SOME EXERCISES CONCERNING LOCALIZED VECTORS IN LOW RANK

Contents

Abstract. We record some exercises whose purpose is to verify that certain classes of vectors in representations of SO(3) and $\mathrm{PGL}_2(\mathbb{R})$ are "localized" in a strong sense under the action of those groups.

1. Overview

The purpose of this note is to record some exercises that aim to convey some computational feeling for "localized vectors" in the precise sense defined in §3 of this note, focusing on low-rank examples.

We let $T \to \infty$ be an asymptotic parameter, and retain the asymptotic notation and conventions of §2.1 of this note concerning "T-dependent elements", "fixed" and "classes".

2. The group SO(3) via weight vectors

2.1. **Lie algebra.** We consider the Lie group SO(3). Its Lie algebra $\mathfrak{so}(3)$ admits a basis $\{R_1, R_2, R_3\}$, where for any angle θ, the element $\exp(\theta R_j)$ defines rotation by θ about the jth axis. These satisfy the commutation relations

$$[R_1, R_2] = R_3, \quad [R_2, R_3] = R_1, \quad [R_3, R_1] = R_2.$$

The center of the universal enveloping algebra is generated by the Casimir element

$$\Omega = -(R_1^2 + R_2^2 + R_3^2).$$

Define the following elements X, Y of the complexified Lie algebra $\mathfrak{so}(3)_{\mathbb{C}}$:

$$X := R_1 + iR_2, \quad Y := -R_1 + iR_2.$$

Then X, Y, R_3 is a basis for $\mathfrak{so}(3)_{\mathbb{C}}$ satisfying the commutation relations

$$[X, Y] = 2iR_3, \quad [iR_3, X] = X, \quad [iR_3, Y] = -Y. \tag{2.1}$$

We observe also that

$$R_1 = \frac{X - Y}{2}, \quad R_2 = \frac{X + Y}{2i},$$

$$\Omega = \frac{XY + YX}{2} - R_3^2.$$

By writing $XY = [X, Y] + YX$ and appealing to the formula (2.1) for $[X, Y]$, we see that

$$\Omega = YX + iR_3(iR_3 + 1). \tag{2.2}$$

Similarly,

$$\Omega = XY + iR_3(iR_3 - 1). \tag{2.3}$$

The imaginary dual of the Lie algebra identifieswith the space of triples of imaginary numbers:

$$\mathfrak{so}(3)^\wedge \cong i\mathbb{R}^3. \tag{2.4}$$

Here $\xi \in i\mathbb{R}^3$ corresponds to the linear map $\mathfrak{so}(3) \to i\mathbb{R}$ given on basis elements by $R_j \mapsto \xi_j$.

2.2. **Representations.** Let π be a (complex) representation of SO(3). It may be decomposed into eigenspaces for R_3. Since $\exp(2\pi R_3) = 1$, the eigenvalues of iR_3 are of the form m with $m \in \mathbb{Z}$:

$$\pi = \oplus_{m \in \mathbb{Z}} \pi(m), \quad \pi(m) := \{v \in \pi : iR_3 v = mv\}.$$

The m for which $\pi(m) \neq 0$ are called the *weights* of π, and the dimensions $\dim \pi(m)$ the corresponding *weight multiplicities*. From the commutation relations (2.1), we see that

$$X : \pi(m) \to \pi(m + 1), \quad Y : \pi(m) \to \pi(m - 1). \tag{2.5}$$

Proposition 2.1. *Let π be an irreducible unitary representation of* SO(3)*. Then Ω acts on π by a scalar of the form*

$$\Omega_\pi = \ell(\ell + 1) \tag{2.6}$$

for some nonnegative integer ℓ. This scalar determines the isomorphism class of π. In fact, there is a basis

$$e_{-\ell}, \qquad e_{-\ell+1}, \qquad \dots, \qquad e_\ell$$

for π on which the Lie algebra acts by the formulas

$$Xe_m = (\Omega_\pi - m(m + 1))^{1/2} e_{m+1} \tag{2.7}$$

$$Ye_{m+1} = (\Omega_\pi - m(m + 1))^{1/2} e_m, \tag{2.8}$$

$$iR_3 e_m = me_m. \tag{2.9}$$

If π is unitary, then this basis is orthonormal.

Proof. Since SO(3) is compact, we know by the Peter–Weyl theorem that π is finite-dimensional. There is thus a largest element $\ell \in \mathbb{Z}_{\geq 0}$ with $\pi(\ell) \neq 0$. For any $v \in \pi(\ell)$, we have $Xv \in \pi(\ell+1) = \{0\}$, hence $Xv = 0$. By (2.2), it follows that

$$0 = YXv = \Omega v - \ell(\ell+1)v. \tag{2.10}$$

Since π is irreducible and Ω commutes with the action of SO(3), we know by Schur's lemma that Ω acts on π by a scalar. Taking v to be a nonzero element of $\pi(\ell)$, we see from (2.10) shows that this scalar must be given by (2.6). We now choose a nonzero vector $e_\ell \in V(\ell)$ and inductively define e_m for integers m with $-\ell \leq m < \ell$ by requiring that (2.8) hold, noting that the square root is positive in the stated range. Using (2.3), we see that $Ye_{-\ell} = 0$. By the mapping property (2.5), we have $e_m \in \pi(m)$, hence (2.9) holds. By the formula (2.3) for Ω, we have

$$XYe_{m+1} = \Omega e_{m+1} - m(m+1)e_{m+1} = (\ell(\ell+1) - m(m+1))e_{m+1},$$

and so (2.7) holds. From the formulas established thus far, we see that $\{e_{-\ell}, e_{-\ell+1}, \dots, e_\ell\}$ spans an invariant subspace of π, which by the irreducibility hypothesis must be π itself.

We have established all assertions except that the basis is may be taken orthonormal when π is unitary. We may assume that e_ℓ was normalized to be a unit vector. We will verify by reverse inductive on m with $-\ell \leq m < \ell$ that e_m is then likewise a unit vector. To that end, observe first by (2.8) that

$$(\Omega_\pi - m(m+1))\|e_m\|^2 = \langle Ye_{m+1}, Ye_{m+1}\rangle .$$

We now use that the adjoint of Y is $-\bar{Y} = X$ to see that

$$\langle Ye_{m+1}, Ye_{m+1}\rangle = \langle XYe_{m+1}, e_{m+1}\rangle .$$

By (**??**), we deduce that e_m and e_{m+1} have the same norm, so the induction follows as claimed. □

The integer ℓ as in the conclusion of Proposition 2.1 is called the *highest weight* of π. The coadjoint orbit for π is given in the optic (2.4) by the sphere of radius $\ell + 1/2$:

$$\mathcal{O}_\pi = \left\{(a,b,c) : a^2 + b^2 + c^2 = (T + \tfrac{1}{2})^2\right\}.$$

2.3. **Localized vectors.** In the following exercises, we assume that the asymptotic parameter $T \to \infty$ is a nonnegative integer, and let π denote the T-dependent representation having highest weight T.

Exercise 1. Let M denote the class of T-dependent vectors v in π of the form

$$v = \sum_{m=-T}^{T} a_m e_m, \tag{2.11}$$

where the coefficients a_m have the following properties:

(1) $a_m = 0$ unless $m = T + \mathrm{O}(1)$.
(2) Each $a_m = \mathrm{O}(1)$.

Clearly, the following conditions hold:

- For each $v \in M$, we have $\|v\| \leq T^{\mathrm{O}(1)}$.
- For all $u, v \in M$, we have $u + v \in M$.
- For all $v \in M$ and $c \in \mathbb{C}$ with $c = \mathrm{O}(1)$, we have $cv \in M$.

Verify the following approximate eigenvector property for the action of the $\mathfrak{g}$ on M: for all $v \in M$, we have

$$Xv \in T^{1/2}M,$$
$$Yv \in T^{1/2}M,$$
$$iR_3v - iTv \in T^{1/2}M.$$

Deduce from Theorem 3.6 of this note that every element of M is localized at the T-dependent element $\tau \in \mathfrak{g}^\wedge$ given in the optic (2.4) by

$$\tau = (0, 0, iT).$$

Exercise 2. Modify Exercise 1 as follows. Change the conditions on the coefficients to the following:

(1) $a_m = 0$ unless $m = T + \mathrm{O}(T^{1/2})$.
(2) $\sum_m |a_m|^2 = \mathrm{O}(1)$.

Verify then that for all $v \in M$,

$$Xv - Tv \in T^{1/2}M,$$
$$Yv - Tv \in T^{1/2}M,$$
$$R_3v \in T^{1/2}M.$$

Deduce that every element of M is localized at the T-dependent element $\tau \in \mathfrak{g}^*$ given in the optic (2.4) by

$$\tau = (0, -iT, 0).$$

Observe that in either case, τ lies quite close to the coadjoint orbit of π (i.e., the sphere of radius $T + \frac{1}{2}$).

3. The group $\mathrm{PGL}_2(\mathbb{R})$ via weight vectors

3.1. **Preliminaries.** We now turn to the group $\mathrm{PGL}_2(\mathbb{R})$. We will use the following notation for a basis of its complexified Lie algebra $\mathfrak{sl}_2(\mathbb{R})_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$:

$$X := \frac{1}{2i}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad Y := \frac{1}{2i}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad H := \frac{1}{2i}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The standard maximal compact connected subgroup K of G, namely the image of $\mathrm{SO}(2)$, is then

$$K = \{\exp(\theta H) : \theta \in \mathbb{R}/2\pi\mathbb{Z}\}.$$

The commutation relations are

$$[X, Y] = -2H, \quad [H, X] = X, \quad [H, Y] = -Y.$$

The center of the universal enveloping algebra is generated by the Casimir element

$$\Omega := H^2 - \frac{XY + YX}{2} = H(H-1) - XY = H(H+1) - YX.$$

The imaginary dual of the Lie algebra identifies with the space of imaginary traceless 2×2 matrices:

$$\mathfrak{sl}_2(\mathbb{R})^\wedge \cong i\,\mathfrak{sl}_2(\mathbb{R}). \tag{3.1}$$

Here $\xi \in i\,\mathfrak{sl}_2(\mathbb{R})$ corresponds to the linear map $\mathfrak{sl}_2(\mathbb{R}) \to i\mathbb{R}$ given by $x \mapsto \mathrm{trace}(x\xi)$.

3.2. Representations.

Proposition 3.1. *Let π be an irreducible unitary representation of* SO(3). *Then Ω acts on π by a scalar, say Ω_π. Then, either:*

(i) π is a one-dimensional representation, either trivial or the sign representation, in which case $\Omega_\pi = 0$.
(ii) π is a discrete series representation $\pi(k)$ for some $k \in \mathbb{Z}_{\geq 1}$, with $\Omega_\pi = k(k-1)$.
(iii) π is a unitary principal series representation $\pi(t, \varepsilon)$, with
- *$t \in \mathbb{R}$ and $\varepsilon \in \{\pm 1\}$, or*
- *$t \in i(\frac{1}{2}, \frac{1}{2}) - \{0\}$ and $\varepsilon = 1$,*

with $\Omega_\pi = -\frac{1}{4} - t^2$.

The only equivalences are that $\pi(t, \varepsilon) \cong \pi(-t, \varepsilon)$.

The representation $\pi = \pi(t, \varepsilon)$ admits a basis e_m, indexed by $m \in \mathbb{Z}$, on which the Lie algebra elements act by the formulas

$$Xe_m = (m(m+1) - \Omega_\pi)^{1/2} e_{m+1},$$

$$Ye_{m+1} = (m(m+1) - \Omega_\pi)^{1/2} e_m,$$

$$He_m = me_m,$$

$$\operatorname{diag}(-1, 1) e_m = (-1)^\varepsilon e_{-m}.$$

The representation $\pi = \pi(k)$ admits a basis e_m, indexed by $\{m \in \mathbb{Z} : |m| \geq k\}$, on which the Lie algebra elements act by the same formulas as above, but with $\varepsilon = 1$.

Proof. Similar to that of Proposition 2.1. □

The tempered irreducible representations are the $\pi(k)$ and the $\pi(t, \varepsilon)$ with $t \in \mathbb{R}$. For either of these, the coadjoint orbit $\mathcal{O}_\pi$ is given in the optic (3.1) by

$$\mathcal{O}_\pi = \left\{ 0 \neq \xi \in i\, \mathfrak{sl}_2(\mathbb{R}) : \det(\xi / i) = \tfrac{1}{4} + \Omega_\pi \right\}. \tag{3.2}$$

3.3. Localized vectors.

Exercise 3. Let π be the T-dependent representation of $\mathrm{PGL}_2(\mathbb{R})$ given by the discrete series representation $\pi_T = \pi(k)$ of lowest weight

$$k = k_T := T.$$

Let M denote the class of T-dependent vectors v in π of the form $v = \sum_m a_m e_m$, where

(1) $a_m = 0$ unless $m = T + \mathrm{O}(1)$, and
(2) each $a_m = \mathrm{O}(1)$.

Verify that for all $v \in M$, we have

$$Xv \in T^{1/2} M,$$

$$Yv \in T^{1/2} M,$$

$$Hv - Tv \in T^{1/2} M.$$

Deduce that every element of M is localized at the T-dependent element $\tau \in \mathfrak{g}^*$ given in the optic (3.1) by

$$\tau = iT \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Exercise 4. Let π be the T-dependent representation of $\mathrm{PGL}_2(\mathbb{R})$ given by tempered principal series representation $\pi_T = \pi(t, \varepsilon)$, where

$$t = t_T := T$$

while $\varepsilon \in \{\pm 1\}$ is fixed. Let M denote the class of T-dependent vectors v in π of the form $v = \sum_m a_m e_m$, where

- $a_m = 0$ unless $m = \mathrm{O}(T^{1/2})$, and
- $\sum_m |a_m|^2 = \mathrm{O}(1)$.

Verify that for all $v \in M$, we have

$$Xv - Tv \in T^{1/2} M,$$
$$Yv - Tv \in T^{1/2} M,$$
$$Hv \in T^{1/2} M.$$

Deduce that every element of M is localized at

$$\tau = iT \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Exercise 5. Let π_1 be any fixed infinite-dimensional irreducible unitary representation of $\mathrm{PGL}_2(\mathbb{R})$ (e.g., the tempered principal series representation $\pi(0, 1)$). Let M denote the class of T-dependent vectors v in π of the form $v = \sum_m a_m e_m$, where

- $a_m = 0$ unless $m = T + \mathrm{O}(T^{1/2})$, and
- $\sum_m |a_m|^2 = \mathrm{O}(1)$.

Verify that for all $v \in M$, we have

$$Xv - Tv \in T^{1/2} M,$$
$$Yv - Tv \in T^{1/2} M,$$
$$Hv - Tv \in T^{1/2} M.$$

Deduce that every element of M is localized at

$$\tau = iT \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Note that in each of these examples, τ lies quite close to $\mathcal{O}_\pi$.

4. The group $\mathrm{PGL}_2(\mathbb{R})$ via the Kirillov model

4.1. **Preliminaries.** Set $G := \mathrm{PGL}_2(\mathbb{R})$. We will work with the subgroups

$$N := \left\{ n(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\},$$

$$A := \left\{ a(y) := \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} : y \in \mathbb{R}^\times \right\},$$

$$B := NA = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

Let $\psi : \mathbb{R} \to \mathrm{U}(1)$ be a nontrivial unitary character. We may identify ψ with a character of the subgroup Set

$$C^\infty((N, \psi)\backslash G) := \{W \in C^\infty(G) : W(n(x)g) = \psi(x) W(g) \text{ for all } (x, g) \in \mathbb{R} \times G\}$$

Let π be an irreducible representation of G. More precisely, we denote here by π the subspace of smooth vectors. We recall that π is *generic* if there is an equivariant embedding $\pi \hookrightarrow C^\infty((N,\psi)\backslash G)$. The space of such embeddings is then one-dimensional, so the image, call it $\mathcal{W}(\pi,\psi)$, is well-defined. Moreover, the restriction map

$$\mathcal{W}(\pi,\psi) \to \{\text{functions } A \to \mathbb{C}\}$$

is injective, and its image contains $C_c^\infty(A)$. Consequently, each $W \in \mathcal{W}(\pi,\psi)$ is determined by the function $W : \mathbb{R}^\times \to \mathbb{C}$ given by

$$W(y) := W(a(y)),$$

and every smooth compactly-supported function arises in this way. We obtain in this way a realization of π as a space of functions on $\mathbb{R}^\times$, called the *Kirillov model*. When π is unitary, an invariant inner product may be given in the Kirillov model by

$$\|W\|^2 := \int_{y\in\mathbb{R}^\times} W(y)\, d^\times y, \quad d^\times y := \frac{dy}{|y|}. \tag{4.1}$$

Standard references for these facts include [2, §6], [1, §10.2], [3].

The action of B on the Kirillov model is completely explicit: we have

$$n(x)W(y) = \psi(yx)W(y), \tag{4.2}$$

$$a(u)W(y) = W(yu). \tag{4.3}$$

Indeed (4.2) follows from the commutation property $a(y)n(x) = n(yx)a(y)$ and the left N-equivariance of W, while (4.3) is obvious.

The infinitesimal generators of N and A are the matrices

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h := \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These act on π by differential operators. The formulas for their action is simplest when

$$\psi(x) := e^{ix}, \tag{4.4}$$

so let's specialize to that case. By differentiating (4.2) and (4.3), we see that

$$eW(y) = iyW(y), \tag{4.5}$$

$$hW(y) = y\partial_y W(y). \tag{4.6}$$

The other standard Lie algebra basis element is

$$f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These elements satisfy

$$[e,f] = 2h, \quad [h,e] = e, \quad [h,f] = -f.$$

The Casimir element Ω is given (with the same normalization as in §3.1) by

$$\Omega = h^2 + \frac{ef+fe}{2} = h^2 - h + ef.$$

Writing Ω_π as before for the eigenvalue by which Ω acts on π, we can solve for the action of f on π:

$$fW(y) = \frac{1}{iy}\left(\Omega_\pi - (y\partial_y)^2 + y\partial_y\right)W(y). \tag{4.7}$$

This formula is the key to verifying the fact recorded above, that any smooth compactly-supported function on $\mathbb{R}^\times$ arises from some (smooth!) vector in π. We refer to [3] and [4, §12] (direct link: §**??**) for further discussion.

4.2. **Localized vectors.** We just give one representative example of the sort of analysis that can be achieved in this way.

Exercise 6. Let π be a T-dependent generic irreducible unitary representation of $\mathrm{PGL}_2(\mathbb{R})$, realized in its Kirillov model with respect to the character (4.4) as above and with unitary structure given by (4.1). Assume that

$$\Omega_\pi = \mathrm{O}(T^2).$$

In other words, in the notation of §3.2, either

(1) $\pi = \pi(t, \varepsilon)$ with $t = \mathrm{O}(T)$, or
(2) $\pi = \pi(k)$ with $k = \mathrm{O}(T)$.

Let ρ be a T-dependent real number with $\rho = \mathrm{O}(T)$. Let M denote the class of all T-dependent elements $W \in \pi$ that are given in the Kirillov model for large enough T by the formula

$$W(y) = T^{1/4} |y|^{i\rho} \phi\left(\frac{y - T}{T^{1/2}}\right),$$

where ϕ belongs to some fixed bounded subset of the space $C_c^\infty(\mathbb{R}^\times)$.

(i) Verify that $\|W\| = \mathrm{O}(1)$ for each $W \in M$.
(ii) Verify that for each $W \in M$, we have

$$eW - iTW \in T^{1/2} M,$$

$$hW - i\rho W \in T^{1/2} M.$$

(iii) Use (4.7) to show that

$$fW - i\beta W \in T^{1/2} M,$$

where

$$\beta := \frac{\rho^2 - \Omega_\pi}{T}.$$

(iv) Deduce that every element of M is localized at the T-dependent element of $\mathfrak{g}^\wedge$ given by

$$\tau = i \begin{pmatrix} \rho & \beta \\ T & -\rho \end{pmatrix},$$

for which

$$\det(\tau / i) = \Omega_\pi.$$

(Compare with (3.2).)

5. TODO

(1) Something for SO(3) using the Borel–Weil model over $\mathbb{CP}^1$.
(2) Something for induced models for $\mathrm{PGL}_2(\mathbb{R})$.
(3) Some follow-up to exercise 4.2 explaining what it says about $wW(y)$, for w the nontrivial Weyl element.

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