Geothermal Reservoir Simulation
1. Mathematical Models for Liquid- and Vapor-Dominated Hydrothermal Systems

CHARLES R. FAUST AND JAMES W. MERCER

U.S. Geological Survey, Reston, Virginia 22092

Two alternative mathematical models are presented that are suitable for numerical simulation of geothermal reservoirs. The general mathematical model describes the three-dimensional flow of single-component water (both one and two phase) and the transport of heat in porous media. It is composed of two nonlinear partial differential equations, posed in terms of fluid pressure and enthalpy, and appropriate boundary conditions. An alternative quasi-three-dimensional model is derived by partial integration (in the vertical dimension) of the three-dimensional equations. The reservoir is assumed to have good vertical communication so that vertical equilibrium (gravity segregation) between steam and water is achieved. The resulting equations, posed in terms of vertically averaged pressure and enthalpy, include effects of an inclined, variable-thickness reservoir and mass and energy leakage to confining beds.

INTRODUCTION

Geothermal energy recently has received substantial attention as an alternative energy source. This has stimulated considerable research in several broad areas: exploration and identification of geothermal resources, technology for extracting energy from different types of geothermal resources, and characterization of geothermal reservoirs and hydrothermal system behavior. Geothermal reservoir simulation, one aspect of the third area, is the subject of this series of three papers. This paper, first in the series, deals with the development of the mathematical models that are the basis of our approach to geothermal reservoir simulation. The second paper introduces the numerical methods used and presents examples to verify the models as well as to demonstrate important characteristics of geothermal reservoir behavior. The final paper in the series presents an application of one of our models to the Wairakei hydrothermal system in New Zealand.

The primary objective of this paper is to provide a rational mathematical description of fluid (liquid water and steam) flow and energy transport in porous hydrothermal systems. This description begins with the balance equations for mass, momentum, and energy in porous media. These equations are combined (invoking appropriate simplifying assumptions) to yield two nonlinear partial differential equations posed in terms of fluid pressure and enthalpy. We then present the consistent boundary and initial conditions, which, along with the partial differential equations, comprise the general three-dimensional mathematical model.

Three-dimensional models are not always practical to use due to constraints imposed by data requirements and computational expense (for numerical solution). As an alternative we propose a quasi-three-dimensional areal model based on vertical integration of the three-dimensional equations. This approach, unique in geothermal simulation, leads to two nonlinear partial differential equations posed in terms of vertically averaged fluid pressures and enthalpies. The resulting areal model accounts for vertical variations in properties (such as the presence of a steam cap) as well as variations in reservoir thickness and slope.

GEOTHERMAL SYSTEMS

The ultimate source of geothermal energy is heat energy generated and stored within the earth. Potential sources of geothermal energy can be divided into three major systems: hydrothermal, geopressured, and hot, dry rock (conduction dominated). In hydrothermal systems, heat from near-surface sources such as magmatic bodies is transferred to porous media and to the fluid within those media by conductive and convective processes. These systems can be further classified as being either liquid dominated or vapor dominated [*White et al.*, 1971]. In geopressured systems, fluid is trapped in geosynclinal accumulations where it is subjected to extreme pressures and high temperatures. Finally, in hot rock systems, low-permeability igneous rocks are heated by sources similar to those associated with hydrothermal systems. These systems are, by nature, fluid independent.

In this study, only geothermal energy from hydrothermal systems is considered. When such a system is utilized for its heat energy, it is called a geothermal reservoir. The best known geothermal reservoir that is liquid dominated is the Wairakei field in New Zealand. Of the vapor-dominated geothermal reservoirs the best known are the Larderello field in Italy and The Geysers field in California.

Although geothermal fluids contain impurities, many reservoirs, as a first approximation, may be treated as pure water systems. Making this assumption, consider the pressure-enthalpy diagram for pure water in Figure 1. Because all the known geothermal reservoirs exist at temperatures below the critical point of water (the temperature above which two phases cannot exist), this diagram may be divided into three regions. The first of these is the compressed-water region, which is the condition existing in liquid-dominated hydrothermal systems. The second is the two-phase (steam-water) region, in which temperature is a function of pressure only. The third region contains superheated steam. The vapor-dominated system described by *White et al.* [1971] is believed to exist in the two-phase region, although the lower part of these systems may have a water table below which the fluid exists as compressed water. In the vapor-dominated systems it is also possible, especially when influenced by production, that parts of the system may contain superheated steam.

From the above description it is apparent that any mathematical model of fluid flow and energy transport in hydro-
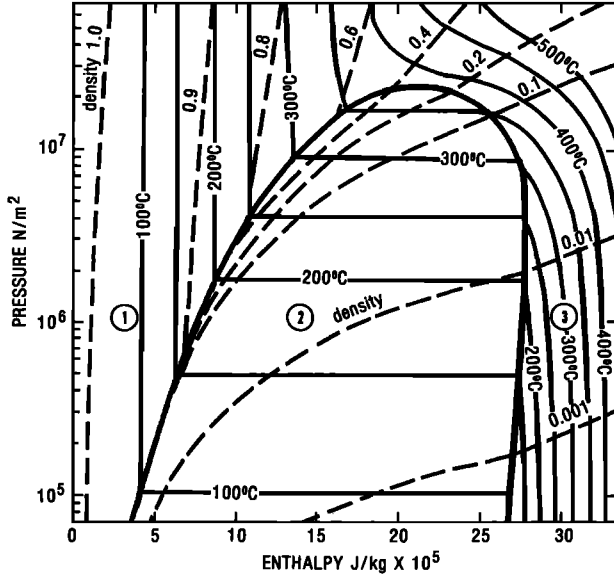
Fig. 1.   Pressure-enthalpy diagram for pure water and vapor showing three states below the critical point (top of two-phase envelope): 1, compressed water; 2, two-phase steam and water; and 3, superheated steam.

thermal systems will be complex. It must account for the flow behavior of one- and two-phase fluids and heat transport in complex natural systems. The remainder of this paper is an attempt to offer a simplified yet realistic description of this behavior subject to appropriate assumptions.

GENERAL MATHEMATICAL MODEL

A general mathematical model of a geothermal system that describes the three-dimensional flow of water and/or steam and transport of heat in a porous medium is the basis for several numerical models developed in this study. Elsewhere, we have presented a heuristic derivation [*Mercer et al.*, 1974] and a more formal derivation [*Faust and Mercer*, 1977a] of this general mathematical model. Other derivations of general geothermal models may also be found in work by *Brownell et al.* [1977] and *Witherspoon et al.* [1975]. Rather than repeat the details of these lengthy derivations, in this paper we only show a brief derivation of the final equations and state major assumptions.

The conservation equations for mass, momentum, and energy must first be considered for each phase in the geothermal system. Using a set of constitutive relationships, these balance equations are reduced to two nonlinear partial differential equations in terms of the dependent variables pressure and enthalpy.

Mass Balance

The mass balance for steam s and water w may be written as

$$\frac{\partial(\phi S_s \rho_s)}{\partial t} + \nabla \cdot (\rho_s \mathbf{v}_s) - q_s' - d_v = 0 \qquad (1)$$

and

$$\frac{\partial(\phi S_w \rho_w)}{\partial t} + \nabla \cdot (\rho_w \mathbf{v}_w) - q_w' + d_v = 0 \qquad (2)$$

where boldface type indicates a vector quantity and ∇ is the vector differential operator. In the above equations, ϕ is the porosity, S is the volumetric saturation, ρ is the density, d_v is the gain of steam mass due to the vaporization (loss of liquid

water), q' is a source term, and $\mathbf{v}$ is the averaged phase velocity.

Momentum Balance

It is assumed that Darcy's equations for multiphase flow may be used as simplified momentum balances. For steam and water these are as follows:

$$\mathbf{v}_s = -\frac{\mathsf{K}k_{rs}}{\mu_s} \cdot (\nabla p_s - \rho_s g \nabla D) \qquad (3)$$

and

$$\mathbf{v}_w = -\frac{\mathsf{K}k_{rw}}{\mu_w} \cdot (\nabla p_w - \rho_w g \nabla D) \qquad (4)$$

where sans serif type indicates a second-order tensor quantity. In these equations, K is the intrinsic permeability tensor of the porous medium, k_r is the relative permeability of the phase, μ is the dynamic viscosity, g is the gravitational constant, D is the depth, and p is phase pressure. The representation of mobility as $\mathsf{K}k_r/\mu$ is thought to be a reasonable assumption, although data on geothermal systems are not available to support it conclusively.

Energy Balance

Simplified energy equations for steam, water, and rock r may be written as

$$\frac{\partial(\phi S_s \rho_s h_s)}{\partial t} + \nabla \cdot (\rho_s h_s \mathbf{v}_s) + \nabla \cdot \lambda_{cs} + \nabla \cdot \lambda_{ds} - \frac{\partial(\phi S_s p_s)}{\partial t}$$
$$- \mathbf{v}_s \cdot \nabla p_s - Q_s' - q_s' h_s' = 0 \qquad (5)$$

$$\frac{\partial(\phi S_w \rho_w h_w)}{\partial t} + \nabla \cdot (\rho_w h_w \mathbf{v}_w) + \nabla \cdot \lambda_{cw} + \nabla \cdot \lambda_{dw} - \frac{\partial(\phi S_w p_w)}{\partial t}$$
$$- \mathbf{v}_w \cdot \nabla p_w + Q_w' - q_w' h_w' = 0 \qquad (6)$$

and

$$\frac{\partial[(1 - \phi)\rho_r h_r]}{\partial t} + \nabla \cdot \lambda_{cr} - Q_r' = 0 \qquad (7)$$

where h is specific enthalpy, λ_c is the heat conduction vector, λ_d is the dispersion (thermal mixing) vector, and Q_w', Q_s', and Q_r' are the interphase energy exchange terms.

In these equations, kinetic energy (viscous dissipation) and potential energy are neglected. *Garg and Pritchett* [1977] have shown that viscous dissipation has negligible effects in one-phase (water) and two-phase geothermal systems.

Constitutive Relationships

The balance equations (1)–(7) are not sufficient to describe the geothermal system, and consequently, additional equations are required. These are in the form of constitutive relationships that are based on the following assumptions:

1.   Capillary pressure effects are negligible.
2.   Thermal equilibrium exists among the steam, water, and rock.
3.   The reservoir fluid is single-component water consisting of either one or two phases.
4.   Relative permeability is a function of liquid volume saturation; hysteresis is neglected.
5.   Viscosities are considered functions of temperature.
6.   Porosity is a linear function of pressure, given by

$$\phi = \phi_t + \beta(p - p_t) \qquad (8)$$

where ϕ_i and p_i are the initial porosity and pressure, respectively, and β is the intergranular vertical compressibility coefficient.

7. Rock density, reservoir thickness, and intrinsic permeability are functions of space.

8. Rock enthalpy is a linear function of temperature, given by

$$h_r = c_r T \qquad (9)$$

where c_r is the specific heat of the rock and T is temperature. We now consider the first three of these assumptions in detail.

Capillary pressure. An expression relating phase pressure is given by

$$p_c = p_s - p_w \qquad (10)$$

where p_c is the capillary pressure. Capillary pressure has the effect of lowering the vapor pressure of water. *Ramey et al.* [1973] point out that the reason for this lowering is that vapor pressure data found in steam tables [*Meyer et al.,* 1968; *Keenan et al.,* 1969] are based on flat steam-water interfaces, whereas the interface in porous media is curved. The amount that the vapor pressure curve is lowered in a geothermal reservoir is not completely understood. The work of *Calhoun et al.* [1949] on consolidated rock does show a lowering with decreased fluid saturation. *Cady* [1969] and *Bilhartz* [1971], however, indicate no significant vapor pressure lowering in experiments using unconsolidated sands. An important difference in these results is that the experiments of Calhoun and others were made at a temperature of 36°C, while those of Cady and Bilhartz were done over a temperature range of 121°–240°C. Further work on the importance of capillary pressure in geothermal reservoirs is required. For this development the only effect of capillary pressure that is considered is the possible occurrence of a residual water saturation (disconnected water in the pore space that is immobile). Other capillary pressure effects are neglected, and capillary pressure is assumed negligible, an assumption which implies that fluid pressures in the steam and water phases are equal. With this assumption, (1) and (2) may be combined to eliminate the vaporization terms:

$$\frac{\partial(\phi\rho)}{\partial t} + \nabla\cdot(\rho_s\mathbf{v}_s) + \nabla\cdot(\rho_w\mathbf{v}_w) - q_s{}' - q_w{}' = 0 \qquad (11)$$

where ρ is the density of the total steam-water mixture, defined as

$$\rho = S_w\rho_w + S_s\rho_s \qquad (12)$$

the volume saturations are defined so that they sum to 1,

$$S_s + S_w = 1 \qquad (13)$$

and phase velocities can now be expressed in terms of a single pressure p.

Thermal equilibrium. The movement of steam and water through porous media is sufficiently slow and the surface areas of all phases are sufficiently large that it is reasonable to assume that local thermal equilibrium among phases is achieved instantaneously. This assumption permits the energy equations for rock, steam, and water to be combined and the medium conduction-dispersion term to be expressed as a function of a single temperature. In this development the lumped conduction-dispersion term is defined by a Fourier-type equation:

$$\lambda_{cr} + \lambda_{cs} + \lambda_{cw} + \lambda_{ds} + \lambda_{dw} = -K_m\nabla T \qquad (14)$$

where the medium conduction-dispersion coefficient K_m is isotropic. This simplifying assumption is made because data on the tensor nature of thermal dispersion are generally unavailable. In addition to the limitation of combining the effects of conduction and dispersion, (14) neglects the important effect of temperature on thermal conductivity. In this regard, *Somerton et al.* [1974] point out that the thermal conductivity of a porous medium is a function of temperature, porosity, and water saturation. This effect may be important in a purely conductive system; however, in this study these effects are neglected.

Invoking these assumptions concerning thermal equilibrium and dispersion, the energy balance equations may be combined, yielding

$$\frac{\partial}{\partial t}[\phi\rho h + (1 - \phi)\rho_r h_r]$$

$$+ \nabla\cdot(\rho_s h_s \mathbf{v}_s) + \nabla\cdot(\rho_w h_w \mathbf{v}_w) - \nabla\cdot(K_m\nabla T) - q_s{}'h_s{}'$$

$$- q_w{}'h_w{}' - \left[\frac{\partial\phi p}{\partial t} + (\mathbf{v}_s + \mathbf{v}_w)\cdot\nabla p\right] = 0 \qquad (15)$$

in which h is the enthalpy of the steam-water mixture and is defined as

$$h = (S_s\rho_s h_s + S_w\rho_w h_w)/\rho \qquad (16)$$

Note that the interphase energy exchange terms in (5), (6), and (7) have been eliminated in (15), because they sum to zero. Finally, the last term in (15) is the pressure material derivative, which in part comprises the compressible work term. *Moench* [1976] points out that the term for compressible work is negligible except for conditions of low water saturation. The same characteristics can be shown for the rest of the pressure material derivative, and for most applications this term is neglected.

Thermodynamic properties. As was previously indicated, it is assumed that the hydrothermal fluid is pure water. Just how unrestrictive this assumption can be is demonstrated by *Haas* [1976a, b], who gives thermodynamic data for an NaCl solution. These data show that in the pressure range of most geothermal reservoirs the vapor pressure curve is lowered by less than 3°C for a 5% (by weight) solution. Although the fluid in geothermal reservoirs contains other dissolved solids in addition to NaCl, in a qualitative sense the effects are similar for other impurities. For geothermal reservoirs such as those at Wairakei, New Zealand, Larderello, Italy, and The Geysers, California, in which the salinities are low [*Koenig,* 1973], the effects of dissolved solids are small.

Additional expressions are needed to relate the thermodynamic properties of pure water and steam to the dependent variables pressure and enthalpy. These expressions are functional in form, were obtained by least squares regressions applied to data from steam tables [*Meyer et al.,* 1968; *Keenan et al.,* 1969], and are described in detail by *Faust and Mercer* [1977b]. This approach avoids the necessity of searching for and interpolating data from tables. Also, derivatives of the functions can be easily obtained by analytical methods. The necessary relationships for this development are as follows:

1. Saturated steam enthalpy h_s and saturated water enthalpy h_w are treated as functions of pressure.

2. Temperature is treated as a function of pressure and enthalpy for the compressed-water region and the superheated-steam region and is treated as a function of pressure in the two-phase region.

3. Total density ρ and steam and water densities ρ_s and ρ_w are considered functions of pressure and enthalpy.

4. Phase saturations are functions of enthalpy and pressure in the two-phase region. Water saturation in the compressed-water region is assumed to be 1 and in the superheated-steam region is assumed to be 0. In the steam-water region, water saturation is obtained from (12), (13), and (16) as

$$S_w = \frac{\rho_s(h_s - h)}{h(\rho_w - \rho_s) - (h_w\rho_w - h_s\rho_s)} \tag{17}$$

and S_s is determined using (13).

Three-Dimensional Equations

The final equations are formulated in terms of the dependent variables pressure and enthalpy, because these two variables uniquely define the thermodynamic state of the system and because they are commonly obtained in a field situation. With the assumptions of zero capillary pressure and thermal equilibrium the number of balance equations was reduced to four (equations (3), (4), (11), and (15)). These can be reduced further by substitution of (3) and (4) into (11) and (15) and by expansion of the temperature derivative in (15) to yield

$$\frac{\partial(\phi\rho)}{\partial t} - \nabla \cdot \left[\frac{Kk_{rs}\rho_s}{\mu_s}\cdot(\nabla p - \rho_s g\nabla D)\right]$$

$$- \nabla \cdot \left[\frac{Kk_{rw}\rho_w}{\mu_w}\cdot(\nabla p - \rho_w g\nabla D)\right] - q_m' = 0 \tag{18}$$

and

$$\frac{\partial}{\partial t}[\phi\rho h + (1 - \phi)\rho_r h_r] - \nabla \cdot \left[\frac{Kk_{rs}\rho_s h_s}{\mu_s}\cdot(\nabla p - \rho_s g\nabla D)\right]$$

$$- \nabla \cdot \left[\frac{Kk_{rw}\rho_w h_w}{\mu_w}\cdot(\nabla p - \rho_w g\nabla D)\right] - \nabla \cdot \left[K_m\left(\frac{\partial T}{\partial p}\right)_h \nabla p\right.$$

$$+ \left. K_m\left(\frac{\partial T}{\partial h}\right)_p \nabla h\right] - q_h' = 0 \tag{19}$$

where the pressure material derivative has been neglected.

These final three-dimensional equations describe the two-phase flow of heat in a steam-water-rock system; however, with minor modification they also describe the flow of heat in a water-rock or a steam-rock system. When either steam or water is absent, the saturation of the absent phase is 0 and that for the existing phase is 1. Further, it is assumed that the relative permeability of the absent phase is 0 and that for the existing phase is 1. Therefore (18) and (19) reduce to the appropriate equations for either the compressed-water region or the superheated-steam region. A solution for these equations will determine whether a specified location contains compressed water, a steam-water mixture, or superheated steam.

Source terms. The mass and energy source terms q_m' and q_h', respectively, represent the amount of mass and heat lost (or gained) to source-sinks. In the two-phase region the mass rate loss to a source-sink is defined as

$$q_m' = q_s' + q_w' \tag{20}$$

and the heat rate loss to a source-sink is

$$q_h' = h_s'q_s' + h_w'q_w' \tag{21}$$

where a negative rate indicates a loss from the reservoir. The steam production rate may be determined by the fractional flow of the steam phase as follows:

$$q_s' = \alpha_s q_m' \tag{22}$$

and

$$\alpha_s = k_{rs}/[k_{rs} + (\rho_w\mu_s/\rho_s\mu_w)k_{rw}]$$

Because h_s' and h_w' are known functions of pressure and the mass flux q_m' is specified, q_s' is calculated using (22), and q_h' is calculated using (21).

Boundary conditions. Equations (18) and (19) together comprise a pair of nonlinear second-order partial differential equations. Two conditions (one in terms of pressure and one in terms of enthalpy) are required at the boundaries. Perhaps the most common boundary condition is the specification of fluxes. Frequently, the flux is specified as zero, indicating a no-flow boundary. For specification of a mass flux the following equation must be satisfied:

$$q_m^* = -\left(\frac{Kk_{rs}\rho_s}{\mu_s} + \frac{Kk_{rw}\rho_w}{\mu_w}\right)\cdot\frac{\partial(p - \rho g D)}{\partial n} \tag{23}$$

where q_m^* is the specified mass flux at the boundary and $\partial/\partial n$ is the outward normal derivative. If a mass flux is specified, a convective energy flux must also be specified according to an equation analogous to (21):

$$q_h^* = q_s^*h_s^* + q_w^*h_w^* \tag{24}$$

where h_s^* and h_w^* are the pressure-dependent saturated water and saturated steam enthalpies and q_s^* and q_w^* are the fractional steam and water fluxes at the boundary. The total energy flux for the general case consists of two parts,

$$q_{th}^* = q_h^* + q_h^{**} \tag{25}$$

where q_h^{**} represents the conductive heat flux at the boundary, determined by

$$q_h^{**} = -K_m\frac{\partial T}{\partial n}\bigg|_{boundary} \tag{26}$$

A constant-pressure condition may also be encountered. Since this implies a mass flux at the boundary, it also implies a convective energy flux. To determine the convective energy flux, the mass flux is calculated from (23) and used in (24).

AREAL MODEL

The general equations (18) and (19) describe three-dimensional flow and heat transport. For many field problems a three-dimensional model is unnecessary and expensive. In order to obtain areal two-dimensional equations the three-dimensional equations must be partially integrated in the z dimension (vertical). The process of vertically integrating the three-dimensional equations is fairly straightforward but is lengthy and tedious. We gave a detailed derivation of the vertically integrated equations in an earlier paper [*Faust and Mercer*, 1977a]. Here, as in the previous section, we will present only the major assumptions and final equations resulting from the derivation.

To perform the averaging procedure, it is necessary to introduce the following definitions. A quantity averaged in the z dimension is given by

$$\langle\psi\rangle = \frac{1}{b}\int_{z_1}^{z_2} \psi\, dz \tag{27}$$

where $z_1 = z_1(x, y, t)$ is the bottom of the reservoir, $z_2 = z_2(x, y, t)$ is the top, $b = b(x, y, t) = z_2 - z_1$ is the thickness, and the angle brackets signify a quantity averaged in the z dimension. Leibnitz's rule is frequently used to reverse the order of in-

tegration and differentiation [*Korn and Korn*, 1961]. This rule has the form [1]

$$\int_{z_1}^{z_2} \frac{\partial \psi}{\partial x} \, dz = \frac{\partial}{\partial x} \int_{z_1}^{z_2} \psi \, dz$$

$$+ \psi(x, y, z_1, t)\frac{\partial z_1}{\partial x} - \psi(x, y, z_2, t)\frac{\partial z_2}{\partial x} \quad (28)$$

In order to proceed with the derivation it is necessary to specify rules for operating on averaged quantities and deviations from the average quantities. First, the relationship between a quantity ψ, its average $\langle \psi \rangle$, and its deviation $\hat{\psi}$ is given by

$$\psi = \langle \psi \rangle + \hat{\psi} \quad (29)$$

which is also shown graphically in Figure 2. Other useful expressions are

$$\langle \psi + \gamma \rangle = \langle \psi \rangle + \langle \gamma \rangle \quad (30)$$

$$\langle \langle \psi \rangle + \gamma \rangle = \langle \psi \rangle \cdot \langle \gamma \rangle \quad (31)$$

$$\langle \psi \cdot \gamma \rangle = \langle \psi \rangle \cdot \langle \gamma \rangle + \langle \hat{\psi}\hat{\gamma} \rangle \quad (32)$$

$$\langle \hat{\psi} \rangle = 0 \quad (33)$$

and

$$\left\langle \frac{\partial \psi}{\partial x} \right\rangle = \frac{\partial \langle \psi \rangle}{\partial x} - \frac{1}{b}[\langle \psi \rangle - \psi(x, y, z_1, t)]\frac{\partial z_1}{\partial x}$$

$$+ \frac{1}{b}[\langle \psi \rangle - \psi(x, y, z_2, t)]\frac{\partial z_2}{\partial x} \quad (34)$$

where γ is an arbitrary quantity and terms such as $\langle \hat{\psi}\hat{\gamma} \rangle$ represent averaged codeviations. These relationships may be obtained directly using (27), (28), and (29), but they are also similar to Reynold's operational rules for turbulent flow [e.g., *Monin and Yaglom*, 1973].

If the Cartesian coordinate system is aligned with the principal directions of the permeability tensor K and the x and y coordinates are horizontal, then (18) and (19) may be rewritten as

$$\frac{\partial (\phi \rho)}{\partial t} - \frac{\partial}{\partial x}\left(\omega_x \frac{\partial p}{\partial x}\right) - \frac{\partial}{\partial y}\left(\omega_y \frac{\partial p}{\partial y}\right)$$

$$- \frac{\partial}{\partial z}\left(\omega_z \frac{\partial p}{\partial z} + \omega_{gz}\right) - q_m' = 0 \quad (35)$$

and

$$\frac{\partial}{\partial t}[\phi \rho h + (1 - \phi)\rho_r h_r] - \frac{\partial}{\partial x}\left[(\omega_{hx} + \omega_{cp})\frac{\partial p}{\partial x}\right]$$

$$- \frac{\partial}{\partial y}\left[(\omega_{hy} + \omega_{cp})\frac{\partial p}{\partial y}\right]$$

$$- \frac{\partial}{\partial z}\left[(\omega_{hz} + \omega_{cp})\frac{\partial p}{\partial z} + \omega_{hgz}\right]$$

$$- \frac{\partial}{\partial x}\left(\omega_{ch}\frac{\partial h}{\partial x}\right) - \frac{\partial}{\partial y}\left(\omega_{ch}\frac{\partial h}{\partial y}\right)$$

$$- \frac{\partial}{\partial z}\left(\omega_{ch}\frac{\partial h}{\partial z}\right) - q_h' = 0 \quad (36)$$

To simplify these equations, we have lumped coefficients. For the x direction (terms for the y and z directions are similar)
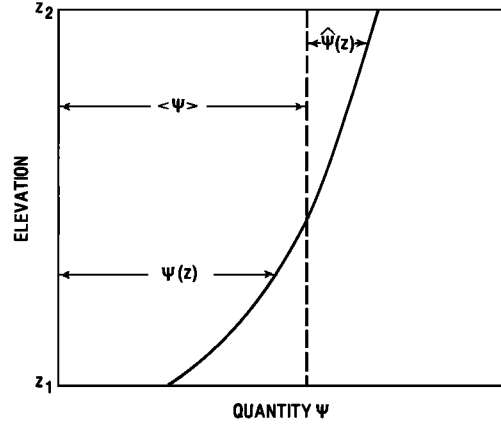

Fig. 2. Graphical representation of a typical quantity $\psi(z)$ as a function of elevation (z dimension) showing the average value $\langle \psi \rangle$ and the deviation $\hat{\psi}$, where z_1 and z_2 are the base and top of the reservoir, respectively.

these terms are

$$\omega_x = \frac{k_x k_{rw}\rho_w}{\mu_w} + \frac{k_x k_{rs}\rho_s}{\mu_s}$$

$$\omega_{hx} = \frac{k_x k_{rw}\rho_w h_w}{\mu_w} + \frac{k_x k_{rs}\rho_s h_s}{\mu_s}$$

$$\omega_{cp} = K_m\left(\frac{\partial T}{\partial p}\right)_h$$

$$\omega_{ch} = K_m\left(\frac{\partial T}{\partial h}\right)_p$$

where k_x is the principal component of the permeability tensor in the x direction. For the z dimension, additional terms are defined as

$$\omega_{gz} = \frac{k_z k_{rw}\rho_w^2 g}{\mu_w} + \frac{k_z k_{rs}\rho_s^2 g}{\mu_s}$$

$$\omega_{hgz} = \frac{k_z k_{rw}\rho_w^2 g h_w}{\mu_w} + \frac{k_z k_{rs}\rho_s^2 g h_s}{\mu_s}$$

The areal (two dimensional) model is obtained by partial integration of (35) and (36) in the vertical dimension. If we assume that variations of reservoir thickness in time are negligible, straightforward substitution of (27) and (29) and application of Leibnitz's rule lead to the following averaged equations:

$$b\frac{\partial}{\partial t}(\langle \phi \rho \rangle) - \frac{\partial}{\partial x}\left[b\left(\langle \omega_x \rangle\left\langle \frac{\partial p}{\partial x} \right\rangle + \left\langle \hat{\omega}_x \frac{\partial \hat{p}}{\partial x} \right\rangle\right)\right]$$

$$- \frac{\partial}{\partial y}\left[b\left(\langle \omega_y \rangle\left\langle \frac{\partial p}{\partial y} \right\rangle + \left\langle \hat{\omega}_y \frac{\partial \hat{p}}{\partial y} \right\rangle\right)\right]$$

$$- b\langle q_m' \rangle + \mathbf{v}|_{z_1}\cdot\nabla(z - z_1) - \mathbf{v}|_{z_2}\cdot\nabla(z - z_2) = 0 \quad (37)$$

$$b\frac{\partial}{\partial t}[(\langle \phi \rho h \rangle + \langle \rho_r h_r \rangle - \langle \phi \rho_r h_r \rangle)]$$

$$- \frac{\partial}{\partial x}\left[b\left(\langle \omega_{hx} \rangle\left\langle \frac{\partial p}{\partial x} \right\rangle + \left\langle \hat{\omega}_{hx} \frac{\partial \hat{p}}{\partial x} \right\rangle\right)\right]$$

$$- \frac{\partial}{\partial y}\left[b\left(\langle \omega_{hy} \rangle\left\langle \frac{\partial p}{\partial y} \right\rangle + \left\langle \hat{\omega}_{hy} \frac{\partial \hat{p}}{\partial y} \right\rangle\right)\right]$$

$$-\frac{\partial}{\partial x}\left[b\left(\langle\omega_{cp}\rangle\left\langle\frac{\partial p}{\partial x}\right\rangle + \langle\omega_{ch}\rangle\left\langle\frac{\partial h}{\partial x}\right\rangle\right.\right.$$

$$+\left.\left.\left\langle\hat{\omega}_{cp}\frac{\partial p}{\partial x}\right\rangle + \left\langle\hat{\omega}_{ch}\frac{\partial h}{\partial x}\right\rangle\right)\right]$$

$$-\frac{\partial}{\partial y}\left[b\left(\langle\omega_{cp}\rangle\left\langle\frac{\partial p}{\partial y}\right\rangle + \langle\omega_{ch}\rangle\left\langle\frac{\partial h}{\partial y}\right\rangle\right.\right.$$

$$+\left.\left.\left\langle\hat{\omega}_{cp}\frac{\partial p}{\partial y}\right\rangle + \left\langle\hat{\omega}_{ch}\frac{\partial h}{\partial y}\right\rangle\right)\right] - b\langle q_h'\rangle$$

$$+ h\mathbf{v}|_{z_1}\cdot\nabla(z-z_1) - h\mathbf{v}|_{z_2}\cdot\nabla(z-z_2)$$

$$+\lambda_m|_{z_1}\cdot\nabla(z-z_1)$$

$$-\lambda_m|_{z_2}\cdot\nabla(z-z_2) = 0 \tag{38}$$

The terms in (37) and (38) with the vertical bars are flux terms that are evaluated at either the top z_2 or the bottom z_1 of the reservoir and are defined (for the reservoir bottom) as follows:

$$\mathbf{v}\bigg|_{z_1}\cdot\nabla(z-z_1) = -\left(\omega_x\frac{\partial p}{\partial x}\right)\bigg|_{z_1}\frac{\partial z_1}{\partial x}$$

$$-\left(\omega_y\frac{\partial p}{\partial y}\right)\bigg|_{z_1}\frac{\partial z_1}{\partial y} + \left(\omega_z\frac{\partial p}{\partial z} + \omega_{gz}\right)\bigg|_{z_1} \tag{39}$$

$$h\mathbf{v}\bigg|_{z_1}\cdot\nabla(z-z_1) = -\left(\omega_{hx}\frac{\partial p}{\partial x}\right)\bigg|_{z_1}\frac{\partial z_1}{\partial x}$$

$$-\left(\omega_{hy}\frac{\partial p}{\partial y}\right)\bigg|_{z_1}\frac{\partial z_1}{\partial y} + \left(\omega_{hz}\frac{\partial p}{\partial z} + \omega_{hgz}\right)\bigg|_{z_1} \tag{40}$$

$$\lambda_m\bigg|_{z_1}\cdot\nabla(z-z_1) = -\left(\omega_{cp}\frac{\partial p}{\partial x} + \omega_{ch}\frac{\partial h}{\partial x}\right)\bigg|_{z_1}\frac{\partial z_1}{\partial x}$$

$$-\left(\omega_{cp}\frac{\partial p}{\partial y} + \omega_{ch}\frac{\partial h}{\partial y}\right)\bigg|_{z_1}\frac{\partial z_1}{\partial y} + \left(\omega_{cp}\frac{\partial p}{\partial z} + \omega_{ch}\frac{\partial h}{\partial z}\right)\bigg|_{z_1} \tag{41}$$

Similar expressions may be written for the reservoir top by exchanging z_2 for z_1 in (39)–(41). These terms account for convective and conductive fluxes at the reservoir top and bottom and include the effects caused by the slope of reservoir bedding and the spatial change in reservoir thickness.

Evaluation of Vertically Averaged Terms

Given (37) and (38), a critical problem remains in evaluating averaged quantities and averages of the pressure and enthalpy derivatives. The relationship between the vertical average of a derivative and the derivative of a vertically averaged quantity is given by (34). For pressure the x derivative becomes

$$\left\langle\frac{\partial p}{\partial x}\right\rangle = \frac{\partial\langle p\rangle}{\partial x} - \frac{1}{b}(\langle p\rangle - p|_{z_1})\frac{\partial z_1}{\partial x} + \frac{1}{b}(\langle p\rangle - p|_{z_2})\frac{\partial z_2}{\partial x} \tag{42}$$

If we assume hydrostatic equilibrium, it can be shown that

$$p|_{z_1} = \langle p\rangle + \langle(z - z_1)\rho g\rangle \tag{43}$$

and

$$p|_{z_2} = \langle p\rangle + \langle(z - z_2)\rho g\rangle - b\langle\rho g\rangle \tag{44}$$

Noting that

$$\langle z - z_1\rangle = b/2 \tag{45}$$

and substituting (32), (43), and (44) into (42) lead to

$$\left\langle\frac{\partial p}{\partial x}\right\rangle = \frac{\partial\langle p\rangle}{\partial x} + \frac{1}{2}\langle\rho g\rangle\left(\frac{\partial z_1}{\partial x} + \frac{\partial z_2}{\partial x}\right)$$

$$+ \frac{1}{b}\langle\hat{z}\hat{\rho}g\rangle\left(\frac{\partial z_1}{\partial x} - \frac{\partial z_2}{\partial x}\right) \tag{46}$$

Further, it may be shown that the derivative of the average depth is

$$\frac{\partial\langle D\rangle}{\partial x} = -\frac{1}{2}\left(\frac{\partial z_1}{\partial x} + \frac{\partial z_2}{\partial x}\right) \tag{47}$$

so that the average of the pressure derivative becomes

$$\left\langle\frac{\partial p}{\partial x}\right\rangle = \frac{\partial\langle p\rangle}{\partial x} - \langle\rho g\rangle\frac{\partial\langle D\rangle}{\partial x} - \frac{1}{b}\langle\hat{z}\hat{\rho}g\rangle\frac{\partial\langle D\rangle}{\partial x} \tag{48}$$

Hence this term includes the effects of a sloping reservoir of variable thickness. Equation (48) and the analogous equation for the y derivative may be used to replace the $\langle\partial p/\partial x\rangle$ and $\langle\partial p/\partial y\rangle$ terms in (37) and (38). For enthalpy the relationship for the x derivative is

$$\left\langle\frac{\partial h}{\partial x}\right\rangle = \frac{\partial\langle h\rangle}{\partial x} - \frac{1}{b}(\langle h\rangle - h|_{z_1})\frac{\partial z_1}{\partial x} + \frac{1}{b}(\langle h\rangle - h|_{z_2})\frac{\partial z_2}{\partial x}$$

$$\tag{49}$$

which, along with the analogous expression for the y derivative, may be substituted into (38) directly.

Having considered the evaluation of averaged derivatives of pressure and enthalpy, we must now address the problem of evaluating the averaged coefficients that also appear in (37) and (38). In general, these terms may be evaluated under two conditions: the fluids are (1) unsegregated and (2) segregated.

If the fluids are not segregated, then their properties are assumed to be uniform throughout the thickness of the reservoir. For this condition, laboratory relative permeability curves may be used in the areal calculations. This leads to the easiest evaluation of the vertically integrated terms, but for two-phase systems it is also a very restrictive assumption, being limited to very thin reservoirs. For single-phase reservoirs, however, the uniform-property assumption is normally used.

A less restrictive condition (one that allows simulation of two-phase reservoirs), and one that conforms with the assumption concerning the absence of significant capillary pressure, is that of segregation. For this condition, it is common to assume vertical equilibrium, a concept that was first used in connection with areal simulation of multiphase petroleum reservoirs [Coats et al., 1967, 1971; Hearn, 1971; Jacks et al., 1973]. It is assumed in applying the concept of vertical equilibrium to petroleum reservoirs that the fluid potentials are uniform throughout the reservoir thickness. This corresponds to a gravity-segregated fluid distribution with the potential of each fluid being uniform in the part of the column occupied by that fluid. This condition requires that the reservoir have good vertical communication. Using this assumption, vertically averaged liquid saturations are related to pressure at some reference level by employing pseudo capillary pressure and pseudo relative permeability curves. Basically, the pseudofunction approach gives results similar to those that would be obtained if vertically averaged pressures were used.

Spivak [1974] has made a detailed study of gravity segregation in two-phase petroleum displacement processes. His conclusions are based on simulation runs in which the parameters

affecting gravity segregation were varied. He found that gravity segregation increases with increasing permeability, increasing fluid density difference, increasing viscosity ratio, and decreasing viscosity for a given ratio. It is obvious that in steam-water geothermal systems, conditions are favorable for gravity segregation, because the density difference and viscosity ratio for steam and water are large and the viscosities are relatively low.

For the geothermal problem, many thermodynamic properties are strongly dependent functions of pressure and enthalpy; hence the pseudofunction approach would be awkward. Instead, the concept of vertical equilibrium is used to average the terms in (37) and (38). To perform this averaging, it is necessary to make certain a priori assumptions concerning the vertical distribution of the dependent variables pressure and enthalpy. In making the following assumptions we have relied upon conceptual descriptions of vapor-dominated reservoirs such as those provided by *White et al.* [1971]. We have also relied upon the results of numerous computer experiments (second paper in this series) using three-dimensional and cross-sectional (vertical) reservoir simulators. Our basic assumption is vertical equilibrium. In the absence of significant capillary pressure, steam and water separate by gravity segregation, producing a steam cap with a water saturation equal to the residual water saturation S_{wr}; below the steam cap the water saturation is 1.0 (see Figure 3).

Because the pressure varies hydrostatically in each phase (see Figure 4), the vertically averaged pressure is

$$\langle p \rangle = [\langle p_w \rangle(z_c - z_1) + \langle p_{sw} \rangle(z_2 - z_c)]/b \qquad (50)$$

Recall that vertically averaged quantities are denoted by angle brackets. In (50) the averaged pressure in the saturated region (below the interface at z_c), $\langle p_w \rangle$, is defined by

$$\langle p_w \rangle = p_{z_c} + \rho_w g(z_c - z_1)/2 \qquad (51)$$

and the averaged pressure in the two-phase region (above the interface at z_c), $\langle p_{sw} \rangle$, is defined by

$$\langle p_{sw} \rangle = p_{z_c} - \rho_x g(z_2 - z_c)/2 \qquad (52)$$

where ρ_x is the density in the two-phase region, defined by

$$\rho_x = \rho_s(1 - S_{wr}) + \rho_w S_{wr} \qquad (53)$$

Note that the steam and water densities in (51)–(53) are considered functions of the interface pressure p_{z_c}.

The assumed enthalpy distribution is also shown in Figure 4. Note that the enthalpy increases with depth in the two-phase zone because the water saturation is assumed constant and pressure increases with depth. The vertically averaged
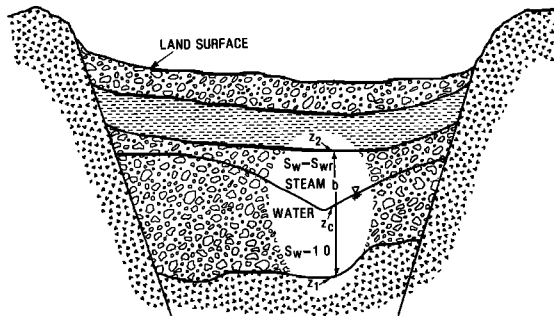
Fig. 3. Idealized cross section showing the steam cap in a confined reservoir; z_c is the steam-water contact, and S_{wr} is the residual water saturation.
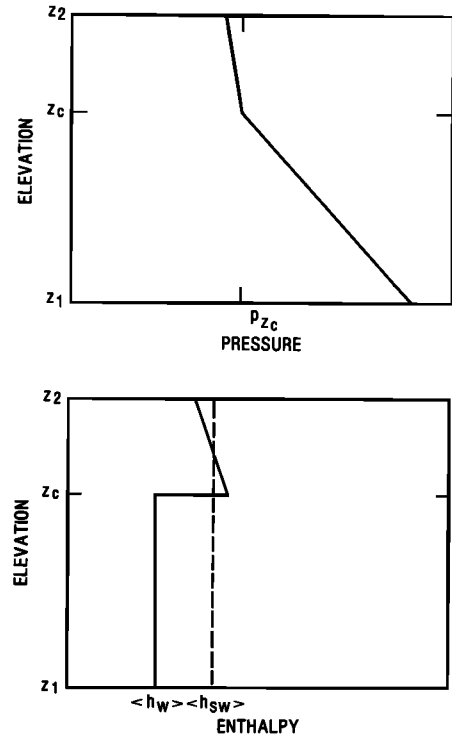
Fig. 4. Pressure and enthalpy profiles through idealized reservoir, showing the pressure at the steam-water contact, p_{z_c}; average water enthalpy $\langle h_w \rangle$; and average steam cap enthalpy $\langle h_{sw} \rangle$.

enthalpy is defined as

$$\langle h \rangle = [\langle h_w \rangle(z_c - z_1) + \langle h_{sw} \rangle(z_2 - z_c)]/b \qquad (54)$$

where $\langle h_w \rangle$ is the averaged enthalpy in the saturated region and is a function of the interface pressure. The averaged enthalpy in the two-phase region above the interface is defined by

$$\langle h_{sw} \rangle = \frac{\rho_s h_s(1 - S_{wr}) + \rho_w h_w S_{wr}}{\rho_s(1 - S_{wr}) + \rho_w S_{wr}} \qquad (55)$$

where the water and steam densities and enthalpies in (55) are considered functions of $\langle p_{sw} \rangle$. As was previously noted, the assumptions concerning the vertical enthalpy distribution are based on results from numerous cross-sectional and three-dimensional simulations.

The assumptions concerning the vertical distribution of pressure and enthalpy expressed explicitly in (50)–(55) permit the determination of all pressure- and enthalpy-dependent parameters in (37) and (38). The procedure used to obtain these parameters is straightforward. Equations (50)–(55) are solved simultaneously, by Newton-Raphson iteration, to obtain the elevation of the interface contact, z_c, and the fluid pressure p_{z_c} at the contact as functions of x-y space. Required for these calculations are the vertically averaged pressure $\langle p \rangle$ and enthalpy $\langle h \rangle$ and the top z_2 and bottom z_1 elevations of the reservoir, all as functions of x-y space. With this information, $\langle p_{sw} \rangle$ is computed using (52), $\langle p_w \rangle$ is computed using (50), p_{z_c} is computed using (51), $\langle h_{sw} \rangle$ is computed using (55), and z_c is computed using (54). If steam is not present at a point (x, y), then averaged pressure and enthalpy are used to calculate the thermodynamic properties of water.

Finally, the averages of relative permeabilities can be obtained by vertical integration of any standard relative per-

meability function. For a segregated system, these averages are given by

$$\langle k_{rw} \rangle = [z_c - z_1]/b \tag{56}$$

and

$$\langle k_{rs} \rangle = (z_2 - z_c)k_{rs}(S_{wr})/b \tag{57}$$

where $k_{rs}(S_{wr})$ is the rock relative permeability of steam at residual water saturation. As may be seen, these result in linear relationships.

Given the assumptions that we have made regarding gravity segregation, it is clear that discontinuities in quantities occur at the steam cap interface z_c. Strict application of Leibnitz's rule to these discontinuities leads to additional interface terms of the form

$$\frac{1}{b}(\psi_{\text{above interface}} - \psi_{\text{below interface}}) \frac{\partial z_c}{\partial x}$$

For practical applications, these are neglected because $(1/b)(\partial z_c/\partial x)$ is usually small. Furthermore, in general, the codeviation terms are also neglected. In paper 2 we outline the conditions for which the vertical equilibrium assumption, the assumption neglecting codeviation terms, and the assumption neglecting interface terms are approximately valid.

CONCLUSIONS

The mathematical development of fluid flow and energy transport in hydrothermal systems presented in this paper serves two general purposes. Mainly, it provides a better understanding of the implicit and explicit assumptions that are necessary to derive tractable governing equations that describe hydrothermal systems. As it emphasizes these assumptions, it also reveals the need for further work to eliminate restrictive approximations.

For two-phase hydrothermal systems the need for experimental data is most evident. Little suitable experimental work has been done on thermal dispersion in steam-water porous systems. Although some very limited data for relative permeability of steam and water are available, they are insufficient for general applications. Additional studies of capillary pressure effects and thermal effects on intrinsic permeability would also be useful.

From practical considerations it is necessary to keep the mathematical models as simple (yet realistic) as possible. It is apparent that general analytical solutions to the multiphase equations for hydrothermal systems are not likely to be obtained due to their complex nonlinear nature. Numerical solutions are also difficult. For three-dimensional problems the expense of numerical solutions is often excessive. A rigorous two-dimensional treatment is a reasonable alternative but also requires more study. Specifically, alternative assumptions for averaging quantities in the vertical dimension should be investigated, and the significance of codeviation terms and interface terms that arise in the partial integration should be determined.

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