1. Holomorphic continuation

Theorem 1 (Identity principle for holomorphic functions). Let $U \subset \mathbb{C}$ be a connected open set. Let $f, g: U \to \mathbb{C}$ be holomorphic functions. If f = g on a set with a limit point in U, then f = g on all of U.

Corollary 2. Let $U \subset \Omega \subseteq \mathbb{C}$ be open subsets, with U nonempty and Ω connected. Let $f: U \to \mathbb{C}$ be a holomorphic function. Then there is at most one extension of f to a holomorphic function $\Omega \to \mathbb{C}$.

2. Cauchy's integral formula

Theorem 3. Let $f: U \to \mathbb{C}$ be a holomorphic function defined on an open subset U. Let γ be a closed rectifiable curve in U. Then $\int_{\gamma} f(z) dz = 0$.

Theorem 4. Let $0 \le a < b \le \infty$. Let f(z) be a holomorphic function on the annulus $\{z \in \mathbb{C} : a < |z| < b\}$ given by a convergent Laurent series

$$f(z) = \sum_{n \in \mathbb{Z}} c_n z^n.$$

(1) For any $r \in (a,b)$ and $n \in \mathbb{Z}$, we have

$$c_n = \oint_{|z|=r} \frac{f(z)}{z^n} \frac{dz}{2\pi i z}$$
$$= \frac{1}{2\pi r^n} \int_{\theta=0}^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta.$$

(2) For each compact subset E of (a,b), there exists $M \geq 0$ so that for all $r \in E$, we have

$$\sum_{n\in\mathbb{Z}} |c_n| r^n \le M. \tag{1}$$

Theorem 5. Let U be an open subset of \mathbb{C} , let $f:U\to\mathbb{C}$ be meromorphic. Let γ be a smooth closed curve in U, oriented counterclockwise, that does not pass through any pole of f. Then

$$\int_{\gamma} f(z) dz = 2\pi i \sum_{\substack{z \in \text{interior}(\gamma) \\ pole \ of \ f}} \text{res}_{z}(f).$$

Remark 6. Let 0 < r < R. Let f be a meromorphic function on a neighborhood of the annulus $\{z : r < |z| < R\}$ that has no poles on either of the circles |z| = r, R. Then

$$\oint_{|z|=R} f(z) dz = \oint_{|z|=r} f(z) dz + 2\pi i \sum_{\substack{r < |z| < R \text{pole of } f}} \operatorname{res}_z(f).$$

3. Holomorphy of limits and series

Theorem 7. Let U be an open subset of the complex plane. Let f_n be a sequence of holomorphic functions on U.

- (1) Suppose that the sequence f_n converges pointwise to some function f, uniformly on compact subsets of U. Then f is holomorphic.
- (2) Suppose that the partial sums $\sum_{n\leq N} f_n$ converge pointwise to some function f, uniformly on compact subsets of U. Then the sum $\sum_n f_n$ is holomorphic.

References