SOME NOTES ON COHERENT STATES

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Abstract. We summarize a paper of Sugita ("Proof of the generalized Lieb-Wehrl conjecture for integer indices larger than one") and Delbourgo–Fox ("Maximum weight vectors possess minimal uncertainty").

1. The generalized Lieb-Wehrl conjecture for integer indices larger than one

We summarize Sugita [2].

Let G be a compact connected Lie group. Let π be an irreducible unitary (complex) representation of G.

Definition 1. We say that a unit vector $v \in \pi$ is a *coherent state* if it is a highest weight vector with respect to some maximal torus and ordering.

For convenience, let us now fix a maximal torus and ordering, so that we may speak of the highest weight λ of π and the line of highest weight vectors. Coherent states are then the G-translates of unit vectors in that line. We fix a highest weight unit vector $v_\lambda \in \pi$.

We equip G with its probability Haar measure dg.

Theorem 2. *Let $q \in \mathbb{Z}_{\geq 2}$. For a unit vector $v \in \pi$, set*

$$I(v) := (\dim \pi) \int_G \left(|\langle gv, v_\lambda \rangle|^2 \right)^q \, dg.$$

(The normalizing factor is for later convenience.) Then $I(v)$ achieves its maximum precisely when v is a coherent state.

Remark 3. Using the convexity of the qth power function $x \mapsto x^q$, one deduces an ostensibly more general result concerning higher rank tensors in place of $v \otimes \overline{v}$.

We turn to the proof of Theorem 2.

Let Π be an irreducible unitary representation of G with highest weight $q\lambda$. Then Π embeds as a subrepresentation of $\pi^{\otimes q}$, with multiplicity one. We thus make the identification

$$\Pi \subseteq \pi^{\otimes q}.$$

Lemma 4. For unit vectors $v \in \pi$, the quantity $I(v)$ is maximized precisely when $v^{\otimes q} \in \Pi$.

Proof. It is enough to show that

$$I(v) = \frac{\dim \pi}{\dim \Pi} \left\langle P v^{\otimes q}, v^{\otimes q} \right\rangle, \tag{1}$$

where $P : \pi^{\otimes q} \to \Pi$ denotes the orthogonal projection, because the right hand side of this formula is clearly maximal precisely when $v^{\otimes q}$ lies in the image Π of P. To establish (1), we first apply the Schur orthogonality relations to see that

$$P = (\dim \Pi) \int_G g v_\lambda^{\otimes q} \otimes \overline{g v_\lambda^{\otimes q}} \, dg.$$

We conclude by inserting this formula for P into the right hand side of (1). $\square$

To complete the proof of Theorem 2, we reduce to establishing the following equivalence:

$$v : \text{coherent} \iff v^{\otimes q} \in \Pi. \tag{2}$$

The forward implication follows from the fact that $v_\lambda^{\otimes q}$ is of weight $q\lambda$, hence lies in Π. It remains only to establish the converse.

To that end, we introduce the Casimir operator Ω for G, given by $\sum_{x \in \mathcal{B}(\mathfrak{g})} x^2$ for an orthonormal basis $\mathcal{B}(\mathfrak{g})$ of $\mathfrak{g} := \mathrm{Lie}(G)$ taken with respect to a G-invariant inner product. This operator lies in the center of the universal enveloping algebra, hence acts on any given irreducible representation as multiplication by some scalar.

Lemma 5. *For unit vectors $v \in \pi$, the inner product*

$$\left\langle \Omega v^{\otimes q}, v^{\otimes q} \right\rangle$$

is maximized precisely when v is coherent.

Proof. We first compute that

$$\left\langle \Omega v^{\otimes q}, v^{\otimes q} \right\rangle = q\Omega_\pi + q(q-1)E(v), \tag{3}$$

where Ω_π denotes the scalar by which Ω acts on π and

$$E(v) := \sum_{x \in \mathcal{B}(\mathfrak{g})} |\langle xv, v \rangle|^2 .$$

Consider for instance the case $q = 2$. For $x \in \mathfrak{g}$, we have

$$\begin{aligned} x^2 v^{\otimes 2} &= x\,(xv \otimes v + v \otimes xv) \\ &= x^2 v \otimes v + 2xv \otimes xv + v \otimes x^2 v. \end{aligned}$$

Summing over $x \in \mathcal{B}(\mathfrak{g})$ gives

$$\begin{aligned} \Omega v^{\otimes 2} &= \Omega v \otimes v + 2 \sum_{x \in \mathcal{B}(\mathfrak{g})} xv \otimes xv + v \otimes \Omega v \\ &= 2\Omega_\pi v \otimes v + 2 \sum_{x \in \mathcal{B}(\mathfrak{g})} xv \otimes xv. \end{aligned}$$

In general, in computing $x^2 v^{\otimes q}$ we get q terms that up to reordering are of the form

$$(x^2 v) \otimes v^{\otimes (q-1)},$$

corresponding to the q instances where the both the operators act on the same index. The remaining $q^2 - q$ terms have the operators acting on distinct indices, and thus up to reordering are of the form

$$(xv)^{\otimes 2} \otimes v^{\otimes (q-2)}.$$

Given that the inner product is preserved by simultaneous reordering of the tensor products, we get

$$\begin{aligned}\langle x^2 v^{\otimes q}, v^{\otimes q}\rangle &= q\langle (x^2 v)\otimes v^{\otimes(q-1)}, v^{\otimes q}\rangle + q\cdot(q-1)\langle (xv)^{\otimes 2}\otimes v^{\otimes(q-2)}, v^{\otimes q}\rangle \\ &= q\langle x^2 v, v\rangle + q\cdot(q-1)\langle xv, v\rangle^2.\end{aligned}$$

After summing over $\mathcal{B}(\mathfrak{g})$, this gives the claimed formula (3).

We conclude via appeal to the following lemma. □

Lemma 6. *For unit vectors v, the quantity $E(v)$ is maximized precisely when v is coherent.*

Proof. See Section 2. □

We may now complete the proof of the backwards implication in (2), hence of Theorem 2. Suppose $v^{\otimes q} \in \Pi$. Then, writing Ω_Π for the scalar by which Ω acts on the irreducible representation Π, we have $\Omega v^{\otimes q} = \Omega_\Pi v^{\otimes q}$ and $\Omega v_\lambda^{\otimes q} = \Omega_\Pi v_\lambda^{\otimes q}$, hence

$$\left\langle \Omega v^{\otimes q}, v^{\otimes q}\right\rangle = \left\langle \Omega v_\lambda^{\otimes q}, v_\lambda^{\otimes q}\right\rangle.$$

By two applications of Lemma 5, we deduce that v is coherent, as required.

2. Coherent states minimize uncertainty

We summarize Delbourgo–Fox [1], and in particular, prove Lemma 6.

Let G be a compact connected Lie group. Let π be an irreducible unitary representation of G. Equip the Lie algebra $\mathfrak{g}$ with a G-invariant inner product, and let $\mathcal{B}(\mathfrak{g})$ be an orthonormal basis.

Definition 7. For each unit vector $v \in \pi$, we define

$$\Delta(v) := \sum_{x\in\mathcal{B}(\mathfrak{g})} \|\bar{x}v\|^2$$

where $\bar{x} := x - \langle xv, v\rangle$.

Theorem 8. *For unit vectors v, the quantity $\Delta(v)$ is minimized precisely when v is a coherent state.*

That Theorem 8 implies Lemma 6 is immediate from the following:

Lemma 9. *We have $\Delta(v) = -\Omega_\pi - E(v)$, where Ω_π denotes the Casimir eigenvalue for π.*

Proof. We note that $\bar{x}v = xv - \langle xv, v\rangle v$, thus $\langle \bar{x}v, v\rangle = 0$ and

$$\|\bar{x}v\|^2 = \langle \bar{x}v, xv\rangle.$$

By expanding the definition of $\bar{x}$, we obtain

$$\langle xv, xv\rangle - \langle xv, v\rangle\langle v, xv\rangle.$$

Using here that x acts via a skew-symmetric operator, we see that the first term equals $-\langle x^2 v, v\rangle$, while the second term may be abbreviated to $|\langle xv, v\rangle|^2$. Summing over x leads to the required identity. □

We turn to the proof of Theorem 8.

Lemma 10. *Let π be a unitary representation of G, let $v \in \pi$ be a unit vector, let $x \in i\mathfrak{g}$. Then the minimum over all (real) scalars c of the quantity*

$$\|(x-c)v\|$$

is achieved by taking $c = \langle xv, v\rangle$.

Proof. Since x lies in $i\mathfrak{g}$, it acts via a self-adjoint operator, so we readily obtain

$$\|(x-c)v\|^2 = \langle xv, xv\rangle - 2c\langle xv, v\rangle + c^2. \tag{4}$$

This is a quadratic polynomial whose minimum is attained at the critical point, which we solve for by taking a derivative. □

The definition of $\Delta(v)$ remains unchanged upon replacing $\mathfrak{g}$ with its imaginary multiple $i\mathfrak{g}$. By Lemma 10, we see that the minimum of $\Delta(v)$ taken over unit vectors v coincides with the minimum of

$$\sum_{x \in \mathcal{B}(i\mathfrak{g})} \|(x-c_x)v\|^2 \tag{5}$$

taken over unit vectors v and tuples of scalars c_x. Such a minimum exists by continuity and compactness. Let us consider one such minimum. The above expression expands, as in (4), to

$$-\Omega_\pi + \sum_x c_x^2 - 2\sum_x c_x \langle xv, v\rangle.$$

With the notation $y := \sum_x c_x x \in \mathfrak{g}$, the above may be written

$$-\Omega_\pi + \|y\|^2 - 2\langle yv, v\rangle.$$

We choose a maximal torus whose Lie algebra contains y, and an ordering with respect to y which is dominant. Our assumptions imply that $\langle yv, v\rangle$ cannot be made larger by changing v, so v must lie in the eigenspace for y with largest eigenvalue. We claim that this eigenspace is one-dimensional. If the claim holds, then v is a highest weight vector, so we are done. Assume the claim fails. Let us then modify v, if necessary, so that it is a highest weight vector for the given torus and ordering; we may do so without changing $\langle yv, v\rangle$, hence without changing our assumption that v and c_x realize the minimum. We see then that by modifying y without changing $\|y\|^2$, we may increase the size of $\langle yv, v\rangle$, contradicting the supposed minimality. This completes the proof of Theorem 8.

3. Determining vectors by their matrix coefficients

It follows from Lemma 10 that for a given unit vector v, the quantity (5) is minimized by taking $c_x = \langle xv, v\rangle$. We record here that *the latter quantities determine v up to multiplication by a unit scalar*. Indeed, let G be a compact connected Lie group, π an irreducible representation, and $u, v \in \pi$ unit vectors with $\langle xu, u\rangle = \langle xv, v\rangle$ for all $x \in \mathfrak{g}$. By expanding the exponential series, we see that $\langle \exp(x)u, u\rangle = \langle \exp(x)v, v\rangle$, so that the $\langle gu, u\rangle = \langle gv, v\rangle$ holds near the identity on G. Since matrix coefficients of finite-dimensional representations define analytic functions, we deduce the equality on all of G. We conclude that u and v are proportional by appeal to the following:

Proposition 11. *Let G be a compact group, let π be an irreducible representation, and let u and v be nonzero vectors in π with the property that*

$$\langle gu, u \rangle = \langle gv, v \rangle \tag{6}$$

for all $g \in G$. Then u and v are proportional.

We record a proof after some lemmas.

Lemma 12 (Schur's lemma). *Let π be an irreducible representation of G. Then $\mathrm{End}_G(\pi) = \mathbb{C}$, that is to say, any linear operator on π that commutes with the action of G is a scalar multiple of the identity.*

Lemma 13. *Let π and σ be unitary representations of G, with π irreducible. Let $u \in \pi$ and $v \in \sigma$ be nonzero vectors such that*

$$\langle gu, u \rangle = \langle gv, v \rangle.$$

Then there is a unique G-equivariant map $T : \pi \to \sigma$ that sends u to v.

Proof. Any vector in π may be written $\sum_{g \in G} c_g g u$ for some finitely-supported coefficients c_g. We have no choice but to attempt to define

$$T\left(\sum_{g \in G} c_g g u\right) := \sum_{g \in G} c_g g v.$$

We need to check that this is well-defined. To see this, we use that the vanishing of the argument of T may be detected via its inner product with itself, and then use that u and v have the same inner products to see that the right hand side must likewise vanish. $\square$

Proof of Proposition 11. By Lemma 13, there is a unique $T \in \mathrm{End}_G(\pi)$ that maps u to v. By Schur's lemma, T is a multiple of the identity. [thinking-face] $\square$

References

[1] R. Delbourgo and J. R. Fox. Maximum weight vectors possess minimal uncertainty. *J. Phys. A*, 10(12):233–235, 1977.

[2] Ayumu Sugita. Proof of the generalized Lieb-Wehrl conjecture for integer indices larger than one. *J. Phys. A*, 35(42):L621–L626, 2002.