1. Introduction

We recall some definitions and background, record proofs of some of the main theorems regarding Krull dimension, and give some of their geometric interpretations. We mainly follow the course reference by Bosch.

2. Basic definitions

Let A be a ring (always commutative and with identity). In what follows, the symbols $\mathfrak{p}$ or $\mathfrak{p}_i$ always denote prime ideals. We set

$$\dim(A) := \sup\{n \geq 0 : \exists \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n\}.$$

For a prime ideal $\mathfrak{p}$ of A, we set

$$\operatorname{height}(\mathfrak{p}) := \sup\{n \geq 0 : \exists \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subseteq \mathfrak{p}\},$$

$$\operatorname{coheight}(\mathfrak{p}) := \sup\{n \geq 0 : \exists \mathfrak{p} \subseteq \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n\}.$$

For a general ideal $\mathfrak{a}$, we set

$$\operatorname{height}(\mathfrak{a}) := \inf_{\mathfrak{p} \supseteq \mathfrak{a}} \operatorname{height}(\mathfrak{p}),$$

$$\operatorname{coheight}(\mathfrak{a}) := \sup_{\mathfrak{p} \supseteq \mathfrak{a}} \operatorname{coheight}(\mathfrak{p}) = \sup\{n \geq 0 : \exists \mathfrak{a} \subseteq \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n\}.$$

Since prime ideals in the localization $A_{\mathfrak{p}}$ correspond to the primes in A contained in $\mathfrak{p}$, we have

$$\operatorname{height}(\mathfrak{p}) = \dim(A_{\mathfrak{p}}).$$

Since prime ideals in the quotient $A/\mathfrak{a}$ correspond to the primes in A containing $\mathfrak{a}$, we have

$$\operatorname{coheight}(\mathfrak{a}) = \dim(A/\mathfrak{a}).$$

We note the following easy inequality:

Lemma 1. $\operatorname{height}(\mathfrak{a}) + \dim(A/\mathfrak{a}) \leq \dim(A)$.

Proof. It suffices to show that if $\operatorname{height}(\mathfrak{a}) \geq r$ and $\dim(A/\mathfrak{a}) \geq s$, then $\dim(A) \geq r + s$. By hypothesis, we may find primes $\mathfrak{a} \subseteq \mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_s$. Then $\operatorname{height}(\mathfrak{q}_0) \geq \operatorname{height}(\mathfrak{a}) \geq r$, so we may find primes $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{q}_0$. Then

$$\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_s$$

is a chain of primes in A of length $r + s$. □

We also note:

Lemma 2. *Let* $(A, \mathfrak{m})$ *be a local ring. Then* $\dim(A) = \operatorname{height}(\mathfrak{m})$.

Proof. Let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ be a chain of primes in A. By enlarging this chain if necessary, we may assume that $\mathfrak{p}_r = \mathfrak{m}$. Thus the suprema in the definitions of $\dim(A)$ and $\operatorname{height}(\mathfrak{m})$ may be taken over the same chains of primes. □

3. Geometric interpretations

Reference for this section: exercises in Chapter 1 of Atiyah–Macdonald.

Let A be a ring. Recall that $\operatorname{Spec}(A)$ denotes the set of prime ideals $\mathfrak{p}$ in A. Each $f \in A$ defines a function

$$f|_{\operatorname{Spec}(A)} : \operatorname{Spec}(A) \to \bigsqcup_{\mathfrak{p} \in \operatorname{Spec}(A)} A/\mathfrak{p}$$

sending $\mathfrak{p}$ to the class of f in the quotient ring $A/\mathfrak{p}$. For $f \in A$ and any subset X of $\operatorname{Spec}(A)$, we may form the restriction $f|_X$ of f to X. For the sake of illustration, note that $f|_{\operatorname{Spec}(A)} = 0$ (i.e., $f|_{\operatorname{Spec}(A)}$ maps each $\mathfrak{p}$ to the zero class in $A/\mathfrak{p}$) if and only if f belongs to the nilradical of A.

For example, we have seen (using the Nullstellensatz) that if $A = \mathbb{C}[X_1, \ldots, X_n]/I$ for some ideal $I \subseteq \mathbb{C}[X_1, \ldots, X_n]$, then the set $\operatorname{Specm}(A)$ of maximal ideals in A is in natural bijection with $V := \{(x_1, \ldots, x_n) \in \mathbb{C}^n : f(x_1, \ldots, x_n) = 0 \text{ for all } f \in I\}$. For each such maximal ideal $\mathfrak{m}$ we may identify $A/\mathfrak{m}$ with $\mathbb{C}$. For $f \in A$, the function $f|_{\operatorname{Specm}(A)}$ then identifies with the obvious map $V \ni (x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n) \in \mathbb{C}$.

For a subset S of A, we set

$$V(S) := \{\mathfrak{p} \in \operatorname{Spec}(A) : \mathfrak{p} \supseteq S\} = \{\mathfrak{p} \in \operatorname{Spec}(A) : f(\mathfrak{p}) = 0 \text{ for each } f \in S\}.$$

For finite sets $S = \{f_1, \ldots, f_n\}$ we write simply $V(f_1, \ldots, f_n) := V(S)$. Note that if S generates an ideal $\mathfrak{a}$, then $V(S) = V(\mathfrak{a})$. Given any subset X of $\operatorname{Spec}(A)$, we set

$$I(X) := \cap_{\mathfrak{p} \in X} \mathfrak{p} = \{f \in A : f|_X = 0\}.$$

Recall that a subset of $\operatorname{Spec}(A)$ is called *closed* if it is of the form $V(S)$ for some S; this defines a topology on $\operatorname{Spec}(A)$. Recall that an ideal $\mathfrak{a}$ is *radical* if $\operatorname{rad}(\mathfrak{a}) = \mathfrak{a}$.

Lemma 3.

- *(i) For each ideal $\mathfrak{a}$ of A, we have $I(V(\mathfrak{a})) = \operatorname{rad}(\mathfrak{a})$.*
- *(ii) For each subset X of $\operatorname{Spec}(A)$, we have $V(I(X)) = \overline{X}$ (the closure of X).*
- *(iii) The maps V and I define mutually-inverse inclusion-reversing bijections between the set of radical ideals of A and the set of closed subsets of $\operatorname{Spec}(A)$.*

Proof. The maps I and V are readily seen to be inclusion-reversing (cf. Exercise Sheet #1).

- (i) By definition, $I(V(\mathfrak{a})) = \cap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p} = \cap_{\mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p} = \operatorname{rad}(\mathfrak{a})$.
- (ii) The set $V(I(X))$ is closed and contains X, so it will suffice to verify for each closed set $V(\mathfrak{a})$ containing X that $V(\mathfrak{a}) \supseteq V(I(X))$. From $V(\mathfrak{a}) \supseteq X$ we see that $f|_X = 0$ for all $f \in \mathfrak{a}$, thus $\mathfrak{a} \subseteq I(X)$. Applying the inclusion-reversing map V, we obtain $V(\mathfrak{a}) \supseteq V(I(X))$, as required.
- (iii) Immediate by the above.

$\square$

Lemma 4. *Let X be a closed subset of $\operatorname{Spec}(A)$. The following are equivalent:*

- *(i) $X = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ of A.*
- *(ii) $I(X)$ is a prime ideal of A.*
- *(iii) X is nonempty and may not be written as $X = X_1 \cup X_2$ for closed subsets X_1, X_2 of $\operatorname{Spec}(A)$ except in the trivial case that either $X \subseteq X_1$ or $X \subseteq X_2$.*

We say that a closed subset X of $\mathrm{Spec}(A)$ is *irreducible* if it satisfies the equivalent conditions of the preceeding lemma. The irreducible closed subsets of $\mathrm{Spec}(A)$ correspond bijectively to the prime ideals of A.

We note that for any ideal $\mathfrak{a}$, we may identify

$$V(\mathfrak{a}) = \mathrm{Spec}(A/\mathfrak{a}).$$

We note also that if $\mathfrak{p}$ is a prime of A, then the primes of the localization $A_{\mathfrak{p}}$ correspond to the primes of A contained in $\mathfrak{p}$, hence the spectrum of $A_{\mathfrak{p}}$ identifies with the set of closed irreducible subsets of $\mathrm{Spec}(A)$ that *contain* $\mathfrak{p}$:

$$\mathrm{Spec}(A_{\mathfrak{p}}) = \{\mathfrak{q} \in \mathrm{Spec}(A) : \mathfrak{q} \subseteq \mathfrak{p}\} = \{\mathfrak{q} \in \mathrm{Spec}(A) : \mathfrak{p} \in V(\mathfrak{q})\}.$$

By an *irreducible component* of a closed subset X of $\mathrm{Spec}(A)$, we shall mean a maximal closed irreducible subset of X, i.e., a closed irreducible subset $Z \subseteq X$ with the property that if $Z' \subseteq X$ is any closed irreducible subset with $Z' \supseteq Z$, then $Z' = Z$. Using the inclusion-reversing bijections noted above, we verify readily that for any ideal $\mathfrak{a}$, the irreducible components of $X = V(\mathfrak{a})$ correspond bijectively to the set (denoted $\mathrm{Ass}'(\mathfrak{a})$ in lecture) of minimal prime ideals $\mathfrak{p} \supseteq \mathfrak{a}$.

We assume henceforth that A is Noetherian. Then the set of minimal primes of any ideal is finite, and any prime containing an ideal contains a minimal prime of that ideal. It follows that the set of irreducible components of any closed subset X of $\mathrm{Spec}(A)$ is a finite set $\{Z_1, \dots, Z_n\}$ for which $X = Z_1 \cup \dots \cup Z_n$.

We define the *dimension* of a closed subset X of $\mathrm{Spec}(A)$ to be

$$\dim(X) = \sup\{n \geq 0 : \exists \text{ closed irreducible subsets } Z_n \subsetneq \dots \subsetneq Z_0 \subseteq X\}$$

and the *codimension* in the special case that Z is closed irreducible to be

$$\mathrm{codim}(Z) := \sup\{n \geq 0 : \exists \text{ closed irreducible subsets } Z_0 \supsetneq \dots \supsetneq Z_n \supset Z\}$$

and then in general by

$$\mathrm{codim}(X) := \inf_{Z \subseteq X:\text{closed irreducible}} \mathrm{codim}(Z).$$

Equivalently, $\mathrm{codim}(X)$ is the smallest codimension of any irreducible component of X. We note also that $\dim(X)$ coincides with the largest dimension of any irreducible component of X. We might write $\mathrm{codim}(X)$ as $\mathrm{codim}_{\mathrm{Spec}(A)}(X)$ when we wish to emphasize the reference space $\mathrm{Spec}(A)$.

Using the inclusion-reversing bijections noted above, we see that

$$\dim \mathrm{Spec}\, A = \dim A$$

and more generally that

$$\dim V(\mathfrak{a}) = \mathrm{coheight}\, \mathfrak{a} = \dim A/\mathfrak{a}, \quad \dim X = \mathrm{coheight}\, I(X) = \dim A/I(X),$$

$$\mathrm{codim}\, V(\mathfrak{a}) = \mathrm{height}\, \mathfrak{a}, \quad \mathrm{codim}\, X = \mathrm{height}\, I(X)$$

for any ideal $\mathfrak{a}$ and any closed $X \subseteq \mathrm{Spec}(A)$. Lemma 1 says that $\dim X + \mathrm{codim}\, X \leq \dim \mathrm{Spec}\, A$.

4. Prime avoidance lemma

Lemma 5. *Let A be a ring, let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be prime ideals, and let $\mathfrak{a}$ be an ideal contained in the union $\cup \mathfrak{p}_j$. Then there exists an index j for which $\mathfrak{a} \subseteq \mathfrak{p}_j$. Equivalently, if $\mathfrak{a} \not\subseteq \mathfrak{p}_j$ for each j, then $\mathfrak{a} \not\subseteq \cup \mathfrak{p}_j$.*

In "geometric" terms, let $Z_1, \dots, Z_n \subseteq \operatorname{Spec}(A)$ be closed irreducible subsets, and let $X = V(\mathfrak{a})$ be a closed irreducible subset of $\operatorname{Spec}(A)$, defined by an ideal $\mathfrak{a}$, with the property that $X \not\supseteq Z_j$ for all j. Then there exists $f \in \mathfrak{a}$ with $f|_{Z_j} \neq 0$ for all j. In particular, we may find $f \in A$ with $f|_X = 0$ but $f|_{Z_j} \neq 0$ for all j.

Proof. We verify that if $\mathfrak{a}$ is not contained in any of the $\mathfrak{p}_j$, then it is not contained in their union. For this we may induct on n. The case $n = 1$ is trivial, so suppose $n > 2$. By our inductive hypothesis, we may find for each $i = 1..n$ an element $a_i \in \mathfrak{a}$ with $a_i \notin \mathfrak{p}_j$ whenever $j \neq i$. If moreover $a_i \notin \mathfrak{p}_i$ for some i, then we are done, so suppose otherwise that $a_i \in \mathfrak{p}_i$ for all i. Set $b_i := \prod_{j:j\neq i} a_j$. Then $b_i \notin \mathfrak{p}_i$ (using that $\mathfrak{p}_i$ is prime) but $b_i \in \mathfrak{p}_j$ for all $j \neq i$. It follows that $x := b_1 + \cdots + b_n$ belongs to $\mathfrak{a}$ but not to $\mathfrak{p}_i$ for any i, hence $\mathfrak{a}$ is not contained in the union of the $\mathfrak{p}_i$. □

5. Artin rings

Theorem 6. *Let A be a ring. The following are equivalent:*

(i) A is an Artin ring.
(ii) A is a Noetherian ring of dimension zero.

6. Krull intersection theorem

Theorem 7. *Let $\mathfrak{a}$ be an ideal contained in the Jacobson radical $\operatorname{Jac}(A)$ of a Noetherian ring A. Then*

$$\cap_{n\geq 0} \mathfrak{a}^n = 0.$$

Corollary 8. *With $A, \mathfrak{a}$ as before, let M be a finitely-generated module. Then $\cap_{n\geq 0} \mathfrak{a}^n M = 0$.*

Corollary 9. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Then $\cap_{n\geq 0} \mathfrak{m}^n = 0$.*

For the proof of Theorem 7, the fact that $\mathfrak{a}$ is contained in the Jacobson radical suggests an application of Nakayama's lemma to the ideal $M' := \cap_{n\geq 0} \mathfrak{a}^n$, for which it is clear that $\mathfrak{a}M' \subseteq M'$ and plausible but non-obvious that $\mathfrak{a}M' = M'$. The key tool in establishing the latter is the following:

Lemma 10 (Artin–Rees lemma). *Let A be Noetherian, let $\mathfrak{a}$ be an ideal, let M be a finitely-generated module, and let $M' \leq M$ be a submodule. There exists $n \geq 0$ so that for all $k \geq 0$,*

$$\mathfrak{a}^k(\mathfrak{a}^n M \cap M') = \mathfrak{a}^{n+k} M \cap M'.$$

Taking $M := A, M' := \cap_{n\geq 0} \mathfrak{a}^n$, $k := 1$ in the Artin–Rees lemma gives $\mathfrak{a}^n M \cap M' = \mathfrak{a}^{n+k} M \cap M' = M'$ and hence $\mathfrak{a}M' = M'$; we then conclude the proof of Theorem 7 by Nakayama, as indicated above.

The proof of Artin–Rees reduces formally to the case $k = 1$, and the containment

$$\mathfrak{a}(\mathfrak{a}^n M \cap M') \subseteq \mathfrak{a}^{n+1} M \cap M'$$

is clear. The proof of the trickier reverse containment is expressed most transparently using the graded ring

$$\tilde{A} := \boxplus_{i\geq 0} A_i = \{a = (a_i)_{i\geq 0} : a_i \in A_i\}, \quad A_i := \mathfrak{a}^i,$$

where the multiplication law extends the bilinear maps $\mathfrak{a}^i \times \mathfrak{a}^j \to \mathfrak{a}^{i+j}$:

$$(a \cdot b)_k = \sum_{i+j=k} a_i b_j.$$

This graded ring acts by the rule $(a \cdot m)_k := \sum_{i+j=k} a_i m_j$ on the graded module

$$\tilde{M} := \boxplus_{i \geq 0} M_i, \quad M_i := \mathfrak{a}^i M,$$

and its graded submodule

$$\tilde{M'} := \boxplus_{i \geq 0} M'_i, \quad M'_i := \mathfrak{a}^i M \cap M'.$$

Since $\mathfrak{a}$ is finitely-generated as a module over A, $\tilde{A}$ is finitely-generated as an algebra over $A_0 = A$; by the Hilbert basis theorem, it follows that $\tilde{A}$ is Noetherian. The module M is finitely-generated over A, from which it follows readily that the graded module $\tilde{M}$ is finitely-generated over $\tilde{A}$; since the ring $\tilde{A}$ is Noetherian, so is the module $\tilde{M}$, hence its submodule $\tilde{M'}$ is finitely-generated. Choose n large enough that the module $\tilde{M'}$ is generated by $\boxplus_{0 \leq i \leq n} M'_i$, thus

$$\tilde{M'} = \tilde{A} \boxplus_{0 \leq i \leq n} M'_i.$$

By taking the degree $n+1$ homogeneous component of this identity, we see that

$$\begin{aligned} \mathfrak{a}^{n+1} M \cap M' = \tilde{M'_{n+1}} &= \sum_{0 \leq i \leq n} A_{n+1-i} M'_i = \sum_{0 \leq i \leq n} \mathfrak{a}^{n+1-i}(\mathfrak{a}^i M \cap M') \\ &\subseteq \sum_{0 \leq i \leq n} \mathfrak{a}(\mathfrak{a}^n M \cap \mathfrak{a}^{n-i} M') \subseteq \mathfrak{a}(\mathfrak{a}^n M \cap M'), \end{aligned}$$

giving the required reverse containment. The proof of Artin–Rees and hence of the Krull intersection theorem is then complete.

7. Kernel of localization with respect to a prime

Let $\mathfrak{p}$ be a prime ideal in a Noetherian ring A. Let $\mathfrak{p}^{(n)}$ denote the nth symbolic power; it is the $\mathfrak{p}$-primary ideal given by $A \cap \mathfrak{p}^n A_{\mathfrak{p}} := \iota^*((\iota_* \mathfrak{p})^n)$, where $\iota : A \to A_{\mathfrak{p}}$ denotes the localization map.

Theorem 11. $\ker(\iota) = \cap_{n \geq 0} \mathfrak{p}^{(n)}$.

Proof. Set $\mathfrak{m} := \iota_* \mathfrak{p}$. We have $\ker(\iota) = \iota^{(-1)}(0)$ and $\iota^{-1}(\cap_{n \geq 0} \mathfrak{m}^n) = \cap_{n \geq 0} \mathfrak{p}^{(n)}$, so it suffices to show that $\cap_{n \geq 0} \mathfrak{m}^n = 0$, which is the content of Corollary 9 of the Krull intersection theorem applied to the Noetherian local ring $(A_{\mathfrak{p}}, \mathfrak{m})$. $\square$

8. Krull's theorems on heights and dimensions

8.1. **Principal ideal theorem.** We start with the special case to which the general one will eventually be reduced:

Lemma 12. *Let* $(A, \mathfrak{m})$ *be a local Noetherian integral domain. Suppose that* $\mathfrak{m}$ *is a minimal prime of some principal ideal* (f), *with* $f \in \mathfrak{m}$. *Then* $\mathfrak{m}$ *and* (0) *are the only primes of* A.

In "geometric" terms: suppose that $\{\mathfrak{m}\} = V(f)$ *for some* $f \in \mathfrak{m}$. *Then* $\operatorname{Spec}(A) = \{\mathfrak{m}, (0)\}$.

Proof. Let $\mathfrak{p}$ be any prime in A other than $\mathfrak{m}$. Necessarily $\mathfrak{p} \subsetneq \mathfrak{m}$; our task is to show that $\mathfrak{p} = (0)$. Since A is a domain, it will suffice to show for some n that $\mathfrak{p}^n = (0)$. Recall that $\mathfrak{p}^{(n)}$ denotes the nth symbolic power of $\mathfrak{p}$, given here with respect to the injective localization map $A \hookrightarrow A_{\mathfrak{p}}$ by $\mathfrak{p}^{(n)} = A \cap \mathfrak{p}^n A_{\mathfrak{p}}$; it is a $\mathfrak{p}$-primary ideal which contains $\mathfrak{p}^n$. It will then suffice to verify that $\mathfrak{p}^{(n)} = (0)$ for some n. By §7, we have $\cap_{n \geq 0} \mathfrak{p}^{(n)} = \ker(A \to A_{\mathfrak{p}}) = (0)$, so it will suffice to verify that the chain of ideals $\mathfrak{p}^{(n)}$ stabilizes, i.e., that $\mathfrak{p}^{(n)} = \mathfrak{p}^{(n+1)}$ for large n.

Set $\overline{A} := A/(f)$, $\overline{\mathfrak{m}} := \mathfrak{m}/(f)$. Our hypotheses imply that $\overline{\mathfrak{m}}$ is the only prime ideal of $\overline{A}$. Thus $\overline{A}$ is a Noetherian ring of dimension 0. By Theorem 6, it follows that $\overline{A}$ is an Artin ring. Thus the descending chain of ideals $\mathfrak{p}^{(n)} + (f)$ must stabilize; in particular,

$$\mathfrak{p}^{(n)} \subseteq \mathfrak{p}^{(n+1)} + (f)$$

for large n. This says that any $x \in \mathfrak{p}^{(n)}$ may be written $x = y + zf$ for some $y \in \mathfrak{p}^{(n+1)}$ and $z \in A$. In that case, $x - y \in \mathfrak{p}^{(n)}$, and so $z \in (\mathfrak{p}^{(n)} : f)$. Since $\mathfrak{p}^{(n)}$ is $\mathfrak{p}$-primary and $f \notin \mathfrak{p}$, we have $(\mathfrak{p}^{(n)} : f) = \mathfrak{p}^{(n)}$, and so in fact $z \in \mathfrak{p}^{(n)}$. Thus

$$\mathfrak{p}^{(n)} \subseteq \mathfrak{p}^{(n+1)} + \mathfrak{p}^{(n)} f,$$

and in fact equality holds, with the reverse containment being clear. This says that $fM = M$ for the finitely-generated module $M := \mathfrak{p}^{(n)}/\mathfrak{p}^{(n+1)}$. Since $f \in \mathfrak{m} = \mathrm{Jac}(A)$, it follows from Nakayama's lemma that $M = 0$. Thus $\mathfrak{p}^{(n)} = \mathfrak{p}^{(n+1)}$ for large n, as was to be shown. □

Theorem 13. *Let A be a Noetherian ring, and let $f \in A$.*

(i) Every minimal prime $\mathfrak{p}$ of (f) satisfies $\mathrm{height}(\mathfrak{p}) \leq 1$.

(ii) If f is a non-zerodivisor, then every minimal prime $\mathfrak{p}$ of (f) satisfies $\mathrm{height}(\mathfrak{p}) = 1$.

In "geometric" terms, $\mathrm{codim}(Z) \leq 1$ *for each irreducible component Z of $V(f) \subseteq$* $\mathrm{Spec}(A)$*; if f is a non-zerodivisor, then* $\mathrm{codim}(Z) = 1$ *for each such Z. (This "generalizes" the fact from linear algebra that the kernel of a linear functional has codimension ≤ 1, with equality whenever the functional is nonzero.)*

Proof. To deduce (ii) from (i), suppose that some minimal prime $\mathfrak{p}$ of (f) has $\mathrm{height}(\mathfrak{p}) = 0$. Then $\mathfrak{p}$ is a minimal prime of (0), hence consists of zero-divisors, and so f is a zerodivisor.

Our main task is thus to establish (i). We must verify that if $\mathfrak{p}_2$ is a minimal prime of (f) and if $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2$ are inclusions of prime ideals, then $\mathfrak{p}_0 = \mathfrak{p}_1$. After replacing A by its quotient $A/\mathfrak{p}_0$, we may reduce to the case that $\mathfrak{p}_0 = (0)$; in particular, A is a local Noetherian domain. After then replacing A by its localization $A_{\mathfrak{p}_2}$, we reduce further to the case that A is a local Noetherian domain whose maximal ideal $\mathfrak{p}_2$ is a minimal prime of (f). We now appeal to the previous lemma. □

We will often apply the above result in a local context:

Corollary 14. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Suppose there exists $f \in A$ for which $\mathfrak{m}$ is the unique prime containing f, thus $V(f) = \{\mathfrak{m}\}$. Then* $\dim(A) = \mathrm{height}(\mathfrak{m}) \leq 1$.

Proof. Given that $\mathfrak{m}$ is maximal, our assumption is equivalent to requiring that $\mathfrak{m}$ be a minimal prime of (f). □

For the sake of illustration, let's reformulate Theorem 13 in the contrapositive. Let A be a Noetherian ring. Let $\mathfrak{p}_0 \subsetneq \mathfrak{p}_2$ be an inclusion of primes in A. By an *intermediary prime* we will mean a prime $\mathfrak{p}_1$ for which $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2$.

Corollary 15. *The following are equivalent:*

(i) There exists an intermediary prime.
(ii) For each $f \in \mathfrak{p}_2$ there exists an intermediary prime containing f.

In "geometric" terms, let $Y_2 \subsetneq Y_0$ be irreducible closed subsets of $\operatorname{Spec}(A)$*. Then either there are no irreducible closed subsets Y_1 contained strictly between Y_2 and Y_0, or for each $f \in I(Y_2)$ there exists an irreducible closed subset $Y_2 \subsetneq Y_1 \subsetneq Y_0$ with $Y_1 \subseteq Z(f)$.*

Proof. We need only show that (i) implies (ii). If (ii) fails, then we may find $f \in \mathfrak{p}_2$ not contained in any intermediary primes. In other words, after replacing A with $A/\mathfrak{p}_0$ as necessary to reduce to the case that $\mathfrak{p}_0$ is a minimal prime of A, we are given that $\mathfrak{p}_2$ is a minimal prime of (f). By Krull's principal ideal theorem, it follows that $\operatorname{height}(\mathfrak{p}_2) \leq 1$; thus there exist no intermediary primes, and so (i) fails. □

8.2. **Dimension theorem.**

Theorem 16. *Let A be a Noetherian ring, and let $f_1, \dots, f_n \in A$. Then each minimal prime $\mathfrak{p}$ of $(f_1, \dots, f_r)$ satisfies* $\operatorname{height}(\mathfrak{p}) \leq r$*. In particular,* $\operatorname{height}(f_1, \dots, f_r) \leq r$.

In "geometric" terms, $\operatorname{codim}(Z) \leq r$ *for each irreducible component Z of $V(f_1, \dots, f_r) \subseteq$* $\operatorname{Spec}(A)$*. (This "generalizes" the fact from linear algebra that the solution set to a system of r linear equations has codimension $\leq r$.)*

Here's a lemma that I think clarifies the key step in the proof.

Lemma 17. *Let $(A, \mathfrak{m})$ be a Noetherian local ring, and let $f_1, \dots, f_r \in \mathfrak{m}$ with $V(f_1, \dots, f_r) = \{\mathfrak{m}\}$. Let $\mathfrak{p} \subsetneq \mathfrak{m}$ be a prime with no prime strictly contained between $\mathfrak{p}$ and $\mathfrak{m}$. Then there exist $g_1, \dots, g_r \in \mathfrak{m}$ for which*

(1) $V(g_1, \dots, g_r) = \{\mathfrak{m}\}$ and
(2) $\mathfrak{p}$ contains and is a minimal prime of $(g_1, \dots, g_{r-1})$.

In "geometric" terms, let Z be a closed irreducible subset of $\operatorname{Spec}(A)$ *that is minimal among the closed irreducible sets that properly contain $\{\mathfrak{m}\}$. Then we may find $g_1, \dots, g_r$ for which $V(g_1, \dots, g_r) = \{\mathfrak{m}\}$ and for which Z is an irreducible component of $V(g_1, \dots, g_{r-1})$.*

Proof. Since $\mathfrak{m}$ is the unique prime ideal containing $(f_1, \dots, f_r)$, we may assume after reindexing $f_1, \dots, f_r$ as necessary that $f_r \notin \mathfrak{p}$. Then the ideal $\mathfrak{p} + (f_r)$ strictly contains $\mathfrak{p}$ and is contained in $\mathfrak{m}$; our hypotheses on $\mathfrak{p}$ imply that $\mathfrak{m}$ is the only prime ideal containing $\mathfrak{p} + (f_r)$, i.e., that $V(\mathfrak{p} + (f_r)) = \{\mathfrak{m}\}$, or that $\operatorname{rad}(\mathfrak{p} + (f_r)) = \mathfrak{m}$. In particular, for each $1 \leq i \leq r - 1$ we may find n_i for which $f_i^{n_i} \in \mathfrak{p} + (f_r)$, say

$$f_i^{n_i} = g_i + z_i f_r \text{ with } g_i \in \mathfrak{p}, z_i \in A.$$

We claim that the conclusion of the lemma is now satisfied with $g_1, \dots, g_{r-1}$ as above and $g_r := f_r$:

(1) The above equation shows that any prime $\mathfrak{q}$ that contains $g_1, \dots, g_{r-1}, f_r$ also contains $f_i^{n_i}$ and hence f_i for $1 \leq i \leq r$, hence $\mathfrak{q} = \mathfrak{m}$. Thus $V(g_1, \dots, g_r) = \{\mathfrak{m}\}$.

(2) It's clear by construction that $\mathfrak{p}$ contains $(g_1, \ldots, g_{r-1})$. There is thus a minimal prime $\mathfrak{p}'$ of $(g_1, \ldots, g_{r-1})$ contained in $\mathfrak{p}$; we must verify that $\mathfrak{p} = \mathfrak{p}'$. (Geometrically, $\mathfrak{p}'$ corresponds to an irreducible component Z' of $V(g_1, \ldots, g_{r-1})$ containing Z.) To see this, consider the quotient ring $\overline{A} := A/(g_1, \ldots, g_{r-1})$. Let

$$\overline{\mathfrak{m}} \supsetneq \overline{\mathfrak{p}} \supseteq \overline{\mathfrak{p}'} \tag{1}$$

denote the chain of primes in $\overline{A}$ given by the image of $\mathfrak{m} \supsetneq \mathfrak{p} \supseteq \mathfrak{p}' \supseteq (g_1, \ldots, g_{r-1})$. Then $(\overline{A}, \overline{\mathfrak{m}})$ is a Noetherian local ring, and our task is equivalent to showing that $\overline{\mathfrak{p}} = \overline{\mathfrak{p}'}$. Let $f \in \overline{A}$ denote the image of f_r. The primes of $\overline{A}$ containing f are in bijection with the primes of A containing $g_1, \ldots, g_{r-1}, f_r$, so $V_{\overline{A}}(f) = \{\overline{\mathfrak{m}}\}$. By Krull's principal ideal theorem (in the form of Corollary 14), it follows that $\operatorname{height}(\overline{\mathfrak{m}}) \leq 1$. From (1) we then deduce that $\overline{\mathfrak{p}} = \overline{\mathfrak{p}'}$, as required. (Intuitively, by choosing f_r not to vanish on any irreducible component of $V(f_1, \ldots, f_{r-1})$, we guarantee that appending it to the set of generators has the effect of knocking down the dimension of each such component by 1.)

□

We now deduce Theorem 16. We must show that if $\mathfrak{p}$ is a minimal prime of $(f_1, \ldots, f_r)$, then $\operatorname{height}(\mathfrak{p}) \leq r$. We may assume without loss of generality (replacing A with $A_{\mathfrak{p}}$ and $\mathfrak{p}$ with $\mathfrak{p}_{\mathfrak{p}}$, which doesn't change the height of or minimality assumption on the latter) that $(A, \mathfrak{p})$ is a Noetherian local ring with $V(f_1, \ldots, f_r) = \{\mathfrak{p}\}$; we must show then that $\operatorname{height}(\mathfrak{p}) \leq r$. We do this by induction on r. The case $r = 1$ is given by Krull's principal ideal theorem, so suppose $r > 1$. Let $\mathfrak{q} \subsetneq \mathfrak{p}$ be a maximal element of the set of primes strictly contained in $\mathfrak{p}$; it will suffice then to show that $\operatorname{height}(\mathfrak{q}) \leq r - 1$. By Lemma 17, we may assume without loss of generality that $\mathfrak{q}$ is a minimal prime of $(f_1, \ldots, f_{r-1})$; the required inequality then follows from our inductive hypothesis.

References