1. Anne-Maria Ernvall-hytönen, *Lattices and modular forms in coset coding*

Wyner (1975). Wiretap channel (Alice, Bob, Eve). Gaussian noise with variance σ^2 (between Alice and Bob) and σ_e^2 (for Eve). We assume $\sigma_e > \sigma$. We would like to do this whole scheme in such a way that we can use the noise so that Bob can still receive the image, but Eve cannot. We aim to do this with lattices $\Lambda \geq \Lambda_e$, where each coset corresponds to a code-letter.

Belfiore and Oggier (2010) (maybe [9]?): secrecy gain. Transmit codeword $x \in \mathbb{R}^n$.

$$\frac{1}{(\sigma_\Lambda\sqrt{2\pi})^n}\int_{V_{\Lambda(x)}} e^{-\|y-x\|^2/2\sigma^2}\,dy.$$

$$\frac{1}{(\sigma_\Lambda\sqrt{2\pi})^n}\sum_{t\in\Lambda_e\cap R}\int_{V_{\Lambda(x+t)}} e^{-\|y-x\|^2/2\sigma^2}\,dy.$$

End up trying to minimize

$$\theta_{\Lambda_e}\left(-\frac{1}{2\pi i\sigma_e^2}\right).$$

Secrecy function

$$\Xi(y) := \frac{\theta_{\lambda\mathbb{Z}^n}(yi)}{\theta_\Lambda(yi)}.$$

Belfiore and Sole: For unimodular Λ, maximum at $y = 1$. Even unimodular: polynomials in

$$E_4 = \frac{1}{2}\left(\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8\right), \qquad \Lambda = \frac{1}{256}\vartheta_2^8\vartheta_3^8\vartheta_4^8.$$

Inverse of $\Xi(y)$ a polynomial in $\frac{\vartheta_4^4\vartheta_2^4}{\vartheta_3^8}$.

(Maybe [6] is a reference.)

ℓ-modular: $1/\sqrt{\ell}$, $\mathbb{Z}\oplus\sqrt{2}\mathbb{Z}\oplus 2\mathbb{Z}$.

The connection to deeper mathematics is, what kinds of representations as polynomials do you have for various theta functions?

2. Jolanta Marzec-ballesteros, *Doubling method for self-dual linear codes*

Garrett, Piatetski-Shapiro–Rallis (1980's). Integral of cusp form against restriction of Siegel-type Eisenstein series equals L-function attached to cusp form times cusp form or Eisenstein series attached to cusp form:

$$\left\langle E\left(\begin{pmatrix} g & \\ & g' \end{pmatrix}, s\right), f(g)\right\rangle = L(f,s)f(g').$$

Done for G symplectic, orthogonal, unitary over global field, also for congreunce subgroups.

Let's start with an overview. Let f be a cusp form on G, and H a subgroup for which $G\times G \hookrightarrow H$. Then form an Eisenstein series on H

$$E(h,s) = \sum_{\gamma\in P\backslash H}\phi(\gamma h, s).$$

Restrict to $h = \mathrm{diag}(g, g')$ and take inner product with $F(g)$. This leads to an unfolding involving a sum over $\gamma \in P\backslash H/(G\times G)$. In favorable cases, only one representative γ_0 contributes, leaving us with

$$\sum_{(k,k')\in G\times G} \left\langle \phi\left(\gamma_0 \begin{pmatrix} kg & \\ & k'g' \end{pmatrix}, s\right), f(g)\right\rangle$$
$$= \sum_{\beta\in\gamma_0(G\times 1)} \psi(f) f|_\beta(g')$$
$$= L(f,s)f(g').$$

We would like to do something similar, but now over finite fields.

A *linear code of length* $2n$ over a finite field $\mathbb{F}$ is a linear subspace $C \subseteq \mathbb{F}^{2n}$. We denote by $\langle , \rangle : C \times C \to \mathbb{F}$ the Euclidean inner product. We say that C is *self-dual* if

$$C = C^\perp := \left\{ v \in \mathbb{F}^{2n} : \langle v, C\rangle = 0 \right\}.$$

Then (the length $2n$ is even and) $\dim C = n$.

The *weight* of a codeword $c = (c_1, \dots, c_{2n}) \in C$ is

$$\mathrm{wt}\, c = \#\left\{ i \in \{1, \dots, 2n\} : c_i \neq 0 \right\}.$$

Weight enumerators are certain homogeneous polynomials of degree $2n$ in variables from the set $V = \{x_\alpha : \alpha \in \mathbb{F}^g\}$, where $g \in \mathbb{N}$ is the genus.

The *genus one weight enumerator* of a code $C \subseteq \mathbb{F}_2^{2n}$ is a polynomial

$$W_1(C, (x_0, x_1)) = \sum_{c\in C} x_0^{2n-\mathrm{wt}\, c} x_1^{\mathrm{wt}\, c}.$$

The *genus g weight enumerator* of a code $C \subseteq \mathbb{F}^{2n}$ is a polynomial

$$W_g(C, x) = \sum_{(c^1,\dots,c^g)\in C^g} \prod_{\alpha\in\mathbb{F}^g} x_\alpha^{w_\alpha(c^1,\dots,c^g)}$$

of degree $2n$, where $x = (x_\alpha)_{\alpha\in\mathbb{F}^g}$ and

$$w_\alpha(c^1, \dots, c^g) = \#\left\{ \text{rows } r \text{ in } (c_i^j)_{i=1..2n}^{j=1..g} : r = \alpha \right\}.$$

As an example, we give a basis for a Hamming code H_8 5nof weight 8, the span over $\mathbb{F}_2$ inside $\mathbb{F}_2^8$ of the vectors

$$\begin{pmatrix}1\\0\\0\\1\\0\\1\\1\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\\0\\1\\0\\1\\0\\1\end{pmatrix} \quad \begin{pmatrix}0\\0\\1\\1\\0\\0\\1\\1\end{pmatrix} \quad \begin{pmatrix}0\\0\\0\\0\\1\\1\\1\\1\end{pmatrix}.$$

Then

$$W_2(H_8, (x_{00}, x_{01}, x_{10}, x_{11})) = \sum_{\alpha\in\mathbb{F}_2^2} x_\alpha^8 + 14 \sum_{\substack{\alpha_1,\alpha_2\in\mathbb{F}_2^2 \\ \alpha_1<\alpha_2}} x_{\alpha_1}^4 x_{\alpha_2}^4 + 168 x_{00}^2 x_{01}^2 x_{10}^2 x_{11}^2$$
$$= (8) + 14(4,4) + 168(2,2,2,2).$$

In general,

$$W_g(C) = \sum_A b_A \cdot (A),$$

where

$$A \in \left\{ (a_0, \ldots, a_{2^g-1}) : \text{admissible tuples}, \sum_{i=0}^{2^g-1} a_i = 2n \right\}.$$

Some analogies with modular forms:

- $W_g(C)$ is like a modular form f of genus g,
- $\sum_A b_A \cdot (A)$ is like a Fourier expansion,
- $(2n)$ is like a constant term $a(0)$.

EXamples of cusp forms:

- $W_1(G_{24}) - W_1(H_8 \times H_8 \times H_8) = -42(20,4) + 168(16,8) - 252(12,12)$ is a cusp form of genus one.
- $W_3(E_{16}) - W_3(H_8 \times H_8) = -2688(9,1,1,1,1,1,1,1) + \cdots$ is a cusp form of genus 3.

Theorem 1 (Runge, 1996; Nebe, Rains, Sloane, 2006)**.** *We have*

$$\langle W_g(C) : C \text{ self-dual, over } \mathbb{F} \rangle = \left(\mathbb{C}[x_\alpha : \alpha \in \mathbb{F}^g]\right)^{\mathcal{C}_g},$$

where

$$\mathcal{C}_g := \langle m_r, d_\phi, h_{\iota,u_\iota,v_\iota} : r \in \mathrm{GL}_g(\mathbb{F}), \phi, \iota \rangle$$

with

$$\begin{aligned} m_r &: x_\alpha \mapsto x_{r\alpha}, \\ d_\phi &: x_\alpha \mapsto e^{2\pi i \phi(\alpha)} x_\alpha, \\ h_{\iota,u_\iota,v_\iota} &: x_\alpha \mapsto (\#\iota\mathbb{F}^g)^{-1/2} \sum_{w \in \iota\mathbb{F}^g} e^{\frac{2\pi i}{p}\langle w, v_\iota \alpha \rangle} x_w + \cdots . \end{aligned}$$

Consider the mean polynomial ("Siegel-type Eisenstein series")

$$\begin{aligned} M_{2g}((2n)) &= \sum_{\gamma \in P_{2g} \backslash \mathcal{C}_{2g}} (2n)^\gamma \\ &= \sum_{\gamma \in P_{2g} \backslash \mathcal{C}_{2g}} \sum_{\alpha \in \mathbb{F}^{2g}} \left((x_\alpha)^\gamma\right)^{2n} = \text{const} \sum_{\substack{C \subset \mathbb{F}^{2n} \\ \text{fixed type}}} W_{2g}(C, x), \end{aligned}$$

and an inner product defined on monomials.

What we prove with Bourganis in 2024 is the following:

Theorem 2. *Let $\mathcal{T}$ be a family of self-dual codes of length $2n$ over a field $\mathbb{F}$. Assume either that $\mathbb{F}$ has odd characteristic or is equal to $\mathbb{F}_2$. Let $C \in \mathcal{T}$ be doubly-even, and fix $g \in \mathbb{N}$. Then there exists an (explicit) constant C such that for a cusp form $f \in \mathcal{T}$ of genus r, with $\deg f = 2n$, we have*

$$\langle M_{2g}((2n))(xy), f(x) \rangle = \begin{cases} 0 & \text{if } r < g, \\ C \cdot f(y) & \text{if } r = g \end{cases}.$$

3. Petru Constantinescu, *Non-vanishing of geodesic periods of automorphic forms*

Preprint: [4], joint with Asbhjørn Nordentoft.

Class groups:

- $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, $K = \mathbb{Q}(\sqrt{D})$,
- Cl_K class group, $h(D) = h_K = |\mathrm{Cl}_K|$ class number,

- $\mathcal{Q}_D$: set of primitive integral binary quadratic forms of discriminant D,
$$\mathcal{Q}_D = \left\{ Q(x,y) = ax^2 + bxy + cy^2 : (a,b,c) = 1,\, b^2 - 4ac = D \right\}.$$
- Gauss: $\Gamma \circlearrowright \mathcal{Q}_D$,
$$(Q.\gamma)\begin{pmatrix} x \\ y \end{pmatrix} = Q\left(\gamma \begin{pmatrix} x \\ y \end{pmatrix}\right),$$
- isomorphism
$$\mathrm{Cl}_K \xrightarrow{\cong} \Gamma\backslash\mathcal{Q}_D$$
$$A \mapsto [a,b,c].$$

Heegner points and closed geodesics:

- $D < 0$: $A \in \mathrm{Cl_K} \rightsquigarrow$ Heegner point $z_A \in \Gamma\backslash\mathbb{H}$,
$$[a,b,c] \rightsquigarrow \frac{-b - i\sqrt{|D|}}{2a}.$$
$$h(D) = |\mathrm{Cl}_K| = |D|^{1/2+o(1)}.$$
- $D > 0$: $A \in \mathrm{Cl}_K^+ \rightsquigarrow$ closed geodesic $C_A \subset \Gamma\backslash\mathbb{H}$, $[a,b,c] \rightsquigarrow$ semicircle with endpoints $\frac{-b\pm\sqrt{D}}{2a}$.
$$h(D)\log\varepsilon_D = D^{1/2+o(1)},$$
$$I(C_A) = 2\log\varepsilon_D.$$

Theorem 3 (Duke '88)**.** *Fix* $\Omega \subset \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$.

- $D < 0$*: equidistribution of* z_A*,* $A \in \mathrm{Cl}_{\mathbb{Q}(\sqrt{D})}$.
- $D > 0$*: similar for closed geodesics.*

Waldspurger formulas. Let f be a nonzero Maass form. Our goal is to study closed geodesic periods $\int_{C_A} f(z)\,\frac{|dz|}{y}$.

$\chi \in \widehat{\mathrm{Cl}_K} \rightsquigarrow \theta_\chi$, the associated theta series (weight one on $\Gamma_0(D)$, nebentypus χ_D).

Theorem 4 (Waldspurger/Zhang/Popa)**.** *Let D be a fundamental discriminant. For* $D < 0$*,*
$$L(f\times\theta_\chi, \tfrac{1}{2}) = \frac{C_f}{D^{1/2}}\left|\sum_{A\in\mathrm{Cl}_K}\chi(A)f(z_A)\right|^2.$$
For $D > 0$*,*
$$L(f\times\theta_\chi, \tfrac{1}{2}) = \frac{C_f}{D^{1/2}}\left|\sum_{A\in\mathrm{Cl}_K^+}\chi(A)\int_{C_A} f(z)\,\frac{|dz|}{y}\right|^2.$$

Theorem 5 (Michel–Venkatesh '05)**.** *Let* $\delta = 1/2700$*. For* $D < 0$*:*
$$\left|\left\{\chi \in \widehat{\mathrm{Cl}_K} : L(f\times\theta_\chi, \tfrac{1}{2}) \neq 0\right\}\right| \gg D^\delta.$$

Sketch of proof. By orthogonality of characters,
$$\frac{1}{h(D)}\sum_{\chi\in\widehat{\mathrm{Cl}_K}} L(f\times\theta_\chi, \tfrac{1}{2}) = \frac{c_f}{D^{1/2}}\sum_{A\in\mathrm{Cl}_K}|\cdots|^2.$$
This converges by Duke's equidistribution theorem. The conclusion then follows from subconvexity for Rankin–Selberg L-functions, due to Harcos–Michel. $\square$

Same proof does not work for geodesics, cannot apply equidistribution and relate to subconvexity (square is outside integral).

Question 6 (Michel, Oberwolfach 2020)**.** Let K be a real quadratic field of discriminant $D > 0$, and assume that $h_K \gg D^\delta$. Does there exist $A \in \mathrm{Cl}_K^+$ such that

$$\int_{C_A} f \frac{|dz|}{y} \neq 0?$$

Equivalently, does there exist $\chi \in \widehat{\mathrm{Cl}_K^+}$ such that $L(f \times \theta_\chi, \frac{1}{2}) \neq 0$?

The prime geodesic theorem. Let $D > 0, A \in \mathrm{Cl}_K^+$. This gives rise to a closed geodesic C_A, with $I(C_A) = 2 \log \varepsilon_D$. Sound–Young have the best estimate for their count.

Theorem 7 (C–Nordentoft 2024)**.** *Let f be a nonzero Maass form for $\mathrm{SL}_2(\mathbb{Z})$. Then*

$$\#\left\{C \in \mathcal{C}(X) : \int_C f(z) \frac{|dz|}{y} = 0\right\} \ll \frac{X}{(\log X)^{5/4}}.$$

Remark 8. We also obtain 100% non-vanishing for periods of weight k holomorphic cusp forms, for any Fuchsian group Γ.

Theorem 9 (C–Nordentoft 2024)**.** *For a positive proportion of positive discriminants $D > 0$ with $\varepsilon_\Delta \leq X$, we get that there exists $\chi \in \widehat{\mathrm{Cl}_{\mathbb{Q}(\sqrt{D})}}$ with $L(f \times \theta_\chi, \frac{1}{2} \neq 0)$.*

We construct a bipartite graph on X_N (double cosets in $\Gamma_\infty \backslash \Gamma / \Gamma_\infty$ with $c \leq N$) times Y_N (conjugacy classes with trace bounded in magnitude by N). This graph relates closed geodesic and vertical geodesics.

4. An excised orthogonal model for families of cusp forms

Talk by Zoe Batterman (Abstract), Akash Narayanan, Christopher Yao. Joint with Owen Barrett, Aditya Jambhale, and Kishan Sharma. Preprint: [2].

Conjecture (Montgomery–Dyson, 1970's): zeros of of zeta vs. GUE.

2005: S.J. Miller noticed a repulsion of the lowest-lying zeros near the central point of a family of even twists of a fixed elliptic curve L-function with finite conductor.

2011: Duenez, Huynh, Keating, Miller, Snaith: proposed an excised orthogonal model to capture the behavior of this repulsion.

Question 10. How accurately do egienvalues of random matrices from classical compact groups model the lowest-lying zeros of families of L-functions associated to a cuspidal newform?

Let

$$S_k^{\mathrm{new}}(M, \chi_f) \ni f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z},$$

$\lambda_f(n) = a_f(n)/n^{(k-1)/2}$.

$$L(s, f) = \sum_{n \geq 1} \lambda_f(n) n^{-s},$$

Various specific families of twists $L(f \otimes \psi_d, s)$, match with classical compact groups:

- principal character, even twists vs. SO(even)
- principal character, odd twists vs. SO(odd)
- non-principal character, self-dual vs. Sp
- generic vs. U

Pictures.

5. On an extension of the Rohrlich-Jensen formula, *Lejla Smajlović*

Joint work with James Cogdell, Jay Jorgensen. Preprint: [3].

What is a Poisson–Jensen formula? We will view it as a way to characterize meromorphic functions in terms of their divisors. Some notation:

- $D_R = \{z = x + iy \in \mathbb{C} : |z| < R\}$
- F: a non-constant meromorphic function on $\overline{D_R}$,

$$F(z) = c_F z^m + \mathrm{O}(z^{m+1}), \quad z \to 0.$$

Then

$$\int_0^{2\pi} \log|F(Re^{i\theta})| \frac{d\theta}{2\pi} + \sum_{D_R} \cdots = F(0).$$

Rohrlich, 1980's: a *modular* generalization, characterizing modular forms via divisors. Given f, a meromorphic function on $\mathbb{H}$ that is invariant by $\mathrm{PSL}(2,\mathbb{Z})$. Assume f is holomorphic at the cusp and that the Fourier expansion of f at ∞ has constant term equal to one. Then

$$\int_{\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}} \log|f(z)| \frac{d\mu(z)}{2\pi} + \sum_{w\in\mathcal{F}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} P(w) = 0.$$

Here

- $\mathrm{ord}_w(f)$ is the order of f at w as a meromorphic function,
- $\mathrm{ord}(w)$ denotes the order of the *point* w with respect to the action of $\mathrm{PSL}(2,\mathbb{Z})$ on $\mathbb{H}$, and
- $P(w) = \log\left(|\eta(w)|^4 \cdot \Im(w)\right)$ is the Kronecker limit function associated to the parabolic Eisenstein series on $\mathrm{PSL}(2,\mathbb{Z})$. This is the function appearing as the next-order term in the expansion of the Eisenstein series as $s \to 1$.

Another way to interpret this formula is as follows. We have

$$\langle 1, \log|f(z)|\rangle = \lim_{Y\to\infty} \int_{\mathcal{F}(Y)} 1 \cdot \log|f(z)|\, d\mu(z),$$

hence the formula reads

$$\langle 1, \log|f(z)|\rangle = -2\pi \sum_{w\in\mathcal{F}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} P(w).$$

Here $\log|f(z)|$ can be replaced by

$$\log\|f(z)\| = \log\left(\Im z^k |f(z)|\right)$$

for weight $2k$ meromorphic modular forms. Generalizations:

- to other Fuchsian groups of the first kind, by Rohrlich.
- to hyperbolic 3-space, by Herrero, Imamoglu, von Pippich, Toth.

Further modular generalization by Bringman and Kane. Keep the modular group setting, but evaluate more general inner products.

- $j(z) = q^{-1} + 744 + \mathrm{O}(q)$: Hauptmodul

- $j_1(z) := j(z) - 744$
- $j_n(z) := j_1|T_n(z)$, for $n \geq 2$
- f: weight $2k$ meromorphic modular form with respect to $\mathrm{PSL}(2,\mathbb{Z})$, normalized so that $f(z) = 1 + \mathrm{O}(q)$

They evaluated the regularized scalar product in terms of the divisor of f, proving that

$$\left\langle j_n(z), \log\left((\Im(z))^k |f(z)|\right)\right\rangle = -2\pi \sum_{w \in \mathcal{F}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} \mathbf{j}_n(w) + \frac{k}{6} c_n,$$

where $\mathbf{j}_n$ is characterized in terms of differential operators by Bringman and Kane; application of our results yields a different expression for the same function, namely

$$\mathbf{j}_n(w) = 2\pi\sqrt{n}\partial_s F_{-n}^{\mathrm{PSL}(2,\mathbb{Z})}(w,s)|_{s=1} - 24\sigma(n)P(w).$$

What is our main goal?

- To extend the point of view that the Rohrlich–Jensen formula is the evaluation of a particular type of inner product.
- To prove the extension of this formula in the setting of an arbitrary, not necessarily arithmetic, Fuchsian group of the first kind with one cusp.

We start with $j_n(z) = j_1|T_n(z)$, which is the unique (up to constants) holomorphic function that is $\mathrm{PSL}(2,\mathbb{Z})$-invariant on $\mathbb{H}$ and whose expansion near ∞ is $q^{-n} + o(q^{-1})$.

These properties hold for the special value $s = 1$ of the Niebur–Poincaré series $F_{-n}^{\Gamma}(z,s)$, defined for any Fuchsian group Γ of the first kind with oen cusp by

$$F_m^{\Gamma}(z,s) = \sum_{\gamma_\infty \backslash \Gamma} e\left(m\Re(\gamma z)\right)\left(\Im(\gamma z)\right)^{1/2} I_{s-1/2}\left(2\pi|m|\Im(\gamma z)\right),$$

for $m \neq 0$.

It is an eigenfucntion of the hyperbolic Laplacian, and may be expressed in terms of j_m.

The term $\log\left(\|f(z)\|\right)$ is a bit complicated, involving findings by Jorgensen, von Pippich and the speaker, plus some additional work done in our paper.

Proposition 11. *Let Γ be a cofinite Fuchsian group with one cusp at ∞ with identity as its scaling matrix. Let $2k \geq 0$ be an even integer, and let f be aw eight $2k$ meromorphic form which is Γ-invariant and with q-expansion at ∞ normalized so its constant term is equal to one. Then, we can express $\log\left(\|f\|(z)\right)$ in terms of parabolic Eisenstein series and Green's functions, as*

$$-2k + 2\pi \sum_{w \in \mathcal{F}_\Gamma} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} \lim_{s \to 1}\left(G_s^{\Gamma}(z,w) + \mathcal{E}_{\Gamma,\infty}^{\mathrm{par}(z,s)}\right) + \cdots.$$

Here

$$\mathcal{E}_\infty^{\mathrm{par}}(z,s) = \sum_{\Gamma_\infty \backslash \Gamma} \Im(\gamma z)^s.$$

The Kronecker limit formula says that

$$\mathcal{E}_\infty^{\mathrm{par}}(z,s) = \frac{1}{(s-1)\,\mathrm{vol}(\Gamma\backslash\mathbb{H})} + \beta - \frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}\log\left(|\eta_\infty^4(z)|\Im(z)\right) + \mathrm{O}(s-1).$$

Then we need to define β, and the Green's function $G_s(z,w)$, which is obtained by averaging the kernel $k_s(z,w)$; both functions are eigenfunctions of the Laplacian with eigenvalue $s(1-s)$, and the Green's function has a specified singularity on the diagonal. The Green's function also has a Laurent series expansion as $s \to 1$ that involves the parabolic Eisenstein series. In particular,

$$\lim_{s\to 1}\left(G_s^{\Gamma}(z,w)+\mathcal{E}^{\mathrm{par}}_{\Gamma,\infty}(z,w)\right)$$

exists, with a logarithmic singularity on the diagonal.

The Rohrlich–Jensen formula can be understood through the study of the regularized inner product of this last limit with $F_{-n}(\cdot,1)$. Regularization is needed only in the cusp (because the logarithmic singularity is integrable). We can thus write

$$\begin{aligned}&\left\langle F_{-n}(\cdot,1),\overline{\lim_{s\to 1}\left(G_s^{\Gamma}(z,w)+\mathcal{E}^{\mathrm{par}}_{\Gamma,\infty}(z,w)\right)}\right\rangle\\ &\quad=\lim_{Y\to\infty}\int_{\mathcal{F}(Y)}F_{-n}(z,1)\lim_{s\to 1}\left(G_s(z,w)+\mathcal{E}^{\mathrm{par}}_{\infty}(z,s)\right)\,d\mu(z).\end{aligned}$$

A key observation is that all terms are eigenfunctions of the Laplacian, hence one can seek to compute the inner product in a manner similar to that which yields the Maass–Selberg formula. The key identity is that

$$\begin{aligned}&\int_{\mathcal{F}(Y)}F_{-n}(z,1)\lim_{s\to 1}\left(G_s(z,w)+\mathcal{E}^{\mathrm{par}}_{\infty}(z,s)\right)\,d\mu(z)\\ &\quad=\partial_s\left(-s(1-s)\int_{\mathcal{F}(Y)}F_{-n}(z,1)\left(G_s(z,w)+\mathcal{E}^{\mathrm{par}}_{\infty}(w,s)\,d\mu(z)\right)\right)|_{s=1}.\end{aligned}$$

We can now absorb $-s(1-s)$ into the integrals after applying the hyperbolic Laplacian Δ to each of the two factors in the integrand.

Main results:

Theorem 12. *For any positive integer n and any point $w\in\mathcal{F}$, we have*

$$\left\langle F_{-n}(\cdot,1),\overline{\lim_{s\to 1}\left(G_s(\cdot,w)+\mathcal{E}^{\mathrm{par}}_{\infty}(\cdot,s)\right)}\right\rangle=-\partial_sF_{-n}(w,s)|_{s=1}.$$

Combined with the previous proposition (describing $\log\|f(z)\|$), we get the Rohrlich–Jensen formula:

$$\langle F_{-n}(\cdot,1),\log\|f\|\rangle=-2\pi\sum_{w\in\mathcal{F}}\frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)}\partial_sF_{-n}(w,s)|_{s=1}.$$

We have further results. For instance, if g is any Γ-invariant analytic function with a pole at ∞, then (Niebur)

$$g(z)=\sum_{n=1}^{K}2\pi\sqrt{n}a_nF_{-n}(z,1)+c(g)$$

for some constants K, a_n, and $c(g)$ depending only upon g. Then, we have the identity

$$\langle g,\log|f|\rangle=-2\pi\sum_{w\in\mathcal{F}}\frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)}\left(2\pi\sum_{n=1}^{K}\sqrt{n}a_n\partial_sF_{-n}(w,s)|_{s=1}-\cdots\right).$$

Further found that the generating series for the Niebur Poincaré series at $s=1$ is, in the z-variable (where we sum over q_z), the holomorphic part of the weight two biharmonic Maass form given by differentiating our linear combination of Eisenstein series and Green's function with respect to z.

6. Shenghao Hua, *Joint Value Distribution of Hecke–Maass Forms*

Preprint: [7]

Motivation: semiclassical limit of solutions to Schroedinger equation. Berry's random wave conjecture. QUE, L^p-norms.

Joint Gaussian moments conjecture: statistical indepndence of values of large eigenfunctions (multiplied together and integrated against some test function).

Theorem 13. *Let f and g be normalized Hecke–MAass cusp forms. Then if $2t_f \le t_g - t_g^\varepsilon$, we have*

$$\int_{\Gamma\backslash\mathbb{H}} \psi f^2 g \ll t_g^{-A},$$

while if $2t_f > t_g - t_g^\varepsilon$, then, assuming GLH, we have

$$\int_{\Gamma\backslash\mathbb{H}} \psi f^2 g = \int_{\Gamma\backslash\mathbb{H}} \psi g + \mathrm{O}(t_f^{-1/4+\varepsilon}(1 + |2t_f - t_g|^{-1/4})).$$

Proof uses spectral expansion of ψ into eigenfunctions u, then spectral expansion of $\int(ug)(f^2)$, reducing to the case $g = u$, then further spectral expansion of $\int(g^2)(f^2)$.

Theorem 14. *Assuming GRH and GRC, we have as $\max(t_f, t_g) \to \infty$*

$$\mathbb{E} f^2 g^2 = 1 + \mathrm{O}((\log(t_f + t_g))^{-1/4+\varepsilon}).$$

Proof eventually applies ultimately Soundararajan's method of moments (which requires GRH).

Work in progress: asymptotic formulas for certain averages of L-functions over toroidal families, going beyond [10].

7. Algebraic proof of modular form inequalities for optimal sphere packings, *Seewoo Lee*

Preprint: [8].

Motivating question: finding densest sphere packings.

- $d = 1$
- $d = 2$: Thue 1910, $\Delta_2 = \frac{\pi}{2\sqrt{3}}$
- $d = 3$: Kepler conjecture 1611, Hales 2005, $\Delta_3 = \frac{\pi}{3\sqrt{2}}$, formally verified 2014 in Isabelle/HOL
- (Korkine–Zolotareff, Blichfeldt, Cohn–Kumar): D_4, D_5, D_6, E_7, E_8 and Leech lattice
- Viazovska, 2016 π-day on arXiv: E_8 lattice packing optimal, with $\Delta_8 = \frac{\pi^4}{388}$
- Cohn–Kumar–Miller–Radchenko–Viazovska, March 21st 2016 on arXiv: $\Delta_{24} = \frac{\pi^{12}}{12!}$ Leech lattice (unique even unimodular with nonzero minimal length 2)

Viazovska et al use:

Theorem 15 (Cohn–Elkies, 2013)**.** *Let $r > 0$. Assume there exists a nice function $f : \mathbb{R}^d \to \mathbb{R}$ satisfying*

- $f(0) = \hat{f}(0) > 0$,
- $f(x) \le 0$ *for all* $\|x\| \ge r$,
- $\hat{f}(y) \ge 0$ *for all* $y \in \mathbb{R}^d$,

then

$$\Delta_d \leq \mathrm{vol}(B^d_{r/2}) = \left(\frac{r}{2}\right)^d \frac{\pi^{d/2}}{(d/2)!}.$$

Proof is not hard (fits in one page).

This leads to the hunt for a "magic function" f satisfying the indicated conditions. BAsed on their numerical experiments, Cohn–Elkies conjectured that the optimal sphere packing can be achieved by a magic function when $d = 2, 8, 24$. The magic function f and its Fourier transform $\hat{f}$ should vanish at the lattice points.

Viazovska et al constructed the magic functions for $d = 8, 24$ using modular forms. Decompose into Fourier eigenfunctions $f = f_+ + f_-$ (parity under Fourier transform). Viazovska writes them as

$$f_\pm(x) = \sin^2\left(\frac{\pi\|x\|^2}{2}\right) \int_0^\infty \varphi_\pm(t) e^{-\|x\|^2 t}\, dt,$$

where the $\sin^2$ factor enforces desired roots.

For $d = 8$, we have $\varphi_\pm(t) = t^2 \psi_\pm(i/t)$, where $\psi_\pm$ are defined in terms of standard modular forms. We also need (for both $d = 8, 24$) a few non-obvious inequalities involving some of these modular forms. One reason these inequalities are difficult is because they're inhomogeneous with respect to the weights of the forms involved.

Original proofs use bounds of Fourier coefficients of the form

$$|c(n)| \leq C_1 e^{C_2 \pi \sqrt{n}},$$

following from the Hardy–Ramanujan formula, reducing the question to finite calculations and interval arithmetic.

Ran Romik (2023) gave an alternative and much simpler proof for $d = 8$ that doe snot use any interval arithmetic, but still requires a "calculator" to check inequalities like

$$e^{3\pi} \frac{9\Gamma(1/4)^{16}}{8192\pi^{12}} < 20480.$$

Question 16. Can we prove such inequalities *algebraically*?

The answer is yes, as we now explain. To that end, we'll develop a theory of *(completely) positive quasimodular forms.*

Definition 17. Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$. We say that a quasimodular form F is *positive* if

$$F(it) \geq 0$$

for all $t > 0$.

We call it *completely positive* if it has nonnegative q-coefficients at ∞.

Note that "completely positive" implies "positive", but the inclusion is strict in general. For instance, the discriminant form has a product formula that tells you that it is positive, but you can check that it's not completely positive.

Easy facts:

Theorem 18.
- *Anti-derivative preserves positivity.*
- *Derivative preserves complete positivity.*
- *Serre derivative preserves complete positivity.*

"Nontrivial" fact (that we won't use, and which also follows directly from Bernstein's theorem):

Theorem 19. *F is completely positive if and only if all its derivatives are positive.*

"Interesting" fact that we will use:

Theorem 20. *Let $F = \sum_{n \geq n_0} a_n q^n$ with $a_{n_0 > 0}$. If some Serre derivative of F is positive, then so is F.*

Examples?

Definition 21 (Kaneko–Koike)**.** For given weight w and depth s, we can speak of *extremal quasimodular forms of weight w and depth s*, which are those whose order of zeros at ∞ is as large as possible.

It's been shown that these have almost all coefficients positive, and conjectured that they're all positive.

Theorem 22. *That conjecture is true for depth 1 extremal forms.*

We also have similar identities proving complete positively of the depth 2 extremal forms of weight $w \leq 14$, but not yet for general w.

This reduces some of the inequalities in the arguments of Viazovska et al to checking some limiting properties of ratios of modular functions. For monotonicity, we check that derivatives are positive, and for this it suffices to check the same for Serre derivatives of derivatives.

Possible future directions:

- Classify the completely positive forms?
- Possible applications in other LP problems?
  - Dual LP, uncertainty principle
  - Any results that are "uniform" in dimensions
- Make a formalization of the proof (e.g., in Lean) easier?

8. Ring of modular forms on certain unitary Shimura variety, *Yuxin Lin*

Recall the definition of modular forms of weight k and level 1, as certain holomorphic functions $f : \mathbb{H} \to \mathbb{C}$.

More generally, let $\mathrm{Sh}(G)$ be a Shimura variety (of level 1), where G is a reductive group over $\mathbb{Q}$. We can understand it as the moduli space of abelian varieties with additional structure. Let $\pi : \mathcal{A} \to \mathrm{Sh}(G)$ be the universal abelian scheme of dimension g.

Let $\underline{\omega} := \pi_* \Omega_{\mathcal{A}/\mathrm{Sh}}$ be the Hodge bundle on $\mathrm{Sh}(G)$, and let $\lambda_{\mathrm{Sh}} := \Lambda^g \underline{\omega}$ be the Hodge line bundle on $\mathrm{Sh}(G)$.

Definition 23. The ring of automorphic forms of scalar-valued weight is the space of global sections of the k-th tensor power of the Hodge line bundle, namely

$$\oplus_k H^0(\mathrm{Sh}, \lambda^{\otimes k}).$$

, 8, 12, 2, 16 An interesting question to ask is, what is the ring structure of such objects? There are some relevant results along these lines.

First, when $G = \mathrm{SL}_2(\mathbb{Z})$, we have that $\mathrm{Sh}(G)$ is the modular curve, and the ring of modular forms is generated by E_4 and E_6 modulo a single relation:

$$\mathcal{M}(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6]/(E_4^3 - E_6^2).$$

Going one dimension higher, for $G = \mathrm{SL}_2(\mathcal{O}_K)$ and $K = \mathbb{Q}(\sqrt{13})$, we have that $\mathrm{Sh}(G)$ is the Hilbert modular surface with real multiplication by $\mathcal{O}_K$. Van Der Geer and Zagier show that the ring of symmetric Hilbert modular forms of even weight has 4 generators at weights $4, 8, 12, 12$ respectively, with a relation at weight 24, while the ring of Hilbert modular forms of even weight has 5 generators of weights $4, 8, 12, 12, 16$, respectively, with two relations at weight 24 and 32. To that end, they find the minimal model Y of the compactification of $\mathrm{Sh}(G)$, then find the canonical divisor of Y and compute the image S of Y under the canonical embedding, then realize sections of $H^0(\mathrm{Sh}, \lambda^{\otimes 2k})$ as meromorphic sections of $\mathcal{O}_S(k)$ with certain multiplicity at the cusps.

This is the strategy we will imitate, but for certain unitary Shimura varieties rather than Hilbert modular surfaces.

To that end, we need to first introduce the moduli space of curves.

Let $G = \mathbb{Z}/d\mathbb{Z}$ be a finite cyclic group.

Definition 24.
- $\mathcal{M}_G$: the moduli space of admissible stable $\mathbb{Z}/d\mathbb{Z}$ covers of $\mathbb{P}^1$. Objects are pairs $(C/S, \iota : G \to \mathrm{Aut}_S(C))$, i.e., a family of curves over the base scheme together with an embedding of G in their automorphism group.
- $\tilde{\mathcal{M}_G}$, the moduli space of admissible stable $\mathbb{Z}/d\mathbb{Z}$ covers of $\mathbb{P}^1$ with an ordering on the ramification points. The objects are tuples $(C/S, \iota : G \to \mathrm{Aut}_S(C), \sigma_i)$.
- The forgetful morphism $\tilde{\mathcal{M}_G} \to \mathcal{M}_G$ is finite étale.

The irreducible connected components of $\mathcal{M}_G$ are indexed by monodromy datum, which is a triple $(d, r, \underline{a})$, where

- r is the number of branching points on $C/\iota(G)$,
- $\underline{a} = (a(1), \dots, a(r)) \in G^r$ is the *inertia type*, which records the character of G at the tangent space of the i-th ramification point.

The *Torelli morphism* T associates the moduli of curves with the moduli of principally polarized abelian varieties:

$$\begin{array}{ccc} \mathcal{M}(d, r, \underline{a})_{\mathbb{C}} & \longrightarrow & \mathrm{Sh}(\mathcal{D})_{\mathbb{C}} \\ \downarrow & & \downarrow \\ \mathcal{M}_{g,\mathbb{C}} & \xrightarrow[T]{} & \mathcal{A}_{g,\mathbb{C}}. \end{array}$$

The image of the Torelli morphism is called the *Torelli locus* inside $\mathcal{A}_{g,\mathbb{C}}$. It turns out that the Jacobians coming out of $\mathcal{M}(d, r, \underline{a})$ have larger endomorphisms than generically, and in particular, at least contain the group algebra generated by G. It's thus reasonable to speculate that the image of T on $\mathcal{M}(d, r, a)$ will sit inside a *special* subvariety of $\mathcal{A}_g$, and this special subvariety will be the focus of our study.

More precisely, $\mathrm{Sh}(\mathcal{D})$ is the smallest PEL type Shimura subvariety of $\mathcal{A}_{g,\mathbb{C}}$ containing $T(\mathcal{M}(d, r, \underline{a})_{\mathbb{C}})$. It parametrizes abelian varieties with multiplication by $\mathbb{Z}[G]$ and signature given by $(d, r, \underline{a})$.

We focus on the family $\mathcal{M}[16] = \mathcal{M}(5, 5, (1, 1, 1, 1, 1))$.

- The associated Shimura variety $\mathrm{Sh}(G)$ is of *unitary type*, whose reductive group $\mathcal{G}$ has $\mathbb{R}$ points given by $\mathrm{GU}(3, 0) \times \mathrm{GU}(2, 1)$. Therefore, it is *compact*. The reason is that the signature of this Shimura variety can be computed, and the signature has a zero in one of the components, which means that the hermitian symmetric domain is compact, and hence this $\mathrm{Sh}(\mathcal{D})$ is compact. In particular, this Shimura variety has no cusp.

- A second reason why this family is so nice is that $\mathrm{Sh}(\mathcal{D})_{\mathbb{C}}$ has dimension 2, which is equal to the dimension of $\mathcal{M}[16]$. Therefore, the Torelli morphism T gives an *isomorphism*

$$T : \mathcal{M}[16] \to \mathrm{Sh}(\mathcal{D}),$$

 such that T gives an isomorphism between the corresponding Hodge line bundles. Thus, in order to compute global sections of the Hodge line bundle on $\mathrm{Sh}(\mathcal{D})$, it suffices to do the analogous thing on $\mathcal{M}[16]$, which is a bit simpler, but still tricky, due to nontrivial stabilizers. To that end, we consider:
- The finite étale cover $\tilde{\mathcal{M}[16]}$, which has a natural forgetful morphism

$$q : \tilde{\mathcal{M}[16]} \to \overline{M_{0,5}},$$

 that is bijective on field-valued points. This is very nice, because we know the geometry of $\overline{M_{0,5}}$: its Picard group is generated by ten divisors, each isomorphic to $\mathbb{P}^1$.

Proposition 25. *Let Δ be the boundary divisor of $\overline{M_{0,5}}$, and let $\lambda_{\tilde{M}}$ be the Hodge divisor on $\tilde{\mathcal{M}[16]}$. Under the forgetful morphism*

$$q : \tilde{\mathcal{M}[16]} \to \overline{M_{0,5}},$$

we have that $q^*\Delta = 5\lambda_{\tilde{M}}$.

Idea of proof.
- Use Grothendieck–Riemann–Roch to relate $\lambda_{\tilde{M}}$ with the *boundary divisor* of $\tilde{\mathcal{M}}$.
- q exhibits Δ as a multiple of the boundary of $\tilde{\mathcal{M}}[16]$.

$\square$

Our next goal is to fill in the lifts

$$\begin{array}{ccc} \tilde{\mathcal{M}[16]} & \xrightarrow{?} & \mathrm{Sh}_{\mathcal{K}}(\mathcal{D}) \\ \downarrow & & \downarrow ? \\ \mathcal{M}[16] & \xrightarrow[T]{} & \mathrm{Sh}(\mathcal{D}). \end{array}$$

Giving such a cover of T is the same as computing the level subgroup corresponding to this cover. That level subgroup is basically a congruence subgroup:

Proposition 26. *The level subgroup $\mathcal{K}$ at the place p satisfies:*

$$\mathcal{K}_p = \begin{cases} \mathcal{G}(\mathbb{Z}_p) & \text{if } p \neq 5, \\ \mathcal{U}(\mathfrak{m}) & \text{if } p = 5. \end{cases}$$

Putting these parts together via the sequence of morphisms

$$\mathrm{Sh}_{\mathcal{K}}(\mathcal{D}) \xleftarrow{T} \tilde{\mathcal{M}[16]} \xrightarrow{q} \overline{M_{0,5}} \xrightarrow{|-K|} S,$$

we obtain the following:

Proposition 27. • *The weight m automorphic forms on $\mathrm{Sh}_{\mathcal{K}}(\mathcal{D})$ is*

$$H^0(\mathrm{Sh}_{\mathcal{K}(\mathcal{D}),\lambda_{\mathrm{Sh}^{\otimes m}}}) \cong H^0(S, \mathcal{O}_S(2\lfloor\frac{m}{5}\rfloor)) \cong (\mathbb{C}[x_0,\dots,x_5])/(Q_{12}-Q_{ij}),$$

with explicit degrees.
- *Similar for the ring of graded automorphic forms.*

From the moduli interpretation, we see that $\mathrm{Sh}(\mathcal{D})$ is the $\mathfrak{S}_5$-invariant of $\mathrm{Sh}_{\mathcal{K}}(\mathcal{D})$. Therefore:

Theorem 28. *The ring of automorphic forms on $\mathrm{Sh}(\mathcal{D})$ is the $\mathfrak{S}_5$-invariant of $\mathrm{Sh}_{\mathcal{K}}(\mathcal{D})$. In particular,*

$$\oplus_m H^0(\mathrm{Sh}(\mathcal{D}), \lambda^{\otimes m}) \cong \mathbb{C}[f_2, f_4, f_6, \varepsilon]/\varepsilon^{10} + \frac{1}{52}(f_2^2 - 4f_4),$$

where the f_{2n} are polynomials of degree $2n$ in x_i, where everything has explicit degrees.

9. Counting modular forms mod p satisfying constraints at p, *Samuele Anni*

Recall the definition of modular forms f of weight k on $\Gamma_0(n)$, with q-expansion $f(z) = \sum a_n q^n$. We also know that there are Hecke operators T_p acting on them. We consider only cuspidal *newforms* ($a_0 = 0$), *normalized* ($a_1 = 1$), which are eigenforms for the Hecke operators and arise from level n and not any smaller level. We use the notation $S_k(n, \mathbb{C})$ and $S_k(n,\mathbb{C})^{\mathrm{new}}$.

Definition 29. The *Hecke algebra* $\mathbb{T}(n,k)$ is the $\mathbb{Z}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(S_k(n,\mathbb{C}))$ generated by Hecke operators T_p for each prime p.

Newforms can be seen as ring homomorphisms $f : \mathbb{T}(n,k) \to \overline{\mathbb{Q}}$.

Theorem 30 (Deligne, Serre, Shimura)**.** *Let n and k be positive integers. Let $\mathbb{F}$ be a finite field of characteristic ℓ, with $\ell \nmid n$, and let $f : \mathbb{T}(n,k) \twoheadrightarrow \mathbb{F}$ be a surjective ring homomorphism. Then there is a unique continuous semisimple representation*

$$\bar{\rho}_f : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}),$$

unramified outside $n\ell$, such that for all p not dividing $n\ell$, the trace of Frobenius at p under $\bar{\rho}_f$ is $f(T_p)$, and the determinant is given in terms of the value of f on the diamond operator (central character).

Computing $\bar{\rho}_f$ is "difficult", but theoretically it *can be done in polynomial time* in $n, k, \#\mathbb{F}$:

- Edixhoven, Couveignes, de Jong, Merkl, Bruin, Bosman: $\#\mathbb{F} \leq 32$.
- Mascot, Zeng, Tian: $\#\mathbb{F} \leq 53$.

Fix a prime $p \geq 5$, a level N prime to p, and a weight $k \geq 2$. Let $S_k := S_k(\Gamma_0(Np), \bar{\mathbb{Q}}_p)$ be the space of p-new (i.e., not coming from level N) cuspidal modular forms of level Np and weight k with coefficients in $\bar{\mathbb{Q}}_p$.

We now discuss the **Atkin–Lehner involution**. There exist $x, y, z, t \in \mathbb{Z}$ for which the matrix

$$W_p = \begin{pmatrix} px & y \\ Npz & pt \end{pmatrix}$$

has determinant p.

The matrix W_p normalizes the group $\Gamma_0(Np)$, and for any weight k, it induces a linear operator w_p on the space of cusp forms S_k that commutes with the Hecke operators T_q for all $q \nmid Np$ and acts as its own inverse.

Any cusp form in S_k that is an eigenform for all T_q with $q \nmid N$ is also an eigenform for w_p, withe eigenvalue ± 1. This involution acts on the modular curve.

The Atkin–Lehner involution w_p splits S_k as a sum of plus and minus spaces:

$$S_k = S_k^+ \oplus S_k^-.$$

Since we have dimension formulas for $s_k := \dim S_k$, in order to understand the dimensions

$$s_k^\pm = \dim S_k^\pm$$

of the Atkin–Lehner eigenspaces, it suffices to understand the difference

$$d_k := s_k^+ - s_k^-.$$

We can get some idea from tables of examples (obtained empirically by looking at many more specific examples):

p	d_k
5	± 1
11	± 2
59	± 6
101	± 7

Table 1. Values of d_k for different p

Classical result that d_k is constant in absolute value and alternates in sign. Need to introduce modification d_k^* taking into account Eisenstein series of weight two. One can show:

Theorem 31 (Fricke, Yamauchi, Momose, Ogg, Wakatsuki, Helfgott, Martin et al.)**.** *We have*

$$d_k^* = (-1)^{k/2} \frac{\#\mathrm{FP}}{2},$$

where #FP *is the number of fixed points of the Atkin–Lehner involution* w_p *on* $X_0(Np)$*.*

The fixed points of w_p on $X_0(Np)$ corresponds to elliptic curves with level structure and CM by $\sqrt{-p}$, in fact the d_k are closely related to class numbers:

$$\#\mathrm{FP} = c_p \cdot h(\sqrt{-p}) \cdot \prod_{q|N,\mathrm{prime}} \left(1 + \left(\frac{-4p}{q}\right)\right).$$

What do we want to do? Systems of mod-p prime-to-Np Hecke eigenvalues correspond to continuous semisimple Galois representations $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$. Can decompose according to these, and then further via Atkin–Lehner, giving spaces $S_{k,\bar\rho}^\pm$ and their dimensions $s_{k,\bar\rho}^\pm$ and differences $d_{k,\bar\rho}$. As before, $k = 2$ and $\bar\rho$ forces us to make an adjustment, so let

$$d_{k,\bar\rho}^* := \begin{cases} d_{k,\bar\rho} - 1 & \text{if } k = 2 \text{ and } \bar\rho = 1 \oplus \omega, \\ d_{k,\bar\rho} & \text{otherwise,} \end{cases}$$

where ω is the cyclotomic character.

Theorem 32 (Anni, Ghitza, Medvedovsky)**.** *For $k \geq 2$ and any $\Gamma_0(Np)$-modular $\bar{\rho}$, we have*

$$d^*_{k+2,\bar{\rho}[1]} = -d^*_{k,\bar{\rho}}.$$

As an example, for $p = 5$, $N = 23$, we can check that for $k > 2$, we have $d_k = \pm 2$.

Bergdall and Pollack use the Ash–Stevens formula, a fundamentally characteristic p technique for filtering cohomology of modular symbols, to derive their dimension formulas. But Ash–Stevens has nothing to say about Atkin–Lehner, in part because Atkin–Lehner operator requires inverting p. On the other hand, the classical complex methods – trace formulas, Gauss–Bonnet, Riemann–Hurwitz – do not know anything about $\bar{\rho}$.

What we do instead is to combine the *trace formula* (Zagier–Cohen–Osterlé–Cohen–Strömberg and Skoruppa–Zagier–Popa) with an *algebra theorem*, an explicit refinement of Brauer–Nesbitt.

What is this explicit Brauer–Nesbitt? See [1].

Theorem 33 (AGM)**.** *Let M and N be two finite free $\mathbb{Z}_p$-modules of the sam erank d, each with an action of an operator T. Then $\bar{M}^{\mathrm{ss}} \cong \bar{N}^{\mathrm{ss}}$ as $\mathbb{F}_p[T]$-modules if and only if for every n with $1 \leq n \leq d$, we have*

$$\mathrm{trace}(T^n|M) = \mathrm{trace}(T^n|N),$$

(...).

Corollary 34. *If we have modules that injective modulo p (but not necessarily in characteristic zero), then we can check that semisimplifications of certain residual quotients are the same by proving an inequality involving traces.*

Now work with the trace formula. We can deduce statements about dimensions.

10. Integer partitions detect the primes, *Jan-Willem Van Ittersum*

Preprint: [5].

We write s_i for the sizes of the parts, and m_i for the multiplicities of the part, with $s_1 > s_2 > \cdots$.

Definition 35 (MacMahon 1920)**.**

$$M_a(n) := \sum_{\substack{n = m_1 s_1 + \cdots + m_a s_a \\ 0 < s_1 < \cdots < s_a}} m_1 \cdots m_a.$$

Remark 36. This is the sum of multiplicity products of partitions of n with a part sizes.

Consider

$$\psi(n) := (n^2 - 3n + 2)M_1(n) - 8M_2(n).$$

Example 37. $n = 3, a = 1$, in which case $\lambda = (3)$ or $\lambda = (1^3)$, in which case $M_1(3) = 1 + 3 = 4$, or $a = 2$, in which case $\lambda = (2, 1)$, so $M_2(3) = 1 \cdot 1 = 1$, and we compute that $\psi(3) = 2M_1(3) - 8M_2(3) = 0$.

Continuing, take $n = 4, a = 1$, in which case $\lambda = (4)$, $\lambda = (2^2)$ $\lambda = (1^4)$, so $M_1(4) = 1 + 2 + 4 = 7$, or $a = 2$, in which case $\lambda = (3, 1)$ or $\lambda = (2, 1^2)$, so $M_2(4) = 1 \cdot 1 + 1 \cdot 2 = 3$, and so $\psi(4) = 6M_1(4) - 8M_2(4) = 18$.

Continuing, one gets the table, for $n = 2..11$, of $\psi(n) = 0, 0, 18, 0, 120, 0, 270, 192, 504, 0$.

Theorem 38 (Craig–vI–Ono)**.** *For positive integers n, we have*

(1) $(n^2-3n+2)M_1(n) - 8M_2(n) \geq 0$,

(2) $(3n^3-13n^2+18n-8)M_1(n) + (12n^2-120n+212)M_2(n) - 960M_3(n) \geq 0$

and for $n \geq 2$ these expressions vanish if and only if n is prime.

Remark 39. We call such an expression *prime-detecting*. The sum of two such expressions is likewise prime-detecting, and multiplying $f(n)$ for a polynomial f yields a prime-detecting expression if $f(n) > 0$ for all n.

We can get a table of prime-detecting expressions: the first two that appear above, then two more, $(126n^5 - \cdots)M_1(n) + \cdots$ and $(300n^8 - \cdots)M_1(n) + \cdots$.

Conjecture 40. *These are all such expressions (up to addition and multiplication as before).*

We then checked whether we could get more results of a similar shape, by considering a generalization of the MacMahon function:

Definition 41. For $\ell \in \mathbb{Z}^a_{\geq 0}$, we define the *generalized MacMahon partition function*

$$M_{\underline{\ell}}(n) := \sum_{\substack{n=m_1s_1+\cdots+m_as_a \\ 0<s_1<\cdots<s_a}} m_1^{\ell_1}\cdots m_a^{\ell_a}.$$

You can think of this as a generalization of the divisor function, but for partitions. For this new generalized partition function, much more is possible:

Theorem 42 (Craig–vI–Ono)**.** *Let $d \geq 4$.*

(1) There exist clu$\underline{\ell} \in \mathbb{Z}$ such that $\sum_{|\ell|<d} c_{\underline{\ell}} M_{\underline{\ell}}(n) \geq 0$, where $|\underline{\ell}| = \ell_1+\cdots+\ell_a$.

(2) There are $\gg d^2$ linearly independent such expressions.

Example 43. $63M_{(2,2)}(n) - 12M_{(3,0)}(n) - \cdots + 12M_{(3,0,0)}(n) = \frac{11}{3}\psi(n)$.

¡++¿

Goal of the talk now is to explain some of the ideas behind these results, which have a lot to do with modular forms.

Prime-detecting quasimodular forms. Let

$$\mathcal{G}_k := -\frac{B_k}{2k} + \sum_{n\geq 1}\sigma_{k-1}(n)q^n, \qquad D := q\frac{\partial}{\partial q}, \qquad q = e^{2\pi i\tau}.$$

Consider

$$\begin{aligned} f_{k,\ell} &:= (D^\ell+1)\mathcal{G}_{k+1} - (D^k+1)\mathcal{G}_{\ell+1} \\ &= * + \sum_{n\geq 1}\sum_{d|n}\left((n^\ell+1)d^k - (n^k+1)d^\ell\right)q^n. \end{aligned}$$

Note that for $d=1$ one gets $n^\ell + 1 - n^k - 1 = n^\ell - n^k$. Calculating similarly with $d = n$, we see that $f_{k,\ell}$ vanishes at prime coefficients. It's also easy to see that its coefficients are positive. But note that $f_{k,\ell}$ is not modular, and not of homogeneous weight. Key features here are quasimodularity and mixed weight.

Theorem 44 (Craig–vI–Ono)**.** *All prime-detecting forms in $\oplus_{k:even}\oplus_{n\geq 0} D^n\mathcal{G}_k$ are linear combinations of D^nH_k, where*

$$H_k = \begin{cases} \frac{1}{6}(D^2-D+1)\mathcal{G}_2 - \mathcal{G}_4 & \text{if } k=6, \\ \frac{1}{24}\left(-D^2\mathcal{G}_{k-6} + (D^2+1)\mathcal{G}_{k-4} - \mathcal{G}_{k-2}\right) & \text{if } k=8. \end{cases}$$

That's the first ingredient, but still need to say something about modularity of MacMahon functions. Let

$$\mathcal{G}(\underline{\ell}) = \sum_{n\geq 0} M_{\underline{\ell}}(n)q^n \xrightarrow{q\to 1} \frac{(1-q)^{|\underline{\ell}|+a}}{\prod_i \ell_i!} \sum_{s_1<\cdots<s_a} \frac{1}{s_1^{\ell_1+1}\cdots s_a^{\ell_a+1}} \quad (\ell_a \geq 1,\, \ell_i \geq 0).$$

The algebra $\mathcal{Z}_q := \langle \mathcal{G}(\underline{\ell})\rangle_{\mathbb{Q}}$ was introduced by Bachmann–Kühn. Facts:

- $\mathcal{Z}_q$ is a differential algebra (closed under differentiation and multiplication)
- $\tilde{M} := \mathbb{Q}[\mathcal{G}_2, \mathcal{G}_4, \mathcal{G}_6] \subseteq \mathcal{Z}_q$ (i.e., it contains the space of quasimodular forms).
- (Hoffman–Ihara) We have

$$\sum_{j\geq 0} \mathcal{G}(\underbrace{1,1,\ldots,1}_{j})x^j = \exp\left(\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} x^n \underbrace{\sum_m \frac{q^m}{(1-q^m)^{2n}}}_{\sum_{m,s\geq 1}\binom{s+n-1}{s-n}q^{ms}\in\tilde{M}}\right),$$

  which is a linear combination of quasimodular Eisenstein series, hence $M_a(n)$ are coefficients of a quasimodular form.

11. Supersingular abelian surfaces and where to find them, *Gabriele Bogo*

Characteristic zero: $\mathrm{End}(E) \cong \mathbb{Z}$ or $\mathcal{O}_D$, order in an imaginary quadratic field.

Characteristic $p > 0$ (Hasse, 1936), can also be $\mathcal{O}_{\infty,p}$, order in a quaternion algebra over $\mathbb{Q}$. In that case, we call it *supersingular*.

Deuring 1941: for $p \geq 5$, the number of supersingular elliptic curves over $\mathbb{F}_{p^2}$ is

$$\lfloor \frac{p-1}{12} \rfloor + \delta + \varepsilon.$$

Another interpretation: writing $\mathcal{M}_{1,1} \cong \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ for the moduli space of elliptic curves over $\mathbb{C}$, there are three special points $\infty, i = \sqrt{-1}, \rho = (-1+\sqrt{-3})/2$, and we can think of Deuring's result as

$$\#(\mathrm{ss}_p) = \lfloor \frac{p-1}{12} \rfloor + \delta + \varepsilon \leq \lfloor \frac{-\chi(\mathbb{H}/\mathrm{SL}_2(\mathbb{Z}))\cdot(p-1)}{2} \rfloor + 2.$$

Let's generalize to higher-dimensional varieties. Let $\mathbb{Q}(\sqrt{D})$ be a real quadratic field, with ring of integers $\mathcal{O}_D$. We consider a family $\mathcal{X} \to \mathcal{C}$ of (p.p.) abelian surfaces, over a curve $\mathcal{C}$, such that:

- the fibers X_c are defined over a number field k, and
- there exists an inclusion of rings $\mathcal{O}_D \hookrightarrow \mathrm{End}_k(X_c)$.

An abelian surface over a field of characteristic p is *supersingular* (resp. *superspecial*) if it is isogenous (resp. isomorphic) to the product of two supersingular elliptic curves.

There are embeddings

$$\mathbb{H}/\Gamma = \mathcal{C} \hookrightarrow X_D(\mathbb{C}) = \mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_D).$$

For example, if $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, this is the inclusion of elliptic curves into the space of abelian surfaces with multiplication. There are also modular curves $\Gamma_0(N)$. But one can have more interesting examples, related to non-arithmetic curves. To give an explicit example, consider the triangle group $\Gamma = \Delta(2,5,\infty) \hookrightarrow X_5$, with explicit equation

$$y^2 = \begin{cases} x^5 - 5x^3 + 5x - 2t & \text{if } t \neq \infty, \\ x^5 - 1 & \text{if } t = \infty. \end{cases}$$

We now take the reductions of the above inclusion modulo p, to a map

$$\overline{\mathcal{C}} \to X_D(\mathbb{F}).$$

We want to study the fibers of this family that, after reduction modulo p, are supersingular.

To that end, consider a smooth algebraic curve in a Hilbert modular surface

$$\mathcal{C} \hookrightarrow X_D$$

with second Lyapunov exponent $\lambda_2 \in \mathbb{Q} \cap (0,1]$. For example, for $\mathrm{SL}_2(\mathbb{Z})$, we have $\lambda_2 = 1$, and for modular curves, $\lambda_2 = 1$, while for $\Delta(2,5,\infty)$ we have $\lambda_2 = 1/3$, but for Teichmuller curves, you can have $1/3, 1/5, 1/7$.

Theorem 45 (B–Li, 2024)**.** *The supersingular locus of $\mathcal{C}$ modulo p has cardinality described by*

$$\lfloor \frac{-\chi(\mathcal{C})(p-\lambda_2)}{2} \rfloor \leq \#\mathrm{ss}_p^{\mathcal{C}} \leq \lfloor \frac{-\chi(\mathcal{C})(p-1)(\lambda_2+1)}{2} \rfloor + r, \qquad \textit{if } \left(\frac{D}{p}\right) = -1,$$

while

$$\#\mathrm{ss}_p^{\mathcal{C}} \leq \lfloor \frac{-\chi(\mathcal{C})(p-1)\lambda_2}{2} \rfloor + r, \qquad \textit{if } \left(\frac{D}{p}\right) = 1.$$

How to find supersingular abelian surfaces? If $\mathcal{C} = \mathbb{H}/\Gamma$ is of genus zero with Hauptmodul j, then the supersingular locus can be described by a polynomial in j:

$$\mathrm{ss}_p^{\mathcal{C}}(j) := \prod_{\substack{c \in \mathcal{C} \\ \overline{X_c} \text{ supersingular}}} (j - j(c)).$$

Theorem 46 (B.–Li, 2024)**.** *There are two families $\{A_{0,n}(j)\}_n$ and $\{A_{1,n}(j)\}_n$ of orthogonal polynomials (Atkin's polynomials) such that for every p of good reduction,*

$$\mathrm{ss}_p^{\mathcal{C}}(j) \equiv \begin{cases} \mathrm{lcm}(A_{0,n_p}, A_{1,\tilde{n}_p}) & \textit{if } \left(\frac{D}{p}\right) = -1, \\ \gcd(A_{0,n_p}, A_{1,\tilde{n}_p}) & \textit{if } \left(\frac{D}{p}\right) = 1 \end{cases}$$

for explicit indices n_p and $\tilde{n}_p$.

For example, consider $\Delta(2,5,\infty)$. Let j be a Hauptmodul, with pole at ∞. The scalar products are defined on $\mathbb{R}[j]$ by

$$\langle f, g \rangle_0 := \int_{\pi/5}^{\pi/2} f(e^{i\theta}) g(e^{i\theta})\, dth,$$

$$\langle f, g \rangle_1 := \int_{2\pi/5}^{\pi/2} f(e^{i\theta}) g(e^{i\theta})\, d\theta.$$

The first polynomials $A_{0,n}(j)$ are: $A_{0,0}(j) = 1$, $A_{0,1}(j) = j - 9/20$, etc. Taking $p = 13$, one finds that

$$\mathrm{ss}_{13}(j) \equiv A_{0,3}(j) \equiv j(j+4)(j-1) \pmod{13}.$$

Next, real multiplication splits $H^1_{\mathrm{dR}}(X_c)$ into two eigenspaces. The eigendifferentials induce two second order differnetial equations, called Picard–Fuchs differential equations. TRuncation of the holomorphic solutions can be related to the supersingular locus of $\mathcal{C}$.

Example 47. For $\mathcal{C} = \mathbb{H}/\Delta(2,5,\infty)$,

$$j^n \cdot {}_2F_1\left(\frac{7}{20}, \frac{3}{20}; 1; \frac{1}{j}\right) = U_{0,n}(j) + \mathrm{O}(j^{-1}),$$

etc. Leads to a formula for $\mathrm{ss}_p^{\mathcal{C}}(j)$.

The partial Hasse invariants h_1 and h_2 are characteristic p Hilbert modular forms of non-parallel weight. They have the following properties:

- The divisor of h_i is the component D_i of the non-ordinary locus of $X_D(\mathbb{F})$.
- Their q-expansion is constant, equal to 1, at every cusp.

In particular, they do not lift to Hilbert modular forms in characteristic zeor.

Theorem 48 (B–Li, 2024)**.** *Let $\mathcal{C} \cong \mathbb{H}/\Gamma \hookrightarrow X_D$ be a smooth algebraic curve with good reduction at p. Then the partial Hasse invariants lift to (twisted) modular forms on Γ in characteristic zero.*

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