These are notes for an ongoing Fall 2023 course on the Riemann zeta function and its generalizations, L-functions. These notes will be filled in as we go.

1. References thus far

- Generating functions and asymptotics: [1, §5.2]
- Mellin transform and asymptotics: [2]

2. Outlines thus far

- Tuesday, 29 Aug: parts of §3; §3 and §3.
- Thursday, 31 Aug: §4.3 and §4.4
- Friday, 1 Sep: §4.5 and §4.6
- Tuesday, 5 Sep: §5

3. Background

3.1. General notation. $\mathbb{R}^+ := (0, \infty)$.

3.2. Asymptotic notation. We use the equivalent notations

$$A = \mathrm{O}(B), \qquad A \ll B, \qquad B \gg A$$

to denote that

$$|A| \leq C|B|$$

for some "constant" C. The precise meaning of "constant" will either be specified or clear from context.

3.3. Holomorphic continuation.

Theorem 3.1 (Identity principle for holomorphic functions)**.** *Let $U \subset \mathbb{C}$ be a connected open set. Let $f, g : U \to \mathbb{C}$ be holomorphic functions. If $f = g$ on a set with a limit point in U, then $f = g$ on all of U.*

Corollary 3.2. *Let $U \subset \Omega \subseteq \mathbb{C}$ be open subsets, with U nonempty and Ω connected. Let $f : U \to \mathbb{C}$ be a holomorphic function. Then there is at most one extension of f to a holomorphic function $\Omega \to \mathbb{C}$.*

3.4. Cauchy's integral formula.

Theorem 3.3. *Let $f : U \to \mathbb{C}$ be a holomorphic function defined on an open subset U. Let γ be a closed rectifiable curve in U. Then $\int_\gamma f(z)\, dz = 0$.*

Theorem 3.4. *Let $0 \leq a < b \leq \infty$. Let $f(z)$ be a holomorphic function on the annulus $\{z \in \mathbb{C} : a < |z| < b\}$ given by a convergent Laurent series*

$$f(z) = \sum_{n \in \mathbb{Z}} c_n z^n.$$

(1) For any $r \in (a, b)$ and $n \in \mathbb{Z}$, we have

$$\begin{aligned} c_n &= \oint_{|z|=r} \frac{f(z)}{z^n} \frac{dz}{2\pi i z} \\ &= \frac{1}{2\pi r^n} \int_{\theta=0}^{2\pi} f(re^{i\theta}) e^{-in\theta}\, d\theta. \end{aligned}$$

(2) For each compact subset E of (a, b), there exists $M \geq 0$ so that for all $r \in E$, we have

$$\sum_{n \in \mathbb{Z}} |c_n| r^n \leq M. \tag{3.1}$$

Theorem 3.5. *Let U be an open subset of $\mathbb{C}$, let $f : U \to \mathbb{C}$ be meromorphic. Let γ be a smooth closed curve in U, oriented counterclockwise, that does not pass through any pole of f. Then*

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{\substack{z \in \mathrm{interior}(\gamma) \\ \text{pole of } f}} \mathrm{res}_z(f).$$

Remark 3.6. Let $0 < r < R$. Let f be a meromorphic function on a neighborhood of the annulus $\{z : r < |z| < R\}$ that has no poles on either of the circles $|z| = r, R$. Then

$$\oint_{|z|=R} f(z)\,dz = \oint_{|z|=r} f(z)\,dz + 2\pi i \sum_{\substack{r<|z|<R \\ \text{pole of } f}} \operatorname{res}_z(f).$$

3.5. **Holomorphy of limits and series.**

Theorem 3.7. *Let U be an open subset of the complex plane. Let f_n be a sequence of holomorphic functions on U.*

1. *Suppose that the sequence f_n converges pointwise to some function f, uniformly on compact subsets of U. Then f is holomorphic.*
2. *Suppose that the partial sums $\sum_{n \le N} f_n$ converge pointwise to some function f, uniformly on compact subsets of U. Then the sum $\sum_n f_n$ is holomorphic.*

3.6. **Definition and basic properties of ζ: overview.** The Riemann zeta function is defined for a complex number s by the series

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

Lemma 3.8. *The series converges absolutely for $\Re(s) > 1$, uniformly for $\Re(s) \geq 1+\varepsilon$ for each $\varepsilon > 0$.*

Proof. Using the identity

$$\left| \frac{1}{n^s} \right| = \frac{1}{n^{\Re(s)}},$$

we reduce to the case that s is real, in which this is a familiar consequence of the integral test. □

Our first main goal in the course is to explain the following basic facts.

Theorem 3.9. *The Riemann zeta function admits a meromorphic continuation to the entire complex plane. It is holomorphic away from a simple pole at $s = 1$, where it has residue 1. It admits a functional equation relating $\zeta(s)$ to $\zeta(1-s)$.*

One historical motivation for considering the zeta function at complex arguments comes from the prime number theorem.

Theorem 3.10 (Prime number theorem)**.** *Let $\pi(x) := \#\{\text{primes } p \leq x\}$ denote the prime counting function. Then*

$$\frac{\pi(x)}{x/\log x} \to 1 \text{ as } x \to \infty.$$

This is related to the following analytic fact concerning the zeros of the zeta function.

Theorem 3.11 (Prime number theorem, formulated in terms of ζ)**.** *We have $\zeta(s) = 0$ only if $\Re(s) < 1$.*

Remark 3.12. Even the statement of Theorem 3.11 is not clear without knowing the meromorphic continuation of ζ. This may offer some motivation for understanding the latter.

We expect stronger nonvanishing properties:

Conjecture 1 (Riemann Hypothesis)**.** *We have $\zeta(s) = 0$ only if $\Re(s) < 1/2$.*

This corresponds to a conjectural stronger form of the prime number theorem, namely that

$$\pi(x) = \int_2^x \frac{t}{\log t}\, dt + \mathrm{O}(x^{1/2} \log x).$$

4. Generating series

Here we study the relationship between asymptotics of sequences $(c_n)_{n \in \mathbb{Z}}$ and meromorphic continuation of the associated generating series. Reference: [1, §5.2].

4.1. **Setup.** We consider a Laurent series

$$f(z) = \sum_{n \in \mathbb{Z}} c_n z^n.$$

Here the c_n are complex coefficients, while z is a nonzero complex argument. We assume that this series converges absolutely for at least one value of z.

Lemma 4.1. *There is a unique maximal open subinterval* (a, b) *of* $\mathbb{R}^+$ *on which* f *converges absolutely. Its endpoints are given explicitly by*

$$a = \inf \left\{ r \in \mathbb{R}^+ : \sum_n |c_n| r^n < \infty \right\},$$

$$b = \sup \left\{ r \in \mathbb{R}^+ : \sum_n |c_n| r^n < \infty \right\}.$$

We refer to the interval (a, b) as the *fundamental interval* for f (or for the c_n).

The fundamental interval controls the growth of the coefficients c_n as $n \to \pm\infty$:

Lemma 4.2. *Let* $b^- < b$ *and* $a^+ > a$. *Then*

$$c_n \ll (b^-)^{-n} \qquad \text{as } n \to \infty$$

and

$$c_n \ll (a^+)^{-n} \qquad \text{as } n \to -\infty.$$

Set

$$\mathcal{C}(a, b) := \{ z \in \mathbb{C} : |z| \in (a, b) \}.$$

Lemma 4.3. $f(z)$ *defines a holomorphic function on* $\mathcal{C}(a, b)$.

Proof. Follows from Theorem 3.7. □

4.2. **Basic study of meromorphic continuation.**

Lemma 4.4. f *does not extend to a holomorphic function on* $\mathcal{C}(A, B)$ *for any strictly larger interval* $(A, B) \supsetneq (a, b)$.

Proof. Suppose otherwise. Let $r \in (A, B) - (a, b)$. Then by Cauchy's integral formula (specifically, the estimate (3.1) of Theorem 3.4), we see that $\sum_{n \in \mathbb{Z}} |c_n| r^n < \infty$. This contradicts the formula for a and b given in Lemma 4.1. □

Note 4.5. It can happen that f extends to a *meromorphic* function on some strictly larger annulus (unique, in view of Corollary 3.2). By Lemma 4.4, this can only happen if f has a pole at some point on the boundary of the fundamental annulus.

Example 4.6. Take

$$c_n = \begin{cases} 2^n & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

Then the fundamental interval is $(a, b) = (0, 1/2)$. However, the function $f(z)$, defined initially for $|z| < 1/2$, evaluates to a rational function:

$$f(z) = \sum_{n \geq 0} 2^n z^n = \frac{1}{1 - 2z}.$$

This is meromorphic on the entire complex plane; the only pole is a simple one at $z = 1/2$, with residue $-1/2$.

The possibility of meromorphically extending f corresponds to the coefficients c_n having asymptotic expansions as $n \to \pm\infty$. For example:

Lemma 4.7 (Meromorphic continuation vs. asymptotic expansion, special case). *Let f and (a, b) be as above. Let $\beta \in \mathbb{C}$ with $|\beta| = b$. Let $B > b$ and $\gamma \in \mathbb{C}$. Then the following are equivalent:*

(i) f extends to a meromorphic function on $\mathcal{C}(a, B)$ with a unique simple pole at $z = \beta$ with residue γ.

(ii) For each $B^- < B$, we have as $n \to \infty$ that

$$c_n = -\gamma\beta^{-n-1} + \mathrm{O}\left((B^-)^{-n}\right), \tag{4.1}$$

Proof. To see that (i) implies (ii), we start with Cauchy's integral formula on the disc of radius b^- for some $b^- \in (a, b)$, then shift the contour, picking up the contribution of the unique pole:

$$\begin{aligned} c_n &= \oint_{|z|=b^-} \frac{f(z)}{z^n} \frac{dz}{2\pi i z} \\ &= \oint_{|z|=B^-} \frac{f(z)}{z^n} \frac{dz}{2\pi i z} - \frac{\gamma}{\beta^{n+1}}. \end{aligned} \tag{4.2}$$

We then estimate this last integral using that f is bounded on compact sets.

Conversely, to verify that (ii) implies (i), we define the coefficients

$$b_n := \begin{cases} -\gamma\beta^{-n-1} & \text{if } n \geq 0, \\ 0 & \text{if } n < 0, \end{cases}$$

The corresponding series

$$f_+(z) := \sum_{n \in \mathbb{Z}} b_n z^n$$

may be evaluated explicitly: a simple geometric series calculation, left to the reader, gives

$$f_+(z) = \frac{\gamma}{z - \beta}.$$

Our hypothesis concerning the c_n reads

$$b_n - c_n = \mathrm{O}\left((B^-)^{-n}\right) \qquad \text{as } n \to \infty. \tag{4.3}$$

On the other hand, because f has fundamental interval (a, b) and b_n vanishes as $n \to -\infty$, we have for each $a^+ > a$ that

$$b_n - c_n = \mathrm{O}\left((a^+)^{-n}\right) \qquad \text{as } n \to -\infty. \tag{4.4}$$

From (4.3) and (4.4), we deduce that the series $f - f_+$ with coefficients $c_n - b_n$ has fundamental interval containing (a, B). This implies that the function

$$f(z) - \frac{\gamma}{z - \beta},$$

defined initially as a holomorphic function on $\mathcal{C}(a, b)$, extends to a holomorphic function on $\mathcal{C}(a, B)$. Equivalently, f extends to a meromorphic function on $\mathcal{C}(a, B)$ with polar behavior as described in (ii). $\square$

Exercise 1. Generalize the above lemma to describe in terms of the coefficients c_n what it means for f to extend to a meromorphic function on $\mathcal{C}(A,B)$ for some $A < a$ and $B > b$, allowing the possibility of multiple poles of arbitrary order.

4.3. **Further examples.**

Example 4.8. Suppose that

$$c_n = \begin{cases} \beta^{-n} & \text{if } n \geq 0, \\ 0 & \text{if } n < 0, \end{cases}$$

so that, initially for $|z| < |\beta|$,

$$f(z) = \sum_{n \geq 0} \beta^{-n} z^n = \frac{1}{1 - z/\beta}.$$

The function f extends meromorphically, having a simple pole at $z = \beta$ with residue $-\beta$. The sequence c_n has the asymptotic behavior indicated in (4.1), in a very strong sense: the sequence is *equal* to the asymptotic.

Exercise 2. Let c_n denote the Fibonacci sequence, thus $c_n = 0$ for $n < 0$ and

$$c_0 = 1, \quad c_1 = 1, \quad c_{n+2} - c_{n+1} - c_n = 0.$$

This exercise rederives a standard formula for this sequence in a way that is intended to illustrate the technique of Lemma 4.7.

(1) Verify by crude estimation that the fundamental interval for the series $f(z) = \sum_n c_n z^n$ contains $(0, 1/2)$.
(2) Show that

$$f(z) = \frac{1}{1 - z - z^2} = \frac{1}{(1 - z/\varphi)(1 - z/\varphi')},$$

where

$$\varphi = \frac{1 + \sqrt{5}}{2} = 1.618\cdots, \quad \varphi' = \frac{1 - \sqrt{5}}{2} = -0.618\cdots.$$

(3) Following the proof of Lemma 4.7, show that

$$c_n = \frac{\varphi^n - (\varphi')^n}{\varphi - \varphi'}.$$

(Use that $f(z) \ll |z|^2$ for $|z| \geq 2$ to show that the "remainder term", namely the integral in (4.2), tends to zero as $B^- \to \infty$.)

Example 4.9. Let $\beta \in \mathbb{C} - \{0\}$ and $a \in \mathbb{Z}_{\geq 0}$. Then one verifies by induction on a, using differentiation, that

$$\frac{1}{(z - \beta)^{a+1}} = (-\beta)^{-a-1} \sum_{n \geq 0} \binom{n+a}{n} \beta^{-n} z^n,$$

where the binomial coefficient expands to a polynomial of degree a in n:

$$\binom{n+a}{a} = \frac{(n+a)!}{a!n!} = \frac{(n+1)(n+2)\cdots(n+a)}{a!}.$$

More generally, given any coefficients $c_0, c_1 \ldots, c_k$, we have

$$\sum_{k=0}^{a} \frac{c_k}{(z-\beta)^{k+1}} = \sum_{n \geq 0} P(n) \beta^{-n} z^n$$

for some polynomial $P(n)$ of degree at most a. Conversely, given such a polynomial, we may find coefficients so that the above identity holds.

Example 4.10. Take

$$c_n := e^{-2^n}.$$

Observe that

$$c_n \to \begin{cases} 0 & \text{if } n \to \infty, \\ 1 & \text{if } n \to -\infty. \end{cases}$$

Moreover, as $n \to \infty$, the convergence of the c_n to zero is rapid in the sense that for each $B < \infty$, we have

$$c_n \ll B^{-n}.$$

The fundamental interval is thus $(1, \infty)$: the series $f(z) = \sum_n c_n z^n$ converges absolutely for $|z| > 1$ and defines a holomorphic function there. We will show that *f extends to a meromorphic function on $\mathbb{C} - \{0\}$, which is holomorphic away from simple poles at $1/2^k$ (for $k \in \mathbb{Z}_{\geq 0}$) with residue $(-1/4)^k/k!$.* To that end, observe first that the contribution to f from $n \geq 0$, namely

$$f_+(z) := \sum_{n>0} c_n z^n,$$

converges absolutely and is thus holomorphic on the entire complex plane. The meromorphic continuation of f thereby reduces to that of the complementary sum

$$f_-(z) := \sum_{n \leq 0} c_n z^n.$$

Inspired by Lemma 4.7, we study the asymptotics of the coefficients c_n as $n \to -\infty$. These are described by the Taylor series of the exponential functions:

$$e^x = \sum_{k \geq 0} \frac{x^k}{k!}.$$

By estimating the tail of this series, we sees that for $x = \mathrm{O}(1)$ and $M = \mathrm{O}(1)$, we have

$$e^x = \sum_{k=0}^{N-1} \frac{x^k}{k!} + \mathrm{O}(x^M).$$

It follows that for $n \leq 0$,

$$c_n = \sum_{k=0}^{N-1} \frac{(-2^n)^k}{k!} + \mathrm{O}(2^{nM}). \tag{4.5}$$

By the method of proof of Lemma 4.7, we deduce from this estimate that f_- the required assertions concerning the meromorphic continuation of f. Let us spell this deduction out for the sake of practice. Set

$$g_k(z) := \sum_{n \leq 0} \frac{(-2^n)^k}{k!} z^n.$$

The estimate (4.5) implies that the modified series

$$f_-(z) - \sum_{k=0}^{N-1} g_k(z) = \sum_{n \leq 0} \left(c_n - \sum_{k=0}^{N-1} \frac{(-2^n)^k}{k!} \right) z^n \tag{4.6}$$

converges absolutely for $|z| > 1/2^M$, hence defines a holomorphic function there. On the other hand, for $|z| > 1$, we see by summing the geometric series that

$$g_k(z) = \frac{(-1)^k}{k!} \frac{1}{1 - 1/2^k z} = \frac{(-1/2)^k}{k!} \frac{z}{z - 1/2^k}.$$

Thus g_k extends to a meromorphic function whose only pole is a simple one at $z = 1/2^k$ with residue $(-1/4)^k/k!$. It follows that f has the claimed meromorphic properties.

4.4. **Regularization.**

Remark 4.11. We can cases view $f(1)$ as the "regularized sum" of the (possibly divergent) series $\sum_n c_n$:

$$\sum_n^{\mathrm{reg}} c_n := \left(\sum_n c_n z^n \right) |_{z=1},$$

keeping in mind here that the series may initially convergent away from the point $z = 1$, so that the specialization is understood as the result of analytic continuation. We make this definition whenever the series is holomorphic at $z = 1$.

For example,

$$\sum_{n \geq 0}^{\mathrm{reg}} (-1)^n = \left(\sum_{n \geq 0} (-1)^n z^n \right) |_{z=1} = \frac{1}{1+z} |_{z=1} = \frac{1}{2}.$$

In this example, we may understand $f(1)$ as the limit of the quantities $f(z)$ for $z < 1$ as $z \to 1$, and also as the Cesaro mean of the partial sums of the series $\sum_{n \geq 0} (-1)^n$, so the interpretation of $f(1)$ as a regularized sum makes intuitive sense.

In other examples, the interpretation may be less clear. For example,

$$\sum_{n \geq 0}^{\mathrm{reg}} 10^n = \left(\sum_{n \geq 0} 10^n z^n \right) |_{z=1} = \frac{1}{1 - 10z} |_{z=1} = \frac{-1}{9},$$

the intuitive meaning of which may be less clear. One way to understand the regularization is as follows: the value $f(1)$ is insensitive to replacing the sequence $(c_n)_n$ by any of its shifts $(c_{n+k})_n$. Setting

$$S := \sum_{n \geq 0}^{\mathrm{reg}} 10^n,$$

we should thus have

$$10S = \sum_{n \geq 0}^{\mathrm{reg}} 10^{n+1} = 1 + S,$$

from which it follows that $S = -1/9$.

In fact, we can define regularized sums even in cases where the series does not converge at any point. The idea is to split the sum into two pieces, one near $+\infty$ and the other near $-\infty$, then to meromorphically continue each part from some initial domain and add together the resulting meromorphic continuations. This is the content of the following definitions and results.

Definition 4.12. Let us say that a function $\mathbb{Z} \to \mathbb{C}$ is *finite* if it is a finite linear combination of functions of the form

$$n \mapsto n^a \beta^{-n},$$

where $a \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathbb{C}^\times$.

Remark 4.13. The intrinsic interpretation of this definition is that the finite functions are those whose translates span a finite-dimensional subspace of the space of all functions $\mathbb{Z} \to \mathbb{C}$.

Definition 4.14. Let us say that a function $c : \mathbb{Z} \to \mathbb{C}$ is *regularizable* if for each $0 < A < B < \infty$, there exist finite functions $c_\pm$ so that

$$c(n) = c_+(n) + \mathrm{O}(B^{-n}) \text{ as } n \to \infty,$$
$$c(n) = c_-(n) + \mathrm{O}(A^{-n}) \text{ as } n \to -\infty.$$

Example 4.15. Any finite function is regularizable.

Definition 4.16. Given a regularizable function c as above, we may define its *regularized generating function*

$$f(z) := \sum_{n \in \mathbb{Z}}^{\text{reg}} c(n) z^n, \tag{4.7}$$

which will be a meromorphic function of $z \in \mathbb{C}^\times$, as follows. Choose $N \in \mathbb{Z}$. Define

$$f_+(z) := \sum_{n \geq N} c(n) z^n,$$

initially for $|z|$ sufficiently small, then in general by meromorphic continuation (with poles described by the asymptotics of c as $n \to \infty$, corresponding to terms of c_+). Similarly, we define

$$f_-(z) := \sum_{n < N} c(n) z^n,$$

initially for $|z|$ sufficiently large, then in general by meromorphic continuation (with poles described by the asymptotics of c as $n \to -\infty$, corresponding to terms of c_-). We then set

$$f(z) := f_+(z) + f_-(z).$$

Lemma 4.17. *The above definition is independent of the choice of N.*

Proof. Modifying N has the effect of adding a polynomial to $f_\pm$ and subtracting the same polynomial from $f_\mp$. □

Lemma 4.18. *Suppose that $c(n)$ is regularizable, and let $k \in \mathbb{Z}$. Then the shifted sequence $d(n) := c(n + k)$ is also regularizable. The regularized generating function g for d is related to the generating function f for c via*

$$g(z) = z^{-k} f(z).$$

In other words,

$$\sum_{n \in \mathbb{Z}}^{\text{reg}} c(n) z^n = z^k \sum_{n \in \mathbb{Z}}^{\text{reg}} c(n + k) z^n.$$

Proof. This may be deduced from Lemma 4.17. □

Exercise 3. Show that if c is a finite function, then its regularized generating function vanishes. [This can be seen via explicit calculation using the definition, or, as in lecture, from Lemma 4.18.]

Example 4.19. Take

$$c_n = n^3 e^{-2^{-n^2}}.$$

Then the series f converges absolutely nowhere: the fundamental interval is empty. But we can still define its regularized generating function, which is actually an entire function of n: it may be written explicitly as the everywhere convergent series

$$\sum_{n \in \mathbb{Z}} (c_n - n^3) z^n. \tag{4.8}$$

Note that the individual pieces $f_\pm$ as in Definition 4.16 are not entire functions of z: they have quadruple poles at $z = 1$, in view of Example 4.9. However, the poles cancel thanks to Exercise 3, so their sum is the entire function (4.8).

4.5. **Permutations without small cycles.** Reference: [1, p176].

Let $S(n)$ denote the symmetric group, consisting of permutations σ of the set $\{1, \dots, n\}$. We have $\#S(n) = n!$.

Each permutation may be written uniquely as a product of disjoint cyclic permutations of some lengths $n_1, \dots, n_k \in \mathbb{Z}_{\geq 1}$, where $n_1 + \dots + n_k = n$.

For each subset S of $\mathbb{N} := \mathbb{Z}_{\geq 1}$ and each $n \geq 0$, let c_n^S denote the number of permutations $\sigma \in S(n)$ each of whose cycle lengths n_j lies in S. We denote by

$$f^S(z) := \sum_{n \geq 0} \frac{c_n^S}{n!} z^n$$

the "exponential generating function" of this sequence. Since $c_n^S \leq n!$, we see that the series converges absolutely for $|z| < 1$.

Example 4.20. We have

$$f^{\mathbb{N}}(z) = \sum_{n \geq 0} z^n = \frac{1}{1-z} = \exp \log \frac{1}{1-z} = \exp \sum_{n \geq 1} \frac{z^n}{n}.$$

Example 4.21. We have

$$f^{\{1\}}(z) = \sum_{n \geq 0} \frac{z^n}{n!} = \exp z = \exp \sum_{n=1} \frac{z^n}{n}.$$

Lemma 4.22. *We have*

$$f^S(z) = \exp \sum_{n \in S} \frac{z^n}{n}.$$

Proof. Using the series definition $\exp x = \sum_{k \geq 0} x^k / k!$, we see that

$$\exp \sum_{n \in S} \frac{z^n}{n} = \sum_{k \geq 0} \frac{1}{k!} \sum_{n_1, \dots, n_k \in S} \frac{z^{n_1 + \dots + n_k}}{n_1 \cdots n_k}.$$

Our task is thus to verify that

$$c_n^S = \sum_{k \geq 0} \sum_{\substack{n_1, \dots, n_k \in S: \\ n_1 + \dots + n_k = n}} \frac{n!}{k! n_1 \cdots n_k}. \tag{4.9}$$

The set of permutations attached to a given multiset of lengths $\{n_1, \dots, n_k\}$ is a conjugacy class in $S(n)$. The size of that conjugacy class is described by the orbit-centralizer formula. The centralizer of a permutation is the group generated by each of the cycles in the decomposition; this group has order $n_1 \cdots n_k$. It follows that the conjugacy class has size

$$\frac{n!}{n_1 \cdots n_k}.$$

Since the order of the n_j doesn't matter, we deduce the claimed formula (4.9). $\square$

As an application, take S to consist of all integers greater than some given integer ℓ. Then

$$f^S(z) = \exp\left(\sum_{n \geq 1} \frac{z^n}{n} - \sum_{n=1}^{\ell} \frac{z^n}{n}\right) = \frac{1}{1-z} \exp\left(-\sum_{n=1}^{\ell} \frac{z^n}{n}\right).$$

This defines a meromorphic function on the entire complex plane, whose only pole is a simple one at $z = 1$. By the usual analysis, we deduce the following asymptotic formula for the coefficients of f^S:

$$\frac{c_n^S}{n!} = \exp\left(-\sum_{n=1}^{\ell} \frac{z^n}{n}\right) + \mathrm{O}(B^{-n}), \tag{4.10}$$

for fixed ℓ and fixed $B < \infty$.

We may interpret the left hand side of (4.10) as the probability that a random permutation of length n has no cycles of length $\leq \ell$.

4.6. **Series that do not admit meromorphic continuations.**

Example 4.23. The series

$$\sum_{n \geq 1} \log(n) z^n$$

converges absolutely for $|z| < 1$, but does not continue meromorphically to $|z| < r$ for any $r > 1$, because $\log(n)$ cannot be approximated up to error $\mathrm{O}(r^{-n})$ by a finite linear combination of the functions $n^a \beta^{-n}$ ($a \in \mathbb{Z}_{\geq 0}$, $\beta \in \mathbb{C}^\times$). Indeed, we would need to have each $\beta = 1$, so the main point is to check that $\log(n)$ cannot be approximated exponentially well by a polynomial. This is because $\log(n)$ grows faster than any constant polynomial but slower than any non-constant polynomials.

Question 1. Does the series in Example 4.23 continue to any open set strictly containing the unit disc?

Exercise 4. Similarly analyze the series

$$\sum_{n \geq 1} n^{1/n} z^n.$$

Example 4.24. For $k \geq 0$ and $n \in \mathbb{Z}_{\geq 1}$, define

$$\sigma_k(n) := \sum_{d \mid n} d^k,$$

where d runs over the positive divisors of n. For example, $\sigma_0(n) = \tau(n)$ is the number of positive divisors of n. The series

$$\sum_{n\geq 0} \sigma_k(n) z^n$$

converges absolutely on the disc $|z| < 1$. It does not extend to a meromorphic function on any strictly larger open set. This can be seen most efficiently using the basic theory of modular forms, which we discuss later in the course.

5. Fourier and Mellin transforms

Thus far we've related sequences $(c_n)_{n\in\mathbb{Z}}$ to their generating functions $\sum c_n z^n$. We can think of such sequences as defining functions $\mathbb{Z} \to \mathbb{C}$, or equivalently, complex measures $\mu = \sum_n c_n \delta_n$ on $\mathbb{Z}$.

We now want to do something similar for measures μ on $\mathbb{R}$. We define the Fourier transform of such a measure as follows, for $s \in \mathbb{C}$:

$$F(s) = \int_{x\in\mathbb{R}} e^{sx}\, d\mu(x).$$

Remark 5.1. Typically the Fourier transform is defined first for a real variable ξ by either $\int_{x\in\mathbb{R}} e^{ix\xi}\, d\mu(x)$ or $\int_{x\in\mathbb{R}} e^{-ix\xi}\, d\mu(x)$, then in general by meromorphic continuation. The convention used above differs mildly (take $s = \pm i\xi$) and is more convenient for our purposes.

Example 5.2. Suppose $\mu = \sum_{n\in\mathbb{Z}} c_n\delta_n$. Then $F(s) = \sum c_n z^n$ with $z = e^s$. This function satisfies the symmetry $F(s + 2\pi i k) = F(s)$ for all $k \in \mathbb{Z}$.

Example 5.3. Suppose $\mu = \sum_{n=1}^{\infty} \delta_{-\log n}$. Then $F(s) = \sum_1^\infty e^{s(-\log n)} = \sum_1^\infty n^{-s} = \zeta(s)$.

The main examples we consider are when $d\mu(x) = f(x)\,dx$ for some function $f : \mathbb{R} \to \mathbb{C}$, so that

$$F(s) = \int_{x\in\mathbb{R}} e^{sx} f(x)\,dx.$$

As in our study of series, there is a maximal open interval (a, b) such that $F(s)$ converges absolutely when $\Re(s) \in (a, b)$, hence defines a holomorphic function there.

Let's recall how the Fourier transform behaves with respect to taking derivatives and multiplying by polynomials.

Lemma 5.4. Given a function $f(x)$ with Fourier transform $F(s)$, the Fourier transform of its derivative $f'(x)$ is given by $-sF(s)$:

$$\int e^{sx} f'(x)\,dx = -sF(s).$$

This formula is valid under sufficient smoothness and decay conditions, e.g., that f is continuously differentiable and

$$\int |e^{sx} f(x)|\,dx < \infty, \quad \int |e^{sx} f'(x)|\,dx < \infty, \quad \lim_{x\to\pm\infty} e^{sx} f(x) = 0.$$

Proof. By integration by parts, we have for $-\infty < a < b < \infty$

$$\int_a^b e^{sx} f'(x)\,dx = e^{sx} f(x)|_a^b - s\int_a^b e^{sx} f(x)\,dx.$$

Now take $(a, b) \to (-\infty, \infty)$. Under the stated conditions, the left hand side converges to $\int e^{sx} f'(x)\,dx$ and the right hand side to $-sF(s)$. $\square$

Lemma 5.5. Given a function $f(x)$ with Fourier transform $F(s)$, the Fourier transform of $xf(x)$ is given by $F'(s)$:

$$\int e^{sx} x f(x)\,dx = F'(s).$$

This formula is valid under sufficient smoothness and decay conditions, e.g., that f is continuously differentiable and

$$\int |e^{sx} f(x)| \, dx < \infty, \quad \int |e^{sx} x f(x)| \, dx < \infty.$$

Proof. We differentiate under the integral sign in the definition of $F(s)$. □

One can iterate the above lemmas to work out how the Fourier transform behaves with respect to taking arbitrarily many derivatives or multiplying by general polynomials. This leads to useful estimates for $F(s)$:

Lemma 5.6. *Let $n \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{C}$. Then*

$$|s|^n |F(s)| \leq \int \left| e^{sx} f^{(n)}(x) \right| \, dx$$

assuming sufficient smoothness and decay conditions, e.g,. that f is n times continuously differentiable, $\lim_{x \to \pm\infty} e^{sx} f^{(j)}(x) = 0$ *for $j \leq n-1$, and* $\left| e^{sx} f^{(j)}(x) \right| \, dx < \infty$ *for $j = 0, 1, \ldots, n$.*

Proof. We see by iterating Lemma 5.4 that $(-s)^n F(s)$ is the Fourier transform of $f^{(n)}(x)$, i.e., that

$$(-s)^n F(s) = \int e^{sx} f^{(n)}(x) \, dx.$$

The conclusion now follows from the triangle inequality. □

The main way to pass from F to f is via the *Fourier inversion formula*, which (when valid) states that

$$f(x) = \int_{(c)} F(s) e^{-sx} \, \frac{ds}{2\pi i}. \tag{5.1}$$

Here $\int_{(c)}$ denote a complex contour integral over $s \in c + i\mathbb{R}$ for some $c \in \mathbb{R}$; in terms of the Lebesgue measure $d\xi$ on $\mathbb{R}$, the above integral may be rewritten

$$\int_{\xi \in \mathbb{R}} F(c + i\xi) e^{-(c+i\xi)x} \, \frac{d\xi}{2\pi}.$$

Theorem 5.7 (L^1 Fourier inversion)**.** *Suppose that the integrals*

$$\int_{x \in \mathbb{R}} e^{cx} |f(x)| \, dx, \tag{5.2}$$

$$\int_{(c)} |F(s)| \, ds \tag{5.3}$$

are both finite. Then the Fourier inversion formula (5.1) *holds.*

It's typically easy to check when the first integral (5.2) is finite. For the second integral (5.3), we can appeal to Lemma 5.6, which implies that under sufficient smoothness and decay conditions,

$$(1 + |s|^2) |F(s)| \leq \int \left(|e^{sx} f(x)| + |e^{sx} f''(x)| \right) \, dx.$$

The finiteness of the right hand side implies that $F(s) \ll (1 + |s|^2)^{-1}$, hence the finiteness of (5.3).

Example 5.8. Let's work out a variant of Example 4.10. Take $f(x) = e^{-e^x}$. As $x \to \infty$, it decays rapidly, namely $f(x) \ll e^{-Cx}$ for each fixed C. As $x \to -\infty$, it admits an asymptotic expansion given by the Taylor series of the expansion function, e.g.,

$$f(x) = 1 - e^x + \frac{e^{2x}}{2!} + \mathrm{O}(e^{3x}).$$

We can argue as in Example 4.10 that the Fourier transform

$$F(s) = \int e^{sx} f(x)\, dx$$

converges absolutely for $\Re(s) > 0$ and extends to a meromorphic function of s, whose only poles are simple poles at $s = -n$ for $n \in \mathbb{Z}_{\geq 0}$, with residue $(-1)^n/n!$. For instance, to check the validity of this statement when $\Re(s) > -3$, we define

$$f_-(x) := \begin{cases} 1 - e^x + \frac{e^{2x}}{2!} & \text{if } x \leq 0, \\ 0 & \text{x ¿ 0.} \end{cases}$$

Its Fourier transform $F_-(s)$ converges absolutely for $\Re(s) > 0$. The Fourier transform $F(s) - F_-(s)$ of the difference $f(x) - f_-(x)$ converges absolutely for $\Re(s) > -3$, hence defines a holomorphic function in that region. But we may evaluate $F_-(s)$ explicitly, and we see that it extends to a meromorphic function:

$$F_-(s) = \int_{-\infty}^{0} \left(1 - e^x + \frac{e^{2x}}{2!}\right) e^{sx}\, dx = \frac{1}{s} - \frac{1}{s+1} + \frac{1}{2!(s+2)}. \tag{5.4}$$

It's often useful to change coordinates. For $x \in \mathbb{R}$, set

$$y := e^x,$$

so that $x = \log y$. Then y lies in $\mathbb{R}^+ = \mathbb{R}_+^\times = (0, \infty)$, and we have

$$y^s = e^{sx}.$$

We have

$$x \to +\infty \iff y \to \infty,$$
$$x \to -\infty \iff y \to 0.$$

Given a function $f(x)$ on $\mathbb{R}$, let us define a function $h(y)$ on $\mathbb{R}^+$ in the evident way, so that

$$h(y) = f(\log y), \quad f(x) = h(e^x).$$

The integrals of these functions are related as follows:

$$\int f(x)\, dx = \int h(y)\, \frac{dy}{y}.$$

We abbreviate

$$d^\times y := \frac{dy}{y},$$

since this normalization occurs often. It has the following useful dilation-invariance: for any $b \in \mathbb{R}^+$,

$$\int h(by)\, d^\times y = \int h(y)\, d^\times y.$$

The Fourier transform of f may be written

$$H(s) := \int y^s h(y)\, d^\times y.$$

It is called the *Mellin transform* of h.

Example 5.9. We have $e^{-e^x} = e^{-y}$, so the Fourier transform $F(s)$ considered in Example 5.8 may be rewritten

$$\Gamma(s) = \int_{\mathbb{R}^+} e^{-y} y^s \, d^\times y.$$

This is the Γ-function. The arguments of Example 5.8 may be rewritten in the new variables. For example, to verify that $\Gamma(s)$, defined initially for $\Re(s) > 0$ by the above absolutely convergent integral, extends to a meromorphic function on $\Re(s) > -3$ with the prescribed polar behavior, we define

$$h_-(y) := \begin{cases} 1 - y + \frac{y^2}{2!} & \text{if } 0 < y \leq 1, \\ 0 & \text{if } y > 1. \end{cases}$$

Then $e^{-y} = h_-(y) + \mathrm{O}(y^3)$ as $y \to 0$, so the difference between $\Gamma(s)$ and the Mellin transform $H_-(s)$ of h_- is defined for $\Re(s) > -3$ by an absolutely convergent integral, hence is holomorphic. On the other hand, we may evaluate $H_-(s)$ explicitly; the result is as in (5.4).

Example 5.10. Take

$$h(y) = \frac{1}{e^y - 1}.$$

The function $yh(y)$ extends to a holomorphic function of y on the disc $\{y \in \mathbb{C} : |y| < 2\pi\}$, so h is represented for small $y > 0$ by an absolutely convergent Laurent series of the following form:

$$h(y) = \frac{1}{y} + \sum_{n=1}^{\infty} \frac{B_n}{n!} y^{n-1}.$$

Here the B_n are complex coefficients, called the *Bernoulli numbers*. On the other hand, h decays rapidly (like $\mathrm{O}(y^N)$ for any fixed N) as $y \to \infty$. By the analysis we've now seen many times, we deduce that the Mellin transform $H(s)$ of $h(y)$ converges absolutely for $\Re(s) > 1$, where it defines a holomorphic function, and extends to a meromorphic function on the complex plane whose only poles are simple poles at $s = 1$ (corresponding to the $1/y$ term in the asymptotic expansion as $y \to 0$) and at $s = -n$ for each $n \in \mathbb{Z}_{\geq 0}$, with residue given by 1 in the former case and by $B_{n+1}/(n+!)!$ in the latter.

On the other hand, we can rewrite

$$h(y) = \frac{e^{-y}}{1 - e^{-y}} = e^{-y} + e^{-2y} + e^{-3y} + \cdots,$$

giving

$$H(s) = \int_{\mathbb{R}^+} y^s \sum_{n=1}^{\infty} e^{-ny} \, d^\times y.$$

The following calculations will show that the doubly integral/sum converges absolutely for $\Re(s) > 1$, so we may rearrange it as

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}^+} y^s e^{-ny} \, d^\times y.$$

The inner integral may be simplified by the substitution $y \mapsto y/n$. This has no effect on the measure $d^\times y$, but replaces y^s by $n^{-s}y^s$, giving

$$\sum_{n=1}^{\infty} n^{-s} \int_{\mathbb{R}^+} y^s e^{-y}\, d^\times y = \zeta(s)\Gamma(s).$$

We see in this way that the ζ function admits a meromorphic continuation. We'll be able to describe its poles once we explain why the Γ-function does notvanish. Also, we may calculate in this way the values of $\zeta(-n)$ for each $n \in \mathbb{Z}_{\geq 0}$. TODO: continue.

References

[1] Herbert S. Wilf. *generatingfunctionology*. A K Peters, Ltd., Wellesley, MA, third edition, 2006.

[2] Don Zagier. The mellin transform and other useful analytic techniques. `http://people.mpim-bonn.mpg.de/zagier/files/tex/MellinTransform/fulltext.pdf`, 2006.