

1. HOLOMORPHIC CONTINUATION

Theorem 1 (Identity principle for holomorphic functions). *Let $U \subset \mathbb{C}$ be a connected open set. Let $f, g : U \rightarrow \mathbb{C}$ be holomorphic functions. If $f = g$ on a set with a limit point in U , then $f = g$ on all of U .*

Corollary 2. *Let $U \subset \Omega \subseteq \mathbb{C}$ be open subsets, with U nonempty and Ω connected. Let $f : U \rightarrow \mathbb{C}$ be a holomorphic function. Then there is at most one extension of f to a holomorphic function $\Omega \rightarrow \mathbb{C}$.*

2. CAUCHY'S INTEGRAL FORMULA

Theorem 3. *Let $f : U \rightarrow \mathbb{C}$ be a holomorphic function defined on an open subset U . Let γ be a closed rectifiable curve in U . Then $\int_{\gamma} f(z) dz = 0$.*

Theorem 4. *Let $0 \leq a < b \leq \infty$. Let $f(z)$ be a holomorphic function on the annulus $\{z \in \mathbb{C} : a < |z| < b\}$ given by a convergent Laurent series*

$$f(z) = \sum_{n \in \mathbb{Z}} c_n z^n.$$

(1) *For any $r \in (a, b)$ and $n \in \mathbb{Z}$, we have*

$$\begin{aligned} c_n &= \oint_{|z|=r} \frac{f(z)}{z^n} \frac{dz}{2\pi i z} \\ &= \frac{1}{2\pi r^n} \int_{\theta=0}^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta. \end{aligned}$$

(2) *For each compact subset E of (a, b) , there exists $M \geq 0$ so that for all $r \in E$, we have*

$$\sum_{n \in \mathbb{Z}} |c_n| r^n \leq M. \quad (1)$$

Theorem 5. *Let U be an open subset of \mathbb{C} , let $f : U \rightarrow \mathbb{C}$ be meromorphic. Let γ be a smooth closed curve in U , oriented counterclockwise, that does not pass through any pole of f . Then*

$$\int_{\gamma} f(z) dz = 2\pi i \sum_{\substack{z \in \text{interior}(\gamma) \\ \text{pole of } f}} \text{res}_z(f).$$

Remark 6. Let $0 < r < R$. Let f be a meromorphic function on a neighborhood of the annulus $\{z : r < |z| < R\}$ that has no poles on either of the circles $|z| = r, R$. Then

$$\oint_{|z|=R} f(z) dz = \oint_{|z|=r} f(z) dz + 2\pi i \sum_{\substack{r < |z| < R \\ \text{pole of } f}} \text{res}_z(f).$$

3. HOLOMORPHY OF LIMITS AND SERIES

Theorem 7. *Let U be an open subset of the complex plane. Let f_n be a sequence of holomorphic functions on U .*

- (1) *Suppose that the sequence f_n converges pointwise to some function f , uniformly on compact subsets of U . Then f is holomorphic.*
- (2) *Suppose that the partial sums $\sum_{n \leq N} f_n$ converge pointwise to some function f , uniformly on compact subsets of U . Then the sum $\sum_n f_n$ is holomorphic.*

4. BLASHKE FACTORS

Definition 8. Let $\alpha \in \mathbb{C}$ with $|\alpha| < 1$. The *Blashke factor* B_α is the function

$$B_\alpha(z) := \frac{\alpha - z}{1 - \bar{\alpha}z}.$$

Lemma 9. The Blashke factors enjoy the following properties:

- (1) $B_\alpha(0) = \alpha$ and $B_\alpha(\alpha) = 0$.
- (2) B_α defines a holomorphic automorphism of the unit disc $\{z : |z| < 1\}$, with inverse B_α (i.e., $B_\alpha(B_\alpha(z)) = z$).
- (3) $|B_\alpha(z)| = 1$ for all z with $|z| = 1$.

5. HARMONIC FUNCTIONS

Let U be an open subset of the plane \mathbb{C} .

5.1. Definition.

Definition 10. We say that a smooth function $u : U \rightarrow \mathbb{R}$ is *harmonic* if $u_{xx} + u_{yy} = 0$.

5.2. Relation with holomorphic functions.

Lemma 11. Let $f : U \rightarrow \mathbb{C}$ be holomorphic. Then, writing $f = u + iv$, with $u, v : U \rightarrow \mathbb{R}$, we have that u is harmonic. Conversely, if U is simply-connected, then every harmonic u arises in this way for some holomorphic f .

Proof. The first claim follows from the Cauchy-Riemann equations. For the second claim, we solve the Cauchy-Riemann equations to find v such that $f = u + iv$ is holomorphic on each connected component of U . \square

5.3. Mean value theorems. For the following results, let $u : U \rightarrow \mathbb{R}$ be harmonic, and suppose that U contains the disc $\{z : |z - z_0| \leq r\}$.

Lemma 12. We have

$$u(z_0) = \int_0^{2\pi} u(z_0 + re^{i\theta}) \frac{d\theta}{2\pi}.$$

Proof. We may assume that U is simply-connected. The claim then follows from the Cauchy integral formula applied to a holomorphic f with $\Re(f) = u$. \square

Lemma 13. For $|z - z_0| \leq r$, we have

$$u(z) = \int_0^{2\pi} u(z_0 + re^{i\theta}) \Re \frac{re^{i\theta} + (z - z_0)}{re^{i\theta} - (z - z_0)} \frac{d\theta}{2\pi}.$$

Proof. We can reduce to the previous lemma using Blashke factors. Alternatively, we can reduce first to the case $z_0 = 0$ and $r = 1$ and $0 < z < 1$. Then Cauchy's integral formula reads

$$f(z) = \int_{w:|w|=r} f(w) \underbrace{\frac{1}{2} \left(\frac{w+z}{w-z} + \frac{w^{-1}+z}{w^{-1}-z} \right)}_{\Re\left(\frac{w+z}{w-z}\right)} \frac{dw}{2\pi i},$$

which yields the required formula upon taking real parts. \square

Lemma 14. For $|z - z_0| \leq r_0 < r$, we have

$$|u(z)| \leq \frac{r + r_0}{r - r_0} \int_0^{2\pi} |u(z_0 + re^{i\theta})| \frac{d\theta}{2\pi}.$$

Proof. We apply Lemma 13 and majorize the integrand in absolute value. \square

6. APPROXIMATE FACTORIZATIONS OF HOLOMORPHIC FUNCTIONS

The next lemma closely follows the presentation of Theorem 21 of <https://terrytao.wordpress.com/2014/12/05/supplement-2-a-little-bit-of-complex-and-fourier-analysis/>.

TODO: maybe revise to follow the presentation of [1, §3.9]

Lemma 15. Fix $0 < c_2 < c_1 < 1$. Let f be a holomorphic function, on a neighborhood of \bar{D} , where $D := \{|z - z_0| < r\}$. Assume that $f(z_0) \neq 0$. Assume given $M \geq 1$ so that whenever $|z - z_0| = r$, we have

$$|f(z)| \leq M|f(z_0)|.$$

Let ρ run over the zeros of f , counted with multiplicity. Then

$$\#\{\rho : |\rho - z_0| \leq c_1 r\} \ll_{c_1} \log M$$

and

$$\frac{f'}{f}(z) = \sum_{|\rho - z_0| \leq c_1 r} \frac{1}{z - \rho} + O_{c_1, c_2} \left(\frac{\log M}{r} \right).$$

Proof. By replacing $f(z)$ with $f(z_0 + z)$, we may assume that $z_0 = 0$. By writing $f(z) = g(rz)$ (so that $f'(z) = rg'(rz)$, hence $\frac{g'}{g}(rz) = \frac{1}{r} \frac{f'}{f}(z)$), we reduce to the case $r = 1$. Our task is then to show that

$$\#\{\rho : |\rho| \leq c_1\} \ll_{c_1} \log M \tag{2}$$

and that for $|z| \leq c_2$, we have

$$\frac{f'}{f}(z) = \sum_{|\rho| \leq c_1} \frac{1}{z - \rho} + O_{c_1, c_2}(\log M). \tag{3}$$

By Jensen's formula, we have

$$0 \leq \sum_{|\rho| \leq 1} \log \frac{1}{|\rho|} = \int_0^{2\pi} \frac{\log |f(e^{i\theta})|}{\log |f(0)|} \frac{d\theta}{2\pi} \leq \log M. \tag{4}$$

On the other hand, for $|\rho| \leq c_1$, we have $\log 1/|\rho| \geq \log 1/c_1$. It follows that

$$\#\{\rho : |\rho| \leq c_1\} \leq \frac{\log M}{\log 1/c_1} \ll_{c_1} \log M,$$

whence (2).

Turning to the proof of (3), we may write

$$f = g \prod_{|\rho| \leq 1} B_\rho,$$

where g is a holomorphic function on a neighborhood of \bar{D} that does not vanish on \bar{D} . We then have

$$\frac{f'}{f} = \frac{g'}{g} + \sum_{|\rho| \leq 1} \frac{B'_\rho}{B_\rho}.$$

Below, we estimate separately the contributions to (3) from g , from $|\rho| \leq c_1$, and from $c_1 < |\rho| \leq 1$. The result follows by combining these estimates.

We first estimate the contribution of g . We will show that

$$\frac{g'}{g}(z) = O_{c_1, c_2}(\log M), \quad (5)$$

We may normalize f so that $g(0) = 1$. Then, since g does not vanish, Jensen's formula reads

$$\int_0^{2\pi} \log|g(e^{i\theta})| \frac{d\theta}{2\pi} = 0. \quad (6)$$

On the other hand, our hypothesis reads

$$|g(e^{i\theta})| = |f(e^{i\theta})| \leq M|f(0)| = M \prod_{\rho} |\rho| \leq M,$$

hence

$$\log|g(e^{i\theta})| \leq \log M.$$

By combining this pointwise upper bound with the mean zero property (6), we deduce that

$$\int_0^{2\pi} |\log|g(e^{i\theta})|| \leq \frac{1}{2} \log M.$$

Since g does not vanish, we may find a holomorphic primitive G for g'/g (i.e., G is a logarithm of g). Lemma 14 gives, for $|z| \leq c_2$, the estimate

$$|\Re(G(z))| \ll_{c_2} \log M,$$

together with a similar bound for the derivatives of the real part of G . By the Cauchy–Riemann equations, the same bound holds for the first derivatives of the imaginary part of G , hence for those of G itself. The estimate (5) follows.

We next estimate the contribution from $|\rho| \leq c_1$. After the calculation

$$\frac{B'_\rho}{B_\rho}(z) = \frac{1}{z - \rho} - \frac{1}{z - 1/\bar{\rho}}, \quad (7)$$

one term of which matches up with a term on the right hand side of (3), we see that it suffices to show that

$$\sum_{|\rho| \leq c_1} \frac{1}{z - 1/\bar{\rho}} \ll_{c_1, c_2} \log M. \quad (8)$$

Since $|z| \leq c_2$, we have

$$\left| \frac{1}{z - 1/\bar{\rho}} \right| \leq \frac{1}{1/c_1 - c_2} \ll_{c_1, c_2} 1.$$

Therefore the required bound (8) follows from (2).

We turn finally to the contribution from $c_1 < |\rho| \leq 1$. For such an element ρ , and z with $|z| \leq c_2$, we have

$$z - \rho \asymp_{c_1, c_2} 1, \quad z - 1/\bar{\rho} \asymp_{c_1, c_2} 1,$$

where we recall that $A \asymp B$ means that $A \ll B \ll A$. By cross-multiplying in (7), it follows that

$$\frac{B'_\rho}{B_\rho}(z) \asymp_{c_1, c_2} \rho - 1/\bar{\rho}.$$

On the other hand, using the Taylor expansion of the logarithm, we have

$$\rho - 1/\bar{\rho} \asymp_{c_1} 1 - |\rho|^2 \asymp_{c_2} \log \frac{1}{|\rho|}.$$

We thereby obtain

$$\sum_{c_1 < |\rho| \leq 1} \frac{B'_\rho}{B_\rho}(z) \ll_{c_1, c_2} \sum_{c_1 < |\rho| \leq 1} \log \frac{1}{|\rho|} \leq \log M, \quad (9)$$

where in the final step we appealed to (4). \square

REFERENCES

- [1] E. C. Titchmarsh. *The theory of the Riemann zeta-function*. The Clarendon Press Oxford University Press, New York, second edition, 1986. Edited and with a preface by D. R. Heath-Brown.