Tesi

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Abstract

abstract

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1 Introduction

Brief intro about the paper. Past work, theory, references.

2 Theoretical background

Explain the concepts of lattice gauge theory, Wilson and Polyakov loops, center symmetry and Sp(2) group. Mostly from Pepe's work. Read the paper by Caselle to see if there is something to pure here (SY conjecture? String tension? Universality with spin model? EST?)

2.1 Lattice gauge theory

In this work, we will be focusing on a (2+1) dimensional lattice gauge theory (two spatial dimensions and one time dimension). We will present a brief summary of lattice gauge theories (LGT).

An LGT is a gauge theory built on a discretized spacetime: a lattice of spacing a with N_s sites in the space directions and N_t sites in the time direction. This discretization is useful, because it introduces a natural cutoff for the momentum p, thus regularizing the theory:

$$p \in \left(-\frac{\pi}{a}, \frac{\pi}{a}\right). \tag{2.1}$$

Starting from the continuum limit action

$$\widetilde{S} = \frac{1}{2g_0^2} \int \mathrm{d}^4x \, \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}), \tag{2.2}$$

where $F_{\mu\nu} = F^a_{\mu\nu}T^a = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$, we define the *link* variable as

$$U_\mu(x) = e^{-iaA_\mu(x)}. \tag{2.3}$$

Given a gauge transformation $G(x)$, a link variable transforms as

$$U_\mu(x) \longrightarrow G(x)U_\mu(x)G^\dagger(x+\hat{\mu}). \tag{2.4}$$

The link variable can be thought of as connecting the site at x to the site at $x + \hat{\mu}$, and it belongs to the gauge group defining the theory. We define a *plaquette* $U_{\mu\nu}(x)$ as the smallest loop of product of link variables:

$$U_{\mu\nu}(x) = U_\mu(x) U_\nu(x + \hat{\mu}) U_\mu^\dagger(x + \hat{\nu}) U_\nu^\dagger(x). \tag{2.5}$$

Note that links are oriented, and the adjoint operation flips a link, inverting its direction.

Finally, we can define the discretized action of an LGT:

$$S = -\frac{\beta}{N} \sum_x \sum_{\mu<\nu} \operatorname{Tr} U_{\mu\nu}(x) = -\frac{2}{g_0^2} \sum_x \sum_{\mu<\nu} \operatorname{Tr} U_{\mu\nu}(x), \tag{2.6}$$

where g_0 is the bare gauge coupling, N is the trace normalization constant related to the gauge group, and $\beta \equiv \frac{2N}{g_0^2}$. This action is invariant under gauge transformations: in fact, despite links not being gauge invariant, a plaquette is indeed gauge invariant.

The path integral is

$$\mathcal{Z} = \int \mathcal{D}U e^{-S} \tag{2.7}$$

for

$$\int \mathcal{D}U = \prod_{x,\mu} \int_{\substack{\text{gauge} \\ \text{group}}} \mathrm{d}U_{x,\mu}\,. \tag{2.8}$$

2.2 Static quarks and Wilson loops

We want to study the confinement-deconfinement phase transition that is typical of QCD. To do so, we imagine instantly creating a static quark-antiquark pair at distance R from one another. "Static" means that these quarks have infinite mass and are thus non-dynamical. We let them evolve with time for a duration of T, and finally we instantly destroy them. In the context of a lattice gauge theory, the line connecting the two quarks has swept an area of $R \times T$. This picture is described by *Wilson loops*.

A Wilson loop is the trace of an ordered product of link variables on an LGT along some path $\mathcal{C}$:

$$W(\mathcal{C}) = \mathrm{Tr} \prod_{t,\vec{x}} U_\mu(t, \vec{x}). \tag{2.9}$$

If the path $\mathcal{C}$ is a $R \times T$ rectangle, we are basically considering the rectangle swept by the quark pair: the Wilson loop is then interpreted as the free energy. In fact, it is possible to relate the potential $V(R)$ between these two quarks to the expectation value of the Wilson loop for the $R \times T$ rectangle:

$$V(R) = -\lim_{L\to\infty} \frac{1}{L} \log \langle W(\mathcal{C}) \rangle. \tag{2.10}$$

In a confining theory, the potential $V(R)$ is linearly rising, and the Wilson loop exhibits a so-called "area law":

$$\langle W(\mathcal{C}) \rangle \sim e^{-\sigma_0 RT}. \tag{2.11}$$

σ_0 is called *zero temperature string tension*. "String tensions" comes from the fact that the interquark potential can be effectively described as a vibrating string [2].

A complete introduction to Wilson loops and their relation to the interquark potential can be found in [4].

2.3 Finite temperature LGT and Polyakov loops

The description of the confining potential through Wilson loops does not take into account the temperature of the system. If we want to study how the confinement-deconfinement transition depends on temperature, we need to define the theory at finite temperature in the first place.

It is well known how to do so for an LGT defined on a $(d+1)$ dimensional lattice of volume $(N_s a)^d (N_t a)$: we consider the lattice regularization and impose *periodic boundary condition* for bosonic fields (which are the fields we will be considering in this work). In particular, the compactified time

direction is interpreted as the temperature T of the system, through the relation

$$T = \frac{1}{N_t a}. \tag{2.12}$$

Thus, the length of the lattice in the time direction is inversely proportional to the temperature of the system.

The Wilson loop is not suitable anymore in describing the interquark potential in a finite temperature context. Fortunately, there is another quantity we can consider: the correlation function of two *Polyakov loops*.

A Polyakov loop is the trace of the ordered product of all time-like links ($U_\mu(t, \vec{x})$ with $\mu = t$) with the same space coordinates:

$$\phi(\vec{x}) = \mathrm{Tr} \prod_{\tau=1}^{N_t} U_t(\tau, \vec{x}), \tag{2.13}$$

where $\vec{x}$ represents the space coordinate of the loop (a d dimensional vector, in the case of a $d+1$ dimensional lattice).

As usual, the correlation function is defined as

$$\langle \phi(\vec{x})\phi(\vec{y}) \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}U \phi(\vec{x})\phi(\vec{y}) e^{-S[U]}. \tag{2.14}$$

Now, the finite temperature interquark potential $V(R, T)$ can be extracted from the relation

$$\langle \phi(\vec{x})\phi(\vec{y}) \rangle = e^{-\frac{1}{T}V(R,T)} = e^{-N_t V(R,T)}, \tag{2.15}$$

where the lattice spacing is set to $a = 1$ for simplicity and $|\vec{x} - \vec{y}| = R$.

Assuming an area law as with the Wilson loops, we have

$$\langle \phi(\vec{x})\phi(\vec{y}) \rangle_{N_t} \sim e^{-\sigma(T) N_t R}, \tag{2.16}$$

where $\sigma(T)$ is the *finite temperature string tension*. Again, refer to [4] for a more extensive overview about Polyakov loops.

2.4 Polyakov loop as order parameter

A compactified time direction with periodic boundary conditions has as consequence the appearance of a new global symmetry in action S: its symmetry group is the center of the gauge group. In the case of an SU(N) gauge group, the center is $\mathbb{Z}_N$; in the case of a Sp(N) gauge group, the center is always $\mathbb{Z}_2$.

Applying a center symmetry transformation has the effect of multiplying all time-like links belonging to the same space-like slice (they have the same coordinate t) by the same element W_0 belonging to the center of the gauge group:

$$U_t(\tau, \vec{x}) \longrightarrow W_0 U_t(\tau, \vec{x}), \quad \forall \vec{x} \tag{2.17}$$

for a fixed value of τ, the time component. From (2.13), this implies that a Polyakov loop $\phi(\vec{x})$ also transforms in the same way:

$$\phi(\vec{x}) \longrightarrow W_0 \phi(\vec{x}). \tag{2.18}$$

Thus, when $\langle \phi(\vec{x}) \rangle = 0$, the center symmetry is *unbroken*; when $\langle \phi(\vec{x}) \rangle \neq 0$, the center symmetry is *broken*. This means that the expectation value of the Polyakov loop is the order parameter of the spontaneous center symmetry breaking phase transition.

Polyakov loops have another interesting property: their expectation value is related to the free energy F of a single, isolated quark:

$$\langle \phi(\vec{x}) \rangle \sim e^{-F/T}. \tag{2.19}$$

We can then distinguish between confinement and deconfinement phase thanks to the expectation value of the Polyakov loop. A confined phase is characterized by $F = \infty$, which implies $\langle \phi(\vec{x}) \rangle = 0$; a deconfined phase is, on the contrary, characterized by a finite F, thus $\langle \phi(\vec{x}) \rangle \neq 0$. Not only is the

Polyakov loop the order parameter of the center symmetry breaking phase transition, but it is also the order parameter of the confinement-deconfinement phase transition. In the confined phase, the center symmetry is unbroken, while is it spontaneously broken in the deconfined phase.

As discussed in [5], while this is true in the infinite volume case, in a finite size scale analysis with a finite volume, the expectation value of a Polyakov loop is always zero: no spontaneous symmetry breaking occurs. We thus use the expectation value of the magnitude of the Polyakov loop $\langle|\phi(\vec{x})|\rangle$ as order parameter. This quantity is always non-zero in a finite volume V, but in the confined phase, for $V \to \infty$, we have $\langle|\phi(\vec{x})|\rangle = 0$.

Recalling the role that β has in the LGT action (2.6), we can define a critical value of β, $\beta_c(N_t)$, which depends on N_t and corresponds to the point at which a phase transition occurs. Inverting $\beta_c(N_t)$ and using (2.12), we can define a critical temperature T_c:

$$T_c = \frac{1}{N_{t,c}(\beta)}, \tag{2.20}$$

which represents the temperature at which the system undergoes a phase transition.

2.5 The Svetitsky–Yaffe conjecture

2.6 Sp(N) group

3 Simulation and algorithm

Probably remove this intro because theory is described above

Define the partition function of the $(2+1)$ dimensional system to be

$$\mathcal{Z} = \int \mathcal{D}U e^{-\beta S(U)} \quad \text{where} \quad \int \mathcal{D}U \equiv \prod_{x,\mu} \int \mathrm{d}U_{x,\mu}\,. \tag{3.1}$$

$U_{x,\mu}$ is an element of the group (a link variable starting at the lattice site x,

pointing along the direction μ), and S is the action, defined as

$$S = \sum_{\square} S_{\square}, \quad S_{\square} \equiv -\frac{1}{4} \operatorname{Tr}\Big(U_{x,\mu} U_{x+\hat{\mu},\nu} U^{\dagger}_{x+\hat{\nu},\mu} U^{\dagger}_{x,\nu} \Big). \tag{3.2}$$

The symbol $\square$ represents the plaquette: the minimum loop possible on the lattice. Note that a link variable has a direction: the adjoint of a link variable is the link connecting the two sites in the opposite direction. In other words:

$$U_{x,\mu} = U^{\dagger}_{x+\hat{\mu},-\mu}. \tag{3.3}$$

Note that the lattice has periodic boundary conditions. In our case, the gauge group of choice is Sp(2).

3.1 Heat-bath algorithm

The heat-bath algorithm is used to calculate each step of the evolution of the system. The basic idea is to generate a new link element U with a Boltzmann probability distribution:

$$P(U) = \frac{1}{\mathcal{Z}} e^{-\beta S(U)} \, \mathrm{d}U \tag{3.4}$$

for each link variable in the system. Updating each link once represents a single step in the algorithm.

While this is fairly easy for the SU(2) gauge group using Creutz's algorithm [3], there is no obvious way to generalize it to other gauge groups, like SU(N) or Sp(N). Thus, to generate new links belonging to Sp(2), we use a more general approach, as designed by Cabibbo and Marinari [1].

We consider a set F of SU(2) subgroups of the gauge group Sp(2). Given a Sp(2) element U of the form

$$U = \begin{pmatrix} W_{11} & W_{12} & X_{11} & X_{12} \\ W_{21} & W_{22} & X_{21} & X_{22} \\ X^*_{22} & -X^*_{21} & W^*_{22} & -W^*_{21} \\ -X^*_{12} & X^*_{11} & -W^*_{12} & W^*_{11} \end{pmatrix}, \tag{3.5}$$

where $W_{ij}, X_{kl} \in \mathbb{C}$ for $i, j, k, l = 1, 2$, we can construct four SU(2) subgroups, by extracting two complex numbers t_1 and t_2:

- $\begin{cases} t_1 = W_{11} \\ t_2 = X_{12} \end{cases}$
- $\begin{cases} t_1 = W_{22} \\ t_2 = X_{21} \end{cases}$
- $\begin{cases} t_1 = W_{11} + W_{22} \\ t_2 = X_{11} - X_{22} \end{cases}$
- $\begin{cases} t_1 = W_{11} + W_{22}^* \\ t_2 = W_{12} - W_{21}^* \end{cases}$

We can then build an SU(2) group element as

$$a_k = \begin{pmatrix} t_1 & t_2 \\ -t_2^* & t_1^* \end{pmatrix}, \quad k = 1, 2, 3, 4, \tag{3.6}$$

where k labels each subgroup. Each choice of t_1 and t_2 above gives a different SU(2) element belonging to a SU(2) subgroup of Sp(2).

We define A_k to be an SU(2) element belonging to the kth SU(2) subgroup, embedded into Sp(2). For each subgroup in F, the Sp(2) embedding is constructed as follows:

- $A_1 = \begin{pmatrix} t_1 & 0 & 0 & t_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -t_2^* & 0 & 0 & t_1^* \end{pmatrix}$
- $A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t_1 & t_2 & 0 \\ 0 & -t_2^* & t_1^* & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
- $A_3 = \begin{pmatrix} t_1 & 0 & t_2 & 0 \\ 0 & t_1 & 0 & -t_2 \\ -t_2^* & 0 & t_1^* & 0 \\ 0 & t_2^* & 0 & t_1^* \end{pmatrix}$

- $A_4 = \begin{pmatrix} t_1 & t_2 & 0 & 0 \\ -t_2^* & t_1^* & 0 & 0 \\ 0 & 0 & t_1 & t_2 \\ 0 & 0 & -t_2^* & t_1^* \end{pmatrix}$

Generating each A_k randomly, we define the new link U' to be

$$U' = A_4 A_3 A_2 A_1 U, \tag{3.7}$$

because, in this case, the set F contains four SU(2) subgroups.

It is proven [1] that this algorithm leads to thermalization, if each A_k is randomly distributed as

$$P(A_k) = \mathrm{d}A_k \, \frac{e^{-\beta S(A_k U_{k-1})}}{\mathcal{Z}_k(U_{k-1})}, \tag{3.8}$$

where $U_n \equiv A_n A_{n-1} \dots A_1 U$ with $U_0 = U$ and

$$\mathcal{Z}_k(U) = \int_{SU(2)_k} \mathrm{d}A \, e^{-\beta S(AU)}. \tag{3.9}$$

The reason for the decomposition into SU(2) subgroups is to efficiently generate A_k according to (3.8). In fact, now that we are dealing with SU(2) elements, we can fall back to Creutz's algorithm [3] to generate each SU(2) element, embed it into Sp(2) as explained above, and left multiply the original link U by it.

Focusing on a single link U to update, we are interested only in the plaquettes that contain U. Defining $\widetilde{U}_i$ to be one of the staples surrounding U (an ordered product of the three links in the plaquettes containing U, that are not the link U itself), we have

$$S(A_k U) = -\frac{1}{4} \operatorname{Tr}\left(A_k U \sum_i \widetilde{U}_i \right) + \text{terms independent of } A_k \tag{3.10}$$

$$= -\frac{1}{4} \operatorname{Tr}\left(a_k u_k \sum_i \tilde{u}_k^i \right) + \text{terms independent of } a_k, \tag{3.11}$$

where a_k, u_k and $\tilde{u}_k$ are SU(2) elements corresponding to the kth subgroup extracted from A_k, U and $\widetilde{U}$, respectively. This implies that we want to generate a_k according to the distribution

$$\mathrm{d}P(a_k) \sim e^{\frac{1}{4}\beta \,\mathrm{Tr}\left(a_k u_k \sum_i \tilde{u}_k^i\right)} \,\mathrm{d}a_k \,. \tag{3.12}$$

We parametrize a_k as

$$a_k = \alpha_0 \mathbb{1} + i\vec{\alpha}\cdot\vec{\sigma}, \tag{3.13}$$

where $\alpha_\mu \in \mathbb{R}\ \forall \mu = 1,2,3,4$ with the constraint that

$$\alpha^2 \equiv \alpha_0^2 + |\vec{\alpha}|^2 = 1 \tag{3.14}$$

and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the three-vector of 2×2 Pauli matrices.

The SU(2) group measure is then

$$\mathrm{d}a_k = \frac{1}{2\pi^2}\delta(\alpha^2 - 1)\,\mathrm{d}^4\alpha \,. \tag{3.15}$$

Since the sum of SU(2) elements is proportional to an SU(2) element, we write

$$u_k \sum_i \tilde{u}_k^i = c\bar{u}_k, \quad \bar{u}_k \in \mathrm{SU}(2) \tag{3.16}$$

where

$$c = \det\left(u_k \sum_i \tilde{u}_k^i\right)^{1/2} . \tag{3.17}$$

The probability distribution for a_k now becomes

$$\mathrm{d}P(a_k) \sim e^{\frac{1}{4}\beta \,\mathrm{Tr}(c a_k \bar{u}_k)} \,\mathrm{d}a_k \,. \tag{3.18}$$

The group measure is invariant under multiplication by another SU(2) element:

$$\mathrm{d}(ba_k) = \mathrm{d}a_k \quad \text{for} \quad b \in \mathrm{SU}(2), \tag{3.19}$$

so we can write

$$\mathrm{d}P(a_k\bar{u}_k^{-1}) \sim e^{\frac{1}{4}\beta c\,\mathrm{Tr}(a_k)}\,\mathrm{d}a_k = \frac{1}{2\pi^2}e^{\frac{\beta}{2}c\alpha_0}\delta(\alpha^2-1)\,\mathrm{d}^4\alpha\,, \tag{3.20}$$

because $\mathrm{Tr}(a_k) = 2\alpha_0$. Noting that $\delta(\alpha^2-1)\,\mathrm{d}^4\alpha = \frac{1}{2}(1-\alpha_0^2)^{1/2}\,\mathrm{d}\alpha_0\,\mathrm{d}\Omega$, we rewrite $\mathrm{d}P(a_k\bar{u}_k^{-1})$ as

$$\mathrm{d}P(a_k\bar{u}_k^{-1}) \sim \frac{1}{2\pi^2}\frac{1}{2}(1-\alpha_0^2)^{1/2}e^{\frac{\beta}{2}c\alpha_0}\,\mathrm{d}\alpha_0\,\mathrm{d}\Omega \tag{3.21}$$

with $\alpha_0 \in (-1,1)$ and $\mathrm{d}\Omega$ is the differential solid angle of the three-vector $\vec{\alpha}$, which is of length $(1-\alpha_0^2)^{1/2}$.

The problem is now about generating the four-vector α_μ according to the distribution above, thus obtaining $a_k \in \mathrm{SU}(2)$. Finally, we obtain A_k by embedding $a_k\bar{u}_k^{-1} \in \mathrm{SU}(2)$ into $\mathrm{Sp}(2)$. Doing this for every $\mathrm{SU}(2)$ subgroup will yield the new link U'.

To generate a_k, we have to randomly generate α_0 according to

$$P(\alpha_0) \sim (1-\alpha_0^2)^{1/2}e^{\frac{\beta}{2}c\alpha_0}. \tag{3.22}$$

The algorithm is quite simple. We uniformly generate x in the range

$$e^{-\beta c} < x < 1 \tag{3.23}$$

and define a trial α_0 distributed according to $e^{\frac{\beta}{2}c\alpha_0}$ as

$$\alpha_0 = 1 + \frac{2}{\beta c}\ln x. \tag{3.24}$$

To account for the term $(1-\alpha_0^2)^{1/2}$ in (3.22), we *reject* this trial α_0 with probability $1-(1-\alpha_0^2)^{1/2}$, generating a new trial α_0 if the rejection is successful. We keep doing this until a trial α_0 is finally accepted.

The unit vector $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ is constructed by uniformly generating

$$\begin{aligned} \phi &\in (0, 2\pi), \\ y &\in (-1, 1) \end{aligned} \tag{3.25}$$

and defining

$$\begin{aligned} \theta &\equiv \arccos(y), \\ r &\equiv (1 - \alpha_0)^{1/2}, \end{aligned} \tag{3.26}$$

α_0 being the trial random number that has been accepted according to (3.8).

We finally have, for $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$,

$$\begin{cases} \alpha_1 = r \sin(\theta) \cos(\phi) \\ \alpha_2 = r \sin(\theta) \sin(\phi) \\ \alpha_3 = r \cos(\theta) \end{cases} . \tag{3.27}$$

a_k will then be constructed using (3.13), and A_k using the embeddings described above, with t_1 and t_2 selected from a_k as of (3.6).

3.2 Overrelaxation

The full algorithm implemented for the simulation also takes advantage of *overrelaxation* [6]. Overrelaxation is used to counter critical slowing down in the vicinity of the phase transition: here, the system's correlation length is so large that lattice updates of the order of the lattice spacing are negligible. This causes the simulation to slow down considerably. Overrelaxation is a way of choosing new link elements to update the lattice with, which prevents critical slowing down.

The idea is to choose a new link element U' "as far as possible" from U (the original link), without actually changing the action S of the system. [6] describes overrelaxation in the case of SU(2) gauge group. For more complex groups, it is not as straightforward to generate new links. Again, we use Cabibbo and Marinari's idea: the new link is chosen left multiplying the old link U by randomly generated A_k elements belonging to Sp(2), which

are in turn built embedding SU(2) elements into Sp(2). k labels the SU(2) subgroups and runs from 1 to 4, as for the heat-bath algorithm.

Note that overrelaxation is applied at each simulation step along with heat-bath: in fact, a single complete simulation step consists of one or more steps of overrelaxation and one step of heat-bath, where a step is the update of the entire lattice exactly once. In our case, overrelaxation is applied three times before each heat-bath step. This is possible because, as mentioned, overrelaxation doesn't actually change the action of the system.

Consider $\widetilde{U}_i$ to be one staple surrounding the link U, and let us define

$$R = \sum_{i=1}^{6} \widetilde{U}_i. \tag{3.28}$$

Evidently, there are six staples surrounding each link in a $(2+1)$ dimensional lattice, thus $i = 1, \ldots, 6$.

[6] defines the new link to be

$$U' = VU^{-1}V, \tag{3.29}$$

where

$$V \equiv \det(R)^{1/2} R^{-1}, \tag{3.30}$$

the inverse of the projection of R onto the SU(2) gauge group. In the case of the Sp(2) gauge group, we define

$$A_k U_{k-1} = V U_{k-1}^{-1} V, \tag{3.31}$$

instead, where

$$U_n \equiv A_n A_{n-1} \ldots A_1 U \quad \text{with} \quad U_0 = U. \tag{3.32}$$

Following Cabibbo and Marinari's prescription, the new link will be

$$U' = A_4 A_3 A_2 A_1 U \tag{3.33}$$

in the case of four SU(2) subgroups.

We then extract the SU(2) elements corresponding to the kth subgroup of Sp(2) from each term in (3.31):

$$\begin{cases} A_k \in \mathrm{Sp}(2) & \longrightarrow a_k \in \mathrm{SU}(2) \\ U_{k-1} \in \mathrm{Sp}(2) & \longrightarrow u_k \in \mathrm{SU}(2) \\ \widetilde{U}_i \in \mathrm{Sp}(2) & \longrightarrow r_k^{(i)} \in \mathrm{SU}(2) \end{cases} . \tag{3.34}$$

We also define

$$r_k \equiv \sum_i r_k^{(i)}. \tag{3.35}$$

Using the fact that a sum of SU(2) elements is proportional to a SU(2) element, we can write

$$u_k r_k = u_k \sum_i r_k^{(i)} = c\bar{u}_k \tag{3.36}$$

with $c = \det(r_k)^{1/2}$ and $\bar{u}_k \in \mathrm{SU}(2)$.

Rewriting (3.31) as

$$A_k = V U^{-1} V U^{-1} \tag{3.37}$$

and applying the equation to the SU(2) elements extracted above, we end up with

$$\begin{aligned} a_k &= \det(r_k) r_k^{-1} u_k^{-1} r_k^{-1} u_k^{-1} \\ &= \det(r_k)(u_k r_k)^{-1}(u_k r_k)^{-1} \\ &= \det(r_k)(r_k u_k)^{-2} \\ &= \det(r_k)(c\bar{u}_k)^{-2}. \end{aligned} \tag{3.38}$$

Given that $c = \det(r_k)^{1/2}$, we finally have

$$a_k = \bar{u}_k^{-2} = \left(\frac{1}{c} u_k \sum_i r_k^{(i)} \right)^{-2} . \tag{3.39}$$

Exactly as with the heat-bath algorithm, once we find a_k, we can embed it into Sp(2) according to the subgroup it belongs to, which yields the corresponding A_k, and find the new link using (3.33).

4 Results

Put plots and results of the fit and simulations: fit of susceptibility peaks, fit of beta vs nt, etc.

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