HORIZONTAL p-ADIC L-FUNCTIONS

Abstract. Talk by Asbjørn Nordentoft at the seminar at Aarhus on 13 May 2024. Joint work with Daniel Kriz.

1. Introduction

Let E be an elliptic curve over $\mathbb{Q}$. Let L/K be a Galois extension of number fields. Then there is the following concept, due to Mazur and Rubin.

Definition 1. We say that E is *diophantine stable* (abbreviated DS) relative to L/K if

$$\operatorname{rank}_{\mathbb{Z}} E(L) = \operatorname{rank}_{\mathbb{Z}} E(K).$$

Question 2. How often is E diophantine stable for L/K?

Remark 3. The notion of diophantine stability has applications to Hilbert's tenth problem. This was part of the original motivation. From our point of view, it's just a natural question.

There is an analytic counterpart to this question. Let $F/\mathbb{Q}$ be an abelian extension of $\mathbb{Q}$. Assuming the BSD conjecture, the question of diophantine stability takes a different form:d

$$\begin{aligned}\operatorname{rank}_{\mathbb{Z}} E(F) &= \operatorname{ord}_{s=1} L(E/F, s) \\ &= \sum_{\chi \in \widehat{\operatorname{Gal}(F/\mathbb{Q})}} \operatorname{ord}_{s=1} L(E, \chi, s) \\ &= \operatorname{rank}_{\mathbb{Z}} E(\mathbb{Q}) \sum_{1 \neq \chi \in \widehat{\operatorname{Gal}(F/\mathbb{Q})}} \operatorname{ord}_{s=1} L(E, \chi, s).\end{aligned}$$

where

$$L(E, \chi, s) = \sum_{n \geq 1} a_E(n) \chi(n) n^{-s}.$$

The upshot is that diophantine stability for $F/\mathbb{Q}$ is related, under BSD, to understanding when $L(E, \chi, 1) \neq 0$ for $\chi \in \widehat{\operatorname{Gal}(F/\mathbb{Q})}$ with $\chi \neq 1$.

Question 4. How often is $L(E, \chi, 1) \neq 0$ for Dirichlet characters χ?

2. Vertical analysis

Let $F_n = \mathbb{Q}(\mu_p)$, $F_\infty = \cup_{n \geq 1} F_n$ (pth cyclotomic extension of $\mathbb{Q}$).

Theorem 5 (Mazur). *If E is good at p, then* $\operatorname{rank}_{\mathbb{Z}} E(F_\infty) < \infty$.

Remark 6. In our langauge of Diophantine stability, this means that if n is sufficiently large, then E is diophantine stable for F_∞/F_n.

Theorem 7 (Rohrlich). *We have $L(E, \chi, 1) \neq 0$ for all but finitely many χ, a Dirichlet character of p-power conductor. This is an exercise in class field theory.*

Remark 8. The results of Mazur and Rohrlich should be equivalent under BSD, but the speaker doesn't know of a direct way to go between them.

3. Horizontal analysis

Fix $d \geq 2$. Let $L/\mathbb{Q}$ be a cyclic extension, of order d. (The corresponding Dirichlet characters χ then have order d.)

Conjecture 9 (Goldfeld)**.** *Suppose $d = 2$, and let χ_D be a quadratic character of conductor D. Then*

$$\operatorname{ord}_{s=1} L(E, \chi_D, s) = \begin{cases} 0 & \textit{with probability } 50\%, \\ 1 & \textit{with probability } 50\%, \\ \geq 2 & \textit{with probability } 0\%. \end{cases}$$

Previous work when $d = 2$:

- Friedberg–Hoffstein, 1990's
- Murty–Murty, Ono–Skinner ($\gg X/\log X$);
- Smith–Kriz

How about when $d > 2$:

Conjecture 10 (David–Fearnley–Kisilenski)**.**

4. Results

We denote by $F_d(X)$ the set of relevant Dirichlet characters $\chi \pmod{D}$ with $D \leq X$ of order τ. There exists $c_d > 0$ such that

$$\# F_d(X) \sim c_d X (\log X)^{\sigma_0(d) - 2}.$$

Here $\sigma_0(d)$ denotes the number of divisors of d.

Theorem 11 (Kriz–N 23)**.** *Let $d \equiv 2 \ (4)$, $d > 2$. Then there exists $\alpha = \alpha(E, d) > 0$ such that*

$$\# \left\{ x \in F_d(X) : L(E, \chi, 1) \neq 0 \right\} \gg \frac{X}{(\log X)^{-\alpha}}.$$

Remark 12. Not previously knwon in this setting for $n = 6$.

We can also study a more general case, by imposing further assumptions.

Theorem 13. *We get the same conclusions in the following cases:*

(1) $L(E, 1) \neq 0$ and there exists $p \mid d$ such that $\overline{\rho}_{E,p}$ is irreducible,
(2) $2 \mid d$ and $\overline{\rho}_{E,2}$ is irreducible.

Remark 14. $d = 2^m, \overline{\rho_{E,2}}$. if and only if $E(\mathbb{Q})[2] = 0$ (satisfied for 100%) and

$$\alpha = \alpha(2^m, E) = \begin{cases} m/3 & \operatorname{im}(\bar{\rho}_{E,2}) \cong S_3, \\ 2m/3 & \cong \mathbb{Z}/3. \end{cases}$$

For $d = 2$, this improves on Ono from 2000's.

Theorem 15 (Simultaneous nonvanishing)**.** *For 100% of tuples of elliptic curves $E_1, \ldots, E_m$ such that $L(E_i, 1) \neq 0$, it holds that: for each d and $\alpha > 0$, we have*

$$\# \left\{ \chi \in F_d(X) : L(E_i, \chi, 1) \neq 0 \, \forall i \right\} \gg \frac{X}{(\log X)^{1-\alpha}}.$$

The proofs uses horizontal p-adic L-functions.

Definition 16. Fix a prime p. Let $E/\mathbb{Q}$ be an elliptic curve, of conductor N. We say that ℓ is a *Taylor–Wiles prime* for (E,p) if the following three conditions are met:

(1) $(\ell, N) = 1$
(2) $\ell \equiv 1\ (p)$
(3) $a_E(\ell) \not\equiv 2\ (p)$.

We say that p is *E-good* if Taylor–Wiles primes have positive density.

Remark 17. The last condition (3) is really one concerning the mod p Galois representation $\overline{\rho}_{E,p}$: if the latter is irreducible, then p is E-good.

Remark 18. If E is non-CM and $p \geq 13$, then p is E-good (Zywina).

Thus, let p be E-good. We consider $\mathcal{L} = (\ell_n)_n$, where $\ell_n \equiv 1\ (p)$ and ℓ_n is a Taylor–Wiles prime for all sufficiently large n. Set

$$m_n := v_p(\ell_n - 1) \geq 1.$$

Defint eh *horizontal Iwasawa algebra*

$$\Lambda^{\text{hor}} := \mathbb{Z}_p\left[\left[\prod_{n\in\mathbb{N}} \mathbb{Z}/p^{mn}\right]\right] := \varprojlim_{\substack{A\leq\mathbb{N}\\ \text{finite}}} \cong \operatorname{Hom}_{\text{cts}}\left(\mathcal{C}(\prod \mathbb{Z}/p^{m_n}, \mathbb{Z}_p), \mathbb{Z}_p\right).$$

Associated to the elliptic curve, we define (using modular symbols) a measure

$$\nu_E \in \Lambda$$

that interpolates the twistsby characters of p-power order, i.e.,

$$\chi : \prod_{n\in\mathbb{N}} (\mathbb{Z}/\ell_n)^* \to \bar{\mathbb{Q}}^\times.$$

¡++¿

Now if χ has p-power order, then it factors through some character

$$\tilde{\chi} : \prod_{n\in\mathbb{N}} \mathbb{Z}/p^{m_n} \to \bar{\mathbb{Q}}^\times.$$

The connection is that

$$\nu_E(\tilde{\chi}) = (\text{Euler factor}) \cdot \frac{L(E,\chi,1)\tau(\bar{\chi})}{\Omega_E}.$$

Pushing forward along

$$\prod \mathbb{Z}/p^{m_n} \twoheadrightarrow \prod \mathbb{Z}/p = (\mathbb{Z}/p)^\infty,$$

get

$$\overline{\nu_E} \in \mathbb{Z}_p[[(\mathbb{Z}/p)^\infty]] = \Lambda^{\text{diag}}.$$

Theorem 19 (K–N)**.** *Let $0 \neq \nu \in \Lambda^{\text{diag}}$. Then $\nu(x) \neq 0$ for a "positive proportion" of characters $\chi : (\mathbb{Z}/p)^\infty \to \bar{\mathbb{Q}}^\times$. There exists $I \subseteq \mathbb{N}$, a finite set, such that for all $\chi : (\mathbb{Z}/p)^\infty \to \bar{\mathbb{Q}}^\times$, there exists $\chi_I : \prod_{n\in I} \mathbb{Z}/p \to \bar{\mathbb{Q}}^\times$ such that $\nu(\chi\chi_I) \neq 0$.*

The *upshot* is that it suffices to prove $\overline{\nu_E} \neq 0$. This follows if we can show any of the following:

$$\begin{cases} & L(E,1) = \bar{\nu}_E(1) \neq 0 \\ p = 2 & \text{Friedberg–Hoffstein,} \\ \text{general } p & \text{use Kurihara's conjecture.} \end{cases}$$

References