Basic Properties

Lemma 0.1:

$$\int_0^{2\pi} e^{imx}\,\mathrm{d}x = \begin{cases} 2\pi (m = 0) \\ 0 \quad (m \in \mathbb{Z} \setminus \{0\}) \end{cases} \tag{1.1}$$

$$p.v. \int_0^{2\pi} \cot\left(\frac{x}{2}\right) e^{imx}\,\mathrm{d}x = 2\pi i \operatorname{sign}(m) \quad m \in \mathbb{Z} \setminus \{0\} \tag{1.2}$$

$$\int_0^{2\pi} \log\left(4 \sin^2 \frac{x}{2}\right) e^{imx}\,\mathrm{d}x = \begin{cases} 0 \quad (m = 0) \\ -\frac{2\pi}{|m|} (m \in \mathbb{Z} \setminus \{0\}) \end{cases} \tag{1.3}$$

Lemma 0.2: $\forall N' \in \mathbb{N}. t_j := \frac{2\pi j}{N'}. \forall m \in \mathbb{Z}.$

$$\frac{2\pi}{N'} \sum_{j=0}^{N'-1} e^{imt_j} = \begin{cases} 2\pi (m \equiv 0 \bmod N') \\ 0 \quad (\text{otherwise}) \end{cases} \tag{2}$$

Subspace U_N

Definition 0.1: $U_N := \operatorname{span}(\{e^{imx} \mid m \in \mathbb{Z}, |m| < N\})$

Theorem 0.1 (Trapezoidal Rule for U_N): $\forall N \in \mathbb{N}. N$-point trapezoidal rule is exact for U_N.

Proof: Lemma 0.2 □

Lemma 0.3 (Trapezoidal Rule for inner product for U_N): Let $N' := \dim U_N = 2N - 1$. Let $t_j := \frac{2\pi j}{N'}$ for $j = 0, ..., N' - 1$. $\forall f, g \in U_N$.

$$\int_0^{2\pi} f(t) g(t)\,\mathrm{d}t = \frac{2\pi}{N'} \sum_{j=0}^{N'-1} f(t_j) \tag{3}$$

Proof: Since, $f(t)g(t) \in U_{2N-1}$ Theorem 0.1 can be applied with $N' = 2N - 1$ points. □

Quadratures

Lemma 0.4: $\forall N \in \mathbb{N}. \forall f \in U_N.$

$$f(x) = \sum_{j=0}^{N'-1} f(t_j) \cdot \left(\frac{1}{N'} \sum_{|m|<N} e^{-imt_j} e^{imx} \right) \tag{4}$$

Proof:

$$f(x) = \sum_{|m|<N} \int_0^{2\pi} f(t) \frac{e^{-imt}}{\sqrt{2\pi}}\,\mathrm{d}t \cdot \frac{e^{imx}}{\sqrt{2\pi}} \tag{5.1}$$

$$\underset{\because \text{ Lemma 0.3}}{=} \frac{1}{2\pi} \sum_{|m|<N} \frac{2\pi}{N'} \sum_{j=0}^{N'-1} f(t_j) e^{-imt_j} \cdot e^{imx} \tag{5.2}$$

$$= \sum_{j=0}^{N'-1} f(t_j) \cdot \left(\frac{1}{N'} \sum_{|m|<N} e^{-imt_j} e^{imx} \right) \tag{5.3}$$

□

Theorem 0.2 (Kussmaul–Martensen (Kress) quadrature for U_N): $\forall N \in \mathbb{N}. \forall f \in U_N.$

$$\int_0^{2\pi} \log\left(4\sin^2\frac{t}{2}\right) f(t)\,\mathrm{d}t = \sum_{j=0}^{N'-1} f(t_j) \cdot \left(-\frac{4\pi}{N'}\sum_{m=1}^{N-1}\frac{\cos mt_j}{m}\right) \tag{6}$$

Proof:

$$\int_0^{2\pi} \log\left(4\sin^2\frac{t}{2}\right) f(t)\,\mathrm{d}t \underset{\because\ \text{Lemma } 0.4}{=} \int_0^{2\pi} \log\left(4\sin^2\frac{t}{2}\right)\left(\sum_{j=0}^{N'-1} f(t_j)\cdot\left(\frac{1}{N'}\sum_{|m|<N} e^{-imt_j}e^{imt}\right)\right)\mathrm{d}t \tag{7.1}$$

$$= \sum_{j=0}^{N'-1} f(t_j)\cdot\left(\frac{1}{N'}\sum_{|m|<N} e^{-imt_j}\int_0^{2\pi}\log\left(4\sin^2\frac{t}{2}\right)e^{imt}\,\mathrm{d}t\right) \tag{7.2}$$

$$= \sum_{j=0}^{N'-1} f(t_j)\cdot\left(\frac{1}{N'}\sum_{0<|m|<N}\left(-\frac{2\pi}{|m|}\right)e^{-imt_j}\right) \tag{7.3}$$

$$= \sum_{j=0}^{N'-1} f(t_j)\cdot\left(-\frac{4\pi}{N'}\sum_{m=1}^{N-1}\frac{\cos mt_j}{m}\right) \tag{7.4}$$

$\square$

Theorem 0.3 (Garrick–Wittich quadrature for U_N): $\forall N\in\mathbb{N}.\forall f\in U_N$.

$$p.v.\int_0^{2\pi}\cot\left(\frac{t}{2}\right)f'(t)\,\mathrm{d}t = \sum_{j=0}^{N'-1} f(t_j)\cdot\left(-\frac{4\pi}{N'}\sum_{m=1}^{N-1} m\cos(mt_j)\right) \tag{8}$$

Proof:

$$p.v.\int_0^{2\pi}\cot\left(\frac{t}{2}\right)f'(t)\,\mathrm{d}t \underset{\because\ \text{Lemma } 0.4}{=} p.v.\int_0^{2\pi}\cot\left(\frac{t}{2}\right)\left(\sum_{j=0}^{N'-1} f(t_j)\cdot\left(\frac{1}{N'}\sum_{|m|<N} e^{-imt_j}\left(e^{imt}\right)'\right)\right)\mathrm{d}t \tag{9.1}$$

$$= p.v.\int_0^{2\pi}\cot\left(\frac{t}{2}\right)\left(\sum_{j=0}^{N'-1} f(t_j)\cdot\left(\frac{1}{N'}\sum_{0<|m|<N} e^{-imt_j}\left(ime^{imt}\right)\right)\right)\mathrm{d}t \tag{9.2}$$

$$= \sum_{j=0}^{N'-1} f(t_j)\cdot\left(\frac{1}{N'}\sum_{0<|m|<N} e^{-imt_j}\left(im\ p.v.\int_0^{2\pi}\cot\left(\frac{t}{2}\right)e^{imt}\,\mathrm{d}t\right)\right) \tag{9.3}$$

$$= \sum_{j=0}^{N'-1} f(t_j)\cdot\left(\frac{1}{N'}\sum_{0<|m|<N} e^{-imt_j}\left(im\cdot 2\pi i\,\mathrm{sign}(m)\right)\right) \tag{9.4}$$

$$= \sum_{j=0}^{N'-1} f(t_j)\cdot\left(\frac{1}{N'}\sum_{0<|m|<N}(-2\pi|m|)e^{-imt_j}\right) \tag{9.5}$$

$$= \sum_{j=0}^{N'-1} f(t_j)\cdot\left(-\frac{4\pi}{N'}\sum_{m=1}^{N-1} m\cos(mt_j)\right) \tag{9.6}$$

$\square$