NOTES ON HASEO KI'S L^4-NORM BOUND

Contents

Abstract. We record a detailed exposition of the proof of Haseo Ki's L^4-norm bound for Maass forms.

1. Notation and setup

Let t traverse a sequence of positive real numbers, tending off to ∞. In what follows, we adopt the convention that everything is allowed by default to depend upon t unless we declare it to be fixed (or, synonymously, write "fix ..."). Moreover, all assertions are understood as holding for t sufficiently large. We adopt the asymptotic notation: if $|A| \leq C|B|$ for some fixed C, then we write

$$A = \mathrm{O}(B), \quad A \ll B \quad \text{or} \quad B \gg A,$$

while if $|A| \leq c|B|$ for each fixed $c > 0$, then we write

$$A = o(B), \quad A \lll B \quad \text{or } B \ggg A.$$

We adopt the following shorthand:

$$A \simeq B \iff A = B + o(1),$$
$$A \asymp B \iff A \ll B \ll A,$$
$$A \prec B \iff A \ll t^{o(1)} B.$$

We abbreviate $e(x) := \exp(2\pi i x)$.

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2. Statement of result

Let $\lambda(n)$ be a sequence of real numbers, indexed by the natural numbers $\mathbb{N} = \{1, 2, \dots\}$, with the property that for each $N \geq 1$, we have

$$\sum_{n \leq N} |\lambda(n)|^4 \prec N. \tag{1}$$

Fix $y_0 > 0$. For each measurable function $W : (0, \infty) \to \mathbb{C}$, we define

$$\mathcal{N}(W) := \int_{y=y_0}^{\infty} \int_{x \in \mathbb{R}/\mathbb{Z}} \left| \sum_n \frac{\lambda(n)}{\sqrt{n}} W(ny) e(nx) \right|^4 \frac{dy}{y^2}. \tag{2}$$

This is the fourth power of a seminorm, and so enjoys a variant of the triangle inequality:

$$\mathcal{N}(W_1 + W_2) \ll \mathcal{N}(W_1) + \mathcal{N}(W_2).$$

We denote by W_t the Whittaker function

$$W_t(y) := \sqrt{y} e^{\pi t/2} K_{it}(y),$$

which enjoys the L^2-normalization property

$$\int_0^{\infty} |W_t(y)|^2 \frac{dy}{y} \asymp 1.$$

The arguments of [6, Theorem 2] establish the following:

Theorem 1. *We have $\mathcal{N}(W_t) \prec 1$.*

Example 2. Let φ be a Hecke–Maass cusp form for $\mathrm{SL}_2(\mathbb{Z})$ of eigenvalue $1/4 + t^2$. Let $\lambda(n)$ denote its normalized Hecke eigenvalues. It is known that (1) holds (see [1, Lemma 3.6]). We may normalize φ so that its Fourier expansion reads

$$\varphi(z) = \sum_{0 \neq n \in \mathbb{Z}} \frac{\lambda(|n|)}{|n|^{1/2}} \operatorname{sgn}(n)^a W(2\pi |n| y) e(nx)$$

for some $a \in \{0, 1\}$. It is known (see [3], [2]) that

$$\|\varphi\|_{L^2}^2 \asymp L(\operatorname{ad} \varphi, 1) = t^{o(1)}.$$

Theorem 1 implies, by integrating over a Siegel domain and considering separately the contributions of positive and negative n, that

$$\|\varphi\|_{L^4} \prec 1.$$

The idea of the proof of Theorem 1 is as follows. We smoothly localize $W_t(y)$ into several regions, according to the size Y of y, the size U of $y - t$, and the sign of $y - t$. In particular, we focus on regions described by the conditions

$$y \asymp Y, \quad y - t \asymp U.$$

We apply Parseval's identity to the x-integral in the definition of $\mathcal{N}(W)$. This introduces a sum over $n_1 + n_2 = n_3 + n_4$ of $\lambda(n_1)\lambda(n_2)\lambda(n_3)\lambda(n_4)$ weighted by an integral over y of a four-fold product of localized copies of W_t. In many cases, a satisfactory estimate will follow already from the size and support properties of the integrand. Such crude arguments suffice away from the following ranges:

(i) $t^{2/3} \lll Y \lll t$ and $U \asymp t$.
(ii) $Y \asymp t$ and $t^{1/2} \lll U \ll t$, with W_t localized on $y < t$.

In these cases, the necessary savings are achieved by integrating by parts once in the integral over y. The partial integration is a bit intricate because it involves the quadratic part of the phase.

3. Parseval

We will prove Theorem 1 by decomposing W_t into at most $\prec 1$ pieces W and showing that $\mathcal{N}(W) \prec 1$ for each piece. The basis for the piecewise estimates will be the following application of Parseval.

Lemma 3. *We have*

$$\mathcal{N}(W) \prec \sum_{n_1+n_2=n_3+n_4} \frac{|\lambda(n_1)|^4}{n_1} |I(\mathfrak{n})|,$$

where

$$I(\mathfrak{n}) := \int_{y=n_1 y_0}^{\infty} W(y) W(\tfrac{n_2}{n_1} y) \overline{W(\tfrac{n_3}{n_1} y) W(\tfrac{n_4}{n_1} y)} \frac{dy}{y^2}. \tag{3}$$

Proof. By Parseval, we have

$$\mathcal{N}(W) = \sum_{n_1+n_2=n_3+n_4} \frac{\lambda(n_1)\cdots\lambda(n_4)}{\sqrt{n_1\cdots n_4}} \int_{y=y_0}^{\infty} W(n_1 y) W(n_2 y) \overline{W(n_3 y) W(n_4 y)} \frac{dy}{y^2}.$$

We majorize each term in absolute value and apply AM-GM:

$$\left| \frac{\lambda(n_1)\cdots\lambda(n_4)}{\sqrt{n_1\cdots n_4}} \right| \leq \frac{1}{4} \sum_{i=1}^{4} \frac{|\lambda(n_i)|^4}{n_i^2}.$$

This gives

$$\mathcal{N}(W) \prec \sum_{n_1+n_2=n_3+n_4} \frac{|\lambda(n_1)|^4}{n_1^2} \left| \int_{y=y_0}^{\infty} W(n_1 y) W(n_2 y) \overline{W(n_3 y) W(n_4 y)} \frac{dy}{y^2} \right|,$$

using the symmetry of the integrand under permutation of the n_j. We conclude by performing the change of variables $y \to y/n_1$. $\square$

4. Estimating via size and support

We next record a crude estimate, taking into account the size and support properties of W but forgoing any cancellation in the integrals (3).

Lemma 4. *Suppose that $W \in C_c(\mathbb{R})$ is supported on the interval $[Y_1, Y_2]$, where $0 < Y_1 < Y_2$. Assume that*

$$Y_2 - Y_1 \gg 1 \tag{4}$$

and

$$Y_1 \asymp Y_2. \tag{5}$$

Then

$$\mathcal{N}(W) \prec \|W\|_\infty^4 \frac{(Y_2 - Y_1)^3}{Y_1^2}.$$

Proof. The support condition for W gives

$$|I(\mathbf{n})| \leq \|W\|_\infty^4 \int_{y=Y_1}^{Y_2} \frac{dy}{y^2} = \|W\|_\infty^4 \frac{Y_2 - Y_1}{Y_1 Y_2}.$$

The integrand in the definition of $I(\mathbf{n})$ vanishes identically unless

- $n_1 y_0 < Y_2$, and
- for each $j = 2,3,4$, the natural number n_j lies in the interval $J_{n_1} := [n_1 Y_1/Y_2, n_1 Y_2/Y_1]$.

Since n_4 is determined by (n_1, n_2, n_3), we deduce from Lemma 3 that

$$\mathcal{N}(W) \ll \|W\|_\infty^4 \frac{Y_2 - Y_1}{Y_1 Y_2} \mathcal{S}, \tag{6}$$

where

$$\mathcal{S} := \sum_{n_1 \leq Y_2/y_0} \frac{|\lambda(n_1)|^4}{n_1} |\mathbb{N} \cap J_{n_1}|^2.$$

The number of integers in an interval is at most one plus the length of that interval, thus

$$|\mathbb{N} \cap J_{n_1}| \leq 1 + n_1 \left(\frac{Y_2}{Y_1} - \frac{Y_1}{Y_2} \right).$$

Using our assumptions (4) and (5), as well as the upper bound $n_1 \ll Y_2$, we deduce that

$$|\mathbb{N} \cap J_{n_1}| \ll Y_2 - Y_1.$$

Invoking now our fourth moment hypothesis (1) to estimate the sum over n_1, we obtain

$$\mathcal{S} \prec (Y_2 - Y_1)^2.$$

Inserting this into the earlier estimate (6) yields the required bound. □

5. Bessel asymptotics

We now recall, following [1, p 1527-1528], the shape of W_t.

For $0 < \xi < \infty$, we define $H(\xi)$ by the formula

$$H(\xi) = \begin{cases} \operatorname{arccosh}(1/\xi) - \sqrt{1-\xi^2} & \text{if } 0 < \xi \leq 1, \\ \sqrt{\xi^2 - 1} - \operatorname{arcsec}(\xi) & \text{if } \xi > 1. \end{cases}$$

It defines a continuous, nonnegative function, vanishing only at $\xi = 1$, with the following properties:

$$H(1+\xi) \asymp |\xi|^{3/2} \text{ if } \xi \lll 1, \tag{7}$$

$$H(\xi) \gg 1 \text{ if } \xi - 1 \asymp 1, \tag{8}$$

$$H(\xi) \gg \xi \text{ if } \xi \ggg 1. \tag{9}$$

The Whittaker function W_t satisfies the following estimates.

Lemma 5. *Let $y > 0$. Write $u := y - t$, so that $y = t + u$ with $-t < u < \infty$. Then*

1. *If $u < 0$ and $u \ggg t^{1/3}$, then*

$$W_t(y) = \frac{\sqrt{2\pi y}}{|t^2 - y^2|^{1/4}} \sin\left(\frac{\pi}{4} + tH\left(\frac{y}{t}\right) \right) \left(1 + \mathrm{O}\left(\frac{1}{tH\left(\frac{y}{t}\right)} \right) \right). \tag{10}$$

2. *If $u > 0$ and $u \ggg t^{1/3}$, then*

$$W_t(y) = \frac{\sqrt{2\pi y}}{|t^2 - y^2|^{1/4}} \exp\left(-tH\left(\frac{y}{t}\right) \right) \left(1 + \mathrm{O}\left(\frac{1}{tH\left(\frac{y}{t}\right)} \right) \right).$$

(3) In general, we have the upper bound

$$W_t(y) \ll \begin{cases} \frac{\sqrt{y}}{|t^2-y^2|^{1/4}} & \text{if } 0 > u \ggg t^{1/3}, \\ \frac{\sqrt{y}}{|t^2-y^2|^{1/4}} \exp\left(-tH\left(\frac{y}{t}\right)\right) & \text{if } 0 < u \ggg t^{1/3}, \\ \sqrt{y}t^{-1/3} \asymp t^{1/6} & \text{if } u \ll t^{1/3}. \end{cases}$$

There are essentially four cases to consider concerning the position of y relative to t, where we again write $y = t + u$:

(i) $0 < y < t$ with $u \asymp t$.
(ii) $0 < y < t$ with $t^{1/3} \lll u \lll t$
(iii) $y = t + \mathrm{O}(t^{1/3})$.
(iv) $y > t$ with $t^{1/3} \lll u$.

In these ranges, the basic upper bound for $W_t(y)$ explicates as follows:

$$W_t(y) \ll \begin{cases} t^{-1/2}y^{1/2} \\ t^{1/4}u^{-1/4} \\ t^{1/6} \\ t^{1/4}u^{-1/4}\exp(-tH(\frac{y}{t})). \end{cases}$$

6. Dyadic decomposition

We fix an even function $V_1 \in C_c^\infty(\mathbb{R}^\times)$ so that for all $y \in \mathbb{R}^\times$, we have

$$\sum_{Y \in \exp(\mathbb{Z})} V_1\left(\frac{y}{Y}\right) = 1.$$

We begin by decomposing

$$W_t = \sum_{Y \in \exp(\mathbb{Z})} W_t^Y,$$

where

$$W_t^Y(y) := V_1\left(\frac{y}{Y}\right) W_t(y).$$

If $Y \lll 1$, then $ny_0 \geq y_0 > Y$ for all natural numbers n, and so $\mathcal{N}(W_t^Y) = 0$. We see also by crude application of the Bessel function asymptotics (see especially (9)) that if $Y \ggg t$, then $\mathcal{N}(W_t^Y) \ll e^{-t}Y^{-100}$, say. Our task thereby reduces to verifying that for each Y in the range

$$1 \ll Y \ll t, \tag{11}$$

we have $\mathcal{N}(W_t^Y) \prec 1$.

Supposing now that $Y \asymp t$, we subdivide further. Fix an even function $V_0 \in C_c^\infty(\mathbb{R})$ taking the value 1 in a neighborhood of the origin. Define $V_\pm \in C^\infty(\mathbb{R})$ by

$$V_+(x) := 1_{x>0} - V_0(x), \quad V_-(x) := 1_{x<0} - V_0(x).$$

Then $1 = V_0 + V_+ + V_-$. We have

$$W_t^Y = W_t^{Y,0} + \sum_\pm \sum_{U \in \exp(\mathbb{Z})} W_t^{Y,\pm,U},$$

where

$$W_t^{Y,0}(y) := V_0\left(\frac{y-t}{t^{1/3}}\right) W_t^Y(y)$$

and

$$W_t^{Y,\pm,U}(y) := V_\pm\left(\frac{y-t}{t^{1/3}}\right) V_1\left(\frac{y-t}{U}\right) W_t^Y(y).$$

We observe that $W_t^{Y,\pm,U} = 0$ unless

$$t^{1/3} \ll U \ll t. \tag{12}$$

The number of relevant Y or T is $\prec 1$, so we reduce to establishing the following.

Proposition 6. *We have the following.*

(i) Assuming (11)*, we have* $\mathcal{N}(W_t^Y) \prec 1$.
(ii) Assuming $Y \asymp t$*, we have* $\mathcal{N}(W_t^{Y,0}) \prec 1$.
(iii) Assuming $Y \asymp t$ *and* (12)*, we have* $\mathcal{N}(W_t^{Y,\pm,U}) \prec 1$.

7. Application of crude bounds

Here we apply Lemma 4 to the various cases. This tells us the bounds for $\mathcal{N}(W_t)$ that follow from the stated *upper bounds* for W_t, without taking into account any oscillation coming from the sin factor.

(i) For $Y \lll t$, we have

$$\|W_t^Y\|_\infty \ll t^{-1/2} Y^{1/2}.$$

The support is contained in $[Y_1, Y_2]$ with $Y_1 \asymp Y_2 \asymp Y$ and $Y_1 - Y_2 \asymp Y$, so we obtain

$$\mathcal{N}(W_t^Y) \prec (t^{-1/2} Y^{1/2})^4 \frac{Y^3}{Y^2} = \frac{Y^3}{t^2}. \tag{13}$$

This gives an acceptable estimate for $Y \prec t^{2/3}$.

(ii) For $Y \asymp t$ and $t^{1/3} \lll U \ll t$, we have

$$\|W_t^{Y,-,U}\|_\infty \ll t^{1/4} U^{-1/4}.$$

The support is contained in $[Y_1, Y_2]$ with $Y_1 \asymp Y_2 \asymp t$ and $Y_1 - Y_2 \asymp U$, hence

$$\mathcal{N}(W_t^{Y,-,U}) \prec (t^{1/4} U^{-1/4})^4 \frac{U^3}{t^2} = \frac{U^2}{t}. \tag{14}$$

This is adequate for $U \prec t^{1/2}$.

(iii) For $Y \asymp t$, we have

$$\|W_t^{Y,0}\|_\infty \ll t^{1/6}.$$

The support is contained in $[Y_1, Y_2]$ with $Y_1 \asymp Y_2 \asymp t$ and $Y_1 - Y_2 \asymp t^{1/3}$, hence

$$\mathcal{N}(W_t^{Y,0}) \prec (t^{1/6})^4 \frac{(t^{1/3})^3}{t^2} = t^{-1/3}.$$

This more than acceptable.

The same argument gives a more-than-satisfactory estimate for $\mathcal{N}(W_t^{Y,\pm,U})$ in the "boundary" range $U \asymp t^{1/3}$ (or indeed, in a somewhat larger range).

(iv) For $Y \asymp t$ and $t^{1/3} \lll U \ll t$, we have

$$\|W_t^{Y,+,U}\|_\infty \ll t^{1/4} U^{-1/4} \exp(-ctH(1 + \tfrac{U}{t}))$$

for some fixed $c > 0$ (depending upon the choice of V_1). The support is as in item (ii) above, so we obtain

$$\mathcal{N}(W_t^{Y,+,U}) \prec \frac{U^2}{t} \exp(-ctH(1 + \tfrac{U}{t})).$$

This is adequate, without taking into account the exponential factor, for $U \prec t^{1/2}$. In the complementary range $U \succ t^{1/2}$, we have $U/t \succ t^{-1/2}$, hence (by (7) and (9)) $H(1+U/t) \succ (t^{-1/2})^{3/2} = t^{-3/4}$; in particular,

$$tH(1+\tfrac{U}{t}) > t^{1/5},$$

say, which leads to the more than adequate bound $\mathcal{N}(W_t^{Y,+,U}) \leq \exp(-t^{1/10})$, say.

The proof of Proposition 6 thereby reduces to that of the following.

Proposition 7. *We have the following.*

(i) For $t^{2/3} \lll Y \lll t$, we have $\mathcal{N}(W_t^Y) \prec 1$.
(ii) For $Y \asymp t$ and $t^{1/2} \lll U \ll t$, we have $\mathcal{N}(W_t^{Y,-,U}) \prec 1$.

8. Smoothening

It will be convenient to "smoothen" the definition of $\mathcal{N}(W)$ a bit, as follows. Retain the setting of Proposition 7. Recall from (2) that $\mathcal{N}(W)$ is defined by an integral over $y \geq y_0$. On the other hand, $W(y)$ is supported on $y \asymp Y$. Let us fix $0 < c < C$ so that $W(y)$ vanishes for $y \notin [cY, CY]$. The product $\lambda(n)W(ny)$ then vanishes for $n \geq CY$. For this reason, modifying $\lambda(n)$ by zeroing it out for $n > CY$ has no effect on $\mathcal{N}(W)$, and no effect on our sole hypothesis (1) concerning λ. Having modified $\lambda(n)$ in this way, we now have $\lambda(n)W(ny) \neq 0$ only if $n \leq CY$ and $ny \geq cY$, which forces $y \geq c/C$. On the other hand, $\mathcal{N}(W)$ only increases if we decrease the fixed quantity $y_0 > 0$. Let us decrease that quantity, if necessary, so that $y_0 \leq c/C$. Having done so, we see that the integration constraints in the definitions of $\mathcal{N}(W)$ and $I(\mathbf{n})$ (see (3)) are now redundant − we might as well integrate over all $y > 0$. In particular,

$$I(\mathbf{n}) = \int_{y=0}^{\infty} W(y)W(\tfrac{n_2}{n_1}y)\overline{W(\tfrac{n_3}{n_1}y)W(\tfrac{n_4}{n_1}y)}\,\frac{dy}{y^2}.$$

This "smoothening" has the effect of eliminating certain boundary terms in the arguments to follow. (The arguments would work anyway, but the details are slightly simplified by having smoothened first.)

9. The diagonal contribution

Recall that $\mathbf{n}$ has denoted a quadruple of natural numbers satisfying $n_1 + n_2 = n_3 + n_4$. Because a pair of natural numbers is determined up to reordering by its sum and product, we have

$$(n_1 n_2 = n_3 n_4) \iff (\{n_1, n_2\} = \{n_3, n_4\}).$$

These conditions describe a "diagonal" case in which there is clearly no hope to obtain cancellation from the integral defining $I(\mathbf{n})$. We pause to estimate that diagonal contribution:

Lemma 8. Suppose that $W \in C_c(\mathbb{R})$ and $[Y_1, Y_2]$ satisfy the conditions of Lemma 4. Then the diagonal contribution

$$\mathcal{N}_{\mathrm{diag}}(W) := \sum_{n_1, n_2 \in \mathbb{N}} \frac{|\lambda(n_1)|^4}{n_1}\,|I(n_1, n_2, n_1, n_2)|$$

satisfies

$$\mathcal{N}_{\mathrm{diag}}(W) \prec \|W\|_\infty^4 \frac{(Y_2 - Y_1)^2}{Y_1^2}.$$

Proof. The proof is the same as that of Lemma 4, except that we no longer need to square the factor $|\mathbb{N} \cap J_{n_1}|$. We thereby improve the earlier estimate by a factor of $Y_2 - Y_1$. □

Return to the setting of Proposition 7, and take $W = W_t^Y$ or $W_t^{Y,-,U}$. We obtain an estimate for the diagonal contribution to $\mathcal{N}(W)$ that improves upon the crude estimates obtained in §7 by the factor $Y_2 - Y_1$, which is roughly Y in first case and U in the second. Dividing (13) by Y and (14) by U, we obtain

$$\mathcal{N}_{\mathrm{diag}}(W_t^Y) \prec \frac{Y^2}{t^2}, \qquad \mathcal{N}_{\mathrm{diag}}(W_t^{Y,-,U}) \prec \frac{U}{t}.$$

In either case, we obtain the adequate estimate $\mathcal{N}_{\mathrm{diag}}(W) \prec 1$.

It remains only to establish the analogue of Proposition 9 for the "off-diagonal" contribution to $\mathcal{N}(W)$, coming from when $n_1 n_2 \neq n_3 n_4$:

$$\mathcal{N}_{\mathrm{off}}(W) := \sum_{\substack{n_1 + n_2 = n_3 + n_4 \\ n_1 n_2 \neq n_3 n_4}} \frac{|\lambda(n_1)|^4}{n_1} |I(\mathbf{n})| . \tag{15}$$

10. Discarding the remainder terms

Retain the hypotheses of Proposition 7. The proof will exploit oscillation coming from the sin factor in the Bessel asymptotics. Let us write those asymptotics as follows:

$$\begin{aligned} W_t^Y &= W_{t,\sharp}^Y + W_{t,\flat}^Y, \\ W_t^{Y,-,U} &= W_{t,\sharp}^{Y,-,U} + W_{t,\flat}^{Y,-,U}, \end{aligned}$$

where the subscript $\sharp$ (resp. $\flat$) denotes the contribution from the main (resp. remainder) terms in (10). We will estimate the contribution of the remainder terms as in §7. The estimate for $\mathcal{N}(W_{t,\flat}^*)$ achieved in this way is obtained by multiplying the bound derived earlier for $\mathcal{N}(W_t^*)$ by the fourth power of the supremum of the absolute value of the factor

$$\frac{1}{tH(\frac{y}{t})} = \frac{1}{tH(1 + \frac{u}{t})}$$

taken over all $y = t + u$ in the support of $W_{t,\flat}^*$.

(i) For $Y \in \mathrm{supp}(W_{t,\flat}^Y)$, we have $y \lll t$, hence $u < 0$ and $u \asymp t$, hence $u/t < 0$ and $u/t \asymp 1$, hence (by (8)) $H(y/t) \gg 1$, hence

$$\frac{1}{tH(\frac{y}{t})} \ll \frac{1}{t}.$$

Thus, multiplying the bound in (13) by t^{-4}, we obtain

$$\mathcal{N}(W_{t,\flat}^Y) \prec \frac{Y^3}{t^6}.$$

This is more than acceptable. The same bound holds for $\mathcal{N}_{\mathrm{off}}$.

(ii) For $Y \in \mathrm{supp}(W_{t,\flat}^{Y,-,U})$, we have $y \asymp t$ and $u < 0$ and $u \asymp U$, hence $u/t < 0$ and $u/t \asymp U/t$. We consider separately two cases.

(a) $U \asymp t$. In this case, $u/t \asymp 1$, hence $H(y/t) \asymp 1$, hence
$$\frac{1}{tH(\frac{y}{t})} \asymp \frac{1}{t}.$$
Thus, multiplying the bound in (14) by t^{-4}, we obtain
$$\mathcal{N}(W_{t,\flat}^{Y,-,U}) \prec \frac{U^2}{t^5}.$$
This is more than acceptable.

(b) $t^{1/2} \prec U \lll t$. In this case, $u/t \lll 1$, hence (by (7)) $H(y/t) \asymp |u/t|^{3/2}$, and so
$$\frac{1}{tH(\frac{y}{t})} \asymp \frac{t^{1/2}}{U^{3/2}}.$$
Thus, multiplying the bound in (14) by t^2/U^6, we obtain
$$\mathcal{N}(W_{t,\flat}^{Y,-,U}) \prec \frac{t}{U^4}.$$
This is acceptable in view of the condition $U \succ t^{1/2}$. The same bound holds for $\mathcal{N}_{\mathrm{off}}$.

We have reduced to establishing the analogue of Proposition 7 but for $\mathcal{N}_{\mathrm{off}}(W_{t,\sharp}^*)$ rather than $\mathcal{N}(W_t^*)$.

11. Absorption of nuisance factors

By definition,
$$W_{t,\sharp}^{Y}(y) = V_1\left(\frac{y}{Y}\right)\frac{\sqrt{2\pi y}}{|t^2-y^2|^{1/4}}\sin\left(\frac{\pi}{4}+tH\left(\frac{y}{t}\right)\right),$$
$$W_{t,\sharp}^{Y,-,U}(y) = V_1\left(\frac{y}{Y}\right)V_-\left(\frac{y-t}{t^{1/3}}\right)V_1\left(\frac{y-t}{U}\right)\frac{\sqrt{2\pi y}}{|t^2-y^2|^{1/4}}\sin\left(\frac{\pi}{4}+tH\left(\frac{y}{t}\right)\right).$$
It will be convenient to write the sin as a sum of exponentials. The other factors present a cosmetic nuisance, so let's discard them, as follows. On the support of $W_{t,\sharp}^{Y}$ (resp. $W_{t,\sharp}^{Y,-,U}$), we have
$$t^2 - y^2 \asymp tY \text{ (resp. } t^2-y^2 \asymp tU).$$
More precisely, we can absorb the factor $|t^2-y^2|^{-1/4}$ into the weight function in a way that does not significantly increase the sizes of the weight functions or their derivatives. We retain the factor $y^{1/2}$, whose fourth power will cancel against the denominator of the factor dy/y^2 against which we integrate in the definition of $I(\mathbf{n})$. Finally, since $U \ggg t^{1/3}$ and since $V_-(x) = 1$ for $x < 0$ with $x \ggg 1$, we have
$$V_-\left(\frac{y-t}{t^{1/3}}\right) = 1 \text{ whenever } y < t \text{ and } V_1\left(\frac{y-t}{U}\right) \neq 0.$$
If $W = cW_0$, then $\mathcal{N}(W) = |c|^4\mathcal{N}(W_0)$, so the estimate $\mathcal{N}(W) \prec 1$ is equivalent to $\mathcal{N}(W_0) \prec |c|^{-4}$. Applying this observation with $c = (tY)^{-1/4}$ (resp. $c = (tU)^{-1/4}$), we thereby reduce the proof of the analogue of Proposition 7 for $W_{t,\sharp}^*$ to that of the following:

Proposition 9. *We have the following.*

(i) *Let $t^{1/3} \prec Y \lll t$. Let V be a nonnegative function belonging to some fixed bounded subset of $C_c^\infty(\mathbb{R}_+^\times)$. Define*

$$W(y) := y^{1/2} V\left(\frac{y}{Y}\right) \exp\left(itH\left(\frac{y}{t}\right)\right).$$

Then

$$\mathcal{N}_{\mathrm{off}}(W) \prec tY.$$

(ii) *Let $t^{1/2} \prec U \ll t$. Let V_1, V_2 be nonnegative functions belonging to some fixed bounded subset of $C_c^\infty(\mathbb{R}_+^\times)$. Define*

$$W(y) := y^{1/2} V_1\left(\frac{y}{t}\right) V_2\left(\frac{t-y}{U}\right) \exp\left(itH\left(\frac{y}{t}\right)\right).$$

Then

$$\mathcal{N}_{\mathrm{off}}(W) \prec tU.$$

12. Reduction to oscillatory integral estimates

Here we state bounds for (mild generalizations of) the integrals $I(\mathbf{n})$ that take into account the oscillation of the integrand, and explain why such bounds suffice.

Proposition 10. *Retain the setting of Proposition 9. Let x_1, x_2, x_3, x_4 be positive real numbers. Define the 4-tuple of signs*

$$(\eta_1, \eta_2, \eta_3, \eta_4) := (1, 1, -1, -1).$$

Define

$$f^Y(y) := \prod_j V\left(\frac{x_j y}{Y}\right), \qquad f^U(y) := \prod_j V_1\left(\frac{x_j y}{t}\right) V_2\left(\frac{t - x_j y}{U}\right)$$

and write $f = f^Y$ or $f = f^U$ according as we are in the first or second case of Proposition 9. Define the phase

$$\Phi(y) := t \sum_j \eta_j H\left(\frac{x_j y}{t}\right).$$

Assume that

$$x_1 \asymp 1, \quad x_1 + x_2 = x_3 + x_4 \quad \text{and} \quad x_1 x_2 \neq x_3 x_4. \tag{16}$$

Then

$$\int f e^{i\Phi} \ll \frac{t}{Y} \frac{1}{x_1 x_2 - x_3 x_4} \qquad \text{if } f = f^Y,$$

$$\int f e^{i\Phi} \ll \left(\frac{U}{t}\right)^{3/2} \frac{1}{x_1 x_2 - x_3 x_4} \qquad \text{if } f = f^U.$$

We now explain how Proposition 10 implies Proposition 9. We specialize to

$$x_1 := 1, \quad x_j = \frac{n_j}{n_1} \text{ for } j = 2, 3, 4,$$

so that

$$I(\mathbf{n}) = \int f(y) e^{i\Phi(y)}\, dy.$$

We estimate $\mathcal{N}_{\mathrm{off}}(W)$ by plugging Proposition 10 into the definition (15). We consider the two cases separately:

(i) In the case $f = f^Y$, we obtain

$$\mathcal{N}_{\text{off}}(W) \ll \frac{t}{Y} \sum_{\substack{n_1,n_2,n_3,n_4 \ll Y \\ n_1+n_2=n_3+n_4 \\ n_1n_2 \neq n_3n_4}} \frac{|\lambda(n_1)|^4}{n_1(x_1x_2 - x_3x_4)}.$$

We have

$$\frac{1}{n_1(x_1x_2 - x_3x_4)} = \frac{n_1}{n_1n_2 - n_3n_4}$$

and, in view of the identity $n_1 + n_2 = n_3 + n_4$,

$$\begin{aligned} n_1n_2 - n_3n_4 &= n_1(n_3 + n_4 - n_1) \\ &= -(n_1 - n_3)(n_1 - n_4), \end{aligned}$$

thus

$$\mathcal{N}_{\text{off}}(W) \ll t \sum_{n_1 \ll Y} |\lambda(n_1)|^4 \sum_{\substack{n_3,n_4 \ll Y \\ n_3,n_4 \neq n_1}} \frac{1}{|(n_1 - n_3)(n_1 - n_4)|} \prec tY,$$

as required.

(ii) In the case $f = f^U$, the same arguments as above, applied with $Y = t$, give

$$\mathcal{N}_{\text{off}}(W) \prec \left(\frac{U}{t}\right)^{3/2} t^2 \prec Ut,$$

using in the final step that $U \ll t$.

Remark 11. In the treatment of f^U, we could have done a bit better, taking into account that $I(\mathbf{n})$ vanishes identically unless $n_3, n_4 \in n_1(1 + \mathrm{O}(U/t))$, but it is apparently unnecessary to do so. In particular, we see that the "hardest" case is that of f^Y for $t^{1-\delta} \ll Y \ll t$.

13. Integration by parts

It remains only to prove Proposition 10. We begin with a general "integration by parts" lemma (compare with [4, Lemma 8.9]).

Lemma 12. Let $f \in C_c^\infty(\mathbb{R})$ and $\Phi \in C^\infty(\mathbb{R})$. Assume that

(i) f is nonnegative,
(ii) Φ is real-valued, and
(iii) there is an interval E containing the support of f such that $\Phi'(x) \neq 0$ and $\Phi''(x) \neq 0$ for all $x \in E$.

Then

$$\left| \int f e^{i\Phi} \right| \leq 2 \int \left| \frac{f'}{\Phi'} \right|.$$

Proof. To improve the cosmetics of the proof, we replace Φ with $i\Phi$, so that it is now imaginary-valued. Set

$$I := \int f e^{\Phi},$$
$$I_1 := \int \frac{f'}{\Phi'} e^{\Phi},$$
$$I_2 := \int \frac{f'}{\Phi'}.$$

It is enough to show that

$$|I + I_1| \le |I_2|.$$

To see this, we begin by differentiating:

$$\left(\frac{f e^{\Phi}}{\Phi'}\right)' = \frac{\Phi'(f'e^{\Phi} + f\Phi' e^{\Phi}) - f e^{\Phi}\Phi''}{(\Phi')^2} = \left(f + \frac{f'}{\Phi'} - \frac{f\Phi''}{(\Phi')^2}\right) e^{\Phi}.$$

By integration by parts, it follows that

$$I + I_1 = I' := \int \frac{f\Phi''}{(\Phi')^2} e^{\Phi}.$$

Since f is nonnegative and both Φ' and Φ'' are nonvanishing on an interval containing the support of f, we see that the ratio $f\Phi''/(\Phi')^2$ is either everywhere nonnegative or everywhere nonpositive. Thus

$$|I'| \le |I''|, \quad I'' := \int \frac{f\Phi''}{(\Phi')^2}.$$

On the other hand,

$$\left(\frac{f}{\Phi'}\right)' = \frac{\Phi' f' - f\Phi''}{(\Phi')^2},$$

hence

$$I'' = I_2.$$

□

To apply Lemma 12 to the situation of interest, we will do the following:

(a) Estimate the L^1-norm of the derivative of f from above by $O(1)$.
(b) Verify that the second derivative of Φ does not vanish on some interval containing the support of f.
(c) Estimate the derivative of Φ from below, on some interval containing the support of f, by the inverse of the right hand side of the respective upper bound stated in Proposition 10

We start with the first of these tasks.

Lemma 13. *Let f be as in either case of Proposition 9. Then $\|f'\|_{L^1(\mathbb{R})} \ll 1$.*

Proof. We consider each case:

(i) Suppose $f = f^Y$. Then

$$f'(y) = \sum_k \frac{x_k}{Y} V'\left(\frac{x_k y}{Y}\right) \prod_{j \ne k} V\left(\frac{x_j y}{Y}\right).$$

Recall our assumption that $x_1 \asymp 1$. We see that $f'(y)$ vanishes unless $y \asymp Y$ and each $x_k \asymp 1$, in which case $f'(y) \ll 1/Y$. The volume of the support is $\ll Y$, so the claim follows.

(ii) Suppose $f = f^U$. Then

$$f'(y) = \sum_k \frac{x_k}{t} V_1'\left(\frac{x_k y}{t}\right) V_2\left(\frac{t - x_k y}{U}\right) \prod_{j \neq k} V_1\left(\frac{x_j y}{t}\right) V_2\left(\frac{t - x_j y}{U}\right)$$
$$- \sum_k \frac{x_k}{U} V_1\left(\frac{x_k y}{t}\right) V_2'\left(\frac{t - x_k y}{U}\right) \prod_{j \neq k} V_1\left(\frac{x_j y}{t}\right) V_2\left(\frac{t - x_j y}{U}\right).$$

We have $f'(y) = 0$ unless $y \asymp t$, $y - t \asymp U$ and each $x_k \asymp 1$, in which case $f'(y) \ll 1/U$. The volume of the support is $\ll U$, so the claim follows.

□

We note that for $j = 1, 2, 3, 4$,

$$f^Y(y) \neq 0 \implies x_j y \asymp Y \lll t, \quad t - x_j y \asymp t, \tag{17}$$

$$f^U(y) \neq 0 \implies x_j y \asymp t, \quad t - x_j y \asymp U \ll t, \quad t > x_j y. \tag{18}$$

In view of Lemmas 12 and 13, the proof of Proposition 10 reduces to the following:

Proposition 14. *Retain the setting of Proposition 10.*

(i) For y as in (17)*, we have the second derivative sign inequality*

$$\frac{\Phi''(y)}{x_1 x_2 - x_3 x_4} < 0 \tag{19}$$

and the first derivative lower bound

$$\frac{\Phi'(y)}{x_1 x_2 - x_3 x_4} \gg \frac{Y}{t}. \tag{20}$$

(ii) For y as in (18)*, we have the second derivative sign inequality* (19) *and the first derivative lower bound*

$$\frac{\Phi'(y)}{x_1 x_2 - x_3 x_4} \gg \left(\frac{t}{U}\right)^{3/2}. \tag{21}$$

To verify the above, we first calculate Φ' explicitly. To describe the result of that calculation, we introduce some notation. Set

$$z_j := \frac{x_j y}{t}.$$

We pause to note that in the respective cases (17) and (18), we have

$$z_j \asymp Y/t \lll 1,$$
$$z_j \asymp 1, \quad z_j - 1 \asymp U/t.$$

Moreover, in either case, we have $z_j \in (0, 1)$. We set[1]

$$h_j := \sqrt{1 - z_j^2} \in (0, 1).$$

Lemma 15. *We have*

$$\Phi'(y) = -\frac{t}{y} \sum_j \eta_j h_j$$

[1] The present definition of h_j differs from that in [6] by a factor of t.

Proof. By the chain rule, we have

$$y\Phi'(y) = \sum_j \eta_j x_j y H'\left(\frac{x_j y}{t}\right).$$
$$= t\sum_j \eta_j z_j H'(z_j).$$

For $x \in (0,1)$, we may check (carefully) that

$$-xH'(x) = \sqrt{1-x^2}.$$

The claimed identity follows. □

We have

$$\frac{-\Phi'(y)}{x_1x_2 - x_3x_4} = \frac{t}{y}\frac{\sum_j \eta_j h_j}{x_1x_2 - x_3x_4}$$
$$= \frac{y}{t}\frac{\sum_j \eta_j h_j}{z_1z_2 - z_3z_4}.$$

Using that $\frac{d}{dx}(1-x^2)^{1/2} = -x(1-x^2)^{-1/2}$, we see that $\Phi''(y)$ is a positive constant multiple of

$$\sum_j \eta_j h_j^{-1}$$

In view of our assumption $x_1 \asymp 1$ (see (16)), we have $y \asymp Y$ on the support of f. The required assertions (19), (20) and (21) concerning Φ' and Φ'' thereby reduce to the following, whose proof is all that remains:

Proposition 16. *Let $z_1, z_2, z_3, z_4 \in (0,1)$ such that*

$$z_1 + z_2 = z_3 + z_4, \quad z_1z_2 \neq z_3z_4.$$

Define $h_j := \sqrt{1-z_j^2} \in (0,1)$. Recall that $(\eta_1,\eta_2,\eta_3,\eta_4) = (1,1,-1,-1)$.

(i) We have

$$\frac{\sum_j \eta_j/h_j}{z_1z_2 - z_3z_4} < 0. \tag{22}$$

(ii) Suppose that each $z_j \lll 1$. Then

$$\frac{\sum_j \eta_j h_j}{z_1z_2 - z_3z_4} \gg 1. \tag{23}$$

(iii) Suppose that each $z_j \asymp 1$ and $z_j - 1 \asymp Z$ for some $Z \ll 1$. Then

$$\frac{\sum_j \eta_j h_j}{z_1z_2 - z_3z_4} \gg 1/Z^{3/2}. \tag{24}$$

14. Fun with square roots

We retain the setting of Proposition 16 and verify its assertions one by one.

We first establish (22). Clearing denominators, we have

$$\sum_j \eta_j/h_j = \frac{(h_2h_3h_4 + h_1h_3h_4) - (h_1h_2h_4 + h_1h_2h_3)}{h_1h_2h_3h_4}.$$

Since each $h_j > 0$, our task reduces to verifying that

$$\frac{a}{z_1 z_2 - z_3 z_4} < 0, \quad a := (h_2 h_3 h_4 + h_1 h_3 h_4)^2 - (h_1 h_2 h_4 + h_1 h_2 h_3)^2.$$

This inequality is a consequence of the following lemma.

Lemma 17. *Set*

$$\begin{aligned} b &:= 2(1 - (z_1 z_2)^2) + (z_1 z_2 + z_3 z_4)(2 - (z_1^2 + z_2^2)) > 0, \\ c &:= 2 h_1 h_2 h_3 h_4 (2 + z_3 z_4 + z_1 z_2) > 0, \\ e &:= h_3 h_4 + h_1 h_2 > 0. \end{aligned}$$

Then

$$\frac{a}{z_3 z_4 - z_1 z_2} = b + \frac{c}{e} > 0.$$

Proof. This is noted in [6]. It may be confirmed by the following SAGE code:

```
h = [var("h" + str(j)) for j in (0..4)]
x = [var("x" + str(j)) for j in (0..4)]
a = (h2*h3*h4 + h1*h3*h4)^2 - (h1*h2*h4 + h1*h2*h3)^2
b = 2*(1 - (x1*x2)^2) + (x1*x2 + x3*x4)*(2 - (x1^2 + x2^2))
c = 2*h1*h2*h3*h4*(2 + x3*x4 + x1*x2)
d = x3*x4 - x1*x2
e = h3*h4 + h1*h2
# To check: a/d = b + c/e
# Rewrite: a = b*d + c*d/e
# Rewrite: a*e = b*d*e + c*d
test0 = a*e - (b*d*e + c*d)
test1 = test0.subs(h1=sqrt(1-x1^2), h2=sqrt(1-x2^2), h3=sqrt(1-x3^2), h4=sqrt(1-x4^2))
test2 = test1.subs(x4 = x1+x2-x3).expand()
test2
```

□

The proofs of (23) and (24) make use of the following algebraic identity.

Lemma 18. *We have*

$$\frac{\sum \eta_j h_j \sum h_j}{z_1 z_2 - z_3 z_4} = 2 + \frac{4}{h_1 h_2 + h_3 h_4} + 2 \frac{z_1 z_2 + z_3 z_4}{h_1 h_2 + h_3 h_4}. \tag{25}$$

Proof. This is a consequence of several identities noted in [6]. I have checked it using computer algebra; the following SAGE code outputs "0".

```
h = [var("h" + str(j)) for j in (0..4)]
x = [var("x" + str(j)) for j in (0..4)]
y = [0,1,1,-1,-1]
a = sum(y[j]*h[j] for j in (1..4))
b = sum(h[j] for j in (1..4))
c = x1*x2 - x3*x4
d = x1*x2 + x3*x4
e = h1*h2 + h3*h4
# To check: a*b/c = 2 + 4/e + 2*d/e.
# Rewrite: a*b*e = 2*c*e + 4*c + 2*d*c.
```

```
test0 = a*b*e - (2*c*e + 4*c + 2*d*c)
test1 = test0.subs(h1=sqrt(1-x1^2), h2=sqrt(1-x2^2), h3=sqrt(1-x3^2), h4=sqrt(1-x4^2))
test2 = test1.subs(x4 = x1+x2-x3).expand()
test2
```

□

We note that each term on the right hand side of (25) is positive. We may thus bound the left hand side from below by any term on the right hand side.

We now verify (23). If each $z_j \lll 1$, then each $h_j \asymp 1$. Bounding the right hand side of (25) from below by the first term on its right hand side gives (23).

We now verify (24). If each $z_j \asymp 1$ and $z_j - 1 \asymp Z$ for some $Z \ll 1$, then each $h_j \asymp Z^{1/2}$, hence $\sum_j h_j \asymp Z^{1/2}$ and $h_1 h_2 + h_3 h_4 \asymp Z$. Bounding the right hand side of (25) from below by the second term on its right hand side gives (24).

The proof of Theorem 1 is complete.

15. Questions

(1) Reduce the "hard" ingredients required in the proof? In this presentation, uniform asymptotics for Bessel functions and some exact identities involving $\sqrt{1-x^2}$ are combined to bound the integrals

$$\int_0^\infty W_t(x_1 y) W_t(x_2 y) W_t(x_3 y) W_t(x_4 y) \, \frac{dy}{y^2},$$

or equivalently (up to normalizing factors),

$$\int_0^\infty K_{it}(x_1 y) K_{it}(x_2 y) K_{it}(x_3 y) K_{it}(x_4 y) \, dy,$$

for positive reals $x_1, \dots, x_4$ satisfying $x_1 + x_2 = x_3 + x_4$, but maybe the bounds are more fundamental than the ingredients.

(2) Geometric way to think of the proof of Proposition 16, using that the points (z_j, h_j) lie in the upper-right quadrant of the unit circle?

(3) Compact quotients, maybe via "theta method"? cf. [7], [8], [5], ...

(4) Other aspects, such as p-adic depth.

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