Part I: Non-Riemannian Hypersquares

Math 256C: From Schemes to Conspiracies **Fall 2020**

Lecture 1: 26 August

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Tutorial

Basics

Here is where lecture content goes, generally a summary or transcription of what is being said or written. Here is a theorem:

Theorem 1.1.1: Kontsevich

The number N_d of rational plane curves of degree d passing through $3d-1$ points in general position is given recursively by

$$N_d = \sum_{d_A+d_B=d} N_{d_A} N_{d_B} d_A^2 d_B \left(d_B \binom{3d-4}{3d_A-2} - d_A \binom{3d-4}{3d_A-1} \right)$$

Here is a margin note: I use these generally to annotate my own thoughts or questions during lecture.

You can have multi-paragraph margin notes, which are configured to not have indented paragraphs.

The above result, is, of course, thoroughly unrelated to the following fact:

Lemma 1.1.2

In a k-free graph on n vertices, there are at most $\binom{k-1}{r}\left(\frac{n}{k-1}\right)^r$ r-cliques.

Setting $r = 2$ in the above, we recover the following result:

Corollary 1.1.3: Turan's Theorem

In a k-free graph on n vertices, there are at most $\frac{k-2}{k-1}\frac{n^2}{2}$ edges.

You can insert a hyperlinked reference for any theorem box if you add a reference tag (see the LaTeX code at Corollary 1.1.3 for formatting, and see the style file for the reference prefixes for each theorem style).

Proof : There is also a proof environment; the proof heading is configured to live in the left margin. ■

Citations live in the right margin[1], but will not work correctly if placed inside a theorem box. The available theorem boxes are `theorem`, `lemma`, `corollary`, `proposition`, `definition`, `example`, `remark`, `question`, `exercise`, and `conjecture`. Unnnumbered versions of all the theorem boxes exist:

[1] R. Hartshorne. *Algebraic Geometry.* Springer, 1977

Proposition: Hurwitz

The group of orientation-preserving conformal automorphisms of a compact Riemann surface of genus $g > 1$ has order at most $84(g-1)$.

References do not work for unnumbered theorems.

Theorem box numbering is not affected by subsection; this is set by `secnumdepth` at the top of the preamble.

Advanced

Using `https://q.uiver.app/` (whose style file is included in our style file for convenience), we can curve arrows more flexibly in commutative diagrams than `tikz-cd` normally allows:

$$\begin{tikzcd}
& \mathbb{P}^n \setminus \{Q\} \arrow[r, "\varphi_0"] & W_0 & \\
X \arrow[ur, hook] \arrow[r, hook] \arrow[rrr, bend right, "\varphi"'] & \mathbb{P}^n \setminus V \arrow[u] \arrow[r, "\varphi_0|_{\mathbb{P}^n \setminus V}"] & W_0 \setminus V' \arrow[u, hook] \arrow[r, "\varphi'"] & W
\end{tikzcd}$$

Finally, there is (in addition to the default environments provided by `tufte-book`) a two-column environment, which I mostly use to spam examples after a big theorem:

Example 1.1.4

In $\mathbb{CP}^2$ with its canonical orientation as a complex manifold, we have $H^2 = H_2 = \mathbb{Z}$. Lines L in $\mathbb{CP}^2$ are embedded copies of $\mathbb{CP}^1 = S^2$, with a single fundamental class $[L] \in H_2$ representing any such line. To compute the self-intersection $[L] \cdot [L]$, choose two representatives L and L' which intersect transversely (in a point, by basic intersection theory); since $\mathbb{CP}^1$ is canonically oriented as well, and the orientations of L and L' agree with that of $\mathbb{CP}^2$ (since they are subspaces of $\mathbb{CP}^2$), we have that $[L] \cdot [L] = 1$.

Example 1.1.5

In $S^2 \times S^2$, the second homology group is $\mathbb{Z}^2$, generated by $h = [S^2 \times \{x\}]$ and $v = [\{x\} \times S^2]$. $S^2 \times \{x\}$ and $\{x\} \times S^2$ intersect transversely at (x, x), and, regarding S^2 with its canonical orientation as a complex manifold ($\mathbb{CP}^1$), the orientations agree to give $h \cdot v = 1$. To compute $h \cdot h$, intersect representatives $S^2 \times \{x\}$ and $S^2 \times \{y\}$ for $x \neq y$, from which it is set-theoretically clear that $h^2 = 0$ (and similarly for v). Thus, the

intersection form as a matrix in the basis (h, v) can be written as $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ This intersection form is therefore *even*, meaning that for any $k = ah + bv$, $k \cdot k = 2ab$ is even.

References

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M. Kontsevich and Y. Manin. Gromov-witten classes, quantum cohomology, and enumerative geometry. *Communications in Mathematical Physics*, 164:525–562, 1994. DOI: https://doi.org/10.1007/BF02101490.

E. Witten. Two-dimensional gravity and intersection theory on moduli space. *Surveys Diff. Geom.*, 1:243–310, 1991. DOI: 10.4310/SDG.1990.v1.n1.a5.