Basic Properties

Theorem 0.1:

$$\int_0^{2\pi} e^{imx}\, dx = \begin{cases} 2\pi & (m = 0) \\ 0 & (m \in \mathbb{Z} \setminus \{0\}) \end{cases} \tag{1.1}$$

$$\text{p.v.} \int_0^{2\pi} \cot\left(\frac{x}{2}\right) e^{imx}\, dx = 2\pi i \operatorname{sign}(m) \quad m \in \mathbb{Z} \setminus \{0\} \tag{1.2}$$

$$\int_0^{2\pi} \log\left(4 \sin^2 \frac{x}{2}\right) e^{imx}\, dx = \begin{cases} 0 & (m = 0) \\ -\frac{2\pi}{|m|} & (m \in \mathbb{Z} \setminus \{0\}) \end{cases} \tag{1.3}$$

Theorem 0.2: $\forall N' \in \mathbb{N}.t_j := \frac{2\pi j}{N'}.\forall m \in \mathbb{Z}.$

$$\frac{2\pi}{N'} \sum_{j=0}^{N'-1} e^{imt_j} = \begin{cases} 2\pi & (m \equiv 0 \bmod N') \\ 0 & (\text{otherwise}) \end{cases} \tag{2}$$

Subspace U_N

Definition 0.1: $U_N := \operatorname{span}(\{e^{imx} \mid m \in \mathbb{Z}, |m| < N\})$

Theorem 0.3 (Trapezoidal Rule for U_N): $\forall N \in \mathbb{N}.N$-point trapezoidal rule is exact for U_N.

Theorem 0.4: Let $N' := \dim U_N = 2N - 1$. Let $t_j := \frac{2\pi j}{N'}$ for $j = 0, ..., N' - 1$. $\forall f, g \in U_N$.

$$\int_0^{2\pi} f(t)g(t)\, dt = \frac{2\pi}{N'} \sum_{j=0}^{N'-1} f(t_j) \tag{3}$$

Proof: $f(t)g(t) \in U_{2N-1}$ $\qquad \square$

Quadratures

Theorem 0.5: $\forall N \in \mathbb{N}.\forall f \in U_N$.

$$f(x) = \sum_{j=0}^{N'-1} f(t_j) \cdot \left(\frac{1}{N'} \sum_{|m|<N} e^{-imt_j} e^{imx}\right) \tag{4}$$

Proof:

$$f(x) = \sum_{|m|<N} \int_0^{2\pi} f(t) \frac{e^{-imt}}{\sqrt{2\pi}}\, dt \cdot \frac{e^{imx}}{\sqrt{2\pi}} \tag{5.1}$$

$$= \frac{1}{2\pi} \sum_{|m|<N} \frac{2\pi}{N'} \sum_{j=0}^{N'-1} f(t_j) e^{-imt_j} \cdot e^{imx} \tag{5.2}$$

$$= \sum_{j=0}^{N'-1} f(t_j) \cdot \left(\frac{1}{N'} \sum_{|m|<N} e^{-imt_j} e^{imx}\right) \tag{5.3}$$

$$\qquad \square$$

Theorem 0.6 (Kussmaul–Martensen (Kress) quadrature): $\forall N \in \mathbb{N}.\forall f \in U_N$.

$$\int_0^{2\pi} \log\left(4 \sin^2 \frac{x}{2}\right) f(t)\, dt = \sum_{j=0}^{N'-1} f(t_j) \cdot \left(-\frac{4\pi}{N'} \sum_{m=1}^{N-1} \frac{\cos mt_j}{m}\right) \tag{6}$$

Proof:

$$\int_0^{2\pi} \log\left(4\sin^2\frac{x}{2}\right) f(t)\,\mathrm{d}t = \int_0^{2\pi} \log\left(4\sin^2\frac{x}{2}\right)\left(\sum_{j=0}^{N'-1} f(t_j)\cdot\left(\frac{1}{N'}\sum_{|m|<N} e^{-imt_j}e^{imt}\right)\right)\mathrm{d}t \quad (7.1)$$

$$= \sum_{j=0}^{N'-1} f(t_j)\cdot\left(\frac{1}{N'}\sum_{|m|<N} e^{-imt_j}\int_0^{2\pi}\log\left(4\sin^2\frac{x}{2}\right)e^{imt}\,\mathrm{d}t\right) \quad (7.2)$$

$$= \sum_{j=0}^{N'-1} f(t_j)\cdot\left(\frac{1}{N'}\sum_{0<|m|<N}\left(-\frac{2\pi}{|m|}\right)e^{-imt_j}\right) \quad (7.3)$$

$$= \sum_{j=0}^{N'-1} f(t_j)\cdot\left(-\frac{4\pi}{N'}\sum_{m=1}^{N-1}\frac{\cos mt_j}{m}\right) \quad (7.4)$$

$$\square$$

Theorem 0.7 (Garrick–Wittich quadrature for U_N): $\forall N\in\mathbb{N}.\forall f\in U_N.$

$$p.v.\int_0^{2\pi}\cot\left(\frac{x}{2}\right)f'(t)\,\mathrm{d}t = \sum_{j=0}^{N'-1} f(t_j)\cdot\left(-\frac{4\pi}{N'}\sum_{m=1}^{N-1} m\cos(mt_j)\right) \quad (8)$$

Proof:

$$\int_0^{2\pi}\cot\left(\frac{x}{2}\right)f'(t)\,\mathrm{d}t = \int_0^{2\pi}\cot\left(\frac{x}{2}\right)\left(\sum_{j=0}^{N'-1} f(t_j)\cdot\left(\frac{1}{N'}\sum_{|m|<N} e^{-imt_j}(e^{imt})'\right)\right)\mathrm{d}t \quad (9.1)$$

$$= \int_0^{2\pi}\cot\left(\frac{x}{2}\right)\left(\sum_{j=0}^{N'-1} f(t_j)\cdot\left(\frac{1}{N'}\sum_{0<|m|<N} e^{-imt_j}(ime^{imt})\right)\right)\mathrm{d}t \quad (9.2)$$

$$= \sum_{j=0}^{N'-1} f(t_j)\cdot\left(\frac{1}{N'}\sum_{0<|m|<N} e^{-imt_j}\left(im\int_0^{2\pi}\cot\left(\frac{x}{2}\right)e^{imt}\,\mathrm{d}t\right)\right) \quad (9.3)$$

$$= \sum_{j=0}^{N'-1} f(t_j)\cdot\left(\frac{1}{N'}\sum_{0<|m|<N} e^{-imt_j}(im\cdot 2\pi i\,\mathrm{sign}(m))\right) \quad (9.4)$$

$$= \sum_{j=0}^{N'-1} f(t_j)\cdot\left(\frac{1}{N'}\sum_{0<|m|<N}(-2\pi|m|)e^{-imt_j}\right) \quad (9.5)$$

$$= \sum_{j=0}^{N'-1} f(t_j)\cdot\left(-\frac{4\pi}{N'}\sum_{m=1}^{N-1} m\cos(mt_j)\right) \quad (9.6)$$

$$\square$$