Abstract

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1 Introduction

Brief intro about the paper. Past work, theory, references.

2 Theoretical background

Explain the concepts of lattice gauge theory, Wilson and Polyakov loops, center symmetry and Sp(2) group. Mostly from Pepe's work. Read the paper by Caselle to see if there is something to pure here (SY conjecture? String tension? Universality with spin model? EST?)

3 Simulation and algorithm

Define the partition function of the $(2+1)$ dimensional system to be

$$\mathcal{Z} = \int \mathcal{D}U e^{-\beta S(U)} \quad \text{where} \quad \int \mathcal{D}U \equiv \prod_{x,\mu} \int \mathrm{d}U_{x,\mu}\,. \tag{3.1}$$

$U_{x,\mu}$ is an element of the group (a link variable starting at the lattice site x, pointing along the direction μ), and S is the action, defined as

$$S = \sum_{\square} S_{\square}, \quad S_{\square} \equiv -\frac{1}{4}\,\mathrm{Tr}\Big(U_{x,\mu}U_{x+\hat{\mu},\nu}U^{\dagger}_{x+\hat{\nu},\mu}U^{\dagger}_{x,\nu}\Big). \tag{3.2}$$

The symbol $\square$ represents the plaquette: the minimum loop possible on the lattice. Note that a link variable has a direction: the adjoint of a link variable is the link connecting the two sites in the opposite direction. In other words:

$$U_{x,\mu} = U^{\dagger}_{x+\hat{\mu},-\mu}. \tag{3.3}$$

Note that the lattice has periodic boundary conditions. In our case, the gauge group of choice is Sp(2).

3.1 Heat-bath algorithm

The heat-bath algorithm is used to calculate each step of the evolution of the system. The basic idea is to generate a new link element U with a Boltzmann probability distribution:

$$P(U) = \frac{1}{\mathcal{Z}} e^{-\beta S(U)} \, \mathrm{d}U \tag{3.4}$$

for each link variable in the system. Updating each link once represents a single step in the algorithm.

While this is fairly easy for the SU(2) gauge group using Creutz's algorithm [2], there is no obvious way to generalize it to other gauge groups, like SU(N) or Sp(N). Thus, to generate new links belonging to Sp(2), we use a more general approach, as designed by Cabibbo and Marinari [1].

We consider a set F of SU(2) subgroups of the gauge group Sp(2). Given a Sp(2) element U of the form

$$U = \begin{pmatrix} W_{11} & W_{12} & X_{11} & X_{12} \\ W_{21} & W_{22} & X_{21} & X_{22} \\ X_{22}^* & -X_{21}^* & W_{22}^* & -W_{21}^* \\ -X_{12}^* & X_{11}^* & -W_{12}^* & W_{11}^* \end{pmatrix}, \tag{3.5}$$

where $W_{ij}, X_{kl} \in \mathbb{C}$ for $i, j, k, l = 1, 2$, we can construct four SU(2) subgroups, by extracting two complex numbers t_1 and t_2:

- $\begin{cases} t_1 = W_{11} \\ t_2 = X_{12} \end{cases}$
- $\begin{cases} t_1 = W_{22} \\ t_2 = X_{21} \end{cases}$
- $\begin{cases} t_1 = W_{11} + W_{22} \\ t_2 = X_{11} - X_{22} \end{cases}$
- $\begin{cases} t_1 = W_{11} + W_{22}^* \\ t_2 = W_{12} - W_{21}^* \end{cases}$

We can then build an SU(2) group element as

$$a_k = \begin{pmatrix} t_1 & t_2 \\ -t_2^* & t_1^* \end{pmatrix}, \quad k = 1, 2, 3, 4, \tag{3.6}$$

where k labels each subgroup. Each choice of t_1 and t_2 above gives a different SU(2) element belonging to a SU(2) subgroup of Sp(2).

We define A_k to be an SU(2) element belonging to the kth SU(2) subgroup, embedded into Sp(2). For each subgroup in F, the Sp(2) embedding is constructed as follows:

- $A_1 = \begin{pmatrix} t_1 & 0 & 0 & t_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -t_2^* & 0 & 0 & t_1^* \end{pmatrix}$
- $A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t_1 & t_2 & 0 \\ 0 & -t_2^* & t_1^* & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
- $A_3 = \begin{pmatrix} t_1 & 0 & t_2 & 0 \\ 0 & t_1 & 0 & -t_2 \\ -t_2^* & 0 & t_1^* & 0 \\ 0 & t_2^* & 0 & t_1^* \end{pmatrix}$
- $A_4 = \begin{pmatrix} t_1 & t_2 & 0 & 0 \\ -t_2^* & t_1^* & 0 & 0 \\ 0 & 0 & t_1 & t_2 \\ 0 & 0 & -t_2^* & t_1^* \end{pmatrix}$

Generating each A_k randomly, we define the new link U' to be

$$U' = A_4 A_3 A_2 A_1 U, \tag{3.7}$$

because, in this case, the set F contains four SU(2) subgroups.

It is proven [1] that this algorithm leads to thermalization, if each A_k is randomly distributed as

$$P(A_k) = \mathrm{d}A_k \, \frac{e^{-\beta S(A_k U_{k-1})}}{\mathcal{Z}_k(U_{k-1})}, \tag{3.8}$$

where $U_n \equiv A_n A_{n-1} \dots A_1 U$ with $U_0 = U$ and

$$\mathcal{Z}_k(U) = \int_{SU(2)_k} \mathrm{d}A \, e^{-\beta S(AU)}. \tag{3.9}$$

The reason for the decomposition into SU(2) subgroups is to efficiently generate A_k according to (3.8). In fact, now that we are dealing with SU(2) elements, we can fall back to Creutz's algorithm [2] to generate each SU(2) element, embed it into Sp(2) as explained above, and left multiply the original link U by it.

Focusing on a single link U to update, we are interested only in the plaquettes that contain U. Defining $\widetilde{U}_i$ to be one of the staples surrounding U (an ordered product of the three links in the plaquettes containing U, that are not the link U itself), we have

$$S(A_k U) = -\frac{1}{4} \operatorname{Tr}\left(A_k U \sum_i \widetilde{U}_i \right) + \text{terms independent of } A_k \tag{3.10}$$

$$= -\frac{1}{4} \operatorname{Tr}\left(a_k u_k \sum_i \tilde{u}_k^i \right) + \text{terms independent of } a_k, \tag{3.11}$$

where a_k, u_k and $\tilde{u}_k$ are SU(2) elements corresponding to the kth subgroup extracted from A_k, U and $\widetilde{U}$, respectively. This implies that we want to generate a_k according to the distribution

$$\mathrm{d}P(a_k) \sim e^{\frac{1}{4}\beta \operatorname{Tr}\left(a_k u_k \sum_i \tilde{u}_k^i\right)} \, \mathrm{d}a_k \, . \tag{3.12}$$

We parametrize a_k as

$$a_k = \alpha_0 \mathbb{1} + i \vec{\alpha} \cdot \vec{\sigma}, \tag{3.13}$$

where $\alpha_\mu \in \mathbb{C}\ \forall \mu = 1,2,3,4$ with the constraint that

$$\alpha^2 \equiv \alpha_0^2 + |\vec{\alpha}|^2 = 1 \tag{3.14}$$

and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the three-vector of 2×2 Pauli matrices.

The SU(2) group measure is then

$$\mathrm{d}a_k = \frac{1}{2\pi^2}\delta(\alpha^2 - 1)\,\mathrm{d}^4\alpha\,. \tag{3.15}$$

Since the sum of SU(2) elements is proportional to an SU(2) element, we write

$$u_k \sum_i \tilde{u}_k^i = c\bar{u}_k, \quad \bar{u}_k \in \mathrm{SU}(2) \tag{3.16}$$

where

$$c = \det\left(u_k \sum_i \tilde{u}_k^i\right)^{1/2}. \tag{3.17}$$

The probability distribution for a_k now becomes

$$\mathrm{d}P(a_k) \sim e^{\frac{1}{4}\beta\,\mathrm{Tr}(ca_k\bar{u}_k)}\,\mathrm{d}a_k\,. \tag{3.18}$$

The group measure is invariant under multiplication by another SU(2) element:

$$\mathrm{d}(ba_k) = \mathrm{d}a_k \quad \text{for} \quad b \in \mathrm{SU}(2), \tag{3.19}$$

so we can write

$$\mathrm{d}P(a_k\bar{u}_k^{-1}) \sim e^{\frac{1}{4}\beta c\,\mathrm{Tr}(a_k)}\,\mathrm{d}a_k = \frac{1}{2\pi^2}e^{\frac{\beta}{2}c\alpha_0}\delta(\alpha^2 - 1)\,\mathrm{d}^4\alpha\,, \tag{3.20}$$

because $\mathrm{Tr}(a_k) = 2\alpha_0$. Noting that $\delta(\alpha^2 - 1)\,\mathrm{d}^4\alpha = \frac{1}{2}(1 - \alpha_0^2)^{1/2}\,\mathrm{d}\alpha_0\,\mathrm{d}\Omega$, we rewrite $\mathrm{d}P(a_k\bar{u}_k^{-1})$ as

$$\mathrm{d}P(a_k\bar{u}_k^{-1}) \sim \frac{1}{2\pi^2}\frac{1}{2}(1 - \alpha_0^2)^{1/2}e^{\frac{\beta}{2}c\alpha_0}\,\mathrm{d}\alpha_0\,\mathrm{d}\Omega \tag{3.21}$$

with $\alpha_0 \in (-1, 1)$ and $\mathrm{d}\Omega$ being the differential solid angle of the three-vector $\vec{a}$, which is of length $(1 - \alpha_0^2)^{1/2}$.

The problem is now generating the four-vector α_μ according to the distribution above, thus obtaining $a_k \in \mathrm{SU}(2)$. Finally, we obtain A_k by embedding $a_k \bar{u}_k^{-1} \in \mathrm{SU}(2)$ into $\mathrm{Sp}(2)$. Doing this for every $\mathrm{SU}(2)$ subgroup will yield the new link U'.

To generate a_k, we have to randomly generate α_0 according to

$$P(\alpha_0) \sim (1 - \alpha_0^2)^{1/2} e^{\frac{\beta}{2} c \alpha_0}. \tag{3.22}$$

The algorithm is quite simple. We uniformly generate x in the range

$$e^{-\beta c} < x < 1 \tag{3.23}$$

and define a trial α_0 distributed according to $e^{\frac{\beta}{2} c \alpha_0}$ as

$$\alpha_0 = 1 + \frac{2}{\beta c} \ln x. \tag{3.24}$$

To account for the term $(1 - \alpha_0^2)^{1/2}$ in (3.22), we *reject* this trial α_0 with probability $1 - (1 - \alpha_0^2)^{1/2}$, generating a new trial α_0 if the rejection is successful. We keep doing this until a trial α_0 is finally accepted.

The unit vector $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ is constructed by uniformly generating

$$\begin{aligned} 0 <& \phi < 2\pi, \\ -1 <& y < 1 \end{aligned} \tag{3.25}$$

and defining

$$\begin{aligned} \theta &\equiv \arccos(y), \\ r &\equiv (1 - \alpha_0)^{1/2}, \end{aligned} \tag{3.26}$$

Of course, α_0 being the trial random number that has been accepted according to (3.8).

We finally have, for $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$,

$$\begin{cases} \alpha_1 = r\sin(\theta)\cos(\phi) \\ \alpha_2 = r\sin(\theta)\sin(\phi) \\ \alpha_3 = r\cos(\theta) \end{cases} . \tag{3.27}$$

a_k will then be constructed using (3.13), and A_k using the embeddings described above, with t_1 and t_2 selected from a_k as of (3.6).

3.2 Overrelaxation

The full algorithm implemented for the simulation also takes advantage of *overrelaxation* [3]. Overrelaxation is used to counter critical slowing down in the vicinity of the phase transition: here, the system's correlation length is so large that lattice updates of the order of the lattice spacing are negligible. This causes the simulation to slow down considerably. Overrelaxation is a way of choosing new link elements to update the lattice with, which prevents critical slowing down.

The idea is to choose a new link element U' "as far as possible" from U (the original link), without actually changing the action S of the system. [3] describes overrelaxation in the case of SU(2) gauge group. For more complex groups, it is not as straightforward to generate new links. Again, we use Cabibbo and Marinari's idea: the new link is chosen left multiplying the old U link by randomly generated A_k elements which belongs to Sp(2), which are in turn built embedding SU(2) elements into Sp(2). k labels the SU(2) subgroups and runs from 1 to 4, as above.

Note that overrelaxation is applied at each simulation step along with heat-bath: in fact, a single complete simulation step consists of one or more steps of overrelaxation and one step of heat-bath, where a step is the update of the entire lattice exactly once. In our case, overrelaxation is applied three times before each heat-bath step. This is possible because, as noted above, overrelaxation doesn't actually change the action of the system.

Consider $\widetilde{U}_i$ to be one staple surrounding the link U, and let us define

$$R = \sum_{i=1}^{6} \widetilde{U}_i. \tag{3.28}$$

Evidently, there are six staples surrounding each link in a $(2+1)$ dimensional lattice, thus $i = 1, \ldots, 6$.

[3] defines the new link to be

$$U' = VU^{-1}V, \tag{3.29}$$

where

$$V \equiv \det(R)^{\,1/2}\, R^{-1}, \tag{3.30}$$

the inverse of the projection of R onto the SU(2) gauge group. In the case of the Sp(2) gauge group, we define

$$A_k U_{k-1} = V U_{k-1}^{-1} V, \tag{3.31}$$

instead, where

$$U_n \equiv A_n A_{n-1} \ldots A_1 U \quad \text{with} \quad U_0 = U. \tag{3.32}$$

Following Cabibbo and Marinari's prescription, the new link will be

$$U' = A_4 A_3 A_2 A_1 U \tag{3.33}$$

in the case of four SU(2) subgroups.

We then extract the SU(2) elements corresponding to the kth subgroup of Sp(2) from each term in (3.31):

$$\begin{cases} A_k \in \mathrm{Sp}(2) & \longrightarrow a_k \in \mathrm{SU}(2) \\ U_{k-1} \in \mathrm{Sp}(2) & \longrightarrow u_k \in \mathrm{SU}(2) \\ \widetilde{U}_i \in \mathrm{Sp}(2) & \longrightarrow r_k^{(i)} \in \mathrm{SU}(2) \end{cases} . \tag{3.34}$$

We also define

$$r_k \equiv \sum_i r_k^{(i)}. \tag{3.35}$$

Using the fact that a sum of SU(2) elements is proportional to a SU(2) element, we can write

$$u_k r_k = u_k \sum_i r_k^{(i)} = c\bar{u}_k, \quad c = \det(r_k)^{1/2}. \tag{3.36}$$

Rewriting (3.31) as

$$A_k = VU^{-1}VU^{-1} \tag{3.37}$$

and applying the equation to the SU(2) elements extracted above, we end up with

$$\begin{aligned} a_k &= \det(r_k) r_k^{-1} u_k^{-1} r_k^{-1} u_k^{-1} \\ &= \det(r_k)(u_k r_k)^{-1}(u_k r_k)^{-1} \\ &= \det(r_k)(r_k u_k)^{-2} \\ &= \det(r_k)(c\bar{u}_k)^{-2}. \end{aligned} \tag{3.38}$$

Given that $c = \det(r_k)^{1/2}$, we finally have

$$a_k = \bar{u}_k^{-2} = \left(\frac{1}{c} u_k \sum_i r_k^{(i)} \right)^{-2}. \tag{3.39}$$

Exactly as with the heat-bath algorithm, once we find a_k, we can embed it into Sp(2) according to the subgroup it belongs to, which yields the corresponding A_k, and find the new link using (3.33).

4 Results

Put plots and results of the fit and simulations: fit of susceptibility peaks, fit of beta vs nt, etc.

References

[1] Nicola Cabibbo and Enzo Marinari. "A new method for updating SU(N) matrices in computer simulations of gauge theories". In: *Physics Letters B* 119.4 (1982), pp. 387–390. ISSN: 0370-2693. DOI: https://doi.org/10.1016/0370-2693(82)90696-7. URL: https://www.sciencedirect.com/science/article/pii/0370269382906967.

[2] Michael Creutz. "Monte Carlo study of quantized SU(2) gauge theory". In: *Phys. Rev. D* 21 (8 Apr. 1980), pp. 2308–2315. DOI: 10.1103/PhysRevD.21.2308. URL: https://link.aps.org/doi/10.1103/PhysRevD.21.2308.

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