

## Basic Properties

*Theorem 0.1:*

$$\int_0^{2\pi} e^{imx} dx = \begin{cases} 2\pi(m=0) \\ 0 \quad (m \in \mathbb{Z} \setminus \{0\}) \end{cases} \quad (1.1)$$

$$\text{p.v.} \int_0^{2\pi} \cot\left(\frac{x}{2}\right) e^{imx} dx = 2\pi i \operatorname{sign}(m) \quad m \in \mathbb{Z} \setminus \{0\} \quad (1.2)$$

$$\int_0^{2\pi} \log\left(4 \sin^2 \frac{x}{2}\right) e^{imx} dx = \begin{cases} 0 & (m=0) \\ -\frac{2\pi}{|m|} & (m \in \mathbb{Z} \setminus \{0\}) \end{cases} \quad (1.3)$$

*Theorem 0.2:*  $\forall N' \in \mathbb{N}. t_j := \frac{2\pi j}{N'}, \forall m \in \mathbb{Z}.$

$$\frac{2\pi}{N'} \sum_{j=0}^{N'-1} e^{imt_j} = \begin{cases} 2\pi(m \equiv 0 \bmod N') \\ 0 \quad (\text{otherwise}) \end{cases} \quad (2)$$

## Subspace $U_N$

*Definition 0.1:*  $U_N := \operatorname{span}(\{e^{imx} \mid m \in \mathbb{Z}, |m| < N\})$

*Theorem 0.3 (Trapezoidal Rule for  $U_N$ ):*  $\forall N \in \mathbb{N}. N\text{-point trapezoidal rule is exact for } U_N.$

*Theorem 0.4:* Let  $N' := \dim U_N = 2N - 1$ . Let  $t_j := \frac{2\pi j}{N'}$  for  $j = 0, \dots, N' - 1$ .  $\forall f, g \in U_N$ .

$$\int_0^{2\pi} f(t)g(t) dt = \frac{2\pi}{N'} \sum_{j=0}^{N'-1} f(t_j)g(t_j) \quad (3)$$

*Proof:*  $f(t)g(t) \in U_{2N-1}$  □

## Quadratures

*Theorem 0.5:*  $\forall N \in \mathbb{N}. \forall f \in U_N.$

$$f(x) = \sum_{j=0}^{N'-1} f(t_j) \cdot \left( \frac{1}{N'} \sum_{|m| < N} e^{-imt_j} e^{imx} \right) \quad (4)$$

*Proof:*

$$f(x) = \sum_{|m| < N} \int_0^{2\pi} f(t) \frac{e^{-imt}}{\sqrt{2\pi}} dt \cdot \frac{e^{imx}}{\sqrt{2\pi}} \quad (5.1)$$

$$= \frac{1}{2\pi} \sum_{|m| < N} \frac{2\pi}{N'} \sum_{j=0}^{N'-1} f(t_j) e^{-imt_j} \cdot e^{imx} \quad (5.2)$$

$$= \sum_{j=0}^{N'-1} f(t_j) \cdot \left( \frac{1}{N'} \sum_{|m| < N} e^{-imt_j} e^{imx} \right) \quad (5.3)$$

□

*Theorem 0.6 (Kusmaul–Martensen (Kress) quadrature):*  $\forall N \in \mathbb{N}. \forall f \in U_N.$

$$\int_0^{2\pi} \log\left(4 \sin^2 \frac{x}{2}\right) f(t) dt = \sum_{j=0}^{N'-1} f(t_j) \cdot \left( -\frac{4\pi}{N'} \sum_{m=1}^{N-1} \frac{\cos mt_j}{m} \right) \quad (6)$$

*Proof:*

$$\int_0^{2\pi} \log\left(4 \sin^2 \frac{x}{2}\right) f(t) \, dt = \int_0^{2\pi} \log\left(4 \sin^2 \frac{x}{2}\right) \left( \sum_{j=0}^{N'-1} f(t_j) \cdot \left( \frac{1}{N'} \sum_{|m|<N} e^{-imt_j} e^{imt} \right) \right) dt \quad (7.1)$$

$$= \sum_{j=0}^{N'-1} f(t_j) \cdot \left( \frac{1}{N'} \sum_{|m|<N} e^{-imt_j} \int_0^{2\pi} \log\left(4 \sin^2 \frac{x}{2}\right) e^{imt} \, dt \right) \quad (7.2)$$

$$= \sum_{j=0}^{N'-1} f(t_j) \cdot \left( \frac{1}{N'} \sum_{0<|m|<N} \left( -\frac{2\pi}{|m|} \right) e^{-imt_j} \right) \quad (7.3)$$

$$= \sum_{j=0}^{N'-1} f(t_j) \cdot \left( -\frac{4\pi}{N'} \sum_{m=1}^{N-1} \frac{\cos mt_j}{m} \right) \quad (7.4)$$

□