

## SOME ALGEBRA-FLAVORED EXERCISES

ABSTRACT. In this evolving note, we record some algebra-flavored exercises relevant for the Oberwolfach seminar.

### 1. CYCLIC AND REGULAR MATRICES

Let  $F$  be a field, let  $V$  be a finite-dimensional vector space over  $F$ , and let  $M := \text{End}(V)$  denote the space of linear maps  $V \rightarrow V$ .

**Definition 1.** Let  $\tau \in M$  and  $v \in V$ . We denote by  $F[\tau]v$  the set of elements of  $V$  that may be written as a polynomial in  $\tau$  applied to  $v$ , or equivalently, the span of the elements

$$v, \quad \tau v, \quad \tau^2 v, \quad (\dots).$$

We say that a vector  $v \in V$  is  $\tau$ -cyclic, or that  $v$  is a *cyclic vector* for  $\tau$ , if

$$F[\tau]v = V.$$

We say that  $\tau$  is *cyclic* if it admits a cyclic vector.

**Exercise 1.** Show that  $\tau$  is cyclic if and only if there is a basis with respect to which it is of the form, e.g., for  $\dim(V) = 4$ ,

$$\begin{pmatrix} 0 & 0 & 0 & * \\ 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}.$$

**Exercise 2.** Show that the set of conjugacy classes consisting of cyclic elements is in bijection with the set of characteristic polynomials, that is to say:

- (i) For each monic polynomial of degree  $\dim(V)$ , there exists a cyclic element  $\tau \in M$  whose characteristic polynomial is that polynomial.
- (ii) Two cyclic elements with the same characteristic polynomial are conjugate.

**Exercise 3.** Show that a matrix given in Jordan form is cyclic precisely when the eigenvalues pertaining to different Jordan blocks are distinct. In particular, a diagonal matrix is cyclic precisely when its diagonal entries are distinct.

**Definition 2.** We say that  $\tau \in M$  is *regular* if  $\dim M_\tau = \dim V$ , where

$$M_\tau := \{x \in M : x\tau = \tau x\}$$

denotes the *centralizer* of  $\tau$  in  $M$ .

**Exercise 4.** Show that for  $\tau \in M$ , the following are equivalent.

- (i)  $\tau$  is cyclic.
- (ii)  $M_\tau = F[\tau]$ .
- (iii)  $\tau$  is regular.

**Definition 3.** We recall that  $\tau \in M$  is *nilpotent* if some power of  $\tau$  vanishes.

**Exercise 5.** Suppose that  $F = \mathbb{R}$ . Fix a norm  $\|\cdot\|$  on  $M$ . Let

$$\mathcal{O} \subseteq M$$

be a conjugacy class consisting of regular (equivalently, cyclic) elements. Let  $x_j \in \mathcal{O}$  be a sequence whose matrix norms  $\|x_j\|$  tend to infinity.

- (i) Show that, after passing to a subsequence if necessary, the normalized limit

$$x := \lim_{j \rightarrow \infty} \frac{x_j}{\|x_j\|}$$

exists, and is nilpotent.

- (ii) Show that there exists a sequence  $x_j$  as above for which the normalized limit  $x$  is regular nilpotent.

**Exercise 6.** Suppose that a matrix of the form

$$\tau = \begin{pmatrix} 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 1 & * & * & * \end{pmatrix}$$

has the property that the standard basis vector  $e_4$  is cyclic. Show that  $\tau$  is invertible.

**Exercise 7.** Show that if a matrix is cyclic, then so is its transpose.

#### REFERENCES