

# Tesi

Nicholas Pini

2022

**Abstract**

abstract

## Contents

<b>1</b>	<b>Introduction . . . . .</b>	<b>1</b>
<b>2</b>	<b>Theoretical background . . . . .</b>	<b>2</b>
2.1	Lattice gauge theory . . . . .	2
2.2	Static quarks and Wilson loops . . . . .	3
2.3	Finite temperature LGT and Polyakov loops . . . . .	4
2.4	Polyakov loop as order parameter . . . . .	6
2.5	The Svetitsky–Yaffe conjecture . . . . .	7
2.6	Effective string theory . . . . .	9
2.7	Sp(N) group . . . . .	9
<b>3</b>	<b>Simulation and algorithm . . . . .</b>	<b>9</b>
3.1	Heat-bath algorithm . . . . .	9
3.2	Overrelaxation . . . . .	15
<b>4</b>	<b>Results . . . . .</b>	<b>17</b>

## 1 Introduction

**Brief intro about the paper. Past work, theory, references.**

## 2 Theoretical background

Explain the concepts of lattice gauge theory, Wilson and Polyakov loops, center symmetry and  $\text{Sp}(2)$  group. Mostly from Pepe's work. Read the paper by Caselle to see if there is something to pure here (SY conjecture? String tension? Universality with spin model? EST?)

### 2.1 Lattice gauge theory

In this work, we will be focusing on a (2+1) dimensional lattice gauge theory (two spatial dimensions and one time dimension). We will present a brief summary of lattice gauge theories (LGT).

An LGT is a gauge theory built on a discretized spacetime: a lattice of spacing  $a$  with  $N_s$  sites in the space directions and  $N_t$  sites in the time direction. This discretization is useful, because it introduces a natural cutoff for the momentum  $p$ , thus regularizing the theory:

$$p \in \left(-\frac{\pi}{a}, \frac{\pi}{a}\right). \quad (2.1)$$

Starting from the continuum limit action

$$\tilde{S} = \frac{1}{2g_0^2} \int d^4x \text{Tr}(F_{\mu\nu}F^{\mu\nu}), \quad (2.2)$$

where  $F_{\mu\nu} = F_{\mu\nu}^a T^a = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$ , we define the *link* variable as

$$U_\mu(x) = e^{-iaA_\mu(x)}. \quad (2.3)$$

Given a gauge transformation  $G(x)$ , a link variable transforms as

$$U_\mu(x) \longrightarrow G(x)U_\mu(x)G^\dagger(x + \hat{\mu}). \quad (2.4)$$

The link variable can be thought of as connecting the site at  $x$  to the site at  $x + \hat{\mu}$ , and it belongs to the gauge group defining the theory. We define a *plaquette*  $U_{\mu\nu}(x)$  as the smallest loop of product of link variables:

$$U_{\mu\nu}(x) = U_{\mu}(x)U_{\nu}(x + \hat{\mu})U_{\mu}^{\dagger}(x + \hat{\nu})U_{\nu}^{\dagger}(x). \quad (2.5)$$

Note that links are oriented, and the adjoint operation flips a link, inverting its direction.

Finally, we can define the discretized action of an LGT:

$$S = -\frac{\beta}{N} \sum_x \sum_{\mu < \nu} \text{Tr} U_{\mu\nu}(x) = -\frac{2}{g_0^2} \sum_x \sum_{\mu < \nu} \text{Tr} U_{\mu\nu}(x), \quad (2.6)$$

where  $g_0$  is the bare gauge coupling,  $N$  is the trace normalization constant related to the gauge group, and  $\beta \equiv \frac{2N}{g_0^2}$ . This action is invariant under gauge transformations: in fact, despite links not being gauge invariant, a plaquette is indeed gauge invariant.

The path integral is

$$\mathcal{Z} = \int \mathcal{D}U e^{-S} \quad (2.7)$$

for

$$\int \mathcal{D}U = \prod_{x,\mu} \int_{\text{gauge group}} dU_{x,\mu}. \quad (2.8)$$

## 2.2 Static quarks and Wilson loops

We want to study the confinement-deconfinement phase transition that is typical of QCD. To do so, we imagine instantly creating a static quark-antiquark pair at distance  $R$  from one another. "Static" means that these quarks have infinite mass and are thus non-dynamical. We let them evolve with time for a duration of  $T$ , and finally we instantly destroy them. In the context of a lattice gauge theory, the line connecting the two quarks has swept an area of  $R \times T$ . This picture is described by *Wilson loops*.

A Wilson loop is the trace of an ordered product of link variables on an LGT along some path  $\mathcal{C}$ :

$$W(\mathcal{C}) = \text{Tr} \prod_{t, \vec{x}} U_\mu(t, \vec{x}). \quad (2.9)$$

If the path  $\mathcal{C}$  is a  $R \times T$  rectangle, we are basically considering the rectangle swept by the quark pair: the Wilson loop is then interpreted as the free energy. In fact, it is possible to relate the potential  $V(R)$  between these two quarks to the expectation value of the Wilson loop for the  $R \times T$  rectangle:

$$V(R) = - \lim_{L \rightarrow \infty} \frac{1}{L} \log \langle W(\mathcal{C}) \rangle. \quad (2.10)$$

In a confining theory, the potential  $V(R)$  is linearly rising, and the Wilson loop exhibits a so-called "area law":

$$\langle W(\mathcal{C}) \rangle \sim e^{-\sigma_0 R T}. \quad (2.11)$$

$\sigma_0$  is called *zero temperature string tension*. "String tensions" comes from the fact that the interquark potential can be effectively described as a vibrating string [3].

A complete introduction to Wilson loops and their relation to the interquark potential can be found in [5].

### 2.3 Finite temperature LGT and Polyakov loops

The description of the confining potential through Wilson loops does not take into account the temperature of the system. If we want to study how the confinement-deconfinement transition depends on temperature, we need to define the theory at finite temperature in the first place.

It is well known how to do so for an LGT defined on a  $(d+1)$  dimensional lattice of volume  $(N_s a)^d (N_t a)$ : we consider the lattice regularization and impose *periodic boundary condition* for bosonic fields (which are the fields we will be considering in this work). In particular, the compactified time

direction is interpreted as the temperature  $T$  of the system, through the relation

$$T = \frac{1}{N_t a}. \quad (2.12)$$

Thus, the length of the lattice in the time direction is inversely proportional to the temperature of the system.

The Wilson loop is not suitable anymore in describing the interquark potential in a finite temperature context. Fortunately, there is another quantity we can consider: the correlation function of two *Polyakov loops*.

A Polyakov loop is the trace of the ordered product of all time-like links ( $U_\mu(t, \vec{x})$  with  $\mu = t$ ) with the same space coordinates:

$$\phi(\vec{x}) = \text{Tr} \prod_{\tau=1}^{N_t} U_t(\tau, \vec{x}), \quad (2.13)$$

where  $\vec{x}$  represents the space coordinate of the loop (a  $d$  dimensional vector, in the case of a  $d + 1$  dimensional lattice).

As usual, the correlation function is defined as

$$\langle \phi(\vec{x}) \phi(\vec{y}) \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}U \phi(\vec{x}) \phi(\vec{y}) e^{-S[U]}. \quad (2.14)$$

Now, the finite temperature interquark potential  $V(R, T)$  can be extracted from the relation

$$\langle \phi(\vec{x}) \phi(\vec{y}) \rangle = e^{-\frac{1}{T} V(R, T)} = e^{-N_t V(R, T)}, \quad (2.15)$$

where the lattice spacing is set to  $a = 1$  for simplicity and  $|\vec{x} - \vec{y}| = R$ .

Assuming an area law as with the Wilson loops, we have

$$\langle \phi(\vec{x}) \phi(\vec{y}) \rangle_{N_t} \sim e^{-\sigma(T) N_t R}, \quad (2.16)$$

where  $\sigma(T)$  is the *finite temperature string tension*. Again, refer to [5] for a more extensive overview about Polyakov loops.

## 2.4 Polyakov loop as order parameter

A compactified time direction with periodic boundary conditions has as consequence the appearance of a new global symmetry in action  $S$ : its symmetry group is the center of the gauge group. In the case of an  $SU(N)$  gauge group, the center is  $\mathbb{Z}_N$ ; in the case of a  $Sp(N)$  gauge group, the center is always  $\mathbb{Z}_2$ .

Applying a center symmetry transformation has the effect of multiplying all time-like links belonging to the same space-like slice (they have the same coordinate  $t$ ) by the same element  $W_0$  belonging to the center of the gauge group:

$$U_t(\tau, \vec{x}) \longrightarrow W_0 U_t(\tau, \vec{x}), \quad \forall \vec{x} \quad (2.17)$$

for a fixed value of  $\tau$ , the time component. From (2.13), this implies that a Polyakov loop  $\phi(\vec{x})$  also transforms in the same way:

$$\phi(\vec{x}) \longrightarrow W_0 \phi(\vec{x}). \quad (2.18)$$

Thus, when  $\langle \phi(\vec{x}) \rangle = 0$ , the center symmetry is *unbroken*; when  $\langle \phi(\vec{x}) \rangle \neq 0$ , the center symmetry is *broken*. This means that the expectation value of the Polyakov loop is the order parameter of the spontaneous center symmetry breaking phase transition.

Polyakov loops have another interesting property: their expectation value is related to the free energy  $F$  of a single, isolated quark:

$$\langle \phi(\vec{x}) \rangle \sim e^{-F/T}. \quad (2.19)$$

We can then distinguish between confinement and deconfinement phase thanks to the expectation value of the Polyakov loop. A confined phase is characterized by  $F = \infty$ , which implies  $\langle \phi(\vec{x}) \rangle = 0$ ; a deconfined phase is, on the contrary, characterized by a finite  $F$ , thus  $\langle \phi(\vec{x}) \rangle \neq 0$ . Not only is the

Polyakov loop the order parameter of the center symmetry breaking phase transition, but it is also the order parameter of the confinement-deconfinement phase transition. In the confined phase, the center symmetry is unbroken, while it is spontaneously broken in the deconfined phase.

As discussed in [6], while this is true in the infinite volume case, in a finite size scale analysis with a finite volume, the expectation value of a Polyakov loop is always zero: no spontaneous symmetry breaking occurs. We thus use the expectation value of the magnitude of the Polyakov loop  $\langle |\phi(\vec{x})| \rangle$  as order parameter. This quantity is always non-zero in a finite volume  $V$ , but in the confined phase, for  $V \rightarrow \infty$ , we have  $\langle |\phi(\vec{x})| \rangle = 0$ .

Recalling the role that  $\beta$  has in the LGT action (2.6), we can define a critical value of  $\beta$ ,  $\beta_c(N_t)$ , which depends on  $N_t$  and corresponds to the point at which a phase transition occurs. Inverting  $\beta_c(N_t)$  and using (2.12), we can define a critical temperature  $T_c$ :

$$T_c = \frac{1}{N_{t,c}(\beta)}, \quad (2.20)$$

which represents the temperature at which the system undergoes a phase transition.

## 2.5 The Svetitsky–Yaffe conjecture

Given the discussion above about the role of Polyakov loops, it is interesting to study the confinement-deconfinement phase transition using an effective action, built by integrating out the spacelike links of the LGT [2] [3]. This effectively amounts to a *dimensional reduction*: we end up with a  $d$  dimensional spin model with global symmetry the center of the gauge group, from a  $(d+1)$  dimensional LGT, as first argued by Svetitsky and Yaffe.

For a second order deconfinement transition, the long range dynamics is dominated by fluctuations in the Polyakov loop. Also, interactions between Polyakov loops are suppressed exponentially in the long distance limit. Thus, in the vicinity of the critical point, the fine details of the Hamiltonian describing the effective spin model can be neglected, and the system will belong to the same universality class of the simplest spin model, with the same spontaneous symmetry breaking and nearest neighbor interactions only.

For example, an  $SU(2)$  Yang-Mills theory in  $(2+1)$ ,  $(3+1)$  dimensions will be mapped to a 2, 3 dimensional spin model, with global symmetry  $\mathbb{Z}_2$ , the center of the  $SU(2)$  gauge group. In particular, the 2 dimensional spin model is exactly integrable, and it is an excellent starting pointing for studying the conjecture.

In particular, the conjecture claims the following:

- The high temperature, deconfined phase of the original Yang-Mills theory corresponds to the low temperature, ordered phase of the effective spin model. In both of these phases, the corresponding order parameter has non-zero expectation value.
- The correlator between Polyakov loops in the confining phase (which is the one we are interested in) is mapped to the spin-spin correlator in the disordered phase of the spin model.

As the second point suggests, the spin-spin correlator will be an important quantity for testing the conjecture. Indeed, in the 2 dimensional spin model they are well known (see [2]). Defining  $R$  to be the distance characterizing the two-point function,  $\xi$  the correlation length of the system and  $\phi(x)$  the spin operator at  $x$ , we have

- for  $R \ll \xi$ :

$$\begin{aligned} \langle \phi(0)\phi(R) \rangle = \frac{k_s}{R^{1/4}} & \left[ 1 + \frac{R}{2\xi} \ln \left( \frac{e^{\gamma_e} R}{8\xi} \right) + \frac{R^2}{16\xi^2} \right. \\ & \left. + \frac{R^3}{32\xi^3} \ln \left( \frac{e^{\gamma_e} R}{8\xi} \right) + O \left( \frac{R^4}{\xi^4} \ln^2 \frac{R}{\xi} \right) \right] \end{aligned} \quad (2.21)$$

- for  $R \gg \xi$ :

$$\langle \phi(0)\phi(R) \rangle = k_l K_0 \left( \frac{R}{\xi} \right) \quad (2.22)$$

where  $k_l$ ,  $k_s$  are constants,  $\gamma_e$  is the Euler-Mascheroni constant and  $K_0$  is the modified Bessel function of order zero. Note that in the analysis of this work, we will be using, in the case  $R \gg \xi$ ,

$$\langle \phi(0)\phi(R) \rangle = k_l \left( K_0 \left( \frac{R}{\xi} \right) - K_0 \left( \frac{L-R}{\xi} \right) \right), \quad (2.23)$$



where  $L$  is the length of the space dimensions (which are all the same length). This is to account for the periodic boundary conditions imposed on the simulated lattice.

## 2.6 Effective string theory

## 2.7 $\text{Sp}(\text{N})$ group

# 3 Simulation and algorithm

**Probably remove this intro because theory is described above**

Define the partition function of the  $(2 + 1)$  dimensional system to be

$$\mathcal{Z} = \int \mathcal{D}U e^{-\beta S(U)} \quad \text{where} \quad \int \mathcal{D}U \equiv \prod_{x,\mu} \int dU_{x,\mu}. \quad (3.1)$$

$U_{x,\mu}$  is an element of the group (a link variable starting at the lattice site  $x$ , pointing along the direction  $\mu$ ), and  $S$  is the action, defined as

$$S = \sum_{\square} S_{\square}, \quad S_{\square} \equiv -\frac{1}{4} \text{Tr} \left( U_{x,\mu} U_{x+\hat{\mu},\nu} U_{x+\hat{\nu},\mu}^{\dagger} U_{x,\nu}^{\dagger} \right). \quad (3.2)$$

The symbol  $\square$  represents the plaquette: the minimum loop possible on the lattice. Note that a link variable has a direction: the adjoint of a link variable is the link connecting the two sites in the opposite direction. In other words:

$$U_{x,\mu} = U_{x+\hat{\mu},-\mu}^{\dagger}. \quad (3.3)$$

Note that the lattice has periodic boundary conditions. In our case, the gauge group of choice is  $\text{Sp}(2)$ .

## 3.1 Heat-bath algorithm

The heat-bath algorithm is used to calculate each step of the evolution of the system. The basic idea is to generate a new link element  $U$  with a Boltzmann probability distribution:

$$P(U) = \frac{1}{\mathcal{Z}} e^{-\beta S(U)} dU \quad (3.4)$$

for each link variable in the system. Updating each link once represents a single step in the algorithm.

While this is fairly easy for the  $SU(2)$  gauge group using Creutz's algorithm [4], there is no obvious way to generalize it to other gauge groups, like  $SU(N)$  or  $Sp(N)$ . Thus, to generate new links belonging to  $Sp(2)$ , we use a more general approach, as designed by Cabibbo and Marinari [1].

We consider a set  $F$  of  $SU(2)$  subgroups of the gauge group  $Sp(2)$ . Given a  $Sp(2)$  element  $U$  of the form

$$U = \begin{pmatrix} W_{11} & W_{12} & X_{11} & X_{12} \\ W_{21} & W_{22} & X_{21} & X_{22} \\ X_{22}^* & -X_{21}^* & W_{22}^* & -W_{21}^* \\ -X_{12}^* & X_{11}^* & -W_{12}^* & W_{11}^* \end{pmatrix}, \quad (3.5)$$

where  $W_{ij}, X_{kl} \in \mathbb{C}$  for  $i, j, k, l = 1, 2$ , we can construct four  $SU(2)$  subgroups, by extracting two complex numbers  $t_1$  and  $t_2$ :

- $\begin{cases} t_1 = W_{11} \\ t_2 = X_{12} \end{cases}$
- $\begin{cases} t_1 = W_{22} \\ t_2 = X_{21} \end{cases}$
- $\begin{cases} t_1 = W_{11} + W_{22} \\ t_2 = X_{11} - X_{22} \end{cases}$
- $\begin{cases} t_1 = W_{11} + W_{22}^* \\ t_2 = W_{12} - W_{21}^* \end{cases}$

We can then build an  $SU(2)$  group element as

$$a_k = \begin{pmatrix} t_1 & t_2 \\ -t_2^* & t_1^* \end{pmatrix}, \quad k = 1, 2, 3, 4, \quad (3.6)$$

where  $k$  labels each subgroup. Each choice of  $t_1$  and  $t_2$  above gives a different  $SU(2)$  element belonging to a  $SU(2)$  subgroup of  $Sp(2)$ .

We define  $A_k$  to be an  $SU(2)$  element belonging to the  $k$ th  $SU(2)$  subgroup, embedded into  $Sp(2)$ . For each subgroup in  $F$ , the  $Sp(2)$  embedding is constructed as follows:

$$\begin{aligned}
\bullet A_1 &= \begin{pmatrix} t_1 & 0 & 0 & t_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -t_2^* & 0 & 0 & t_1^* \end{pmatrix} \\
\bullet A_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t_1 & t_2 & 0 \\ 0 & -t_2^* & t_1^* & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
\bullet A_3 &= \begin{pmatrix} t_1 & 0 & t_2 & 0 \\ 0 & t_1 & 0 & -t_2 \\ -t_2^* & 0 & t_1^* & 0 \\ 0 & t_2^* & 0 & t_1^* \end{pmatrix} \\
\bullet A_4 &= \begin{pmatrix} t_1 & t_2 & 0 & 0 \\ -t_2^* & t_1^* & 0 & 0 \\ 0 & 0 & t_1 & t_2 \\ 0 & 0 & -t_2^* & t_1^* \end{pmatrix}
\end{aligned}$$

Generating each  $A_k$  randomly, we define the new link  $U'$  to be

$$U' = A_4 A_3 A_2 A_1 U, \quad (3.7)$$

because, in this case, the set  $F$  contains four  $SU(2)$  subgroups.

It is proven [1] that this algorithm leads to thermalization, if each  $A_k$  is randomly distributed as

$$P(A_k) = dA_k \frac{e^{-\beta S(A_k U_{k-1})}}{\mathcal{Z}_k(U_{k-1})}, \quad (3.8)$$

where  $U_n \equiv A_n A_{n-1} \dots A_1 U$  with  $U_0 = U$  and

$$\mathcal{Z}_k(U) = \int_{SU(2)_k} dA e^{-\beta S(AU)}. \quad (3.9)$$

The reason for the decomposition into  $SU(2)$  subgroups is to efficiently generate  $A_k$  according to (3.8). In fact, now that we are dealing with  $SU(2)$  elements, we can fall back to Creutz's algorithm [4] to generate each  $SU(2)$  element, embed it into  $Sp(2)$  as explained above, and left multiply the original link  $U$  by it.

Focusing on a single link  $U$  to update, we are interested only in the plaquettes that contain  $U$ . Defining  $\tilde{U}_i$  to be one of the staples surrounding  $U$  (an ordered product of the three links in the plaquettes containing  $U$ , that are not the link  $U$  itself), we have

$$S(A_k U) = -\frac{1}{4} \text{Tr} \left( A_k U \sum_i \tilde{U}_i \right) + \text{terms independent of } A_k \quad (3.10)$$

$$= -\frac{1}{4} \text{Tr} \left( a_k u_k \sum_i \tilde{u}_k^i \right) + \text{terms independent of } a_k, \quad (3.11)$$

where  $a_k$ ,  $u_k$  and  $\tilde{u}_k$  are  $SU(2)$  elements corresponding to the  $k$ th subgroup extracted from  $A_k$ ,  $U$  and  $\tilde{U}$ , respectively. This implies that we want to generate  $a_k$  according to the distribution

$$dP(a_k) \sim e^{\frac{1}{4}\beta \text{Tr}(a_k u_k \sum_i \tilde{u}_k^i)} da_k. \quad (3.12)$$

We parametrize  $a_k$  as

$$a_k = \alpha_0 \mathbb{1} + i\vec{\alpha} \cdot \vec{\sigma}, \quad (3.13)$$

where  $\alpha_\mu \in \mathbb{R} \ \forall \mu = 1, 2, 3, 4$  with the constraint that

$$\alpha^2 \equiv \alpha_0^2 + |\vec{\alpha}|^2 = 1 \quad (3.14)$$

and  $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$  is the three-vector of  $2 \times 2$  Pauli matrices.

The SU(2) group measure is then

$$da_k = \frac{1}{2\pi^2} \delta(\alpha^2 - 1) d^4\alpha. \quad (3.15)$$

Since the sum of SU(2) elements is proportional to an SU(2) element, we write

$$u_k \sum_i \tilde{u}_k^i = c \bar{u}_k, \quad \bar{u}_k \in \text{SU}(2) \quad (3.16)$$

where

$$c = \det \left( u_k \sum_i \tilde{u}_k^i \right)^{1/2}. \quad (3.17)$$

The probability distribution for  $a_k$  now becomes

$$dP(a_k) \sim e^{\frac{1}{4}\beta \text{Tr}(ca_k \bar{u}_k)} da_k. \quad (3.18)$$

The group measure is invariant under multiplication by another SU(2) element:

$$d(ba_k) = da_k \quad \text{for } b \in \text{SU}(2), \quad (3.19)$$

so we can write

$$dP(a_k \bar{u}_k^{-1}) \sim e^{\frac{1}{4}\beta c \text{Tr}(a_k)} da_k = \frac{1}{2\pi^2} e^{\frac{\beta}{2} c \alpha_0} \delta(\alpha^2 - 1) d^4\alpha, \quad (3.20)$$

because  $\text{Tr}(a_k) = 2\alpha_0$ . Noting that  $\delta(\alpha^2 - 1) d^4\alpha = \frac{1}{2}(1 - \alpha_0^2)^{1/2} d\alpha_0 d\Omega$ , we rewrite  $dP(a_k \bar{u}_k^{-1})$  as

$$dP(a_k \bar{u}_k^{-1}) \sim \frac{1}{2\pi^2} \frac{1}{2} (1 - \alpha_0^2)^{1/2} e^{\frac{\beta}{2} c \alpha_0} d\alpha_0 d\Omega \quad (3.21)$$

with  $\alpha_0 \in (-1, 1)$  and  $d\Omega$  is the differential solid angle of the three-vector  $\vec{\alpha}$ , which is of length  $(1 - \alpha_0^2)^{1/2}$ .

The problem is now about generating the four-vector  $\alpha_\mu$  according to the distribution above, thus obtaining  $a_k \in \text{SU}(2)$ . Finally, we obtain  $A_k$  by embedding  $a_k \bar{u}_k^{-1} \in \text{SU}(2)$  into  $\text{Sp}(2)$ . Doing this for every SU(2) subgroup will yield the new link  $U'$ .

To generate  $a_k$ , we have to randomly generate  $\alpha_0$  according to

$$P(\alpha_0) \sim (1 - \alpha_0^2)^{1/2} e^{\frac{\beta}{2} c \alpha_0}. \quad (3.22)$$

The algorithm is quite simple. We uniformly generate  $x$  in the range

$$e^{-\beta c} < x < 1 \quad (3.23)$$

and define a trial  $\alpha_0$  distributed according to  $e^{\frac{\beta}{2} c \alpha_0}$  as

$$\alpha_0 = 1 + \frac{2}{\beta c} \ln x. \quad (3.24)$$

To account for the term  $(1 - \alpha_0^2)^{1/2}$  in (3.22), we *reject* this trial  $\alpha_0$  with probability  $1 - (1 - \alpha_0^2)^{1/2}$ , generating a new trial  $\alpha_0$  if the rejection is successful. We keep doing this until a trial  $\alpha_0$  is finally accepted.

The unit vector  $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$  is constructed by uniformly generating

$$\begin{aligned} \phi &\in (0, 2\pi), \\ y &\in (-1, 1) \end{aligned} \quad (3.25)$$

and defining

$$\begin{aligned} \theta &\equiv \arccos(y), \\ r &\equiv (1 - \alpha_0)^{1/2}, \end{aligned} \quad (3.26)$$

$\alpha_0$  being the trial random number that has been accepted according to (3.8).

We finally have, for  $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ ,

$$\begin{cases} \alpha_1 = r \sin(\theta) \cos(\phi) \\ \alpha_2 = r \sin(\theta) \sin(\phi) \\ \alpha_3 = r \cos(\theta) \end{cases} . \quad (3.27)$$

$a_k$  will then be constructed using (3.13), and  $A_k$  using the embeddings described above, with  $t_1$  and  $t_2$  selected from  $a_k$  as of (3.6).

## 3.2 Overrelaxation

The full algorithm implemented for the simulation also takes advantage of *overrelaxation* [7]. Overrelaxation is used to counter critical slowing down in the vicinity of the phase transition: here, the system's correlation length is so large that lattice updates of the order of the lattice spacing are negligible. This causes the simulation to slow down considerably. Overrelaxation is a way of choosing new link elements to update the lattice with, which prevents critical slowing down.

The idea is to choose a new link element  $U'$  "as far as possible" from  $U$  (the original link), without actually changing the action  $S$  of the system. [7] describes overrelaxation in the case of  $SU(2)$  gauge group. For more complex groups, it is not as straightforward to generate new links. Again, we use Cabibbo and Marinari's idea: the new link is chosen left multiplying the old link  $U$  by randomly generated  $A_k$  elements belonging to  $Sp(2)$ , which are in turn built embedding  $SU(2)$  elements into  $Sp(2)$ .  $k$  labels the  $SU(2)$  subgroups and runs from 1 to 4, as for the heat-bath algorithm.

Note that overrelaxation is applied at each simulation step along with heat-bath: in fact, a single complete simulation step consists of one or more steps of overrelaxation and one step of heat-bath, where a step is the update of the entire lattice exactly once. In our case, overrelaxation is applied three times before each heat-bath step. This is possible because, as mentioned, overrelaxation doesn't actually change the action of the system.

Consider  $\tilde{U}_i$  to be one staple surrounding the link  $U$ , and let us define

$$R = \sum_{i=1}^6 \tilde{U}_i. \quad (3.28)$$

Evidently, there are six staples surrounding each link in a  $(2+1)$  dimensional lattice, thus  $i = 1, \dots, 6$ .

[7] defines the new link to be

$$U' = VU^{-1}V, \quad (3.29)$$

where

$$V \equiv \det(R)^{1/2} R^{-1}, \quad (3.30)$$

the inverse of the projection of  $R$  onto the  $SU(2)$  gauge group. In the case of the  $Sp(2)$  gauge group, we define

$$A_k U_{k-1} = V U_{k-1}^{-1} V, \quad (3.31)$$

instead, where

$$U_n \equiv A_n A_{n-1} \dots A_1 U \quad \text{with} \quad U_0 = U. \quad (3.32)$$

Following Cabibbo and Marinari's prescription, the new link will be

$$U' = A_4 A_3 A_2 A_1 U \quad (3.33)$$

in the case of four  $SU(2)$  subgroups.

We then extract the  $SU(2)$  elements corresponding to the  $k$ th subgroup of  $Sp(2)$  from each term in (3.31):

$$\begin{cases} A_k \in Sp(2) & \longrightarrow a_k \in SU(2) \\ U_{k-1} \in Sp(2) & \longrightarrow u_k \in SU(2) \\ \tilde{U}_i \in Sp(2) & \longrightarrow r_k^{(i)} \in SU(2) \end{cases} . \quad (3.34)$$

We also define

$$r_k \equiv \sum_i r_k^{(i)}. \quad (3.35)$$

Using the fact that a sum of  $SU(2)$  elements is proportional to a  $SU(2)$  element, we can write

$$u_k r_k = u_k \sum_i r_k^{(i)} = c \bar{u}_k \quad (3.36)$$



with  $c = \det(r_k)^{1/2}$  and  $\bar{u}_k \in \text{SU}(2)$ .

Rewriting (3.31) as

$$A_k = VU^{-1}VU^{-1} \quad (3.37)$$

and applying the equation to the  $\text{SU}(2)$  elements extracted above, we end up with

$$\begin{aligned} a_k &= \det(r_k) r_k^{-1} u_k^{-1} r_k^{-1} u_k^{-1} \\ &= \det(r_k) (u_k r_k)^{-1} (u_k r_k)^{-1} \\ &= \det(r_k) (r_k u_k)^{-2} \\ &= \det(r_k) (c \bar{u}_k)^{-2}. \end{aligned} \quad (3.38)$$

Given that  $c = \det(r_k)^{1/2}$ , we finally have

$$a_k = \bar{u}_k^{-2} = \left( \frac{1}{c} u_k \sum_i r_k^{(i)} \right)^{-2}. \quad (3.39)$$

Exactly as with the heat-bath algorithm, once we find  $a_k$ , we can embed it into  $\text{Sp}(2)$  according to the subgroup it belongs to, which yields the corresponding  $A_k$ , and find the new link using (3.33).

## 4 Results

**Put plots and results of the fit and simulations: fit of susceptibility peaks, fit of beta vs nt, etc.**

## References

- [1] Nicola Cabibbo and Enzo Marinari. “A new method for updating  $\text{SU}(N)$  matrices in computer simulations of gauge theories”. In: *Physics Letters B* 119.4 (1982), pp. 387–390. ISSN: 0370-2693. DOI: [https://doi.org/10.1016/0370-2693\(82\)90696-7](https://doi.org/10.1016/0370-2693(82)90696-7). URL: <https://www.sciencedirect.com/science/article/pii/0370269382906967>.

- [2] F. Caristo et al. “Fine corrections in the effective string describing  $SU(2)$  Yang-Mills theory in three dimensions”. In: *Journal of High Energy Physics* 2022.3 (Mar. 2022). DOI: 10.1007/jhep03(2022)115. URL: <https://doi.org/10.1007%2Fjhep03%282022%29115>.
- [3] Michele Caselle. *Effective string description of the confining flux tube at finite temperature*. 2021. DOI: 10.48550/ARXIV.2104.10486. URL: <https://arxiv.org/abs/2104.10486>.
- [4] Michael Creutz. “Monte Carlo study of quantized  $SU(2)$  gauge theory”. In: *Phys. Rev. D* 21 (8 Apr. 1980), pp. 2308–2315. DOI: 10.1103/PhysRevD.21.2308. URL: <https://link.aps.org/doi/10.1103/PhysRevD.21.2308>.
- [5] Christof Gattringer and Christian B. Lang. *Quantum chromodynamics on the lattice*. Vol. 788. Berlin: Springer, 2010. ISBN: 978-3-642-01849-7, 978-3-642-01850-3. DOI: 10.1007/978-3-642-01850-3.
- [6] K. Holland, M. Pepe, and U.-J. Wiese. “The deconfinement phase transition of  $Sp(2)$  and  $Sp(3)$  Yang–Mills theories in 2+1 and 3+1 dimensions”. In: *Nuclear Physics B* 694.1 (2004), pp. 35–58. ISSN: 0550-3213. DOI: <https://doi.org/10.1016/j.nuclphysb.2004.06.026>. URL: <https://www.sciencedirect.com/science/article/pii/S0550321304004250>.
- [7] Istvan Montvay and Gernot Münster. *Quantum Fields on a Lattice*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1994. DOI: 10.1017/CB09780511470783.