These are notes for an ongoing Fall 2023 course on the Riemann zeta function and its generalizations, L-functions. These notes will be filled in as we go.

1. References thus far

- Generating functions and asymptotics: [1, §5.2]
- Mellin transform and asymptotics: [2]

2. Outlines thus far

(Some links here refer to external files; I'll have to think of a good way to notate that.)

- Tuesday, 29 Aug: parts of §4; §1, §2
- Thursday, 31 Aug: §3, §4
- Friday, 1 Sep: §5 and §6
- Tuesday, 5 Sep: §5
- Thursday, 7 Sep: §5.2, §2.1

3. Course notes that I've since split off into separate files

- Complex-analytic preliminaries
- Generating functions and asymptotics
- Fourier and Mellin transforms
- Bernoulli numbers and Euler–Maclaurin summation

4. Background

4.1. General notation. $\mathbb{R}^+ := (0, \infty)$.

4.2. Asymptotic notation. We use the equivalent notations

$$A = \mathrm{O}(B), \qquad A \ll B, \qquad B \gg A$$

to denote that

$$|A| \le C|B|$$

for some "constant" C. The precise meaning of "constant" will either be specified or clear from context.

4.3. **Definition and basic properties of ζ: overview.** The Riemann zeta function is defined for a complex number s by the series

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

Lemma 4.1. *The series converges absolutely for $\Re(s) > 1$, uniformly for $\Re(s) \geq 1+\varepsilon$ for each $\varepsilon > 0$.*

Proof. Using the identity

$$\left| \frac{1}{n^s} \right| = \frac{1}{n^{\Re(s)}},$$

we reduce to the case that s is real, in which this is a familiar consequence of the integral test. $\square$

Our first main goal in the course is to explain the following basic facts.

Theorem 4.2. *The Riemann zeta function admits a meromorphic continuation to the entire complex plane. It is holomorphic away from a simple pole at $s = 1$, where it has residue 1. It admits a functional equation relating $\zeta(s)$ to $\zeta(1-s)$.*

One historical motivation for considering the zeta function at complex arguments comes from the prime number theorem.

Theorem 4.3 (Prime number theorem)**.** *Let $\pi(x) := \# \{primes\ p \leq x\}$ denote the prime counting function. Then*

$$\frac{\pi(x)}{x/\log x} \to 1 \text{ as } x \to \infty.$$

This is related to the following analytic fact concerning the zeros of the zeta function.

Theorem 4.4 (Prime number theorem, formulated in terms of ζ)**.** *We have $\zeta(s) = 0$ only if $\Re(s) < 1$.*

Remark 4.5. Even the statement of Theorem 4.4 is not clear without knowing the meromorphic continuation of ζ. This may offer some motivation for understanding the latter.

We expect stronger nonvanishing properties:

Conjecture 1 (Riemann Hypothesis)**.** *We have $\zeta(s) = 0$ only if $\Re(s) < 1/2$.*

This corresponds to a conjectural stronger form of the prime number theorem, namely that

$$\pi(x) = \int_2^x \frac{t}{\log t}\, dt + \mathrm{O}(x^{1/2} \log x).$$

5. Basic analytic properties of ζ

5.1. Meromorphic continuation and evaluation at negative integers. Take

$$h(y) = \frac{1}{e^y - 1}.$$

The function $yh(y)$ extends to a holomorphic function of y on the disc $\{y \in \mathbb{C} : |y| < 2\pi\}$, so h is represented for small $y > 0$ by an absolutely convergent Laurent series of the following form:

$$h(y) = \frac{1}{y} + \sum_{n=1}^{\infty} \frac{B_n}{n!} y^{n-1}.$$

Here the B_n are complex coefficients, called the *Bernoulli numbers* (see this note). On the other hand, h decays rapidly (like $\mathrm{O}(y^N)$ for any fixed N) as $y \to \infty$. By the analysis we've now seen many times, we deduce that the Mellin transform $H(s)$ of $h(y)$ converges absolutely for $\Re(s) > 1$, where it defines a holomorphic function, and extends to a meromorphic function on the complex plane whose only poles are simple poles at $s = 1$ (corresponding to the $1/y$ term in the asymptotic expansion as $y \to 0$) and at $s = -n$ for each $n \in \mathbb{Z}_{\geq 0}$, with residue given by 1 in the former case and by $B_{n+1}/(n+!)!$ in the latter.

On the other hand, we can rewrite

$$h(y) = \frac{e^{-y}}{1 - e^{-y}} = e^{-y} + e^{-2y} + e^{-3y} + \cdots ,$$

giving

$$H(s) = \int_{\mathbb{R}^+} y^s \sum_{n=1}^{\infty} e^{-ny} \, d^{\times} y.$$

The following calculations will show that the doubly integral/sum converges absolutely for $\Re(s) > 1$, so we may rearrange it as

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}^+} y^s e^{-ny} \, d^{\times} y.$$

The inner integral may be simplified by the substitution $y \mapsto y/n$. This has no effect on the measure $d^{\times} y$, but replaces y^s by $n^{-s} y^s$, giving

$$\sum_{n=1}^{\infty} n^{-s} \int_{\mathbb{R}^+} y^s e^{-y} \, d^{\times} y = \zeta(s)\Gamma(s).$$

We see in this way that the ζ function admits a meromorphic continuation. Since the Γ-function does not vanish, we deduce that the only pole of ζ is the simple one at $s = 1$, with residue $s = 0$. Also, since we have computed the residues of both $\Gamma(s)$ and $\zeta(s)\Gamma(s)$ at the nonpositive integers, we may calculate in this way the values of $\zeta(-n)$ for each $n \in \mathbb{Z}_{\geq 0}$:

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}.$$

5.2. **Another perspective on the meromorphic continuation.** (TODO: think of better section titles)

Given $f : \mathbb{R} \to \mathbb{C}$ of sufficient decay at infinity, we define $g : \mathbb{R}^+ \to \mathbb{C}$ by

$$g(y) := \sum_{n \in \mathbb{Z}} f(ny).$$

We assume henceforth that f lies in the Schwartz space $\mathcal{S}(\mathbb{R})$.

The asymptotics of $g(y)$ are described as follows.

Lemma 5.1. *Let $f \in \mathcal{S}(R)$.*

(1) As $y \to \infty$, we have

$$g(y) = f(0) + \mathrm{O}(y^{-N}) \tag{5.1}$$

for each fixed N.

(2) As $y \to 0$, we have

$$g(y) = y^{-1} \int_{\mathbb{R}} f(x)\, dx + \mathrm{O}(y^N) \tag{5.2}$$

for each fixed N.

Proof. The first estimate (5.1) is an easy exercise using the definition of the Schwartz space. The second estimate (5.2) may be proved either via Euler–Maclaurin summation (see external §2.1) or Poisson summation (see external §**??**). □

Assuming that $f(0)$ and $\int f$ are nonzero, it follows that the Mellin transform of g does not converge absolutely at any point. We can still define a regularized Mellin transform

$$G(s) = \int_{\mathbb{R}^+}^{\mathrm{reg}} y^s g(y)\, d^{\times} y,$$

like in §4, by splitting the integral into two pieces (e.g, the contributions of $(0, 1)$ and $(1, \infty)$), meromorphically continuing each piece, and then summing the results on their domain of overlap (if any).

Lemma 5.2. *$G(s)$ defines a meromorphic function on the complex plane whose only poles at simple ones:*

- *at $s = 1$, with residue $f(0)$, and*
- *at $s = 0$, with residue $-\int_{\mathbb{R}} f(x)\, dx$.*

Proof. Let's carry this out in detail, applying the recipe of external §2.

- We see from (5.1) that the integral

  $$G_{-}(s) := \int_0^1 y^s g(y)\, d^{\times} y$$

  converges absolutely for $\Re(s) < 0$ and extends to a meromorphic function of s, whose only pole is a simple pole at $s = 1$ of residue $f(0)$. Indeed,

  $$G_{-}(s) = \int_0^1 y^s \left(g(y) - f(0)\right)\, d^{\times} y + f(0) \underbrace{\int_0^1 y^s\, d^{\times} y}_{\frac{1}{s}},$$

  where the first integral on the right hand side converges absolutely for all s.

- We see from (5.2) that the integral
$$G_+(s) := \int_1^\infty y^s g(y)\, d^\times y$$
converges absolutely for $\Re(s) > 1$ and extends to a meromorphic function of s, whose only pole is a simple pole at $s = 0$ of residue $-I_f$, where $I_f := \int_{\mathbb{R}} f(x)\, dx$. Indeed,
$$G_+(s) = \int_0^1 y^s \left(g(y) - y^{-1} I_f\right)\, d^\times y + (I_f) \underbrace{\int_1^\infty y^{s-1}\, d^\times y}_{\frac{-1}{s-1}}.$$

□

We denote henceforth by $\mathcal{F}f$ the (normalized) Fourier transform
$$\mathcal{F}f(\xi) := \int_{x \in \mathbb{R}} f(x) e^{-2\pi i x \xi}\, dx.$$

Lemma 5.3. *Write $G(s) = G_f(s)$ to indicate the dependence upon f. Then we have the following functional equation:*
$$G_f(s) = G_{\mathcal{F}f}(1 - s).$$

Proof. This is a consequence of the Poisson summation formula (see external (**??**))
$$\sum_{n \in \mathbb{Z}} f(ny) = y^{-1} \sum_{n \in \mathbb{Z}} \mathcal{F}f(n/y).$$
Writing g_f to indicate the dependence of g upon f, the above identity reads
$$g_f(y) = y^{-1} g_{\mathcal{F}f}(1/y).$$
Taking the (regularized) Mellin transform of both sides yields
$$G_f(s) = \int_{\mathbb{R}^+}^{\mathrm{reg}} y^{s-1} g_{\mathcal{F}f}(1/y)\, d^\times y.$$
To evaluate this last integral, we substitute $y \mapsto 1/y$, which leaves the measure $d^\times y$ invariant (and is unaffected by the regularization). This gives
$$G_f(s) = \int_{\mathbb{R}^+}^{\mathrm{reg}} y^{1-s} g_{\mathcal{F}f}(y)\, d^\times y = G_{\mathcal{F}f}(1 - s),$$
as required. □

We suppose henceforth that f is even. This is without much loss of generality – any function can be written as a sum of even and odd functions, and if f is odd, then g vanishes identically.

Exercise 1. Let $f \in \mathcal{S}(\mathbb{R})$ be even. Show that the Mellin transform
$$F(s) := \int y^s f(y)\, d^\times y$$
converges initially for $\Re(s) > 0$ and extends to a meromorphic function on the complex plane, whose only poles are simple ones at $s = -2n$ for $n \in \mathbb{Z}_{\geq 0}$ with residue
$$\mathrm{res}_{s=-2n} F(s) = \frac{1}{(2n)!} f^{(2n)}(0).$$

[Use Taylor's theorem with remainder, and note that the odd Taylor coefficients vanish in view of the evenness assumption on f.]

Example 5.4. Take $f(x) := e^{-\pi x^2}$. Then

$$F(s) = \pi^{-s/2}\Gamma(s/2), \tag{5.3}$$

as one sees by substituting $y \mapsto \sqrt{y}$ and then $y \mapsto y/\pi$ in the defining integral. This has poles in the expected places.

For f even, we have

$$g(y) = f(0) + 2\sum_{n=1}^{\infty} f(ny).$$

Using that the constant function 1 has vanishing regularized Mellin transform (for reasons similar to external Exercise 3), we see that $G(s)$ admits the following absolutely convergent integral representation for $\Re(s) > 1$:

$$G(s) = 2\int_{\mathbb{R}^+} y^s \sum_{n=1}^{\infty} f(ny)\, d^\times y.$$

By substituting $y \mapsto y/n$, we see that

$$G(s) = 2\zeta(s)F(s), \qquad F(s) := \int y^s f(y)\, d^\times y.$$

This gives another proof of the meromorphic behavior of ζ. TODO: say more.

We can also deduce the functional equation:

Theorem 5.5. *We have*

$$\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s) = \xi(1-s).$$

Proof. Take $f(x) := e^{-\pi x^2}$. By (5.3), we then have $G_f(s) = \xi(s)$. On the other hand, by the well-known formula for the Fourier transform of a Gaussian, we have $\mathcal{F}f = f$. By Lemma 5.3, it follows that $G_f(s) = G_f(1-s)$. The claimed formula follows. □

References

[1] Herbert S. Wilf. *generatingfunctionology*. A K Peters, Ltd., Wellesley, MA, third edition, 2006.

[2] Don Zagier. The mellin transform and other useful analytic techniques. `http://people.mpim-bonn.mpg.de/zagier/files/tex/MellinTransform/fulltext.pdf`, 2006.