

# SOME NOTES ON COHERENT STATES

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ABSTRACT. We summarize a paper of Sugita (“Proof of the generalized Lieb-Wehrl conjecture for integer indices larger than one”) and Delbourgo–Fox (“Maximum weight vectors possess minimal uncertainty”).

## 1. THE GENERALIZED LIEB-WEHRL CONJECTURE FOR INTEGER INDICES LARGER THAN ONE

We summarize Sugita [2].

Let  $G$  be a compact connected Lie group. Let  $\pi$  be an irreducible unitary (complex) representation of  $G$ .

**Definition 1.** We say that a unit vector  $v \in \pi$  is a *coherent state* if it is a highest weight vector with respect to some maximal torus and ordering.

For convenience, let us now fix a maximal torus and ordering, so that we may speak of the highest weight  $\lambda$  of  $\pi$  and the line of highest weight vectors. Coherent states are then the  $G$ -translates of unit vectors in that line.

We equip  $G$  with its probability Haar  $dg$ .

**Theorem 2.** Let  $q \in \mathbb{Z}_{\geq 2}$ . For a unit vector  $v \in \pi$ , set

$$I(v) := (\dim \pi) \int_G (|\langle gv, v_\lambda \rangle|^2)^q dg.$$

(The normalizing factor is for later convenience.) Then  $I(v)$  achieves its maximum precisely when  $v$  is a coherent state.

**Remark 3.** Using the convexity of the  $q$ th power function  $x \mapsto x^q$ , one deduces an ostensibly more general result concerning higher rank tensors in place of  $v \otimes \bar{v}$ .

We turn to the proof of Theorem 2.

Let  $\Pi$  be an irreducible unitary representation of  $G$  with highest weight  $q\lambda$ . Then  $\Pi$  embeds as a subrepresentation of  $\pi^{\otimes q}$ , with multiplicity one. We identify  $\Pi$  with a subrepresentation of  $\pi^{\otimes q}$ :

$$\pi^{\otimes q} \supseteq \Pi.$$

We denote by

$$P : \pi^{\otimes q} \rightarrow \Pi$$

the orthogonal projection.

*Lemma 4.* For each unit vector  $v \in \pi$ , we have

$$I(v) = \frac{\dim \pi}{\dim \Pi} \langle P v^{\otimes q}, v^{\otimes q} \rangle.$$

*Proof.* Using the Schur orthogonality relations, we obtain the following integral formula for  $P$ :

$$P = (\dim \Pi) \int_G gv_\lambda^{\otimes q} \otimes \overline{gv_\lambda^{\otimes q}} dg.$$

Inserting this formula for  $P$  into the right hand side of the required identity gives what we want.  $\square$

It follows that  $I(v)$  is maximized precisely when  $v^{\otimes q} \in \Pi$ . We will show that

$$v : \text{coherent} \iff v^{\otimes q} \in \Pi. \quad (1)$$

The forward implication follows from the fact that  $v^{\otimes q} \in \Pi$ . It remains only to establish the converse.

To that end, we introduce the Casimir operator  $\Omega$  for  $G$ , given by  $\sum_{x \in \mathcal{B}(\mathfrak{g})} x^2$  for an orthonormal basis  $\mathcal{B}(\mathfrak{g})$  of  $\mathfrak{g} := \text{Lie}(G)$  taken with respect to a  $G$ -invariant inner product. This operator lies in the center of the universal enveloping algebra, hence acts on any given irreducible representation as multiplication by some scalar.

*Lemma 5.* For unit vectors  $v \in \pi$ , the inner product

$$\langle \Omega v^{\otimes q}, v^{\otimes q} \rangle$$

is maximized precisely when  $v$  is coherent.

*Proof.* We first compute that

$$\langle \Omega v^{\otimes q}, v^{\otimes q} \rangle = q\Omega_\pi + q(q-1)E(v), \quad (2)$$

where  $\Omega_\pi$  denotes the scalar by which  $\Omega$  acts on  $\pi$  and

$$E(v) := \sum_{x \in \mathcal{B}(\mathfrak{g})} |\langle xv, v \rangle|^2.$$

Consider for instance the case  $q = 2$ . For  $x \in \mathfrak{g}$ , we have

$$\begin{aligned} x^2 v^{\otimes 2} &= x(xv \otimes v + v \otimes xv) \\ &= x^2 v \otimes v + 2xv \otimes xv + v \otimes x^2 v. \end{aligned}$$

Summing over  $x \in \mathcal{B}(\mathfrak{g})$  gives

$$\begin{aligned} \Omega v^{\otimes 2} &= \Omega v \otimes v + 2 \sum_{x \in \mathcal{B}(\mathfrak{g})} xv \otimes xv + v \otimes \Omega v \\ &= 2\Omega_\pi v \otimes v + 2 \sum_{x \in \mathcal{B}(\mathfrak{g})} xv \otimes xv. \end{aligned}$$

When  $q = 3$ , we obtain instead

$$\Omega v^{\otimes 3} = 3\Omega_\pi v^{\otimes 3} + 2 \sum_{x \in \mathcal{B}} (xv \otimes xv \otimes v + xv \otimes v \otimes xv + v \otimes xv \otimes xv).$$

We get a similar expression in general, which, after pairing with  $v^{\otimes q}$ , gives the claimed formula (2).

We conclude via appeal to the following lemma.  $\square$

*Lemma 6.* For unit vectors  $v$ , the quantity  $E(v)$  is maximized precisely when  $v$  is coherent.

*Proof.* See Section 2.  $\square$

We may now complete the proof of the backwards implication in (1), hence of Theorem 2. Suppose  $v^{\otimes q} \in \Pi$ . Then, writing  $\Omega_\Pi$  for the scalar by which  $\Omega$  acts on the irreducible representation  $\Pi$ , we have  $\Omega v^{\otimes q} = \Omega_\Pi v^{\otimes q}$  and  $\Omega v_\lambda^{\otimes q} = \Omega_\Pi v_\lambda^{\otimes q}$ , hence

$$\langle \Omega v^{\otimes q}, v^{\otimes q} \rangle = \langle \Omega v_\lambda^{\otimes q}, v_\lambda^{\otimes q} \rangle.$$

By two applications of Lemma 5, we deduce that  $v$  is coherent, as required.

## 2. COHERENT STATES MINIMIZE UNCERTAINTY

We summarize Delbourgo–Fox [1], and in particular, prove Lemma 6.

Let  $G$  be a compact connected Lie group. Let  $\pi$  be an irreducible unitary representation of  $G$ . Equip the Lie algebra  $\mathfrak{g}$  with a  $G$ -invariant inner product, and let  $\mathcal{B}(\mathfrak{g})$  be an orthonormal basis.

**Definition 7.** For each unit vector  $v \in \pi$ , we define

$$\Delta(v) := \sum_{x \in \mathcal{B}(\mathfrak{g})} \|\bar{x}v\|^2$$

where  $\bar{x} := x - \langle xv, v \rangle$ .

**Theorem 8.** For unit vectors  $v$ , the quantity  $\Delta(v)$  is minimized precisely when  $v$  is a coherent state.

That Theorem 8 implies Lemma 6 is immediate from the following:

*Lemma 9.* We have  $\Delta(v) = -\Omega_\pi - E(v)$ , where  $\Omega_\pi$  denotes the Casimir eigenvalue for  $\pi$ .

*Proof.* We note that  $\bar{x}v = xv - \langle xv, v \rangle v$ , thus  $\langle \bar{x}v, v \rangle = 0$  and

$$\|\bar{x}v\|^2 = \langle \bar{x}v, xv \rangle.$$

By expanding the definition of  $\bar{x}$ , we obtain

$$\langle xv, xv \rangle - \langle xv, v \rangle \langle v, xv \rangle.$$

Using here that  $x$  acts via a skew-symmetric operator, we see that the first term equals  $-\langle x^2v, v \rangle$ , while the second term may be abbreviated to  $|\langle xv, v \rangle|^2$ . Summing over  $x$  leads to the required identity.  $\square$

We turn to the proof of Theorem 8.

*Lemma 10.* Let  $\pi$  be a unitary representation of  $G$ , let  $v \in \pi$  be a unit vector, let  $x \in i\mathfrak{g}$ . Then the minimum over all (real) scalars  $c$  of the quantity

$$\|(x - c)v\|$$

is achieved by taking  $c = \langle xv, v \rangle$ .

*Proof.* Since  $x$  lies in  $i\mathfrak{g}$ , it acts via a self-adjoint operator, so we readily obtain

$$\|(x - c)v\|^2 = \langle xv, xv \rangle - 2c\langle xv, v \rangle + c^2. \quad (3)$$

This is a quadratic polynomial whose minimum is attained at the critical point, which we solve for by taking a derivative.  $\square$

The definition of  $\Delta(v)$  remains unchanged upon replacing  $\mathfrak{g}$  with its imaginary multiple  $i\mathfrak{g}$ . By Lemma 10, we see that the minimum of  $\Delta(v)$  taken over unit vectors  $v$  coincides with the minimum of

$$\sum_{x \in \mathcal{B}(i\mathfrak{g})} \|(x - c_x)v\|^2 \quad (4)$$

taken over unit vectors  $v$  and tuples of scalars  $c_x$ . Such a minimum exists by continuity and compactness. Let us consider one such minimum. The above expression expands, as in (3), to

$$-\Omega_\pi + \sum_x c_x^2 - 2 \sum_x c_x \langle xv, v \rangle.$$

With the notation  $y := \sum_x c_x x \in \mathfrak{g}$ , the above may be written

$$-\Omega_\pi + \|y\|^2 - 2\langle yv, v \rangle.$$

We choose a maximal torus whose Lie algebra contains  $y$ , and an ordering with respect to  $y$  which is dominant. Our assumptions imply that  $\langle yv, v \rangle$  cannot be made larger by changing  $v$ , so  $v$  must lie in the eigenspace for  $y$  with largest eigenvalue. We claim that this eigenspace is one-dimensional. If the claim holds, then  $y$  is a highest weight vector, so we are done. Assume the claim fails. Let us then modify  $v$ , if necessary, so that it is a highest weight vector for the given torus and ordering; we may do so without changing  $\langle yv, v \rangle$ , hence without changing our assumption that  $v$  and  $c_x$  realize the minimum. We see then that by modifying  $y$  without changing  $\|y\|^2$ , we may increase the size of  $\langle yv, v \rangle$ , contradicting the supposed minimality. This completes the proof of Theorem 8.

### 3. DETERMINING VECTORS BY THEIR MATRIX COEFFICIENTS

It follows from Lemma 10 that for a given unit vector  $v$ , the quantity (4) is minimized by taking  $c_x = \langle xv, v \rangle$ . We record here that *the latter quantities determine  $v$  up to multiplication by a unit scalar*. Indeed, let  $G$  be a compact connected Lie group,  $\pi$  an irreducible representation, and  $u, v \in \pi$  unit vectors with  $\langle xu, u \rangle = \langle xv, v \rangle$  for all  $x \in \mathfrak{g}$ . By expanding the exponential series, we see that  $\langle \exp(x)u, u \rangle = \langle \exp(x)v, v \rangle$ , so that the  $\langle gu, u \rangle = \langle gv, v \rangle$  holds near the identity on  $G$ . Since matrix coefficients of finite-dimensional representations define analytic functions, we deduce the equality on all of  $G$ . We conclude that  $u$  and  $v$  are proportional by appeal to the following:

**Proposition 11.** *Let  $G$  be a compact group, let  $\pi$  be an irreducible representation, and let  $u$  and  $v$  be nonzero vectors in  $\pi$  with the property that*

$$\langle gu, u \rangle = \langle gv, v \rangle \quad (5)$$

*for all  $g \in G$ . Then  $u$  and  $v$  are proportional.*

We record a proof after some lemmas.

**Lemma 12** (Schur's lemma). *Let  $\pi$  be an irreducible representation of  $G$ . Then  $\text{End}_G(\pi) = \mathbb{C}$ , that is to say, any linear operator on  $\pi$  that commutes with the action of  $G$  is a scalar multiple of the identity.*

**Lemma 13.** *Let  $\pi$  and  $\sigma$  be unitary representations of  $G$ , with  $\pi$  irreducible. Let  $u \in \pi$  and  $v \in \sigma$  be nonzero vectors such that*

$$\langle gu, u \rangle = \langle gv, v \rangle.$$

Then there is a unique  $G$ -equivariant map  $T : \pi \rightarrow \sigma$  that sends  $u$  to  $v$ .

*Proof.* Any vector in  $\pi$  may be written  $\sum_{g \in G} c_g g u$  for some finitely-supported coefficients  $c_g$ . We have no choice but to attempt to define

$$T \left( \sum_{g \in G} c_g g u \right) := \sum_{g \in G} c_g g v.$$

We need to check that this is well-defined. To see this, we use that the vanishing of the argument of  $T$  may be detected via its inner product with itself, and then use that  $u$  and  $v$  have the same inner products to see that the right hand side must likewise vanish.  $\square$

*Proof of Proposition 11.* By Lemma 13, there is a unique  $T \in \text{End}_G(\pi)$  that maps  $u$  to  $v$ . By Schur's lemma,  $T$  is a multiple of the identity. [thinking-face]  $\square$

#### REFERENCES

- [1] R. Delbourgo and J. R. Fox. Maximum weight vectors possess minimal uncertainty. *J. Phys. A*, 10(12):233–235, 1977.
- [2] Ayumu Sugita. Proof of the generalized Lieb-Wehrl conjecture for integer indices larger than one. *J. Phys. A*, 35(42):L621–L626, 2002.