# Part I: Non-Riemannian Hypersquares

**Math 256C: From Schemes to Machinations** **Fall 2020**

# Lecture 1: 26 August

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# Tutorial

## Basics

Here is where lecture content goes, generally a summary or transcription of what is being said or written. Here is a theorem:

**Theorem 1.1.1: Kontsevich**

The number $N_d$ of rational plane curves of degree $d$ passing through $3d-1$ points in general position is given recursively by

$$N_d = \sum_{d_A+d_B=d} N_{d_A} N_{d_B} d_A^2 d_B \left( d_B \binom{3d-4}{3d_A-2} - d_A \binom{3d-4}{3d_A-1} \right)$$

Here is a margin note: I use these generally to annotate my own thoughts or questions during lecture. If used for producing some kind of document with external consumption in mind, margin notes are good for imparting intuition and introducing individual, idiosyncratic, and informal, but, illuminating and illustrative ideologies.

You can have multi-paragraph margin notes, which are configured to not have indented paragraphs.

The above result, is, of course, thoroughly unrelated to the following fact:

**Lemma 1.1.2**

In a $k$-free graph on $n$ vertices, there are at most $\binom{k-1}{r}(\frac{n}{k-1})^r$ $r$-cliques.

Setting $r = 2$ in the above, we recover the following result:

**Corollary 1.1.3: Turan's Theorem**

In a $k$-free graph on $n$ vertices, there are at most $\frac{k-2}{k-1}\frac{n^2}{2}$ edges.

You can insert a hyperlinked reference for any theorem box if you add a reference tag (see the LaTeX code at Corollary 1.1.3 for formatting, and see the style file for the reference prefixes for each theorem style).

Proof : There is also a proof environment; the proof heading is configured to live in the left margin. ■

Citations live in the right margin [1], but will not work correctly if placed

[1] R. Hartshorne. *Algebraic Geometry.* Springer, 1977

inside a theorem box. The available theorem boxes are `theorem`, `lemma`, `corollary`, `proposition`, `definition`, `example`, `remark`, `question`, `exercise`, and `conjecture`. Unnnumbered versions of all the theorem boxes exist:

**Proposition: Hurwitz**

The group of orientation-preserving conformal automorphisms of a compact Riemann surface of genus $g > 1$ has order at most $84(g-1)$.

References do not work for unnumbered theorems.

## Advanced

Theorem box numbering is not affected by subsection; this is set by `secnumdepth` at the top of the preamble.

Using `https://q.uiver.app/` (whose style file is included in our style file for convenience), we can curve arrows more flexibly in commutative diagrams than `tikz-cd` normally allows:

$$\begin{tikzcd}
& \mathbb{P}^n \setminus \{Q\} \arrow[r, "\varphi_0"] & W_0 & \\
X \arrow[ur, hook] \arrow[r, hook] \arrow[rrr, bend right, "\varphi"'] & \mathbb{P}^n \setminus V \arrow[u] \arrow[r, "\varphi_0|_{\mathbb{P}^n \setminus V}"] & W_0 \setminus V' \arrow[u, hook] \arrow[r, "\varphi'"] & W
\end{tikzcd}$$

Finally, there is (in addition to the default environments provided by `tufte-book`) a two-column environment, which I mostly use to spam examples after a big theorem:

**Example 1.1.4**

In $\mathbb{CP}^2$ with its canonical orientation as a complex manifold, we have $H^2 = H_2 = \mathbb{Z}$. Lines $L$ in $\mathbb{CP}^2$ are embedded copies of $\mathbb{CP}^1 = S^2$, with a single fundamental class $[L] \in H_2$ representing any such line. To compute the self-intersection $[L] \cdot [L]$, choose two representatives $L$ and $L'$ which intersect transversely (in a point, by basic intersection theory); since $\mathbb{CP}^1$ is canonically oriented as well, and the orientations of $L$ and $L'$ agree with that of $\mathbb{CP}^2$ (since they are subspaces of $\mathbb{CP}^2$), we have that $[L] \cdot [L] = 1$.

**Example 1.1.5**

In $S^2 \times S^2$, the second homology group is $\mathbb{Z}^2$, generated by $h = [S^2 \times \{x\}]$ and $v = [\{x\} \times S^2]$. $S^2 \times \{x\}$ and $\{x\} \times S^2$ intersect transversely at $(x, x)$, and, regarding $S^2$ with its canonical orientation as a complex manifold ($\mathbb{CP}^1$), the orientations agree to give $h \cdot v = 1$. To compute $h \cdot h$, intersect representatives $S^2 \times \{x\}$ and $S^2 \times \{y\}$ for $x \neq y$, from which it is set-theoretically clear that $h^2 = 0$ (and similarly for $v$). Thus, the intersection form as a matrix in the basis $(h, v)$ can be written as $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ This intersection form is therefore *even*, meaning that for any $k = ah + bv$, $k \cdot k = 2ab$ is even.

# References

B. Fantechi. Stacks for everybody. In C. Casacuberta, R. M. Miró-Roig, J. Verdera, and S. Xambó-Descamps, editors, *European Congress of Mathematics*, pages 349–359, Basel, 2001. Birkhäuser Basel. ISBN 978-3-0348-8268-2.

R. Hartshorne. *Algebraic Geometry*. Springer, 1977.

M. Kontsevich and Y. Manin. Gromov-witten classes, quantum cohomology, and enumerative geometry. *Communications in Mathematical Physics*, 164:525–562, 1994. DOI: https://doi.org/10.1007/BF02101490.

E. Witten. Two-dimensional gravity and intersection theory on moduli space. *Surveys Diff. Geom.*, 1:243–310, 1991. DOI: 10.4310/SDG.1990.v1.n1.a5.