1. ANNE-MARIA ERNVALL-HYTÖNEN, *Lattices and modular forms in coset coding*

Wyner (1975). Wiretap channel (Alice, Bob, Eve). Gaussian noise with variance σ^2 (between Alice and Bob) and σ_e^2 (for Eve). We assume $\sigma_e > \sigma$. We would like to do this whole scheme in such a way that we can use the noise so that Bob can still receive the image, but Eve cannot. We aim to do this with lattices $\Lambda \geq \Lambda_e$, where each coset corresponds to a code-letter.

Belfiore and Oggier (2010) (maybe [14]?): secrecy gain. Transmit codeword $x \in \mathbb{R}^n$.

$$\frac{1}{(\sigma_\Lambda \sqrt{2\pi})^n} \int_{V_{\Lambda(x)}} e^{-\|y-x\|^2/2\sigma^2} \, dy.$$

$$\frac{1}{(\sigma_\Lambda \sqrt{2\pi})^n} \sum_{t \in \Lambda_e \cap R} \int_{V_{\Lambda(x+t)}} e^{-\|y-x\|^2/2\sigma^2} \, dy.$$

End up trying to minimize

$$\theta_{\Lambda_e} \left(-\frac{1}{2\pi i \sigma_e^2} \right).$$

Secrecy function

$$\Xi(y) := \frac{\theta_{\lambda \mathbb{Z}^n}(yi)}{\theta_\Lambda(yi)}.$$

Belfiore and Sole: For unimodular Λ, maximum at $y = 1$. Even unimodular: polynomials in

$$E_4 = \frac{1}{2} \left(\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8 \right), \qquad \Lambda = \frac{1}{256} \vartheta_2^8 \vartheta_3^8 \vartheta_4^8.$$

Inverse of $\Xi(y)$ a polynomial in $\frac{\vartheta_4^4 \vartheta_2^4}{\vartheta_3^8}$.

(Maybe [8] is a reference.)

ℓ-modular: $1/\sqrt{\ell}$, $\mathbb{Z} \oplus \sqrt{2}\mathbb{Z} \oplus 2\mathbb{Z}$.

The connection to deeper mathematics is, what kinds of representations as polynomials do you have for various theta functions?

2. JOLANTA MARZEC-BALLESTEROS, *Doubling method for self-dual linear codes*

Garrett, Piatetski-Shapiro–Rallis (1980's). Integral of cusp form against restriction of Siegel-type Eisenstein series equals L-function attached to cusp form times cusp form or Eisenstein series attached to cusp form:

$$\left\langle E \left(\begin{pmatrix} g & \\ & g' \end{pmatrix}, s \right), f(g) \right\rangle = L(f,s) f(g').$$

Done for G symplectic, orthogonal, unitary over global field, also for congreunce subgroups.

Let's start with an overview. Let f be a cusp form on G, and H a subgroup for which $G \times G \hookrightarrow H$. Then form an Eisenstein series on H

$$E(h,s) = \sum_{\gamma \in P \backslash H} \phi(\gamma h, s).$$

Restrict to $h = \text{diag}(g, g')$ and take inner product with $F(g)$. This leads to an unfolding involving a sum over $\gamma \in P \backslash H / (G \times G)$. In favorable cases, only one

representative γ_0 contributes, leaving us with

$$\sum_{(k,k')\in G\times G} \left\langle \phi\left(\gamma_0\begin{pmatrix} kg & \\ & k'g' \end{pmatrix}\right), s\right\rangle, f(g)\right\rangle$$

$$= \sum_{\beta\in\gamma_0(G\times 1)} \psi(f)f|_\beta(g')$$

$$= L(f,s)f(g').$$

We would like to do something similar, but now over finite fields.

A *linear code of length* $2n$ over a finite field $\mathbb{F}$ is a linear subspace $C\subseteq\mathbb{F}^{2n}$. We denote by $\langle,\rangle : C\times C\to\mathbb{F}$ the Euclidean inner product. We say that C is *self-dual* if

$$C = C^\perp := \left\{v\in\mathbb{F}^{2n} : \langle v,C\rangle = 0\right\}.$$

Then (the length $2n$ is even and) $\dim C = n$.

The *weight* of a codeword $c = (c_1,\ldots,c_{2n})\in C$ is

$$\operatorname{wt} c = \#\left\{i\in\{1,\ldots,2n\} : c_i\neq 0\right\}.$$

Weight enumerators are certain homogeneous polynomials of degree $2n$ in variables from the set $V = \{x_\alpha : \alpha\in\mathbb{F}^g\}$, where $g\in\mathbb{N}$ is the genus.

The *genus one weight enumerator* of a code $C\subseteq\mathbb{F}_2^{2n}$ is a polynomial

$$W_1(C,(x_0,x_1)) = \sum_{c\in C} x_0^{2n-\operatorname{wt} c}x_1^{\operatorname{wt} c}.$$

The *genus g weight enumerator* of a code $C\subseteq\mathbb{F}^{2n}$ is a polynomial

$$W_g(C,x) = \sum_{(c^1,\ldots,c^g)\in C^g}\prod_{\alpha\in\mathbb{F}^g} x_\alpha^{w_\alpha(c^1,\ldots,c^g)}$$

of degree $2n$, where $x = (x_\alpha)_{\alpha\in\mathbb{F}^g}$ and

$$w_\alpha(c^1,\ldots,c^g) = \#\left\{\text{rows } r \text{ in } (c_i^j)_{i=1..2n}^{j=1..g} : r = \alpha\right\}.$$

As an example, we give a basis for a Hamming code H_8 5nof weight 8, the span over $\mathbb{F}_2$ inside $\mathbb{F}_2^8$ of the vectors

$$\begin{pmatrix}1\\0\\0\\1\\0\\1\\1\\0\end{pmatrix}\quad\begin{pmatrix}0\\1\\0\\1\\0\\1\\0\\1\end{pmatrix}\quad\begin{pmatrix}0\\0\\1\\1\\0\\0\\1\\1\end{pmatrix}\quad\begin{pmatrix}0\\0\\0\\0\\1\\1\\1\\1\end{pmatrix}.$$

Then

$$W_2(H_8,(x_{00},x_{01},x_{10},x_{11})) = \sum_{\alpha\in\mathbb{F}_2^2} x_\alpha^8 + 14\sum_{\substack{\alpha_1,\alpha_2\in\mathbb{F}_2^2\\\alpha_1<\alpha_2}} x_{\alpha_1}^4 x_{\alpha_2}^4 + 168 x_{00}^2 x_{01}^2 x_{10}^2 x_{11}^2$$

$$= (8) + 14(4,4) + 168(2,2,2,2).$$

In general,

$$W_g(C) = \sum_A b_A\cdot(A),$$

where
$$A \in \left\{ (a_0, \ldots, a_{2^g-1}) : \text{admissible tuples}, \sum_{i=0}^{2^g-1} a_i = 2n \right\}.$$

Some analogies with modular forms:

- $W_g(C)$ is like a modular form f of genus g,
- $\sum_A b_A \cdot (A)$ is like a Fourier expansion,
- $(2n)$ is like a constant term $a(0)$.

EXamples of cusp forms:

- $W_1(G_{24}) - W_1(H_8 \times H_8 \times H_8) = -42(20,4) + 168(16,8) - 252(12,12)$ is a cusp form of genus one.
- $W_3(E_{16}) - W_3(H_8 \times H_8) = -2688(9,1,1,1,1,1,1,1) + \cdots$ is a cusp form of genus 3.

Theorem 1 (Runge, 1996; Nebe, Rains, Sloane, 2006). *We have*
$$\langle W_g(C) : C \text{ self-dual, over } \mathbb{F} \rangle = (\mathbb{C}[x_\alpha : \alpha \in \mathbb{F}^g])^{\mathcal{C}_g},$$

where
$$\mathcal{C}_g := \langle m_r, d_\phi, h_{\iota, u_\iota, v_\iota} : r \in \mathrm{GL}_g(\mathbb{F}), \phi, \iota \rangle$$

with
$$m_r : x_\alpha \mapsto x_{r\alpha},$$
$$d_\phi : x_\alpha \mapsto e^{2\pi i \phi(\alpha)} x_\alpha,$$
$$h_{\iota, u_\iota, v_\iota} : x_\alpha \mapsto (\# \iota \mathbb{F}^g)^{-1/2} \sum_{w \in \iota \mathbb{F}^g} e^{\frac{2\pi i}{p} \langle w, v_\iota \alpha \rangle} x_w + \cdots.$$

Consider the mean polynomial ("Siegel-type Eisenstein series")
$$M_{2g}((2n)) = \sum_{\gamma \in P_{2g} \backslash \mathcal{C}_{2g}} (2n)^\gamma$$
$$= \sum_{\gamma \in P_{2g} \backslash \mathcal{C}_{2g}} \sum_{\alpha \in \mathbb{F}^{2g}} ((x_\alpha)^\gamma)^{2n} = \text{const} \sum_{\substack{C \subset \mathbb{F}^{2n} \\ \text{fixed type}}} W_{2g}(C, x),$$

and an inner product defined on monomials.

What we prove with Bourganis in 2024 is the following:

Theorem 2. *Let $\mathcal{T}$ be a family of self-dual codes of length $2n$ over a field $\mathbb{F}$. Assume either that $\mathbb{F}$ has odd characteristic or is equal to $\mathbb{F}_2$. Let $C \in \mathcal{T}$ be doubly-even, and fix $g \in \mathbb{N}$. Then there exists an (explicit) constant C such that for a cusp form $f \in \mathcal{T}$ of genus r, with $\deg f = 2n$, we have*
$$\langle M_{2g}((2n))(xy), f(x) \rangle = \begin{cases} 0 & \text{if } r < g, \\ C \cdot f(y) & \text{if } r = g \end{cases}.$$

3. PETRU CONSTANTINESCU, *Non-vanishing of geodesic periods of automorphic forms*

Preprint: [6], joint with Asbhjørn Nordentoft.
Class groups:

- $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, $K = \mathbb{Q}(\sqrt{D})$,
- Cl_K class group, $h(D) = h_K = |\mathrm{Cl}_K|$ class number,

- $\mathcal{Q}_D$: set of primitive integral binary quadratic forms of discriminant D,
$$\mathcal{Q}_D = \left\{ Q(x,y) = ax^2 + bxy + cy^2 : (a,b,c) = 1,\, b^2 - 4ac = D \right\}.$$
- Gauss: $\Gamma \circlearrowright \mathcal{Q}_D$,
$$(Q.\gamma)\begin{pmatrix} x \\ y \end{pmatrix} = Q\left(\gamma \begin{pmatrix} x \\ y \end{pmatrix} \right),$$
- isomorphism
$$\mathrm{Cl}_K \xrightarrow{\cong} \Gamma \backslash \mathcal{Q}_D$$
$$A \mapsto [a,b,c].$$

Heegner points and closed geodesics:

- $D < 0$: $A \in \mathrm{Cl}_K \rightsquigarrow$ Heegner point $z_A \in \Gamma \backslash \mathbb{H}$,
$$[a,b,c] \rightsquigarrow \frac{-b - i\sqrt{|D|}}{2a}.$$
$$h(D) = |\mathrm{Cl}_K| = |D|^{1/2 + o(1)}.$$
- $D > 0$: $A \in \mathrm{Cl}_K^+ \rightsquigarrow$ closed geodesic $C_A \subset \Gamma \backslash \mathbb{H}$, $[a,b,c] \rightsquigarrow$ semicircle with endpoints $\frac{-b \pm \sqrt{D}}{2a}$.
$$h(D) \log \varepsilon_D = D^{1/2 + o(1)},$$
$$I(C_A) = 2 \log \varepsilon_D.$$

Theorem 3 (Duke '88)**.** *Fix $\Omega \subset \mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}$.*

- *$D < 0$: equidistribution of z_A, $A \in \mathrm{Cl}_{\mathbb{Q}(\sqrt{D})}$.*
- *$D > 0$: similar for closed geodesics.*

Waldspurger formulas. Let f be a nonzero Maass form. Our goal is to study closed geodesic periods $\int_{C_A} f(z) \frac{|dz|}{y}$.

$\chi \in \widehat{\mathrm{Cl}_K} \rightsquigarrow \theta_\chi$, the associated theta series (weight one on $\Gamma_0(D)$, nebentypus χ_D).

Theorem 4 (Waldspurger/Zhang/Popa)**.** *Let D be a fundamental discriminant. For $D < 0$,*
$$L(f \times \theta_\chi, \tfrac{1}{2}) = \frac{C_f}{D^{1/2}} \left| \sum_{A \in \mathrm{Cl}_K} \chi(A) f(z_A) \right|^2.$$

For $D > 0$,
$$L(f \times \theta_\chi, \tfrac{1}{2}) = \frac{C_f}{D^{1/2}} \left| \sum_{A \in \mathrm{Cl}_K^+} \chi(A) \int_{C_A} f(z) \frac{|dz|}{y} \right|^2.$$

Theorem 5 (Michel–Venkatesh '05)**.** *Let $\delta = 1/2700$. For $D < 0$:*
$$\left| \left\{ \chi \in \widehat{\mathrm{Cl}_K} : L(f \times \theta_\chi, \tfrac{1}{2}) \neq 0 \right\} \right| \gg D^\delta.$$

Sketch of proof. By orthogonality of characters,
$$\frac{1}{h(D)} \sum_{\chi \in \widehat{\mathrm{Cl}_K}} L(f \times \theta_\chi, \tfrac{1}{2}) = \frac{c_f}{D^{1/2}} \sum_{A \in \mathrm{Cl}_K} |\cdots|^2.$$

This converges by Duke's equidistribution theorem. The conclusion then follows from subconvexity for Rankin–Selberg L-functions, due to Harcos–Michel. $\qquad\square$

Same proof does not work for geodesics, cannot apply equidistribution and relate to subconvexity (square is outside integral).

Question 6 (Michel, Oberwolfach 2020)**.** Let K be a real quadratic field of discriminant $D > 0$, and assume that $h_K \gg D^\delta$. Does there exist $A \in \mathrm{Cl}_K^+$ such that
$$\int_{C_A} f \frac{|dz|}{y} \neq 0?$$
Equivalently, does there exist $\chi \in \widehat{\mathrm{Cl}_K^+}$ such that $L(f \times \theta_\chi, \frac{1}{2}) \neq 0$?

The prime geodesic theorem. Let $D > 0, A \in \mathrm{Cl}_K^+$. This gives rise to a closed geodesic C_A, with $I(C_A) = 2\log \varepsilon_D$. Sound–Young have the best estimate for their count.

Theorem 7 (C–Nordentoft 2024)**.** *Let* f *be a nonzero Maass form for* $\mathrm{SL}_2(\mathbb{Z})$. *Then*
$$\#\left\{ C \in \mathcal{C}(X) : \int_C f(z) \frac{|dz|}{y} = 0 \right\} \ll \frac{X}{(\log X)^{5/4}}.$$

Remark 8. We also obtain 100% non-vanishing for periods of weight k holomorphic cusp forms, for any Fuchsian group Γ.

Theorem 9 (C–Nordentoft 2024)**.** *For a positive proportion of positive discriminants* $D > 0$ *with* $\varepsilon_\Delta \leq X$, *we get that there exists* $\chi \in \widehat{\mathrm{Cl}_{\mathbb{Q}(\sqrt{D})}}$ *with* $L(f \times \theta_\chi, \frac{1}{2} \neq 0)$.

We construct a bipartite graph on X_N (double cosets in $\Gamma_\infty \backslash \Gamma / \Gamma_\infty$ with $c \leq N$) times Y_N (conjugacy classes with trace bounded in magnitude by N). This graph relates closed geodesic and vertical geodesics.

4. AN EXCISED ORTHOGONAL MODEL FOR FAMILIES OF CUSP FORMS

Talk by Zoe Batterman (Abstract), Akash Narayanan, Christopher Yao. Joint with Owen Barrett, Aditya Jambhale, and Kishan Sharma. Preprint: [3].

Conjecture (Montgomery–Dyson, 1970's): zeros of of zeta vs. GUE.

2005: S.J. Miller noticed a repulsion of the lowest-lying zeros near the central point of a family of even twists of a fixed elliptic curve L-function with finite conductor.

2011: Duenez, Huynh, Keating, Miller, Snaith: proposed an excised orthogonal model to capture the behavior of this repulsion.

Question 10. How accurately do egienvalues of random matrices from classical compact groups model the lowest-lying zeros of families of L-functions associated to a cuspidal newform?

Let
$$S_k^{\mathrm{new}}(M, \chi_f) \ni f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z},$$
$\lambda_f(n) = a_f(n)/n^{(k-1)/2}$.
$$L(s, f) = \sum_{n \geq 1} \lambda_f(n) n^{-s},$$
Various specific families of twists $L(f \otimes \psi_d, s)$, match with classical compact groups:

- principal character, even twists vs. SO(even)
- principal character, odd twists vs. SO(odd)
- non-principal character, self-dual vs. Sp
- generic vs. U

Pictures.

5. On an extension of the Rohrlich-Jensen formula, *Lejla Smajlović*

Joint work with James Cogdell, Jay Jorgensen. Preprint: [5].

What is a Poisson–Jensen formula? We will view it as a way to characterize meromorphic functions in terms of their divisors. Some notation:

- $D_R = \{z = x + iy \in \mathbb{C} : |z| < R\}$
- F: a non-constant meromorphic function on $\overline{D_R}$,

$$F(z) = c_F z^m + O(z^{m+1}), \quad z \to 0.$$

Then

$$\int_0^{2\pi} \log|F(Re^{i\theta})| \, \frac{d\theta}{2\pi} + \sum_{D_R} \cdots = F(0).$$

Rohrlich, 1980's: a *modular* generalization, characterizing modular forms via divisors. Given f, a meromorphic function on $\mathbb{H}$ that is invariant by $\mathrm{PSL}(2, \mathbb{Z})$. Assume f is holomorphic at the cusp and that the Fourier expansion of f at ∞ has constant term equal to one. Then

$$\int_{\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}} \log|f(z)| \, \frac{d\mu(z)}{2\pi} + \sum_{w \in \mathcal{F}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} P(w) = 0.$$

Here

- $\mathrm{ord}_w(f)$ is the order of f at w as a meromorphic function,
- $\mathrm{ord}(w)$ denotes the order of the *point* w with respect to the action of $\mathrm{PSL}(2, \mathbb{Z})$ on $\mathbb{H}$, and
- $P(w) = \log\left(|\eta(w)|^4 \cdot \Im(w)\right)$ is the Kronecker limit function associated to the parabolic Eisenstein series on $\mathrm{PSL}(2, \mathbb{Z})$. This is the function appearing as the next-order term in the expansion of the Eisenstein series as $s \to 1$.

Another way to interpret this formula is as follows. We have

$$\langle 1, \log|f(z)| \rangle = \lim_{Y \to \infty} \int_{\mathcal{F}(Y)} 1 \cdot \log|f(z)| \, d\mu(z),$$

hence the formula reads

$$\langle 1, \log|f(z)| \rangle = -2\pi \sum_{w \in \mathcal{F}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} P(w).$$

Here $\log|f(z)|$ can be replaced by

$$\log\|f(z)\| = \log\left(\Im z^k |f(z)|\right)$$

for weight $2k$ meromorphic modular forms. Generalizations:

- to other Fuchsian groups of the first kind, by Rohrlich.
- to hyperbolic 3-space, by Herrero, Imamoglu, von Pippich, Toth.

Further modular generalization by Bringman and Kane. Keep the modular group setting, but evaluate more general inner products.

- $j(z) = q^{-1} + 744 + O(q)$: Hauptmodul

- $j_1(z) := j(z) - 744$
- $j_n(z) := j_1|T_n(z)$, for $n \geq 2$
- f: weight $2k$ meromorphic modular form with respect to $\mathrm{PSL}(2,\mathbb{Z})$, normalized so that $f(z) = 1 + \mathrm{O}(q)$

They evaluated the regularized scalar product in terms of the divisor of f, proving that

$$\left\langle j_n(z), \log\left((\Im(z))^k |f(z)|\right)\right\rangle = -2\pi \sum_{w \in \mathcal{F}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} \mathbf{j}_n(w) + \frac{k}{6} c_n,$$

where $\mathbf{j}_n$ is characterized in terms of differential operators by Bringman and Kane; application of our results yields a different expression for the same function, namely

$$\mathbf{j}_n(w) = 2\pi\sqrt{n}\partial_s F_{-n}^{\mathrm{PSL}(2,\mathbb{Z})}(w,s)|_{s=1} - 24\sigma(n)P(w).$$

What is our main goal?

- To extend the point of view that the Rohrlich–Jensen formula is the evaluation of a particular type of inner product.
- To prove the extension of this formula in the setting of an arbitrary, not necessarily arithmetic, Fuchsian group of the first kind with one cusp.

We start with $j_n(z) = j_1|T_n(z)$, which is the unique (up to constants) holomorphic function that is $\mathrm{PSL}(2,\mathbb{Z})$-invariant on $\mathbb{H}$ and whose expansion near ∞ is $q^{-n} + o(q^{-1})$.

These properties hold for the special value $s = 1$ of the Niebur–Poincaré series $F_{-n}^\Gamma(z,s)$, defined for any Fuchsian group Γ of the first kind with oen cusp by

$$F_m^\Gamma(z,s) = \sum_{\gamma_\infty \backslash \Gamma} e\left(m\Re(\gamma z)\right) (\Im(\gamma z))^{1/2} I_{s-1/2}\left(2\pi|m|\Im(\gamma z)\right),$$

for $m \neq 0$.

It is an eigenfucntion of the hyperbolic Laplacian, and may be expressed in terms of j_m.

The term $\log\left(\|f(z)\|\right)$ is a bit complicated, involving findings by Jorgensen, von Pippich and the speaker, plus some additional work done in our paper.

Proposition 11. *Let Γ be a cofinite Fuchsian group with one cusp at ∞ with identity as its scaling matrix. Let $2k \geq 0$ be an even integer, and let f be aw eight $2k$ meromorphic form which is Γ-invariant and with q-expansion at ∞ normalized so its constant term is equal to one. Then, we can express $\log\left(\|f\|(z)\right)$ in terms of parabolic Eisenstein series and Green's functions, as*

$$-2k + 2\pi \sum_{w \in \mathcal{F}_\Gamma} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} \lim_{s \to 1}\left(G_s^\Gamma(z,w) + \mathcal{E}_{\Gamma,\infty}^{\mathrm{par}(z,s)}\right) + \cdots.$$

Here

$$\mathcal{E}_\infty^{\mathrm{par}}(z,s) = \sum_{\Gamma_\infty \backslash \Gamma} \Im(\gamma z)^s.$$

The Kronecker limit formula says that

$$\mathcal{E}_\infty^{\mathrm{par}}(z,s) = \frac{1}{(s-1)\,\mathrm{vol}(\Gamma\backslash\mathbb{H})} + \beta - \frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}\log\left(|\eta_\infty^4(z)|\Im(z)\right) + \mathrm{O}(s-1).$$

Then we need to define β, and the Green's function $G_s(z,w)$, which is obtained by averaging the kernel $k_s(z,w)$; both functions are eigenfunctions of the Laplacian with eigenvalue $s(1-s)$, and the Green's function has a specified singularity on the

diagonal. The Green's function also has a Laurent series expansion as $s \to 1$ that involves the parabolic Eisenstein series. In particular,

$$\lim_{s \to 1} \left(G_s^\Gamma(z, w) + \mathcal{E}_{\Gamma, \infty}^{\mathrm{par}}(z, w) \right)$$

exists, with a logarithmic singularity on the diagonal.

The Rohrlich–Jensen formula can be understood through the study of the regularized inner product of this last limit with $F_{-n}(\cdot, 1)$. Regularization is needed only in the cusp (because the logarithmic singularity is integrable). We can thus write

$$\left\langle F_{-n}(\cdot, 1), \overline{\lim_{s \to 1} \left(G_s^\Gamma(z, w) + \mathcal{E}_{\Gamma, \infty}^{\mathrm{par}}(z, w) \right)} \right\rangle$$

$$= \lim_{Y \to \infty} \int_{\mathcal{F}(Y)} F_{-n}(z, 1) \lim_{s \to 1} \left(G_s(z, w) + \mathcal{E}_\infty^{\mathrm{par}}(z, s) \right) d\mu(z).$$

A key observation is that all terms are eigenfunctions of the Laplacian, hence one can seek to compute the inner product in a manner similar to that which yields the Maass–Selberg formula. The key identity is that

$$\int_{\mathcal{F}(Y)} F_{-n}(z, 1) \lim_{s \to 1} \left(G_s(z, w) + \mathcal{E}_\infty^{\mathrm{par}}(z, s) \right) d\mu(z)$$

$$= \partial_s \left(-s(1 - s) \int_{\mathcal{F}(Y)} F_{-n}(z, 1) \left(G_s(z, w) + \mathcal{E}_\infty^{\mathrm{par}}(w, s) \, d\mu(z) \right) \right) |_{s=1}.$$

We can now absorb $-s(1-s)$ into the integrals after applying the hyperbolic Laplacian Δ to each of the two factors in the integrand.

Main results:

Theorem 12. *For any positive integer n and any point $w \in \mathcal{F}$, we have*

$$\left\langle F_{-n}(\cdot, 1), \overline{\lim_{s \to 1} \left(G_s(\cdot, w) + \mathcal{E}_\infty^{\mathrm{par}}(\cdot, s) \right)} \right\rangle = -\partial_s F_{-n}(w, s)|_{s=1}.$$

Combined with the previous proposition (describing $\log \|f(z)\|$), we get the Rohrlich–Jensen formula:

$$\langle F_{-n}(\cdot, 1), \log \|f\| \rangle = -2\pi \sum_{w \in \mathcal{F}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} \partial_s F_{-n}(w, s)|_{s=1}.$$

We have further results. For instance, if g is any Γ-invariant analytic function with a pole at ∞, then (Niebur)

$$g(z) = \sum_{n=1}^{K} 2\pi \sqrt{n} a_n F_{-n}(z, 1) + c(g)$$

for some constants K, a_n, and $c(g)$ depending only upon g. Then, we have the identity

$$\langle g, \log |f| \rangle = -2\pi \sum_{w \in \mathcal{F}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} \left(2\pi \sum_{n=1}^{K} \sqrt{n} a_n \partial_s F_{-n}(w, s)|_{s=1} - \cdots \right).$$

Further found that the generating series for the Niebur Poincaré series at $s = 1$ is, in the z-variable (where we sum over q_z), the holomorphic part of the weight two biharmonic Maass form given by differentiating our linear combination of Eisenstein series and Green's function with respect to z.

6. Shenghao Hua, *Joint Value Distribution of Hecke–Maass Forms*

Preprint: [10]

Motivation: semiclassical limit of solutions to Schroedinger equation. Berry's random wave conjecture. QUE, L^p-norms.

Joint Gaussian moments conjecture: statistical indepndence of values of large eigenfunctions (multiplied together and integrated against some test function).

Theorem 13. *Let f and g be normalized Hecke–MAass cusp forms. Then if $2t_f \leq t_g - t_g^\varepsilon$, we have*

$$\int_{\Gamma \backslash \mathbb{H}} \psi f^2 g \ll t_g^{-A},$$

while if $2t_f > t_g - t_g^\varepsilon$, then, assuming GLH, we have

$$\int_{\Gamma \backslash \mathbb{H}} \psi f^2 g = \int_{\Gamma \backslash \mathbb{H}} \psi g + \mathrm{O}(t_f^{-1/4+\varepsilon}(1 + |2t_f - t_g|^{-1/4})).$$

Proof uses spectral expansion of ψ into eigenfunctions u, then spectral expansion of $\int (ug)(f^2)$, reducing to the case $g = u$, then further spectral expansion of $\int (g^2)(f^2)$.

Theorem 14. *Assuming GRH and GRC, we have as $\max(t_f, t_g) \to \infty$*

$$\mathbb{E} f^2 g^2 = 1 + \mathrm{O}((\log(t_f + t_g))^{-1/4+\varepsilon}).$$

Proof eventually applies ultimately Soundararajan's method of moments (which requires GRH).

Work in progress: asymptotic formulas for certain averages of L-functions over toroidal families, going beyond [17].

7. Algebraic proof of modular form inequalities for optimal sphere packings, *Seewoo Lee*

Preprint: [12].

Motivating question: finding densest sphere packings.

- $d = 1$
- $d = 2$: Thue 1910, $\Delta_2 = \frac{\pi}{2\sqrt{3}}$
- $d = 3$: Kepler conjecture 1611, Hales 2005, $\Delta_3 = \frac{\pi}{3\sqrt{2}}$, formally verified 2014 in Isabelle/HOL
- (Korkine–Zolotareff, Blichfeldt, Cohn–Kumar): D_4, D_5, D_6, E_7, E_8 and Leech lattice
- Viazovska, 2016 π-day on arXiv: E_8 lattice packing optimal, with $\Delta_8 = \frac{\pi^4}{388}$
- Cohn–Kumar–Miller–Radchenko–Viazovska, March 21st 2016 on arXiv: $\Delta_{24} = \frac{\pi^{12}}{12!}$ Leech lattice (unique even unimodular with nonzero minimal length 2)

Viazovska et al use:

Theorem 15 (Cohn–Elkies, 2013). *Let $r > 0$. Assume there exists a nice function $f : \mathbb{R}^d \to \mathbb{R}$ satisfying*

- $f(0) = \hat{f}(0) > 0$,
- $f(x) \leq 0$ for all $\|x\| \geq r$,
- $\hat{f}(y) \geq 0$ for all $y \in \mathbb{R}^d$,

then

$$\Delta_d \leq \mathrm{vol}(B_{r/2}^d) = \left(\frac{r}{2}\right)^d \frac{\pi^{d/2}}{(d/2)!}.$$

Proof is not hard (fits in one page).

This leads to the hunt for a "magic function" f satisfying the indicated conditions. BAsed on their numerical experiments, Cohn–Elkies conjectured that the optimal sphere packing can be achieved by a magic function when $d = 2, 8, 24$. The magic function f and its Fourier transform $\hat{f}$ should vanish at the lattice points.

Viazovska et al constructed the magic functions for $d = 8, 24$ using modular forms. Decompose into Fourier eigenfunctions $f = f_+ + f_-$ (parity under Fourier transform). Viazovska writes them as

$$f_\pm(x) = \sin^2\left(\frac{\pi\|x\|^2}{2}\right) \int_0^\infty \varphi_\pm(t) e^{-\|x\|^2 t}\, dt,$$

where the $\sin^2$ factor enforces desired roots.

For $d = 8$, we have $\varphi_\pm(t) = t^2 \psi_\pm(i/t)$, where $\psi_\pm$ are defined in terms of standard modular forms. We also need (for both $d = 8, 24$) a few non-obvious inequalities involving some of these modular forms. One reason these inequalities are difficult is because they're inhomogeneous with respect to the weights of the forms involved.

Original proofs use bounds of Fourier coefficients of the form

$$|c(n)| \leq C_1 e^{C_2 \pi \sqrt{n}},$$

following from the Hardy–Ramanujan formula, reducing the question to finite calculations and interval arithmetic.

Ran Romik (2023) gave an alternative and much simpler proof for $d = 8$ that doe snot use any interval arithmetic, but still requires a "calculator" to check inequalities like

$$e^{3\pi} \frac{9\Gamma(1/4)^{16}}{8192\pi^{12}} < 20480.$$

Question 16. Can we prove such inequalities *algebraically*?

The answer is yes, as we now explain. To that end, we'll develop a theory of *(completely) positive quasimodular forms*.

Definition 17. Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$. We say that a quasimodular form F is *positive* if

$$F(it) \geq 0$$

for all $t > 0$.

We call it *completely positive* if it has nonnegative q-coefficients at ∞.

Note that "completely positive" implies "positive", but the inclusion is strict in general. For instance, the discriminant form has a product formula that tells you that it is positive, but you can check that it's not completely positive.

Easy facts:

Theorem 18.
- *Anti-derivative preserves positivity.*
- *Derivative preserves complete positivity.*
- *Serre derivative preserves complete positivity.*

"Nontrivial" fact (that we won't use, and which also follows directly from Bernstein's theorem):

Theorem 19. *F is completely positive if and only if all its derivatives are positive.*

"Interesting" fact that we will use:

Theorem 20. *Let $F = \sum_{n \geq n_0} a_n q^n$ with $a_{n_0} > 0$. If some Serre derivative of F is positive, then so is F.*

Examples?

Definition 21 (Kaneko–Koike)**.** For given weight w and depth s, we can speak of *extremal quasimodular forms of weight w and depth s*, which are those whose order of zeros at ∞ is as large as possible.

It's been shown that these have almost all coefficients positive, and conjectured that they're all positive.

Theorem 22. *That conjecture is true for depth 1 extremal forms.*

We also have similar identities proving complete positively of the depth 2 extremal forms of weight $w \leq 14$, but not yet for general w.

This reduces some of the inequalities in the arguments of Viazovska et al to checking some limiting properties of ratios of modular functions. For monotonicity, we check that derivatives are positive, and for this it suffices to check the same for Serre derivatives of derivatives.

Possible future directions:

- Classify the completely positive forms?
- Possible applications in other LP problems?
  - Dual LP, uncertainty principle
  - Any results that are "uniform" in dimensions
- Make a formalization of the proof (e.g., in Lean) easier?

8. Ring of modular forms on certain unitary Shimura variety, *Yuxin Lin*

Recall the definition of modular forms of weight k and level 1, as certain holomorphic functions $f : \mathbb{H} \to \mathbb{C}$.

More generally, let $\mathrm{Sh}(G)$ be a Shimura variety (of level 1), where G is a reductive group over $\mathbb{Q}$. We can understand it as the moduli space of abelian varieties with additional structure. Let $\pi : \mathcal{A} \to \mathrm{Sh}(G)$ be the universal abelian scheme of dimension g.

Let $\underline{\omega} := \pi_* \Omega_{\mathcal{A}/\mathrm{Sh}}$ be the Hodge bundle on $\mathrm{Sh}(G)$, and let $\lambda_{\mathrm{Sh}} := \Lambda^g \underline{\omega}$ be the Hodge line bundle on $\mathrm{Sh}(G)$.

Definition 23. The ring of automorphic forms of scalar-valued weight is the space of global sections of the k-th tensor power of the Hodge line bundle, namely

$$\oplus_k H^0(\mathrm{Sh}, \lambda^{\otimes k}).$$

, 8, 12, 2, 16 An interesting question to ask is, what is the ring structure of such objects? There are some relevant results along these lines.

First, when $G = \mathrm{SL}_2(\mathbb{Z})$, we have that $\mathrm{Sh}(G)$ is the modular curve, and the ring of modular forms is generated by E_4 and E_6 modulo a single relation:

$$\mathcal{M}(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6]/(E_4^3 - E_6^2).$$

Going one dimension higher, for $G = \mathrm{SL}_2(\mathcal{O}_K)$ and $K = \mathbb{Q}(\sqrt{13})$, we have that $\mathrm{Sh}(G)$ is the Hilbert modular surface with real multiplication by $\mathcal{O}_K$. Van Der Geer and Zagier show that the ring of symmetric Hilbert modular forms of even weight has 4 generators at weights $4, 8, 12, 12$ respectively, with a relation at weight 24, while the ring of Hilbert modular forms of even weight has 5 generators of weights $4, 8, 12, 12, 16$, respectively, with two relations at weight 24 and 32. To that end, they find the minimal model Y of the compactification of $\mathrm{Sh}(G)$, then find the canonical divisor of Y and compute the image S of Y under the canonical embedding, then realize sections of $H^0(\mathrm{Sh}, \lambda^{\otimes 2k})$ as meromorphic sections of $\mathcal{O}_S(k)$ with certain multiplicity at the cusps.

This is the strategy we will imitate, but for certain unitary Shimura varieties rather than Hilbert modular surfaces.

To that end, we need to first introduce the moduli space of curves.

Let $G = \mathbb{Z}/d\mathbb{Z}$ be a finite cyclic group.

Definition 24.
- $\mathcal{M}_G$: the moduli space of admissible stable $\mathbb{Z}/d\mathbb{Z}$ covers of $\mathbb{P}^1$. Objects are pairs $(C/S, \iota : G \to \mathrm{Aut}_S(C))$, i.e., a family of curves over the base scheme together with an embedding of G in their automorphism group.
- $\tilde{\mathcal{M}}_G$, the moduli space of admissible stable $\mathbb{Z}/d\mathbb{Z}$ covers of $\mathbb{P}^1$ with an ordering on the ramification points. The objects are tuples $(C/S, \iota : G \to \mathrm{Aut}_S(C), \sigma_i)$.
- The forgetful morphism $\tilde{\mathcal{M}}_G \to \mathcal{M}_G$ is finite étale.

The irreducible connected components of $\mathcal{M}_G$ are indexed by monodromy datum, which is a triple $(d, r, \underline{a})$, where
- r is the number of branching points on $C/\iota(G)$,
- $\underline{a} = (a(1), \ldots, a(r)) \in G^r$ is the *inertia type*, which records the character of G at the tangent space of the i-th ramification point.

The *Torelli morphism* T associates the moduli of curves with the moduli of principally polarized abelian varieties:

$$
\begin{array}{ccc}
\mathcal{M}(d, r, \underline{a})_{\mathbb{C}} & \longrightarrow & \mathrm{Sh}(\mathcal{D})_{\mathbb{C}} \\
\downarrow & & \downarrow \\
\mathcal{M}_{g,\mathbb{C}} & \xrightarrow{\ T\ } & \mathcal{A}_{g,\mathbb{C}}.
\end{array}
$$

The image of the Torelli morphism is called the *Torelli locus* inside $\mathcal{A}_{g,\mathbb{C}}$. It turns out that the Jacobians coming out of $\mathcal{M}(d, r, \underline{a})$ have larger endomorphisms than generically, and in particular, at least contain the group algebra generated by G. It's thus reasonable to speculate that the image of T on $\mathcal{M}(d, r, a)$ will sit inside a *special* subvariety of $\mathcal{A}_g$, and this special subvariety will be the focus of our study.

More precisely, $\mathrm{Sh}(\mathcal{D})$ is the smallest PEL type Shimura subvariety of $\mathcal{A}_{g,\mathbb{C}}$ containing $T(\mathcal{M}(d, r, \underline{a})_{\mathbb{C}})$. It parametrizes abelian varieties with multiplication by $\mathbb{Z}[G]$ and signature given by $(d, r, \underline{a})$.

We focus on the family $\mathcal{M}[16] = \mathcal{M}(5, 5, (1, 1, 1, 1, 1))$.
- The associated Shimura variety $\mathrm{Sh}(G)$ is of *unitary type*, whose reductive group $\mathcal{G}$ has $\mathbb{R}$ points given by $\mathrm{GU}(3, 0) \times \mathrm{GU}(2, 1)$. Therefore, it is *compact*. The reason is that the signature of this Shimura variety can be computed, and the signature has a zero in one of the components, which means that the

hermitian symmetric domain is compact, and hence this $\text{Sh}(\mathcal{D})$ is compact. In particular, this Shimura variety has no cusp.

- A second reason why this family is so nice is that $\text{Sh}(\mathcal{D})_{\mathbb{C}}$ has dimension 2, which is equal to the dimension of $\mathcal{M}[16]$. Therefore, the Torelli morphism T gives an *isomorphism*

$$T : \mathcal{M}[16] \to \text{Sh}(\mathcal{D}),$$

such that T gives an isomorphism between the corresponding Hodge line bundles. Thus, in order to compute global sections of the Hodge line bundle on $\text{Sh}(\mathcal{D})$, it suffices to do the analogous thing on $\mathcal{M}[16]$, which is a bit simpler, but still tricky, due to nontrivial stabilizers. To that end, we consider:

- The finite étale cover $\tilde{\mathcal{M}}[16]$, which has a natural forgetful morphism

$$q : \tilde{\mathcal{M}}[16] \to \overline{M_{0,5}},$$

that is bijective on field-valued points. This is very nice, because we know the geometry of $\overline{M_{0,5}}$: its Picard group is generated by ten divisors, each isomorphic to $\mathbb{P}^1$.

Proposition 25. *Let Δ be the boundary divisor of $\overline{M_{0,5}}$, and let $\lambda_{\tilde{M}}$ be the Hodge divisor on $\tilde{\mathcal{M}}[16]$. Under the forgetful morphism*

$$q : \tilde{\mathcal{M}}[16] \to \overline{M_{0,5}},$$

we have that $q^*\Delta = 5\lambda_{\tilde{M}}$.

Idea of proof.
- Use Grothendieck–Riemann–Roch to relate $\lambda_{\tilde{M}}$ with the *boundary divisor* of $\tilde{\mathcal{M}}$.
- q exhibits Δ as a multiple of the boundary of $\tilde{\mathcal{M}}[16]$.

$\square$

Our next goal is to fill in the lifts

$$
\begin{array}{ccc}
\tilde{\mathcal{M}}[16] & \xrightarrow{\;?\;} & \text{Sh}_{\mathcal{K}}(\mathcal{D}) \\
\downarrow & & \downarrow{\scriptstyle ?} \\
\mathcal{M}[16] & \xrightarrow[T]{} & \text{Sh}(\mathcal{D}).
\end{array}
$$

Giving such a cover of T is the same as computing the level subgroup corresponding to this cover. That level subgroup is basically a congruence subgroup:

Proposition 26. *The level subgroup $\mathcal{K}$ at the place p satisfies:*

$$\mathcal{K}_p = \begin{cases} \mathcal{G}(\mathbb{Z}_p) & \text{if } p \neq 5, \\ \mathcal{U}(\mathfrak{m}) & \text{if } p = 5. \end{cases}$$

Putting these parts together via the sequence of morphisms

$$\text{Sh}_{\mathcal{K}}(\mathcal{D}) \xleftarrow{T} \tilde{\mathcal{M}}[16] \xrightarrow{q} \overline{M_{0,5}} \xrightarrow{|-K|} S,$$

we obtain the following:

Proposition 27. • *The weight m automorphic forms on $\mathrm{Sh}_{\mathcal{K}}(\mathcal{D})$ is*

$$H^0(\mathrm{Sh}_{\mathcal{K}(\mathcal{D}),\lambda_{\mathrm{Sh}\otimes m}}) \cong H^0(S, \mathcal{O}_S(2\lfloor \frac{m}{5}\rfloor)) \cong (\mathbb{C}[x_0,\ldots,x_5])/(Q_{12}-Q_{ij}),$$

with explicit degrees.
   • *Similar for the ring of graded automorphic forms.*

From the moduli interpretation, we see that $\mathrm{Sh}(\mathcal{D})$ is the $\mathfrak{S}_5$-invariant of $\mathrm{Sh}_{\mathcal{K}}(\mathcal{D})$. Therefore:

Theorem 28. *The ring of automorphic forms on $\mathrm{Sh}(\mathcal{D})$ is the $\mathfrak{S}_5$-invariant of $\mathrm{Sh}_{\mathcal{K}}(\mathcal{D})$. In particular,*

$$\oplus_m H^0(\mathrm{Sh}(\mathcal{D}),\lambda^{\otimes m}) \cong \mathbb{C}[f_2,f_4,f_6,\varepsilon]/\varepsilon^{10} + \frac{1}{52}(f_2^2 - 4f_4),$$

where the f_{2n} are polynomials of degree $2n$ in x_i, where everything has explicit degrees.

9. Counting modular forms mod p satisfying constraints at p, Samuele Anni

Recall the definition of modular forms f of weight k on $\Gamma_0(n)$, with q-expansion $f(z) = \sum a_n q^n$. We also know that there are Hecke operators T_p acting on them. We consider only cuspidal *newforms* ($a_0 = 0$), *normalized* ($a_1 = 1$), which are eigenforms for the Hecke operators and arise from level n and not any smaller level. We use the notation $S_k(n,\mathbb{C})$ and $S_k(n,\mathbb{C})^{\mathrm{new}}$.

Definition 29. The *Hecke algebra* $\mathbb{T}(n,k)$ is the $\mathbb{Z}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(S_k(n,\mathbb{C}))$ generated by Hecke operators T_p for each prime p.

Newforms can be seen as ring homomorphisms $f : \mathbb{T}(n,k) \to \overline{\mathbb{Q}}$.

Theorem 30 (Deligne, Serre, Shimura). *Let n and k be positive integers. Let $\mathbb{F}$ be a finite field of characteristic ℓ, with $\ell \nmid n$, and let $f : \mathbb{T}(n,k) \twoheadrightarrow \mathbb{F}$ be a surjective ring homomorphism. Then there is a unique continuous semisimple representation*

$$\bar\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}),$$

unramified outside $n\ell$, such that for all p not dividing $n\ell$, the trace of Frobenius at p under $\bar\rho_f$ is $f(T_p)$, and the determinant is given in terms of the value of f on the diamond operator (central character).

Computing $\bar\rho_f$ is "difficult", but theoretically it *can be done in polynomial time* in $n, k, \#\mathbb{F}$:

   • Edixhoven, Couveignes, de Jong, Merkl, Bruin, Bosman: $\#\mathbb{F} \leq 32$.
   • Mascot, Zeng, Tian: $\#\mathbb{F} \leq 53$.

Fix a prime $p \geq 5$, a level N prime to p, and a weight $k \geq 2$. Let $S_k := S_k(\Gamma_0(Np),\bar{\mathbb{Q}}_p)$ be the space of p-new (i.e., not coming from level N) cuspidal modular forms of level Np and weight k with coefficients in $\bar{\mathbb{Q}}_p$.

We now discuss the **Atkin–Lehner involution**. There exist $x,y,z,t \in \mathbb{Z}$ for which the matrix

$$W_p = \begin{pmatrix} px & y \\ Npz & pt \end{pmatrix}$$

has determinant p.

The matrix W_p normalizes the group $\Gamma_0(Np)$, and for any weight k, it induces a linear operator w_p on the space of cusp forms S_k that commutes with the Hecke operators T_q for all $q \nmid Np$ and acts as its own inverse.

Any cusp form in S_k that is an eigenform for all T_q with $q \nmid N$ is also an eigenform for w_p, withe eigenvalue ± 1. This involution acts on the modular curve.

The Atkin–Lehner involution w_p splits S_k as a sum of plus and minus spaces:

$$S_k = S_k^+ \oplus S_k^-.$$

Since we have dimension formulas for $s_k := \dim S_k$, in order to understand the dimensions

$$s_k^\pm = \dim S_k^\pm$$

of the Atkin–Lehner eigenspaces, it suffices to understand the difference

$$d_k := s_k^+ - s_k^-.$$

We can get some idea from tables of examples (obtained empirically by looking at many more specific examples):

p	d_k
5	± 1
11	± 2
59	± 6
101	± 7

TABLE 1. Values of d_k for different p

Classical result that d_k is constant in absolute value and alternates in sign. Need to introduce modification d_k^* taking into account Eisenstein series of weight two. One can show:

Theorem 31 (Fricke, Yamauchi, Momose, Ogg, Wakatsuki, Helfgott, Martin et al.). *We have*

$$d_k^* = (-1)^{k/2} \frac{\#\mathrm{FP}}{2},$$

where $\#\mathrm{FP}$ is the number of fixed points of the Atkin–Lehner involution w_p on $X_0(Np)$.

The fixed points of w_p on $X_0(Np)$ corresponds to elliptic curves with level structure and CM by $\sqrt{-p}$, in fact the d_k are closely related to class numbers:

$$\#\mathrm{FP} = c_p \cdot h(\sqrt{-p}) \cdot \prod_{q | N, \text{prime}} \left(1 + \left(\frac{-4p}{q} \right) \right).$$

What do we want to do? Systems of mod-p prime-to-Np Hecke eigenvalues correspond to continuous semisimple Galois representations $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$. Can decompose according to these, and then further via Atkin–Lehner, giving spaces $S_{k,\bar{\rho}}^\pm$ and their dimensions $s_{k,\bar{\rho}}^\pm$ and differences $d_{k,\bar{\rho}}$. As before, $k = 2$ and $\bar{\rho}$ forces us to make an adjustment, so let

$$d_{k,\bar{\rho}}^* := \begin{cases} d_{k,\bar{\rho}} - 1 & \text{if } k = 2 \text{ and } \bar{\rho} = 1 \oplus \omega, \\ d_{k,\bar{\rho}} & \text{otherwise,} \end{cases}$$

where ω is the cyclotomic character.

Theorem 32 (Anni, Ghitza, Medvedovsky)**.** *For $k \geq 2$ and any $\Gamma_0(Np)$-modular $\bar{\rho}$, we have*

$$d^*_{k+2,\bar{\rho}[1]} = -d^*_{k,\bar{\rho}}.$$

As an example, for $p = 5$, $N = 23$, we can check that for $k > 2$, we have $d_k = \pm 2$.

Bergdall and Pollack use the Ash–Stevens formula, a fundamentally characteristic p technique for filtering cohomology of modular symbols, to derive their dimension formulas. But Ash–Stevens has nothing to say about Atkin–Lehner, in part because Atkin–Lehner operator requires inverting p. On the other hand, the classical complex methods – trace formulas, Gauss–Bonnet, Riemann–Hurwitz – do not know anything about $\bar{\rho}$.

What we do instead is to combine the *trace formula* (Zagier–Cohen–Osterlé–Cohen–Strömberg and Skoruppa–Zagier–Popa) with an *algebra theorem*, an explicit refinement of Brauer–Nesbitt.

What is this explicit Brauer–Nesbitt? See [2].

Theorem 33 (AGM)**.** *Let M and N be two finite free $\mathbb{Z}_p$-modules of the sam erank d, each with an action of an operator T. Then $\bar{M}^{\mathrm{ss}} \cong \bar{N}^{\mathrm{ss}}$ as $\mathbb{F}_p[T]$-modules if and only if for every n with $1 \leq n \leq d$, we have*

$$\mathrm{trace}(T^n|M) = \mathrm{trace}(T^n|N),$$

(...).

Corollary 34. *If we have modules that injective modulo p (but not necessarily in characteristic zero), then we can check that semisimplifications of certain residual quotients are the same by proving an inequality involving traces.*

Now work with the trace formula. We can deduce statements about dimensions.

10. INTEGER PARTITIONS DETECT THE PRIMES, *Jan-Willem Van Ittersum*

Preprint: [7].

We write s_i for the sizes of the parts, and m_i for the multiplicities of the part, with $s_1 > s_2 > \cdots$.

Definition 35 (MacMahon 1920)**.**

$$M_a(n) := \sum_{\substack{n = m_1 s_1 + \cdots + m_a s_a \\ 0 < s_1 < \cdots < s_a}} m_1 \cdots m_a.$$

Remark 36. This is the sum of multiplicity products of partitions of n with a part sizes.

Consider

$$\psi(n) := (n^2 - 3n + 2)M_1(n) - 8M_2(n).$$

Example 37. $n = 3, a = 1$, in which case $\lambda = (3)$ or $\lambda = (1^3)$, in which case $M_1(3) = 1 + 3 = 4$, or $a = 2$, in which case $\lambda = (2, 1)$, so $M_2(3) = 1 \cdot 1 = 1$, and we compute that $\psi(3) = 2M_1(3) - 8M_2(3) = 0$.

Continuing, take $n = 4, a = 1$, in which case $\lambda = (4)$, $\lambda = (2^2)$ $\lambda = (1^4)$, so $M_1(4) = 1 + 2 + 4 = 7$, or $a = 2$, in which case $\lambda = (3, 1)$ or $\lambda = (2, 1^2)$, so $M_2(4) = 1 \cdot 1 + 1 \cdot 2 = 3$, and so $\psi(4) = 6M_1(4) - 8M_2(4) = 18$.

Continuing, one gets the table, for $n = 2..11$, of $\psi(n) = 0, 0, 18, 0, 120, 0, 270, 192, 504, 0$.

Theorem 38 (Craig–vI–Ono). *For positive integers n, we have*

(1) $(n^2 - 3n + 2)M_1(n) - 8M_2(n) \geq 0$,

(2) $(3n^3 - 13n^2 + 18n - 8)M_1(n) + (12n^2 - 120n + 212)M_2(n) - 960M_3(n) \geq 0$

and for $n \geq 2$ these expressions vanish if and only if n is prime.

Remark 39. We call such an expression *prime-detecting*. The sum of two such expressions is likewise prime-detecting, and multiplying $f(n)$ for a polynomial f yields a prime-detecting expression if $f(n) > 0$ for all n.

We can get a table of prime-detecting expressions: the first two that appear above, then two more, $(126n^5 - \cdots)M_1(n) + \cdots$ and $(300n^8 - \cdots)M_1(n) + \cdots$.

Conjecture 40. *These are all such expressions (up to addition and multiplication as before).*

We then checked whether we could get more results of a similar shape, by considering a generalization of the MacMahon function:

Definition 41. For $\ell \in \mathbb{Z}_{\geq 0}^a$, we define the *generalized MacMahon partition function*

$$M_{\underline{\ell}}(n) := \sum_{\substack{n = m_1 s_1 + \cdots + m_a s_a \\ 0 < s_1 < \cdots < s_a}} m_1^{\ell_1} \cdots m_a^{\ell_a}.$$

You can think of this as a generalization of the divisor function, but for partitions. For this new generalized partition function, much more is possible:

Theorem 42 (Craig–vI–Ono). *Let $d \geq 4$.*

(1) *There exist $c_{\underline{\ell}} \in \mathbb{Z}$ such that $\sum_{|\underline{\ell}| < d} c_{\underline{\ell}} M_{\underline{\ell}}(n) \geq 0$, where $|\underline{\ell}| = \ell_1 + \cdots + \ell_a$.*

(2) *There are $\gg d^2$ linearly independent such expressions.*

Example 43. $63M_{(2,2)}(n) - 12M_{(3,0)}(n) - \cdots + 12M_{(3,0,0)}(n) = \frac{11}{3}\psi(n)$.

¡++¿

Goal of the talk now is to explain some of the ideas behind these results, which have a lot to do with modular forms.

Prime-detecting quasimodular forms. Let

$$\mathcal{G}_k := -\frac{B_k}{2k} + \sum_{n \geq 1} \sigma_{k-1}(n)q^n, \qquad D := q\frac{\partial}{\partial q}, \qquad q = e^{2\pi i \tau}.$$

Consider

$$f_{k,\ell} := (D^\ell + 1)\mathcal{G}_{k+1} - (D^k + 1)\mathcal{G}_{\ell+1}$$

$$= * + \sum_{n \geq 1} \sum_{d \mid n} \left((n^\ell + 1)d^k - (n^k + 1)d^\ell \right) q^n.$$

Note that for $d = 1$ one gets $n^\ell + 1 - n^k - 1 = n^\ell - n^k$. Calculating similarly with $d = n$, we see that $f_{k,\ell}$ vanishes at prime coefficients. It's also easy to see that its coefficients are positive. But note that $f_{k,\ell}$ is not modular, and not of homogeneous weight. Key features here are quasimodularity and mixed weight.

Theorem 44 (Craig–vI–Ono). *All prime-detecting forms in $\oplus_{k:even} \oplus_{n \geq 0} D^n \mathcal{G}_k$ are linear combinations of $D^n H_k$, where*

$$H_k = \begin{cases} \frac{1}{6}(D^2 - D + 1)\mathcal{G}_2 - \mathcal{G}_4 & \text{if } k = 6, \\ \frac{1}{24}\left(-D^2\mathcal{G}_{k-6} + (D^2 + 1)\mathcal{G}_{k-4} - \mathcal{G}_{k-2} \right) & \text{if } k = 8. \end{cases}$$

That's the first ingredient, but still need to say something about modularity of MacMahon functions. Let

$$\mathcal{G}(\underline{\ell}) = \sum_{n \geq 0} M_{\underline{\ell}}(n) q^n \xrightarrow{q \to 1} \frac{(1-q)^{|\underline{\ell}|+a}}{\prod_i \ell_i!} \sum_{s_1 < \cdots < s_a} \frac{1}{s_1^{\ell_1+1} \cdots s_a^{\ell_a+1}} \quad (\ell_a \geq 1, \ell_i \geq 0).$$

The algebra $\mathcal{Z}_q := \langle \mathcal{G}(\underline{\ell}) \rangle_{\mathbb{Q}}$ was introduced by Bachmann–Kühn. Facts:

- $\mathcal{Z}_q$ is a differential algebra (closed under differentiation and multiplication)
- $\tilde{M} := \mathbb{Q}[\mathcal{G}_2, \mathcal{G}_4, \mathcal{G}_6] \subseteq \mathcal{Z}_q$ (i.e., it contains the space of quasimodular forms).
- (Hoffman–Ihara) We have

$$\sum_{j \geq 0} \mathcal{G}(\underbrace{1, 1, \ldots, 1}_{j}) x^j = \exp\left(\sum_{n \geq 1} \frac{(-1)^{n+1}}{n} x^n \underbrace{\sum_m \frac{q^m}{(1-q^m)^{2n}}}_{\sum_{m,s \geq 1} \binom{s+n-1}{s-n} q^{ms} \in \tilde{M}} \right),$$

which is a linear combination of quasimodular Eisenstein series, hence $M_a(n)$ are coefficients of a quasimodular form.

11. Supersingular abelian surfaces and where to find them, *Gabriele Bogo*

Characteristic zero: $\text{End}(E) \cong \mathbb{Z}$ or $\mathcal{O}_D$, order in an imaginary quadratic field.

Characteristic $p > 0$ (Hasse, 1936), can also be $\mathcal{O}_{\infty,p}$, order in a quaternion algebra over $\mathbb{Q}$. In that case, we call it *supersingular*.

Deuring 1941: for $p \geq 5$, the number of supersingular elliptic curves over $\mathbb{F}_{p^2}$ is

$$\lfloor \frac{p-1}{12} \rfloor + \delta + \varepsilon.$$

Another interpretation: writing $\mathcal{M}_{1,1} \cong \mathbb{H}/\text{SL}_2(\mathbb{Z})$ for the moduli space of elliptic curves over $\mathbb{C}$, there are three special points $\infty, i = \sqrt{-1}, \rho = (-1 + \sqrt{-3})/2$, and we can think of Deuring's result as

$$\#(\text{ss}_p) = \lfloor \frac{p-1}{12} \rfloor + \delta + \varepsilon \leq \lfloor \frac{-\chi(\mathbb{H}/\text{SL}_2(\mathbb{Z})) \cdot (p-1)}{2} \rfloor + 2.$$

Let's generalize to higher-dimensional varieties. Let $\mathbb{Q}(\sqrt{D})$ be a real quadratic field, with ring of integers $\mathcal{O}_D$. We consider a family $\mathcal{X} \to \mathcal{C}$ of (p.p.) abelian surfaces, over a curve $\mathcal{C}$, such that:

- the fibers X_c are defined over a number field k, and
- there exists an inclusion of rings $\mathcal{O}_D \hookrightarrow \text{End}_k(X_c)$.

An abelian surface over a field of characteristic p is *supersingular* (resp. *superspecial*) if it is isogenous (resp. isomorphic) to the product of two supersingular elliptic curves.

There are embeddings

$$\mathbb{H}/\Gamma = \mathcal{C} \hookrightarrow X_D(\mathbb{C}) = \mathbb{H}^2/\text{SL}_2(\mathcal{O}_D).$$

For example, if $\Gamma = \text{SL}_2(\mathbb{Z})$, this is the inclusion of elliptic curves into the space of abelian surfaces with multiplication. There are also modular curves $\Gamma_0(N)$. But one can have more interesting examples, related to non-arithmetic curves. To give

an explicit example, consider the triangle group $\Gamma = \Delta(2,5,\infty) \hookrightarrow X_5$, with explicit equation

$$y^2 = \begin{cases} x^5 - 5x^3 + 5x - 2t & \text{if } t \neq \infty, \\ x^5 - 1 & \text{if } t = \infty. \end{cases}$$

We now take the reductions of the above inclusion modulo p, to a map

$$\overline{\mathcal{C}} \to X_D(\mathbb{F}).$$

We want to study the fibers of this family that, after reduction modulo p, are supersingular.

To that end, consider a smooth algebraic curve in a Hilbert modular surface

$$\mathcal{C} \hookrightarrow X_D$$

with second Lyapunov exponent $\lambda_2 \in \mathbb{Q} \cap (0,1]$. For example, for $\mathrm{SL}_2(\mathbb{Z})$, we have $\lambda_2 = 1$, and for modular curves, $\lambda_2 = 1$, while for $\Delta(2,5,\infty)$ we have $\lambda_2 = 1/3$, but for Teichmuller curves, you can have $1/3, 1/5, 1/7$.

Theorem 45 (B–Li, 2024). *The supersingular locus of $\mathcal{C}$ modulo p has cardinality described by*

$$\lfloor \frac{-\chi(\mathcal{C})(p-\lambda_2)}{2} \rfloor \leq \#\mathrm{ss}_p^{\mathcal{C}} \leq \lfloor \frac{-\chi(\mathcal{C})(p-1)(\lambda_2+1)}{2} \rfloor + r, \quad \text{if } \left(\frac{D}{p} \right) = -1,$$

while

$$\#\mathrm{ss}_p^{\mathcal{C}} \leq \lfloor \frac{-\chi(\mathcal{C})(p-1)\lambda_2}{2} \rfloor + r, \quad \text{if } \left(\frac{D}{p} \right) = 1.$$

How to find supersingular abelian surfaces? If $\mathcal{C} = \mathbb{H}/\Gamma$ is of genus zero with Hauptmodul j, then the supersingular locus can be described by a polynomial in j:

$$\mathrm{ss}_p^{\mathcal{C}}(j) := \prod_{\substack{c \in \mathcal{C} \\ \overline{X_c} \text{ supersingular}}} (j - j(c)).$$

Theorem 46 (B–Li, 2024). *There are two families $\{A_{0,n}(j)\}_n$ and $\{A_{1,n}(j)\}_n$ of orthogonal polynomials (Atkin's polynomials) such that for every p of good reduction,*

$$\mathrm{ss}_p^{\mathcal{C}}(j) \equiv \begin{cases} \mathrm{lcm}(A_{0,n_p}, A_{1,\tilde{n}_p}) & \text{if } \left(\frac{D}{p} \right) = -1, \\ \gcd(A_{0,n_p}, A_{1,\tilde{n}_p}) & \text{if } \left(\frac{D}{p} \right) = 1 \end{cases}$$

for explicit indices n_p and $\tilde{n}_p$.

For example, consider $\Delta(2,5,\infty)$. Let j be a Hauptmodul, with pole at ∞. The scalar products are defined on $\mathbb{R}[j]$ by

$$\langle f, g \rangle_0 := \int_{\pi/5}^{\pi/2} f(e^{i\theta}) g(e^{i\theta}) \, dth,$$

$$\langle f, g \rangle_1 := \int_{2\pi/5}^{\pi/2} f(e^{i\theta}) g(e^{i\theta}) \, d\theta.$$

The first polynomials $A_{0,n}(j)$ are: $A_{0,0}(j) = 1$, $A_{0,1}(j) = j - 9/20$, etc. Taking $p = 13$, one finds that

$$\mathrm{ss}_{13}(j) \equiv A_{0,3}(j) \equiv j(j+4)(j-1) \pmod{13}.$$

Next, real multiplication splits $H^1_{\mathrm{dR}}(X_c)$ into two eigenspaces. The eigendifferentials induce two second order differnetial equations, called Picard–Fuchs differential equations. TRuncation of the holomorphic solutions can be related to the supersingular locus of $\mathcal{C}$.

Example 47. For $\mathcal{C} = \mathbb{H}/\Delta(2,5,\infty)$,

$$j^n \cdot {}_2F_1\left(\frac{7}{20}, \frac{3}{20}; 1; \frac{1}{j}\right) = U_{0,n}(j) + \mathrm{O}(j^{-1}),$$

etc. Leads to a formula for $\mathrm{ss}_p^{\mathcal{C}}(j)$.

The partial Hasse invariants h_1 and h_2 are characteristic p Hilbert modular forms of non-parallel weight. They have the following properties:

- The divisor of h_i is the component D_i of the non-ordinary locus of $X_D(\mathbb{F})$.
- Their q-expansion is constant, equal to 1, at every cusp.

In particular, they do not lift to Hilbert modular forms in characteristic zeor.

Theorem 48 (B–Li, 2024). *Let $\mathcal{C} \cong \mathbb{H}/\Gamma \hookrightarrow X_D$ be a smooth algebraic curve with good reduction at p. Then the partial Hasse invariants lift to (twisted) modular forms on Γ in characteristic zero.*

12. Congruences for the number of 3- and 6-regular partitions and quadratic forms, *Cristina Ballantine*

Definition 49. A *partition* of a positive number n is a non-increasings equence of positive integers, named *parts*, that add up to n.

We denote by $p(n)$ the number of partitions of n.

Example 50. $p(4) = 5$.

Ramanujan congruences:

$$p(5n + 4) \equiv 0 \pmod 5,$$
$$p(7n + 5) \equiv 0 \pmod 7,$$
$$p(11n + 6) \equiv 0 \pmod{11}.$$

Ramanujan: it appears there are no equally simple properties for any moduli involving primes other than these.

Ahlgren and Ono (2001): there are such Ramanujan congruences modulo every prime $\ell \geq 5$ (actually mod ℓ^m).

How about $\ell = 2, 3$? Radu (2012, formerly Subbarao Conjecture): for $\ell \in \{2,3\}$, there is no arithmetic progression $An + B$ such that $p(An + B) \equiv 0 \pmod \ell$ for all n.

Conjecture (Parkin and Shank): approximately half the values of $p(n)$ are even.

Bellaiche and Nicolas 2016: the number of $n \leq x$ with $p(n)$ even is at least

$$0.069\sqrt{x}\log\log x,$$

while the number that are odd is at least a constant multiple of $\sqrt{x}/\log x$.

The reason we can do this using modular forms is that

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{i=1}^{\infty} \frac{1}{1 - q^i} = \frac{q^{1/24}}{\eta(q)}.$$

Let's write $Q(n)$ for the number of partitions with distinct parts. Then

$$\sum_{n=0}^{\infty} Q(n)q^n = \prod_{i=1}^{\infty}(1+q^i) \equiv \prod_{i=1}^{\infty}(1-q^i) \pmod{2} = \sum_{j\in\mathbb{Z}}(-1)^j q^{j(3j-1)/2},$$

so $Q(n)$ is even asymptotically 100% of the time.

Definition 51. An *r-regular partition* is a partition in which no parts are divisible by r.

Notation 52. $b_r(n) :=$ the number of r-regular partitions of n.

$b_2(n)$ is even asymptotically 100% of the time.

Theorem 53 (Keith–Zanello 2022). *If $p \equiv 13, 17, .19, 23 \pmod{24}$ is prime, then*
$$b_3\left(2(p^2 n + pk - 24^{-1})\right) \equiv 0 \pmod{2}$$
for $1 \le k \le p-1$, where the inverse is taken modulo p.

Theorem 54 (Yao 2022). *Let $p \ge 5$ for a prime. Then we can construct more arithmetic progressions where this happens, different from the above.*

(1) If $b_3\left(\frac{p^2-1}{12}\right) \equiv 1 \pmod{2}$, then we get a whole arithmetic progression.
(2) (...)

Experimentally, we found a bunch of arithmetic progressions on which b_3 was even. For instance, for $\alpha \in \{1, 51, 76, 101\}$,
$$b_3\left(2(5^3 n + \alpha)\right) \equiv 0 \pmod{2}.$$
Conjectured that one can do this for every prime $p \ge 5$.

The bad news is that the congruences we found for $p = 5, 7, 11$ are special cases of Yao's theorem. My coauthor did more experience and found other primes $p \in \{29, 59, \ldots, 683\}$ where we have
$$b_3\left(2(p^2 n + p\alpha - 24^{-1})\right) \equiv 0 \pmod{2},$$
where $0 \le \alpha < p$, $\alpha \ne \lfloor p/24 \rfloor$, and 24^{-1} taken modulo p, that do not fit into Yao's theorem.

We proved the congruences algorithmically, using a result of Radu involving modular forms and Sturm's bound.

Can we describe these primes? What is the general theorem?

Let $\mathcal{P}$ be the set of primes p such that for some $j \in \{1, 4, 8\}$, the equation $x^2 + 24 \cdot 9y^2 = jp$ has primitive solutions.

13. Exact formulae for ranks of partitions, *Qihang Sun*

Preprint: [15].

$p(n)$: partition function. Ramanujan's congruence mod 5 and 7.

Dyson's rank: for each partition $\Lambda = \{\Lambda_1 \ge \Lambda_2 \ge \cdots \ge \Lambda_k\}$, define $\mathrm{rank}(\Lambda) := \Lambda_1 - \kappa$ to be the largest part minus the number of parts. Rank generating function $N(a, b, n)$. Explains Ramanujan congruences, e.g.,

$$N(a, 5, 5n+4) = \frac{1}{5}p(5n+4),$$

$$N(a, 7, 7n+5) = \frac{1}{7}p(7n+5).$$

These conjectures were all proved by Swinnerton-Dyer in 1954.

We'll focus on the analytic side of those rank functions. For $q = e^{2\pi i z}$ and $w = \zeta_b^a$,

$$\mathcal{R}(w, q) = 1 + \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} N(m, n) w^m q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq, q)_n (w^{-1}q, q)_n} =: 1 + \sum_{n=1}^{\infty} A\left(\frac{a}{b}, n\right) q^n.$$

Examples:
- $R(1, q) = 1 + \sum p(n) q^n = q^{1/24}/\eta(z)$
- $R(-1, q) = f(q)$ (Ramanujan's 3rd other mock theta function)
- $R(\zeta_3, q) = R(\zeta_3^2, q) = \gamma(q)$ (another mock theta function)

Hardy–Ramanujan (1918):

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi \sqrt{\frac{2n}{3}}\right),$$

together with more precise result. Rademacher (1937): exact formula.

$A(\frac{1}{2}, n) = N(0, 2, n) - N(1, 2, n)$. Ramanujan claimed, Dragonette (1952) proved:

$$A(\tfrac{1}{2}, n) = \frac{(-1)^{n-1} e^{\pi \sqrt{\tilde{n}/6}}}{2\sqrt{\tilde{n}}} + \cdots.$$

Andrews (1966).

Conjectures by Andrews and Lewis, proved by Bringmann (2009).

$$A(\tfrac{1}{3}, n) = \frac{4\sqrt{3} i}{(24n - 1)^{1/2}} \sum_{\substack{3|k}}^{\lfloor \sqrt{n} \rfloor} \cdots.$$

Should be an exact formula.

More generally, Bringmann gave asymptotics for $A(\frac{\ell}{u}, n)$.

Proofs use Maass Poincaré series.

14. The rational torsion subgroups of the Drinfeld modular Jacobians for prime-power levels, *Sheng-Yang Kevin Ho*

Preprint: [9].

$X_0(N)$: modular curve,

$$J_0(N)(\mathbb{Q}) \cong \mathbb{Z}^r \oplus \mathcal{T}(N).$$

Can we compute the rational torsion subgroup?

Let $\mathcal{C}(N)$ denote the rational cuspidal divisor class group for $X_0(N)$. By a theorem of Manin and Drinfeld, we have

$$\mathcal{C}(N) \subseteq \mathcal{T}(N).$$

Conjecture (generalized Ogg's conjecture): for any positive integer N, we have $\mathcal{C}(N) = \mathcal{T}(N)$.

Strategy:

(1) Compute $\mathcal{C}(N)$ explicitly to give a lower bound for $\mathcal{T}(N)$.
(2) Study the Eisenstein ideal of the Hecke algebra of level N to give an upper bound for $\mathcal{T}(N)$.

Pass to the function field setting.

15. A variation of a theme after Dirichlet, *Chung Hang Kwan*

$$\int_0^1 \int_0^\infty \Phi \left[\begin{pmatrix} y_0 & & \\ & y_0 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] d^\times y_0 e(-u) \, du,$$

$$\int_0^\infty \int_0^1 \Phi \left[\begin{pmatrix} 1 & u & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 & 0 & 0 \\ 0 & y_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] e(-u) \, du \, d^\times y_0.$$

16. On Hecke eigenvalues of Ikeda lifts, *Nagarjuna Chary Addanki*

Preprint: [1].

$S_k(\Gamma_2)$ decomposes into a Maass space and its complement. The Maass space is generated by Saito–Kurokawa lifts. These lifts are characterized by the signs of their eigenvalues: F is a Saito–Kurokawa lift if and only if all the eigenvalues are positive. The question is whether there is a similar space in higher dimensions. Ikeda lifts are generalizations of these Saito–Kurokawa lifts, so we can: are the eigenvalues of such lifts positive?

We're going to talk about Siegel modular forms, which are defined on the Siegel upper half-space

$$\mathbb{H}_n := \left\{ Z \in M_n(\mathbb{C}), \; {}^t Z = Z, \; \Im(Z) > 0 \right\}.$$

The symplectic group $\mathrm{GSp}_{2n}(R) \leq \mathrm{GL}_{2n}(R)$ is given by the condition ${}^t g J g = \mu(g) J$, where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We set $\Gamma_n := \mathrm{Sp}_{2n}(\mathbb{Z})$. A Siegel modular form F of weight k over Γ_n is defined in the usual way. The space of such forms, $M_k(\Gamma_n)$, is finite-dimensional. Elements F admit a Fourier expansion

$$F(Z) = \sum_{\substack{T = T^t, \\ T \geq 0, \\ \text{half-integral}}} A(T) e^{2\pi i \, \mathrm{trace}(TZ)}.$$

We can define a notion of cusp form ($A(T) = 0$ unless $T > 0$) and Petersson inner product.

For each $g \in G(\mathbb{Q})^+ \cap M_{2n}(\mathbb{Z})$, there is an associated Hecke operator $T(g)$ on $M_k(\Gamma_n)$. The Hecke algebra, $\mathcal{H}_n$ is generated by $T(g)$.

- $\mathcal{H}_n$ is commutative.
- To each $m \in \mathbb{N}$ we may associate $T(m) \in \mathcal{H}_n$ such that $T(mn) = T(m)T(n)$ when $(m, n) = 1$.
- Each $T(g)$ is self-adjoint with respect to the Petersson inner product.

Andrianov 1973: basis of simultaneous eigenfunctions.

Eigenvalues may be given in terms of the Satake parameters. Let F be a Hecke eigenform and $\lambda_F(g)$ the eigenvalue corresponding to the operator $T(g)$. For any g with $\mu(g) = p^r$, depending on F, there are $n + 1$ complex numbers

$$\left(a_{0,p}^{(F)}, a_{1,p}^{(F)}, \ldots, a_{n,p}^{(F)} \right)$$

satisfying, with

$$\Gamma_n g \Gamma_n = \sqcup_i \Gamma_n g_i, \quad g_i = \begin{pmatrix} A_i & B_i \\ 0 & D_i \end{pmatrix}, \quad D_i = \begin{pmatrix} p^{d_{i1}} & \cdots & * \\ 0 & \cdots & * \\ 0 & \cdots & p^{d_{in}} \end{pmatrix},$$

we can explicitly describe the Hecke eigenvalue by

$$\lambda(g) = a_{0,p}^r \sum_i \prod_{j=1}^n (a_{j,p}^{-j})^{d_{ij}}.$$

.

Andrianov, 1974: if $F \in S_k(\Gamma_n)$ is a Hecke eigenform, then

$$\sum_{r=0}^\infty \lambda_F(p^r) p^{-rs} = \frac{P_{F,p}(p^{-s})}{Q_{F,p}(p^{-s})},$$

$$Q_{F,p}(p^{-s}) = \cdots.$$

Let $f \in S_k(\Gamma_1)$ be a Hecke eigenform.

Theorem 55 (Ikeda, 2001). *Assume that $n \equiv k \pmod 2$. Then*

$$F_f(Z) = \sum_{B \in \mathcal{S}_{2n}(\mathbb{Z})^+} A(B) e^{2\pi i \operatorname{trace}(BZ)}, \quad Z \in \mathbb{H}_{2n}$$

is a Hecke eigenform in $S_{k+n}(\Gamma_{2n})$ and the standard L-function of F is equal to

$$\zeta(s) \prod_{i=1}^{2n} L(s + k + n - i, f, \operatorname{spin}).$$

Here $\mathcal{S}_{2n}(\mathbb{Z})^+$ is the set of all positive definite half-integral matrices of size $2n$.

To compute the spin L-function of the Ikeda lift, move to the language of automorphic representations. Let $F_f \in S_{k+n}(\Gamma_{2n})$ be the Ikeda lift of $f \in S_{2k}(\Gamma_1)$. Let Π and π be the associated automorphic representations.

Theorem 56 (Schmidt, 2003). *We have*

$$L_p(s, \Pi, \operatorname{spin}) = \prod_{j=0}^n \prod_{\substack{r=j(j-2n) \\ \text{Step } 2}}^{j(2n-j)} L_p(s + \tfrac{r}{2}, \pi, \operatorname{Sym}^{n-j})^{\beta(r,j,n)}.$$

Here $\beta(r, j, n)$ is given explicitly.

Theorem 57 (A, 2024). *For $n \equiv k \pmod 2$, let $F_f \in S_{k+n}(\Gamma_{2n})$ be the Ikeda lift of $f \in S_{2k}(\Gamma_1)$. For all large enough primes p, we have $\lambda_{F_f}(p) \geq 0$.*

Lemma 58. If $F_f \in S_{k+2}(\Gamma_4)$ is the Ikeda lift of $f \in S_{2k}(\Gamma_1)$, then $\lambda_{F_f}(p)$ is positive for all primes p.

For $\lambda_F(p^r)$ with $r > 1$, we use

$$\sum_{r=0}^\infty \lambda_F(p^r) p^{-rs} = \frac{P_{F,p}(p^{-s})}{Q_{F,p}(p^{-s})},$$

and we need knowledge of both numerator and denominator. The denominator can be computed using Schmidt's formluaf or the L-function, but the numerator is known only for genus at most 4.

Let $T(p^4)$ be Hecke operators in $\mathcal{H}_4$. Vankov (2011) found polynomials P_p and Q_p over the Hecke algebra such that

$$\sum_{r \geq 0} T(p^r)x^r = \frac{P_p(x)}{Q_p(x)}.$$

$P_{p,F}(x)$ can be retrieved from $P_p(x)$.

$\lambda_F(p^r)$ for genus 4 Ikeda lifts? We do the following.

- Express $\frac{1}{Q_{p,F}(p^{-s})}$ as partial fractions.
- Write each partial fraction as a power series in p^{-s}.
- Multiply by $P_{p,F}(p^{-s})$.
- Compare the coefficients of p^{-rs} on both sides.

This yields a formula for $\lambda_{F_f}(p^r)$ in terms of some $c_i \in \mathbb{Z}[a, a^{-1}]$ that are bounded, uniformly in r. This lead sto:

Theorem 59 (A, 2024). *Let F_f be a Hecke eigenform in $S_{k+2}(\Gamma_4)$, the Ikeda lift of $f \in S_{2k}(\Gamma_1)$. FOr a fixed r, the number $\lambda_{F_f}(p^r)$ is positive for all large enough primes p.*

17. Abelian covers of P1 of p-ordinary Ekedahl–Oort type, *Deepesh Singhal*

Preprint: [13].

Fix a prime p. Let A be a g-dimensional principally polarized abelian variety defined over $\bar{\mathbb{F}}_p$. There are three invariants attached to A:

- p-rank: $f(A) = \dim_{\bar{\mathbb{F}}_p}(\mu_p, A)$. Note that
  - $\#A[p](\bar{\mathbb{F}}_p) = p^f$,
  - $0 \leq f \leq g$.
- Newton polygon: the isogeny class of the p-divisible group $A[p^\infty]$, which can be presented as a sequence of $2g$ rational numbers $NP(A) := (\nu_1, \ldots, \nu_{2g})$.
- Ekedahl–Oort type: the isomorphism class of the p-kernel $A[p]$, which can be presented as an element $w(A) \in \mathfrak{S}_{2g}$.

When $g = 1$, A is an elliptic curve:

- $f(A) = 0$ iff $NP(A) = (\frac{1}{2}, \frac{1}{2})$ iff $w(A) = \text{id}$ iff A is supersingular.
- $f(A) = 1$ iff $NP(A) = (0, 1)$ iff $w(A) = (21)$ iff A is ordinary.

When $g \geq 2$, these are distinct invariants with some relation among each other:

- p-rank can be computed from either the Newton polygon or the EO-type.
- $f(A) = \#\{i : v_i = 0\}$.

One has the *Torelli morphism* from the moduli of curves to the moduli of abelian varieties:

$$\text{Hurwitz space} \xrightarrow{T} \text{Shimura variety}$$

$$C \xrightarrow{T} J(C).$$

- Fact: $\dim(\text{im}(T)) < \dim(\text{ambient Shimura variety})$.
- For $S = \mathcal{A}_{g,1}$, it is known which p-rank, Newton polygons and EO-types give *nonempty* stratum.

Question 60. Which stratum in $\mathcal{S}$ has nonempty intersection with the Torelli locus?

We turn to the Hurwitz space of cyclic covers of $\mathbb{P}^1$. Fix m such that $p \nmid m$. $\mathcal{M}_{\mu_m}$ is the moduli space of μ_m covers of $\mathbb{P}^1$.

- $\mathcal{M}_{\mu_m} = \cup \mathcal{M}(m, r, \underline{a})$ is the decomposition of irreducible components.
- r is the number of branching points.
- $\underline{a} = (a(1), a(2), \ldots, a(r))$ is the monodromy data of each branching point.
- Looking at one of the irreducible components, we have have the restricted Torelli morphism

$$\mathcal{M}(m, r, \underline{a}) \xrightarrow{T} \mathrm{Sh}(H, \mu).$$

Here

- $\mathrm{Sh}(H, \mu)$ is the smallest PEL type Shimura variety containing the image of T, with μ determined by the data $\underline{a}$.
- The set of the Newton polygons occurring is known and denoted by $B(H_{\mathbb{Q}_p}, \mu)$. However, very little is known about which of these Newton polygons have a stratum that intersects the Torelli locus.
- The set $B(H_{\mathbb{Q}_p}, \mu)$ has a partial ordering, and $\nu(H, \mu)$ is the unique maximal element in this set. It is called the μ-*ordinary* Newton polygon.

Example 61. Consider the moduli space of μ_7-covers of $\mathbb{P}^1$, with $r = 4$ branching points, and monodromy datum $(3, 2, 1, 1)$. If $p \equiv 6 \pmod{7}$, then there are three Newton polygons:

- the μ-ordinary $\mu = (0, 1)^4 \oplus (\frac{1}{2}, \frac{1}{2})^2$, with p-rank 4.
- $\nu = (0, 1)^2 \oplus (\frac{1}{2}, \frac{1}{2})^4$, with p-rank 2.
- the basic $\beta = (\frac{1}{2}, \frac{1}{2})^6$, with p-rank 0.

We refine Bouw's result and show the sharpness of the upper bound on Newton polygon given by $B(\mathbb{H}_{\mathbb{Q}_p}, \mu)$:

Theorem 62 (Y. Lin, E. Mantovan, S., 2023). *Fix a prime p and let $m \in \mathbb{N}$ such that $p \nmid m$ and $p > m(r - 2)$. Then for any $\mathcal{M}(m, r, \underline{a})$ such that $r \leq 5$, we have*

$$T(\mathcal{M}(G, r, \underline{a}))[\nu(H, \mu)] \neq \emptyset.$$

In addition, we also show (...).

Comments:

- Same conclusion holds for abelian covers.
- The statement of the theorem likely remains true without the assumptions that $p > m(r - 2)$ and $r \leq 5$. However, these conditions are needed for our technique to work.
- (...)

Proof strategy:

Theorem 63 (Moonen, 2004). *For a PEL-type Shimura variety $\mathrm{Sh}(H, \mu)$, we have*

$$\nu(H, \mu) \ stratum \ = \ maximal \ EO \ stratum \ \subseteq \ maximal \ p\text{-}rank \ stratum.$$

To prove the non-emptiness of the intersection with maximal EO stratum, we use the equivalence of categories between

- p-kernel of polarized abelian variety,
- polarized mod p Dieudonné module,
- Hasse–Witt triples.

We translate the condition of having maximal EO type to linear algebra conditions on the Hasse–Witt triple. By explicitly computing them, we are able to verify the rank conditions.

18. Hyperbolic Counting Problems, *Marius Voskou*

Preprint: [16].

We start with an interlude on hyperbolic geometry. Work with $\mathrm{PSL}_2(\mathbb{R})$ (orientation-preserving isometries), lattice points $\gamma \cdot z$, where $\gamma \in \Gamma$, a discrete cofinite subgroup of $\mathrm{PSL}_2(\mathbb{R})$, and $z \in \mathbb{H}$.

Consider a hyperbolic circle of center w, radius R.

This looks like a Euclidean circle; the only difference is that the center is slightly lower than what you'd expect, which has to do with the fact that distances blow up as you get closer to the x-axis.

We're now ready to formulate our first hyperbolic counting problem: take

$$N(z, w, R) := \# \left\{ \gamma z : \gamma \in \Gamma, \mathrm{dist}(w, \gamma z) \leq R \right\}.$$

The same method as in the Euclidean counting problem does, in principle work. We get a main term comparable to the area, an error term comparable to the circumference. The problem is that in hyperbolic geometry, these are comparable – both are comparable to e^R – so the error term is not smaller than the main term. Instead, we use the spectral theory of automorphic forms.

To that end, the first step is to write our count in terms of the automorphic kernel:

$$N(z, w, R) = K(z, w) = \sum_{\gamma} \mathbf{1}(\mathrm{dist}(w, \gamma z) < R).$$

We know that automorphic kernels have nice spectral expansions in terms of eigenfunctions of the hyperbolic Laplacian. Since it's a 20 minute talk, we'll just take spectral theory as a black box and not say much about it. The thing we want to note is that the hyperbolic Laplacian has two kinds of eigenvalues:

- large eigenvalues: $\lambda \geq 1/4$, $\lambda = s(1-s), s = 1/2 + it$.
- small eigenvalues: $\lambda < 1/4, \lambda = s(1-s), s \in (1/2, 1]$, of which there are finitely many.

Using this spectral expansion, Selberg in the 70's proved an asymptotic formula for $N(z, w, R)$:

Theorem 64 (Selberg 1970, Gunther 1980, Good 1983)**.** *We have*

$$N(z, w, \log X) = \frac{2\pi}{\mathrm{area}(\mathbb{H}/\Gamma)}X + \sum_{s_j} c_j(z, w)X^{s_j} + \mathrm{O}(X^{2/3}),$$

where the sum is over finitely many real numbers $s_j \in (1/2, 1)$, corresponding to the small eigenvalues.

Now, let's move to a slightly more complicated problem: let's ask what happens if instead of points, we have geodesics. For fixed (closed) geodesics ℓ_1, ℓ_2, how many geodesics $\gamma \cdot \ell_1$ have distance $\leq R$ from ℓ_2? For simplicity, take $\ell_1 = \ell_2 = \ell$.

[Picture of a pair of closed geodeics intersecting.]

The distance between closed geodesics is the length of the common perpendicular. We want to count how many closed geodesics are at given distance from one closed geodesic.

Our strategy is again to use the spectral theory of automorphic forms. We can write the count as the integral of an automorphic kernel:

$$N(\ell_1, \ell_2, R) = \int_{\ell_1} \int_{\ell_2} K(z, w) \, ds(z) \, ds(w).$$

This leads to an asymptotic formula:

Theorem 65 (Good 1983, Lekkas–Petridis 2024)**.** *We have*

$$N(\ell, \log X) = \frac{2(\operatorname{len} \ell)^2}{\pi \operatorname{area}(\mathbb{H}/\Gamma)} X + \sum_{s_j} (\ell) X^{s_j} + E(X),$$

where $E(X) = \mathrm{O}(X^{2/3})$.

Let's say a few words about why it's necessary to have a second proof of this. The first proof by Good was hard to read, and some people still question whether it was valid. In particular, if you want to do refinements of this or apply his methods to obtain related results, it's difficult. Petridis–Lekkas found simpler methods that also apply to other problems.

Conjecture 66. *We have* $E(X) = \mathrm{O}(X^{1/2+\varepsilon})$. *(And similarly for the previous problem, and many similar problems.)*

This has been around for many decades, and we have zero improvements on $X^{2/3}$.

What we're going to try to do now is to motivate this conjecture. To that end, we'll start by saying why we can't do any better:

Theorem 67 (V. 2024)**.** *We have*

$$E(X) = \Omega \left(X^{1/2} (\log \log X)^{1/4 - \delta} \right).$$

Theorem 68 (Lekkas–Petridis 2024)**.** *We have*

$$\left(\frac{1}{X} \int_X^{2X} |E(u)|^2 \, du \right)^{1/2} \ll X^{1/2} \log X.$$

Now, let's restrict to the case of a primitive hyperbolic element

$$h = \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}.$$

For $\ell = \mathcal{I}/\langle h \rangle$, we have

$$\cosh \operatorname{dist}(\ell, \gamma \ell) = \max \left(|ad + bc|, 1 \right),$$

so our counting problem can be rephrased as follows: how many double cosets $\gamma \in \langle h \rangle \backslash \Gamma / \langle h \rangle$ with $|ad + bc| < X$? We can also ask, are there more γ with $ad + bc > 0$ than $ad + bc < 0$, or less? Equivalently, are there more a, d having the same sign than different, or less? Geometric interpretation: the sign corresponds to the direction of the image $\gamma \cdot \ell$, and whether it lies on the left or the right of ℓ. This behaves differently as we tilde ℓ to some ℓ_θ. Idea: consider

$$\frac{\partial^2}{\partial\theta\partial\phi} N(\ell_\theta, \ell_\phi, R).$$

Let's finish with a nice arithmetic application. Fix a prime p. For Γ an appropriate quaternion order, and $m = \left(1 + \sqrt{2} \right)^2$:

Theorem 69 (V. 2023, $p = 5$; Hejhal 1982)**.** *We have*

$$\sum_{n \leq X} \mathcal{N}(n)\mathcal{N}(pn \pm 1) = \frac{p}{p + \left(\frac{2}{p}\right)} \cdot \left(\frac{\log m}{\pi}\right)^2 X + \sum_{1/2 < s_j < 1} a_j^{\pm} X^{s_j} + \mathrm{O}(X^{2/3}),$$

where $a_j^{\pm}$ is real and $\mathcal{N}(n)$ is the number of $\mathfrak{a} \leq \mathbb{Z}[\sqrt{2}]$ with $\|(\|\mathfrak{a}) = n$.

That's all.

19. False indefinite theta functions and partitions separated by parity, *Kathrin Bringmann*

Today we'll first say a bit about modular forms and partitions, but we'll probably skip most of that, since we've already seen it. In particular, we need the Dedekind eta function $\eta(q) = q^{1/24} \prod (1 - q^n)$ and the theta function $\Theta(q) = \sum q^{n^2}$, together with the partition generating function $P(q) = \sum p(n)q^n$. By work of Hardy and Ramanujan who developed the circle method, we have asymptotics for $p(n)$, which Rademacher improved to an exact formula: letting

$$A_k(n) := \sum_{h(k)^*} \omega_{h,k} e^{-\frac{2\pi i n h}{k}},$$

and the Bessel function I_α, Rademacher obtained

$$p(n) = \frac{2\pi}{(24n - 1)^{3/4}} \sum_{k \geq 1} \frac{A_k(n)}{n} I_{3/2}\left(\frac{4\pi\sqrt{n}}{k}\right).$$

Goal: determine asymptotic behavior of $a(n)$ as $n \to \infty$.

Let $A(q) := \sum_{n \geq 0} a(n)q^n$ denote the generating function. Then we can recover $a(n)$ via Cauchy. The Hardy–Ramanujan asymptotics are obtained using

$$I_\kappa(x) \sim \frac{e^x}{\sqrt{2\pi x}}.$$

Rademacher–Zuckerman obtained exact formulas for Fourier coefficients of modular forms of negative weights, and Bringmann–Ono addressed the case of weight zero.

Ramanujan's lost notebook introduced the q-hypergeometric series $\sigma(q)$, with coefficients $S(n)$.

Theorem 70 (Andrews–Dyson–Hickerson)**.** *We have $S(n) = T(24n + 1)$, where $T(m)$ is the number of solutions to a certain Pell's equation.*

Definition 71. We say that a q-series is *lacunary* if its coefficients are almost always zero.

Corollary 72. *$\sigma(q)$ is lacunary. $S(n)$ obtains every integer infinitely often.*

Key identity: an expression for $\sigma(q)$ as a false indefinite theta function (not quite a modular form).

Now, what are the partition statistics that we're interested in? We're interested in partitions separated by parity: even parts larger than odd parts (or reversed). You can also require that parts be or not be distinct,. In particular, we'r einterested in $p_{od}^{eu}(n)$, the number of partitions with odd parts smaller than even parts and odd parts distinct. For instance, for $n = 5$, we get $5, 4 + 1, 2 + 2 + 1$, so $p_{od}^{eu}(5) = 3$.

The associated generating function $F_{od}^{eu}(q)$ is related to σ via

$$\frac{1}{(q^2, q^2)_\infty} \left(1 - \cdots\right),$$

where σ is a simple expression involving σ.

Theorem 73 (B–Craig–Nazaroglu). *As $n \to \infty$, we have*

$$\sim \frac{e^{\pi\sqrt{n/3}}}{2\sqrt{3}n}.$$

Proof. Use that coefficients are monotonic, Tauberian theorem, Euler–Maclaurin.
$\square$

Let's now tweak the combinatorics a little bit: take $p_{eu}^{ou}(n)$ denote the number of partitions with even parts smaller than odd parts, then the corresponding generating function is

$$F_{eu}^{ou}(q) = \frac{1}{(1-q)(q^2;q^2)_\infty}.$$

We can also define p_{eu}^{od}, whose generating function is

$$\frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} q^{n^2}.$$

Next, we look at p_{ed}^{ou} (even parts smaller than odd parts, parts distinct), where the generating function is

$$\frac{1}{2(q;q^2)_\infty} \left((q;-q)_\infty + 1 - \sum_{n\geq 0}(1-(-1)^n q^n)(-1)^{\cdots}q^{\cdots} \right).$$

This isn't modular, but you can improve it to obtain some sort of modular by introducing, for $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$,

$$\psi(z,\tau) := i \sum_{n\in\mathbb{Z}} \operatorname{sgn}\left(n+\tfrac{1}{2}\right)(-1)^n \zeta^{n+1/2} q^{\frac{1}{2}(n+\frac{1}{2})^2}.$$

We can recover this as the limit as $t \to \infty$ of a $\hat\psi(z;\tau,\tau+it)$, where:

Theorem 74 (B.–Nazaroglu). $\hat\psi$ *has modular transformation properties.*

Next, we want to look at p_{ed}^{od}: partitions where even parts are smaller than odd parts, odd parts distinct. The generating function is

$$\frac{(-q;q^2)_\infty}{1-q} - \frac{q(-q^2;q^2)_\infty}{1-q},$$

which is just some modular object, so you can use the circle method.

Similarly for a related case.

Next, let's look at p_{ou}^{ed}: odd parts smaller than even parts. The generating function involves Ramanujan's mock theta function $f(q)$, which Zwegers showed how to understand (by completing by adding a second piece, given by an integral of a theta function of weight 3/2).

Harmonic Maass forms are $\hat{f}$ that transform like a modular form of weight $1/2$, annihilated by hyperbolic Laplacian of weight $1/2$. B.–Mahlburg developed a theory of coefficients of mock modular forms multiplied by a modular form. B.–Craig–Nazaroglu obtained leading asymptotics for all coefficients; can one obtain refinements for all of them?

σ is not enough; one needs also its companion

$$\sigma^*(q) = \sum_{n \geq 1} \frac{(-1)^n q^{n^2}}{(q; q^2)_n}.$$

Let $a(n)$ its coefficients.

Theorem 75 (Cohen). *These are the coefficients of a Maass form.*

Definition 76 (Zagier). $f : \mathbb{Q} \to \mathbb{C}$ is a quantum modular form if $h_\gamma(x) := f(x) - f|_k \gamma(x)$ is nice on some nice subsets of $\mathbb{R}$.

For q a root of unity, we have $\sigma(q) = -\sigma^*(q^{-1})$. We define f in terms of σ and σ^*, and also a function $R_\tau(z)$, and a measure defined by a bracket. Then we can concisely express the transformation law of f in terms of these definitions. In particular, we deduce that f is a quantum modular form. This leads to improved asymptotics for the coefficients. First, we split off components that we understand. Let $r_o(n)$ denote the number of partitions of n into distinct odd parts. Its generating function is $\prod_{n \geq 0}(1 + q^{2n+1})$. We can decompose into even and odd parts.

Our main theorem:

Theorem 77 (B.–Craig–Nazaroglu). *Very good asymptotics for $\alpha_j(n)$.*

Work in progress: recover determine asymptotic expansion.

Idea of proof: use Maass forms. Build a vector-valued form out of σ. Circle method.

20. On the local functional equation for archimedean exterior square L-function of GL_n, *Ravi Raghunathan*

The purpose of the talk is to complete the archimedean theory of the exterior square L-function of GL_r via the theory of integral representations due to Jacquet and Shalika.

- F: local, characteristic zero
- $\psi_F : F \to \mathbb{C}$, nontrivial
- π: irreducible admissible representation of $G_r := \mathrm{GL}_r(F)$.

Focus on the case F archimedean, since the results are new only in that case.

The proofs are usually simpler when r is odd, so we'll focus on the case $r = 2n$.

The basic idea is as in Tate's thesis. To each irreducible admissible representation π of G_r, the local Langlands correspondence attaches an n-dimensional representation $\rho(\pi)$ of the Weil group (resp. Weil–Deligne group) when F is archimedean (resp. p-adic). For a character $\chi : F^\times \to \mathbb{C}$, the twisted exterior square L-function and ε-factor of π are defined via the local Langlands correspondence:

$$L(s, \pi, \wedge^2, \chi) := L(s, \wedge^2(\rho(\pi)) \otimes \chi),$$
$$\varepsilon(s, \pi, \wedge^2, \chi) := \varepsilon(s, \wedge^2(\rho(\pi)) \otimes \chi, \psi_F).$$

The right hand sides are the Artin L-function and ε-factor. When the character χ is trivial, we drop it from the notation.

Example 78. If π_p is an unramified representation of $G_n(\mathbb{Q}_p)$, then $\rho(\pi) = \chi_1 \oplus \cdots \oplus \chi_n$, $\chi_i = |.|_p^{\lambda_i}$, then with $\alpha_i(p) = p^{-\lambda_i}$, we have

$$L_G(s, \pi_p, \wedge^2) = \prod_{i<j} \frac{1}{(1 - \alpha_i(p)\alpha_j(p)p^{-s})}.$$

In fact, the above formula is valid (even for finite extensions of $\mathbb{Q}_p$) as long as $\chi_i\chi_j$ is ramified, whenever χ_i and χ_j are both ramified.

If F is p-adic, we denote by $\mathfrak{o}_F$ the ring of integers, and q the cardinality of the residue field. We let N_r denote the standard maximal unipotent subgroup, and Z_r the center of G_r. Let M_r denote the space of all $r \times r$ matrices, and V_r the subspace of upper-triangular $r \times r$ matrices.

We set

$$w_n = \begin{pmatrix} & & 1 \\ & \cdots & \\ 1 & & \end{pmatrix}, \quad w_{n,n} := \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}.$$

We view these as elements of G_n or G_r, respectively.

Okay, one more thing. We will assume our representations are generic, i.e., have a Whittaker model, which means you can embed the representation into a space of functions on G. We extend ψ_F to a character of N_r by setting $\psi(u) := \prod_{j=1}^{r-1} \psi_F(u_{j,j+1})$ for $u \in N_r$. We will assume that ψ is an irreducible generic representation of G_n with respect to ψ as above. We denote the Harish–Chandra Schwartz space of Whittaker functions on π by $\mathcal{W}(\pi, \psi)$, and refer to it in this talk simply as the Whittaker model of π.

When F is archimedean, the space $\mathcal{W}(\pi, \psi)$ consists of smooth functions satisfying a moderate growth condition, transformation property with respect to N_r and ψ, and is a Frechét space.

We turn to the integral representation of Jacquet and Shalika. For each $W \in \mathcal{W}(\pi, \psi)$, we define the function $\tilde{W}$ on G_r by

$$\tilde{W}(g) := W(w_n \, {}^\iota g), \quad {}^\iota g := {}^t g^{-1}.$$

The Whittaker model of $\tilde{\pi}_v$, $\mathcal{W}(\tilde{\pi}, \bar{\psi})$, consists precisely of the set of functions $\{\tilde{W} : W \in \mathcal{W}(\pi, \psi)\}$. Let $\mathcal{S}(F^n)$ be the space of Schwartz functions. Use self-dual Haar measures. Then the integral is

$$J(s, W, \phi) = \int_{N_n \backslash G_n} \int_{V_n \backslash M_n} W\left(\sigma \begin{pmatrix} 1_n & X \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}\right) \psi(-\operatorname{trace} X) \, dX \phi(e_n g) |\det g|^s \, dg$$

for $W \in \mathcal{W}(\pi, \psi)$ and $\phi \in \mathcal{S}(F^n)$ and $s \in \mathbb{C}$ and σ the permutation matrix given by

$$\begin{pmatrix} 1 & 2 & \cdots & n & | & n+1 & n+2 & \cdots & 2n \\ 1 & 3 & \cdots & 2n-1 & | & 2 & 4 & \cdots & 2n \end{pmatrix}$$

Proposition 79 (Jacquet–Shalika 1990). *(1) For a generic irreducible representation π of G_n, the integrals $J(s, W, \phi)$ converge absolutely for $\Re(s) > 1 - \eta$ for some $\eta > 0$.*
(2) The integrals admit meromorphic continuation.
(3) The integrals define elements of $\mathbb{C}(q^{-s})$.

We can thus take the fractional ideal of $\mathbb{C}[q^s, q^{-s}]$ generated by these integrals. The generator can be taken (by results of Belt) to be of the form $1/P(q^{-s})$. This

is then defined to be the Jacquet–Shalika exterior square L-value. Are these the same as the Artin L-functions? Yes:

Theorem 80 (Kewat–R, Y. Jo). *Let F be a p-adic field. Then with π as above,*
$$L_{JS}(s, \pi, \wedge^2) = L(s, \pi, \wedge^2).$$

We now want to do the same thing in the archimedean setting, where less is known. Implicit in Jacquet–Shalika is that these integrals admit meromorphic continuation.

First main result:

Theorem 81. *For $r = 2n$, the quotients*
$$\Xi(s, W, \phi) := \frac{J(s, W, \phi)}{L(s, \pi, \wedge^2)}$$
are entire and of finite order in vertical strips. Moreover, if π is "nearly tempered", then for every $s_0 \in \mathbb{C}$, we can choose $W \in \mathcal{W}(\pi, \psi)$ and $\phi \in \mathcal{S}(F^n)$ such that $\Xi(s_0, W, \phi) \neq 0$.

Here "nearly tempered" means that every pole of the L-function is realized by one of the integrals, so that the integrals really do capture the L-function more-or-less completely.

Second main result is the local functional equation:

Theorem 82. *With conditions as above, we have*
$$\frac{J(1 - s, \rho(w_{n,n})\tilde{W}, \hat{\phi})}{L(1 - s, \tilde{\pi}, \wedge^2)} = \varepsilon_{JS}(s, \pi, \wedge^2, \psi) \frac{J(s, W, \phi)}{L(s, \pi, \wedge^2)},$$
where ε_{JS} is of the form $A_0 B_0^s$.

For F p-adic this was proved by Matringe, and in the odd case by Cogdell–???.

The main new tool we will use in the proofs of our results is the following very nice observation of Beuzart–Plessis (2021), which is deduced from Theorem 2 of Finis–Lapid–Muller (2015).

Theorem 83 (Beuzart–Plessis). *Let k be a number field and let w_0 and w_1 be places of k with w_1 non-archimedean. Let V be any open subset of $\mathrm{Temp}(\mathrm{GL}_n(k_{w_0}))$. There exists a cuspidal automorphic representation $\Pi = \otimes'_w \Pi_v$ of $\mathrm{GL}_n(\mathbb{A}_k)$ such that $\Pi_{w_0} \in V$ and Π_w is unramified for every finite place $w \neq w_1$.*

The nice new part of this theorem is that you can embed archimedean representations as well. Moreover, there is this density that happens – the moment you have an open set, you can find something there that can be pushed inside a global representation. Once you have this, you're in great shape, because we were already (in Kewat–R) able to do this whenever we could push our representations into global ones. Indeed, Beuzart–Plessis used the above theorem to prove analogues of our main results (Theorems 81 and 80) for the Asai L-function.

Let's say a bit about the proof. Directly imitating Beuzart–Plessis allows us to prove a functional equation, but we don't' know that the ε-factor that we get matches the Artin ε-factor. However, using the embedding theorem for different pairs of places (three times) allows us to show that the two ε-factors are in fact the same. What remains to show:

- Nonvanishing of $\Xi(s_0, W_v, \phi_v)$ in general.

- Showing that there exist finitely many Whittaker and Schwartz functions that realize the L-factor.

Humphries and Jo [11] prove the second assertion up to a polynomial factor, using a single test vector.

21. Traces of Poincare series at square discriminants and Fourier coefficients of mock modular forms, *Vaibhav Kalia*

(Joint work with PhD supervisor Balesh Kumar.)

[Oops, started late]

$\varepsilon_\delta = 1$ if $\delta \equiv 1(4)$, and $\varepsilon_d = i$ otherwise.

Fix $d, D \equiv 0, 1(4)$; call them *discriminants* if they are nonzero.

Fix $\tau \in \mathbb{H}$.

Let $k \in \frac{1}{2}\mathbb{Z}$. Assume $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ if $k \in \mathbb{Z}$, else $\Gamma = \Gamma_0(4)$.

We define the slash operator and Kohnen's space $M_k^{!,+}$.

Let j be Klein's j-function with q-expansion

$$j(\tau) = q^{-1} + 744 + 196884q + \cdots.$$

Zagier 2002: we can give a basis for $M_{3/2}^{!,+}$ as $\mathcal{G}_d$, indexed by $0 < d \equiv 0, 1(4)$, where

$$\mathcal{G}_d(\tau) = q^{-d} + \sum_{0 > D \equiv 0,1(4)} a(d, D)q^{-D},$$

where

$$a(d, D) = \mathrm{Tr}_{D,d}(j - 744) := \frac{1}{\sqrt{d}} \sum_{Q \in \mathrm{PSL}_2(\mathbb{Z}) \backslash \mathcal{Q}_{dD}} \frac{\chi_d(Q)}{|\Gamma_Q|}(j - 744)(\tau_Q).$$

Further, if for $0 > d \equiv 0, 1(4)$ we define

$$\mathcal{F}_d(\tau) := q^d + \sum_{0 < D \equiv 0,1(4)} a(D, d)q^D,$$

then Borcherds (1995) basis for $M_{1/2}^{!,+}$ is given by $\mathcal{F}_d$ for $0 > d \equiv 0, 1(4)$.

Question 84. Are there analogues of the functions $\mathrm{Tr}_{D,d}(\bullet)$ when $dD > 0$ for weight zero functions? Are they related to any famiy of modular forms?

Answer 1. Yes: harmonic weak Maass forms and mock modular forms.

We recall the definition of a weight k harmonic weak Maass form on Γ. It's any smooth function $f : \mathbb{H} \to \mathbb{C}$ that satisfies $f|_k A(\tau) = f(\tau)$, annihilated by Δ_k (hyperbolic Laplacian), and has an exponential growth condition at all cusps. We denote by $\mathcal{H}_k^!$ the space of such weight k harmonic Maass forms on Γ.

By a *sesqui* harmonic Maasss form of weight k, we mean the same thing, but without the second condition concerning Δ_k, and instead $\xi_k \circ \Delta_k(f) = 0$, where ξ_k is as in the definition of the shadow.

The Fourier expansion of $f \in \mathcal{H}_k^!$ at ∞ with $k \neq 1$ is given by

$$f(\tau) = \sum_{\substack{n \in \mathbb{Z} \\ n \gg -\infty}} c_f^+(n)q^n + c_f^-(0)y^{1-k} + \sum_{\substack{n \in \mathbb{Z}-\{0\} \\ n \ll \infty}} c_f^-(n)W_k(2\pi ny)q^n.$$

Following Zagier (2007), a *mock modular form* of weight k for Γ is the holomorphic part f^+ of $f \in \mathcal{H}_k^!$. It turns out that $\xi_k(f)$ is a weakly holmorphic of weight $2 - k$, called the *shadow* of f.

Duke–Imamoglu–Toth (2011) constructed a basis $\mathcal{F}_d$, indexed by $d < 0$, for mock modular forms, where

$$\mathcal{F}_d = \sum_{D>0} a(D,d) q^D$$

with $a(d, D)$ defined in terms of the cycle integrals

$$(2\pi)^{-1} \sum_{Q \in \Gamma \backslash \mathcal{Q}_{dD}} \chi_d(Q) \int_{C_Q} (j - 744)(\tau) \frac{d\tau}{Q(\tau, 1)}$$

for $d, d > 0$ with $dD \neq \square$.

To interpret $a(D, d)$ when $dD = \square$, appeal to BFI2015 and Andersen2015.

JKK2013–2014: basis $\mathcal{G}_d$, indexed by $d > 0$, with coefficients $b(d, D)$ given in terms of twisted traces of cycle integrals of $\hat{\mathbb{J}}_1(\tau)$, where $\Delta_0(\hat{\mathbb{J}}_1) = -j + 720$ (done in the case where $d, D < 0$ and $dD \neq \square$).

What should $b(d, D)$ be when $d, D < 0$ and $dD = \square$? We'll see in a moment.

We'll need the notion of Niebur Poincaré series (1973), which we've already seen, defined for $\Re(s) > 1$ by

$$G_{-1}(\tau, s) = \sum_{A \in \Gamma_\infty \backslash \Gamma} e\left(-\Re(A\tau)\right) \phi_{-1,s}\left(\Im(A\tau)\right).$$

Here

$$\phi_{-1,s}(y) := 2\pi y^{1/2} I_{s-1/2}(2\pi y).$$

It defines a smooth weight zero function for Γ.

The traces of cycle integrals of these Niebur Poincaré series we'll discuss now.

- If $0 < d, D$ with D fundamental and $dD \neq \square$, we have (DIT 2011)

$$\mathrm{Tr}_{d,D}(G_{-1}(\tau, s)) := \frac{1}{2\pi} \sum_{Q \in \Gamma \backslash \mathcal{Q}_{dD}} \chi_D(Q) \int_{C_Q} G_{-1}(\tau, s) \frac{d\tau}{Q(\tau, 1)}.$$

- With $d, D < 0$ and D fundamental and $dD \neq \square$, due to $\chi_D(-Q) = \mathrm{sgn}(D)\chi_D(Q)$, we can define

$$\mathrm{Tr}_{d,D}(G_{-1}(\tau, s)) = \frac{1}{\pi} \sum_{Q \in \Gamma_\infty \backslash \mathcal{Q}_{dD}^+} \chi_D(Q) \int_{S_Q} e(-\Re(\tau)) \phi_{-1,s}(\Im(\tau)) \frac{d\tau}{Q(\tau, 1)}.$$

- Superscripted $+$ means $a > 0$.

If $0 < d$, $D = \square$ and D fundamental, then the above trace is not defined. Andersen 2015:

$$\mathrm{Tr}_{d,D}(G_{-1}(\tau, s)) := \frac{1}{2\pi} \sum_{Q \in \Gamma \backslash \mathcal{Q}_{dD}} \chi_D(Q) \int_{C_Q} G_{-1,Q}(\cdots).$$

ETC.

Modified traces:

$$\mathrm{Tr}_{d,D}(G_{-1}(\tau, s)) = \cdots,$$

$$\tilde{\mathrm{Tr}}_{d,D}(G_{-1}(\tau, s)) := \frac{1}{\pi} \sum_{Q \in \Gamma_\infty \backslash \mathcal{Q}_{dD}^+} \chi_D(Q) \int_{C_Q} e(-\Re(\tau)) \phi_{-1,s}(\Im \tau) \frac{d\tau}{Q(\tau, 1)}.$$

JKK2014: $\hat{\mathbb{J}}_1(\tau, s) := \partial_s G_{-1}(\tau, s)$ and proved that $\hat{\mathbb{J}}_1(\tau) := \hat{\mathbb{J}}_1(\tau, 1)$ is a sesqui-harmonic Maass form of weight zero for Γ. We define

$$\tilde{\mathrm{Tr}}_{d,D}(\hat{\mathbb{J}}_1(\tau)) := \partial\P\tilde{\mathrm{Tr}}_{d,D}(G_{-1}(\tau, s))|_{s=1}$$

when $d, D < 0$ with $dD = \square$.

Theorem 85 (-, Kumar 2024). *For d, D negative with D fundamental and $dD = \square$, we have*

$$b(d, D) = -8\sqrt{dD}\tilde{\mathrm{Tr}}_{d,D}(\cdots),$$

and also ETC.

Idea of the proof:

- $\tilde{\mathrm{Tr}}_{d,D}(G_{-1}(\tau, s)) = \alpha(s, dD) \sum_{c=1}^{\infty} \frac{S(d, D, 4c)}{c^{1/2}} J_{s-1/2}\left(\frac{\pi\sqrt{dD}}{c}\right)$, where

$$S(d, D, 4a) := \sum_{\substack{b(4a) \\ (b^2 - dD)/4a \in \mathbb{Z}}} \chi_D\left(\left[a, b, \frac{b^2 - dD}{4a}\right]\right) e\left(\frac{b}{2a}\right).$$

- $\tilde{\mathrm{Tr}}_{d,D}(\hat{\mathbb{J}}_1 \cdots)$, etc.

We can further relate these traces to the regularized inner product. Let $f, g \in M_{1/2}^{!,+}$ (resp. $M_0^!$). There exists $G \in \mathcal{H}_{3/2}^{!,+}$ (resp. $\mathcal{H}_2^!$) such that $\xi_-(G) = g$ (BDS2017). Then

$$\langle f, g \rangle = \rho_k \sum_{n \in \mathbb{Z}} c_f(n) \cdots.$$

ETC. Relate to Borcherds.

Idea of the proof:

- Since mock modular forms $\mathcal{G}_d$ with shadow $\mathcal{F}_d$ has

$$b(d, D) = 4\sqrt{dD}c_{|d|}(|D|) + 192\pi H(|d|)H(|D|),$$

we get by a computation

$$\langle \mathcal{F}_d, \mathcal{F}_D \rangle = 2\sqrt{|Dd|}c_{|d|}(|D|) + 96\pi H(\cdots),$$

we get ETC.

Next:

Corollary 86. *Let D be a negative fundamental discriminant. Then we have*

$$\tilde{\mathrm{Tr}}_{D,D}(\hat{\mathbb{J}}_1(\tau)) = -\frac{1}{4\pi\sqrt{|D|}} \sum_{n>0} \left(\frac{D}{n}\right) \frac{\langle j_n + 24\sigma(n), j_1 + 24 \rangle}{n} e^{-2\pi n/|D|} - \cdots.$$

Idea of the proof: ANS2021 computed inner products of $\mathcal{F}_D$ in terms of exponential integrals, and then explicitly computed $\tilde{J}$, defined such that $\xi_2(\tilde{J}) = j_1 = j - 744$, to be given by $4\pi h_1 - 8\pi E_2^*$, where $h_1 \in H_2^!$ (DIT2016) satisfies $\xi_2(4\pi h_1) = j_1 + 24$, and

$$E_2^*(\tau) := 1 - 24\sum_{n \geq 1} \sigma(n)q^n.$$

We can thus write this inner product in terms of the Fourier coefficients of the harmonic Maass forms h_1. We then just need to use that $\xi_2(4\pi h_1) = j_1 + 24$ to write $c_{4\pi h_1}^-(n) = 4\pi n c_{j_1+24}(-n)$, then similarly for $+$.

As final remarks:

- For $n < 0$, the Fourier coefficients $c_{j_1+24}(-n)$ of $j_1 + 24$ is the Rademacher–Petersson formula

$$\frac{2\pi}{\sqrt{-n}} \sum_{c>0} \frac{K(-1, -n, c)}{c} I_1\left(\frac{4\pi\sqrt{-n}}{c}\right).$$

- The regularized inner product $\langle j_n + 24\sigma(n), j_1 + 24 \rangle$ can also be interpreted in terms of the Rademacher–ETC.

22. Limiting behaviour and modular completions of MacMahon-like q-series, *Badri Vishal Pandey*

Preprint: [4].

Definition 87 (MacMahon's q-series)**.**

$$\mathcal{A}_a(q) := \sum_{1 \le n_1 < \cdots < n_a} \frac{q^{n_1 + \cdots + n_a}}{(1 - q^{n_1})^{a_1} \cdots}.$$

Example 88.

$$\mathcal{A}_1(q) = \sum_{n \ge 1} \frac{q^n}{(1 - q^n)^2} = \frac{1}{24} - \frac{1}{24} E_2(\tau).$$

This is a quasimodular form.

Motivation of our work is the following theorem:

Theorem 89 (Amdeberhan–Ono–Singh)**.** *Let $a \in \mathbb{N}$. We have*

$$q^{-\frac{a(a+1)}{2}} \mathcal{A}_a(q) = \prod_{n \ge 1} \frac{1}{(1 - q^n)^3} + \mathrm{O}(\cdots).$$

Example 90. We have

$$\prod_{n \ge 1} \frac{1}{(1 - q^n)^3} = 1 + 3q + 9q^2 + 22q^4 + \cdots,$$

and these agree to increasing order with $q^{-1}\mathcal{A}_1$, $q^{-3}\mathcal{A}_2$, $q^{-6}\mathcal{A}_3$, $q^{-10}\mathcal{A}_4$, $q^{-15}\mathcal{A}_5$, $q^{-21}\mathcal{A}_6$, etc.

Let $p_k(n)$ count the partitions of n with parts having k-colors. Its generating function is

$$\sum_{n \ge 0} p_k(n) q^n = \prod_{n \ge 1} \frac{1}{(1 - q^n)^k}.$$

Question 91 (Ono)**.** Find a family of functions that approximate k-colored partitions.

Definition 92. For $a, r, s \in \mathbb{N}$ and $k \in \mathbb{Z}$, define the q-series

$$\mathcal{A}_{a,k,r}(q) := \sum_{1 \le n_1 < \cdots < n_a} \frac{q^{r(n_1 + \cdots + n_a)}}{(1 - q^{n_1})^k \cdots} =: \sum_{n \ge 0} c_{a,k,r}(n) q^n,$$

$$\mathcal{B}_{a,k,r,s}(q) = \sum_{\cdots} \frac{q^{r(n_1^2 + \cdots + n_a^2) + s(n_1 + \cdots + n_a)}}{\cdots} =: \sum_{n \ge 0} d_{a,k,r,s}(n) q^n.$$

Theorem 93 (Bringmann, Craig, van-Ittersum, P.)**.** *Let $a, r, s \in \mathbb{N}$ and $k \in \mathbb{Z}$. We have*

$$q^{-\frac{ra(a+1)}{2}} \mathcal{A}_{a,k,r}(q) = \prod_{n \geq 1} \frac{1}{(1 - q^{rn})(1 - q^n)^k} + \mathrm{O}(q^{a+1}),$$

and similarly for $\mathcal{B}$.

Example 94. We have

$$\frac{1}{(1 - q^n)^4} = 1 + 4q + 14q^2 + 40q^3 + \cdots,$$

approximately to increasing accuracy by $q^{-1} \mathcal{A}_{1,3,1}$, $q^{-3} \mathcal{A}_{2,3,1}$, $q^{-6} \mathcal{A}_{3,3,1}$, and so on.

We want to study these functions $\mathcal{A}$ and $\mathcal{B}$.

Theorem 95 (Bringmann, Craig, van-Ittersum, P.)**.** *If $a, r \in \mathbb{N}$, then $\mathcal{A}_{a,2r,r}$ is a quasimodular form of (mixed) weight $2ar$.*

Example 96. Formulas for $\mathcal{A}_{2,4,2}$ and $\mathcal{A}_{2,4,3}$ in terms of $G_4, G_6, G_2, G_3, G_7, G_5$.

Remark 97. The proof uses the quasi-shuffle algebra.

Remark 98. If $k \neq 2r$, then odd Eisenstein series are involved.

Corollary 99 (Bringmann, Craig, van-Ittersum, P.)**.** *Let $a, r, m \in \mathbb{N}$. There are infinitely many non-nested AP $An + B$ with*

$$c_{a,2r,r}(An + B) \equiv 0 \pmod{m}.$$

Conjecture 100 (Bringmann, Craig, van-Ittersum, P.)**.** *Let $a, k, r, s \in \mathbb{N}$. For any prime p, we have*

$$c_{a,k,r}(p^{\alpha+1}n + p^\alpha \beta) \equiv 0 \pmod{p^{\nu_p(k)-\alpha}}.$$

ETC.

We move now to results on $\mathcal{B}$, which turns out to have modular properties.

Definition 101. We call a function $\hat{f}(\tau, \bar{\tau})$ a *completion* of $f(\tau)$ if

$$\lim_{\bar{\tau} \to -i\infty} \hat{f}(\tau, \bar{\tau}) = f(\tau).$$

Theorem 102. *Let $k, r, s \in \mathbb{N}$.*

(1) *The function $\mathcal{B}_{a,k,r,s}$ is a polynomial in $\mathcal{B}_{k,r,s} := \mathcal{B}_{1,k,r,s}$ of degree a.*
(2) *$\mathcal{B}_{k,r,s} + \mathcal{B}_{k,r,k-s}$ are a mixed weight linear combination of:*
    (a) *functions admitting a modular completion*
    (b) *powers of q times theta functions*
(3) *If $k = s + 1$, then (2b) in the above linear combination vanishes.*

Define Zwegers' multivariable Appell function for $\ell \in \mathbb{N}$ by

$$A_\ell(z_1, z_2 \tau) = \zeta_1^{\ell/2} \sum_{n \in \mathbb{Z}} \frac{(-1)^{\ell n} q^{\frac{\ell n(n+1)2}{\cdots}} \cdots}{\cdots}.$$

Define some R. Then $\hat{A}_\ell$ is given in terms of A_ℓ and ϑ and R.
Theorem of Zwegers describes transformation of $\hat{A}_\ell$.

Now we briefly describe the idea of the proof of our result. Taylor expansion where Taylor coefficients turn out to be similar functions:

$$\exp\left(\sum_{n\geq 1}\frac{(-1)^{n+1}}{n}\mathcal{B}_{nk,nr,ns}(q)X^n\right) = \sum_{j\geq 0}\mathcal{B}_{j,k,r,s}(q)X^j.$$

From this we get that the coefficients on the right hand side are indeed polynomials in terms of the coefficients on the left hand side. This gives the first part. To get the other parts, we use that

$$\mathcal{B}_{j,r,s-k+j}(q) + \mathcal{B}_{j,r,k-s}(q) = \sum_{n\in\mathbb{Z}-\{0\}}\frac{q^{rn^2+(s-k+j)n}}{(1-q^n)^j}$$

$$= -\frac{1}{j!}\left[\frac{\partial^i}{\partial\zeta^j}(1-\zeta)\sum_{n\in\mathbb{Z}}\frac{q^{rn^2+(s+1-k)n}}{1-\zeta q^n}\right]\big|_{z=0}$$

$$= -\frac{1}{j!}\left[\frac{\partial^j}{\partial\zeta^j}(1-\zeta)\zeta^{-r}A_{2r}(z,(-k-r+s+1)\tau\tau)\right]\big|_{z=0}.$$

We get

$$\mathcal{B}_{k,r,k-1}(q) + \mathcal{B}_{k,r,1}(q) = -\frac{1}{k!}\sum_{\ell=0}^{k}\frac{\alpha_{1/2}(k,\ell)}{(2\pi i)^\ell}f_{r,\ell}(\tau),$$

where

$$f_{r,\ell}(\tau) := \left[\frac{\partial^\ell}{\partial z^\ell}\left(\left(\zeta^{-1/2}-\zeta^{1/2}\right)\zeta^{-r}A_{2r}(z,-r\tau,\tau)\right)\right]_{z=0}$$

and $\alpha_{1/2}(k,\ell)$ is a generalized Stirling number defined by a recurrence. Enough to show that $f_{r,\ell}(\tau)$ can be completed.

Fact: Taylor coefficients of a Jacobi form are quasi-modular.

Example 103 $(k=2=s+1, r=1)$.   (1) We write $\mathcal{B}_{2,2,1,1}(q)$ as the coefficient of X^2 in

$$\exp\left(\mathcal{B}_{2,1,1}(q)X - \tfrac{1}{2}\mathcal{B}_{4,2,2}(q)X^2 + \mathrm{O}(X^3)\right),$$

which we compute to be $\frac{1}{2}\mathcal{B}_{2,1,1}^2(q) - \frac{1}{2}\mathcal{B}_{4,2,2}(q)$.
(2) ETC.

23. Explicit lower bounds on the conductor of abelian varieties with specified bad reduction, *Pierre Tchamitchian*

The first part is about L-functions. Consider a "good" L-function: there exists

- $N > 0$, the conductor,
- (d_1, d_2) with $d = d_1 + 2d_2$, the signature,
- μ_j and ν_k, the spectral parameters $(j \leq d_1, k \leq d_2)$

such that the completed L-function

$$\Lambda(s) = N^{s/2}\prod_{j\leq d_1}\Gamma_{\mathbb{R}}(s+\mu_j)\prod_{k\leq d_2}\Gamma_{\mathbb{C}}(s+\nu_k)L(s)$$

satisfies

$$\Lambda(s) = \varepsilon\tilde{\Lambda}(1-s)$$

for some ε, and $L(s)$ admits an Euler product

$$L(s) = \prod_p F_p(p^{-s})^{-1} = \prod_p \left(1 - \alpha_{1,p} p^{-s}\right)^{-1} \cdots \left(1 - \alpha_{d_p,p} p^{-s}\right)^{-1},$$

absolutely convergent for $\sigma > 1$.

Definition 104. We say that $F : \mathbb{R} \to \mathbb{R}$ is a *Mestre test function* if there exists $\varepsilon > 0$ such that

(i) $F(x) \exp\left((\frac{1}{2} + \varepsilon)|x|\right)$ is integrable
(ii) $F(x) \exp\left((\frac{1}{2} + \varepsilon)x\right)$ has bounded variation, is integrable, and the value at each point is the average of its right and left limits
(iii) $(F(x) - F(0))/x$ has bounded variation.

Then we get the Weil–Mestre explicit formula: for $A = \pi^{-d/2} 2^{-d_2} N^{1/2}$ and $\Phi(s) = \int F(x) e^{(s-1/2)x} \, dx$, we have

$$\sum_\rho \Phi(\rho) - \sum_\mu \Phi(\mu) + \sum_{j \leq d_1} I(\tfrac{1}{2}, \mu_j/2) + I(1/2, \overline{\mu_j}/2) + \sum_{k \leq d_2} J(1, \nu_k) + J(1, \overline{\nu_k})$$

$$= 2F(0) \log(A) - \sum_{p,i,m \geq 1} \left(\alpha_{i,p}^m F(m \log p) + \overline{\alpha_{i,p}}^m F(-m \log p)\right) \frac{\log p}{p^{m/2}}.$$

Morally, the proof is a Fourier transform, as in the prime number theorem. You integrate the function $\Phi(s)\Lambda'(s)/\Lambda(s)$ along the rectangular contour with real parts $-\varepsilon$ and $1 + \varepsilon$. Relate left and right hadn sides using functional equation. Compute explicitly the right hand side using the Euler product, and use a Fourier transform. Because of this, we use GRH.

Trick: choose $F(x)$ such that $\Re(\Phi(s)) > 0$ for s on the critical line. Equivalently, choose $F(x)$ with a positive Fourier transform. Then we have the *Weil–Mestre explicit inequality*

$$2F(0) \log(A) \geq \sum_{j \leq d_1} I(\tfrac{1}{2}, \mu_j/2) + I(\tfrac{1}{2}, \overline{\mu_j}/2) + \sum_{k \leq d_2} J(1, \nu_k) + J(1, \overline{\nu_k})$$

$$+ \sum_{p,i,m \geq 1} \left(\alpha_{i,p}^m F(m \log p) + \overline{\alpha_{i,p}}^m F(-m \log p)\right) \frac{\log p}{p^{m/2}}.$$

Theorem 105 (Poitou, Odlyzko, 1976)**.** *Let K be a numebr field of degree n and discriminant d_K, with r_1 real embeddings and r_2 complex embeddings. Then, for n big enough,*

$$|d_K|^{1/n} > (215.3)^{r_1/n} (44.7)^{r_2/n}.$$

Let's talk about bad reduction of elliptic curves. Take E over $\mathbb{Q}$, say

$$y^2 = x^3 + ax + b$$

with $4a^3 + 27b^2 \neq 0$. By reducing the coefficients (of a minimal model at p) modulo p, one gets a curve E_p over $\mathbb{F}_p$. Then we end up in one of the following cases:

- E_p is an elliptic curve over $\mathbb{F}_p$
- E_p is a singular curve, with $E_{p,ns} \cong \mathbb{G}_m$
- E_p is a singular curve, and $E_{p,ns} \cong \mathbb{G}_a$

Set respectively $n_p = 0, 1, 2$ depending upon the reduction (unless $p \in \{2, 3\}$).

The *conductor* of E of the elliptic curve is defined by $N_E := \prod p^{n_p}$.

We define the L-series in terms of the $a_p := p + 1 - \#E_p(\mathbb{F}_p)$, by $L_p(s) = 1 - a_p p^{-s} + p^{1-2s}$ if E has good redcution, omitting the final term if bad reduction. We multiply these together to get $L_E(s)$. Then we have the modularity theorem, saying that these admit an analytic continuation. We also have the Mordell–Weil theorem and BSD conjecture.

If F is even, has positive Fourier transform, and support in $[-1, 1]$, and write $F_\lambda(x) = F(x/\lambda)$ for $\lambda > 0$, then we have the Mestre inequality. It involves

$$M_{\lambda,F} = 2\log(2\pi) + 2I_{F_\lambda,2} = 2\,(\cdots).$$

Instead of looking at all elliptic curves, we can only look at the ones with some prescribed reduction.

We can now do the same things for abelian varieties. Looking at the bad reduction, there's a smooth part, torus part and unipotent part. We associate L-functions and Satake parameters. We have a Mestre inequality (1986): the conductor satisfies $N_A \geq 10.323^g$.

LMFDB.

Everything that we described here over $\mathbb{Q}$ also works over number fields.

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