1. Some things to think about in the afternoons and beyond

(1) Understand the details of the consequences of subconvexity mentioned in lecture (reference: [19, §5]):
- Subconvexity vs. geometry of numbers.
- Distinguishing modular forms.
- Duke theorem. Reduction of supersingular elliptic curves.
- Quantum unique ergodicity. One exercise here is to understand how this follows from subconvexity in the special case of Eisenstein series, as in [17, §2].

(2) Some "classic" papers (non-exhaustive):
- Bounds for Fourier coefficients of half-integral weight, applications to quadratic forms: [15], [9], [11]
- Subconvexity for GL_2: [10] [12] [13].
- Moments and amplification via periods: [27], [20], [16], [24].
- Shifted convolution sums:
  - via δ-symbol: [10]
  - via periods: [25][4], [5].
- Papers emphasizing variation of the test vector: [23, 2, 27].

One exercise is draw parallels, e.g., between
- [20][Thm 5.1] and [10],
- [27, §4] and [12], or
- [20][Thm 5.2] and [13].

Another is to reprove some results using different methods, e.g., by working out a "classical" proof in the style of [13] for subconvexity for Maass forms at special points, namely, for $L(1/2+it_f, f)$ with f on $\mathrm{SL}_2(\mathbb{Z})$ of eigenvalue $1/4+t_f^2$, by estimating an amplified fourth moment, e.g.,

$$\sum_{f : t_f \in [T, T+1]} \left| \sum_{\ell \asymp L} c_\ell \lambda_f(\ell) \right|^2 \left| L(\tfrac{1}{2} + it_f, f) \right|^4 .$$

(3) Study the proof of the convexity bound. There are two steps:
- The Phragmen–Lindelöf convexity principle, to reduce estimates for $\Re(s) = 1/2$ to estiamtes for $\Re(s) = 1 + \varepsilon$ and $\Re(s) = -\varepsilon$.
- The functional equation, to reduce further to estimates for $\Re(s) = 1+\varepsilon$.
- Establishing the necessary bounds for $\Re(s) = 1 + \varepsilon$, for which see `https://www.math.wsu.edu/faculty/scliu/papers/Convexity.pdf` and references.

(4) Some recent papers, concerning subconvexity or related problems, that haven't been fully explored (e.g., interpreted via integral representations):
- δ-method papers such as [26] and [1]
- Higher moments over very large families, as in [7], [8]
- Rankin–Selberg when the rank difference is larger than one, as in [6]

(5) Higher rank subconvex bounds [3, 18, 22, 21, 14]. There are many "exercises" implicit in these papers; for instance, a half-dozen are suggested in [22, Remark 1.4]. Some other questions:
- These have all proceeded via arithmetic amplification. Is it possible to succeed in some cases via "family shortening" (as in, e.g., [25])? A natural case to try would be the t-aspect. Some experiments with GL_2

suggest this is difficult (see `https://ultronozm.github.io/math/20230522T174726__shrinking-archimedean-families-second-moment-gl2.html`).

"Purely horizontal" aspects remain open, e.g., twists by Dirichlet characters of prime conductor on GL_4.

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