LU Decomposition

Theory

LU DECOMPOSITION METHOD

OVERVIEW:

LU Decomposition factors a square matrix A into the product of a lower triangular matrix L and an upper triangular matrix U, such that A = LU. This decomposition is particularly useful for solving multiple systems with the same coefficient matrix.

MATHEMATICAL FOUNDATION:

For matrix A: A = LU

Where:

- L is a lower triangular matrix (elements above diagonal are 0)
- U is an upper triangular matrix (elements below diagonal are 0)

To solve Ax = b:

1. Decompose: A = LU
2. Solve LY = B (forward substitution)
3. Solve UX = Y (back substitution)

DOOLITTLE'S METHOD:

This is the most common LU decomposition where:

- L has 1's on the diagonal
- U has the actual pivot values on diagonal

Algorithm: For k = 1 to n: For j = k to n: u[k][j] = a[k][j] - sum(l[k][p] * u[p][j]) for p = 1 to k-1

For i = k+1 to n: l[i][k] = (a[i][k] - sum(l[i][p] * u[p][k])) / u[k][k] for p = 1 to k-1

CROUT'S METHOD:

Alternative where:

- U has 1's on the diagonal
- L has the actual pivot values on diagonal

FORWARD SUBSTITUTION (Ly = b):

For i = 1 to n: y[i] = (b[i] - sum(l[i][j] * y[j])) / l[i][i] for j = 1 to i-1

BACK SUBSTITUTION (Ux = y):

For i = n down to 1: x[i] = (y[i] - sum(u[i][j] * x[j])) / u[i][i] for j = i+1 to n

TIME COMPLEXITY:

- Decomposition: O(n^3)
- Each solve: O(n^2)
- Efficient for multiple right-hand sides

ADVANTAGES:

- Efficient for multiple systems with same A
- Decomposition done once, reused multiple times
- Can compute determinant easily: det(A) = product of U diagonal elements
- Numerically stable with partial pivoting

DISADVANTAGES:

- Requires additional storage for L and U
- Not all matrices have LU decomposition
- May need pivoting for numerical stability