These are notes for an ongoing Fall 2023 course on the Riemann zeta function and its generalizations, L-functions. These notes will be filled in as we go.

1. References thus far

- Generating functions and asymptotics: [1, §5.2]
- Mellin transform and asymptotics: [2]

2. Outlines thus far

- Tuesday, 29 Aug: parts of §4; §4 and §4.
- Thursday, 31 Aug: §**??** and §**??**
- Friday, 1 Sep: §**??** and §**??**
- Tuesday, 5 Sep: §5

3. Course notes that I've since split off into separate files

- Complex-analytic preliminaries
- Generating functions and asymptotics
- Fourier and Mellin transforms

4. Background

4.1. General notation. $\mathbb{R}^+ := (0, \infty)$.

4.2. Asymptotic notation. We use the equivalent notations

$$A = \mathrm{O}(B), \qquad A \ll B, \qquad B \gg A$$

to denote that

$$|A| \leq C|B|$$

for some "constant" C. The precise meaning of "constant" will either be specified or clear from context.

4.3. **Definition and basic properties of ζ: overview.** The Riemann zeta function is defined for a complex number s by the series

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

Lemma 4.1. *The series converges absolutely for $\Re(s) > 1$, uniformly for $\Re(s) \geq 1+\varepsilon$ for each $\varepsilon > 0$.*

Proof. Using the identity

$$\left| \frac{1}{n^s} \right| = \frac{1}{n^{\Re(s)}},$$

we reduce to the case that s is real, in which this is a familiar consequence of the integral test. □

Our first main goal in the course is to explain the following basic facts.

Theorem 4.2. *The Riemann zeta function admits a meromorphic continuation to the entire complex plane. It is holomorphic away from a simple pole at $s = 1$, where it has residue 1. It admits a functional equation relating $\zeta(s)$ to $\zeta(1-s)$.*

One historical motivation for considering the zeta function at complex arguments comes from the prime number theorem.

Theorem 4.3 (Prime number theorem)**.** *Let $\pi(x) := \#\{\text{primes } p \leq x\}$ denote the prime counting function. Then*

$$\frac{\pi(x)}{x/\log x} \to 1 \text{ as } x \to \infty.$$

This is related to the following analytic fact concerning the zeros of the zeta function.

Theorem 4.4 (Prime number theorem, formulated in terms of ζ)**.** *We have $\zeta(s) = 0$ only if $\Re(s) < 1$.*

Remark 4.5. Even the statement of Theorem 4.4 is not clear without knowing the meromorphic continuation of ζ. This may offer some motivation for understanding the latter.

We expect stronger nonvanishing properties:

Conjecture 1 (Riemann Hypothesis)**.** *We have $\zeta(s) = 0$ only if $\Re(s) < 1/2$.*

This corresponds to a conjectural stronger form of the prime number theorem, namely that

$$\pi(x) = \int_2^x \frac{t}{\log t}\, dt + \mathrm{O}(x^{1/2} \log x).$$

5. Basic analytic properties of ζ

Example 5.1. Take

$$h(y) = \frac{1}{e^y - 1}.$$

The function $yh(y)$ extends to a holomorphic function of y on the disc $\{y \in \mathbb{C} : |y| < 2\pi\}$, so h is represented for small $y > 0$ by an absolutely convergent Laurent series of the following form:

$$h(y) = \frac{1}{y} + \sum_{n=1}^{\infty} \frac{B_n}{n!} y^{n-1}.$$

Here the B_n are complex coefficients, called the *Bernoulli numbers*. On the other hand, h decays rapidly (like $\mathrm{O}(y^N)$ for any fixed N) as $y \to \infty$. By the analysis we've now seen many times, we deduce that the Mellin transform $H(s)$ of $h(y)$ converges absolutely for $\Re(s) > 1$, where it defines a holomorphic function, and extends to a meromorphic function on the complex plane whose only poles are simple poles at $s = 1$ (corresponding to the $1/y$ term in the asymptotic expansion as $y \to 0$) and at $s = -n$ for each $n \in \mathbb{Z}_{\geq 0}$, with residue given by 1 in the former case and by $B_{n+1}/(n+!)!$ in the latter.

On the other hand, we can rewrite

$$h(y) = \frac{e^{-y}}{1 - e^{-y}} = e^{-y} + e^{-2y} + e^{-3y} + \cdots,$$

giving

$$H(s) = \int_{\mathbb{R}^+} y^s \sum_{n=1}^{\infty} e^{-ny} \, d^{\times} y.$$

The following calculations will show that the doubly integral/sum converges absolutely for $\Re(s) > 1$, so we may rearrange it as

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}^+} y^s e^{-ny} \, d^{\times} y.$$

The inner integral may be simplified by the substitution $y \mapsto y/n$. This has no effect on the measure $d^{\times} y$, but replaces y^s by $n^{-s} y^s$, giving

$$\sum_{n=1}^{\infty} n^{-s} \int_{\mathbb{R}^+} y^s e^{-y} \, d^{\times} y = \zeta(s)\Gamma(s).$$

We see in this way that the ζ function admits a meromorphic continuation. Since the Γ-function does not vanish, we deduce that the only pole of ζ is the simple one at $s = 1$, with residue $s = 0$. Also, since we have computed the residues of both $\Gamma(s)$ and $\zeta(s)\Gamma(s)$ at the nonpositive integers, we may calculate in this way the values of $\zeta(-n)$ for each $n \in \mathbb{Z}_{\geq 0}$:

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}.$$

References

[1] Herbert S. Wilf. *generatingfunctionology*. A K Peters, Ltd., Wellesley, MA, third edition, 2006.

[2] Don Zagier. The mellin transform and other useful analytic techniques. `http://people.mpim-bonn.mpg.de/zagier/files/tex/MellinTransform/fulltext.pdf`, 2006.