

Goddard Rocket Problem

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Homework 5
CSI747

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1 Physics Equations

Let $h(t)$ describe the height of a rocket at time t , $v(t)$ describe its velocity, $a(t)$ its acceleration, $T(t)$ the thrust produced by its engine, $m(t)$ its mass, and $R(t)$ the force of air resistance acting on the rocket.

$$\begin{aligned}v(t) &= \frac{dh(t)}{dt} \\ a(t) &= \frac{dv(t)}{dt}\end{aligned}\tag{1}$$

Equation 1 describes the relationship between $h(t)$, $v(t)$, and $a(t)$ (which comes from basic calculus and the definition of velocity and acceleration).

$$m(t)a(t) = T(t) - R(t) - m(t)g\tag{2}$$

Newton's second law states that the net force on an object is equal to its mass times its acceleration. This allows us to describe the relationship between the gravitational, thrust, and air resistance forces acting on the rocket (where g is the acceleration due to gravity). This relationship is described by equation 2.

$$R(t) = \sigma v(t)^2 e^{-\frac{h(t)}{d}}\tag{3}$$

The force of air resistance $R(t)$ is proportional to the square of the velocity of the rocket with a proportionality constant σ and decays exponentially with height $h(t)$ (d is an air density adjustment parameter). This force is described by equation 3.

$$T(t) = -c \frac{dm(t)}{dt}\tag{4}$$

Finally, the overall thrust of the rocket is inversely proportional to the rate of change in the rocket's mass (as fuel is consumed) with proportionality constant c . This relationship is described by equation 4.

2 Discretization

To model the above equations in AMPL, the problem was divided into n time steps with variable tf representing the unknown final time. Velocity was modeled as the difference in rocket height at adjacent steps divided by the time difference between steps. Acceleration was similarly modeled and indexed such that indices for height and acceleration corresponded to values for the same time step. Velocity values were averaged to arrive at velocity values corresponding to the same time steps as the height and acceleration values.

3 AMPL Model

```
reset;

model;

param g := 32.174; # in ft/s^2
param Tmax := 200; #maximum thrust capable of rocket
param sigma := 5.4915e-5; #air resistance parameter
param c := 1580.9425; #thrust coeff
param d := 23800; #air density adjustment parameter
param m0 := 3; #initial mass (measured in units)
param mempty := 0.01; # mass of the rocket without the fuel

# discretization
param n := 300;

# final moment of time
var tf >= 80, <=220, := 200;

# height of rocket
var h {j in 0..n} >= 0;

# velocity of rocket
var v {j in 1..n} = ( h[j] - h[j-1] ) / ( tf / n );

# average velocity at h[1] through h[n-1]
var v_avg {j in 1..n-1} = ( v[j] + v[j+1] ) / 2;

# acceleration of rocket
var a {j in 1..n-1} = ( v[j+1] - v[j] ) / ( tf / n );

# thrust of rocket
var T {j in 0..n} >= 0, <= Tmax;
```

```

# mass of rocket
var m {j in 0..n} >= mempty, <= m0;

# derivative of mass with respect to time
var d_m {j in 1..n} = ( m[j] - m[j-1] ) / ( tf / n );

# average derivative of mass at m[1] through m[n-1]
var d_m_avg {j in 1..n-1} = ( d_m[j] + d_m[j+1] ) / 2;

# air resistance acting against rocket
var R {j in 0..n};

# maximize the final altitude of the rocket
maximize altitude: h[n];

# constrained by Newton's second law (F=MA)
s.t. newton {j in 1..n-1}: m[j] * a[j] = T[j] - R[j] - m[j] * g;

# where air resistance is defined as follows
s.t. air_resistance {j in 1..n-1}: R[j] = sigma *
                                     ( v_avg[j] )^2 *
                                     exp( -h[j] / d );

# trust of the rocket is proportional to the
# amount of fuel being burned
s.t. thrust {j in 1..n-1}: T[j] = -c * d_m_avg[j];

# initial conditions
s.t. initial_height: h[0] = 0;
s.t. initial_velocity: v[1] = 0;
s.t. initial_mass: m[0] = m0;

s.t. neg_d_m {j in 1..n}: d_m[j] <= 0;
s.t. neg_d_m_avg {j in 1..n-1}: d_m_avg[j] <= 0;

option solver loqo;

option loqo_options "iterlim=20000";

solve;

display tf;
display altitude;
display a;
display v;

```

```
display h;
display m;
display T;
display R;
```

4 AMPL Results

```
*****

NEOS Server Version 5.0
Job#       : 309789
Password   : JPsdjLDC
Solver     : nco:LOQO:AMPL
Start      : 2012-09-27 21:55:12
End        : 2012-09-27 21:56:21
Host       : neos-1.chtc.wisc.edu

Disclaimer:

This information is provided without any express or
implied warranty. In particular, there is no warranty
of any kind concerning the fitness of this
information for any particular purpose.
*****
Job 309789 sent to neos-1.chtc.wisc.edu
password: JPsdjLDC
----- Begin Solver Output -----
Executing /opt/neos/Drivers/loqo-ampl/loqo-driver.py at time: 2012-09-27 21:55:12.793343
File exists
You are using the solver loqo.

%% YOUR COMMENTS %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Job 1
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Executing AMPL.
processing data.
processing commands.

Presolve eliminates 3 constraints and 7 variables.
Substitution eliminates 1496 variables.
Adjusted problem:
1199 variables:
601 nonlinear variables
598 linear variables
1497 constraints, all nonlinear; 7176 nonzeros
898 equality constraints
599 inequality constraints
1 linear objective; 1 nonzero.

LOQO 6.07: iterlim=20000
LOQO 6.07: optimal solution (1019 iterations, 3065 evaluations)
primal objective 121642.9087
dual objective 121642.908

tf = 177.751

altitude = 121643

a [*] :=
1  34.4908    61 -54.9812    121  55.3808    181  46.3049    241 -52.9132
2  37.9799    62 -14.0376    122 -54.8391    182 -54.7421    242  49.5761
3  37.9428    63 -12.6346    123 -52.6532    183 -52.5334    243  54.8359
```

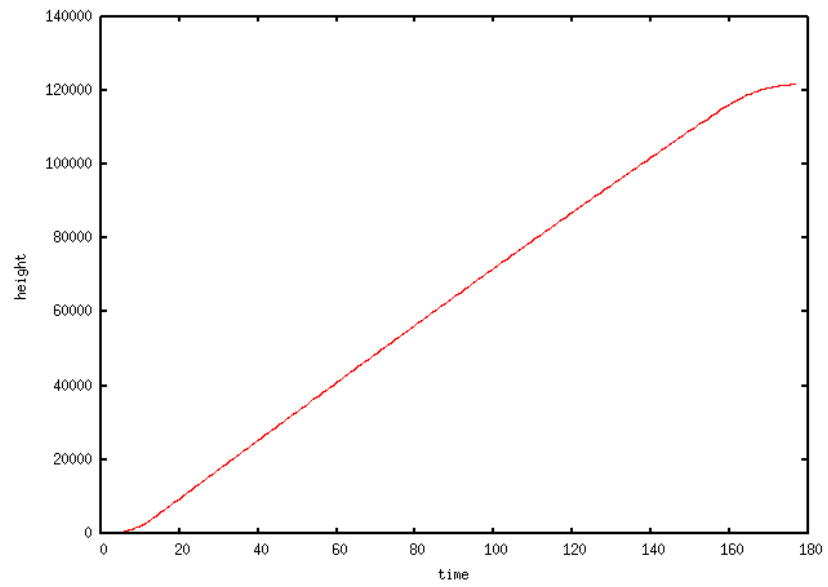


Figure 1: Rocket Height Over Time

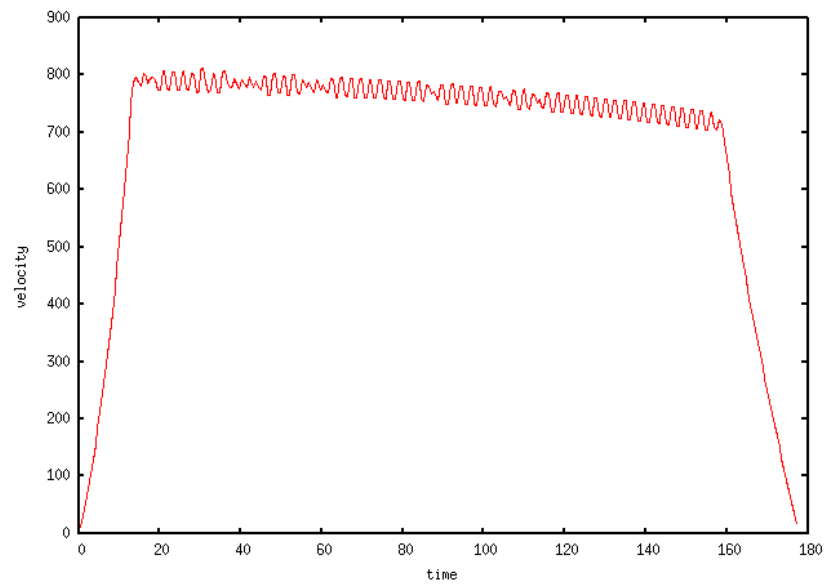


Figure 2: Rocket Velocity Over Time

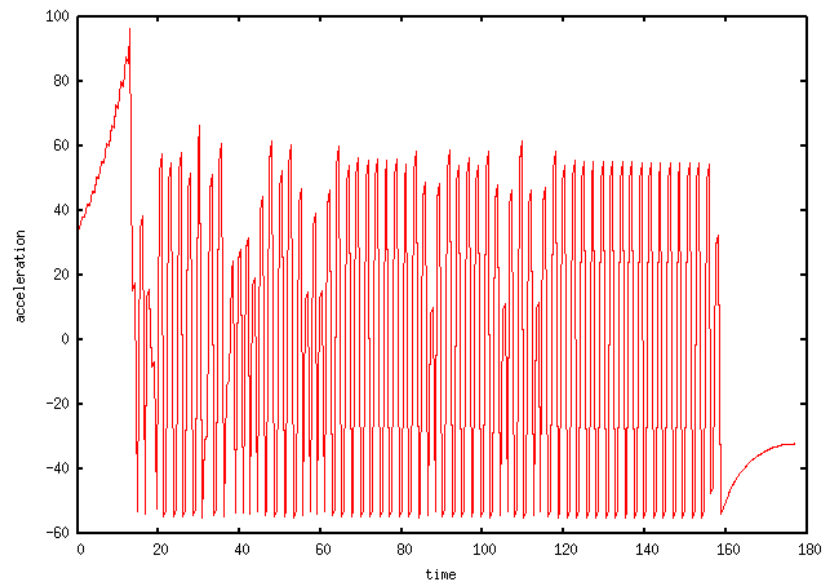


Figure 3: Rocket Acceleration Over Time

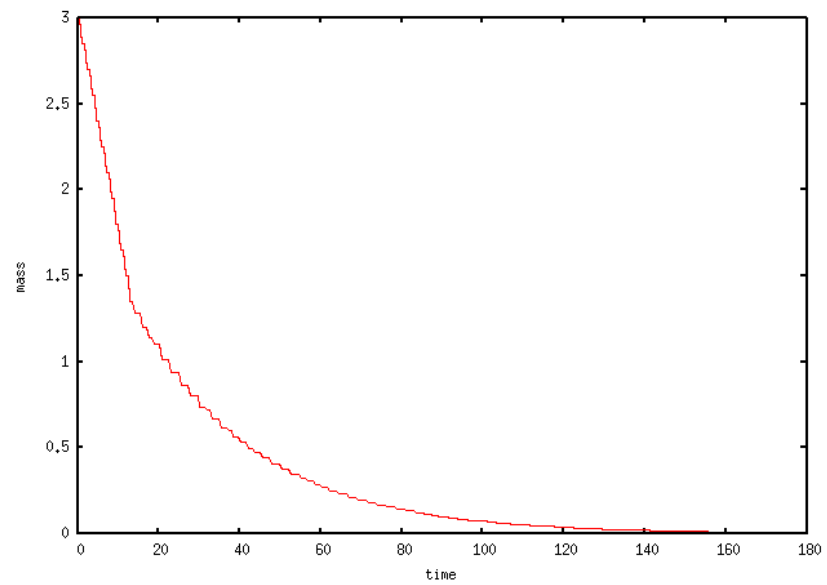


Figure 4: Rocket Mass Over Time

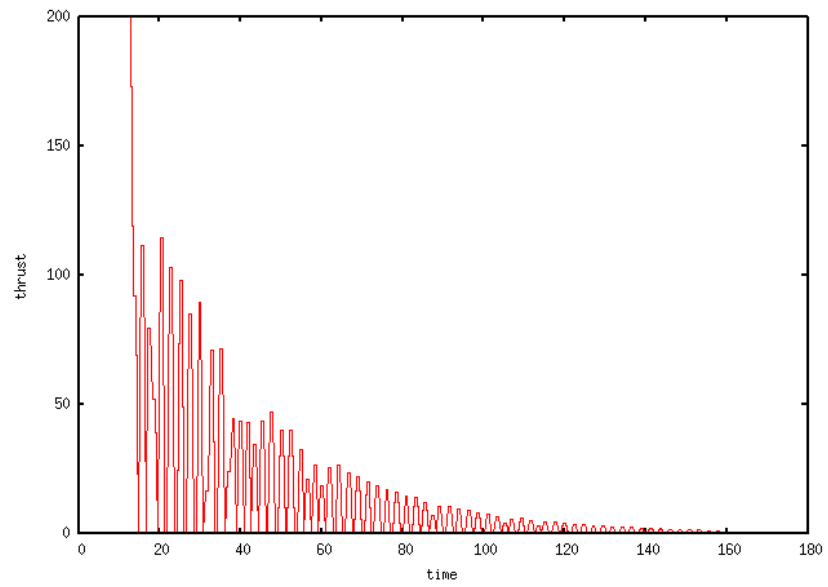


Figure 5: Rocket Thrust Over Time

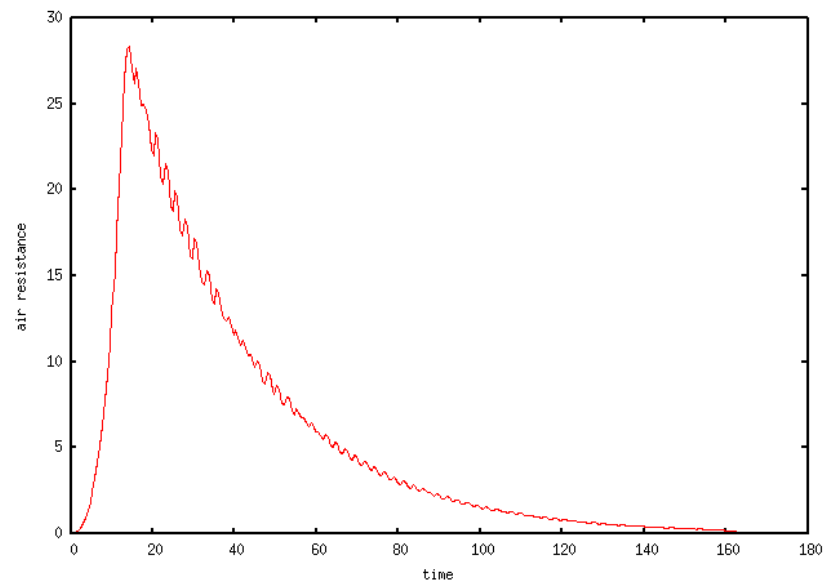


Figure 6: Rocket Air Resistance Over Time

4	41.7725	64	21.5862	124	50.4155	184	55.3291	244	-55.2457
5	41.6826	65	24.4341	125	55.9088	185	61.362	245	-52.9259
6	45.8968	66	-53.599	126	-54.8641	186	-55.2606	246	49.5436
7	45.7339	67	24.8261	127	-52.6739	187	-52.9947	247	54.7923
8	50.3841	68	27.8974	128	50.0035	188	42.0189	248	-55.2656
9	50.1224	69	-53.5979	129	55.4458	189	46.5348	249	-52.9388
10	55.2703	70	28.3982	130	-54.8531	190	-54.5434	250	49.5176
11	54.8769	71	31.7508	131	-52.6623	191	9.4839	251	54.7558
12	60.5974	72	-53.893	132	50.6188	192	11.593	252	-55.2867
13	60.0304	73	16.6597	133	56.1308	193	-53.2488	253	-52.9524
14	66.4166	74	19.178	134	-54.8996	194	42.4716	254	49.4705
15	65.6234	75	-53.2142	135	-52.7021	195	47.0391	255	54.6955
16	72.7936	76	39.8399	136	49.1969	196	-54.8487	256	-55.3069
17	71.707	77	44.2202	137	54.5393	197	-52.6185	257	-52.9651
18	79.8158	78	-54.4819	138	-54.8191	198	52.7318	258	49.4817
19	78.3505	79	-52.3482	139	-52.628	199	58.4261	259	54.6998
20	87.6055	80	55.32	140	52.5651	200	-55.1375	260	-55.3319
21	85.6511	81	61.4354	141	58.3045	201	-52.8737	261	-52.9819
22	96.3405	82	-54.9405	142	-55.0459	202	48.7264	262	49.3232
23	15.0062	83	-52.7585	143	-52.8294	203	53.952	263	54.515
24	17.4412	84	47.0728	144	44.1295	204	-55.0436	264	-47.7026
25	-53.2484	85	52.2062	145	48.9015	205	-52.7863	265	-45.6775
26	34.4574	86	-54.6479	146	-54.5123	206	50.2607	266	29.0435
27	38.334	87	-52.4948	147	7.96878	207	55.6551	267	32.302
28	-54.0422	88	54.3807	148	10.0187	208	-55.1007	268	-54.1401
29	13.1391	89	60.3784	149	-53.0946	209	-52.8339	269	-51.9066
30	15.462	90	-55.0173	150	43.5646	210	49.6318	270	-49.907
31	-8.31443	91	-52.8248	151	48.2832	211	54.9489	271	-48.1112
32	-6.78914	92	42.1946	152	-54.7379	212	-55.0998	272	-46.4941
33	-52.505	93	46.7927	153	-52.5481	213	-52.8295	273	-45.0343
34	51.8809	94	-54.3005	154	52.8187	214	49.8387	274	-43.7136
35	57.5928	95	12.5371	155	58.5727	215	55.1737	275	-42.5165
36	-54.7749	96	14.8266	156	-55.0029	216	-55.122	276	-41.4298
37	-52.6215	97	-53.2657	157	-52.7834	217	-52.8456	277	-40.4418
38	49.255	98	35.1089	158	48.8326	218	49.7228	278	-39.5426
39	54.6549	99	39.0223	159	54.117	219	55.0387	279	-38.7237
40	-54.6701	100	-54.1414	160	-54.8955	220	-55.1359	280	-37.9774
41	-52.5266	101	12.7995	161	-52.6848	221	-52.8542	281	-37.2972
42	52.3692	102	15.0944	162	50.9183	222	49.7297	282	-36.6774
43	58.1392	103	-53.1391	163	56.4399	223	55.0403	283	-36.1128
44	-54.8438	104	41.767	164	-54.9852	224	-55.1537	284	-35.599
45	-52.6816	105	46.3286	165	-52.7628	225	-52.8661	285	-35.1322
46	46.3965	106	-54.5883	166	48.848	226	49.689	286	-34.7088
47	51.4661	107	-52.4345	167	54.1253	227	54.9888	287	-34.3259
48	-54.4899	108	53.8659	168	-54.8859	228	-55.1707	288	-33.9807
49	-52.363	109	59.7831	169	-52.6714	229	-52.8772	289	-33.671
50	59.4803	110	-54.9264	170	52.6674	230	49.6655	290	-33.3946
51	66.1624	111	-52.7363	171	58.3915	231	54.9562	291	-33.1499
52	-55.3019	112	48.7278	172	-55.14	232	-55.1888	292	-32.9351
53	-31.17	113	54.0344	173	-52.8964	233	-52.889	293	-32.749
54	-29.7758	114	-54.7904	174	43.3234	234	49.6349	294	-32.5904
55	46.2286	115	-52.6128	175	47.986	235	54.9153	295	-32.4584
56	51.2806	116	50.9256	176	-54.5419	236	-55.2071	296	-32.3521
57	-54.5854	117	56.484	177	9.17056	237	-52.9009	297	-32.271
58	-52.4467	118	-54.8597	178	11.27	238	49.6058	298	-32.2145
59	54.7809	119	-52.6733	179	-53.221	239	54.8761	299	-32.1824
60	60.8428	120	49.9401	180	41.7938	240	-55.2261		

;

v [*] :=

1	0	61	822.575	121	775.101	181	759.588	241	731.985
2	20.4359	62	789.998	122	807.915	182	787.024	242	700.634
3	42.9391	63	781.681	123	775.422	183	754.589	243	730.008
4	65.4203	64	774.195	124	744.225	184	723.463	244	762.498
5	90.1707	65	786.985	125	774.097	185	756.245	245	729.765
6	114.868	66	801.462	126	807.223	186	792.603	246	698.406
7	142.062	67	769.704	127	774.716	187	759.86	247	727.761
8	169.159	68	784.414	128	743.506	188	728.461	248	760.226

9	199.012	69	800.943	129	773.133	189	753.357	249	727.481
10	228.71	70	769.186	130	805.985	190	780.929	250	696.114
11	261.457	71	786.012	131	773.484	191	748.612	251	725.453
12	293.972	72	804.825	132	742.282	192	754.231	252	757.896
13	329.876	73	772.893	133	772.274	193	761.1	253	725.139
14	365.444	74	782.764	134	805.531	194	729.55	254	693.764
15	404.797	75	794.127	135	773.003	195	754.715	255	723.076
16	443.679	76	762.597	136	741.777	196	782.585	256	755.483
17	486.809	77	786.203	137	770.926	197	750.087	257	722.714
18	529.296	78	812.403	138	803.241	198	718.911	258	691.332
19	576.587	79	780.123	139	770.761	199	750.155	259	720.65
20	623.01	80	749.106	140	739.578	200	784.772	260	753.06
21	674.916	81	781.883	141	770.723	201	752.103	261	720.275
22	725.665	82	818.284	142	805.269	202	720.775	262	688.883
23	782.747	83	785.732	143	772.654	203	749.646	263	718.107
24	791.638	84	754.472	144	741.352	204	781.613	264	750.408
25	801.972	85	782.363	145	767.499	205	748.999	265	722.144
26	770.422	86	813.295	146	796.474	206	717.723	266	695.08
27	790.838	87	780.916	147	764.175	207	747.503	267	712.288
28	813.551	88	749.813	148	768.896	208	780.478	268	731.427
29	781.531	89	782.033	149	774.833	209	747.831	269	699.349
30	789.316	90	817.808	150	743.374	210	716.527	270	668.594
31	798.477	91	785.21	151	769.186	211	745.934	271	639.024
32	793.551	92	753.911	152	797.794	212	778.491	272	610.518
33	789.528	93	778.911	153	765.362	213	745.844	273	582.97
34	758.419	94	806.636	154	734.227	214	714.543	274	556.287
35	789.159	95	774.463	155	765.522	215	744.072	275	530.387
36	823.283	96	781.891	156	800.226	216	776.763	276	505.196
37	790.828	97	790.676	157	767.637	217	744.103	277	480.648
38	759.65	98	759.116	158	736.363	218	712.792	278	456.686
39	788.834	99	779.918	159	765.296	219	742.253	279	433.257
40	821.217	100	803.039	160	797.361	220	774.863	280	410.313
41	788.825	101	770.96	161	764.835	221	742.195	281	387.812
42	757.702	102	778.544	162	733.619	222	710.879	282	365.713
43	788.731	103	787.487	163	763.788	223	740.344	283	343.982
44	823.179	104	756.002	164	797.229	224	772.955	284	322.585
45	790.684	105	780.749	165	764.65	225	740.277	285	301.492
46	759.47	106	808.199	166	733.388	226	708.953	286	280.676
47	786.96	107	775.855	167	762.331	227	738.394	287	260.111
48	817.454	108	744.788	168	794.4	228	770.975	288	239.773
49	785.168	109	776.703	169	761.88	229	738.287	289	219.639
50	754.143	110	812.125	170	730.672	230	706.957	290	199.689
51	789.385	111	779.581	171	761.878	231	736.384	291	179.903
52	828.587	112	748.335	172	796.475	232	768.945	292	160.261
53	795.82	113	777.206	173	763.804	233	736.246	293	140.747
54	777.352	114	809.222	174	732.463	234	704.909	294	121.343
55	759.71	115	776.758	175	758.132	235	734.318	295	102.033
56	787.1	116	745.585	176	786.564	236	766.855	296	82.8016
57	817.484	117	775.758	177	754.248	237	734.145	297	63.6329
58	785.142	118	809.225	178	759.681	238	702.801	298	44.5122
59	754.067	119	776.721	179	766.359	239	732.192	299	25.425
60	786.525	120	745.512	180	734.825	240	764.707	300	6.35686

;

h [*] :=

0	0	101	40835.5	202	86735.3
1	1.75884e-14	102	41296.8	203	87179.5
2	12.1083	103	41763.3	204	87642.6
3	37.5499	104	42211.3	205	88086.4
4	76.3116	105	42673.9	206	88511.6
5	129.738	106	43152.7	207	88954.5
6	197.797	107	43612.4	208	89417
7	281.969	108	44053.7	209	89860
8	382.197	109	44513.9	210	90284.6
9	500.112	110	44995.1	211	90726.6
10	635.623	111	45457	212	91187.8
11	790.538	112	45900.4	213	91629.7
12	964.717	113	46360.9	214	92053.1

13	1160.17	114	46840.4	215	92494
14	1376.7	115	47300.6	216	92954.2
15	1616.54	116	47742.4	217	93395.1
16	1879.42	117	48202	218	93817.4
17	2167.86	118	48681.5	219	94257.2
18	2481.47	119	49141.7	220	94716.3
19	2823.1	120	49583.4	221	95156.1
20	3192.23	121	50042.6	222	95577.3
21	3592.12	122	50521.3	223	96015.9
22	4022.08	123	50980.8	224	96473.9
23	4485.86	124	51421.7	225	96912.5
24	4954.91	125	51880.4	226	97332.6
25	5430.08	126	52358.7	227	97770.1
26	5886.55	127	52817.7	228	98226.9
27	6355.13	128	53258.2	229	98664.3
28	6837.16	129	53716.3	230	99083.2
29	7300.22	130	54193.8	231	99519.5
30	7767.89	131	54652.1	232	99975.1
31	8240.99	132	55091.9	233	100411
32	8711.17	133	55549.5	234	100829
33	9178.97	134	56026.8	235	101264
34	9628.34	135	56484.8	236	101718
35	10095.9	136	56924.3	237	102153
36	10583.7	137	57381.1	238	102570
37	11052.3	138	57857	239	103004
38	11502.4	139	58313.7	240	103457
39	11969.8	140	58751.9	241	103890
40	12456.3	141	59208.5	242	104306
41	12923.7	142	59685.7	243	104738
42	13372.7	143	60143.5	244	105190
43	13840	144	60582.7	245	105622
44	14327.7	145	61037.5	246	106036
45	14796.2	146	61509.4	247	106467
46	15246.2	147	61962.2	248	106918
47	15712.5	148	62417.7	249	107349
48	16196.8	149	62876.8	250	107761
49	16662	150	63317.3	251	108191
50	17108.9	151	63773	252	108640
51	17576.6	152	64245.7	253	109070
52	18067.5	153	64699.2	254	109481
53	18539	154	65134.2	255	109909
54	18999.6	155	65587.8	256	110357
55	19449.8	156	66061.9	257	110785
56	19916.1	157	66516.8	258	111195
57	20400.5	158	66953.1	259	111622
58	20865.7	159	67406.5	260	112068
59	21312.5	160	67878.9	261	112495
60	21778.5	161	68332.1	262	112903
61	22265.9	162	68766.8	263	113328
62	22733.9	163	69219.3	264	113773
63	23197.1	164	69691.7	265	114201
64	23655.8	165	70144.7	266	114613
65	24122.1	166	70579.3	267	115035
66	24597	167	71031	268	115468
67	25053	168	71501.6	269	115882
68	25517.8	169	71953.1	270	116279
69	25992.3	170	72386	271	116657
70	26448.1	171	72837.4	272	117019
71	26913.8	172	73309.3	273	117364
72	27390.7	173	73761.9	274	117694
73	27848.6	174	74195.9	275	118008
74	28312.4	175	74645.1	276	118307
75	28782.9	176	75111.1	277	118592
76	29234.7	177	75558	278	118863
77	29700.6	178	76008.1	279	119120
78	30181.9	179	76462.2	280	119363
79	30644.2	180	76897.6	281	119592
80	31088	181	77347.6	282	119809

81	31551.3	182	77813.9	283	120013
82	32036.1	183	78261	284	120204
83	32501.7	184	78689.7	285	120383
84	32948.7	185	79137.8	286	120549
85	33412.2	186	79607.4	287	120703
86	33894.1	187	80057.6	288	120845
87	34356.8	188	80489.2	289	120975
88	34801.1	189	80935.6	290	121094
89	35264.4	190	81398.3	291	121200
90	35749	191	81841.8	292	121295
91	36214.2	192	82288.7	293	121379
92	36660.9	193	82739.7	294	121450
93	37122.4	194	83171.9	295	121511
94	37600.4	195	83619.1	296	121560
95	38059.2	196	84082.8	297	121598
96	38522.5	197	84527.2	298	121624
97	38991	198	84953.2	299	121639
98	39440.8	199	85397.6	300	121643
99	39902.9	200	85862.6		
100	40378.7	201	86308.3		

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m [*] :=

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1	3	62	0.614182	123	0.18177	184	0.0549748	245	0.0159504
2	2.85009	63	0.596053	124	0.18177	185	0.0505433	246	0.0159504
3	2.85009	64	0.596053	125	0.167784	186	0.0505433	247	0.0147303
4	2.70018	65	0.562752	126	0.167784	187	0.0505433	248	0.0147303
5	2.70018	66	0.562752	127	0.167784	188	0.0505433	249	0.0147303
6	2.55027	67	0.562752	128	0.167784	189	0.0469646	250	0.0147303
7	2.55027	68	0.530034	129	0.154923	190	0.0469646	251	0.0136036
8	2.40035	69	0.530034	130	0.154923	191	0.0469646	252	0.0136036
9	2.40035	70	0.530034	131	0.154923	192	0.044752	253	0.0136036
10	2.25044	71	0.497777	132	0.154923	193	0.044752	254	0.0136036
11	2.25044	72	0.497777	133	0.142977	194	0.044752	255	0.0125635
12	2.10053	73	0.497777	134	0.142977	195	0.0415598	256	0.0125635
13	2.10053	74	0.471828	135	0.142977	196	0.0415598	257	0.0125635
14	1.95062	75	0.471828	136	0.142977	197	0.0415598	258	0.0125635
15	1.95062	76	0.471828	137	0.132103	198	0.0415598	259	0.0116027
16	1.80071	77	0.439133	138	0.132103	199	0.0382892	260	0.0116027
17	1.80071	78	0.439133	139	0.132103	200	0.0382892	261	0.0116027
18	1.6508	79	0.439133	140	0.132103	201	0.0382892	262	0.0116027
19	1.6508	80	0.439133	141	0.121724	202	0.0382892	263	0.0107165
20	1.50089	81	0.403801	142	0.121724	203	0.0353858	264	0.0107165
21	1.50089	82	0.403801	143	0.121724	204	0.0353858	265	0.010654
22	1.35098	83	0.403801	144	0.121724	205	0.0353858	266	0.010654
23	1.35098	84	0.403801	145	0.112924	206	0.0353858	267	0.01
24	1.28207	85	0.373727	146	0.112924	207	0.0326634	268	0.01
25	1.28207	86	0.373727	147	0.112924	208	0.0326634	269	0.01
26	1.28207	87	0.373727	148	0.107736	209	0.0326634	270	0.01
27	1.19845	88	0.373727	149	0.107736	210	0.0326634	271	0.01
28	1.19845	89	0.343886	150	0.107736	211	0.030165	272	0.01
29	1.19845	90	0.343886	151	0.099974	212	0.030165	273	0.01
30	1.13906	91	0.343886	152	0.099974	213	0.030165	274	0.01
31	1.13906	92	0.343886	153	0.099974	214	0.030165	275	0.01
32	1.10025	93	0.319535	154	0.099974	215	0.0278529	276	0.01
33	1.10025	94	0.319535	155	0.0921056	216	0.0278529	277	0.01
34	1.10025	95	0.319535	156	0.0921056	217	0.0278529	278	0.01
35	1.01448	96	0.303789	157	0.0921056	218	0.0278529	279	0.01
36	1.01448	97	0.303789	158	0.0921056	219	0.0257203	280	0.01
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38	1.01448	99	0.283821	160	0.0851203	221	0.0257203	282	0.01
39	0.937342	100	0.283821	161	0.0851203	222	0.0257203	283	0.01
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41	0.937342	102	0.26981	163	0.0785357	224	0.0237506	285	0.01
42	0.937342	103	0.26981	164	0.0785357	225	0.0237506	286	0.01
43	0.863912	104	0.26981	165	0.0785357	226	0.0237506	287	0.01
44	0.863912	105	0.250732	166	0.0785357	227	0.0219322	288	0.01

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45 0.863912    106 0.250732    167 0.0725798    228 0.0219322    289 0.01
46 0.863912    107 0.250732    168 0.0725798    229 0.0219322    290 0.01
47 0.800066    108 0.250732    169 0.0725798    230 0.0219322    291 0.01
48 0.800066    109 0.23082     170 0.0725798    231 0.0202533    292 0.01
49 0.800066    110 0.23082     171 0.0668692    232 0.0202533    293 0.01
50 0.800066    111 0.23082     172 0.0668692    233 0.0202533    294 0.01
51 0.733154    112 0.23082     173 0.0668692    234 0.0202533    295 0.01
52 0.733154    113 0.213342    174 0.0668692    235 0.0187032    296 0.01
53 0.733154    114 0.213342    175 0.0620729    236 0.0187032    297 0.01
54 0.720915    115 0.213342    176 0.0620729    237 0.0187032    298 0.01
55 0.720915    116 0.213342    177 0.0620729    238 0.0187032    299 0.01
56 0.667675    117 0.196849    178 0.0591632    239 0.0172719    300 0.01
57 0.667675    118 0.196849    179 0.0591632    240 0.0172719
58 0.667675    119 0.196849    180 0.0591632    241 0.0172719
59 0.667675    120 0.196849    181 0.0549748    242 0.0172719
60 0.614182    121 0.18177     182 0.0549748    243 0.0159504
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2 200         78 2.59397e-07    154 10.4975       230 2.23989
3 200         79 2.52582e-07    155 10.4975       231 2.23989
4 200         80 47.1369       156 6.04003e-08    232 3.79659e-08
5 200         81 47.1369       157 6.25757e-08    233 3.82455e-08
6 200         82 1.3069e-07     158 9.31912       234 2.06804
7 200         83 1.34978e-07    159 9.31912       235 2.06804
8 200         84 40.1217       160 7.19598e-08    236 3.71105e-08
9 200         85 40.1217       161 7.0637e-08     237 3.73631e-08
10 200        86 1.36636e-07    162 8.78474       238 1.90944
11 200        87 1.33953e-07    163 8.78474       239 1.90944
12 200        88 39.8127       164 6.80312e-08    240 3.62578e-08
13 200        89 39.8127       165 7.02742e-08    241 3.65e-08
14 200        90 1.96447e-07    166 7.94591       242 1.76302
15 200        91 2.02812e-07    167 7.94591       243 1.76302
16 200        92 32.4868       168 5.47209e-08    244 3.55242e-08
17 200        93 32.4868       169 5.25897e-08    245 3.57933e-08
18 200        94 5.67063e-08    170 7.61864       246 1.62782
19 200        95 21.0065       171 7.61864       247 1.62782
20 200        96 21.0065       172 1.01799e-07    248 3.48069e-08
21 200        97 7.09747e-08    173 1.0452e-07     249 3.50218e-08
22 200        98 26.6403       174 6.39881       250 1.5031
23 91.9229    99 26.6403       175 6.39881       251 1.5031
24 91.9229    100 6.8124e-08    176 2.28952e-08    252 3.43705e-08
25 1.59622e-07 101 18.6918      177 3.88186       253 3.44363e-08
26 111.558    102 18.6918      178 3.88186       254 1.38768
27 111.558    103 5.1298e-08    179 2.42684e-08    255 1.38768
28 1.60748e-07 104 25.4528      180 5.58781       256 3.32555e-08
29 79.2336    105 25.4528      181 5.58781       257 3.49372e-08
30 79.2336    106 1.9061e-07    182 7.42463e-08    258 1.28188
31 51.79      107 1.86792e-07    183 7.02909e-08    259 1.28188
32 51.79      108 26.5651     184 5.91221       260 3.32131e-08
33 2.20653e-07 109 26.5651     185 5.91221       261 2.46993e-08
34 114.421    110 9.37424e-08    186 6.66345e-08    262 1.18218
35 114.421    111 9.61772e-08    187 7.12126e-08    263 1.18218
36 2.52001e-07 112 23.3178      188 4.77438       264 0.0834388
37 2.54864e-07 113 23.3178      189 4.77438       265 0.0834389
38 102.911    114 1.17822e-07    190 2.07533e-08    266 0.872522
39 102.911    115 1.17242e-07    191 2.95194       267 0.872522
40 2.22368e-07 116 22.0032      192 2.95194       268 4.93958e-09
41 2.20905e-07 117 22.0032      193 1.8363e-08     269 5.10682e-09
42 97.9646    118 9.94938e-08    194 4.25879       270 1.03763e-09
43 97.9646    119 1.00543e-07    195 4.25879       271 4.81504e-10
44 2.83143e-07 120 20.1167      196 1.04177e-07    272 2.40862e-10
45 2.89961e-07 121 20.1167      197 1.02963e-07    273 1.66305e-10
46 85.1795    122 1.0065e-07     198 4.36334       274 1.16752e-10
47 85.1795    123 1.00758e-07    199 4.36334       275 8.82946e-11
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50 89.2685 126 9.54483e-08 202 3.8735 278 4.57333e-11
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52 2.52045e-06 128 17.1575 204 5.48914e-08 280 3.37204e-11
53 16.3277 129 17.1575 205 5.24598e-08 281 3.03373e-11
54 16.3277 130 8.96435e-08 206 3.63203 282 2.62638e-11
55 71.0291 131 8.96162e-08 207 3.63203 283 2.37315e-11
56 71.0291 132 15.9375 208 4.1391e-08 284 2.14592e-11
57 2.20418e-07 133 15.9375 209 4.31297e-08 285 1.97668e-11
58 2.0878e-07 134 9.30949e-08 210 3.33323 286 1.8115e-11
59 71.3653 135 9.43491e-08 211 3.33323 287 1.69657e-11
60 71.3653 136 14.5076 212 4.53868e-08 288 1.5958e-11
61 2.0525e-07 137 14.5076 213 4.48604e-08 289 1.51184e-11
62 24.1863 138 7.44361e-08 214 3.08451 290 1.43405e-11
63 24.1863 139 7.34679e-08 215 3.08451 291 1.36544e-11
64 44.4283 140 13.8469 216 4.15423e-08 292 1.33991e-11
65 44.4283 141 13.8469 217 4.22204e-08 293 1.28409e-11
66 9.59333e-08 142 1.17962e-07 218 2.84527 294 1.30418e-11
67 43.6497 143 1.20413e-07 219 2.84527 295 1.25552e-11
68 43.6497 144 11.7394 220 4.15175e-08 296 1.36427e-11
69 7.64311e-08 145 11.7394 221 4.1588e-08 297 1.30553e-11
70 43.0345 146 3.45708e-08 222 2.62782 298 1.68047e-11
71 43.0345 147 6.92141 223 2.62782 299 1.59833e-11
72 1.01167e-07 148 6.92141 224 3.98584e-08 300 0
73 34.619 149 3.42536e-08 225 4.02227e-08
74 34.619 150 10.356 226 2.42589
75 6.64194e-08 151 10.356 227 2.42589
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R [*] :=
0 0 76 9.64175 152 2.2558 228 0.504369
1 0.00573347 77 10.0727 153 2.03688 229 0.454066
2 0.0551119 78 9.79612 154 2.00042 230 0.444971
3 0.160946 79 8.85914 155 2.1392 231 0.47522
4 0.331289 80 8.71549 156 2.10267 232 0.466124
5 0.574029 81 9.33737 157 1.89824 233 0.419546
6 0.898772 82 9.19314 158 1.85796 234 0.411145
7 1.31408 83 8.31205 159 1.974 235 0.439197
8 1.83129 84 8.12173 160 1.93406 236 0.430793
9 2.45939 85 8.58648 161 1.74589 237 0.387659
10 3.21159 86 8.39913 162 1.7119 238 0.379901
11 4.09698 87 7.59444 163 1.82539 239 0.405919
12 5.1308 88 7.46488 164 1.79149 240 0.398154
13 6.32166 89 7.9853 165 1.61696 241 0.358207
14 7.6871 90 7.85547 166 1.58279 242 0.351042
15 9.23445 91 7.10152 167 1.68232 243 0.375178
16 10.9839 92 6.91255 168 1.64842 244 0.368003
17 12.9405 93 7.25424 169 1.4877 245 0.331002
18 15.1275 94 7.07018 170 1.46088 246 0.324385
19 17.5463 95 6.71979 171 1.56261 247 0.346776
20 20.2246 96 6.72827 172 1.53572 248 0.340147
21 23.1579 97 6.40742 173 1.38569 249 0.305872
22 26.3801 98 6.20049 174 1.35036 250 0.299761
23 28.1836 99 6.43332 175 1.42305 251 0.320541
24 28.3126 100 6.23481 176 1.38844 252 0.314417
25 27.0189 101 5.9274 177 1.31548 253 0.282662
26 26.1316 102 5.9383 178 1.31157 254 0.277015
27 27.0573 103 5.65659 179 1.24521 255 0.296293
28 26.2081 104 5.50276 180 1.21163 256 0.29063
29 24.928 105 5.76968 181 1.27344 257 0.261209
30 24.9731 106 5.61998 182 1.24068 258 0.256003
31 24.6124 107 5.07996 183 1.11926 259 0.273917
32 23.8604 108 4.9922 184 1.10174 260 0.268693
33 22.3691 109 5.33961 185 1.18459 261 0.241427
34 21.9402 110 5.25171 186 1.16688 262 0.236592
35 23.3546 111 4.74619 187 1.05235 263 0.253169
36 22.9282 112 4.64412 188 1.02443 264 0.249852
37 20.7436 113 4.92599 189 1.07785 265 0.227306

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38	20.3026	114	4.82502	190	1.05057	266	0.22031
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41	19.0774	117	4.551	193	0.943141	269	0.197326
42	18.7186	118	4.46565	194	0.918248	270	0.17733
43	19.9418	119	4.03527	195	0.966707	271	0.159372
44	19.5847	120	3.95264	196	0.942356	272	0.143201
45	17.7168	121	4.20185	197	0.849667	273	0.128603
46	17.3015	122	4.11985	198	0.834672	274	0.115396
47	18.262	123	3.72252	199	0.894331	275	0.103425
48	17.8542	124	3.64781	200	0.879257	276	0.0925576
49	16.1525	125	3.88127	201	0.792575	277	0.0826777
50	15.9391	126	3.80703	202	0.775885	278	0.0736862
51	17.1728	127	3.43955	203	0.825859	279	0.0654968
52	16.9563	128	3.36944	204	0.80926	280	0.058034
53	15.5916	129	3.58313	205	0.729383	281	0.0512322
54	14.5987	130	3.51352	206	0.715005	282	0.0450336
55	14.5075	131	3.17411	207	0.763228	283	0.0393878
56	15.3085	132	3.11101	208	0.748864	284	0.0342502
57	14.9635	133	3.31199	209	0.674822	285	0.0295818
58	13.5356	134	3.24923	210	0.661175	286	0.0253481
59	13.3077	135	2.93504	211	0.705172	287	0.0215188
60	14.2361	136	2.87344	212	0.691556	288	0.0180673
61	14.0078	137	3.05254	213	0.623072	289	0.01497
62	13.0472	138	2.99148	214	0.610602	290	0.0122064
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64	12.3844	140	2.65261	216	0.639169	292	0.00761086
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66	12.057	142	2.78406	218	0.564202	294	0.00416407
67	11.5728	143	2.51425	219	0.602136	295	0.00284386
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70	10.9292	146	2.52254	222	0.521239	298	0.000405221
71	11.2142	147	2.38831	223	0.556435	299	8.36294e-05
72	10.8112	148	2.37572	224	0.545781	300	0
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