Abstract

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1 Introduction

Brief intro about the paper. Past work, theory, references.

2 Theoretical background

Explain the concepts of lattice gauge theory, Wilson and Polyakov loops, center symmetry and Sp(2) group. Mostly from Pepe's work. Read the paper by Caselle to see if there is something to pure here (SY conjecture? String tension? Universality with spin model? EST?)

3 Simulation and algorithm

Define the partition function of the system to be

$$\mathcal{Z} = \int \mathcal{D}U e^{-\beta S(U)} \quad \text{where} \quad \int \mathcal{D}U \equiv \prod_{x,\mu} \int \mathrm{d}U_{x,\mu}\,. \tag{3.1}$$

$U_{x,\mu}$ is an element of the group (a link variable starting at the lattice site x, pointing along the direction μ), and S is the action, defined as

$$S = \sum_{\Box} S_{\Box}, \quad S_{\Box} \equiv -\frac{1}{4}\,\mathrm{Tr}\Big(U_{x,\mu} U_{x+\hat{\mu},\nu} U^{\dagger}_{x+\hat{\nu},\mu} U^{\dagger}_{x,\nu}\Big). \tag{3.2}$$

The symbol $\Box$ represents the plaquette: the minimum loop possible on the lattice. Note that a link variable has a direction: the adjoint of a link variable is the link connecting the two sites in the opposite direction. In other words:

$$U_{x,\mu} = U^{\dagger}_{x+\hat{\mu},-\mu}. \tag{3.3}$$

Note that the lattice has periodic boundary conditions.

The algorithm basic idea is to generate a new link element U with a Boltzmann probability distribution:

$$P(U) = \frac{1}{\mathcal{Z}} e^{-\beta S(U)} \,\mathrm{d}U \,. \tag{3.4}$$

While this is fairly easy for the SU(2) gauge group using Creutz's algorithm [2], there is no obvious way to generalize it to other gauge groups, like SU(N) or Sp(N). Thus, to generate new links belonging to Sp(2), we use a more general approach, as designed by Cabibbo and Marinari [1].

We consider a set F of SU(2) subgroups of the gauge group Sp(2). Given a Sp(2) element U of the form

$$U = \begin{pmatrix} W_{11} & W_{12} & X_{11} & X_{12} \\ W_{21} & W_{22} & X_{21} & X_{22} \\ X_{22}^* & -X_{21}^* & W_{22}^* & -W_{21}^* \\ -X_{12}^* & X_{11}^* & -W_{12}^* & W_{11}^* \end{pmatrix}, \tag{3.5}$$

where $W_{ij}, X_{kl} \in \mathbb{C}$ for $i, j, k, l = 1, 2$, we can construct four SU(2) subgroups, by extracting two complex numbers t_1 and t_2:

- $\begin{cases} t_1 = W_{11} \\ t_2 = X_{12} \end{cases}$
- $\begin{cases} t_1 = W_{22} \\ t_2 = X_{21} \end{cases}$
- $\begin{cases} t_1 = W_{11} + W_{22} \\ t_2 = X_{11} - X_{22} \end{cases}$
- $\begin{cases} t_1 = W_{11} + W_{22}^* \\ t_2 = W_{12} - W_{21}^* \end{cases}$

We can then build an SU(2) group element as

$$a_k = \begin{pmatrix} t_1 & t_2 \\ -t_2^* & t_1^* \end{pmatrix}, \quad k = 1, 2, 3, 4, \tag{3.6}$$

where k labels each subgroup. Each choice of t_1 and t_2 above gives a different SU(2) element belonging to a SU(2) subgroup of Sp(2).

We define A_k to be an SU(2) element belonging to the kth SU(2) subgroup, embedded into Sp(2). For each subgroup in F, the Sp(2) embedding is constructed as follows:

- $A_1 = \begin{pmatrix} t_1 & 0 & 0 & t_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -t_2^* & 0 & 0 & t_1^* \end{pmatrix}$
- $A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t_1 & t_2 & 0 \\ 0 & -t_2^* & t_1^* & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
- $A_3 = \begin{pmatrix} t_1 & 0 & t_2 & 0 \\ 0 & t_1 & 0 & -t_2 \\ -t_2^* & 0 & t_1^* & 0 \\ 0 & t_2^* & 0 & t_1^* \end{pmatrix}$
- $A_4 = \begin{pmatrix} t_1 & t_2 & 0 & 0 \\ -t_2^* & 1 & 0 & 0 \\ 0 & 0 & t_1 & t_2 \\ 0 & 0 & -t_2^* & 1 \end{pmatrix}$

Generating each A_k randomly, we define the new link U' to be

$$U' = \left(\prod_k A_k \right) U. \tag{3.7}$$

It is proven [1] that this algorithm leads to thermalization, if each a_k is randomly distributed as

$$P(A_k) = \mathrm{d}A_k \, \frac{e^{-\beta S(A_k U_{k-1})}}{\mathcal{Z}_k(U_{k-1})}, \tag{3.8}$$

where $U_{k-1} = \left(\prod_1^{k-1} A_k \right) U$ with $U_0 = U$ and

$$\mathcal{Z}_k(U) = \int_{SU(2)_k} \mathrm{d}A \, e^{-\beta S(AU)}. \tag{3.9}$$

The reason for the decomposition into SU(2) subgroups is to efficiently generate A_k according to (3.8). In fact, now that we are dealing with SU(2) elements, we can fall back to Creutz's algorithm [2] to generate each SU(2) element, embed it into Sp(2) as explained above, and left multiply the original link U by it.

Focusing on a single link U to update, we are interested only in the plaquettes that contain U. Defining $\widetilde{U}$ to be the product of the staples surrounding U (an ordered product of the three links in the plaquettes that are not U itself), we have

$$S(A_kU) = -\frac{1}{4}\,\mathrm{Tr}\left(A_kU\sum_i \widetilde{U}_i\right) + \text{terms independent of } A_k \tag{3.10}$$

$$= -\frac{1}{4}\,\mathrm{Tr}\left(a_ku_k\sum_i \tilde{u}_k^i\right) + \text{terms independent of } a_k, \tag{3.11}$$

where a_k, u_k and $\tilde{u}_k$ are SU(2) elements corresponding to the kth subgroup extracted from A_k, U and $\widetilde{U}$, respectively. This implies that we want to generate a_k according to the distribution

$$\mathrm{d}P(a_k) \sim e^{\frac{1}{4}\beta\,\mathrm{Tr}\left(a_ku_k\sum_i\tilde{u}_k^i\right)}\,\mathrm{d}a_k\,. \tag{3.12}$$

We parametrize a_k as

$$a_k = \alpha_0\mathbb{1} + i\vec{\alpha}\cdot\vec{\sigma}, \tag{3.13}$$

where $\alpha_\mu \in \mathbb{C}\ \forall\mu = 1,2,3,4$ with the constraint that

$$\alpha^2 \equiv \alpha_0^2 + |\vec{\alpha}|^2 = 1 \tag{3.14}$$

and $\vec{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ is the three-vector of the 2×2 Pauli matrices.

The SU(2) group measure is

$$\mathrm{d}a_k = \frac{1}{2\pi^2}\delta(\alpha^2-1)\,\mathrm{d}^4\alpha\,. \tag{3.15}$$

Since the sum of SU(2) elements is proportional to an SU(2) element, we write

$$u_k \sum_i \tilde{u}_k^i = c\bar{u}_k, \quad \bar{u}_k \in \mathrm{SU}(2) \tag{3.16}$$

where

$$c = \det\left(u_k \sum_i \tilde{u}_k^i\right)^{1/2}. \tag{3.17}$$

The probability distribution for a_k now becomes

$$\mathrm{d}P(a_k) \sim e^{\frac{1}{4}\beta\,\mathrm{Tr}(c a_k \bar{u}_k)}\,\mathrm{d}a_k\,. \tag{3.18}$$

The group measure is invariant under multiplication by another SU(2) element:

$$\mathrm{d}(b a_k) = \mathrm{d}a_k \quad \text{for} \quad b \in \mathrm{SU}(2), \tag{3.19}$$

so that we can write

$$\mathrm{d}P(a_k \bar{u}_k^{-1}) \sim e^{\frac{1}{4}\beta c\,\mathrm{Tr}(a_k)}\,\mathrm{d}a_k = \frac{1}{2\pi^2} e^{\frac{\beta}{2} c\alpha_0} \delta(\alpha^2 - 1)\,\mathrm{d}^4\alpha\,, \tag{3.20}$$

because $\mathrm{Tr}(a_k) = 2\alpha_0$. Noting that $\delta(\alpha^2 - 1)\,\mathrm{d}^4\alpha = \frac{1}{2}(1-\alpha_0^2)^{1/2}\,\mathrm{d}\alpha_0\,\mathrm{d}\Omega$, we rewrite $\mathrm{d}P(a_k\bar{u}_k^{-1})$ as

$$\mathrm{d}P(a_k \bar{u}_k^{-1}) \sim \frac{1}{2\pi^2}\frac{1}{2}(1-\alpha_0^2)^{1/2}\, e^{\frac{\beta}{2}c\alpha_0}\,\mathrm{d}\alpha_0\,\mathrm{d}\Omega \tag{3.21}$$

with $\alpha_0 \in (-1, 1)$ and $\vec{\alpha}$ is a totally random unit three-vector.

The problem is now generating the four-vector α_μ according to the distribution above, thus obtaining $a_k \in \mathrm{SU}(2)$. Finally, we obtain A_k by embedding $a_k\bar{u}_k^{-1} \in \mathrm{SU}(2)$ into Sp(2). Doing this for every SU(2) subgroup will yield the new link U'.

To generate a_k, we have to randomly generate α_0 according to

$$P(\alpha_0) \sim (1-\alpha_0^2)^{1/2}\, e^{\frac{\beta}{2}c\alpha_0}. \tag{3.22}$$

The algorithm is quite simple. We uniformly generate x in the range

$$e^{-\beta c} < x < 1 \tag{3.23}$$

and define a trial α_0 distributed according to $e^{\frac{\beta}{2}c\alpha_0}$ as

$$\alpha_0 = 1 + \frac{2}{\beta c}\ln x. \tag{3.24}$$

To account for the term $(1-\alpha_0^2)^{1/2}$ in (3.22), we *reject* this trial α_0 with probability $1-(1-\alpha_0^2)^{1/2}$, generating a new trial α_0 if the rejection is successful. We keep doing this until a trial α_0 is finally accepted.

The unit vector $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ is constructed by uniformly generating

$$0 < \phi < 2\pi, \tag{3.25}$$
$$-1 < y < 1 \tag{3.26}$$

and defining

$$\theta \equiv \arccos(y), \tag{3.27}$$
$$r \equiv (1-\alpha_0)^{1/2}. \tag{3.28}$$

We finally have

$$\begin{cases} \alpha_1 = r\sin(\theta)\cos(\phi) \\ \alpha_2 = r\sin(\theta)\sin(\phi) \\ \alpha_3 = r\cos(\theta) \end{cases} . \tag{3.29}$$

4 Results

Put plots and results of the fit and simulations: fit of susceptibility peaks, fit of beta vs nt, etc.

References

[1] Nicola Cabibbo and Enzo Marinari. "A new method for updating SU(N) matrices in computer simulations of gauge theories". In: *Physics Letters B* 119.4 (1982), pp. 387–390. ISSN: 0370-2693. DOI: https://doi.org/10.1016/0370-2693(82)90696-7. URL: https://www.sciencedirect.com/science/article/pii/0370269382906967.

[2] Michael Creutz. "Monte Carlo study of quantized SU(2) gauge theory". In: *Phys. Rev. D* 21 (8 Apr. 1980), pp. 2308–2315. DOI: 10.1103/PhysRevD.21.2308. URL: https://link.aps.org/doi/10.1103/PhysRevD.21.2308.