

Basic Properties

Theorem 0.1:

$$\int_0^{2\pi} e^{imx} dx = \begin{cases} 2\pi(m=0) \\ 0 \quad (m \in \mathbb{Z} \setminus \{0\}) \end{cases} \quad (1.1)$$

$$\text{p.v.} \int_0^{2\pi} \cot\left(\frac{x}{2}\right) e^{imx} dx = 2\pi i \operatorname{sign}(m) \quad m \in \mathbb{Z} \setminus \{0\} \quad (1.2)$$

$$\int_0^{2\pi} \log\left(4 \sin^2 \frac{x}{2}\right) e^{imx} dx = \begin{cases} 0 \quad (m=0) \\ -\frac{2\pi}{|m|} (m \in \mathbb{Z} \setminus \{0\}) \end{cases} \quad (1.3)$$

Theorem 0.2: $\forall N' \in \mathbb{N}. t_j := \frac{2\pi j}{N'}. \forall m \in \mathbb{Z}.$

$$\frac{2\pi}{N'} \sum_{j=0}^{N'-1} e^{imt_j} = \begin{cases} 2\pi(m \equiv 0 \pmod{N'}) \\ 0 \quad (\text{otherwise}) \end{cases} \quad (2)$$

Subspace U_N

Definition 0.1: $U_N := \operatorname{span}(\{e^{imx} \mid m \in \mathbb{Z}, |m| < N\})$

Theorem 0.3 (Trapezoidal Rule for U_N): $\forall N \in \mathbb{N}. N$ -point trapezoidal rule is exact for U_N .

Theorem 0.4: Let $N' := \dim U_N = 2N - 1$. Let $t_j := \frac{2\pi j}{N'}$ for $j = 0, \dots, N' - 1$. $\forall f, g \in U_N$.

$$\int_0^{2\pi} f(t)g(t) dt = \frac{2\pi}{N'} \sum_{j=0}^{N'-1} f(t_j) \quad (3)$$

Proof: $f(t)g(t) \in U_{2N-1}$

□

Quadratures

Theorem 0.5: $\forall N \in \mathbb{N}. \forall f \in U_N$.

$$f(x) = \sum_{j=0}^{N'-1} f(t_j) \cdot \left(\frac{1}{N'} \sum_{|m| < N} e^{-imt_j} e^{imx} \right) \quad (4)$$

Proof:

$$f(x) = \sum_{|m| < N} \int_0^{2\pi} f(t) \frac{e^{-imt}}{\sqrt{2\pi}} dt \cdot \frac{e^{imx}}{\sqrt{2\pi}} \quad (5.1)$$

$$= \frac{1}{2\pi} \sum_{|m| < N} \frac{2\pi}{N'} \sum_{j=0}^{N'-1} f(t_j) e^{-imt_j} \cdot e^{imx} \quad (5.2)$$

$$= \sum_{j=0}^{N'-1} f(t_j) \cdot \left(\frac{1}{N'} \sum_{|m| < N} e^{-imt_j} e^{imx} \right) \quad (5.3)$$

□

Theorem 0.6 (Kussmaul–Martensen (Kress) quadrature): $\forall N \in \mathbb{N}. \forall f \in U_N$.

$$\int_0^{2\pi} \log\left(4 \sin^2 \frac{x}{2}\right) f(t) dt = \sum_{j=0}^{N'-1} f(t_j) \cdot \left(-\frac{4\pi}{N'} \sum_{m=1}^{N-1} \frac{\cos mt_j}{m} \right) \quad (6)$$

Proof:

$$\int_0^{2\pi} \log\left(4 \sin^2 \frac{x}{2}\right) f(t) dt = \int_0^{2\pi} \log\left(4 \sin^2 \frac{x}{2}\right) \left(\sum_{j=0}^{N'-1} f(t_j) \cdot \left(\frac{1}{N'} \sum_{|m|< N} e^{-imt_j} e^{imt} \right) \right) d\mathbf{t})$$

$$= \sum_{j=0}^{N'-1} f(t_j) \cdot \left(\frac{1}{N'} \sum_{|m|< N} e^{-imt_j} \int_0^{2\pi} \log\left(4 \sin^2 \frac{x}{2}\right) e^{imt} dt \right) \quad (7.2)$$

$$= \sum_{j=0}^{N'-1} f(t_j) \cdot \left(\frac{1}{N'} \sum_{0<|m|< N} \left(-\frac{2\pi}{|m|} \right) e^{-imt_j} \right) \quad (7.3)$$

$$= \sum_{j=0}^{N'-1} f(t_j) \cdot \left(-\frac{4\pi}{N'} \sum_{m=1}^{N-1} \frac{\cos mt_j}{m} \right) \quad (7.4)$$

□