## 1. Notation

We work with

$$G := \mathrm{PGL}_2(\mathbb{R}), \quad \Gamma := \mathrm{PGL}_2(\mathbb{Z}),$$
$$n(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N := \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \leq G,$$
$$a(y) := \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}.$$

## 2. Fourier expansions of automorphic functions

A nice enough function $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ has a Fourier expansion $f(x) = \sum_n e(nx)\hat{f}(n)$, with Fourier coefficients $\hat{f}(n) = \int_{\mathbb{R}/\mathbb{Z}} f(x)e(-nx)\,dx$.

Given an automorphic function

$$\varphi : \mathrm{PGL}_2(\mathbb{Z})\backslash \mathrm{PGL}_2(\mathbb{R}) \to \mathbb{C},$$

we may view the function $\varphi(\bullet g) : N(\mathbb{Z})\backslash N(\mathbb{R}) \to \mathbb{C}$ as an a function on $\mathbb{R}/\mathbb{Z}$, so it admits a Fourier expansion:

$$\varphi(n(x)g) = \sum_{n \in \mathbb{Z}} e(nx) W_{\varphi,n}(x),$$

where

$$W_{\varphi,n}(x) := \int_{\mathbb{R}/\mathbb{Z}} \varphi(n(u)g)e(-nu)\,du.$$

If $\varphi$ is a Hecke eigenform, then we can write

$$W_{\varphi,n}(g) = \frac{\lambda(n)}{|n|^{1/2}} W_{\varphi,1}(a(n)g), \tag{1}$$

where $\lambda(n)$ is the Hecke eigenvalue. Thus

$$\varphi(g) = \sum_{n \in \mathbb{Z}_{\neq 0}} \frac{\lambda(n)}{|n|^{1/2}} W_\varphi(a(n)g). \tag{2}$$

**Example 1.** Classical holomorphic modular form $f$ of weight $k$.

$$f(x+iy) = \sum_{n \geq 1} a_n e(nz), \quad e(z) := e^{2\pi i z}.$$
$$y^{k/2} f(x+iy) = \sum_{n \geq 1} a_n n^{-k/2} W_k(ny) e(nx), \quad W_k(y) := y^{k/2} e^{-2\pi y},$$
$$a_n n^{-k/2} = \frac{\lambda(n)}{|n|^{1/2}}.$$

If we define $\varphi : \mathrm{PGL}_2(\mathbb{Z})\backslash \mathrm{PGL}_2(\mathbb{R}) \to \mathbb{C}$ by $\varphi(g) := f|_k g(i)$, where in general

$$f|_k g(z) := \frac{(\det g)^{k/2}}{(cz+d)^k} f\left(\frac{az+b}{cz+d}\right), \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then, since $n(x)a(y) \cdot i = x + iy$, we have

$$y^{k/2} f(x+iy) = \varphi(n(x)a(y)),$$
$$W_k(y) = W_{\varphi,1}(a(y)).$$

**Remark 2.** The idea of the proof of (1) is that by "uniqueness of Whittaker functionals", we know that the two quantities $W_{\varphi,n}$ and $W_{\varphi,1}(a(n)\bullet)$ are proportional. On the other hand, by studying how Hecke operators affect Fourier (or Whittaker) expansions, we see that the former is $\frac{\lambda(n)}{|n|^{1/2}}$ times the latter. If the latter were to vanish identically, then it would follow that the former vanishes identically, and so $\varphi$ would vanish. Therefore the latter does not vanish identically, and we get the desired relation. For the case of (2) relevant for Example 1, see [3, Chapter VII, §5.4, Thm 7].

## 3. Modularity

We restrict now to the case that $\varphi$ lies inside some irreducible subrepresentation $\pi$ of the space of cusp forms on $\Gamma\backslash G$, consisting of Hecke eigenfunctions with eigenvalues $\lambda(n)$. Thus $\lambda(n)$ depends only upon $\pi$, not the choice of $\varphi \in \pi$.

The theory of the Kirillov model says that for any function $h$ on $\mathbb{R}^\times$, there is a unique $\varphi \in \pi$ so that

$$W_\varphi(a(\bullet)) := [y \mapsto W_\varphi(a(y))] = h.$$

This is useful, because we then can write

$$\varphi(1) = \sum_{n \neq 0} \frac{\lambda(n)}{|n|^{1/2}} W_\varphi(a(n)).$$

On the other hand, $\varphi$ is invariant under $\Gamma$, so for any $\gamma \in \Gamma$ (e.g., $\gamma = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$), we may write $\varphi(1) = \varphi(\gamma)$, hence

$$\sum_{n \neq 0} \frac{\lambda(n)}{|n|^{1/2}} W_\varphi(a(n)) = \sum_{n \neq 0} \frac{\lambda(n)}{|n|^{1/2}} W_\varphi(a(n)\gamma).$$

**Remark 3.** Assuming $\pi$ is tempered, any Whittaker function enjoys the decay estimates

$$W_\varphi(a(y)) \ll_\varphi \min(y^{1/2+\varepsilon}, e^{-2\pi y}).$$

Then the problem arises: given a function $h : \mathbb{R}^\times \to \mathbb{C}$, having chosen $\varphi$ so that $W_\varphi(a(\bullet)) = h$, can we understand the new function

$$h_\gamma := W_\varphi(a(\bullet)\gamma)?$$

This is typically understood using the local functional equation and/or Bessel transform.

A very convenient case is when we start with some fixed $\varphi$, and just use that for all $g \in G$ and $\gamma \in \Gamma$, we have $\varphi(\gamma g) = \varphi(g)$, hence

$$\sum_{n \neq 0} \frac{\lambda(n)}{|n|^{1/2}} W_\varphi(a(n)g) = \sum_{n \neq 0} \frac{\lambda(n)}{|n|^{1/2}} W_\varphi(a(n)\gamma g). \tag{3}$$

We can just pretend at first approximation that $W_\varphi(a(\bullet)k)$ looks like some fixed bump function for all $k$ in some fixed compact subset of $G$. It's useful to recall that any element of $G$ can be written using the Cartan decomposition as $k_1 a(y) k_2$, or using the Iwasawa decomposition as $n(x)a(y)k$.

## 4. Case studies

Here we give some examples of how to apply (3). We'll focus on cases where $\varphi$ is *fixed*. This case is the relevant one if one would like to try (as we did in our internal seminar) understanding papers like [1] from a geometric perspective; that's the motivation for this whole discussion.

4.1. **Varying archimedean frequency.** Let's say we're given $h \in C_c^\infty(\mathbb{R}^\times)$ and we want to understand what the following sums look like:

$$\sum_{n \neq 0} \frac{\lambda(n)}{|n|^{1/2}} h(n/Y) e(\xi n). \tag{4}$$

To do so, we choose $\varphi$ so that $W_\varphi(a(\bullet)) = h$. Then we observe that the above sum is exactly

$$\varphi(n(\xi)a(1/Y)).$$

Note that the image of the argument of $\varphi$ in $\mathbb{H}$ is $\xi + i/Y$.

One simple thing one can do is take $\xi = 0$ and use this to see that we get massive decay when $Y \to \infty$. How to see that? Well, the point in question, $i/Y$, is tending to zero. So we should apply some element $\gamma \in \Gamma$ that maps 0 to $\infty$. We can take

$$\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then we get

$$\varphi(a(1/Y)) = \varphi(\gamma a(1/Y)) = \varphi(a(Y)\gamma).$$

By passing to Fourier expansions, we deduce that

$$\sum_{n \neq 0} \frac{\lambda(n)}{|n|^{1/2}} h(n/Y) = \sum_{n \neq 0} \frac{\lambda(n)}{|n|^{1/2}} W_\varphi(a(nY)\gamma).$$

Since $\gamma$ lies in some fixed compact collection, we can think of $W_\varphi(a(\bullet)\gamma)$ as roughly a fixed bump function (more precisely, it decays like $y^{1/2}$ near zero and exponentially near $\infty$, but whatever). In particular, the above sum is essentially restricted to $nY \ll 1$. But if $Y \to \infty$ and $n$ is a nonzero integer, then this last condition is never satisfied, and so the right hand side is vanishingly small.

4.2. **Varying non-archimedean frequency.** Another basic interesting example: for a natural number $q$, suppose we want to understand (4) in the special case that $\xi = a/q$, where $a$ is an integer coprime to $q$. That is to say, we care about

$$\sum_{n \neq 0} \frac{\lambda(n)}{|n|^{1/2}} h(n/Y) e_q(an),$$

where $e_q : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}^\times$, $e_q(x) := e(x/q)$. Welp, with the same choice of $\varphi$ as above, the above sum is

$$\varphi(n(a/q)a(1/Y)).$$

The image in $\mathbb{H}$ is $a/q + i/Y$.

Suppose we're in the regime where $Y$ is quite large (we can figure out later exactly how large we need it to be). Then the argument of $\varphi$ is tending to the cusp $a/q$. We should choose $\gamma$ that maps this cusp to $\infty$. We can take

$$\gamma = \begin{pmatrix} -\bar{a} & * \\ q & -a \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Then, we try using that

$$\varphi(n(a/q)a(1/Y)) = \varphi(\gamma n(a/q)a(1/Y)).$$

This begs the question of understanding

$$\gamma n(a/q)a(1/Y).$$

Welp, if we have

$$\gamma\left(\frac{a}{q} + \frac{i}{Y}\right) = x' + iy',$$

then we know that

$$\gamma n(a/q)a(1/Y) = n(x')a(y')k$$

for some $k \in \mathrm{SO}(2)$.

We might try factoring

$$\gamma = n(-\bar{a}/q)\begin{pmatrix} 0 & -1/q \\ q & 0 \end{pmatrix} n(-a/q),$$

then

$$\gamma n(a/q)a(1/Y) = n(-\bar{a}/q)\begin{pmatrix} 0 & -1/q \\ q/Y & 0 \end{pmatrix} = n(-\bar{a}/q)a(Y/q^2)w,$$

where

$$w := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This tells us that

$$\sum_{n \neq 0} \frac{\lambda(n)}{|n|^{1/2}} h(n/Y) e_q(an) = \sum_{n \neq 0} \frac{\lambda(n)}{|n|^{1/2}} e_q(-\bar{a}n) W_\varphi(a(nY/q^2)w).$$

The right hand side is essentially restricted to $n/(q^2/Y) \ll 1$. So, for example:

- If $Y \ggg q^2$, then the dual sum (the RHS of the above) is of length $q^2/Y \lll 1$, so it is negligible.
- If $Y \approx q$, then the original sum and the dual sum are both of length roughly $q$, so we "gain nothing" (at least as far as trivial bounds are concerned) by dualizing (i.e., applying the above formula).

## 5. Varying archimedean frequency, but using Dirichlet approximation

A third example along such lines, and maybe the most relevant: take a general $\xi$ as in (4), but let's do Dirichlet approximation on it first to approximate it by some rational $a/q$. We should get, for some $q \leq Q$, that

$$|\xi - a/q| \leq \frac{1}{qQ}.$$

Write $\xi = a/q + \delta$, where $|\delta| \leq 1/(qQ)$. The sum of interest may then be written

$$\varphi(n(a/q)n(\delta)a(1/Y)) = \varphi(n(a/q)a(1/Y)) = (n(Y\delta)\varphi)(n(a/q)a(1/Y)).$$

(Here we are using the notation $(g\varphi)(h) := \varphi(hg)$.)

Now we apply the above analysis, but with $\varphi$ replaced by its right translate $n(Y\delta)\varphi$. This gives that the sum of interest is

$$\sum_{n \neq 0} \frac{\lambda(n)}{|n|^{1/2}} e_q(-\bar{a}n) W_\varphi(a(nY/q^2)wn(Y\delta)).$$

We are led to the problem of understanding the above argument. If it happens that $Y\delta \ll 1$, then there is no change from before. We know that $\delta \ll 1/qQ$, so this happens if $Y \ll qQ$.

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$wn(-1/x) = n(x)a(x^2)wn(x)w.$$

This is useful in the limit as $x \to 0$.

$$W_\varphi(a(y)w) = \int_{t \in \mathbb{R}^\times} W_\varphi(a(t))J(ty)\, d^\times t.$$

## References


[1] Keshav Aggarwal. Weyl bound for GL(2) in $t$-aspect via a simple delta method. *J. Number Theory*, 208:72–100, 2020.

[2] E. Kowalski, P. Michel, and J. VanderKam. Rankin-Selberg $L$-functions in the level aspect. *Duke Math. J.*, 114(1):123–191, 2002.

[3] J.-P. Serre. *A course in arithmetic*. Springer-Verlag, New York, 1973. Translated from the French, Graduate Texts in Mathematics, No. 7.