1. Holomorphic continuation

Theorem 1 (Identity principle for holomorphic functions)**.** *Let $U \subset \mathbb{C}$ be a connected open set. Let $f, g : U \to \mathbb{C}$ be holomorphic functions. If $f = g$ on a set with a limit point in U, then $f = g$ on all of U.*

Corollary 2. *Let $U \subset \Omega \subseteq \mathbb{C}$ be open subsets, with U nonempty and Ω connected. Let $f : U \to \mathbb{C}$ be a holomorphic function. Then there is at most one extension of f to a holomorphic function $\Omega \to \mathbb{C}$.*

2. Cauchy's integral formula

Theorem 3. *Let $f : U \to \mathbb{C}$ be a holomorphic function defined on an open subset U. Let γ be a closed rectifiable curve in U. Then $\int_\gamma f(z)\,dz = 0$.*

Theorem 4. *Let $0 \leq a < b \leq \infty$. Let $f(z)$ be a holomorphic function on the annulus $\{z \in \mathbb{C} : a < |z| < b\}$ given by a convergent Laurent series*

$$f(z) = \sum_{n \in \mathbb{Z}} c_n z^n.$$

(1) For any $r \in (a, b)$ and $n \in \mathbb{Z}$, we have

$$\begin{aligned} c_n &= \oint_{|z|=r} \frac{f(z)}{z^n} \frac{dz}{2\pi i z} \\ &= \frac{1}{2\pi r^n} \int_{\theta=0}^{2\pi} f(re^{i\theta}) e^{-in\theta}\,d\theta. \end{aligned}$$

(2) For each compact subset E of (a, b), there exists $M \geq 0$ so that for all $r \in E$, we have

$$\sum_{n \in \mathbb{Z}} |c_n| r^n \leq M. \tag{1}$$

Theorem 5. *Let U be an open subset of $\mathbb{C}$, let $f : U \to \mathbb{C}$ be meromorphic. Let γ be a smooth closed curve in U, oriented counterclockwise, that does not pass through any pole of f. Then*

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{\substack{z \in \text{interior}(\gamma) \\ \textit{pole of } f}} \operatorname{res}_z(f).$$

Remark 6. Let $0 < r < R$. Let f be a meromorphic function on a neighborhood of the annulus $\{z : r < |z| < R\}$ that has no poles on either of the circles $|z| = r, R$. Then

$$\oint_{|z|=R} f(z)\,dz = \oint_{|z|=r} f(z)\,dz + 2\pi i \sum_{\substack{r < |z| < R \\ \text{pole of } f}} \operatorname{res}_z(f).$$

3. Holomorphy of limits and series

Theorem 7. *Let U be an open subset of the complex plane. Let f_n be a sequence of holomorphic functions on U.*

(1) Suppose that the sequence f_n converges pointwise to some function f, uniformly on compact subsets of U. Then f is holomorphic.

(2) Suppose that the partial sums $\sum_{n \leq N} f_n$ converge pointwise to some function f, uniformly on compact subsets of U. Then the sum $\sum_n f_n$ is holomorphic.

4. Blashke factors

Definition 8. Let $\alpha \in \mathbb{C}$ with $|\alpha| < 1$. The *Blashke factor* B_α is the function

$$B_\alpha(z) := \frac{\alpha - z}{1 - \bar{\alpha} z}.$$

Lemma 9. *The Blashke factors enjoy the following properties:*

1. $B_\alpha(0) = \alpha$ *and* $B_\alpha(\alpha) = 0$.
2. B_α *defines a holomorphic automorphism of the unit disc* $\{z : |z| < 1\}$*, with inverse* B_α *(i.e.,* $B_\alpha(B_\alpha(z)) = z$*).*
3. $|B_\alpha(z)| = 1$ *for all* z *with* $|z| = 1$.

5. Harmonic functions

Let U be an open subset of the plane $\mathbb{C}$.

5.1. **Definition.**

Definition 10. We say that a smooth function $u : U \to \mathbb{R}$ is *harmonic* if $u_{xx} + u_{yy} = 0$.

5.2. **Relation with holomorphic functions.**

Lemma 11. *Let* $f : U \to \mathbb{C}$ *be holomorphic. Then, writing* $f = u + iv$*, with* $u, v : U \to \mathbb{R}$*, we have that* u *is harmonic. Conversely, if* U *is simply-connected, then every harmonic* u *arises in this way for some holomorphic* f.

Proof. The first claim follows from the Cauchy-Riemann equations. For the second claim, we solve the Cauchy-Riemann equations to find v such that $f = u + iv$ is holomorphic on each connected component of U. $\square$

5.3. **Mean value theorems.** For the following results, let $u : U \to \mathbb{R}$ be harmonic, and suppose that U contains the disc $\{z : |z - z_0| \leq r\}$.

Lemma 12. *We have*

$$u(z_0) = \int_0^{2\pi} u(z_0 + re^{i\theta}) \frac{d\theta}{2\pi}.$$

Proof. We may assume that U is simply-connected. The claim then follows from the Cauchy integral formula applied to a holomorphic f with $\Re(f) = u$. $\square$

Lemma 13. *For* $|z - z_0| \leq r$*, we have*

$$u(z) = \int_0^{2\pi} u(z_0 + re^{i\theta}) \Re \frac{re^{i\theta} + (z - z_0)}{re^{i\theta} - (z - z_0)} \frac{d\theta}{2\pi}.$$

Proof. We can reduce to the previous lemma using Blashke factors. Alternatively, we can reduce first to the case $z_0 = 0$ and $r = 1$ and $0 < z < 1$. Then Cauchy's integral formula reads

$$f(z) = \int_{w:|w|=r} f(w) \underbrace{\frac{1}{2}\left(\frac{w+z}{w-z} + \frac{w^{-1}+z}{w^{-1}-z}\right)}_{\Re\left(\frac{w+z}{w-z}\right)} \frac{dw}{2\pi i},$$

which yields the required formula upon taking real parts. $\square$

Lemma 14. *For $|z-z_0| \leq r_0 < r$, we have*

$$|u(z)| \leq \frac{r+r_0}{r-r_0} \int_0^{2\pi} |u(z_0 + re^{i\theta})| \frac{d\theta}{2\pi}.$$

Proof. We apply Lemma 13 and majorize the integrand in absolute value. □

5.4. **Approximate factorizations of holomorphic functions.** The next lemma closely follows the presentation of Theorem 21 of https://terrytao.wordpress.com/2014/12/05/245a-supplement-2-a-little-bit-of-complex-and-fourier-analysis/.

Lemma 15. *Fix $0 < c_2 < c_1 < 1$. Let f be a holomorphic function, on a neighborhood of $\bar{D}$, where $D := \{|z-z_0| < r\}$. Assume that $f(z_0) \neq 0$. Assume given $M \geq 1$ so that whenever $|z-z_0| = r$, we have*

$$|f(z)| \leq M|f(z_0)|.$$

Let ρ run over the zeros of f, counted with multiplicity. Then

$$\#\{\rho : |\rho - z_0| \leq c_1 r\} \ll_{c_1} \log M$$

and

$$\frac{f'}{f}(z) = \sum_{|\rho - z_0| \leq c_1 r} \frac{1}{z-\rho} + \mathrm{O}_{c_1,c_2}\left(\frac{\log M}{r}\right).$$

Proof. By replacing $f(z)$ with $f(z_0+z)$, we may assume that $z_0 = 0$. By writing $f(z) = g(rz)$ (so that $f'(z) = rg'(rz)$, hence $\frac{g'}{g}(rz) = \frac{1}{r}\frac{f'}{f}(z)$), we reduce to the case $r = 1$. Our task is then to show that

$$\#\{\rho : |\rho| \leq c_1\} \ll_{c_1} \log M \tag{2}$$

and that for $|z| \leq c_2$, we have

$$\frac{f'}{f}(z) = \sum_{|\rho| \leq c_1} \frac{1}{z-\rho} + \mathrm{O}_{c_1,c_2}(\log M). \tag{3}$$

By Jensen's formula, we have

$$0 \leq \sum_{|\rho| \leq 1} \log \frac{1}{|\rho|} = \int_0^{2\pi} \frac{\log|f(e^{i\theta})|}{\log|f(0)|} \frac{d\theta}{2\pi} \leq \log M. \tag{4}$$

On the other hand, for $|\rho| \leq c_1$, we have $\log 1/|\rho| \geq \log 1/c_1$. It follows that

$$\#\{\rho : |\rho| \leq c_1\} \leq \frac{\log M}{\log 1/c_1} \ll_{c_1} \log M,$$

whence (2).

Turning to the proof of (3), we may write

$$f = g \prod_{|\rho| \leq 1} B_\rho,$$

where g is a holomorphic function on a neighborhood of $\bar{D}$ that does not vanish on $\bar{D}$. We then have

$$\frac{f'}{f} = \frac{g'}{g} + \sum_{|\rho| \leq 1} \frac{B'_\rho}{B_\rho}.$$

Below, we estimate separately the contributions to (3) from g, from $|\rho| \leq c_1$, and from $c_1 < |\rho| \leq 1$. The result follows by combining these estimates.

We first estimate the contribution of g. We will show that

$$\frac{g'}{g}(z) = \mathrm{O}_{c_1,c_2}(\log M), \tag{5}$$

We may normalize f so that $g(0) = 1$. Then, since g does not vanish, Jensen's formula reads

$$\int_0^{2\pi} \log|g(e^{i\theta})| \, \frac{d\theta}{2\pi} = 0. \tag{6}$$

On the other hand, our hypothesis reads

$$|g(e^{i\theta})| = |f(e^{i\theta})| \leq M|f(0)| = M\prod_\rho |\rho| \leq M,$$

hence

$$\log|g(e^{i\theta})| \leq \log M.$$

By combining this pointwise upper bound with the mean zero property (6), we deduce that

$$\int_0^{2\pi} \big|\log|g(e^{i\theta})|\big| \leq \frac{1}{2}\log M.$$

Since g does not vanish, we may find a holomorphic primitive G for g'/g (i.e., G is a logarithm of g). The Poisson integral formula gives, for $|z| \leq c_2$, the estimate

$$|\Re(G(z))| \ll_{c_2} \log M,$$

together with a similar bound for the derivatives of the real part of G. By the Cauchy–Riemann equations, the same bound holds for the first derivatives of the imaginary part of G, hence for those of G itself. The estimate (5) follows.

We next estimate the contribution from $|\rho| \leq c_1$. After the calculation

$$\frac{B'_\rho}{B_\rho}(z) = \frac{1}{z-\rho} - \frac{1}{z - 1/\bar{\rho}}, \tag{7}$$

one term of which matches up with a term on the right hand side of (3), we see that it suffices to show that

$$\sum_{|\rho|\leq c_1} \frac{1}{z - 1/\bar{\rho}} \ll_{c_1,c_2} \log M. \tag{8}$$

Since $|z| \leq c_2$, we have

$$\left|\frac{1}{z-1/\bar{\rho}}\right| \leq \frac{1}{1/c_1 - c_2} \ll_{c_1,c_2} 1.$$

Therefore the required bound (8) follows from (2).

We turn finally to the contribution from $c_1 < |\rho| \leq 1$. For such an element ρ, and z with $|z| \leq c_2$, we have

$$z - \rho \asymp_{c_1,c_2} 1, \qquad z - 1/\bar{\rho} \asymp_{c_1,c_2} 1,$$

where we recall that $A \asymp B$ means that $A \ll B \ll A$. By cross-multiplying in (7), it follows that

$$\frac{B'_\rho}{B_\rho}(z) \asymp_{c_1,c_2} \rho - 1/\bar{\rho}.$$

On the other hand, using the Taylor expansion of the logarithm, we have

$$\rho - 1/\bar{\rho} \asymp_{c_1} 1 - |\rho|^2 \asymp_{c_2} \log\frac{1}{|\rho|}.$$

We thereby obtain

$$\sum_{c_1<|\rho|\leq 1} \frac{B'_\rho}{B_\rho}(z) \ll_{c_1,c_2} \sum_{c_1<|\rho|\leq 1} \log\frac{1}{|\rho|} \leq \log M, \tag{9}$$

where in the final step we appealed to (4). $\square$

References