PART I:	Non-Riemannia	n Hypersquares

Math 256C: From Schemes to Machinations

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## **Tutorial**

### **Basics**

Here is where lecture content goes, generally a summary or transcription of what is being said or written. Here is a theorem:

### Theorem 1.1.1: Kontsevich

The number  $N_d$  of rational plane curves of degree d passing through 3d-1 points in general position is given recursively by

$$N_{d} = \sum_{d_{A} + d_{B} = d} N_{d_{A}} N_{d_{B}} d_{A}^{2} d_{B} \left( d_{B} \binom{3d - 4}{3d_{A} - 2} - d_{A} \binom{3d - 4}{3d_{A} - 1} \right)$$

The above result, is, of course, thoroughly unrelated to the following fact:

#### Lemma 1.1.2

In a k-free graph on n vertices, there are at most  $\binom{k-1}{r}(\frac{n}{k-1})^r$  r-cliques.

Setting r=2 in the above, we recover the following result:

### Corollary 1.1.3: Turan's Theorem

In a k-free graph on n vertices, there are at most  $\frac{k-2}{k-1} \frac{n^2}{2}$  edges.

You can insert a hyperlinked reference for any theorem box if you add a reference tag (see the LATEX code at Corollary 1.1.3 for formatting, and see the style file for the reference prefixes for each theorem style).

PROOF: There is also a proof environment; the proof heading is configured to live in the left margin.

Citations live in the right margin <sup>1</sup>, but will not work correctly if placed

Here is a margin note: I use these generally to annotate my own thoughts or questions during lecture. If used for producing some kind of document with external consumption in mind, margin notes are good for imparting intuition and introducing individual, idiosyncratic, and informal, but, illuminating and illustrative ideologies.

You can have multi-paragraph margin notes, which are configured to not have indented paragraphs.

<sup>&</sup>lt;sup>1</sup> R. Hartshorne. *Algebraic Geometry*. Springer, 1977

inside a theorem box. The available theorem boxes are theorem, lemma, corollary, proposition, definition, example, remark, question, exercise, and conjecture. Unnnumbered versions of all the theorem boxes exist:

### Proposition: Hurwitz

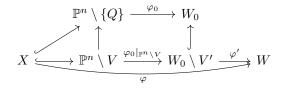
The group of orientation-preserving conformal automorphisms of a compact Riemann surface of genus g > 1 has order at most 84(g-1).

#### Advanced

Using https://q.uiver.app/ (whose style file is included in our style file for convenience), we can curve arrows more flexibly in commutative diagrams than tikz-cd normally allows:

References do not work for unnumbered theorems.

Theorem box numbering is not affected by subsection; this is set by secnumdepth at the top of the pream-



Finally, there is (in addition to the default environments provided by tufte-book) a two-column environment, which I mostly use to spam examples after a big theorem:

In  $\mathbb{CP}^2$  with its canonical orientation as a complex manifold, we have  $H^2 = H_2 = \mathbb{Z}$ . Lines L in  $\mathbb{CP}^2$  are embedded copies of  $\mathbb{CP}^1 = S^2$ , with a single fundamental class  $[L] \in H_2$  representing any such line. To compute the self-intersection  $[L] \cdot [L]$ , choose two representatives L and L' which intersect transversely (in a point, by basic intersection theory); since  $\mathbb{CP}^1$  is canonically oriented as well, and the orientations of L and L'agree with that of  $\mathbb{CP}^2$  (since they are subspaces of  $\mathbb{CP}^2$ ), we have that  $[L] \cdot [L] = 1$ .

In  $S^2 \times S^2$ , the second homology group is  $\mathbb{Z}^2$ , generated by  $h = [S^2 \times \{x\}]$  and  $v = [\{x\} \times S^2]$ .  $S^2 \times \{x\}$  and  $\{x\} \times S^2$  intersect transversely at (x,x), and, regarding  $S^2$  with its canonical orientation as a complex manifold ( $\mathbb{CP}^1$ ), the orien-

tations agree to give  $h \cdot v = 1$ . To compute  $h \cdot h$ , intersect representatives  $S^2 \times \{x\}$  and  $S^2 \times \{y\}$ for  $x \neq y$ , from which it is set-theoretically clear that  $h^2 = 0$  (and similarly for v). Thus, the intersection form as a matrix in the basis (h, v)can be written as  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  This intersection form is therefore *even*, meaning that for any k = ah + bv,  $k \cdot k = 2ab$  is even.

# References

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- M. Kontsevich and Y. Manin. Gromov-witten classes, quantum cohomology, and enumerative geometry. *Communications in Mathematical Physics*, 164:525–562, 1994. DOI: https://doi.org/10.1007/BF02101490.
- E. Witten. Two-dimensional gravity and intersection theory on moduli space. Surveys Diff. Geom., 1:243–310, 1991. DOI: 10.4310/SDG.1990.v1.n1.a5.