

These are notes for an ongoing Fall 2023 course on the Riemann zeta function and its generalizations, L -functions. These notes will be filled in as we go.

1. BACKGROUND

1.1. **General notation.** $\mathbb{R}^+ := (0, \infty)$.

1.2. **Asymptotic notation.** We use the equivalent notations

$$A = O(B), \quad A \ll B, \quad B \gg A$$

to denote that

$$|A| \leq C|B|$$

for some “constant” C . The precise meaning of “constant” will either be specified or clear from context.

1.3. **Holomorphic continuation.**

Theorem 1.1 (Identity principle for holomorphic functions). *Let $U \subset \mathbb{C}$ be a connected open set. Let $f, g : U \rightarrow \mathbb{C}$ be holomorphic functions. If $f = g$ on a set with a limit point in U , then $f = g$ on all of U .*

Corollary 1.2. *Let $U \subset \Omega \subseteq \mathbb{C}$ be open subsets, with U nonempty and Ω connected. Let $f : U \rightarrow \mathbb{C}$ be a holomorphic function. Then there is at most one extension of f to a holomorphic function $\Omega \rightarrow \mathbb{C}$.*

1.4. **Cauchy’s integral formula.**

Theorem 1.3. *Let $f : U \rightarrow \mathbb{C}$ be a holomorphic function defined on an open subset U . Let γ be a closed rectifiable curve in U . Then $\int_{\gamma} f(z) dz = 0$.*

Theorem 1.4. *Let $0 \leq a < b \leq \infty$. Let $f(z)$ be a holomorphic function on the annulus $\{z \in \mathbb{C} : a < |z| < b\}$ given by a convergent Laurent series*

$$f(z) = \sum_{n \in \mathbb{Z}} c_n z^n.$$

(1) *For any $r \in (a, b)$ and $n \in \mathbb{Z}$, we have*

$$\begin{aligned} c_n &= \oint_{|z|=r} \frac{f(z)}{z^n} \frac{dz}{2\pi iz} \\ &= \frac{1}{2\pi r^n} \int_{\theta=0}^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta. \end{aligned}$$

(2) *For each compact subset E of (a, b) , there exists $M \geq 0$ so that for all $r \in E$, we have*

$$\sum_{n \in \mathbb{Z}} |c_n| r^n \leq M. \tag{1.1}$$

1.5. **Holomorphy of limits and series.**

Theorem 1.5. *Let U be an open subset of the complex plane. Let f_n be a sequence of holomorphic functions on U .*

- (1) *Suppose that the sequence f_n converges pointwise to some function f , uniformly on compact subsets of U . Then f is holomorphic.*
- (2) *Suppose that the partial sums $\sum_{n \leq N} f_n$ converge pointwise to some function f , uniformly on compact subsets of U . Then the sum $\sum_n f_n$ is holomorphic.*

2. ASYMPTOTICS AND MEROMORPHIC CONTINUATION

We consider a Laurent series

$$f(z) = \sum_{n \in \mathbb{Z}} c_n z^n.$$

Here the c_n are complex coefficients, while z is a nonzero complex argument. We assume that this series converges absolutely for at least one value of z .

Lemma 2.1. *There is a unique maximal open subinterval (a, b) of \mathbb{R}^+ on which f converges absolutely. Its endpoints are given explicitly by*

$$a = \inf \left\{ r \in \mathbb{R}^+ : \sum_n |c_n| r^n < \infty \right\},$$

$$b = \sup \left\{ r \in \mathbb{R}^+ : \sum_n |c_n| r^n < \infty \right\}.$$

We refer to the interval (a, b) as the *fundamental interval* for f (or for the c_n).

The fundamental interval controls the growth of the coefficients c_n as $n \rightarrow \pm\infty$:

Lemma 2.2. *Let $b^- < b$ and $a^+ > a$. Then*

$$c_n \ll (b^-)^{-n} \quad \text{as } n \rightarrow \infty$$

and

$$c_n \ll (a^+)^{-n} \quad \text{as } n \rightarrow -\infty.$$

Set

$$\mathcal{C}(a, b) := \{z \in \mathbb{C} : |z| \in (a, b)\}.$$

Lemma 2.3. *$f(z)$ defines a holomorphic function on $\mathcal{C}(a, b)$.*

Proof. Follows from Theorem 1.5. □

Lemma 2.4. *f does not extend to a holomorphic function on $\mathcal{C}(A, B)$ for any strictly larger interval $(A, B) \supsetneq (a, b)$.*

Proof. Suppose otherwise. Let $r \in (A, B) - (a, b)$. Then by Cauchy's integral formula (specifically, the estimate (1.1) of Theorem 1.4), we see that $\sum_{n \in \mathbb{Z}} |c_n| r^n < \infty$. This contradicts the formula for a and b given in Lemma 2.1. □

Note 2.5. It can happen that f extends to a *meromorphic* function on some strictly larger annulus (unique, in view of Corollary 1.2). By Lemma 2.4, this can only happen if f has a pole at some point on the boundary of the fundamental annulus.

Example 2.6. Take

$$c_n = \begin{cases} 2^n & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

Then the fundamental interval is $(a, b) = (0, 1/2)$. However, the function $f(z)$, defined initially for $|z| < 1/2$, evaluates to a rational function:

$$f(z) = \sum_{n \geq 0} 2^n z^n = \frac{1}{1 - 2z}.$$

This is meromorphic on the entire complex plane; the only pole is a simple one at $z = 1/2$, with residue $-1/2$.

The possibility of meromorphically extending f corresponds to the coefficients c_n having asymptotic expansions as $n \rightarrow \pm\infty$. For example:

Lemma 2.7 (Meromorphic continuation vs. asymptotic expansion, special case). *Let f and (a, b) be as above. Let $\beta \in \mathbb{C}$ with $|\beta| = b$. Let $B > b$ and $\gamma \in \mathbb{C}$. Then the following are equivalent:*

- (i) *f extends to a meromorphic function on $\mathcal{C}(a, B)$ with a unique simple pole at $z = \beta$ with residue γ .*
- (ii) *For each $B^- < B$, we have as $n \rightarrow \infty$ that*

$$c_n = -\gamma\beta^{-n-1} + O\left((B^-)^{-n}\right),$$

Proof. To see that (i) implies (ii), we start with Cauchy's integral formula on the disc of radius b^- for some $b^- \in (a, b)$, then shift the contour, picking up the contribution of the unique pole:

$$\begin{aligned} c_n &= \oint_{|z|=b^-} \frac{f(z)}{z^n} \frac{dz}{2\pi iz} \\ &= \oint_{|z|=B^-} \frac{f(z)}{z^n} \frac{dz}{2\pi iz} - \frac{\gamma}{\beta^{n+1}}. \end{aligned} \quad (2.1)$$

We then estimate this last integral using that f is bounded on compact sets.

Conversely, to verify that (ii) implies (i), we define the coefficients

$$b_n := \begin{cases} -\gamma\beta^{-n-1} & \text{if } n \geq 0, \\ 0 & \text{if } n < 0, \end{cases}$$

The corresponding series

$$f_+(z) := \sum_{n \in \mathbb{Z}} b_n z^n$$

may be evaluated explicitly: a simple geometric series calculation, left to the reader, gives

$$f_+(z) = \frac{\gamma}{z - \beta}.$$

Our hypothesis concerning the c_n reads

$$b_n - c_n = O\left((B^-)^{-n}\right) \quad \text{as } n \rightarrow \infty. \quad (2.2)$$

On the other hand, because f has fundamental interval (a, b) and b_n vanishes as $n \rightarrow -\infty$, we have for each $a^+ > a$ that

$$b_n - c_n = O\left((a^+)^{-n}\right) \quad \text{as } n \rightarrow -\infty. \quad (2.3)$$

From (2.2) and (2.3), we deduce that the series $f - f_+$ with coefficients $c_n - b_n$ has fundamental interval containing (a, B) . This implies that the function

$$f(z) - \frac{\gamma}{z - \beta},$$

defined initially as a holomorphic function on $\mathcal{C}(a, b)$, extends to a holomorphic function on $\mathcal{C}(a, B)$. Equivalently, f extends to a meromorphic function on $\mathcal{C}(a, B)$ with polar behavior as described in (ii). \square

Exercise 1. Generalize the above lemma to describe in terms of the coefficients c_n what it means for f to extend to a meromorphic function on $\mathcal{C}(A, B)$ for some $A < a$ and $B > b$, allowing the possibility of multiple poles of arbitrary order.

Exercise 2. Let c_n denote the Fibonacci sequence, thus $c_n = 0$ for $n < 0$ and

$$c_0 = 1, \quad c_1 = 1, \quad c_{n+2} - c_{n+1} - c_n = 0.$$

This exercise rederives a standard formula for this sequence in a way that is intended to illustrate the technique of Lemma 2.7.

- (1) Verify by crude estimation that the fundamental interval for the series $f(z) = \sum_n c_n z^n$ contains $(0, 1/2)$.
- (2) Show that

$$f(z) = \frac{1}{1 - z - z^2} = \frac{1}{(1 - z/\varphi)(1 - z/\varphi')},$$

where

$$\varphi = \frac{1 + \sqrt{5}}{2} = 1.618\ldots, \quad \varphi' = \frac{1 - \sqrt{5}}{2} = -0.618\ldots.$$

- (3) Following the proof of Lemma 2.7, show that

$$c_n = \frac{\varphi^n - (\varphi')^n}{\varphi - \varphi'}.$$

(Use that $f(z) \ll |z|^{-2}$ for $|z| \geq 2$ to show that the “remainder term”, namely the integral in (2.1), tends to zero as $B^- \rightarrow \infty$.)

3. THE ZETA FUNCTION

3.1. Overview. The Riemann zeta function is defined for a complex number s by the series

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

Lemma 3.1. *The series converges absolutely for $\Re(s) > 1$, uniformly for $\Re(s) \geq 1 + \varepsilon$ for each $\varepsilon > 0$.*

Proof. Using the identity

$$\left| \frac{1}{n^s} \right| = \frac{1}{n^{\Re(s)}},$$

we reduce to the case that s is real, in which this is a familiar consequence of the integral test. \square

Our first main goal in the course is to explain the following basic facts.

Theorem 3.2. *The Riemann zeta function admits a meromorphic continuation to the entire complex plane. It is holomorphic away from a simple pole at $s = 1$, where it has residue 1. It admits a functional equation relating $\zeta(s)$ to $\zeta(1 - s)$.*

One historical motivation for considering the zeta function at complex arguments comes from the prime number theorem.

Theorem 3.3 (Prime number theorem). *Let $\pi(x) := \#\{\text{primes } p \leq x\}$ denote the prime counting function. Then*

$$\frac{\pi(x)}{x/\log x} \rightarrow 1 \text{ as } x \rightarrow \infty.$$

This is related to the following analytic fact concerning the zeros of the zeta function.

Theorem 3.4 (Prime number theorem, formulated in terms of ζ). *We have $\zeta(s) = 0$ only if $\Re(s) < 1$.*

Remark 3.5. Even the statement of Theorem 3.4 is not clear without knowing the meromorphic continuation of ζ . This may offer some motivation for understanding the latter.

We expect stronger nonvanishing properties:

Conjecture 1 (Riemann Hypothesis). *We have $\zeta(s) = 0$ only if $\Re(s) < 1/2$.*

This corresponds to a conjectural stronger form of the prime number theorem, namely that

$$\pi(x) = \int_2^x \frac{t}{\log t} dt + O(x^{1/2} \log x).$$

REFERENCES