These are notes for an ongoing Fall 2023 course on the Riemann zeta function and its generalizations, L-functions. These notes will be filled in as we go.

1. Background

1.1. General notation. $\mathbb{R}^+ := (0, \infty)$.

1.2. Asymptotic notation. We use the equivalent notations

$$A = \mathrm{O}(B), \qquad A \ll B, \qquad B \gg A$$

to denote that

$$|A| \leq C|B|$$

for some "constant" C. The precise meaning of "constant" will either be specified or clear from context.

1.3. Holomorphic continuation.

Theorem 1.1 (Identity principle for holomorphic functions)**.** *Let $U \subset \mathbb{C}$ be a connected open set. Let $f, g : U \to \mathbb{C}$ be holomorphic functions. If $f = g$ on a set with a limit point in U, then $f = g$ on all of U.*

Corollary 1.2. *Let $U \subset \Omega \subseteq \mathbb{C}$ be open subsets, with U nonempty and Ω connected. Let $f : U \to \mathbb{C}$ be a holomorphic function. Then there is at most one extension of f to a holomorphic function $\Omega \to \mathbb{C}$.*

1.4. Cauchy's integral formula.

Theorem 1.3. *Let $f : U \to \mathbb{C}$ be a holomorphic function defined on an open subset U. Let γ be a closed rectifiable curve in U. Then $\int_\gamma f(z)\, dz = 0$.*

Theorem 1.4. *Let $0 \leq a < b \leq \infty$. Let $f(z)$ be a holomorphic function on the annulus $\{z \in \mathbb{C} : a < |z| < b\}$ given by a convergent Laurent series*

$$f(z) = \sum_{n \in \mathbb{Z}} c_n z^n.$$

(1) For any $r \in (a, b)$ and $n \in \mathbb{Z}$, we have

$$\begin{aligned} c_n &= \oint_{|z|=r} \frac{f(z)}{z^n} \frac{dz}{2\pi i z} \\ &= \frac{1}{2\pi r^n} \int_{\theta=0}^{2\pi} f(re^{i\theta}) e^{-in\theta}\, d\theta. \end{aligned}$$

(2) For each compact subset E of (a, b), there exists $M \geq 0$ so that for all $r \in E$, we have

$$\sum_{n \in \mathbb{Z}} |c_n| r^n \leq M. \tag{1.1}$$

Theorem 1.5. *Let U be an open subset of $\mathbb{C}$, let $f : U \to \mathbb{C}$ be meromorphic. Let γ be a smooth closed curve in U, oriented counterclockwise, that does not pass through any pole of f. Then*

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{\substack{z \in \mathrm{interior}(\gamma) \\ \text{pole of } f}} \mathrm{res}_z(f).$$

Remark 1.6. Let $0 < r < R$. Let f be a meromorphic function on a neighborhood of the annulus $\{z : r < |z| < R\}$ that has no poles on either of the circles $|z| = r, R$. Then

$$\oint_{|z|=R} f(z)\,dz = \oint_{|z|=r} f(z)\,dz + 2\pi i \sum_{\substack{r<|z|<R \\ \text{pole of } f}} \operatorname{res}_z(f).$$

1.5. **Holomorphy of limits and series.**

Theorem 1.7. *Let U be an open subset of the complex plane. Let f_n be a sequence of holomorphic functions on U.*

(1) *Suppose that the sequence f_n converges pointwise to some function f, uniformly on compact subsets of U. Then f is holomorphic.*
(2) *Suppose that the partial sums $\sum_{n\leq N} f_n$ converge pointwise to some function f, uniformly on compact subsets of U. Then the sum $\sum_n f_n$ is holomorphic.*

2. Asymptotics and meromorphic continuation

Reference: generatingfunctionology, section 5.2.

We consider a Laurent series

$$f(z) = \sum_{n \in \mathbb{Z}} c_n z^n.$$

Here the c_n are complex coefficients, while z is a nonzero complex argument. We assume that this series converges absolutely for at least one value of z.

Lemma 2.1. *There is a unique maximal open subinterval (a, b) of $\mathbb{R}^+$ on which f converges absolutely. Its endpoints are given explicitly by*

$$a = \inf \left\{ r \in \mathbb{R}^+ : \sum_n |c_n| r^n < \infty \right\},$$

$$b = \sup \left\{ r \in \mathbb{R}^+ : \sum_n |c_n| r^n < \infty \right\}.$$

We refer to the interval (a, b) as the *fundamental interval* for f (or for the c_n).

The fundamental interval controls the growth of the coefficients c_n as $n \to \pm\infty$:

Lemma 2.2. *Let $b^- < b$ and $a^+ > a$. Then*

$$c_n \ll (b^-)^{-n} \qquad \textit{as } n \to \infty$$

and

$$c_n \ll (a^+)^{-n} \qquad \textit{as } n \to -\infty.$$

Set

$$\mathcal{C}(a, b) := \{z \in \mathbb{C} : |z| \in (a, b)\}.$$

Lemma 2.3. *$f(z)$ defines a holomorphic function on $\mathcal{C}(a, b)$.*

Proof. Follows from Theorem 1.7. □

Lemma 2.4. *f does not extend to a holomorphic function on $\mathcal{C}(A, B)$ for any strictly larger interval $(A, B) \supsetneq (a, b)$.*

Proof. Suppose otherwise. Let $r \in (A, B) - (a, b)$. Then by Cauchy's integral formula (specifically, the estimate (1.1) of Theorem 1.4), we see that $\sum_{n \in \mathbb{Z}} |c_n| r^n < \infty$. This contradicts the formula for a and b given in Lemma 2.1. □

Note 2.5. It can happen that f extends to a *meromorphic* function on some strictly larger annulus (unique, in view of Corollary 1.2). By Lemma 2.4, this can only happen if f has a pole at some point on the boundary of the fundamental annulus.

Example 2.6. Take

$$c_n = \begin{cases} 2^n & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

Then the fundamental interval is $(a, b) = (0, 1/2)$. However, the function $f(z)$, defined initially for $|z| < 1/2$, evaluates to a rational function:

$$f(z) = \sum_{n \geq 0} 2^n z^n = \frac{1}{1 - 2z}.$$

This is meromorphic on the entire complex plane; the only pole is a simple one at $z = 1/2$, with residue $-1/2$.

The possibility of meromorphically extending f corresponds to the coefficients c_n having asymptotic expansions as $n \to \pm\infty$. For example:

Lemma 2.7 (Meromorphic continuation vs. asymptotic expansion, special case). *Let f and (a, b) be as above. Let $\beta \in \mathbb{C}$ with $|\beta| = b$. Let $B > b$ and $\gamma \in \mathbb{C}$. Then the following are equivalent:*

(i) f extends to a meromorphic function on $\mathcal{C}(a, B)$ with a unique simple pole at $z = \beta$ with residue γ.

(ii) For each $B^- < B$, we have as $n \to \infty$ that

$$c_n = -\gamma\beta^{-n-1} + \mathrm{O}\left((B^-)^{-n}\right), \tag{2.1}$$

Proof. To see that (i) implies (ii), we start with Cauchy's integral formula on the disc of radius b^- for some $b^- \in (a, b)$, then shift the contour, picking up the contribution of the unique pole:

$$\begin{aligned} c_n &= \oint_{|z|=b^-} \frac{f(z)}{z^n} \frac{dz}{2\pi i z} \\ &= \oint_{|z|=B^-} \frac{f(z)}{z^n} \frac{dz}{2\pi i z} - \frac{\gamma}{\beta^{n+1}}. \end{aligned} \tag{2.2}$$

We then estimate this last integral using that f is bounded on compact sets.

Conversely, to verify that (ii) implies (i), we define the coefficients

$$b_n := \begin{cases} -\gamma\beta^{-n-1} & \text{if } n \geq 0, \\ 0 & \text{if } n < 0, \end{cases}$$

The corresponding series

$$f_+(z) := \sum_{n \in \mathbb{Z}} b_n z^n$$

may be evaluated explicitly: a simple geometric series calculation, left to the reader, gives

$$f_+(z) = \frac{\gamma}{z - \beta}.$$

Our hypothesis concerning the c_n reads

$$b_n - c_n = \mathrm{O}\left((B^-)^{-n}\right) \qquad \text{as } n \to \infty. \tag{2.3}$$

On the other hand, because f has fundamental interval (a, b) and b_n vanishes as $n \to -\infty$, we have for each $a^+ > a$ that

$$b_n - c_n = \mathrm{O}\left((a^+)^{-n}\right) \qquad \text{as } n \to -\infty. \tag{2.4}$$

From (2.3) and (2.4), we deduce that the series $f - f_+$ with coefficients $c_n - b_n$ has fundamental interval containing (a, B). This implies that the function

$$f(z) - \frac{\gamma}{z - \beta},$$

defined initially as a holomorphic function on $\mathcal{C}(a, b)$, extends to a holomorphic function on $\mathcal{C}(a, B)$. Equivalently, f extends to a meromorphic function on $\mathcal{C}(a, B)$ with polar behavior as described in (ii). $\square$

Example 2.8. Suppose that

$$c_n = \begin{cases} \beta^{-n} & \text{if } n \geq 0, \\ 0 & \text{if } n < 0, \end{cases}$$

so that, initially for $|z| < |\beta|$,

$$f(z) = \sum_{n \geq 0} \beta^{-n} z^n = \frac{1}{1 - z/\beta}.$$

The function f extends meromorphically, having a simple pole at $z = \beta$ with residue $-\beta$. The sequence c_n has the asymptotic behavior indicated in (2.1), in a very strong sense: the sequence is *equal* to the asymptotic.

Exercise 1. Generalize the above lemma to describe in terms of the coefficients c_n what it means for f to extend to a meromorphic function on $\mathcal{C}(A, B)$ for some $A < a$ and $B > b$, allowing the possibility of multiple poles of arbitrary order.

Exercise 2. Let c_n denote the Fibonacci sequence, thus $c_n = 0$ for $n < 0$ and

$$c_0 = 1, \quad c_1 = 1, \quad c_{n+2} - c_{n+1} - c_n = 0.$$

This exercise rederives a standard formula for this sequence in a way that is intended to illustrate the technique of Lemma 2.7.

(1) Verify by crude estimation that the fundamental interval for the series $f(z) = \sum_n c_n z^n$ contains $(0, 1/2)$.
(2) Show that

$$f(z) = \frac{1}{1 - z - z^2} = \frac{1}{(1 - z/\varphi)(1 - z/\varphi')},$$

where

$$\varphi = \frac{1 + \sqrt{5}}{2} = 1.618\cdots, \quad \varphi' = \frac{1 - \sqrt{5}}{2} = -0.618\cdots.$$

(3) Following the proof of Lemma 2.7, show that

$$c_n = \frac{\varphi^n - (\varphi')^n}{\varphi - \varphi'}.$$

(Use that $f(z) \ll |z|^2$ for $|z| \geq 2$ to show that the "remainder term", namely the integral in (2.2), tends to zero as $B^- \to \infty$.)

Example 2.9. Let $\beta \in \mathbb{C} - \{0\}$ and $a \in \mathbb{Z}_{\geq 0}$. Then one verifies by induction on a, using differentiation, that

$$\frac{1}{(z - \beta)^{a+1}} = (-\beta)^{-a-1} \sum_{n \geq 0} \binom{n+a}{n} \beta^{-n} z^n,$$

where the binomial coefficient expands to a polynomial of degree a in n:

$$\binom{n+a}{a} = \frac{(n+a)!}{a!n!} = \frac{(n+1)(n+2)\cdots(n+a)}{a!}.$$

More generally, given any coefficients $c_0, c_1 \ldots, c_k$, we have

$$\sum_{k=0}^{a} \frac{c_k}{(z - \beta)^{k+1}} = \sum_{n \geq 0} P(n) \beta^{-n} z^n$$

for some polynomial $P(n)$ of degree at most a. Conversely, given such a polynomial, we may find coefficients so that the above identity holds.

Example 2.10. Take

$$c_n := e^{-2^n}.$$

Observe that

$$c_n \to \begin{cases} 0 & \text{if } n \to \infty, \\ 1 & \text{if } n \to -\infty. \end{cases}$$

Moreover, as $n \to \infty$, the convergence of the c_n to zero is rapid in the sense that for each $B < \infty$, we have

$$c_n \ll B^{-n}.$$

The fundamental interval is thus $(1, \infty)$: the series $f(z) = \sum_n c_n z^n$ converges absolutely for $|z| > 1$ and defines a holomorphic function there. We will show that *f extends to a meromorphic function on $\mathbb{C} - \{0\}$, which is holomorphic away from simple poles at $1/2^k$ (for $k \in \mathbb{Z}_{\geq 0}$) with residue $(-1/4)^k/k!$.* To that end, observe first that the contribution to f from $n \geq 0$, namely

$$f_+(z) := \sum_{n>0} c_n z^n,$$

converges absolutely and is thus holomorphic on the entire complex plane. The meromorphic continuation of f thereby reduces to that of the complementary sum

$$f_-(z) := \sum_{n \leq 0} c_n z^n.$$

Inspired by Lemma 2.7, we study the asymptotics of the coefficients c_n as $n \to -\infty$. These are described by the Taylor series of the exponential functions:

$$e^x = \sum_{k \geq 0} \frac{x^k}{k!}.$$

By estimating the tail of this series, we sees that for $x = \mathrm{O}(1)$ and $M = \mathrm{O}(1)$, we have

$$e^x = \sum_{k=0}^{N-1} \frac{x^k}{k!} + \mathrm{O}(x^M).$$

It follows that for $n \leq 0$,

$$c_n = \sum_{k=0}^{N-1} \frac{(-2^n)^k}{k!} + \mathrm{O}(2^{nM}). \tag{2.5}$$

By the method of proof of Lemma 2.7, we deduce from this estimate that f_- the required assertions concerning the meromorphic continuation of f. Let us spell this deduction out for the sake of practice. Set

$$g_k(z) := \sum_{n \leq 0} \frac{(-2^n)^k}{k!} z^n.$$

The estimate (2.5) implies that the modified series

$$f_-(z) - \sum_{k=0}^{N-1} g_k(z) = \sum_{n \leq 0} \left(c_n - \sum_{k=0}^{N-1} \frac{(-2^n)^k}{k!} \right) z^n \tag{2.6}$$

converges absolutely for $|z| > 1/2^M$, hence defines a holomorphic function there. On the other hand, for $|z| > 1$, we see by summing the geometric series that

$$g_k(z) = \frac{(-1)^k}{k!} \frac{1}{1 - 1/2^k z} = \frac{(-1/2)^k}{k!} \frac{z}{z - 1/2^k}.$$

Thus g_k extends to a meromorphic function whose only pole is a simple one at $z = 1/2^k$ with residue $(-1/4)^k/k!$. It follows that f has the claimed meromorphic properties.

TODO: I'll add some more systematic notes on regularization later.

Remark 2.11. We can cases view $f(1)$ as the "regularized sum" of the (possibly divergent) series $\sum_n c_n$:

$$\sum_n^{\text{reg}} c_n := \left(\sum_n c_n z^n \right) |_{z=1},$$

keeping in mind here that the series may initially convergent away from the point $z = 1$, so that the specialization is understood as the result of analytic continuation. We make this definition whenever the series is holomorphic at $z = 1$.

For example,

$$\sum_{n \geq 0}^{\text{reg}} (-1)^n = \left(\sum_{n \geq 0} (-1)^n z^n \right) |_{z=1} = \frac{1}{1+z} |_{z=1} = \frac{1}{2}.$$

In this example, we may understand $f(1)$ as the limit of the quantities $f(z)$ for $z < 1$ as $z \to 1$, and also as the Cesaro mean of the partial sums of the series $\sum_{n \geq 0} (-1)^n$, so the interpretation of $f(1)$ as a regularized sum makes intuitive sense.

In other examples, the interpretation may be less clear. For example,

$$\sum_{n \geq 0}^{\text{reg}} 10^n = \left(\sum_{n \geq 0} 10^n z^n \right) |_{z=1} = \frac{1}{1 - 10z} |_{z=1} = \frac{-1}{9},$$

the intuitive meaning of which may be less clear. One way to understand the regularization is as follows: the value $f(1)$ is insensitive to replacing the sequence $(c_n)_n$ by any of its shifts $(c_{n+k})_n$. Setting

$$S := \sum_{n \geq 0}^{\text{reg}} 10^n,$$

we should thus have

$$10S = \sum_{n \geq 0}^{\text{reg}} 10^{n+1} = 1 + S,$$

from which it follows that $S = -1/9$.

Example 2.12. Take

$$c_n = n^3 e^{-2^{-n^2}}.$$

Then the series f converges absolutely nowhere: the fundamental interval is empty. But we can regularize it to get a well-defined f. TODO: expand.

Idea: meromorphically continue positive and negative parts separately, take the sum of these on intersections of domains of definition.

TODO: expansion of c_n.

Only pole for both f_+ and f_- is at one; quadruple pole, with residues that cancel.

Example 2.13. TODO. For $a \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathbb{C}^\times$, take

$$c_n := n^a \beta^{-n}.$$

Then we claim that the regularized generating function f vanishes identically.

TODO: one way to see this is to derive that $f(z) = (z/\beta) f(z)$.

3. The zeta function

3.1. **Overview.** The Riemann zeta function is defined for a complex number s by the series

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

Lemma 3.1. *The series converges absolutely for* $\Re(s) > 1$*, uniformly for* $\Re(s) \geq 1+\varepsilon$ *for each* $\varepsilon > 0$.

Proof. Using the identity

$$\left| \frac{1}{n^s} \right| = \frac{1}{n^{\Re(s)}},$$

we reduce to the case that s is real, in which this is a familiar consequence of the integral test. □

Our first main goal in the course is to explain the following basic facts.

Theorem 3.2. *The Riemann zeta function admits a meromorphic continuation to the entire complex plane. It is holomorphic away from a simple pole at* $s = 1$*, where it has residue* 1*. It admits a functional equation relating* $\zeta(s)$ *to* $\zeta(1-s)$.

One historical motivation for considering the zeta function at complex arguments comes from the prime number theorem.

Theorem 3.3 (Prime number theorem)**.** *Let* $\pi(x) := \# \{primes\ p \leq x\}$ *denote the prime counting function. Then*

$$\frac{\pi(x)}{x / \log x} \rightarrow 1 \text{ as } x \rightarrow \infty.$$

This is related to the following analytic fact concerning the zeros of the zeta function.

Theorem 3.4 (Prime number theorem, formulated in terms of ζ)**.** *We have* $\zeta(s) = 0$ *only if* $\Re(s) < 1$.

Remark 3.5. Even the statement of Theorem 3.4 is not clear without knowing the meromorphic continuation of ζ. This may offer some motivation for understanding the latter.

We expect stronger nonvanishing properties:

Conjecture 1 (Riemann Hypothesis)**.** *We have* $\zeta(s) = 0$ *only if* $\Re(s) < 1/2$.

This corresponds to a conjectural stronger form of the prime number theorem, namely that

$$\pi(x) = \int_2^x \frac{t}{\log t}\, dt + \mathrm{O}(x^{1/2} \log x).$$

References