1. Statement

Let f be an element of the Schwartz space $\mathcal{S}(\mathbb{R})$. Denote by $\mathcal{F}f$ the (normalized) Fourier transform

$$\mathcal{F}f(\xi) := \int_{x \in \mathbb{R}} f(x) e^{-2\pi i x \xi} \, dx.$$

Theorem 1. *We have*

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \mathcal{F}f(n).$$

Proof. TODO □

It is often useful to apply this formula to the dilated function

$$f_y(x) := f(xy)$$

for some $y > 0$. The Fourier transform of this dilate is given by

$$\mathcal{F}f_y(x) = y^{-1} \mathcal{F}f(x/y).$$

Thus

$$\sum_{n \in \mathbb{Z}} f(ny) = y^{-1} \sum_{n \in \mathbb{Z}} \mathcal{F}f(n/y). \tag{1}$$

2. Application to asymptotics of Riemann sums

(Compare with external §2.1.)

Lemma 2. Let $f \in \mathcal{S}(\mathbb{R})$. For $y \in (0,1)$, we have

$$\sum_{n \in \mathbb{Z}} f(ny) = y^{-1} \int f + \mathrm{O}(y^N)$$

for each fixed N.

Proof. We apply Poisson summation in the form (1). We simplify the contribution from $n = 0$ to the right hand side of (1) using that

$$\mathcal{F}f(0) = \int f.$$

We leave to the reader the exercise of verifying that the contribution from $n \neq 0$ is acceptable. □

TODO: etc

References