1. Some things to think about in the afternoons and beyond

(1) Understand the details of the consequences of subconvexity mentioned in lecture (reference: [19, §5]):
  - Subconvexity vs. geometry of numbers.
  - Distinguishing modular forms.
  - Duke theorem. Reduction of supersingular elliptic curves.
  - Quantum unique ergodicity. One exercise here is to understand how this follows from subconvexity in the special case of Eisenstein series, as in [17, §2].

(2) Some "classic" papers (non-exhaustive):
  - Bounds for Fourier coefficients of half-integral weight, applications to quadratic forms: [15], [9], [11]
  - Subconvexity for GL_2: [10] [12] [13].
  - Moments and amplification via periods: [27], [20], [16], [24].
  - Shifted convolution sums:
    - via δ-symbol: [10]
    - via periods: [25], [4], [5].
  - Papers emphasizing variation of the test vector: [23, 2, 27].

  One exercise is draw parallels, e.g., between
  - [20, Thm 5.1] and [10],
  - [27, §4] and [12], or
  - [20, Thm 5.2] and [13].

  Another is to reprove some results using different methods, e.g., by working out a "classical" proof in the style of [13] for subconvexity for Maass forms at special points, namely, for $L(1/2+it_f, f)$ with f on $\mathrm{SL}_2(\mathbb{Z})$ of eigenvalue $1/4+t_f^2$, by estimating an amplified fourth moment, e.g.,

$$\sum_{f:t_f\in[T,T+1]} \left|\sum_{\ell\asymp L} c_\ell \lambda_f(\ell)\right|^2 \left|L(\tfrac{1}{2}+it_f, f)\right|^4 .$$

(3) Study the proof of the convexity bound. There are two steps:
  - The Phragmen–Lindelöf convexity principle, to reduce estimates for $\Re(s) = 1/2$ to estiamtes for $\Re(s) = 1+\varepsilon$ and $\Re(s) = -\varepsilon$.
  - The functional equation, to reduce further to estimates for $\Re(s) = 1+\varepsilon$.
  - Establishing the necessary bounds for $\Re(s) = 1+\varepsilon$, for which see `https://www.math.wsu.edu/faculty/scliu/papers/Convexity.pdf` and references.

(4) Some recent papers, concerning subconvexity or related problems, that haven't been fully explored (e.g., interpreted via integral representations):
  - δ-method papers such as [26] and [1]
  - Higher moments over very large families, as in [7], [8]
  - Rankin–Selberg when the rank difference is larger than one, as in [6]

(5) Higher rank subconvex bounds [3], [18], [22], [21], [14]. There are many "exercises" implicit in these papers; for instance, a half-dozen are suggested in [22, Remark 1.4]. Some other questions:
  - These have all proceeded via arithmetic amplification. Is it possible to succeed in some cases via "family shortening" (as in, e.g., [25])? A natural case to try would be the t-aspect. Some experiments with GL_2

suggest this is difficult (see `https://ultronozm.github.io/math/20230522T174726__shrinking-archimedean-families-second-moment-gl2.html`).

"Purely horizontal" aspects remain open, e.g., twists by Dirichlet characters of prime conductor on GL_4.

References

[1] Keshav Aggarwal, Wing Hong Leung, and Ritabrata Munshi. Short second moment bound and subconvexity for gl(3) l-functions. 06 2022. arXiv:2206.06517.

[2] Joseph Bernstein and Andre Reznikov. Subconvexity bounds for triple L-functions and representation theory. *Ann. of Math. (2)*, 172(3):1679–1718, 2010.

[3] Valentin Blomer and Jack Buttcane. On the subconvexity problem for L-functions on GL(3). *Ann. Sci. Éc. Norm. Supér. (4)*, 53(6):1441–1500, 2020.

[4] Valentin Blomer and Gergely Harcos. The spectral decomposition of shifted convolution sums. *Duke Math. J.*, 144(2):321–339, 2008.

[5] Valentin Blomer, Subhajit Jana, and Paul D. Nelson. Local integral transforms and global spectral decomposition. 04 2024. arXiv:2404.10692.

[6] Valentin Blomer, Xiaoqing Li, and Stephen D. Miller. A spectral reciprocity formula and non-vanishing for L-functions on $\mathrm{GL}(4) \times \mathrm{GL}(2)$. *J. Number Theory*, 205:1–43, 2019.

[7] Vorrapan Chandee and Xiannan Li. The 8th moment of the family of $\Gamma_1(q)$-automorphic L-functions. *Int. Math. Res. Not. IMRN*, (22):8443–8485, 2020.

[8] Vorrapan Chandee and Xiannan Li. The second moment of $GL(4) \times GL(2)$ L-functions at special points. *Adv. Math.*, 365:107060, 39, 2020.

[9] W. Duke. Hyperbolic distribution problems and half-integral weight Maass forms. *Invent. Math.*, 92(1):73–90, 1988.

[10] W. Duke, J. Friedlander, and H. Iwaniec. Bounds for automorphic L-functions. *Invent. Math.*, 112(1):1–8, 1993.

[11] W. Duke, J. Friedlander, and H. Iwaniec. Bilinear forms with Kloosterman fractions. *Invent. Math.*, 128(1):23–43, 1997.

[12] W. Duke, J. B. Friedlander, and H. Iwaniec. Bounds for automorphic L-functions. II. *Invent. Math.*, 115(2):219–239, 1994.

[13] W. Duke, J. B. Friedlander, and H. Iwaniec. Bounds for automorphic L-functions. III. *Invent. Math.*, 143(2):221–248, 2001.

[14] Yueke Hu and Paul D Nelson. Subconvex bounds for $u_{n+1} \times u_n$ in horizontal aspects. 09 2023. arXiv:2309.06314.

[15] Henryk Iwaniec. Fourier coefficients of modular forms of half-integral weight. *Invent. Math.*, 87(2):385–401, 1987.

[16] Henryk Iwaniec and Peter Sarnak. L^∞ norms of eigenfunctions of arithmetic surfaces. *Ann. of Math. (2)*, 141(2):301–320, 1995.

[17] Wenzhi Luo and Peter Sarnak. Quantum ergodicity of eigenfunctions on $\mathrm{PSL}_2(\mathbf{Z})\backslash\mathbf{H}^2$. *Inst. Hautes Études Sci. Publ. Math.*, (81):207–237, 1995.

[18] Simon Marshall. Subconvexity for l-functions on $\mathrm{u}(n) \times \mathrm{u}(n+1)$ in the depth aspect. 09 2023. arXiv:2309.16667.

[19] Philippe Michel. Analytic number theory and families of automorphic L-functions. In *Automorphic forms and applications*, volume 12 of *IAS/Park City Math. Ser.*, pages 181–295. Amer. Math. Soc., Providence, RI, 2007.

[20] Philippe Michel and Akshay Venkatesh. The subconvexity problem for GL_2. *Publ. Math. Inst. Hautes Études Sci.*, (111):171–271, 2010.

[21] Paul D. Nelson. Bounds for standard L-functions. *arXiv e-prints*, page arXiv:2109.15230, September 2021.

[22] Paul D. Nelson. Spectral aspect subconvex bounds for $U_{n+1} \times U_n$. *Invent. Math.*, 232(3):1273–1438, 2023.

[23] Andre Reznikov. Rankin-Selberg without unfolding and bounds for spherical Fourier coefficients of Maass forms. *J. Amer. Math. Soc.*, 21(2):439–477, 2008.

[24] Peter Sarnak. Fourth moments of Grössencharakteren zeta functions. *Comm. Pure Appl. Math.*, 38(2):167–178, 1985.

[25] Peter Sarnak. Estimates for Rankin-Selberg L-functions and quantum unique ergodicity. *J. Funct. Anal.*, 184(2):419–453, 2001.

[26] Prahlad Sharma. Subconvexity for $GL(3) \times GL(2)$ twists. *Adv. Math.*, 404:Paper No. 108420, 47, 2022. With an appendix by Will Sawin.

[27] Akshay Venkatesh. Sparse equidistribution problems, period bounds and subconvexity. *Ann. of Math. (2)*, 172(2):989–1094, 2010.