1. Holomorphic continuation

Theorem 1 (Identity principle for holomorphic functions)**.** *Let $U \subset \mathbb{C}$ be a connected open set. Let $f, g : U \to \mathbb{C}$ be holomorphic functions. If $f = g$ on a set with a limit point in U, then $f = g$ on all of U.*

Corollary 2. *Let $U \subset \Omega \subseteq \mathbb{C}$ be open subsets, with U nonempty and Ω connected. Let $f : U \to \mathbb{C}$ be a holomorphic function. Then there is at most one extension of f to a holomorphic function $\Omega \to \mathbb{C}$.*

2. Cauchy's integral formula

Theorem 3. *Let $f : U \to \mathbb{C}$ be a holomorphic function defined on an open subset U. Let γ be a closed rectifiable curve in U. Then $\int_\gamma f(z)\,dz = 0$.*

Theorem 4. *Let $0 \le a < b \le \infty$. Let $f(z)$ be a holomorphic function on the annulus $\{z \in \mathbb{C} : a < |z| < b\}$ given by a convergent Laurent series*

$$f(z) = \sum_{n \in \mathbb{Z}} c_n z^n.$$

(1) For any $r \in (a, b)$ and $n \in \mathbb{Z}$, we have

$$\begin{aligned} c_n &= \oint_{|z|=r} \frac{f(z)}{z^n} \frac{dz}{2\pi i z} \\ &= \frac{1}{2\pi r^n} \int_{\theta=0}^{2\pi} f(re^{i\theta}) e^{-in\theta}\,d\theta. \end{aligned}$$

(2) For each compact subset E of (a, b), there exists $M \ge 0$ so that for all $r \in E$, we have

$$\sum_{n \in \mathbb{Z}} |c_n| r^n \le M. \tag{1}$$

Theorem 5. *Let U be an open subset of $\mathbb{C}$, let $f : U \to \mathbb{C}$ be meromorphic. Let γ be a smooth closed curve in U, oriented counterclockwise, that does not pass through any pole of f. Then*

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{\substack{z \in \text{interior}(\gamma) \\ \textit{pole of } f}} \operatorname{res}_z(f).$$

Remark 6. Let $0 < r < R$. Let f be a meromorphic function on a neighborhood of the annulus $\{z : r < |z| < R\}$ that has no poles on either of the circles $|z| = r, R$. Then

$$\oint_{|z|=R} f(z)\,dz = \oint_{|z|=r} f(z)\,dz + 2\pi i \sum_{\substack{r<|z|<R \\ \text{pole of } f}} \operatorname{res}_z(f).$$

3. Holomorphy of limits and series

Theorem 7. *Let U be an open subset of the complex plane. Let f_n be a sequence of holomorphic functions on U.*

(1) Suppose that the sequence f_n converges pointwise to some function f, uniformly on compact subsets of U. Then f is holomorphic.

(2) Suppose that the partial sums $\sum_{n \le N} f_n$ converge pointwise to some function f, uniformly on compact subsets of U. Then the sum $\sum_n f_n$ is holomorphic.

References