50.007
Machine Learning

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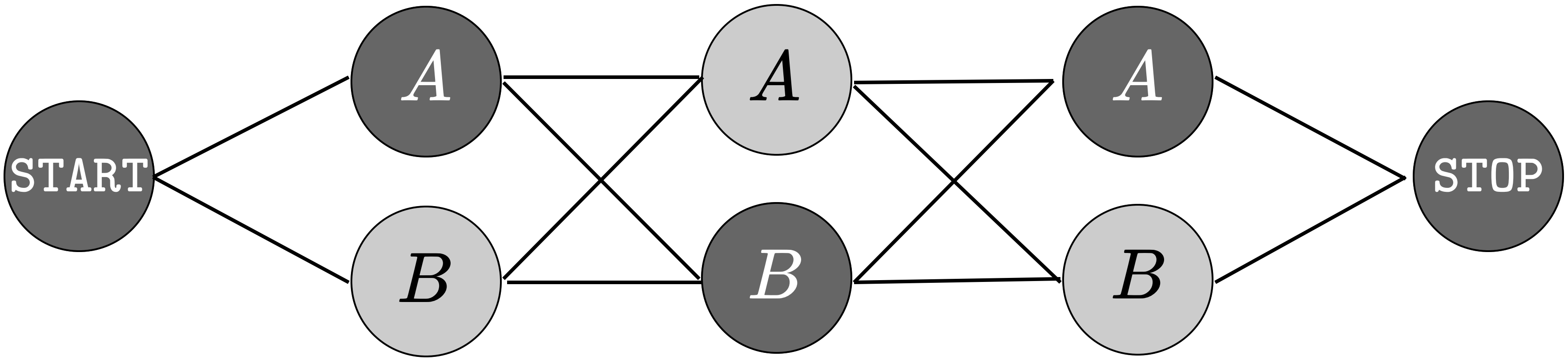
TECHNOLOGY AND DESIGN

Hidden Markov Model (IV)

Hidden Markov Model
Unsupervised Learning

We don't know the model parameters,
but only know there are two possible states: A, B.



$$\mathbf{x} = \text{the}, \text{dog}, \text{the}$$

What is the most probable $\mathbf{y}$ sequence for the given $\mathbf{x}$ sequence?

Hard EM for HMM

E-Step

Run Viterbi, and then collect counts from each instance

M-Step

$$a_{u,v} = \frac{\text{count}(u,v)}{\text{count}(u)} \qquad b_u(o) = \frac{\text{count}(u \to o)}{\text{count}(u)}$$

Soft EM for HMM

E-Step

Run forward-backward algorithm

to collect fractional counts

from each instance

M-Step

$$a_{u,v} = \frac{\text{count}(u,v)}{\text{count}(u)} \qquad b_u(o) = \frac{\text{count}(u \to o)}{\text{count}(u)}$$

Soft EM for HMM

$$\boxed{\text{Finding the fractional count}}$$

$$\text{count}(u, v) = \sum_{i=1}^{m} \text{count}^{(i)}(u, v)$$

Soft EM for HMM

$$\boxed{\text{Finding the fractional count}}$$

$$\text{count}(u, v) = \sum_{i=1}^{m} \text{count}^{(i)}(u, v)$$

$$= \sum_{i=1}^{m} \sum_{\mathbf{y}} p(\mathbf{y}|\mathbf{x}^{(i)})\text{count}(\mathbf{x}^{(i)}, \mathbf{y}, u \to v)$$

Soft EM for HMM

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$$\text{count}(u, v) = \sum_{i=1}^{m} \text{count}^{(i)}(u, v)$$

$$= \sum_{i=1}^{m} \sum_{\mathbf{y}} p(\mathbf{y}|\mathbf{x}^{(i)})\text{count}(\mathbf{x}^{(i)}, \mathbf{y}, u \to v)$$

$$= \sum_{i=1}^{m} \boxed{\sum_{j=0}^{n} p(y_j = u, y_{j+1} = v|\mathbf{x}^{(i)})}$$

Soft EM for HMM

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$$= \sum_{i=1}^{m} \boxed{\sum_{j=0}^{n} p(y_j = u, y_{j+1} = v|\mathbf{x}^{(i)})}$$

$$\text{count}(u) = \sum_{i=1}^{m} \text{count}^{(i)}(u)$$

$$= \sum_{i=1}^{m} \boxed{\sum_{j=0}^{n} p(y_j = u|\mathbf{x}^{(i)})}$$

n here is the length of the input sentence, which may be different for a different input

Soft EM for HMM

$$\boxed{\text{Finding the fractional count}}$$

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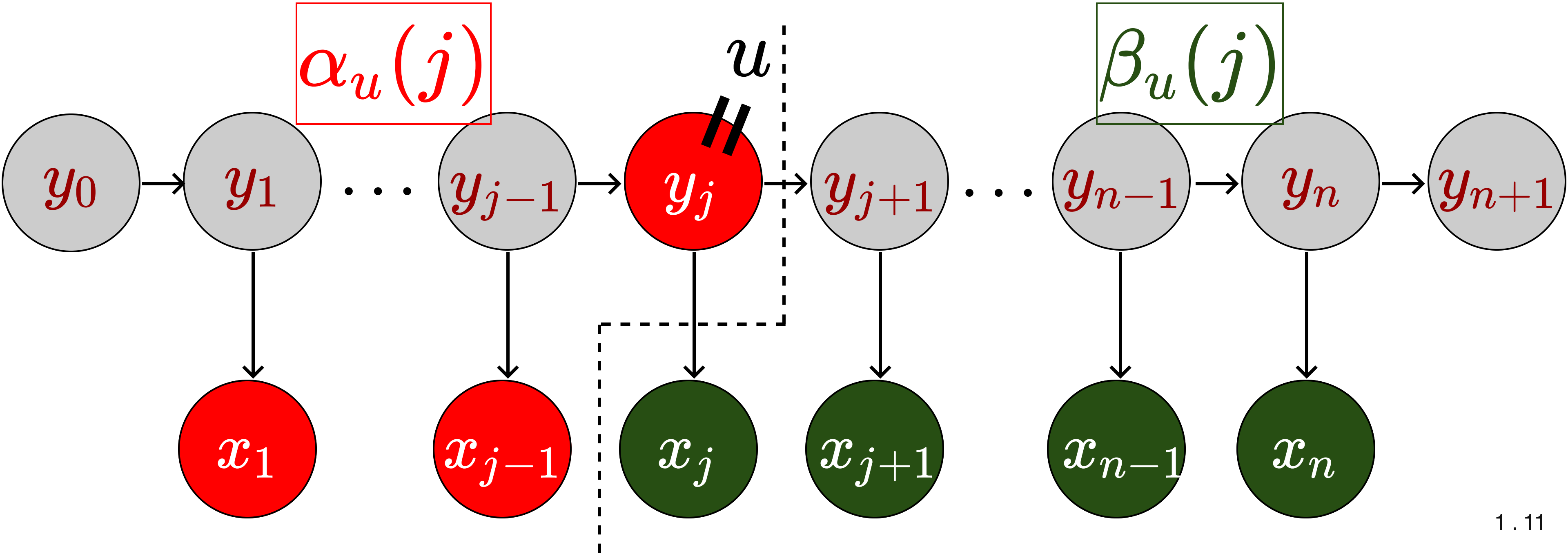
$$\text{count}(u) = \sum_{i=1}^{m} \text{count}^{(i)}(u)$$

$$= \sum_{i=1}^{m} \sum_{j=0}^{n} p(y_j = u|\mathbf{x})$$

Inference for HMM

$$p(y_j = u|\mathbf{x})$$

$$= \frac{p(x_1,x_2,...,x_{j-1},y_j{=}u,x_j,x_{j+1},...,x_n;\theta)}{\sum_v p(x_1,x_2,...,x_{k-1},y_k{=}v,x_k,x_{k+1},...,x_n;\theta)}$$



Inference for HMM

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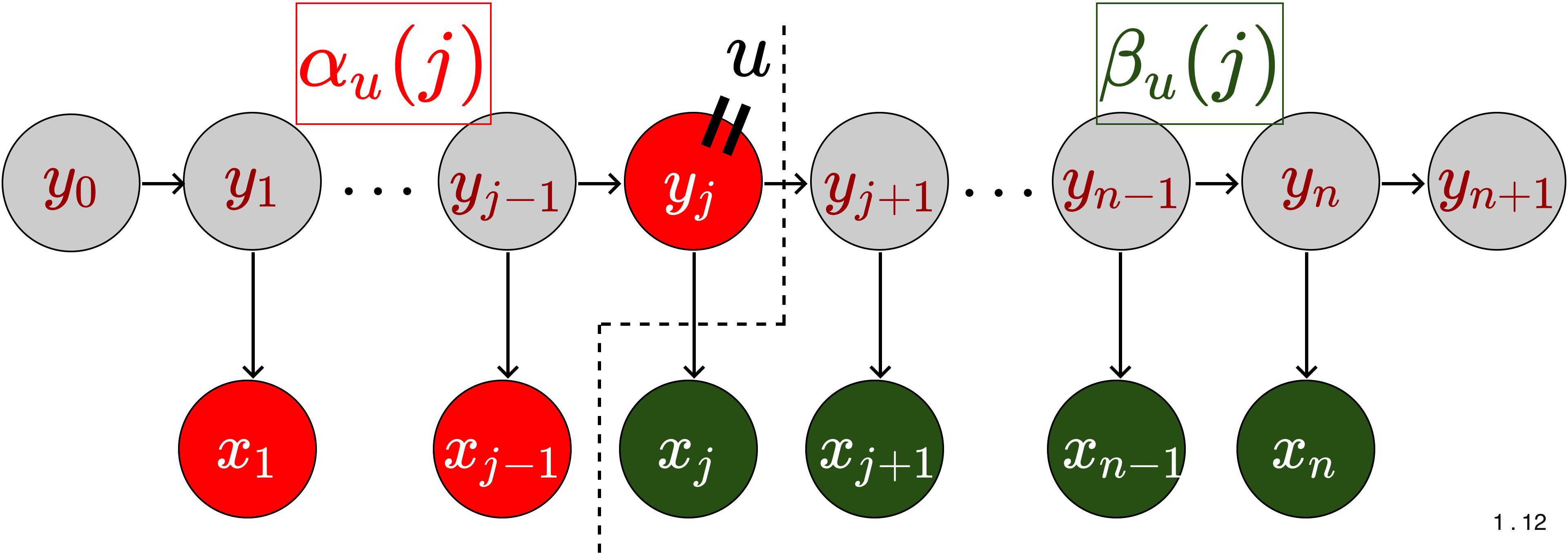
$$= \frac{\alpha_u(j)\beta_u(j)}{}$$



Inference for HMM

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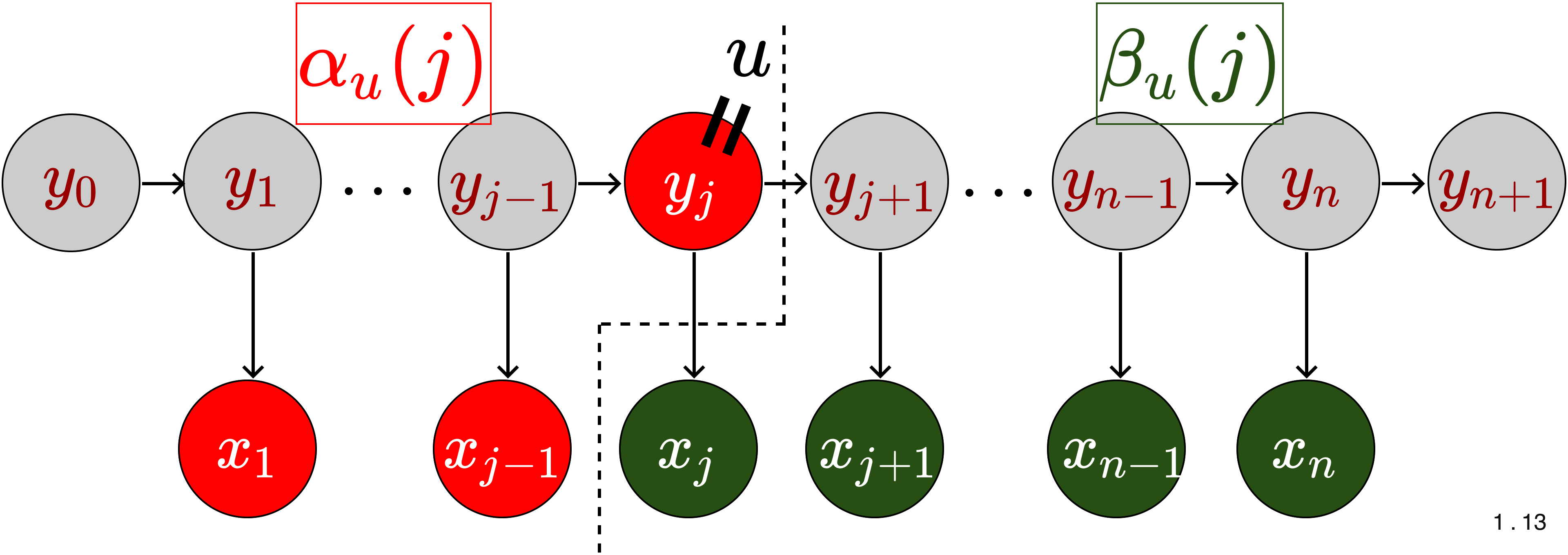
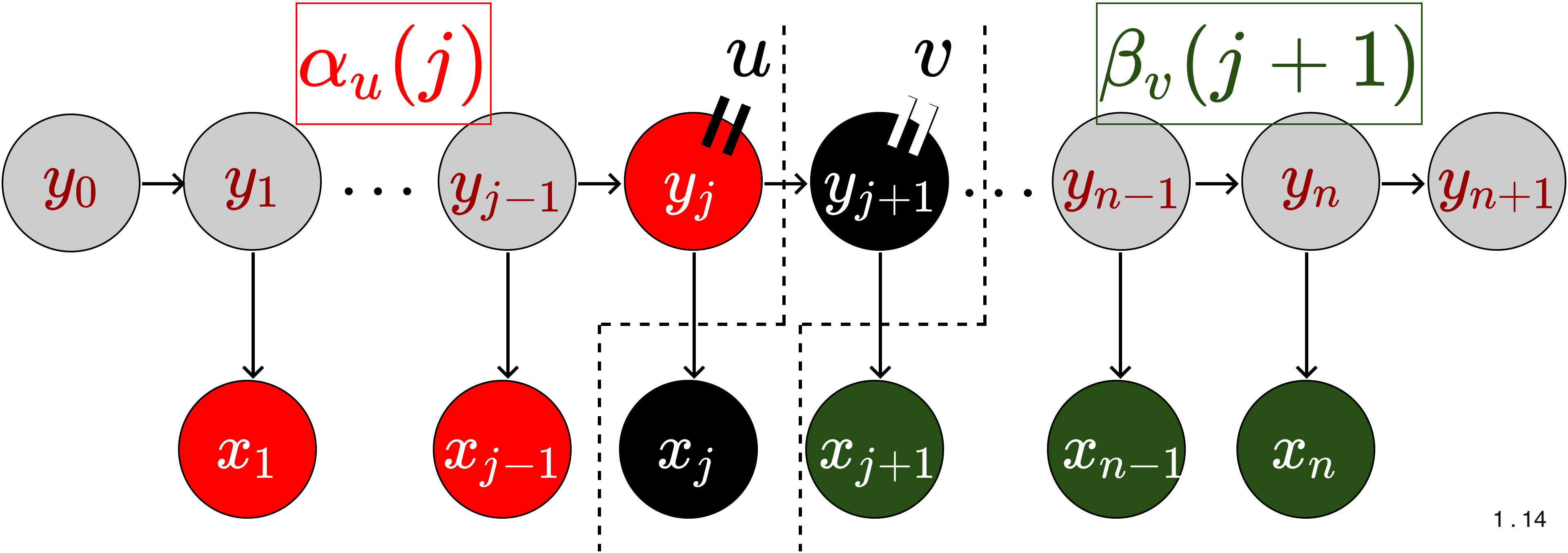


Inference for HMM

$$p(y_j = u, y_{j+1} = v | \mathbf{x})$$

$$= \frac{p(x_1, x_2, \ldots, x_{j-1}, y_j = u, x_j, y_{j+1} = v, x_{j+1}, \ldots, x_n; \theta)}{\sum_v p(x_1, x_2, \ldots, x_{k-1}, y_k = v, x_k, x_{k+1}, \ldots, x_n; \theta)}$$

$$= \frac{\alpha_u(j) \cdot b_u(x_j) \cdot a_{u,v} \cdot \beta_v(j+1)}{\sum_v \alpha_v(k) \beta_v(k)}$$



Soft EM for HMM

$\boxed{\text{Finding the fractional count}}$

$$\text{count}(u,v) = \sum_{i=1}^{m} \text{count}^{(i)}(u,v)$$

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$$\text{count}(u) = \sum_{i=1}^{m} \text{count}^{(i)}(u)$$

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$$= \sum_{i=1}^{m} \sum_{j=0}^{n} p(y_j = u, y_{j+1} = v | \mathbf{x})$$

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$$= \sum_{i=1}^{m} \sum_{j=0}^{n} \frac{\alpha_u(j)\beta_u(j)}{\sum_v \alpha_v(k)\beta_v(k)}$$

In the M-Step:

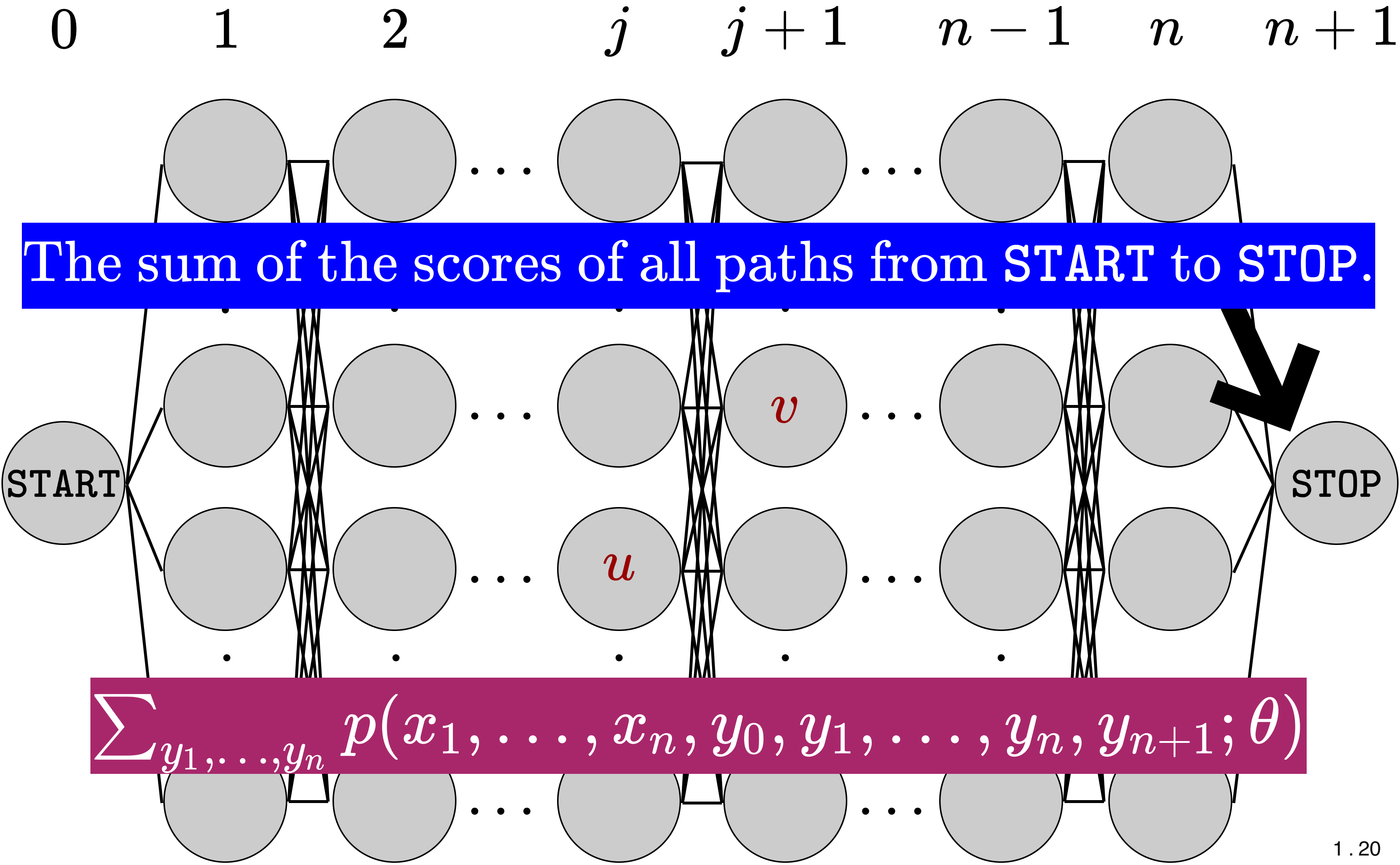
$$a_{u,v} = \frac{\text{count}(u,v)}{\text{count}(u)}$$

Question

How to find an efficient procedure to calculate forward and backward probabilities?

Calculate forward/backward scores efficiently → Perform inference efficiently → Calculate the expected counts efficiently → Perform the soft-EM efficiently

Inference in HMM



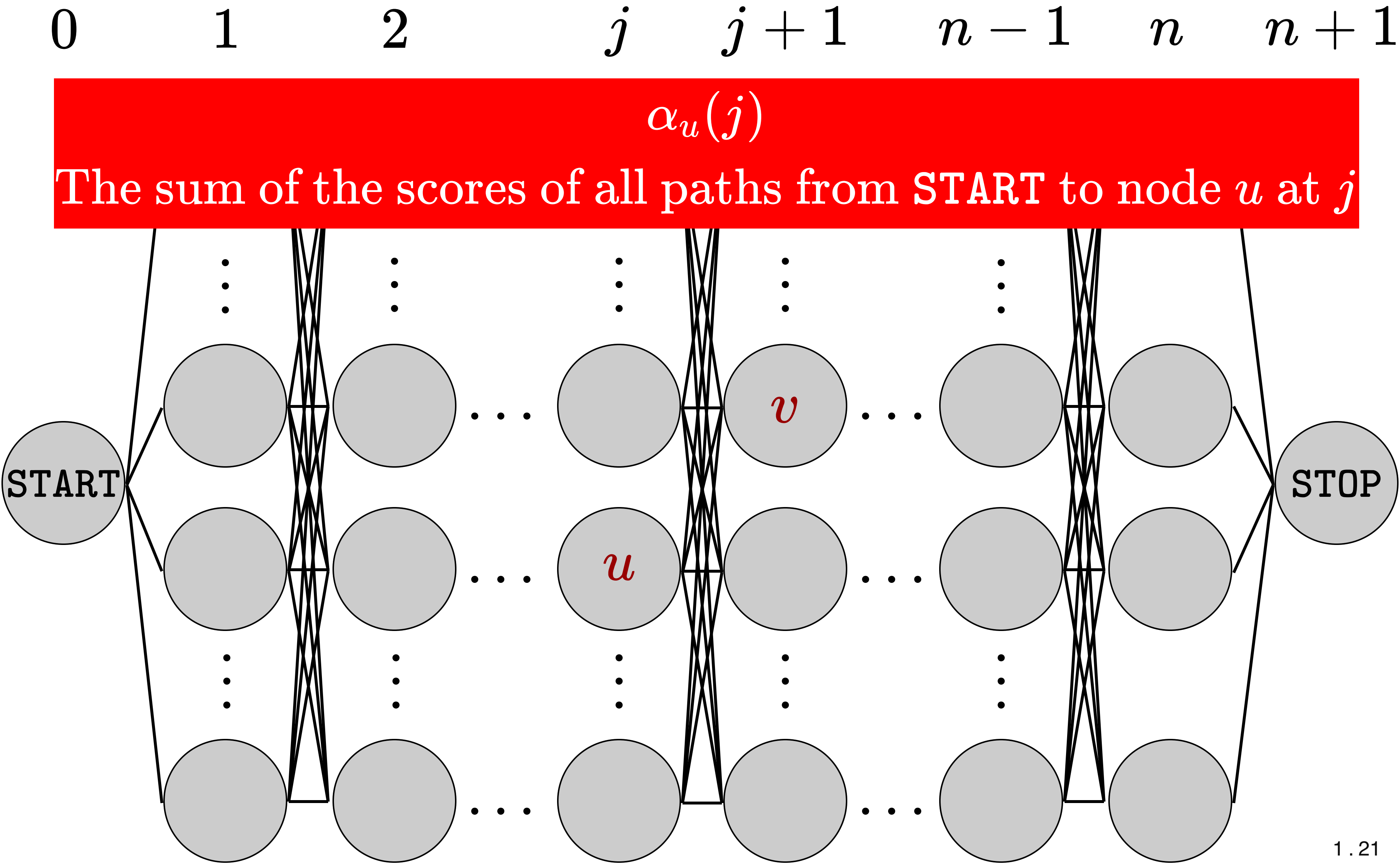
The sum of the scores of all paths from START to STOP.

$$\sum_{y_1,\ldots,y_n} p(x_1,\ldots,x_n,y_0,y_1,\ldots,y_n,y_{n+1};\theta)$$

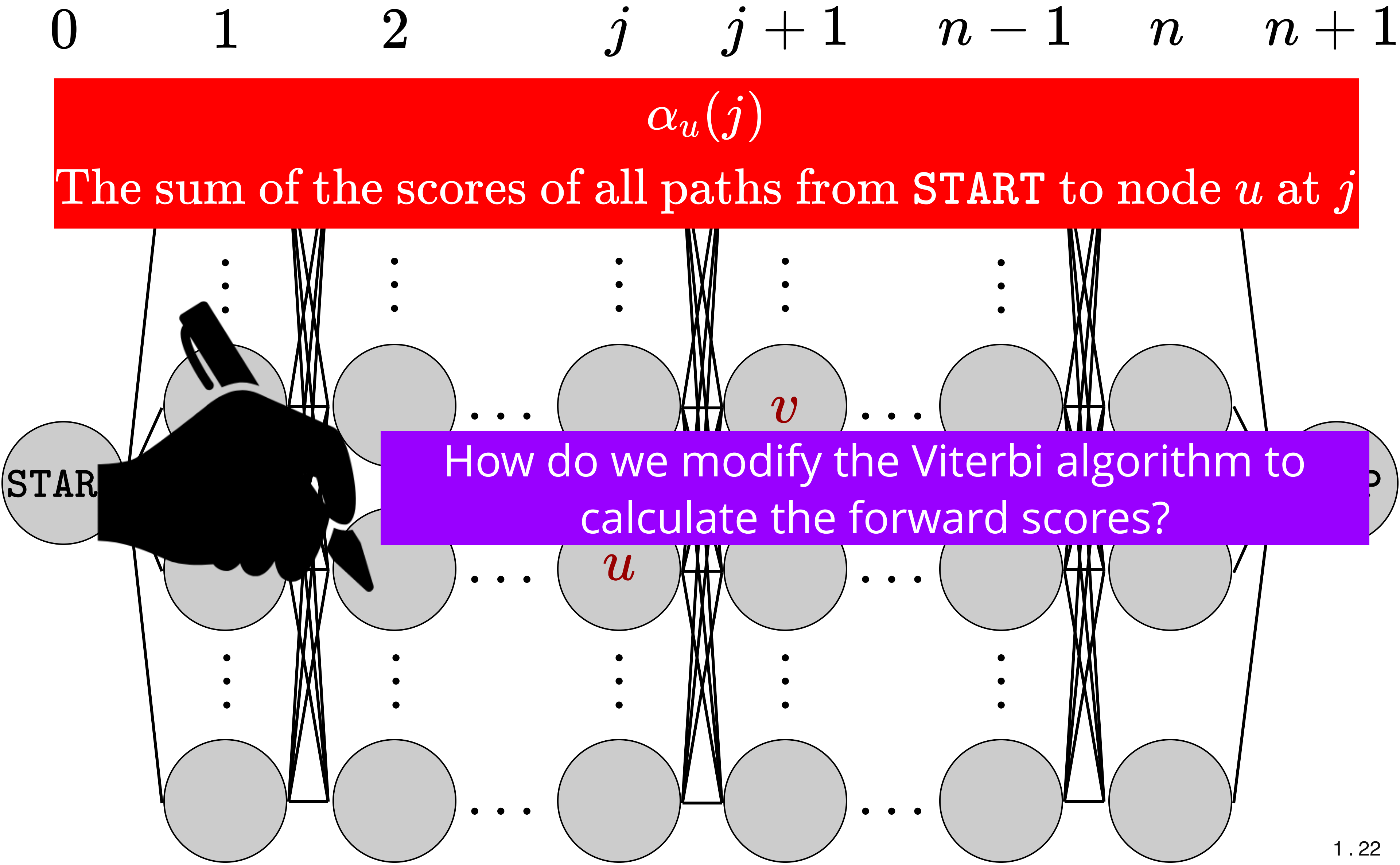
Forward-Backward Algorithm

$\alpha_u(j)$

The sum of the scores of all paths from **START** to node u at j



Forward-Backward Algorithm



$\alpha_u(j)$

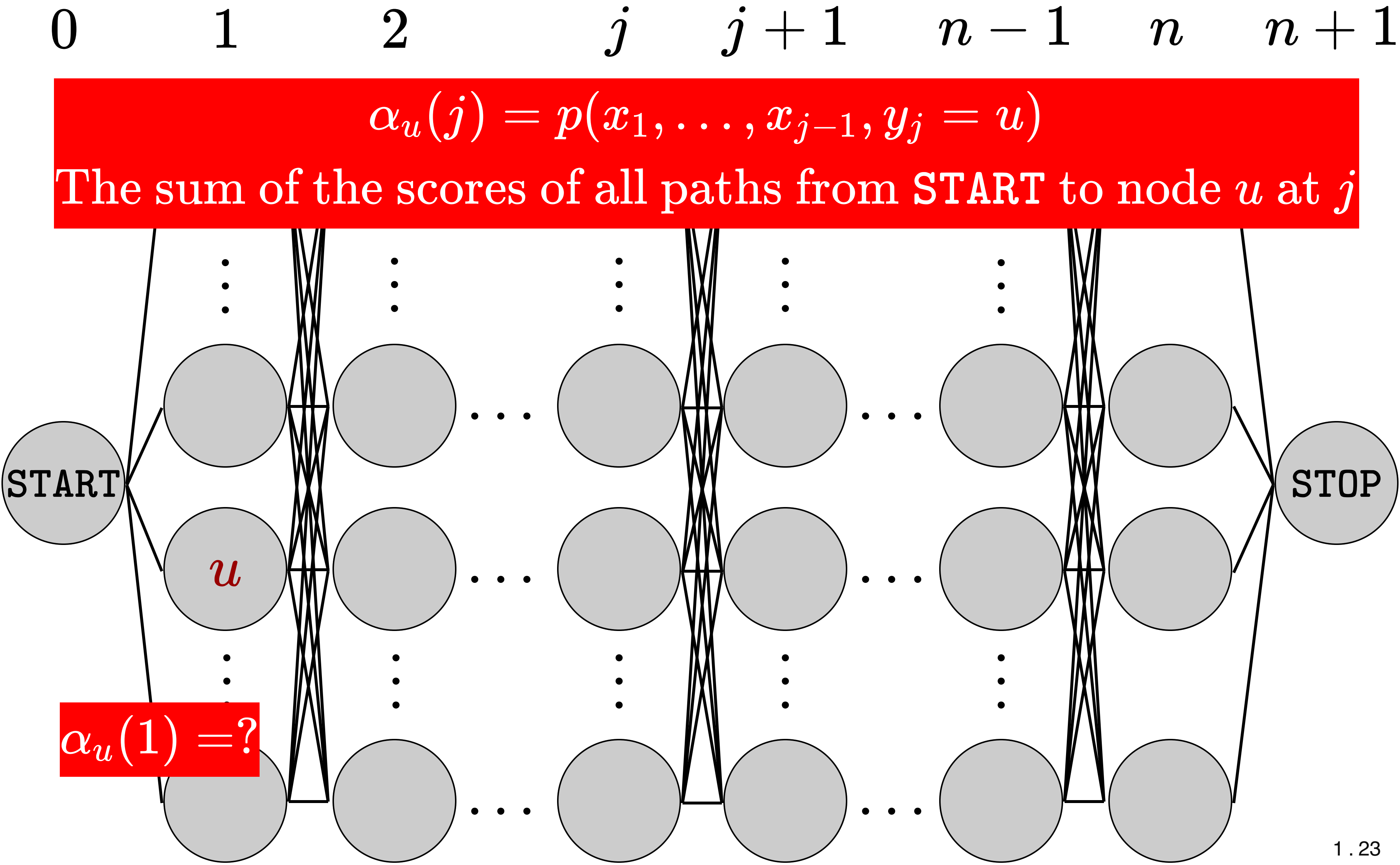
The sum of the scores of all paths from **START** to node u at j

How do we modify the Viterbi algorithm to calculate the forward scores?

Forward-Backward Algorithm

$$\alpha_u(j) = p(x_1, \ldots, x_{j-1}, y_j = u)$$

The sum of the scores of all paths from **START** to node u at j

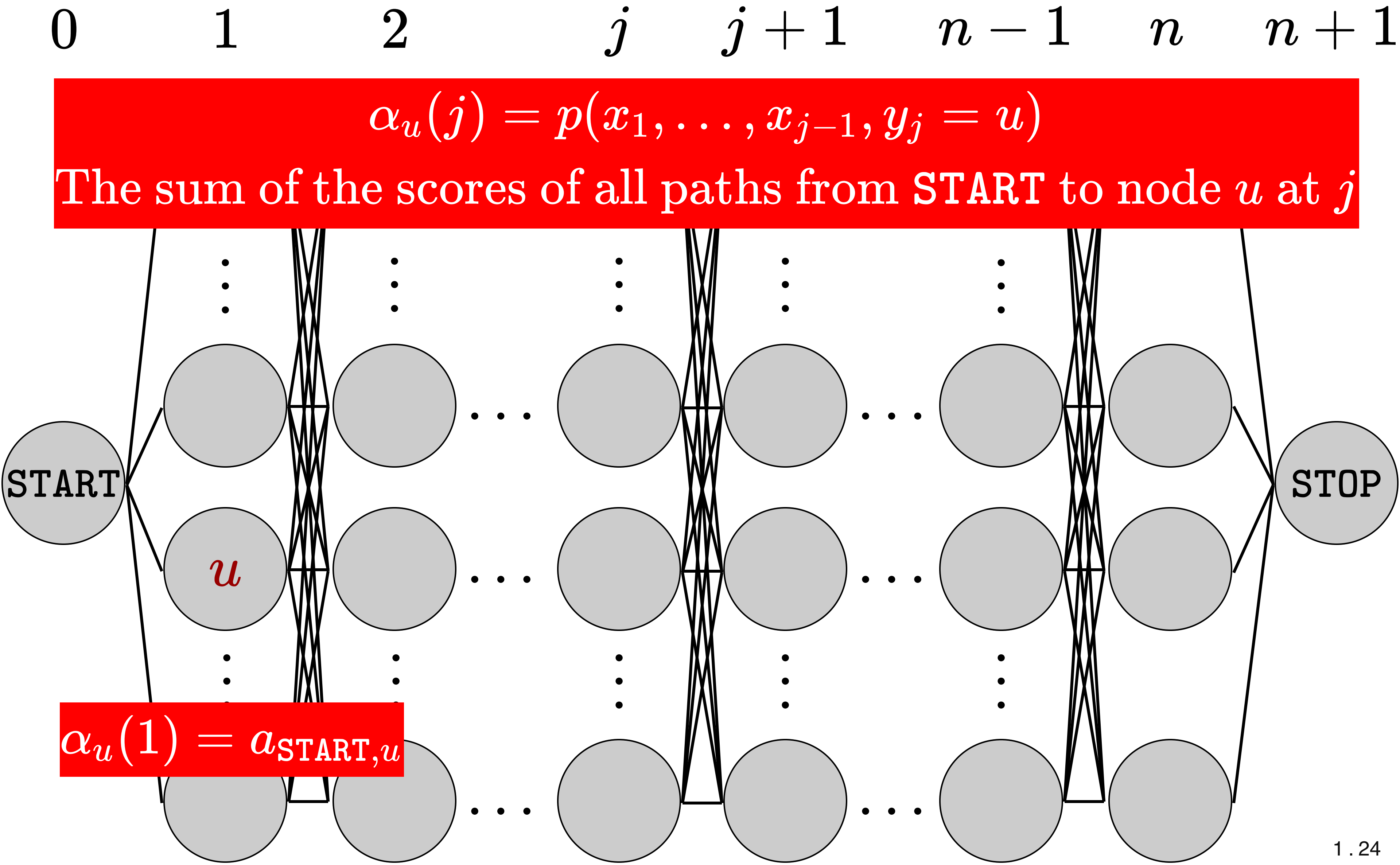


$\alpha_u(1) = ?$

Forward-Backward Algorithm

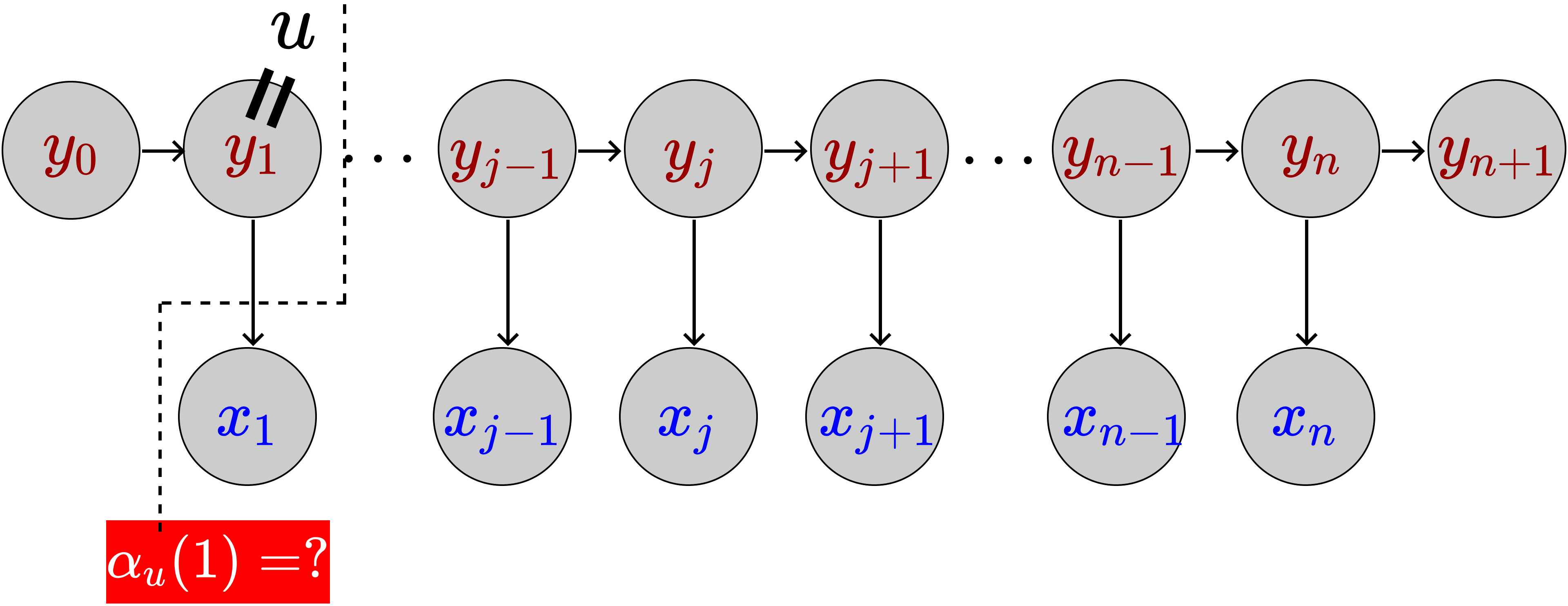
$$\alpha_u(j) = p(x_1, \ldots, x_{j-1}, y_j = u)$$

The sum of the scores of all paths from **START** to node u at j



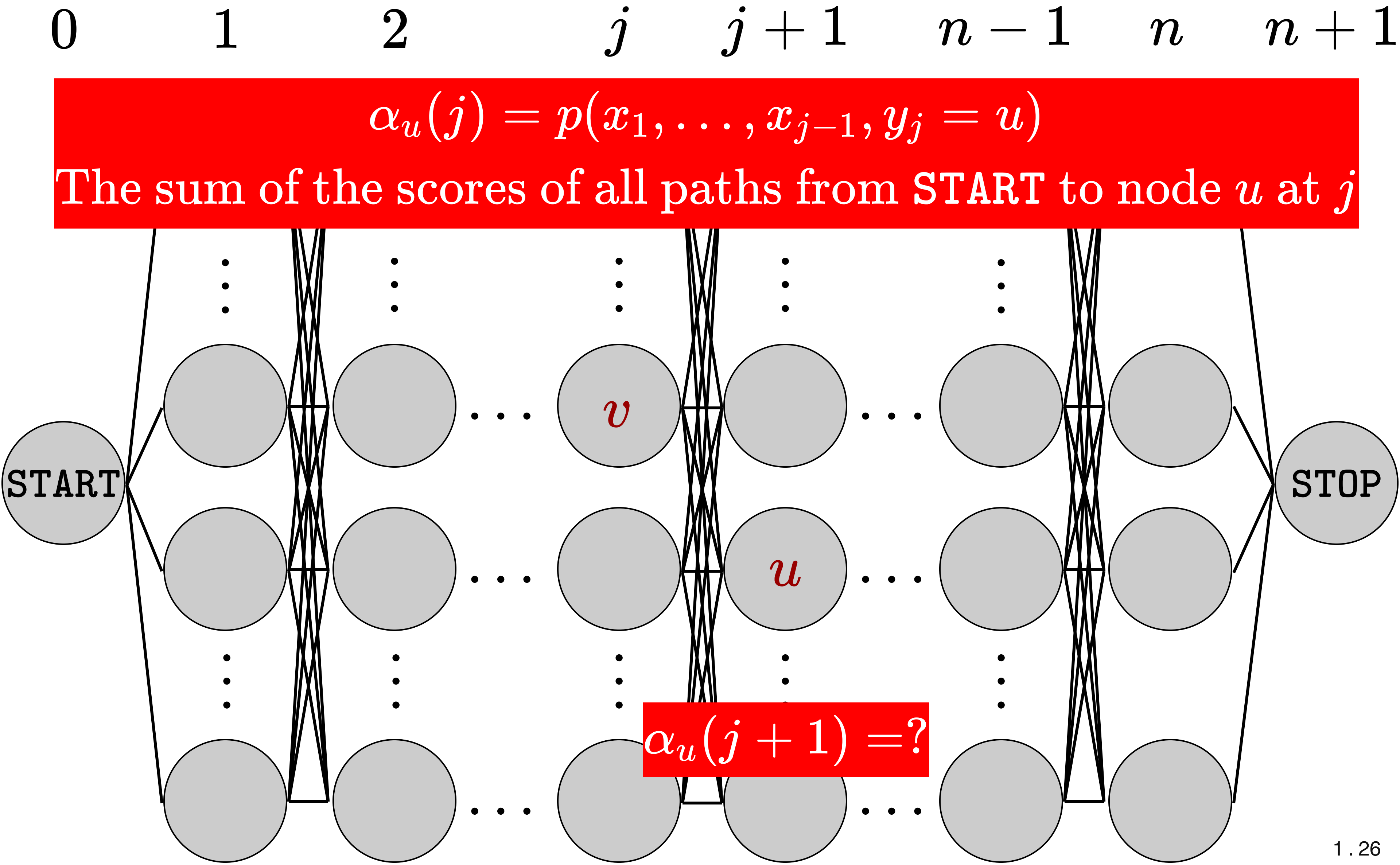
$$\alpha_u(1) = a_{\texttt{START},u}$$

Forward-Backward Algorithm



$$\boxed{\alpha_u(1) = a_{\mathtt{START},u}}$$

Forward-Backward Algorithm

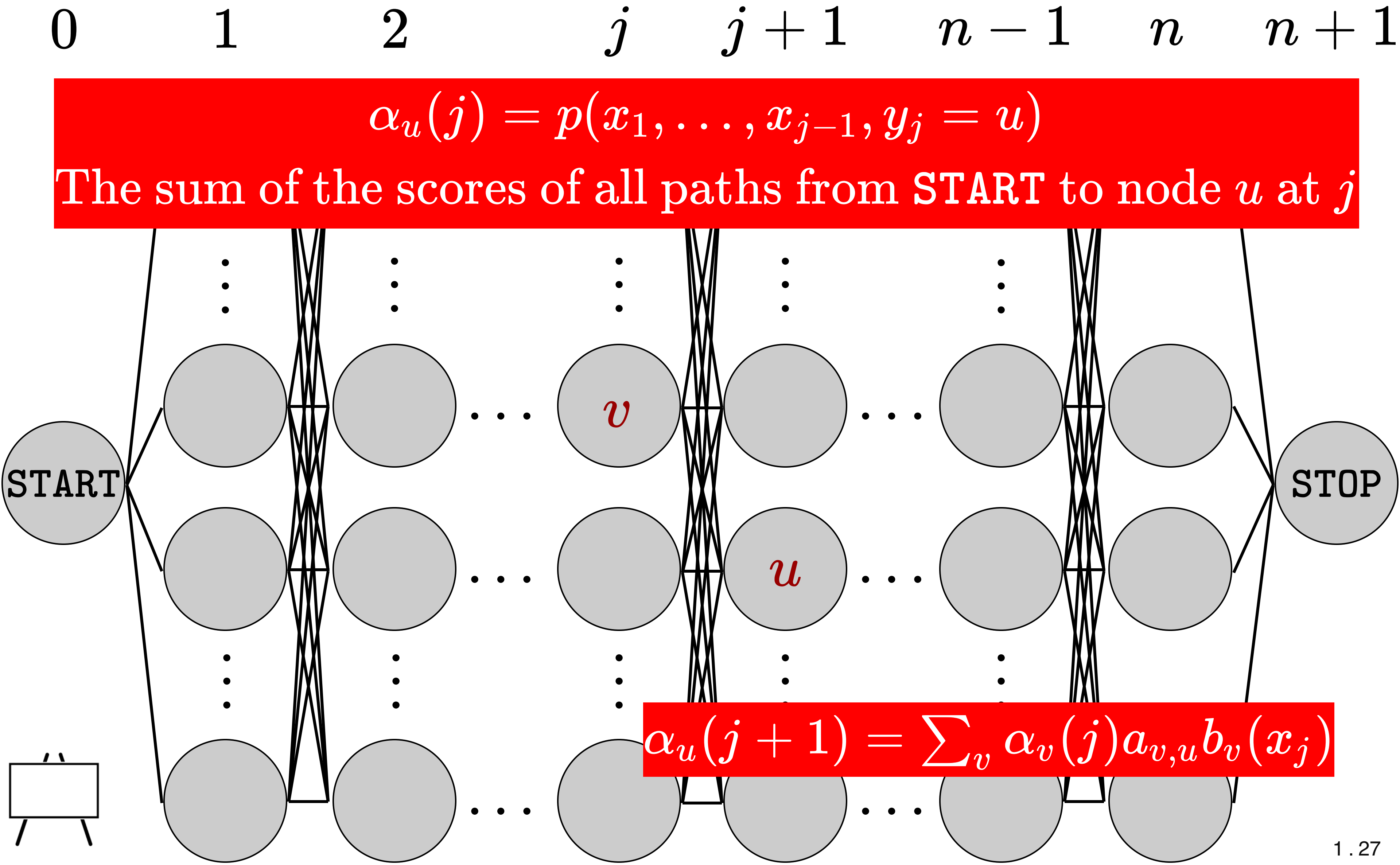


$$\alpha_u(j) = p(x_1, \ldots, x_{j-1}, y_j = u)$$

The sum of the scores of all paths from **START** to node u at j

$$\alpha_u(j+1) = ?$$

Forward-Backward Algorithm

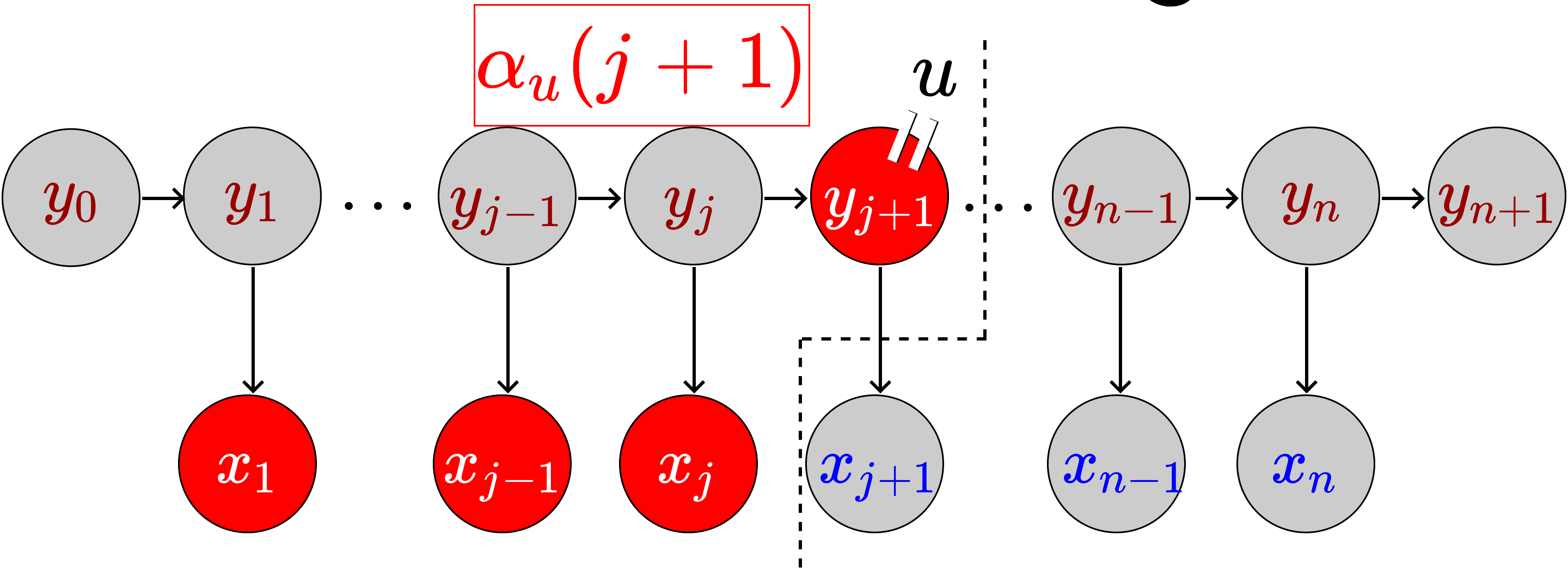


$$\alpha_u(j) = p(x_1, \ldots, x_{j-1}, y_j = u)$$

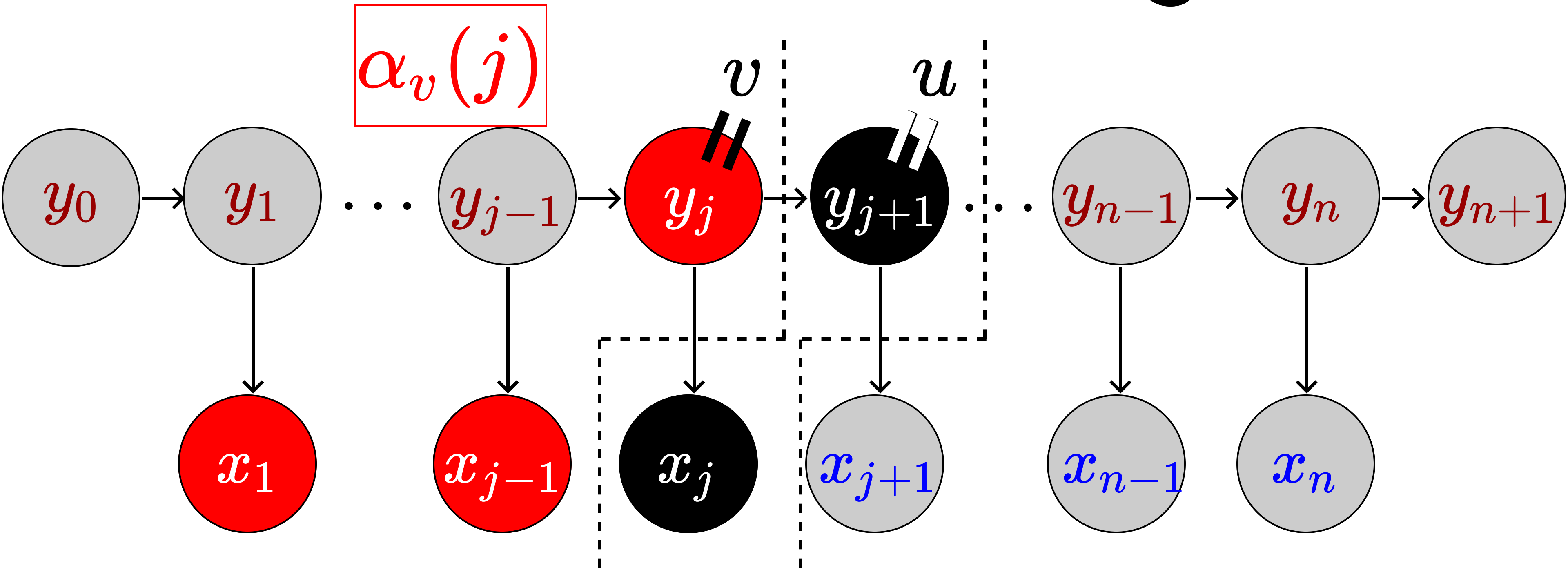
The sum of the scores of all paths from **START** to node u at j

$$\alpha_u(j+1) = \sum_v \alpha_v(j) a_{v,u} b_v(x_j)$$

Forward-Backward Algorithm



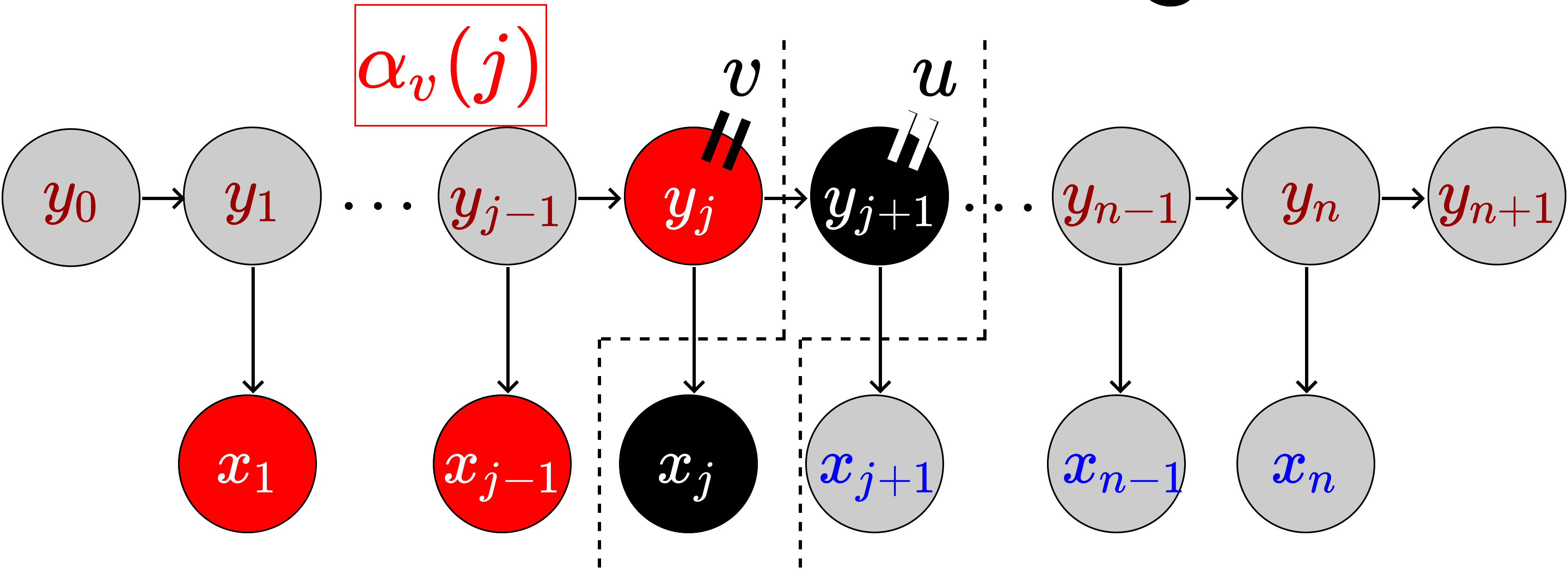
Forward-Backward Algorithm



Assuming the previous state is v

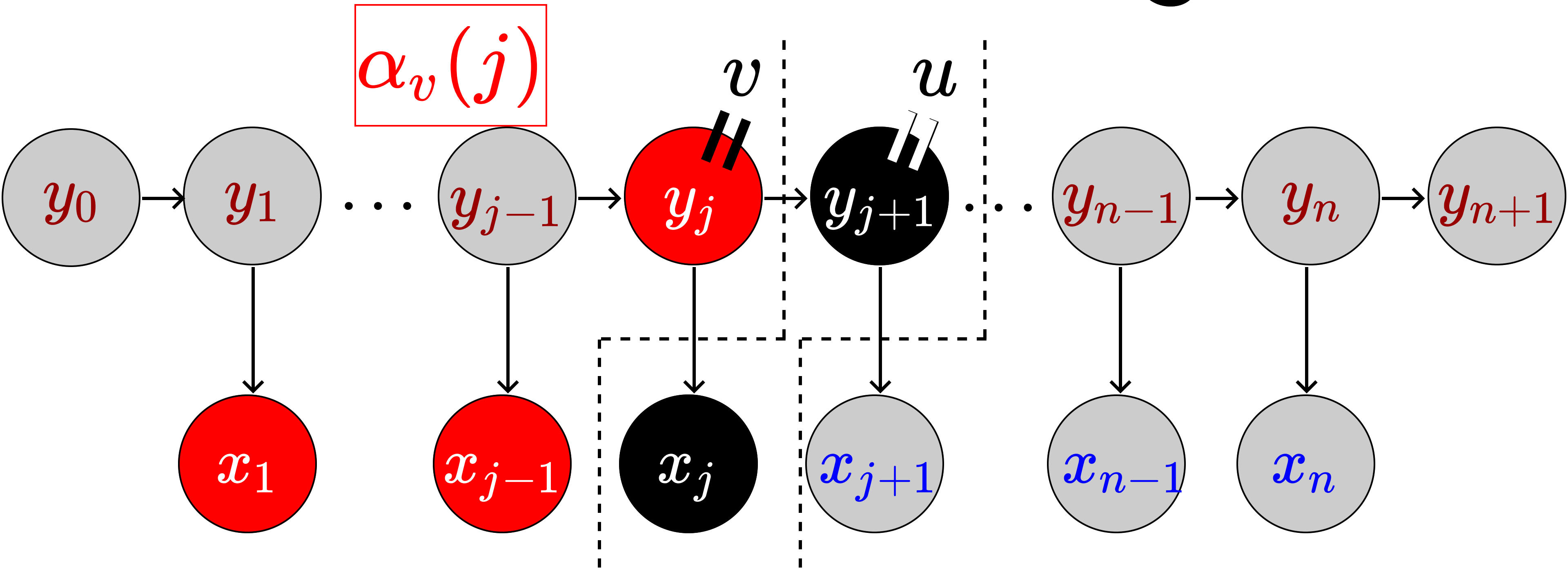
How do we generate these two nodes in black?

Forward-Backward Algorithm



$$\alpha_v(j)a_{v,u}b_v(x_j)$$

Forward-Backward Algorithm

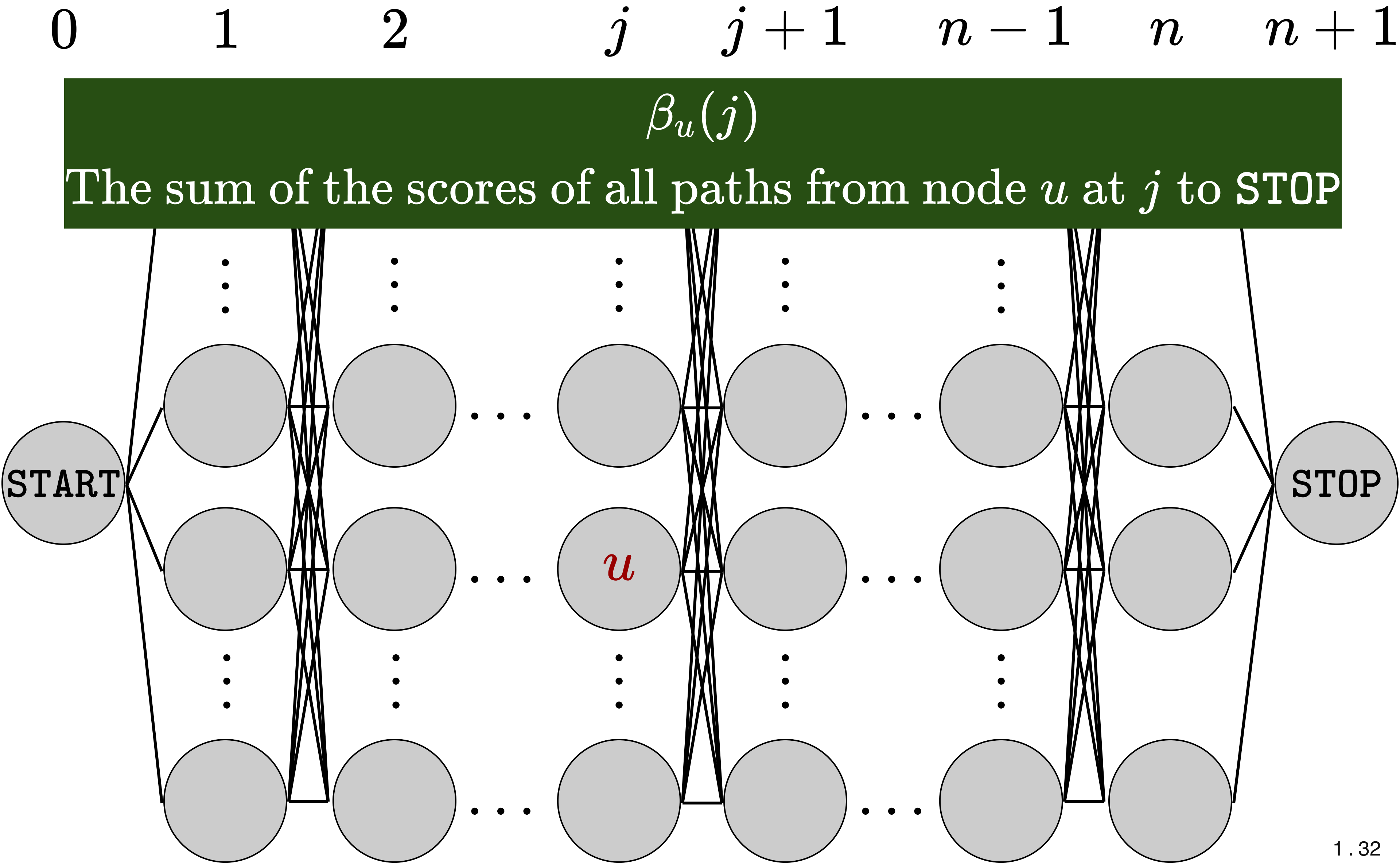


$$\boxed{\alpha_u(j+1) = \sum_v \alpha_v(j) a_{v,u} b_v(x_j)}$$

Forward-Backward Algorithm

$\beta_u(j)$

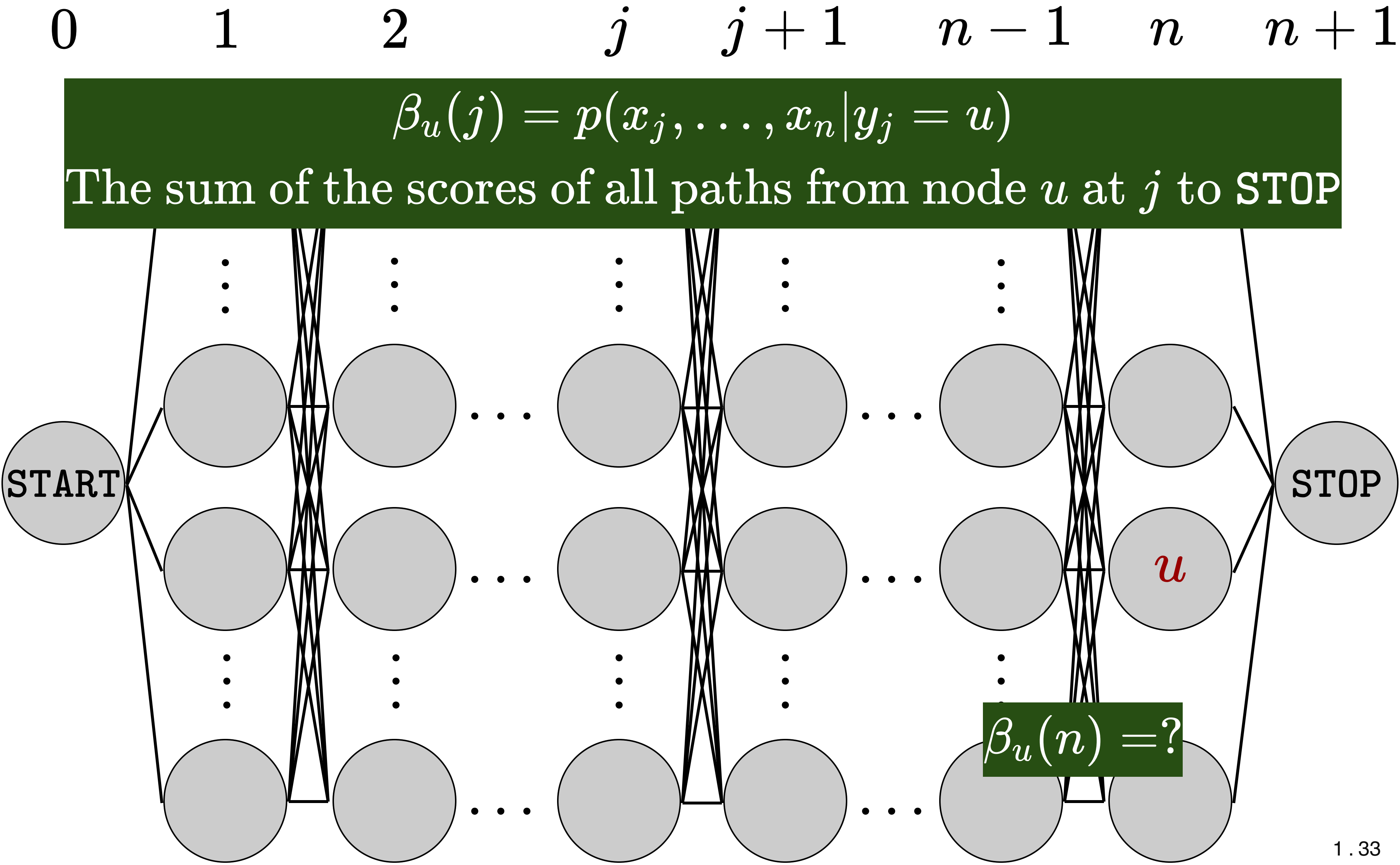
The sum of the scores of all paths from node u at j to STOP



Forward-Backward Algorithm

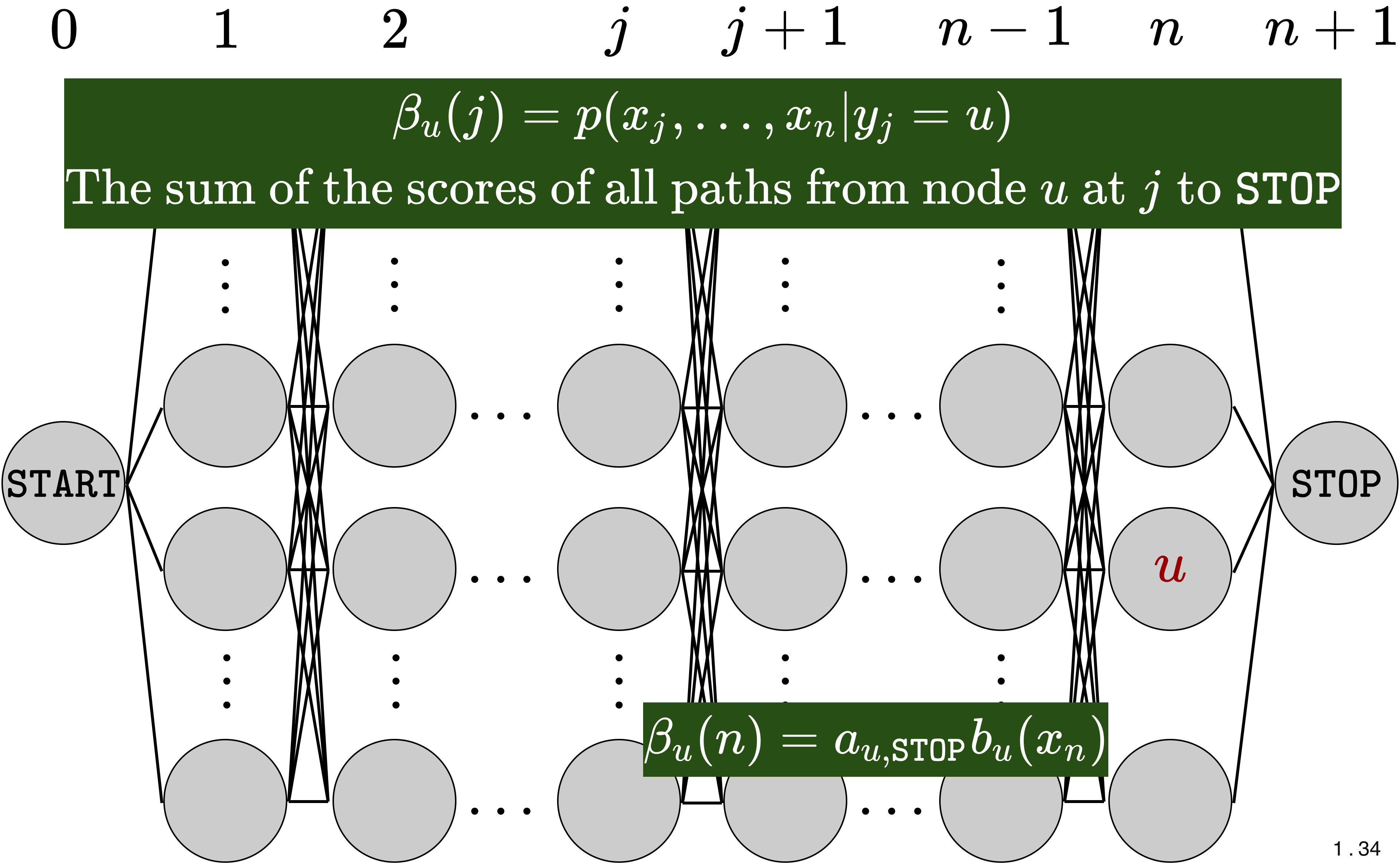
$$\beta_u(j) = p(x_j, \ldots, x_n | y_j = u)$$

The sum of the scores of all paths from node u at j to STOP



$\beta_u(n) = ?$

Forward-Backward Algorithm

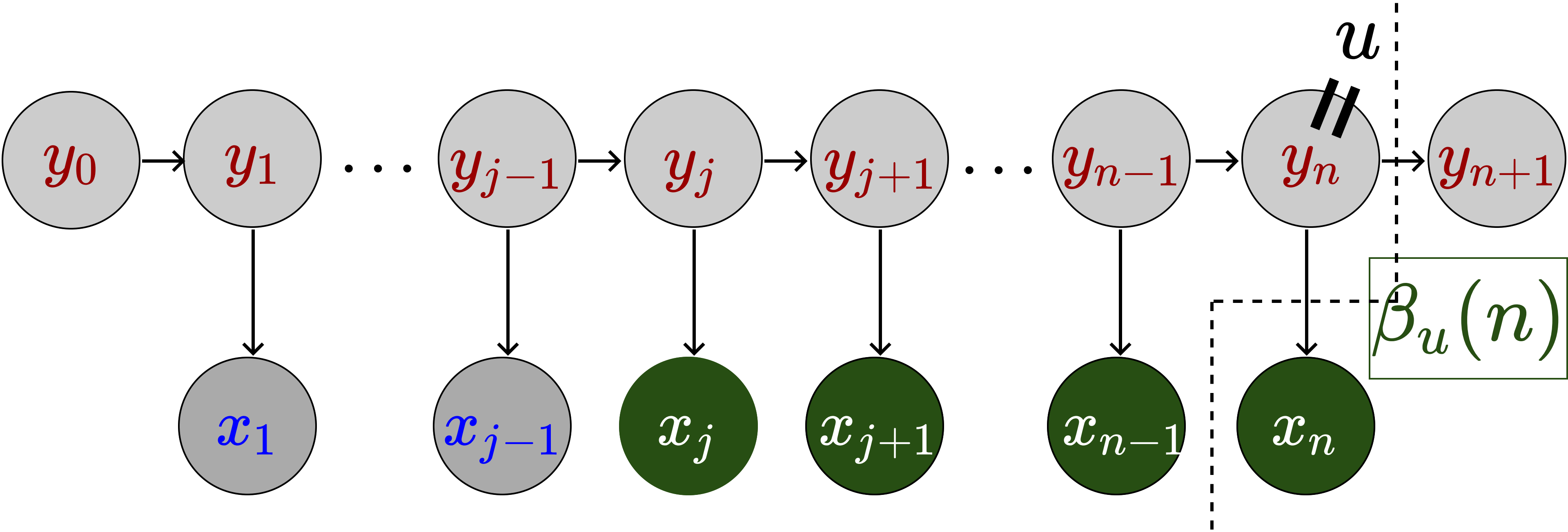


$$\beta_u(j) = p(x_j, \ldots, x_n | y_j = u)$$

The sum of the scores of all paths from node u at j to STOP

$$\beta_u(n) = a_{u,\texttt{STOP}} b_u(x_n)$$

Forward-Backward Algorithm

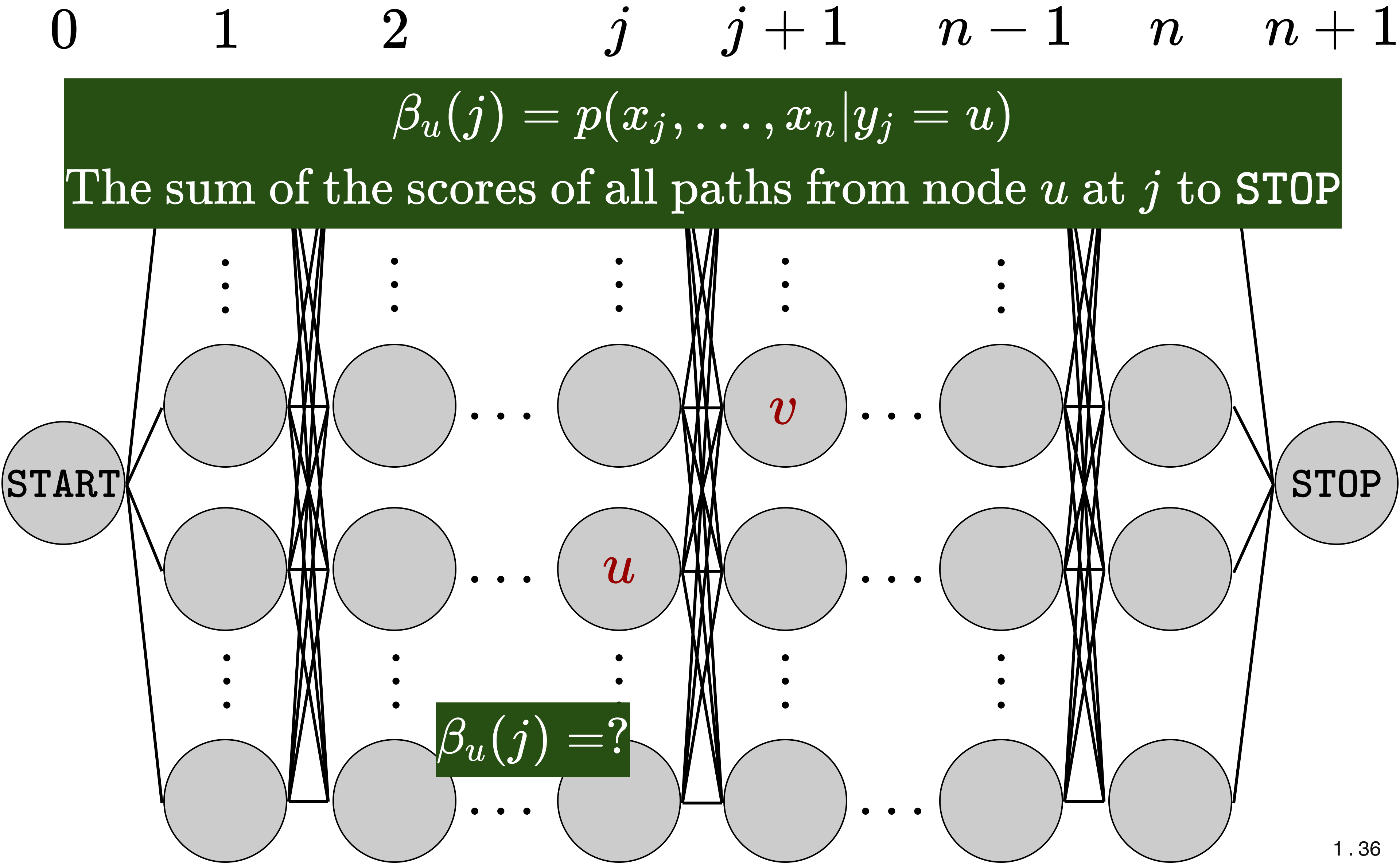


$$\boxed{\beta_u(n) = a_{u,\texttt{STOP}} b_u(x_n)}$$

Forward-Backward Algorithm

$$\beta_u(j) = p(x_j, \ldots, x_n | y_j = u)$$

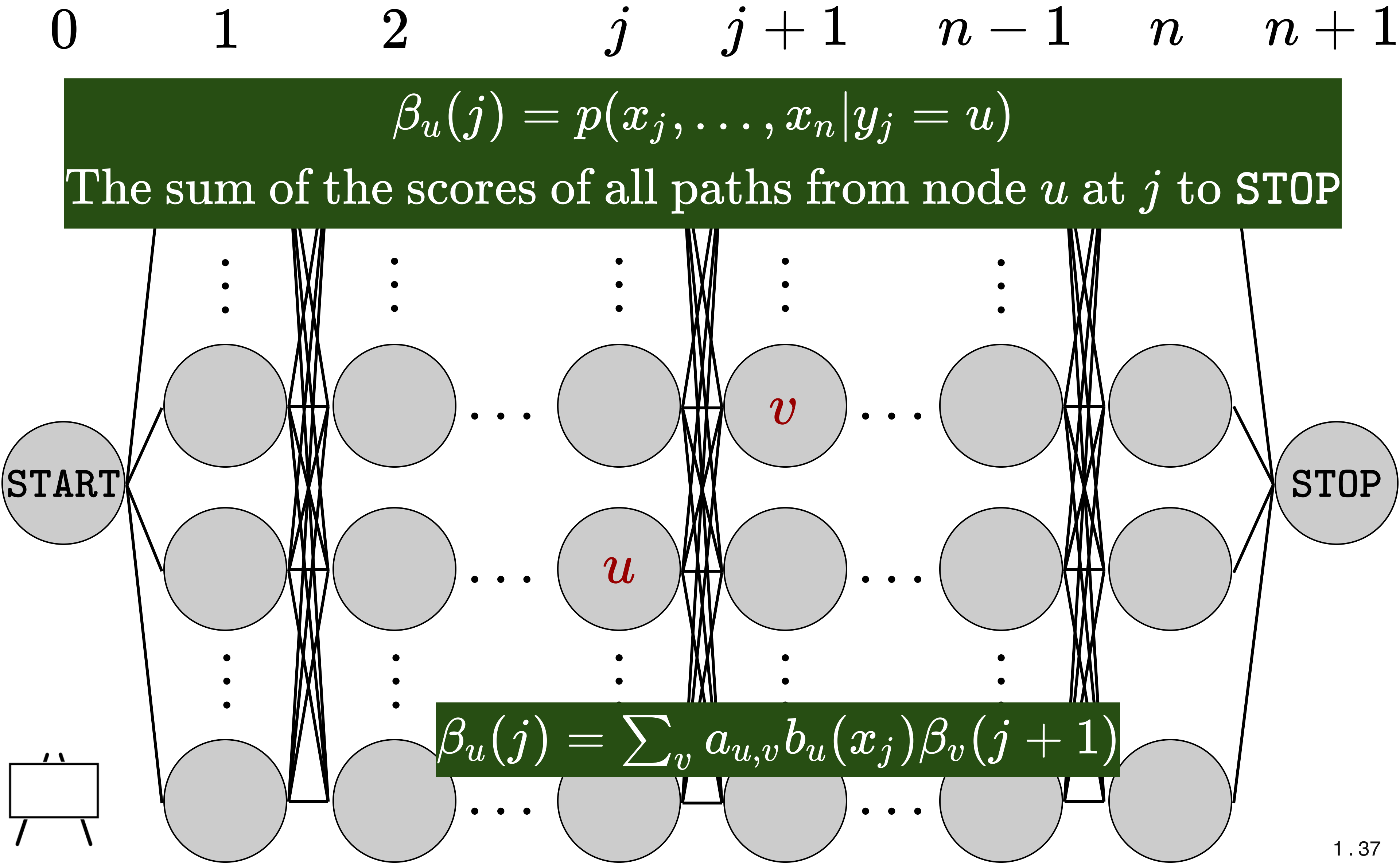
The sum of the scores of all paths from node u at j to STOP



Forward-Backward Algorithm

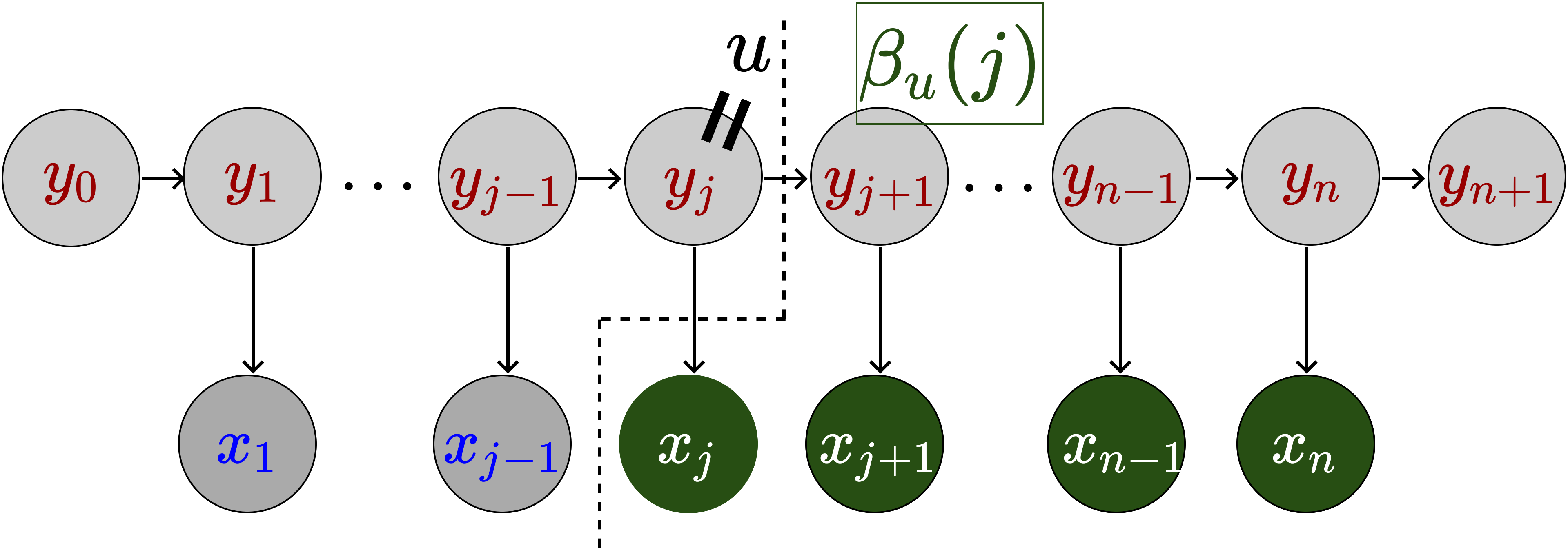
$$\beta_u(j) = p(x_j, \ldots, x_n | y_j = u)$$

The sum of the scores of all paths from node u at j to STOP

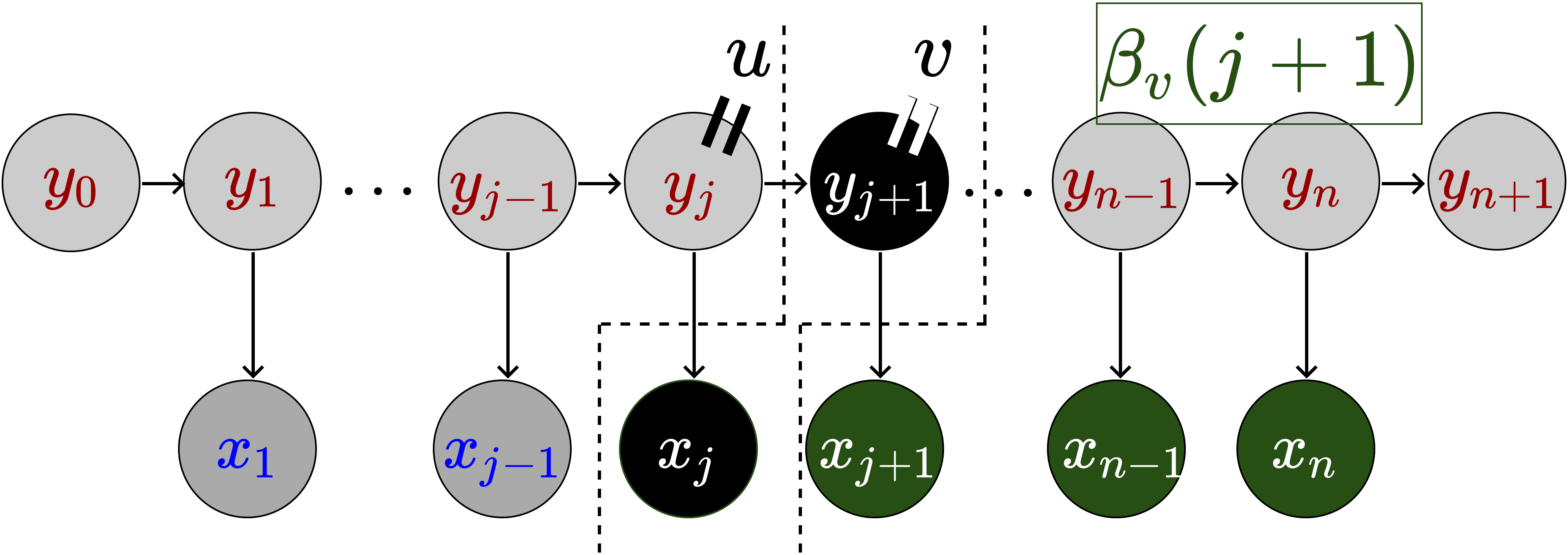


$$\beta_u(j) = \sum_v a_{u,v} b_u(x_j) \beta_v(j+1)$$

Forward-Backward Algorithm



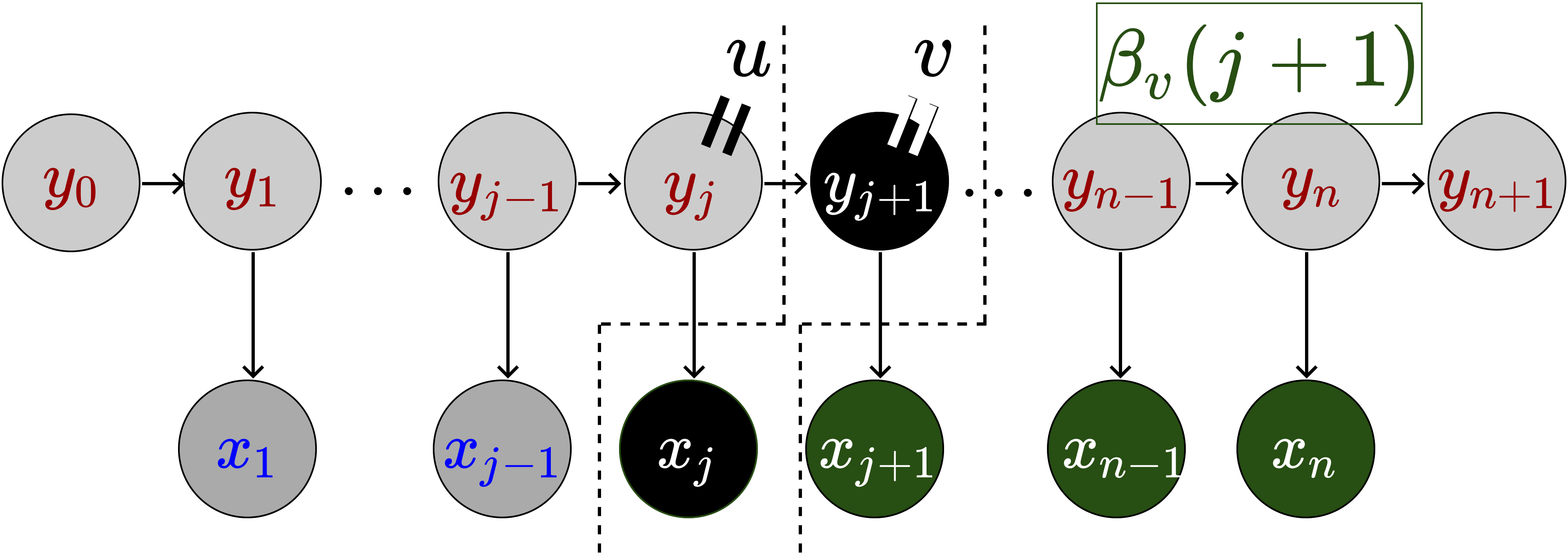
Forward-Backward Algorithm



Assuming the next state is v

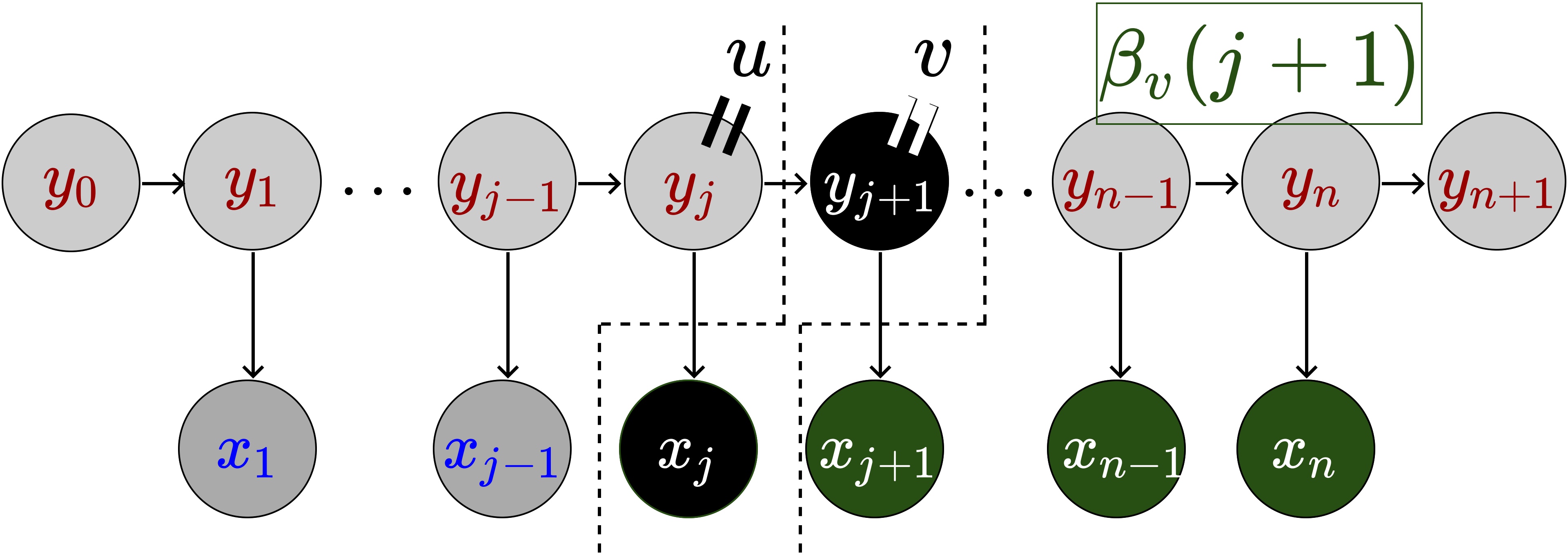
How do we generate these two nodes in black?

Forward-Backward Algorithm



$$a_{u,v} b_u(x_j) \beta_v(j+1)$$

Forward-Backward Algorithm

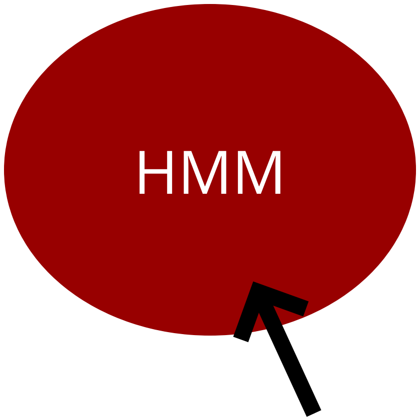


$$\boxed{\beta_u(j) = \sum_v a_{u,v} b_u(x_j) \beta_v(j+1)}$$

Question

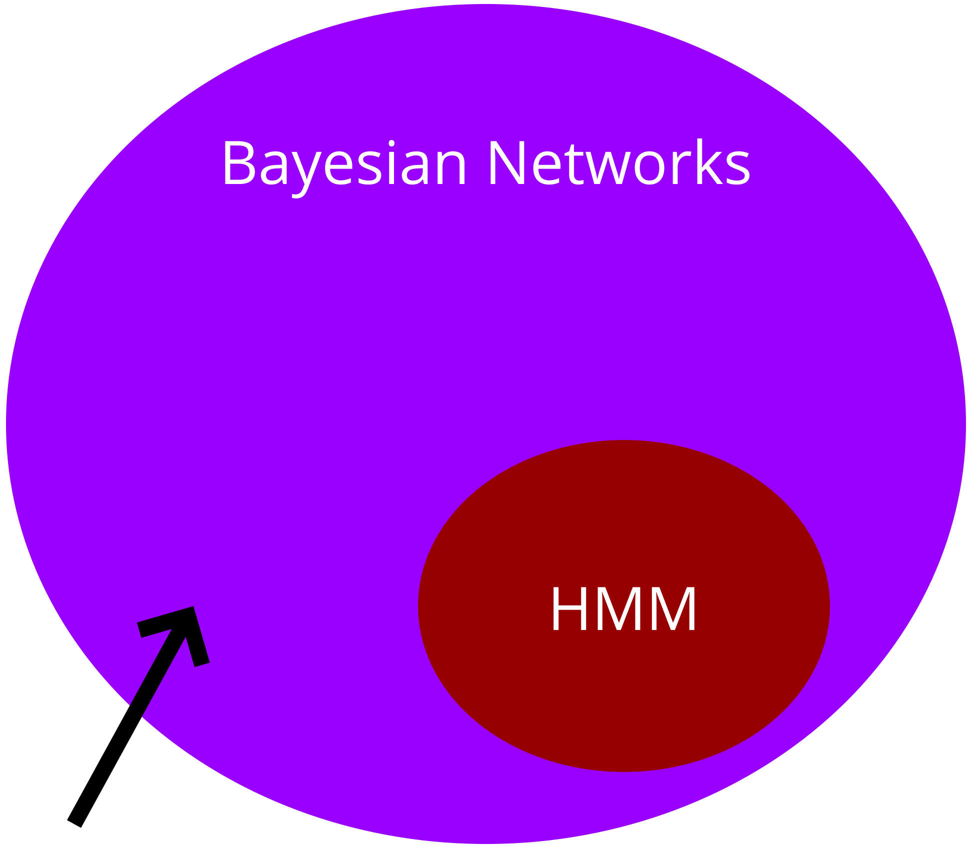
What is the time complexity of the forward backward algorithm?

Hidden Markov Model



Where we are now

Bayesian Networks



Where we will be next