SOME EXERCISES CONCERNING LOCALIZED VECTORS IN LOW RANK

Contents

Abstract. We record some exercises whose purpose is to verify that certain classes of vectors in representations of SO(3) and $\mathrm{PGL}_2(\mathbb{R})$ are "localized" in a strong sense under the action of those groups.

1. Overview

In this note, we give exercises that aim to convey some computational feeling for "localized vectors" in the precise sense defined in §3 of this note, focusing on low-rank examples. Along the way, we recall the basic representation theory for such examples.

2. Setup

We let $T \to \infty$ be an asymptotic parameter, and retain the asymptotic notation and conventions of §2.1 of this note concerning "T-dependent elements", "fixed" (equivalently, "T-independent") and "classes". In particular, we recall that "class" means "collection of T-dependent sets". A typical example is the class $\mathrm{O}(1)$ inside $\mathbb{C}$, consisting of all T-dependent subsets $S = S_T \subseteq \mathbb{C}$ for which there is a fixed $C \geq 0$ so that for all T, we have $\|c_T\| \leq C$ for all $c_T \in S_T$.

Let G be a fixed real Lie group, and let $\pi = \pi_T$ be a T-dependent unitary representation. We recall Theorem 3.6 from this note:

Theorem 1. *Let $\tau \in \mathfrak{g}^\wedge$ with $\tau = \mathrm{O}(1)$. Let M be a class of T-dependent vectors $v = v_T$ in $\pi = \pi_T$ with the following properties:*

(i) For each $v \in M$, we have $\|v\| = \mathrm{O}(1)$.
(ii) For all $u, v \in M$, we have $u + v \in M$.
(iii) For all $v \in M$ and $c \in \mathbb{C}$ with $c = \mathrm{O}(1)$, we have $cv \in M$.
(iv) For each fixed $\varepsilon > 0$, fixed $x \in \mathfrak{g}$ and each $v \in M$, we have

$$xv - \langle x, \tau \rangle v \in T^{1/2+\varepsilon} M. \tag{2.1}$$

That is to say, the T-dependent vector on the left hand side may be written $T^{1/2+\varepsilon}u$, where u belongs to the class M.

Then each $v \in M$ is localized at τ in the sense of Definition 3.3 of this note.

The purpose of the present note is to give some examples of classes M satisfying the above conditions, hence, in particular, examples of localized vectors. In each case, the first three properties will clearly hold, so we do not mention them; the main point is to verify the approximate eigenvector property (2.1) for elements x of a fixed basis of $\mathfrak{g}$.

Exercise 1. Let τ be a T-dependent element of $\mathfrak{g}^\wedge$ with $\tau = \mathrm{O}(T)$. For $x \in \mathfrak{g}$, set

$$\bar{x} := x - \tau(x).$$

Consider the following condition (*) on a T-dependent vector $v \in \pi$:

- for all fixed $\varepsilon > 0$, fixed $k \in \mathbb{Z}_{\geq 0}$, and fixed $x_1, \dots, x_k \in \mathfrak{g}$, we have

$$\|\bar{x}_1 \cdots \bar{x}_k v\| \ll T^{k/2+\varepsilon}.$$

(i) Let M be a class satisfying the hypotheses of Theorem 1. Show that each $v \in M$ satisfies (*).
(ii) Let v be a T-dependent vector in $\pi = \pi_T$ satisfying (*). Fix a basis $\mathcal{B}$ of $\mathfrak{g}$, and let M denote the class of T-dependent vectors in π of the form

$$\sum_{x_1, \dots, x_k \in \mathcal{B}} c_{x_1, \dots, x_k} \bar{x}_1 \cdots \bar{x}_k v,$$

where $k = \mathrm{O}(1)$ and each $c_{x_1, \dots, x_k} = \mathrm{O}(1)$. Verify that M satisfies the hypotheses of Theorem 1.

3. The group SO(3) via weight vectors

3.1. **Lie algebra.** We consider the Lie group SO(3). Its Lie algebra $\mathfrak{so}(3)$ admits a basis $\{R_1, R_2, R_3\}$, where for any angle θ, the element $\exp(\theta R_j)$ defines rotation by θ about the jth axis. These satisfy the commutation relations

$$[R_1, R_2] = R_3, \quad [R_2, R_3] = R_1, \quad [R_3, R_1] = R_2.$$

The center of the universal enveloping algebra is generated by the Casimir element

$$\Omega = -(R_1^2 + R_2^2 + R_3^2).$$

Define the following elements X, Y of the complexified Lie algebra $\mathfrak{so}(3)_{\mathbb{C}}$:

$$X := R_1 + iR_2, \quad Y := -R_1 + iR_2.$$

Then X, Y, R_3 is a basis for $\mathfrak{so}(3)_{\mathbb{C}}$ satisfying the commutation relations

$$[X, Y] = 2iR_3, \quad [iR_3, X] = X, \quad [iR_3, Y] = -Y. \tag{3.1}$$

We observe also that

$$R_1 = \frac{X - Y}{2}, \quad R_2 = \frac{X + Y}{2i},$$
$$\Omega = \frac{XY + YX}{2} - R_3^2.$$

By writing $XY = [X, Y] + YX$ and appealing to the formula (3.1) for $[X, Y]$, we see that

$$\Omega = YX + iR_3(iR_3 + 1). \tag{3.2}$$

Similarly,

$$\Omega = XY + iR_3(iR_3 - 1). \tag{3.3}$$

The imaginary dual of the Lie algebra identifies with the space of triples of imaginary numbers:

$$\mathfrak{so}(3)^\wedge \cong i\mathbb{R}^3. \tag{3.4}$$

Here $\xi \in i\mathbb{R}^3$ corresponds to the linear map $\mathfrak{so}(3) \to i\mathbb{R}$ given on basis elements by $R_j \mapsto \xi_j$.

3.2. **Representations.** Let π be a (complex) representation of SO(3). It may be decomposed into eigenspaces for R_3. Since $\exp(2\pi R_3) = 1$, the eigenvalues of iR_3 are integers:

$$\pi = \oplus_{m \in \mathbb{Z}} \pi(m), \quad \pi(m) := \{v \in \pi : iR_3 v = mv\}.$$

The m for which $\pi(m) \neq 0$ are called the *weights* of π, and the dimensions $\dim \pi(m)$ the corresponding *weight multiplicities*. From the commutation relations (3.1), we see that

$$X : \pi(m) \to \pi(m+1), \quad Y : \pi(m) \to \pi(m-1). \tag{3.5}$$

Proposition 2. *Let π be an irreducible unitary representation of* SO(3). *Then Ω acts on π by a scalar of the form*

$$\Omega_\pi = \ell(\ell + 1) \tag{3.6}$$

for some nonnegative integer ℓ. This scalar determines the isomorphism class of π. In fact, there is a basis

$$e_{-\ell}, \qquad e_{-\ell+1}, \qquad \dots, \qquad e_\ell$$

for π on which the Lie algebra acts by the formulas

$$Xe_m = (\Omega_\pi - m(m+1))^{1/2} e_{m+1} \tag{3.7}$$
$$Ye_{m+1} = (\Omega_\pi - m(m+1))^{1/2} e_m, \tag{3.8}$$
$$iR_3 e_m = me_m. \tag{3.9}$$

If π is unitary, then this basis is orthonormal.

Exercise 2. Attempt to work out the proof of Proposition 2, or as many parts of it as you can, on your own, without reading what's written here in detail.

Proof. Since SO(3) is compact, we know by the Peter–Weyl theorem that π is finite-dimensional. There is thus a largest element $\ell \in \mathbb{Z}_{\geq 0}$ with $\pi(\ell) \neq 0$. For any $v \in \pi(\ell)$, we have $Xv \in \pi(\ell + 1) = \{0\}$, hence $Xv = 0$. By (3.2), it follows that

$$0 = YXv = \Omega v - \ell(\ell + 1)v. \tag{3.10}$$

Since π is irreducible and Ω commutes with the action of SO(3), we know by Schur's lemma that Ω acts on π by a scalar. Taking v to be a nonzero element of $\pi(\ell)$, we see from (3.10) that this scalar must be given by (3.6). We now choose a nonzero vector $e_\ell \in V(\ell)$ and define e_m by reverse induction for integers m with $-\ell \leq m < \ell$ by requiring that (3.8) hold, noting that the square root is positive in the stated range. Using (3.3), we see that $Ye_{-\ell} = 0$. By the mapping property (3.5), we have $e_m \in \pi(m)$, hence (3.9) holds. By the formula (3.3) for Ω, we have

$$XYe_{m+1} = \Omega e_{m+1} - m(m+1)e_{m+1} = (\Omega - m(m+1))e_{m+1}, \tag{3.11}$$

and so (3.7) holds. From the formulas established thus far, we see that the e_m ($-\ell \leq m \leq \ell$) span an invariant subspace of π, which by the irreducibility hypothesis must be π itself.

We have established all assertions except that the basis may be taken orthonormal when π is unitary. We may assume that e_ℓ was normalized to be a unit vector, and will verify then by reverse inductive on $m < \ell$ that e_m is then likewise a unit vector. To that end, observe first by (3.8) that

$$(\Omega_\pi - m(m+1))\|e_m\|^2 = \langle Ye_{m+1}, Ye_{m+1}\rangle,$$

then use that the adjoint of Y is $-\bar{Y} = X$ to see that

$$\langle Ye_{m+1}, Ye_{m+1}\rangle = \langle XYe_{m+1}, e_{m+1}\rangle.$$

By (3.11), we deduce that e_m and e_{m+1} have the same norm, so the induction follows as claimed. □

The integer ℓ as in the conclusion of Proposition 2 is called the *highest weight* of π. The coadjoint orbit for π turns out to be given in the optic (3.4) by the sphere of radius $\ell + 1/2$:

$$\mathcal{O}_\pi = \left\{(a,b,c) : a^2 + b^2 + c^2 = (T + \tfrac{1}{2})^2\right\}.$$

3.3. **Localized vectors.** In the following exercises, we assume that the asymptotic parameter $T \to \infty$ is valued in the nonnegative integers, and let π denote the T-dependent representation having highest weight T.

Remark 3. The purpose of the following exercises is not to overload on analysis of localized vectors using weight vector bases, but rather to illustrate the strong parallel with what happens for SO(3) as explained above in §3.3.

Exercise 3. Let M denote the class of T-dependent vectors v in π given in terms of a basis as in Proposition 2 by $v = \sum_m a_m e_m$, where the coefficients have the following properties:

(1) $a_m = 0$ unless $m = T + \mathrm{O}(1)$.
(2) Each $a_m = \mathrm{O}(1)$.

Verify that for all $v \in M$, we have

$$Xv \in T^{1/2}M,$$
$$Yv \in T^{1/2}M,$$
$$iR_3v - iTv \in T^{1/2}M.$$

Deduce from Theorem 1 that every element of M is localized at the T-dependent element $\tau \in \mathfrak{g}^\wedge$ given in the optic (3.4) by

$$\tau = (0, 0, iT).$$

Exercise 4. Let M be the class of T-dependent vectors in π of the form $\sum a_m e_m$, where the coefficients have the following properties:

(1) $a_m = 0$ unless $m = \mathrm{O}(T^{1/2})$.
(2) The function of $\theta \in \mathbb{R}/\mathbb{Z}$ defined by

$$a(\theta) := \sum_n a(n) e(n\theta), \qquad e(\theta) := e^{2\pi i \theta}$$

is an L^2-normalized bump of width $T^{-1/2}$, in the following sense: for fixed $k, \ell \in \mathbb{Z}_{\geq 0}$,

$$a^{(\ell)}(\theta) \ll T^{1/4+\ell/2} \left(1 + \frac{\|\theta\|}{T^{-1/2}}\right)^{-k}, \tag{3.12}$$

where $a^{(\ell)}$ denotes the ℓth derivative and $\|\theta\|$ the distance to the nearest integer.

We note that, by Parseval, the second condition implies that $\sum_m |a_m|^2 = \mathrm{O}(1)$.

(i) Show that if $f \in C_c^\infty(\mathbb{R})$ is fixed, then the T-dependent vector $\sum_m a_m e_m$ with coefficients

$$a_n := T^{-1/4} f\left(\frac{n}{T^{1/2}}\right) \tag{3.13}$$

belongs to M.

(ii) Show that for all $v \in M$,

$$Xv - Tv \in T^{1/2} M,$$
$$Yv - Tv \in T^{1/2} M,$$
$$R_3 v \in T^{1/2} M.$$

Deduce that every element of M, and in particular, the element defined by (3.13), is localized at the T-dependent element $\tau \in \mathfrak{g}^*$ given in the optic (3.4) by

$$\tau = (0, -iT, 0).$$

4. The group $\mathrm{PGL}_2(\mathbb{R})$ via weight vectors

4.1. **Preliminaries.** We now turn to the group

$$G := \mathrm{PGL}_2(\mathbb{R}).$$

We will use the following notation for a basis of its complexified Lie algebra $\mathfrak{sl}_2(\mathbb{R})_\mathbb{C} = \mathfrak{sl}_2(\mathbb{C})$:

$$X := \frac{1}{2i}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix},$$
$$Y := \frac{1}{2i}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix},$$
$$H := \frac{1}{2i}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The standard maximal compact connected subgroup K of G, namely the image of $\mathrm{SO}(2)$, is then

$$K = \{\exp(\theta H) : \theta \in \mathbb{R}/2\pi\mathbb{Z}\}.$$

The commutation relations are

$$[X, Y] = -2H, \quad [H, X] = X, \quad [H, Y] = -Y.$$

The center of the universal enveloping algebra is generated by the Casimir element

$$\begin{aligned}\Omega &:= H^2 - \frac{XY+YX}{2} \\ &= H(H-1) - XY \\ &= H(H+1) - YX.\end{aligned}$$

The imaginary dual of the Lie algebra identifies with the space of imaginary traceless 2×2 matrices:

$$\mathfrak{sl}_2(\mathbb{R})^\wedge \cong i\,\mathfrak{sl}_2(\mathbb{R}). \tag{4.1}$$

Here $\xi \in i\,\mathfrak{sl}_2(\mathbb{R})$ corresponds to the linear map $\mathfrak{sl}_2(\mathbb{R}) \to i\mathbb{R}$ given by $x \mapsto \operatorname{trace}(x\xi)$.

4.2. **Representations.**

Proposition 4. *Let π be an irreducible unitary representation of $\mathrm{PGL}_2(\mathbb{R})$. Then Ω acts on π by a scalar, say Ω_π. Then, either:*

(i) π is a one-dimensional representation, either trivial or the sign representation, in which case $\Omega_\pi = 0$.

(ii) π is a discrete series representation $\pi(k)$ for some $k \in \mathbb{Z}_{\geq 1}$, with $\Omega_\pi = k(k-1)$. (We have chosen the numbering so that such representations correspond to holomorphic modular forms of weight $2k$ in the traditional sense.)

(iii) π is a unitary principal series representation $\pi(t, \varepsilon)$, with
- *$t \in \mathbb{R}$ and $\varepsilon \in \{\pm 1\}$, or*
- *$t \in i(\frac{1}{2}, \frac{1}{2}) - \{0\}$ and $\varepsilon = 1$,*

with $\Omega_\pi = -\frac{1}{4} - t^2$.

The only equivalences are that $\pi(t, \varepsilon) \cong \pi(-t, \varepsilon)$.

The representation $\pi = \pi(t, \varepsilon)$ admits a basis e_m, indexed by $m \in \mathbb{Z}$, on which the Lie algebra elements act by the formulas

$$\begin{aligned} Xe_m &= (m(m+1) - \Omega_\pi)^{1/2} e_{m+1}, \\ Ye_{m+1} &= (m(m+1) - \Omega_\pi)^{1/2} e_m, \\ He_m &= me_m, \\ \operatorname{diag}(-1,1)e_m &= (-1)^\varepsilon e_{-m}. \end{aligned}$$

The representation $\pi = \pi(k)$ admits a basis e_m, indexed by $\{m \in \mathbb{Z} : |m| \geq k\}$, on which the Lie algebra elements act by the same formulas as above, but with $\varepsilon = 1$.

Proof. Similar to that of Proposition 2. $\square$

The tempered irreducible representations are the $\pi(k)$ and the $\pi(t, \varepsilon)$ with $t \in \mathbb{R}$. For either of these, the coadjoint orbit $\mathcal{O}_\pi$ is given in the optic (4.1) by

$$\mathcal{O}_\pi = \left\{0 \neq \xi \in i\,\mathfrak{sl}_2(\mathbb{R}) : \det(\xi/i) = \tfrac{1}{4} + \Omega_\pi\right\}. \tag{4.2}$$

4.3. **Localized vectors.**

Exercise 5. Let π be the T-dependent representation of $\mathrm{PGL}_2(\mathbb{R})$ given by the discrete series representation $\pi_T = \pi(k)$ of lowest weight

$$k = k_T := T.$$

Let M denote the class of T-dependent vectors v in π of the form $v = \sum_m a_m e_m$, where

(1) $a_m = 0$ unless $m = T + \mathrm{O}(1)$, and

(2) each $a_m = \mathrm{O}(1)$.

Verify that for all $v \in M$, we have

$$Xv \in T^{1/2} M,$$

$$Yv \in T^{1/2} M,$$

$$Hv - Tv \in T^{1/2} M.$$

Deduce that every element of M is localized at the T-dependent element $\tau \in \mathfrak{g}^*$ given in the optic (4.1) by

$$\tau = iT \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Exercise 6. Let π be the T-dependent representation of $\mathrm{PGL}_2(\mathbb{R})$ given by the tempered principal series representation $\pi_T = \pi(t, \varepsilon)$, where

$$t = t_T := T$$

while $\varepsilon \in \{\pm 1\}$ is fixed. Let M denote the class of T-dependent vectors v in π of the form $v = \sum_m a_m e_m$, where the coefficients satisfy the support condition

$$a_m \neq 0 \implies m = \mathrm{O}(T^{1/2})$$

as well as the Fourier series condition (3.12) enunciated in Exercise 4. Verify that for all $v \in M$, we have

$$Xv - Tv \in T^{1/2} M,$$

$$Yv - Tv \in T^{1/2} M,$$

$$Hv \in T^{1/2} M.$$

Deduce that every element of M is localized at

$$\tau = iT \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Remark 5. Suppose that π as in Exercise 6 comes equipped with a equivariant isometric embedding

$$\iota : \pi \hookrightarrow L^2(\Gamma \backslash G)$$

for some finite volume quotient $\Gamma \backslash G$ by a discrete subgroup $\Gamma < G = \mathrm{PGL}_2(\mathbb{R})$, such as $\Gamma = \mathrm{PGL}_2(\mathbb{Z})$. Under such an embedding, the spherical vector e_0 maps to an L^2-normalized Maass form

$$\varphi_0 := \iota(e_0) \in \pi^K \subseteq C^\infty(\Gamma \backslash G / K)$$

of eigenvalue $1/4 + t^2$. The image of the class M is closely related to the microlocal lift of Zelditch et al (see [10, 6, 9, 1]). More precisely, for each unit vector $v \in M$, its image

$$\varphi := \iota(v) \in \pi \subseteq C^\infty(\Gamma \backslash G)$$

may be shown to have the following properties

- For each fixed $\Psi \in C_c^\infty(\Gamma \backslash G / K)$, we have

$$\int_{\Gamma \backslash G} |\varphi|^2 \Psi = \int_{\Gamma \backslash G} |\varphi_0|^2 \Psi + \mathrm{O}(T^{1/2+\varepsilon}).$$

- Let $G_\tau \leq G$ denote the centralizer of τ, thus G_τ is the diagonal subgroup
$$G_\tau = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$
For each fixed $\Psi \in C_c^\infty(\Gamma \backslash G)$ and fixed $g \in G_\tau$, we have
$$\int_{x \in \Gamma \backslash G} |\varphi(xg)|^2 \Psi(x) \, d\mu(x) = \int_{x \in \Gamma \backslash G} |\varphi(x)|^2 \Psi(x) \, d\mu(x) + \mathrm{O}(T^{1/2+\varepsilon}).$$

For some more precise assertions, see [7, §7.1, Lemma 2] (direct link: 2) for further discussion.

Exercise 7. Let π be any fixed infinite-dimensional irreducible unitary representation of $\mathrm{PGL}_2(\mathbb{R})$ (e.g., the tempered principal series representation $\pi(0,1)$). Let M denote the class of T-dependent vectors v in π of the form $v = \sum_m a_m e_m$, where the coefficients satisfy the support condition
$$a_m \neq 0 \implies m = T + \mathrm{O}(T^{1/2})$$
as well as the Fourier series condition (3.12) enunciated in Exercise 4. Verify that for all $v \in M$, we have
$$Xv - Tv \in T^{1/2} M,$$
$$Yv - Tv \in T^{1/2} M,$$
$$Hv - Tv \in T^{1/2} M.$$
Deduce that every element of M is localized at
$$\tau = iT \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

5. The group $\mathrm{PGL}_2(\mathbb{R})$ via the Kirillov model

5.1. **Preliminaries.** Set $G := \mathrm{PGL}_2(\mathbb{R})$. We will work with the subgroups
$$N := \left\{ n(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\},$$
$$A := \left\{ a(y) := \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} : y \in \mathbb{R}^\times \right\},$$
$$B := NA = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

Let $\psi : \mathbb{R} \to \mathrm{U}(1)$ be a nontrivial unitary character. We may identify ψ with a character of the subgroup Set
$$C^\infty((N,\psi)\backslash G) := \{ W \in C^\infty(G) : W(n(x)g) = \psi(x) W(g) \text{ for all } (x,g) \in \mathbb{R} \times G \}$$
Let π be an irreducible representation of G. More precisely, we denote here by π the subspace of smooth vectors. We recall that π is *generic* if there is an equivariant embedding $\pi \hookrightarrow C^\infty((N,\psi)\backslash G)$. The space of such embeddings is then one-dimensional, so the image, call it $\mathcal{W}(\pi,\psi)$, is well-defined. Moreover, the restriction map
$$\mathcal{W}(\pi,\psi) \to \{\text{functions } A \to \mathbb{C}\}$$
is injective, and its image contains $C_c^\infty(A)$. Consequently, each $W \in \mathcal{W}(\pi,\psi)$ is determined by the function $W : \mathbb{R}^\times \to \mathbb{C}$ given by
$$W(y) := W(a(y)),$$

and every smooth compactly-supported function arises in this way. We obtain in this way a realization of π as a space of functions on $\mathbb{R}^\times$, called the *Kirillov model*. When π is unitary, an invariant inner product may be given in the Kirillov model by

$$\langle W_1, W_2 \rangle := \int_{y \in \mathbb{R}^\times} W_1(y) \overline{W_2(y)} \, d^\times y, \quad d^\times y := \frac{dy}{|y|}. \tag{5.1}$$

Standard references for these facts include [3, §6], [2, §10.2], [4].

The action of B on the Kirillov model is completely explicit: we have

$$n(x)W(y) = \psi(yx)W(y), \tag{5.2}$$

$$a(u)W(y) = W(yu). \tag{5.3}$$

Indeed (5.2) follows from the commutation property $a(y)n(x) = n(yx)a(y)$ and the left N-equivariance of W, while (5.3) is obvious.

The infinitesimal generators of N and A are the matrices

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h := \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These act on π by differential operators. The formulas for their action is simplest when

$$\psi(x) := e^{ix}, \tag{5.4}$$

so let's specialize to that case. By differentiating (5.2) and (5.3), we see that

$$eW(y) = iyW(y), \tag{5.5}$$

$$hW(y) = y\partial_y W(y). \tag{5.6}$$

The other standard Lie algebra basis element is

$$f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These elements satisfy

$$[e, f] = 2h, \quad [h, e] = e, \quad [h, f] = -f. \tag{5.7}$$

The Casimir element Ω is given (with the same normalization as in §4.1) by

$$\Omega = h^2 + \frac{ef + fe}{2} = h^2 - h + ef.$$

Writing Ω_π as before for the eigenvalue by which Ω acts on π, we can solve for the action of f on π:

$$fW(y) = \frac{1}{iy} \left(\Omega_\pi - (y\partial_y)^2 + y\partial_y \right) W(y). \tag{5.8}$$

This formula is the key to verifying the fact recorded above, that any smooth compactly-supported function on $\mathbb{R}^\times$ arises from some (smooth!) vector in π. We refer to [4] and [8, §12] (direct link: §12) for further discussion.

5.2. **Localized vectors.** We just give one representative example of the sort of analysis that can be achieved in this way.

Exercise 8. Let π be a T-dependent generic irreducible unitary representation of $\mathrm{PGL}_2(\mathbb{R})$, realized in its Kirillov model with respect to the character (5.4) as above and with unitary structure given by (5.1). Assume that

$$\Omega_\pi = \mathrm{O}(T^2).$$

In other words, in the notation of §4.2, either

(1) $\pi = \pi(t, \varepsilon)$ with $t = \mathrm{O}(T)$, or
(2) $\pi = \pi(k)$ with $k = \mathrm{O}(T)$.

Let α and ρ be T-dependent real numbers with $\rho = \mathrm{O}(T)$ and $\alpha \asymp T$ (see §2.1 of this note for notation). Let M denote the class of all T-dependent elements $W \in \pi$ that are given in the Kirillov model for large enough T by the formula

$$W(y) = T^{1/4}|y|^{i\rho}\phi\left(\frac{y-\alpha}{T^{1/2}}\right), \tag{5.9}$$

where ϕ belongs to some fixed bounded subset of the space $C_c^\infty(\mathbb{R}^\times)$. (We recall that a subset $\mathfrak{B}$ of this space is bounded if there are constants $C_n \geq 0$ and a compact set $E \subseteq \mathbb{R}^\times$ such that each $\phi \in \mathfrak{B}$ is supported in E and has nth derivative bounded in L^∞-norm by C_n.)

(i) Verify that $\|W\| = \mathrm{O}(1)$ for each $W \in M$.
(ii) Verify that for each $W \in M$, we have
$$eW - i\alpha W \in T^{1/2}M,$$
$$hW - i\rho W \in T^{1/2}M.$$
(iii) Use (5.8) to show that
$$fW - i\beta W \in T^{1/2}M,$$
where
$$\beta := \frac{\rho^2 - \Omega_\pi}{\alpha}.$$
(iv) Deduce that every element of M is localized at the T-dependent element of $\mathfrak{g}^\wedge$ given by
$$\tau = i\begin{pmatrix} \rho & \beta \\ \alpha & -\rho \end{pmatrix},$$
for which
$$\det(\tau/i) = \Omega_\pi.$$
(Compare with (4.2).)
(v) Define the Weyl group element
$$w := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in G,$$
which satisfies
$$\mathrm{Ad}(w)e = f, \qquad \mathrm{Ad}(w)h = -h, \qquad \mathrm{Ad}(w)f = e.$$
Show that for each $W \in M$, the w-translate wW of W, given in the Kirillov model by
$$wW(y) := W(a(y)w), \tag{5.10}$$

is localized at

$$w \cdot \tau = i \begin{pmatrix} -\rho & \alpha \\ \beta & \rho \end{pmatrix}.$$

(vi) Show that fixed $k, \ell \in \mathbb{Z}_{\geq 0}$, we have

$$\int_{y \in \mathbb{R}^\times} \left| \frac{(y\partial_y)^k |y|^{i\rho} wW(y)}{T^{k/2}} \right|^2 \left| \frac{y - \beta}{T^{1/2}} \right|^\ell d^\times y \ll 1.$$

Informally, and at least when $\beta \asymp T$, this says that $wW(y)$ looks roughly like $T^{1/4}|y|^{-i\rho}$ times a bump function on $\beta + \mathrm{O}(T^{1/2})$.

[Hint: consider the identity

$$(e - i\beta)^k (h + i\rho)^\ell wW = w(f - i\beta)^k(-h + i\rho)^\ell W$$

and use that the norm defined by (5.1) is G-invariant and that each element of M has norm $\mathrm{O}(1)$.]

Remark 6. Set $\Gamma := \mathrm{PGL}_2(\mathbb{Z}) < G$. Suppose that the representation π of G considered in Exercise 8 comes with an embedding $\pi \hookrightarrow L^2_{\mathrm{cusp}}(\Gamma\backslash G)$ as a Hecke eigenspace, with Hecke eigenvalues $\lambda : \mathbb{N} \to \mathbb{C}$. Then, as explained in §2.2 of this note, each $\varphi \in \pi$ (regarded as a function $\varphi : \Gamma\backslash G \to \mathbb{C}$) admits a Whittaker expansion

$$\varphi(g) = \sum_{n \neq 0} \frac{\lambda(|n|)}{|n|^{1/2}} W(a(2\pi n) g) \tag{5.11}$$

where $W \in \mathcal{W}(\pi, \psi)$ denotes the image of φ under the equivariant map defined by integrating over $(\Gamma \cap N)\backslash N$ against ψ^{-1}. (The factor 2π appears because we are using the unconventional choice (5.4) for ψ.) The left-invariance of φ under w implies a case of the Voronoi summation formula (see this note of Cogdell for a more general discussion): by applying the expansion (5.11) to both sides of the identity $\varphi(1) = \varphi(w)$, and with the same abbreviations

$$W(y) := W(a(y)), \qquad wW(y) := W(a(y)w)$$

as before, we obtain

$$\sum_{n \neq 0} \frac{\lambda(|n|)}{|n|^{1/2}} W(2\pi n) = \sum_{n \neq 0} \frac{\lambda(|n|)}{|n|^{1/2}} wW(2\pi n).$$

The values $W(y)$ and $wW(y)$ are related by an integral transform involving the Bessel function attached to π (see §2.2 of Cogdell's note, or [5, Appendix A]). When $W(y)$ is given by (5.9), one can derive asymptotics for $wW(y)$ using stationary phase analysis; derivations like this may be found in many analytic number theory papers. One consequence of Exercise 8 is that in many cases, the shapes of $W(y)$ and $wW(y)$ may be related by "pure thought".

6. Relative character asymptotics

6.1. **The case** $(\mathrm{SO}(3), \mathrm{SO}(2))$.

7. TODO

(1) Something for SO(3) using the Borel–Weil model over $\mathbb{CP}^1$.
(2) Something for induced models for $\mathrm{PGL}_2(\mathbb{R})$.
(3) ~~Some follow-up to exercise 5.2 explaining what it says about $wW(y)$, for w the nontrivial Weyl element.~~

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