!A \otimes !B$ is freely generated by the precubical set C given by

$$\begin{aligned} C_n = \big\{ (a_1, \ldots, a_p, b_1, \ldots, b_q) \;\big|\; & p+q = n, a_i \leq_A a_j \in A_1, b_k \leq_B b_\ell \in B_1 \\ & \text{for } 1 \leq i \leq j \leq p \leq k \leq \ell \leq q \big\} \\ \delta_i^\nu(c_1, \ldots, c_n) = {} & (c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n) \end{aligned}$$

and $!(A \vee B)$ is freely generated by the precubical set D given as

$$\begin{aligned} D_n &= \big\{ (d_1, \ldots, d_n) \;\big|\; d_i \leq d_j \in A_1 \cup B_1 \text{ for } 1 \leq i \leq j \leq n \big\} \\ \delta_i^\nu(d_1, \ldots, d_n) &= (d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n) \end{aligned}$$

As C and D are isomorphic, so are $!A \otimes !B$ and $!(A \vee B)$. □

5.2 Higher-Dimensional Automata

5.9 Definitions.

- A *higher-dimensional automaton* is an object $\langle * \to A \rangle$ of the comma category $\langle * \downarrow \mathsf{Cub} \rangle$ where A is freely generated by a *geometric* precubical set.

- A *labeled higher-dimensional automaton* is an object $\langle * \to A \xrightarrow{\lambda} !\Sigma \rangle$ of the arrow category $\langle * \to \mathsf{Cub} \to \mathsf{Cub} \rangle$ where $\langle * \to A \rangle$ is a higher-dimensional automaton, Σ is a one-dimensional one-point precubical set, and λ is freely generated by a precubical map.

5.10 Remark. The restriction to *geometric* precubical sets is necessary for making our later arguments work. It is also quite natural, cf. Remark 4.8.

5.11 Definitions.

- The *tensor product* of two labeled higher-dimensional automata $\langle * \xrightarrow{i} A \xrightarrow{\lambda} !\Sigma \rangle$, $\langle * \xrightarrow{j} B \xrightarrow{\mu} !\Xi \rangle$ is the labeled higher-dimensional automaton

$$* \xrightarrow{i \otimes j} A \otimes B \xrightarrow{\lambda \otimes \mu} !\Sigma \otimes !\Xi$$

- The *relabeling* of a labeled higher-dimensional automaton $\langle * \to A \xrightarrow{\lambda} !\Sigma \rangle$ by a mapping $\sigma : !\Sigma \to !\Xi$ which is freely generated by a precubical morphism is the labeled higher-dimensional automaton

$$* \to A \xrightarrow{\sigma \circ \lambda} !\Xi$$

- The *restriction* of a labeled higher-dimensional automaton $\langle * \to B \xrightarrow{\mu} !\Xi \rangle$ by a mapping $\sigma : !\Sigma \to !\Xi$ which is injective and freely generated by a precubical morphism is $\langle * \to A \xrightarrow{\lambda} !\Sigma \rangle$ as defined by the pullback diagram

$$\begin{tikzcd}
* \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& A \arrow[r, dotted, "f"] \arrow[d, dotted, "\lambda"] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "\mu"] \\
& !\Sigma \arrow[r, "\sigma"'] & !\Xi
\end{tikzcd}$$

5.12 Remark. The above operations generalise the respective constructions for transition systems, cf. [WN95]. As noted therein, these three operations are sufficient for expressing arbitrary parallel compositions.

5.13 Lemma. *The above operations on higher-dimensional automata are well-defined.*

Proof: Tensor product is well-defined because of Proposition 5.8, and for relabeling there is nothing to prove. As to restriction being well-defined, we note first that the arrow $* \to !\Sigma$ above is unique because $!\Sigma$ is a one-point cubical set, and that the outer square in the diagram commutes, hence $\langle * \to A \to !\Sigma \rangle$ exists and is unique up to isomorphism.

We miss to show that λ is freely generated by a precubical morphism. Let $a \in A_n$, and assume that $\lambda(a)$ is degenerate, *i.e.* $\lambda(a) = \varepsilon_i \delta_i^0 \lambda(a)$ for some $i \in \mathbb{N}$. Then $\mu(f(a)) = \sigma(\lambda(a)) = \varepsilon_i \delta_i^0 \mu(f(a))$, and as μ is freely generated by a precubical morphism, $f(a) = \varepsilon_i \delta_i^0 f(a)$. But then $\lambda(a) = \lambda(\varepsilon_i \delta_i^0 a)$ and $f(a) = f(\varepsilon_i \delta_i^0 a)$, hence $a = \varepsilon_i \delta_i^0 a$ by the uniqueness part of Lemma 3.6. □

5.3 A First Notion of Simulation

Our first category of higher-dimensional automata is a straight-forward generalisation of the category of transition systems.

5.14 Remark. Pointed cube paths in a higher-dimensional automaton express *computations*, cf. [Gla06].

5.15 Definition. The category $\ell\mathsf{HDA}$ of *labeled higher-dimensional automata and simulations* is the full subcategory of the arrow category $\langle * \to \mathsf{Cub} \to \mathsf{Cub} \rangle$ spanned by the labeled higher-dimensional automata.

5.16 Remarks.

- The morphisms in $\ell\mathsf{HDA}$ are commuting diagrams of the form

$$\begin{tikzcd} * \arrow[r, equal] \arrow[d] & * \arrow[d] \\ A \arrow[r, red, "f"'] \arrow[d] & B \arrow[d] \\ !\Sigma \arrow[r, red, "\sigma"'] & !\Xi \end{tikzcd}$$

  where the black arrows represent cubical morphisms which are freely generated by precubical morphisms, and the red arrows indicate cubical morphisms which may not be freely generated by precubical morphisms. We shall denote these morphisms by $(f, \sigma) : \langle * \to A \to !\Sigma\rangle \to \langle * \to B \to !\Xi\rangle$.

- By Corollary 5.7, the label part $\sigma : !\Sigma \to !\Xi$ of a simulation (f, σ) is induced by a partial function $\Sigma \to \Xi$. Hence a simulation (f, σ) consists of a structural morphism f and a partial label function σ, in the spirit of [WN95].

5.17 Definitions.

- The *tensor product* of $(f, \sigma) : \langle * \to A_1 \to !\Sigma_1\rangle \to \langle * \to A_2 \to !\Sigma_2\rangle \in \ell\mathsf{HDA}$ and $(g, \xi) : \langle * \to B_1 \to !\Xi_1\rangle \to \langle * \to B_2 \to !\Xi_2\rangle \in \ell\mathsf{HDA}$ is the simulation

$$(f \otimes g, \sigma \otimes \xi) : \langle * \to A_1 \otimes B_1 \to !\Sigma_1 \otimes !\Xi_1\rangle \to \langle * \to A_2 \otimes B_2 \to !\Sigma_2 \otimes \Xi_2\rangle$$

- The *relabeling* of $(f, \mathsf{id}) : \langle * \to A \xrightarrow{\lambda} !\Sigma\rangle \to \langle * \to B \xrightarrow{\mu} !\Sigma\rangle \in \ell\mathsf{HDA}$ by a mapping $\sigma : !\Sigma \to !\Xi$ which is freely generated by a precubical morphism is the simulation

$$(f, \mathsf{id}) : \langle * \to A \xrightarrow{\sigma\circ\lambda} !\Xi\rangle \to \langle * \to B \xrightarrow{\mu\circ\lambda} !\Xi\rangle$$

- The *restriction* of $(g, \mathsf{id}) : \langle * \to B_1 \to !\Xi\rangle \to \langle * \to B_2 \to !\Xi\rangle \in \ell\mathsf{HDA}$ by a mapping $\sigma : !\Sigma \to !\Xi$ which is injective and freely generated by a precubical morphism is the simulation $(f, \mathsf{id}) : \langle * \to A_1 \to !\Sigma\rangle \to \langle * \to$

$A_2 \to !\Sigma\rangle$ as defined by the double pullback diagram

$$\begin{tikzcd}
 & \ast \arrow[dl] \arrow[dd, bend right] \arrow[ddr, bend left] \arrow[drr] & & \\
A_1 \arrow[rrr, dotted] \arrow[dr, dotted, "f"] \arrow[ddr, dotted, bend right] \arrow[dr, phantom, "\lrcorner", very near start] & & & B_1 \arrow[dl, "g"'] \arrow[ddl, bend left] \\
 & A_2 \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & B_2 \arrow[d] & \\
 & !\Sigma \arrow[r, "\sigma"'] & !\Xi &
\end{tikzcd}$$

5.18 Remark. Restriction and relabeling of morphisms as above can also be expressed using the *cartesian* respectively *cocartesian* property of these operations on objects, cf. [WN95]. For tensor product the analogy breaks, as tensor product is not the categorical product.

5.4 Bisimilarity

We express bisimilarity through spans of certain bisimulation maps. Bisimilar higher-dimensional automata share a labeling set, so the label part of a bisimulation map is the identity.

5.19 Definition. A simulation $(f, \sigma) : \langle \ast \to A \to !\Sigma\rangle \to \langle \ast \to B \to !\Xi\rangle \in \ell\mathsf{HDA}$ is *non-contracting* if f and σ are freely generated by precubical morphisms.

5.20 Definition. A non-contracting simulation $(f, \mathsf{id}) : \langle \ast \to A \to !\Sigma\rangle \to \langle \ast \to B \to !\Sigma\rangle \in \ell\mathsf{HDA}$ is a *bisimulation map* provided that for any reachable $a_1 \in A$ and any cube path $(b_1, \ldots, b_m)$ in B such that $b_1 = f(a_1)$, there exists a cube path $(a_1, \ldots, a_m)$ in A such that $b_i = f(a_i)$ for all $i = 1, \ldots, m$.

5.21 Remark. *Any* simulation $(f, \mathsf{id}) : \langle * \to A \xrightarrow{\lambda} !\Sigma\rangle \to \langle * \to B \xrightarrow{\mu} !\Sigma\rangle$ gives rise to a diagram in Cub

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\lambda"'] & B \arrow[d, "\mu"] \\
!\Sigma \arrow[r, "\mathsf{id}"'] & !\Sigma
\end{tikzcd}$$

where λ and μ are freely generated by precubical morphisms, hence so is f. So the condition on (f, id) to be non-contracting in the above definition is redundant.

5.22 Proposition. *A non-contracting simulation* $(f, \mathsf{id}) : \langle * \to A \to !\Sigma\rangle \to \langle * \to B \to !\Sigma\rangle \in \ell\mathsf{HDA}$ *is a bisimulation map if and only if it satisfies the property that for any reachable* $a_1 \in A$ *and for any* $b_2 \in B$ *such that* $f(a_1) = \delta^0_j b_2$ *for some* $j \in \mathbb{N}$*, there exists* $a_2 \in A$ *such that* $a_1 = \delta^0_j a_2$ *and* $b_2 = f(a_2)$*.*

Proof: For the "only if" part, assume that f is a bisimulation map and let $a_1 \in A$, $b_2 \in B$, and $j \in \mathbb{N}$ such that a_1 is reachable and $f(a_1) = \delta^0_j b_2$. Hence we have a cube path (a_1, a_2) in A such that $b_2 = f(a_2)$. Supposing that $a_2 \lhd^*_+ a_1$ leads to the contradiction that $b_2 \lhd^* f(a_1) \lhd b_2$, thus $a_1 \lhd^*_- a_2$, and as B is non-selflinked, $a_1 = \delta^0_j a_2$.

To show the reverse implication, assume that f satisfies the property of the lemma; by induction it is enough to show that f lifts cube paths of length 2. Let $a_1 \in A$ be reachable, $b_1 = f(a_1)$, and let (b_1, b_2) be a cube path in B.

If $b_2 \lhd^*_+ b_1$, *i.e.* $b_2 = \delta^1_{j_1} \cdots \delta^1_{j_\ell} b_1$, then the cube path $(a_1, \delta^1_{j_1} \cdots \delta^1_{j_\ell} a_1)$ has the property required by the definition. If $b_1 \lhd^*_- b_2$, *i.e.* $b_1 = \delta^0_{j_1} \cdots \delta^0_{j_\ell} b_2$, we can apply the property of the lemma ℓ times to get a cube $a_2 \in A$ such that $a_1 = \delta^0_{j_1} \cdots \delta^0_{j_\ell} a_2$ and $b_2 = f(a_2)$. $\square$

5.23 Definition (cf. [JNW96]**)** Two labeled higher-dimensional automata $\langle * \to A \to !\Sigma\rangle$, $\langle * \to B \to !\Sigma\rangle$ are *bisimilar* if there exists a third higher-dimensional automaton $\langle * \to C \to !\Sigma \rangle$ and a span of bisimulation maps

$$\begin{tikzcd}
* \arrow[r, equal] \arrow[d] & * \arrow[r, equal] \arrow[d] & * \arrow[d] \\
A \arrow[d] & C \arrow[l] \arrow[r] \arrow[d] & B \arrow[d] \\
!\Sigma & !\Sigma \arrow[l, "\mathsf{id}"] \arrow[r, "\mathsf{id}"'] & !\Sigma
\end{tikzcd}$$

5.24 Proposition. *Two labeled higher-dimensional automata $\langle * \xrightarrow{i} A \xrightarrow{\lambda} !\Sigma \rangle$, $\langle * \xrightarrow{j} B \xrightarrow{\mu} !\Sigma \rangle$ are bisimilar if and only if there exists a precubical set (a* bisimulation relation*) $R \subseteq A \times B$ such that $(i*, j*) \in R_0$, $(a, b) \in R$ implies $\lambda a = \mu b$, and for all reachable $a_1 \in A$, $b_1 \in B$ such that $(a_1, b_1) \in R$,*

- *if $a_1 = \delta^0_j a_2$ for some $a_2 \in A$ and some $j \in \mathbb{N}$, then also $b_1 = \delta^0_j b_2$ for some $b_2 \in B$ such that $(a_2, b_2) \in R$, and*
- *if $b_1 = \delta^0_j b_2$ for some $b_2 \in B$ and some $j \in \mathbb{N}$, then also $a_1 = \delta^0_j a_2$ for some $a_2 \in A$ such that $(a_2, b_2) \in R$.*

Proof of proposition: Note first that if $R_1 \subseteq A \times B$, $R_2 \subseteq B \times C$ are bisimulation relations, for higher-dimensional automata $\langle * \to A \to !\Sigma \rangle$, $\langle * \to B \to !\Sigma \rangle$, $\langle * \to C \to !\Sigma \rangle$, then the relation $R_2 \circ R_1 \subseteq A \times C$ given by

$$R_2 \circ R_1 = \big\{ (a, c) \subseteq A \times C \bigm| \exists\, b \in B : (a, b) \in R_1, (b, c) \in R_2 \big\}$$

is again a bisimulation relation. Hence being related by a bisimulation relation is itself a transitive relation; it is also reflexive and symmetric.

Now if $f : C \to A$ is a bisimulation map, then $R = \{(c, f(c)) \mid c \in C\}$ is a bisimulation relation as required, hence by the above remark on transitivity, we are done. For the converse, assume $R \subseteq A \times B$ to be a bisimulation relation, and let $f : R \to A$, $g : R \to B$ be the restrictions of the projection morphisms $A \leftarrow A \times B \to B$. Then f and g are bisimulation maps by Proposition 5.22. $\square$

5.5 Bisimilarity is Compositional

5.25 Lemma. *If* $(f, \mathsf{id}) : \langle * \to A_1 \to !\Sigma\rangle \to \langle * \to A_2 \to !\Sigma\rangle$ *and* $(g, \mathsf{id}) : \langle * \to B_1 \to !\Xi\rangle \to \langle * \to B_2 \to !\Xi\rangle$ *are bisimulation maps, then so is* $(f \otimes g, \mathsf{id}) : \langle * \to A_1 \otimes B_1 \to !\Sigma \otimes !\Xi\rangle \to \langle * \to A_2 \otimes B_2 \to !\Sigma \otimes !\Xi\rangle$.

Proof: We use Proposition 5.22. Let $a_1 \otimes b_1 \in A_1 \otimes B_1$ be reachable, and let $a_2 \otimes b_2 \in A_2 \otimes B_2$ such that $f \otimes g(a_1 \otimes b_1) = f(a_1) \otimes g(b_1) = \delta_i^0(a_2 \otimes b_2)$ for some i. Assume first that $\delta_i^0(a_2 \otimes b_2) = \delta_i^0 a_2 \otimes b_2$, then $f(a_1) = \delta_i^0 a_2$ and $g(b_1) = b_2$. As f is a bisimulation map, we have $c \in A_1$ such that $a_1 = \delta_i^0 c$ and $a_2 = f(c)$, hence $f \otimes g(c \otimes b_1) = a_2 \otimes b_2$ and $\delta_i^0(c \otimes b_1) = a_1 \otimes b_1$. The proof for the case $\delta_i^0(a_2 \otimes b_2) = a_2 \otimes \delta_{i-p}^0 b_2$ is similar, using the bisimulation property of g. □

5.26 Lemma. *Given a bisimulation map* $(g, \mathsf{id}) : \langle * \xrightarrow{j_1} B_1 \xrightarrow{\lambda} !\Xi\rangle \to \langle * \xrightarrow{j_2} B_2 \xrightarrow{\mu} !\Xi\rangle$ *and* $\sigma : !\Sigma \to !\Xi$ *injective and freely generated by a precubical morphism, then the restriction* $(f, \mathsf{id}) : \langle * \xrightarrow{i_1} A_1 \xrightarrow{\iota} !\Sigma\rangle \to \langle * \xrightarrow{i_2} A_2 \xrightarrow{\kappa} !\Sigma\rangle$ *given by the below double pullback diagram is again a bisimulation map:*

$$\begin{tikzcd}
 & & \ast \arrow[dll, "i_1"'] \arrow[drr, "j_1"] \arrow[ddl, bend right, "i_2"'] \arrow[ddr, bend left, "j_2"] & & \\
A_1 \arrow[rrrr, dotted, "k"] \arrow[dr, dotted, "f"] \arrow[ddr, dotted, bend right, "\iota"'] \arrow[dr, phantom, very near start, "\lrcorner"] & & & & B_1 \arrow[dl, "g"'] \arrow[ddl, bend left, "\lambda"] \\
 & A_2 \arrow[rr, dotted, "h"] \arrow[d, dotted, "\kappa"] \arrow[dr, phantom, very near start, "\lrcorner"] & & B_2 \arrow[d, "\mu"] & \\
 & !\Sigma \arrow[rr, "\sigma"'] & & !\Xi &
\end{tikzcd}$$

Proof: Note [Bor94, Prop. 2.5.3] that injectivity of σ implies that also h and k are injective.

Let $a_1 \in A_1$, $a_2 \in A_2$, and $i \in \mathbb{N}_+$ such that $f(a_1) = \delta_i^0 a_2$. Then $g(k(a_1)) = h(f(a_1)) = \delta_i^0 h(a_2)$, and as g is a bisimulation map, we have $b_1 \in B_1$ such that $k(a_1) = \delta_i^0 b_1$ and $g(b_1) = h(a_2)$.

Now $\lambda(b_1) = \mu(g(b_1)) = \mu(h(a_2)) = \sigma(\kappa(a_2))$, so an application of Lemma 3.6 to the outer pullback square gives $c \in A_1$ such that $k(c) = b_1$. Then $k(\delta_i^0 c) = \delta_i^0 b_1 = k(a_1)$, hence $\delta_i^0 c = a_1$ by injectivity of k, and $h(f(c)) = g(k(c)) = g(b_1) = h(a_2)$, hence $f(c) = a_2$ by injectivity of h. □

5.27 Corollary. *Tensor product, restriction, and relabeling respect bisimilarity.*

Proof: The last claim is clear, and the other two follow from the lemmas above, using spans of bisimulation maps. □

5.6 Bisimulation Maps Lift Directed Paths

Theorem B. For a non-contracting simulation $(f, \mathsf{id}) : \langle * \xrightarrow{i} A \to !\Sigma\rangle \to \langle * \to B \to !\Sigma\rangle \in \ell\mathsf{HDA}$ to be a bisimulation map it is necessary and sufficient that for any $x \in \uparrow[i*]$ and any d-path $q : \vec{I} \to |B|$ such that $q(0) = |f|(x)$, there exists a d-path $p : \vec{I} \to |A|$ such that $p(0) = x$ and $q = |f| \circ p$.

5.28 Remark. If $r : \vec{I} \to |A|$ is a d-path with $r(1) = x$, we can let

$$q' : \overrightarrow{[0,2]} \to |B| \qquad p' : \overrightarrow{[0,2]} \to |A|$$

be the d-maps

$$q'(s) = \begin{cases} |f|(r(s)) & (0 \le s \le 1) \\ q(s) & (1 \le s \le 2) \end{cases} \qquad\qquad p'(s) = \begin{cases} r(s) & (0 \le s \le 1) \\ p(s) & (1 \le s \le 2) \end{cases}$$

and encode the above property in the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, hook] & {|*|} \arrow[d, "|i|"] \\
\vec{I} \arrow[r, "r"] \arrow[d, hook] & {|A|} \arrow[d, "|f|"] \\
\overrightarrow{[0,2]} \arrow[r, "q'"'] \arrow[ur, dotted, "p'"] & {|B|}
\end{tikzcd}$$

Hence, in case all of A is reachable, f is a bisimulation map if and only if $|f|$ has the *right-lifting property* with respect to the inclusion $\vec{I} \hookrightarrow \overrightarrow{[0,2]}$, or equivalently, to the inclusion $0 \hookrightarrow \vec{I}$.

Proof of theorem. To show necessity, let f be a bisimulation map, and let $x \in \uparrow[i*]$ and $q : \vec{I} \to |B|$ be such that $q(0) = |f|(x)$. Let $\mathsf{carrs}\, q = (b_1, \dots, b_m)$ and $a_1 = \mathsf{carr}\, x$, then a_1 is reachable by Lemma 4.21, and $b_1 = f(a_1)$ by Lemma 4.23. Hence we have a cube path $(a_1, \dots, a_m)$ in A such that $b_i = f(a_i)$ for all $i = 1, \dots, m$, and by Lemma 4.26 there exists $p : \vec{I} \to |A|$ such that $q = |f| \circ p$. The construction in the proof of that lemma entails that $p(0) = x$.

For the other direction we use Proposition 5.22. Assume $|f|$ to have the lifting property of the theorem, let $a_1 \in A$ be reachable and $b_2 \in B$, $j \in \mathbb{N}$ such that $f(a_1) = \delta_j^0 b_2$. Let $x \in]a_1[$, then $x \in \uparrow[i*]$ by Lemma 4.21. Let $q : \vec{I} \to |B|$ such that $\mathsf{carrs}\, q = (f(a_1), b_2)$ and $q(0) = |f|(x)$; existence of q follows from Corollary 4.20.

The lifting property now gives us $p : \vec{I} \to |A|$ fulfilling $p(0) = x$ and $q = |f| \circ p$. By Corollary 4.24, $\mathsf{carrs}\, p = (a_1, a_2)$, with $f(a_2) = b_2$, and Lemma 4.12 entails that $a_1 = \delta_j^0 a_2$. □

5.29 Remark. Using Theorem B, we can devise new proofs of Lemmas 5.25 and 5.26 on the compositionality of bisimilarity:

Lemma 5.25 now follows easily from Lemma 3.22 and the fact that if $|f| : |A_1| \to |A_2|$ and $|g| : |B_1| \to |B_2|$ have the lifting property of the theorem, then so has $|f| \times |g| : |A_1| \times |B_1| \to |A_2| \times |B_2|$.

As to Lemma 5.26, let $x \in [i_1*]$ and $q : \vec{I} \to |A_2|$ be such that $q(0) = |f|(x)$. Let $q' = |h| \circ q : \vec{I} \to |B_2|$, then $q'(0) = |g|\big(|k|(x)\big)$ and $|k|(x) \in [j_1*]$, and by the lifting property of $|g|$, we have a d-path $p' : \vec{I} \to |B_1|$ such that $p'(0) = |k|(x)$ and $q' = |g| \circ p'$. Hence $|h| \circ q = |g| \circ p'$, and an application of Lemma 4.27 provides us with a d-path $p : \vec{I} \to |A_1|$ filling in the diagram

$$
\begin{tikzcd}
\vec{I} \arrow[dr, dotted, "p"] \arrow[drr, bend left, "p'"] \arrow[ddr, bend right, "q"'] & & \\
& |A_1| \arrow[r, "|k|"'] \arrow[d, "|f|"] & |B_1| \arrow[d, "|g|"] \\
& |A_2| \arrow[r, "|h|"'] & |B_2|
\end{tikzcd}
$$

By the last part of that lemma, $p(0) = x$. □

5.7 Simulation up to Homotopy

Our second category of higher-dimensional automata takes homotopy of computations into account. Here simulations and bisimulations are not exact; they merely match computations up to homotopy.

5.30 Definition. Given labeled higher-dimensional automata $\langle * \to A \xrightarrow{\lambda} !\Sigma \rangle$, $\langle * \to B \xrightarrow{\mu} !\Xi \rangle$ and cubical morphisms $f : \langle * \to A \rangle \to \langle * \to B \rangle$, $\sigma : !\Sigma \to !\Xi$, then (f, σ) is a *homotopy simulation* provided that for any pointed cube path $(a_1, \dots, a_n)$ in A with $\dim a_n = 0$, there exists a pointed cube path $(b_1, \dots, b_n) \sim \big(f(a_1), \dots, f(a_n)\big)$ in B such that $\mu(b_i) = \sigma(\lambda(a_i))$ for all $i = 1, \dots, n$. The category of labeled higher-dimensional automata and homotopy simulations is denoted $\ell\mathsf{HDA}^{\mathrm{h}}$.

5.31 Remark. If the condition $\dim a_n = 0$ were omitted, we would have $\mu(f(a_n)) = \sigma(\lambda(a_n))$ for any pointed cube path $(a_1, \dots, a_n)$, hence for the reachable part of A, (f, σ) would be a simulation as of Definition 5.15. The present condition expresses the property that for any computation $(a_1, \dots, a_n)$ in A, there exists a computation $(b_1, \dots, b_n)$ in B for which $\mu(b_i) = \sigma(\lambda(a_i))$, and whose *completion* $(b_1, \dots, b_n, c^+(b_n))$ (where $c^+(b_n)$ denotes the upper corner of b_n) is homotopic to the completion of the image of $(a_1, \dots, a_n)$.

Also note that both cube paths $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ may contain degenerate cubes, cf. Remark 4.34.

5.32 Corollary. *There is a forgetful functor* $\ell\mathsf{HDA} \longrightarrow \ell\mathsf{HDA}^{\mathrm{h}}$.

5.33 Definitions.

- The *tensor product* of $(f, \sigma) : \langle * \to A_1 \to !\Sigma_1 \rangle \to \langle * \to A_2 \to !\Sigma_2 \rangle \in \ell\mathsf{HDA}^{\mathrm{h}}$ and $(g, \xi) : \langle * \to B_1 \to !\Xi_1 \rangle \to \langle * \to B_2 \to !\Xi_2 \rangle \in \ell\mathsf{HDA}^{\mathrm{h}}$ is the homotopy simulation

$$(f \otimes g, \sigma \otimes \xi) : \langle * \to A_1 \otimes B_1 \to !\Sigma_1 \otimes !\Xi_1 \rangle \to \langle * \to A_2 \otimes B_2 \to !\Sigma_2 \otimes \Xi_2 \rangle$$

- The *relabeling* of $(f, \mathsf{id}) : \langle * \to A \xrightarrow{\lambda} !\Sigma \rangle \to \langle * \to B \xrightarrow{\mu} !\Sigma \rangle \in \ell\mathsf{HDA}^{\mathrm{h}}$ by a mapping $\sigma : !\Sigma \to !\Xi$ which is freely generated by a precubical morphism is the homotopy simulation

$$(f, \mathsf{id}) : \langle * \to A \xrightarrow{\sigma \circ \lambda} !\Xi \rangle \to \langle * \to B \xrightarrow{\mu \circ \lambda} !\Xi \rangle$$

5.34 Remark. Contrary to the similar definitions in 5.17, the present ones require some checking that the constructed mappings actually constitute homotopy simulations; this is however straight-forward.

To define *restriction* of homotopy simulations poses some problems: If

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_1 \arrow[d] \\
!\Sigma \arrow[r] & !\Xi
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_2 \arrow[d] \\
!\Sigma \arrow[r] & !\Xi
\end{tikzcd}$$

are restriction diagrams, and $(g, \mathsf{id}) : \langle * \to B_1 \to !\Xi \rangle \to \langle * \to B_2 \to !\Xi \rangle$ is a homotopy simulation, then the diagram

$$\begin{tikzcd}
A_1 \arrow[rrr, dotted] \arrow[ddr, dotted, bend right] & & & B_1 \arrow[dl, "g"'] \arrow[ddl, bend left, "\neq"'] \\
& A_2 \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & B_2 \arrow[d] & \\
& !\Sigma \arrow[r, "\sigma"'] & !\Xi &
\end{tikzcd}$$

does not commute, hence there appears to be no natural way to define $f : A_1 \to A_2$.

5.8 Bisimulation up to Homotopy

5.35 Definition. A homotopy simulation $(f, \sigma) : \langle * \to A \to !\Sigma\rangle \to \langle * \to B \to !\Xi\rangle \in \ell\mathsf{HDA}^{\mathrm{h}}$ is *non-contracting* if f and σ are freely generated by precubical morphisms.

5.36 Definition. A non-contracting homotopy simulation

$$(f, \mathsf{id}) : \langle * \to A \xrightarrow{\lambda} !\Sigma\rangle \to \langle * \to B \xrightarrow{\mu} !\Sigma\rangle \in \ell\mathsf{HDA}^{\mathrm{h}}$$

is a *homotopy bisimulation map* provided that for any pointed cube path $(a_1, \dots, a_n)$ in A and for any cube path $(b_1, \dots, b_m)$ in B for which $b_1 = f(a_n)$ and $\dim b_m = 0$, there exists a cube path $(a_n, \dots, a_\ell)$ in A such that $\big(f(a_1), \dots, f(a_\ell)\big) \sim \big(f(a_1), \dots, f(a_n), b_2, \dots, b_m\big)$.

5.37 Remark. For the condition $\dim b_m = 0$, cf. Remark 5.31; we only want *completions* of computations to match up to homotopy.

5.38 Definition. The *unfolding* of a higher-dimensional automaton $\langle * \xrightarrow{i} A \xrightarrow{\lambda} !\Sigma\rangle$ is the higher-dimensional automaton

$$* \xrightarrow{\tilde{i}} \tilde{A} \xrightarrow{\tilde{\lambda}} \widetilde{!\Sigma}$$

where $\tilde{A}$ and $\widetilde{!\Sigma}$ are the universal coverings of $i : * \to A$ and $\lambda \circ i : * \to !\Sigma$, $\tilde{\lambda}$ is the induced morphism, and $\tilde{i}* = [i*]$.

5.39 Proposition. *A non-contracting homotopy simulation* $(f, \mathsf{id}) : \langle * \to A \xrightarrow{\lambda} !\Sigma\rangle \to \langle * \to B \xrightarrow{\mu} !\Sigma\rangle \in \ell\mathsf{HDA}^{\mathrm{h}}$ *is a homotopy bisimulation map if and only if* $\tilde{f} : \tilde{A} \to \tilde{B}$ *has the property that for any* $\tilde{a} \in \tilde{A}$, *and for any* $\tilde{b} \in \tilde{B}_0$ *which is reachable from* $\tilde{f}(\tilde{a})$, *there exists* $\tilde{c} \in \tilde{A}_0$ *reachable from* $\tilde{a}$ *such that* $\tilde{b} = \tilde{f}(\tilde{c})$.

Proof: To prove the "only if" part, let $\tilde{a} = [a_1, \dots, a_n] \in \tilde{A}$ and $\tilde{b} = [b_1, \dots, b_m] \in \tilde{B}_0$ such that $\tilde{b}$ is reachable from $\tilde{f}(\tilde{a})$. By Lemma 4.51, there is $k \le m$ such that $(b_1, \dots, b_k) \sim (f(a_1), \dots, f(a_n))$, hence $b_k = f(a_n)$. Consequently, we have a cube path $(a_n, \dots, a_\ell)$ in A such that $\big(f(a_1), \dots, f(a_\ell)\big) \sim \big(f(a_1), \dots, f(a_n), b_{k+1}, \dots, b_m\big)$. By Lemma 4.51, $\tilde{c} = [a_1, \dots, a_\ell] \in \tilde{A}_0$ is reachable from $\tilde{a}$, and

$$\begin{aligned}\tilde{f}(\tilde{c}) = [f(a_1), \dots, f(a_\ell)] &= [f(a_1), \dots, f(a_n), b_{k+1}, \dots, b_m] \\ &= [b_1, \dots, b_k, b_{k+1}, \dots, b_m] = \tilde{b}\end{aligned}$$

To show the reverse implication, let $(a_1, \dots, a_n)$ be a pointed cube path in A and $(b_1, \dots, b_m)$ be a cube path in B such that $b_1 = f(a_n)$ and $\dim b_m = 0$. Let $\tilde{a} = [a_1, \dots, a_n] \in \tilde{A}$, $\tilde{b} = [f(a_1), \dots, f(a_n), b_2, \dots, b_m] \in \tilde{B}_0$, then $\tilde{b}$ is reachable from $\tilde{f}(\tilde{a})$ by Lemma 4.51. Hence we have $\tilde{c} = [c_1, \dots, c_k] \in \tilde{A}_0$ reachable from $\tilde{a}$ such that $\tilde{b} = \tilde{f}(\tilde{c})$. By Lemma 4.51, $(c_1, \dots, c_\ell) \sim (a_1, \dots, a_n)$ for some $\ell \le k$, hence $c_\ell = a_n$, and the cube path $(c_\ell, \dots, c_k)$ is as required:

$$\begin{aligned}\big(f(a_1), \dots, f(a_n), f(c_{\ell+1}), \dots, f(c_k)\big) &\sim \big(f(c_1), \dots, f(c_k)\big) \\ &\sim \big(f(a_1), \dots, f(a_n), b_2, \dots, b_m\big) \;\Box\end{aligned}$$

5.40 Corollary. *If $(f, \mathsf{id}) : \langle * \to A \xrightarrow{\lambda} !\Sigma\rangle \to \langle * \to B \xrightarrow{\mu} !\Sigma\rangle \in \ell\mathsf{HDA}^{\mathrm{h}}$ is a non-contracting homotopy simulation such that $\tilde{f} : \tilde{A} \to \tilde{B}$ is an isomorphism, then (f, id) is a homotopy bisimulation.*

5.41 Definition (cf. [JNW96]) Two labeled higher-dimensional automata $\langle * \to A \to !\Sigma\rangle$, $\langle * \to B \to !\Sigma\rangle$ are *homotopy bisimilar* if there exists $\langle * \to C \to !\Sigma\rangle \in \ell\mathsf{HDA}$ and a span of homotopy bisimulation maps

$$\langle * \to A \to !\Sigma\rangle \longleftarrow \langle * \to C \to !\Sigma\rangle \longrightarrow \langle * \to B \to !\Sigma\rangle$$

5.42 Corollary. *Any higher-dimensional automaton $\langle * \to A \xrightarrow{\lambda} !\Sigma\rangle$ is homotopy bisimilar to a relabeling of its unfolding via the homotopy bisimulation map*

$$(\pi_A, \mathsf{id}) : \langle * \to \tilde{A} \xrightarrow{\pi_{!\Sigma} \circ \tilde{\lambda}} !\Sigma\rangle \longrightarrow \langle * \to A \xrightarrow{\lambda} !\Sigma\rangle$$

Proof: Indeed, $\tilde{\pi}_A : \tilde{\tilde{A}} \to \tilde{A}$ is an isomorphism by Corollary 4.54. $\square$

Theorem C. A non-contracting homotopy simulation $(f, \mathsf{id}) : \langle * \xrightarrow{i} A \to\, !\Sigma\rangle \to \langle * \xrightarrow{j} B \to\, !\Sigma\rangle \in \ell\mathsf{HDA}^{\mathrm{h}}$ is a homotopy bisimulation map if and only if one of the following equivalent properties holds:

- For any d-path $r : \vec{I} \to |A|$ with $r(0) = [i*]$ and any d-path $q : \vec{I} \to |B|$ with $q(0) = |f|(r(1))$ and $\mathsf{carr}\, q(1) \in B_0$, there exists a d-path $p : \vec{I} \to |A|$ such that $p(0) = r(1)$ and $|f| \circ (r * p) \sim (|f| \circ r) * q$ rel $([j*], [\mathsf{carr}\, q(1)])$.
- For any $x \in |\tilde{A}|$ and any $y \in {\uparrow}|\tilde{f}|(x) \subseteq |\tilde{B}|$ with $\mathsf{carr}\, y \in \tilde{B}_0$, there is $z \in {\uparrow}x$ such that $y = |\tilde{f}|(z)$.

5.43 Remarks.

- With q' and p' as in the remark following Theorem B, we can encode the first property above in the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, hook] & {|*|} \arrow[d, "|i|"] \\
\vec{I} \arrow[r, "r"] \arrow[d, hook] & {|A|} \arrow[d, "|f|"] \\
\overrightarrow{[0,2]} \arrow[r, "q'"'] \arrow[ur, dotted, "p'"] & {|B|} \arrow[ul, phantom, "\sim"]
\end{tikzcd}$$

  where the lower triangle commutes *up to d-homotopy*. That is, $|f|$ lifts d-paths up to d-homotopy.

- Equivalence of the two above properties also follows from $|\tilde{A}|$ being the universal directed covering space of $|A|$, cf. Remark 4.49.

Proof of theorem. To show that the first property is necessary, let f be a homotopy bisimulation map, and let $r : \vec{I} \to |A|$, $q : \vec{I} \to |B|$ be such that $r(0) = [i*]$, $q(0) = |f|(r(1))$, and $\mathsf{carr}\, q(1) \in B_0$. Let $\mathsf{carrs}\, r = (a_1, \dots, a_n)$, $\mathsf{carrs}\, q = (b_1, \dots, b_m)$, then $b_1 = f(a_n)$ by Lemma 4.23, and $b_m \in B_0$.

Hence we have a cube path $(a_n, \dots, a_\ell)$ in A such that $\big(f(a_1), \dots, f(a_\ell)\big) \sim \big(f(a_1), \dots, f(a_n), b_2, \dots, b_m\big)$.

Apply Corollary 4.20 to get a d-path $p : \vec{I} \to |A|$ such that $\mathsf{carrs}\, p = (a_n, \dots, a_\ell)$ and $p(0) = r(1)$, then $\mathsf{carrs}\, \big(|f| \circ (r * p)\big) = \big(f(a_1), \dots, f(a_\ell)\big)$ and $\mathsf{carrs}\, \big((|f| \circ r) * q\big) = \big(f(a_1), \dots, f(a_n), b_2, \dots, b_m\big)$ by Corollary 4.24, hence $|f| \circ (r * p) \sim (|f| \circ r) * q$ rel $([j*], [\mathsf{carr}\, q(1)])$ by Theorem A.

For sufficiency of the first property, let $(a_1, \dots, a_n)$ be a pointed cube path in A and $(b_1, \dots, b_m)$ a cube path in B such that $b_1 = f(a_n)$ and $\dim b_m = 0$. By Corollary 4.20 we have d-paths $r : \vec{I} \to |A|$, $q : \vec{I} \to |B|$ such that $\mathsf{carrs}\, p = (a_1, \dots, a_n)$, $\mathsf{carrs}\, q = (b_1, \dots, b_m)$, and $q(0) = |f|(r(1))$. Then $\mathsf{carr}\, q(1) \in B_0$, hence there exists $p : \vec{I} \to |A|$ with $p(0) = r(1)$ and $|f| \circ (r * p) \sim (|f| \circ r) * q$ rel $([j*], \mathsf{carr}\, q(1))$. Letting $(a_n, \dots, a_\ell) = \mathsf{carrs}\, p$, Theorem A entails that $\big(f(a_1), \dots, f(a_\ell)\big) \sim \big(f(a_1), \dots, f(a_n), b_2, \dots, b_m\big)$.

To show necessity of the second property, let again f be an h-bisimulation map, and let $x \in |\tilde{A}|$, $y \in \uparrow|\tilde{f}|(x)$ such that $\mathsf{carr}\, y \in \tilde{B}_0$. Then $\mathsf{carr}\, y$ is reachable from $\mathsf{carr}(|\tilde{f}|(x)) = \tilde{f}(\mathsf{carr}\, x)$ by Lemma 4.21, so by Proposition 5.39 we have $\tilde{c} \in \tilde{A}_0$ reachable from $\mathsf{carr}\, x$ such that $\tilde{f}(\tilde{c}) = \mathsf{carr}\, y$. Let $z \in [\tilde{c}] = \{z\}$, then $z \in \uparrow x$ by Lemma 4.21, and $|\tilde{f}|(z) = y$.

To show that the second property implies that (f, id) is an h-bisimulation map, we use again Proposition 5.39. Let $\tilde{a} \in \tilde{A}$, $\tilde{b} \in \tilde{B}_0$ such that $\tilde{b}$ is reachable from $\tilde{f}(\tilde{a})$, let $y \in [\tilde{b}] = \{y\}$, and let $x \in [\tilde{a}]$. By Lemma 4.21, $y \in \uparrow|\tilde{f}|(x)$, hence we have $z \in \uparrow x$ with $|\tilde{f}|(z) = y$. Then $\mathsf{carr}\, z \in \tilde{A}_0$, $\mathsf{carr}\, z$ is reachable from $\tilde{a} = \mathsf{carr}\, x$ by Lemma 4.21, and $\tilde{f}(\mathsf{carr}\, z) = \mathsf{carr}(|\tilde{f}|(z)) = \tilde{b}$. □

5.44 Corollary. *Tensor product and relabeling respect homotopy bisimilarity.*

Proof: For relabeling this is clear. For tensor product, if $|f| : |A_1| \to |A_2|$ and $|g| : |B_1| \to |B_2|$ have the lifting property of Theorem C, then so has $|f| \times |g| : |A_1| \times |B_1| \to |A_2| \times |B_2|$, hence using spans of homotopy bisimulation maps, we are done. □

6 Conclusion and Future Work

We have in Chapter 5 introduced two different categories of higher-dimensional automata, each with their own notion of bisimulation morphisms. Other notions of simulations and bisimulations can be found in the literature, notably van Glabbeek's [Gla91, Gla06], Cattani-Sassone's [CS96], and Worytkiewicz's [Wor04] (see also [HPTW03]). There is work to be done in comparing these different notions.

We have seen that our bisimulation morphisms are characterised by a certain d-path lifting property of their geometric realisations; bisimulation maps lift d-paths, and homotopy bisimulation maps lift d-paths up to d-homotopy. Maps with this kind of lifting property are "morally" expected to be some kind of fibrations, so there are some connections to model category theory here. It would certainly be a worthwhile task to set out to find a model category structure on d-spaces, or on cubical sets, where the fibrations are exactly the (homotopy) bisimulation maps.

The developments in Chapter 5 are based on the notion of cube paths as combinatorial counterparts to directed paths in precubical sets. We have seen in Chapter 4 that d-homotopy of d-paths is closely related to homotopy of cube paths, and using recent results by Fajstrup [Faj06], that the universal covering of a precubical set provides a combinatorial analogue to the universal directed covering space of its realisation. It is natural to ask how far this analogy bears; one should attempt to develop a general covering theory for precubical sets.

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