

## Basic Properties

Lemma 0.1:

$$\int_0^{2\pi} e^{imx} dx = \begin{cases} 2\pi(m=0) \\ 0 \quad (m \in \mathbb{Z} \setminus \{0\}) \end{cases} \quad (1.1)$$

$$\text{p.v.} \int_0^{2\pi} \cot\left(\frac{x}{2}\right) e^{imx} dx = 2\pi i \operatorname{sign}(m) \quad m \in \mathbb{Z} \setminus \{0\} \quad (1.2)$$

$$\int_0^{2\pi} \log\left(4 \sin^2 \frac{x}{2}\right) e^{imx} dx = \begin{cases} 0 & (m=0) \\ -\frac{2\pi}{|m|} & (m \in \mathbb{Z} \setminus \{0\}) \end{cases} \quad (1.3)$$

Lemma 0.2:  $\forall N' \in \mathbb{N}. t_j := \frac{2\pi j}{N'}. \forall m \in \mathbb{Z}.$

$$\frac{2\pi}{N'} \sum_{j=0}^{N'-1} e^{imt_j} = \begin{cases} 2\pi(m \equiv 0 \bmod N') \\ 0 \quad (\text{otherwise}) \end{cases} \quad (2)$$

## Subspace $U_N$

Definition 0.1:  $U_N := \operatorname{span}(\{e^{imx} \mid m \in \mathbb{Z}, |m| < N\})$

Theorem 0.1 (Trapezoidal Rule for  $U_N$ ):  $\forall N \in \mathbb{N}. N$ -point trapezoidal rule is exact for  $U_N$ .

Proof: Lemma 0.2 □

Lemma 0.3 (Trapezoidal Rule for inner product for  $U_N$ ): Let  $N' := \dim U_N = 2N - 1$ . Let  $t_j := \frac{2\pi j}{N'}$  for  $j = 0, \dots, N' - 1$ .  $\forall f, g \in U_N$ .

$$\int_0^{2\pi} f(t)g(t) dt = \frac{2\pi}{N'} \sum_{j=0}^{N'-1} f(t_j)g(t_j) \quad (3)$$

Proof: Since,  $f(t)g(t) \in U_{2N-1}$  Theorem 0.1 can be applied with  $N' = 2N - 1$  points. □

## Quadratures

Lemma 0.4:  $\forall N \in \mathbb{N}. \forall f \in U_N$ .

$$f(x) = \sum_{j=0}^{N'-1} f(t_j) \cdot \left( \frac{1}{N'} \sum_{|m| < N} e^{-imt_j} e^{imx} \right) \quad (4)$$

Proof:

$$f(x) = \sum_{|m| < N} \int_0^{2\pi} f(t) \frac{e^{-imt}}{\sqrt{2\pi}} dt \cdot \frac{e^{imx}}{\sqrt{2\pi}} \quad (5.1)$$

$$\stackrel{\text{Lemma 0.3}}{=} \frac{1}{2\pi} \sum_{|m| < N} \frac{2\pi}{N'} \sum_{j=0}^{N'-1} f(t_j) e^{-imt_j} \cdot e^{imx} \quad (5.2)$$

$$= \sum_{j=0}^{N'-1} f(t_j) \cdot \left( \frac{1}{N'} \sum_{|m| < N} e^{-imt_j} e^{imx} \right) \quad (5.3)$$

□

Theorem 0.2 (Kusmaul–Martensen (Kress) quadrature for  $U_N$ ):  $\forall N \in \mathbb{N}. \forall f \in U_N$ .

$$\int_0^{2\pi} \log\left(4 \sin^2 \frac{x}{2}\right) f(t) dt = \sum_{j=0}^{N'-1} f(t_j) \cdot \left(-\frac{4\pi}{N'} \sum_{m=1}^{N-1} \frac{\cos mt_j}{m}\right) \quad (6)$$

*Proof:*

$$\begin{aligned} \int_0^{2\pi} \log\left(4 \sin^2 \frac{x}{2}\right) f(t) dt &\stackrel{\text{Lemma 0.4}}{=} \int_0^{2\pi} \log\left(4 \sin^2 \frac{x}{2}\right) \left(\sum_{j=0}^{N'-1} f(t_j) \cdot \left(\frac{1}{N'} \sum_{|m|<N} e^{-imt_j} e^{imt}\right)\right) dt \\ &= \sum_{j=0}^{N'-1} f(t_j) \cdot \left(\frac{1}{N'} \sum_{|m|<N} e^{-imt_j} \int_0^{2\pi} \log\left(4 \sin^2 \frac{x}{2}\right) e^{imt} dt\right) \end{aligned} \quad (7.2)$$

$$= \sum_{j=0}^{N'-1} f(t_j) \cdot \left(\frac{1}{N'} \sum_{0<|m|<N} \left(-\frac{2\pi}{|m|}\right) e^{-imt_j}\right) \quad (7.3)$$

$$= \sum_{j=0}^{N'-1} f(t_j) \cdot \left(-\frac{4\pi}{N'} \sum_{m=1}^{N-1} \frac{\cos mt_j}{m}\right) \quad (7.4)$$

□

*Theorem 0.3 (Garrick–Wittich quadrature for  $U_N$ ):*  $\forall N \in \mathbb{N}. \forall f \in U_N$ .

$$p.v. \int_0^{2\pi} \cot\left(\frac{x}{2}\right) f'(t) dt = \sum_{j=0}^{N'-1} f(t_j) \cdot \left(-\frac{4\pi}{N'} \sum_{m=1}^{N-1} m \cos(mt_j)\right) \quad (8)$$

*Proof:*

$$\int_0^{2\pi} \cot\left(\frac{x}{2}\right) f'(t) dt \stackrel{\text{Lemma 0.4}}{=} \int_0^{2\pi} \cot\left(\frac{x}{2}\right) \left(\sum_{j=0}^{N'-1} f(t_j) \cdot \left(\frac{1}{N'} \sum_{|m|<N} e^{-imt_j} (e^{imt})'\right)\right) dt$$

$$= \int_0^{2\pi} \cot\left(\frac{x}{2}\right) \left(\sum_{j=0}^{N'-1} f(t_j) \cdot \left(\frac{1}{N'} \sum_{0<|m|<N} e^{-imt_j} (ime^{imt})'\right)\right) dt \quad (9.2)$$

$$= \sum_{j=0}^{N'-1} f(t_j) \cdot \left(\frac{1}{N'} \sum_{0<|m|<N} e^{-imt_j} \left(im \int_0^{2\pi} \cot\left(\frac{x}{2}\right) e^{imt} dt\right)\right) \quad (9.3)$$

$$= \sum_{j=0}^{N'-1} f(t_j) \cdot \left(\frac{1}{N'} \sum_{0<|m|<N} e^{-imt_j} (im \cdot 2\pi i \operatorname{sign}(m))\right) \quad (9.4)$$

$$= \sum_{j=0}^{N'-1} f(t_j) \cdot \left(\frac{1}{N'} \sum_{0<|m|<N} (-2\pi|m|) e^{-imt_j}\right) \quad (9.5)$$

$$= \sum_{j=0}^{N'-1} f(t_j) \cdot \left(-\frac{4\pi}{N'} \sum_{m=1}^{N-1} m \cos(mt_j)\right) \quad (9.6)$$

□