1. Gebhard Boeckle's lectures

1.1. **Galois representations and congruences.** We first discuss profinite groups. Let G be a topological group.

Theorem 1. *The following are equivalent:*

(a) G is compact, Hausdorff, and totally disconnected.
(b) G is compact, and admits a neighborhood basis of the identity by open normal subgroups.
(c) There is a directed poset I and an inverse system (G_i) of finite (discrete) groups such that $G = \varprojlim_I G_i$.

We say that G is *profinite* if the above conditions hold. The topology on $\varprojlim G_i$ is that obtained by regarding it as a closed subgroup of the product $\prod G_i$.

Constructions:

(a) If G is discrete, then we equip it with the profinite topology $G^{\mathrm{pf}} := \varprojlim G/N$, where N runs over the finite index subgroups.
(b) If $G = \varprojlim G_i$ is profinite, then
  (i) The abelianization is given by
$$G^{ab} = G/\overline{[G,G]} = \varprojlim G_i^{ab},$$
  and in particular, is profinite.
  (ii) For H finite, write H_p for its maximal p-group quotient. Then
$$G_p = \varprojlim (G_i)_p$$
  is a pro-p-group (and in particular, profinite).
  (iii) If $N \leq G$ is closed and normal, then G/N is profinite.

Example 2. (a) Let F be a field. Set $G_F := \mathrm{Aut}_F(F^{\mathrm{sep}}) = \mathrm{Gal}(F^{\mathrm{sep}}/F)$ profinite. Define the poset
$$\mathcal{I}_F := \{L \subseteq F^{\mathrm{sep}} : L \supseteq F \text{ finite Galois}, \subseteq\}.$$
Then
$$G_F \xrightarrow{\cong} \varprojlim_{L \in \mathcal{I}_F} \mathrm{Gal}(L/F).$$

(b) Let $F' \subseteq F^{\mathrm{sep}}$ be a normal extension of F. Then $G_{F'} \leq G_F$ is closed and normal. We may thus write
$$\mathrm{Gal}(F'/F) \cong G_F/G_{F'} = \varprojlim_{\substack{L \in \mathcal{I}_F, \\ L \subseteq F'}} \mathrm{Gal}(L/F).$$

(c) Let $\mathbb{N}$ denote the natural numbers, ordered by divisibility. Then
$$\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n = \prod_p \mathbb{Z}_p,$$
where the last step is the Chinese remainder theorem. We sometimes need a slight modification:
$$\hat{\mathbb{Z}}^{(p)} = \varprojlim_{p \nmid n} \mathbb{Z}/n = \prod_{\ell \text{ prime}, \ell \neq p} \mathbb{Z}_\ell.$$

Let's fix some notation:

(1) Let K be a number field, $\mathcal{O}_K$ its ring of integers. Let $\mathrm{Pl}_K = \mathrm{Pl}_K^\infty \sqcup \mathrm{Pl}_K^{\mathrm{fin}}$ denote the set of places v of K. Let v be a finite place. We may then attach to it a maximal ideal $\mathfrak{q}_v$ of $\mathcal{O}_K$, giving a bijection

$$\mathrm{Pl}_K^{\mathrm{fin}} \leftrightarrow \mathrm{Max}(\mathcal{O}_K).$$

We may form the residue field $k_v := \mathcal{O}_K/\mathfrak{q}_v$. We denote q_v for the cardinality of k_v. We write $\mathrm{char}(v)$ for the characteristic of k_v. We denote by $\mathcal{O}_v = \varprojlim \mathcal{O}/\mathfrak{q}_v^n$, with fraction field K_v. Also, we have a short exact sequence

$$1 \to I_v \to G_v := \mathrm{Gal}_{K_v} \to \mathrm{Gal}_{k_v} \to 1.$$

A topological generator for Gal_{k_v} is given by

$$\mathrm{Fr}_v : \alpha \mapsto \alpha^{q_v}.$$

We denote by $\mathrm{Frob}_v \in G_v$ some lift of Fr_v.

We write $S_\infty := \mathrm{Pl}_K^\infty$ for the set of archimedean places, so that $K \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{v \in S_\infty} K_v$. For a rational prime p, we write S_p for the set of places v of K such that $v \mid p$.

(2) We also need some local analogues for $E \supseteq \mathbb{Q}_p$ a p-adic field.

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References