

# SPECTRAL ASPECT SUBCONVEXITY FOR $\mathrm{PGL}_2$ : A COUPLE APPROACHES

## CONTENTS

1.	Goal	1
2.	Automorphic background	2
2.1.	Cuspidal automorphic representations	2
2.2.	Whittaker expansion	2
2.3.	Kirillov model	2
3.	Integral representation	3
4.	Coadjoint orbits	3
5.	Choice of vector	4
6.	Truncation	5
7.	Symmetries	6
8.	The convexity threshold	6
9.	Amplification	7
10.	Comparison with Iwaniec	7
	References	8

ABSTRACT. We discuss and compare two ways to estimate a short second moment of  $L$ -functions on  $\mathrm{PGL}_2$  in the spectral aspect: that of Iwaniec [7], using the approximate functional equation and Kuznetsov formula, and that of [12] using period integrals.

**Remark 0.1.** This is a lightly edited version of “log-2022-03-08-a.tex”. It needs a lot of polishing, but might be useful to someone even in its current state.

## 1. GOAL

Take

$$\begin{aligned}
 G &:= \mathrm{PGL}_2(\mathbb{R}), \\
 \Gamma &:= \mathrm{PGL}_2(\mathbb{Z}) \hookrightarrow G, \\
 H &:= \mathrm{GL}_1(\mathbb{R}) \cong \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \hookrightarrow G, \\
 \Gamma_H &:= \mathrm{GL}_1(\mathbb{Z}) = \{\pm 1\} \hookrightarrow H.
 \end{aligned}$$

Let  $\pi \in L_{\mathrm{cusp}}^2(\Gamma \backslash G)$  be a cuspidal automorphic representation corresponding to a Maass form of eigenvalue  $1/4 + T^2$ . We aim to show that

$$L(\pi, \tfrac{1}{2}) \ll T^{1/2-\delta} \tag{1.1}$$

for some fixed  $\delta > 0$ .

Such an estimate was first shown by Iwaniec [7] (initially under an additional hypothesis, which Iwaniec later observed could be removed using identity  $\lambda(p)^2 - \lambda(p^2) = 1$ ). Iwaniec’s method consists of estimating an amplified second moment over  $T$  in an interval of length a bit more than 1. This method was adapted to

$\mathrm{PGL}_3$  by Blomer–Buttcane [3, 4], using an amplified fourth moment, and then to  $\mathrm{GL}_n$  in [12] (direct link), using an amplified  $2(n-1)$ th moment. In the works of Iwaniec and Blomer–Buttcane, the moment is estimated using the approximate functional equation, Kuznetsov formula, and several applications of Poisson summation. In [12], the moment is estimated implicitly by direct analysis of an integral representation for the  $L$ -function, with vectors chosen as in [14] and the averaging implemented via the pretrace formula like in [8].

**Remark 1.1.** The best known bound for (1.1) is to due to Ivic [6], who showed that (1.1) holds for any fixed  $\delta < 1/6$ . Ivic’s method consists of estimating an unamplified fourth moment over  $T$  in an interval of length a bit more than  $T^{1/3}$  (compare with [10, 11, 5, 2]). Such approaches using “higher moments” have no known extension beyond  $\mathrm{GL}_2$ . It would be significant to identify such an extension.

## 2. AUTOMORPHIC BACKGROUND

**2.1. Cuspidal automorphic representations.** The space  $L^2_{\mathrm{cusp}}(\Gamma \backslash G)$  consists of square-integrable functions on  $\Gamma \backslash G$  whose constant term vanishes. It is simultaneously a representation for the group  $G$ , acting via right translation, and the Hecke algebra, acting via left translation with respect to double cosets in  $\Gamma \backslash \mathrm{PGL}_2(\mathbb{Q})/\Gamma$ . Under these actions, it decomposes as a direct sum of irreducible representations, each occurring with multiplicity one. We denote by  $\pi$  the space of smooth vectors inside one such representation.

**2.2. Whittaker expansion.** The Hecke eigenvalues of  $\pi$  are described by a multiplicative function  $\lambda_\pi : \mathbb{N} \rightarrow \mathbb{C}$ , which specifies the eigenvalues for a spanning set of double cosets. For  $\varphi \in \pi$ , one defines the Whittaker function  $W_\varphi : G \rightarrow \mathbb{C}$  by

$$W_\varphi(g) := \int_{x \in \mathbb{R}/\mathbb{Z}} \varphi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) e(-x) dx,$$

where we employ the standard abbreviation  $e(t) := e^{2\pi it}$ . Conversely,  $\varphi$  may be recovered from  $\lambda_\pi$  and  $W_\varphi$  via the formula

$$\varphi(g) = \sum_{n \neq 0} \frac{\lambda_\pi(|n|)}{|n|^{1/2}} W_\varphi \left( \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} g \right), \quad (2.1)$$

with the sum taken over all nonzero integers  $n$ .

**2.3. Kirillov model.** The Whittaker model  $\mathcal{W}(\pi)$  for  $\pi$  is defined to consist of all functions of the form  $W_\varphi$ . The theory of the Kirillov model implies that the restriction map

$$\mathcal{W}(\pi) \rightarrow \{\text{functions } H \rightarrow \mathbb{C}\}$$

is injective, and that its image contains  $C_c^\infty(H)$ . Combining this fact with (2.1) tells us that

- any  $\varphi \in \pi$  is determined by  $W_\varphi : H \rightarrow \mathbb{C}$ , and
- every smooth compactly-supported function  $H \rightarrow \mathbb{C}$  determines a unique  $\varphi \in \pi$ .

## 3. INTEGRAL REPRESENTATION

The  $L$ -function  $L(\pi, s)$  may be defined for  $\Re(s) > 1$  by the absolutely convergent Dirichlet series  $\sum_{n \in \mathbb{N}} \lambda_\pi(n) n^{-s}$  and in general by meromorphic continuation. By unfolding (2.1), one obtains the Hecke/Jacquet–Langlands integral representation

$$\int_{\Gamma_H \backslash H} \varphi = L(\pi, \tfrac{1}{2}) \int_H W_\varphi. \quad (3.1)$$

## 4. COADJOINT ORBITS

We denote by  $\mathfrak{g}$  the Lie algebra of  $G$  and by  $\mathfrak{g}^*$  its linear dual. Using the trace pairing, we may identify  $\mathfrak{g}^*$  with  $\mathfrak{g}$ , which we identify further with the space  $\mathfrak{sl}_2(\mathbb{R})$  of traceless  $2 \times 2$  matrices  $\xi$ . The coadjoint orbit

$$\mathcal{O}_\pi \subseteq \mathfrak{g}^*$$

attached to  $\pi$  is the one-sheeted hyperboloid cut out by the equation

$$\det(\xi) = -T^2.$$

(See for instance §6.2–§6.3 of Quantum variance III.) It comes with a natural symplectic volume form that describes the character of  $\pi$  via the Kirillov formula (see [14, §6]). We note that in the coordinates

$$\xi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} x & y - z \\ y + z & -x \end{pmatrix},$$

we have

$$\det(\xi) = -a^2 - bc = z^2 - y^2 - x^2.$$

In particular,  $\mathcal{O}_\pi$  contains the circle  $\{(x, y, 0) : x^2 + y^2 = T^2\}$ . The circular strip  $\{(x, y, z) \in \mathcal{O}_\pi : |z| \leq 1/2\}$  has symplectic volume one; in the orbit method heuristic described in [14, §1.7], it corresponds to the weight zero vector in  $\pi$ .

The integrals (3.1) may be understood as describing how  $\pi$  oscillates against the trivial character of  $H$ . The orbit method suggests [14, §1.9] that such oscillation may be understood in terms of the intersection

$$\mathcal{O}_\pi \cap \mathfrak{h}^\perp,$$

i.e., the preimage in  $\mathcal{O}_\pi$  of the trivial element 0 of  $\mathfrak{h}^*$ . That intersection is given in  $(x, y, z)$  coordinates by

$$\{(0, y, z) : y^2 - z^2 = T^2\}$$

and in  $(a, b, c)$  coordinates by

$$\{(0, b, c) : bc = T^2\}.$$

It is a closed  $H$ -orbit, with trivial stabilizer.

We pick a point  $\tau \in \mathcal{O}_\pi \cap \mathfrak{h}^\perp$  of size comparable to  $T$ . For concreteness, let us take

$$\tau = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}.$$

**Remark 4.1.** For the variant problem concerning  $\pi$  attached to a holomorphic form of weight  $2k$ , we would take  $T := k - 1/2$ , we would take  $\mathcal{O}_\pi$  cut out by  $\det(\xi) = T^2$ , and we would take

$$\tau = \begin{pmatrix} 0 & -T \\ T & 0 \end{pmatrix}.$$

See for instance §8.4 of Quantum variance III.

## 5. CHOICE OF VECTOR

We seek a unit vector  $\varphi \in \pi$  that is “localized at  $\tau$ .” Informally, this means that for each fixed Lie algebra element  $X \in \mathfrak{g}$ , we have

$$X\varphi = i\langle X, \tau \rangle \varphi + O(T^{1/2}). \quad (5.1)$$

For further informal discussion of this concept, see [14, §1.7] and [13, §2.5]. For some precise definitions, see §7.1 of Quantum variance III, or [12, §14] (direct link: §14).

Such a vector  $\varphi$  may be described readily in the Kirillov model for  $\pi$ . Recall (from §2.3) that this model consists of the restrictions to  $H$  of elements  $W$  of the Whittaker model  $\mathcal{W}(\pi)$ . It will be convenient to think of such restricted elements  $W$  as functions on  $\mathbb{R}^\times$  via the abbreviation

$$W(y) := W \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right).$$

We consider the following basis elements for  $\mathfrak{g}$ :

$$\partial_a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \partial_b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \partial_c = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The action of the first two of these elements on the Kirillov model is given very simply:

$$\partial_a W(y) = 2yW'(y), \quad \partial_b W(y) = 2\pi i y W(y).$$

From now on, we will be very vague and informal with asymptotic notation. For a precise discussion, we refer to Exercise 6 of this note. Since  $\langle \partial_a, \tau \rangle = 0$  and  $\langle \partial_b, \tau \rangle = T$ , the condition (5.1) says in particular that

$$\partial_a W_\varphi = O(T^{1/2}), \quad \partial_b W_\varphi = iTW_\varphi + O(T^{1/2}).$$

These formulas suggest taking the following smoothened  $L^2$ -normalized characteristic function:

$$W_\varphi(y) = T^{1/4} 1_{T/2\pi + O(T^{1/2})}^{\text{smooth}}(y). \quad (5.2)$$

Using that  $W_\varphi$  is an eigenfunction under the Casimir operator, one can check also that

$$\partial_c W_\varphi = iTW_\varphi + O(T^{1/2}).$$

This is the content of Exercise 6 of this note. For  $\mathrm{GL}_n$ , precise forms of all these estimates are established in [12, Part 3] (direct link: Part 3).

**Remark 5.1.** This choice of  $\varphi$  is closely related to the “microlocal lift” of Zelditch [17] et al., see [9, 16, 1]. More precisely, it is (asymptotically) a  $G$ -translate of the usual definition; the limit invariance for  $L^2$ -masses will be with respect to the stabilizer  $G_\tau$  of  $\tau$ , namely

$$G_\tau = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \right\}, \quad (5.3)$$

rather than with respect to the diagonal subgroup.

By choosing the smooth bump function implicit in (5.2) to be nonnegative, we may arrange that

$$\int_H W_\varphi \asymp T^{-1/4},$$

so that (3.1) reads

$$T^{-1/4} L(\pi, \tfrac{1}{2}) \asymp \int_{\Gamma_H \backslash H} \varphi. \quad (5.4)$$

Our task is to show that the right hand side is  $\ll T^{1/4-\delta}$ .

## 6. TRUNCATION

We now indicate why the integral on the right hand side of (5.4) may be effectively truncated to a fixed compact set. For a detailed discussion of this point, see [12, §5.3] (direct link: 5.3) or [10, §5.1.4] or [15, §3] (or §3 of this informal note).

Consider, for a fixed even “truncation” function  $\mathcal{T} \in C_c^\infty(H)$ , the map

$$I : H \rightarrow \mathbb{C}$$

$$I(Y) := \int_{y \in \Gamma_H \backslash H} \mathcal{T}(y/Y) \varphi(y) d^\times y$$

assigning to a parameter  $Y$  the smoothened integral of  $\varphi|_H$  over the corresponding dyadic range. We have

$$\int_{\Gamma_H \backslash H} \varphi|_H|^{s-1/2} = L(\pi, s) Z(W_\varphi, s),$$

where  $Z(W_\varphi, s)$  denotes the local zeta integral

$$Z(W_\varphi, s) := \int_H W_\varphi|_H|^{s-1/2}.$$

For  $s = O(1)$ , we see by explicit calculation that

$$Z(W_\varphi, s) \approx T^{s-3/4}.$$

If moreover  $|\Re(s)| \leq 1/2$ , then the convexity bound reads

$$L(\pi, s) \ll T^{1-s}.$$

These bounds become only polynomially worse if we relax the condition  $s = O(1)$  to  $\Re(s) = O(1)$ . Multiplying them together and convolving against the rapidly-decaying Mellin transform of the fixed test function  $\mathcal{T}$ , we deduce the Mellin transform estimate

$$|\Re(s)| \leq 1/2 \implies \tilde{I}(s) \ll T^{1/4} (1 + |s|)^{-\infty}.$$

It follows readily that for fixed  $\kappa > 0$ , we incur the acceptable error  $O(T^{1/4-\kappa/2})$  by smoothly truncating the integral  $\int_H \varphi$  to the range  $\{T^{-\kappa} < |y| < T^\kappa\}$ . If we seek only a qualitative subconvex bound, then we can take  $\kappa$  as small as we like, so there is no harm in truncating to  $\{|y| = T^{o(1)}\}$ . (Note that these arguments are specific to  $\varphi$ : for general vectors, one cannot always truncate to the range  $|y| = T^{o(1)}$ .)

The model problem is thus to bound

$$\int_{y \in H, y \asymp 1} \varphi(y) d^\times y. \quad (6.1)$$

## 7. SYMMETRIES

We use a convolution kernel  $\omega \in C_c^\infty(G)$  to “remember” many of the symmetries satisfied by  $\varphi$ . (When amplifying, we really take  $\omega \in C_c^\infty(\mathrm{PGL}_2(\mathbb{A}))$ .) Roughly speaking, we take  $\omega$  to be a character multiple of an approximate subgroup of  $G$ :

$$\omega := \mathrm{vol}(J)^{-1} 1_J^{\mathrm{smooth}} \chi_\tau^{-1},$$

where:

- $J$  is a subset of  $G$  roughly of the shape

$$J = (1 + \mathrm{O}(T^{-\varepsilon})) \cap (G_\tau + \mathrm{O}(T^{-1/2-\varepsilon})),$$

with  $G_\tau$  the stabilizer of  $\tau$ , as described in (5.3).

- $1_J^{\mathrm{smooth}}$  is a smoothened characteristic function of  $J$ .
- $\chi_\tau$  is the “approximate character” of  $J$  attached to  $\tau$ , given near the identity in exponential coordinates by

$$\chi_\tau(\exp(X)) = e^{i\langle X, \tau \rangle}.$$

We may also describe  $\omega$  in terms of the function  $\mathfrak{g}^* \rightarrow \mathbb{C}$  obtained by taking the Fourier transform of the pullback  $\omega \circ \exp$ . This function is roughly a smoothened characteristic function of a “coin-shaped” neighborhood of  $\tau$ , of thickness  $T^\varepsilon$  (resp.  $T^{1/2+\varepsilon}$ ) in directions transverse (resp. tangential) to the coadjoint orbit  $\mathcal{O}_\pi$  at  $\tau$ . The intersection of this neighborhood with  $\mathcal{O}_\pi$  has symplectic volume  $\approx T^{2\varepsilon} \approx 1$ . The orbit method heuristic suggests that for an irreducible representation  $\sigma$  of  $G$ , we have  $\sigma(\omega) \approx 0$  unless  $\sigma$  is a principal series representation of parameter  $T + \mathrm{O}(T^\varepsilon)$ , in which case  $\sigma(\omega)$  is approximately a projection onto a rank  $\approx T^{2\varepsilon} \approx 1$  “subspace” of vectors microlocalized at  $\tau$ . In particular,

$$\pi(\omega)\varphi \approx \varphi. \tag{7.1}$$

These heuristics and definitions can be made precise, and the above approximation holds in an extremely strong sense (i.e., up to  $\mathrm{O}(T^{-\infty})$  with respect to any fixed seminorm).

For further informal discussion concerning  $\omega$  in a general setting, see [13, §2]. For a precise discussion in the current rank one example, see §7.1 of Quantum variance III.

## 8. THE CONVEXITY THRESHOLD

Let’s explain how to recover the convexity bound from here. Our task is to show that

$$\int_{y \in H, y \asymp 1} \pi(\omega)\varphi(y) d^\times y \ll T^{1/4}.$$

We view the square of the left hand side as one term arising from an integrated pretrace formula, like in the sup norm story [8]. Alternatively, we write the left hand side as the inner product of  $\varphi$  against a Poincaré series and apply Cauchy–Schwarz; see [13, §5.3]. Either way, we reduce to checking that

$$\int_{y_1, y_2 \asymp 1} \sum_{\gamma \in \Gamma} \omega(y_1^{-1} \gamma y_2) \ll T^{1/2}. \tag{8.1}$$

Since  $\omega$  is supported on  $1 + O(T^{-\varepsilon})$ , we see that the only  $\gamma$  that contribute are those in  $\Gamma_H$ . Combining the  $y_1$  and  $y_2$  integrals, it remains to check that

$$\int_{y \in H: y \asymp 1} \omega(y) \ll T^{1/2}. \quad (8.2)$$

To that end, we observe that

$$H \cap G_\tau = \{1\},$$

i.e., that no nontrivial matrices are simultaneously diagonal and of the form (5.3). (This is a baby case of the “stability” feature explained in [14, §1.9, §14].) It follows that (up to some  $\varepsilon$ ’s in the exponents)

$$H \cap J \subseteq O(T^{-1/2}),$$

and so the volume of the integral in (8.2) is  $O(T^{-1/2})$ .

On the other hand, the magnitude of the integrand is

$$\mathrm{vol}(J)^{-1} \approx T.$$

Indeed,  $J$  has dimensions roughly 1 along one direction and  $T^{-1/2}$  along the remaining two directions.

These observations combine to give the required estimate (8.2).

**Remark 8.1.** It’s clear in retrospect that we should have obtained such an estimate: the orbit method heuristic applied to  $\omega$  suggests that the left hand side of (8.1) is a proxy for the sum of  $|L(\pi, \frac{1}{2})|^2$  over  $T$  in an interval of width roughly  $O(T^\varepsilon)$  (see [13, §2.3]), which is of the appropriate size for an averaged Lindelöf estimate to recover convexity.

## 9. AMPLIFICATION

This section is just a stub; see [13, §1.5, §2.7-2.10] for details and pictures relevant for this example. To carry out the amplification step, we basically need to know that the vector  $\varphi'$  obtained by averaging the translates of  $\varphi$  under elements of  $H$  of size  $\asymp 1$  satisfies a matrix coefficient estimate  $\langle g\varphi', \varphi' \rangle \ll T^{-\delta}$  except when  $g$  is very close to  $H$ . This is word-for-word what happens in the sup norm problem [8], where, if we replace  $H$  with  $\mathrm{SO}(2)$ , then  $\varphi'$  becomes something like the weight zero vector.

## 10. COMPARISON WITH IWANIEC

The “two arguments” are “equivalent.” Recall that Iwaniec [7] starts with the approximate functional equation, applies Kuznetsov, and then applies Poisson summation to both variables. The reduction to (6.1) is essentially the approximate functional equation (TODO: explain this in some detail?). In the passage to (8.1), rather than applying the pretrace formula, we could have instead applied the Fourier expansion of  $\varphi$  and averaged each term in the resulting double sum using Kuznetsov. A couple applications of Poisson to the geometric side of Kuznetsov would then bring us right back to (8.1).

It’s more interesting to compare the generalization of this argument to  $\mathrm{GL}_3$  with Blomer–Buttcane [3]. The arguments are again ultimately “equivalent,” but the present approach seems less miraculous.

## REFERENCES

- [1] Nalini Anantharaman and Steve Zelditch. Patterson-Sullivan distributions and quantum ergodicity. *Ann. Henri Poincaré*, 8(2):361–426, 2007.
- [2] Olga Balkanova, Dmitry Frolenkov, and Han Wu. On weyl’s subconvex bound for cube-free hecke characters: Totally real case, 2022.
- [3] Valentin Blomer and Jack Buttcane. On the subconvexity problem for  $L$ -functions on  $\mathrm{GL}(3)$ . *Ann. Sci. Éc. Norm. Supér. (4)*, 53(6):1441–1500, 2020.
- [4] Valentin Blomer and Jack Buttcane. Subconvexity for  $L$ -functions of non-spherical cusp forms on  $\mathrm{GL}(3)$ . *Acta Arith.*, 192(1):31–62, 2020.
- [5] Valentin Blomer, Subhajit Jana, and Paul D. Nelson. The Weyl bound for triple product  $L$ -functions. *Duke Math. J.*, 172(6):1173–1234, 2023.
- [6] Aleksandar Ivić. On sums of Hecke series in short intervals. *J. Théor. Nombres Bordeaux*, 13(2):453–468, 2001.
- [7] Henryk Iwaniec. The spectral growth of automorphic  $L$ -functions. *Journal für die reine und angewandte Mathematik*, 428:139–160, 1992.
- [8] Henryk Iwaniec and Peter Sarnak.  $L^\infty$  norms of eigenfunctions of arithmetic surfaces. *Ann. of Math. (2)*, 141(2):301–320, 1995.
- [9] Elon Lindenstrauss. On quantum unique ergodicity for  $\Gamma \backslash \mathbb{H} \times \mathbb{H}$ . *Internat. Math. Res. Notices*, (17):913–933, 2001.
- [10] Philippe Michel and Akshay Venkatesh. The subconvexity problem for  $\mathrm{GL}_2$ . *Publ. Math. Inst. Hautes Études Sci.*, (111):171–271, 2010.
- [11] Paul D. Nelson. Eisenstein series and the cubic moment for  $\mathrm{PGL}_2$ . arxiv preprint, 2019.
- [12] Paul D. Nelson. Bounds for standard  $L$ -functions. *arXiv e-prints*, page arXiv:2109.15230, September 2021.
- [13] Paul D. Nelson. Spectral aspect subconvex bounds for  $U_{n+1} \times U_n$ . *Invent. Math.*, 232(3):1273–1438, 2023.
- [14] Paul D. Nelson and A. Venkatesh. The orbit method and analysis of automorphic forms. *Acta Math.*, 226(1):1–209, 2021.
- [15] Raphael Schumacher. Subconvexity for  $\mathrm{GL}_3(R)$   $L$ -Functions via Integral Representations. *arXiv e-prints*, page arXiv:2004.06791, April 2020.
- [16] Lior Silberman and Akshay Venkatesh. On quantum unique ergodicity for locally symmetric spaces. *Geom. Funct. Anal.*, 17(3):960–998, 2007.
- [17] Steven Zelditch. Uniform distribution of eigenfunctions on compact hyperbolic surfaces. *Duke Math. J.*, 55(4):919–941, 1987.