

1. ANNE-MARIA ERNVALL-HYTÖNEN, *Lattices and modular forms in coset coding*

Wyner (1975). Wiretap channel (Alice, Bob, Eve). Gaussian noise with variance σ^2 (between Alice and Bob) and σ_e^2 (for Eve). We assume $\sigma_e > \sigma$. We would like to do this whole scheme in such a way that we can use the noise so that Bob can still receive the image, but Eve cannot. We aim to do this with lattices $\Lambda \geq \Lambda_e$, where each coset corresponds to a code-letter.

Belfiore and Oggier (2010) (maybe [?]): secrecy gain. Transmit codeword $x \in \mathbb{R}^n$.

$$\frac{1}{(\sigma_\Lambda \sqrt{2\pi})^n} \int_{V_{\Lambda(x)}} e^{-\|y-x\|^2/2\sigma^2} dy.$$

$$\frac{1}{(\sigma_\Lambda \sqrt{2\pi})^n} \sum_{t \in \Lambda_e \cap R} \int_{V_{\Lambda(x+t)}} e^{-\|y-x\|^2/2\sigma^2} dy.$$

End up trying to minimize

$$\theta_{\Lambda_e} \left(-\frac{1}{2\pi i \sigma_e^2} \right).$$

Secrecy function

$$\Xi(y) := \frac{\theta_{\Lambda \mathbb{Z}^n}(yi)}{\theta_{\Lambda}(yi)}.$$

Belfiore and Sole: For unimodular Λ , maximum at $y = 1$. Even unimodular: polynomials in

$$E_4 = \frac{1}{2} (\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8), \quad \Lambda = \frac{1}{256} \vartheta_2^8 \vartheta_3^8 \vartheta_4^8.$$

Inverse of $\Xi(y)$ a polynomial in $\frac{\vartheta_4^4 \vartheta_2^4}{\vartheta_3^8}$.

(Maybe [?] is a reference.)

ℓ -modular: $1/\sqrt{\ell}$, $\mathbb{Z} \oplus \sqrt{2}\mathbb{Z} \oplus 2\mathbb{Z}$.

The connection to deeper mathematics is, what kinds of representations as polynomials do you have for various theta functions?

2. JOLANTA MARZEC-BALLESTEROS, *Doubling method for self-dual linear codes*

Garrett, Piatetski-Shapiro–Rallis (1980's). Integral of cusp form against restriction of Siegel-type Eisenstein series equals L -function attached to cusp form times cusp form or Eisenstein series attached to cusp form:

$$\left\langle E \left(\begin{pmatrix} g & \\ & g' \end{pmatrix}, s \right), f(g) \right\rangle = L(f, s) f(g').$$

Done for G symplectic, orthogonal, unitary over global field, also for congruence subgroups.

Let's start with an overview. Let f be a cusp form on G , and H a subgroup for which $G \times G \hookrightarrow H$. Then form an Eisenstein series on H

$$E(h, s) = \sum_{\gamma \in P \backslash H} \phi(\gamma h, s).$$

Restrict to $h = \text{diag}(g, g')$ and take inner product with $F(g)$. This leads to an unfolding involving a sum over $\gamma \in P \backslash H / (G \times G)$. In favorable cases, only one

representative γ_0 contributes, leaving us with

$$\begin{aligned} & \sum_{(k,k') \in G \times G} \left\langle \phi \left(\gamma_0 \begin{pmatrix} kg & \\ & k'g' \end{pmatrix}, s \right), f(g) \right\rangle \\ &= \sum_{\beta \in \gamma_0(G \times 1)} \psi(f) f|_{\beta}(g') \\ &= L(f, s) f(g'). \end{aligned}$$

We would like to do something similar, but now over finite fields.

A *linear code of length* $2n$ over a finite field \mathbb{F} is a linear subspace $C \subseteq \mathbb{F}^{2n}$. We denote by $\langle, \rangle : C \times C \rightarrow \mathbb{F}$ the Euclidean inner product. We say that C is *self-dual* if

$$C = C^\perp := \{v \in \mathbb{F}^{2n} : \langle v, C \rangle = 0\}.$$

Then (the length $2n$ is even and) $\dim C = n$.

The *weight* of a codeword $c = (c_1, \dots, c_{2n}) \in C$ is

$$\text{wt } c = \#\{i \in \{1, \dots, 2n\} : c_i \neq 0\}.$$

Weight enumerators are certain homogeneous polynomials of degree $2n$ in variables from the set $V = \{x_\alpha : \alpha \in \mathbb{F}^g\}$, where $g \in \mathbb{N}$ is the genus.

The *genus one weight enumerator* of a code $C \subseteq \mathbb{F}_2^{2n}$ is a polynomial

$$W_1(C, (x_0, x_1)) = \sum_{c \in C} x_0^{2n - \text{wt } c} x_1^{\text{wt } c}.$$

The *genus g weight enumerator* of a code $C \subseteq \mathbb{F}^{2n}$ is a polynomial

$$W_g(C, x) = \sum_{(c^1, \dots, c^g) \in C^g} \prod_{\alpha \in \mathbb{F}^g} x_\alpha^{w_\alpha(c^1, \dots, c^g)}$$

of degree $2n$, where $x = (x_\alpha)_{\alpha \in \mathbb{F}^g}$ and

$$w_\alpha(c^1, \dots, c^g) = \#\left\{\text{rows } r \text{ in } (c_i^j)_{i=1..2n}^{j=1..g} : r = \alpha\right\}.$$

As an example, we give a basis for a Hamming code H_8 of weight 8, the span over \mathbb{F}_2 inside \mathbb{F}_2^8 of the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Then

$$\begin{aligned} W_2(H_8, (x_{00}, x_{01}, x_{10}, x_{11})) &= \sum_{\alpha \in \mathbb{F}_2^2} x_\alpha^8 + 14 \sum_{\substack{\alpha_1, \alpha_2 \in \mathbb{F}_2^2 \\ \alpha_1 < \alpha_2}} x_{\alpha_1}^4 x_{\alpha_2}^4 + 168 x_{00}^2 x_{01}^2 x_{10}^2 x_{11}^2 \\ &= (8) + 14(4, 4) + 168(2, 2, 2, 2). \end{aligned}$$

In general,

$$W_g(C) = \sum_A b_A \cdot (A),$$

where

$$A \in \left\{ (a_0, \dots, a_{2^g-1}) : \text{admissible tuples, } \sum_{i=0}^{2^g-1} a_i = 2n \right\}.$$

Some analogies with modular forms:

- $W_g(C)$ is like a modular form f of genus g ,
- $\sum_A b_A \cdot (A)$ is like a Fourier expansion,
- $(2n)$ is like a constant term $a(0)$.

EXamples of cusp forms:

- $W_1(G_{24}) - W_1(H_8 \times H_8 \times H_8) = -42(20, 4) + 168(16, 8) - 252(12, 12)$ is a cusp form of genus one.
- $W_3(E_{16}) - W_3(H_8 \times H_8) = -2688(9, 1, 1, 1, 1, 1, 1, 1) + \dots$ is a cusp form of genus 3.

Theorem 1 (Runge, 1996; Nebe, Rains, Sloane, 2006). *We have*

$$\langle W_g(C) : C \text{ self-dual, over } \mathbb{F} \rangle = (\mathbb{C}[x_\alpha : \alpha \in \mathbb{F}^g])^{C_g},$$

where

$$C_g := \langle m_r, d_\phi, h_{\iota, u_\iota, v_\iota} : r \in \text{GL}_g(\mathbb{F}), \phi, \iota \rangle$$

with

$$\begin{aligned} m_r : x_\alpha &\mapsto x_{r\alpha}, \\ d_\phi : x_\alpha &\mapsto e^{2\pi i \phi(\alpha)} x_\alpha, \\ h_{\iota, u_\iota, v_\iota} : x_\alpha &\mapsto (\#\iota\mathbb{F}^g)^{-1/2} \sum_{w \in \iota\mathbb{F}^g} e^{\frac{2\pi i}{p} \langle w, v_\iota \alpha \rangle} x_w + \dots \end{aligned}$$

Consider the mean polynomial (“Siegel-type Eisenstein series”)

$$\begin{aligned} M_{2g}((2n)) &= \sum_{\gamma \in P_{2g} \backslash C_{2g}} (2n)^\gamma \\ &= \sum_{\gamma \in P_{2g} \backslash C_{2g}} \sum_{\alpha \in \mathbb{F}^{2g}} ((x_\alpha)^\gamma)^{2n} = \text{const} \sum_{\substack{C \subset \mathbb{F}^{2n} \\ \text{fixed type}}} W_{2g}(C, x), \end{aligned}$$

and an inner product defined on monomials.

What we prove with Bourganis in 2024 is the following:

Theorem 2. *Let \mathcal{T} be a family of self-dual codes of length $2n$ over a field \mathbb{F} . Assume either that \mathbb{F} has odd characteristic or is equal to \mathbb{F}_2 . Let $C \in \mathcal{T}$ be doubly-even, and fix $g \in \mathbb{N}$. Then there exists an (explicit) constant C such that for a cusp form $f \in \mathcal{T}$ of genus r , with $\deg f = 2n$, we have*

$$\langle M_{2g}((2n))(xy), f(x) \rangle = \begin{cases} 0 & \text{if } r < g, \\ C \cdot f(y) & \text{if } r = g \end{cases}.$$

3. PETRU CONSTANTINESCU, *Non-vanishing of geodesic periods of automorphic forms*

Preprint: [?], joint with Asbjørn Nordentoft.

Class groups:

- $\Gamma = \text{PSL}_2(\mathbb{Z})$, $K = \mathbb{Q}(\sqrt{D})$,
- Cl_K class group, $h(D) = h_K = |\text{Cl}_K|$ class number,

- \mathcal{Q}_D : set of primitive integral binary quadratic forms of discriminant D ,
 $\mathcal{Q}_D = \{Q(x, y) = ax^2 + bxy + cy^2 : (a, b, c) = 1, b^2 - 4ac = D\}.$

- Gauss: $\Gamma \curvearrowright \mathcal{Q}_D$,

$$(Q \cdot \gamma) \begin{pmatrix} x \\ y \end{pmatrix} = Q \left(\gamma \begin{pmatrix} x \\ y \end{pmatrix} \right),$$

- isomorphism

$$\begin{aligned} \text{Cl}_K &\xrightarrow{\cong} \Gamma \backslash \mathcal{Q}_D \\ A &\mapsto [a, b, c]. \end{aligned}$$

Heegner points and closed geodesics:

- $D < 0$: $A \in \text{Cl}_K \rightsquigarrow$ Heegner point $z_A \in \Gamma \backslash \mathbb{H}$,

$$[a, b, c] \rightsquigarrow \frac{-b - i\sqrt{|D|}}{2a}.$$

$$h(D) = |\text{Cl}_K| = |D|^{1/2+o(1)}.$$

- $D > 0$: $A \in \text{Cl}_K^+ \rightsquigarrow$ closed geodesic $C_A \subset \Gamma \backslash \mathbb{H}$, $[a, b, c] \rightsquigarrow$ semicircle with endpoints $\frac{-b \pm \sqrt{D}}{2a}$.

$$h(D) \log \varepsilon_D = D^{1/2+o(1)},$$

$$I(C_A) = 2 \log \varepsilon_D.$$

Theorem 3 (Duke '88). *Fix $\Omega \subset \text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}$.*

- $D < 0$: *equidistribution of z_A , $A \in \text{Cl}_{\mathbb{Q}(\sqrt{D})}$.*
- $D > 0$: *similar for closed geodesics.*

Waldspurger formulas. Let f be a nonzero Maass form. Our goal is to study closed geodesic periods $\int_{C_A} f(z) \frac{|dz|}{y}$.

$\chi \in \widehat{\text{Cl}_K} \rightsquigarrow \theta_\chi$, the associated theta series (weight one on $\Gamma_0(D)$, nebentypus χ_D).

Theorem 4 (Waldspurger/Zhang/Popa). *Let D be a fundamental discriminant. For $D < 0$,*

$$L(f \times \theta_\chi, \tfrac{1}{2}) = \frac{C_f}{D^{1/2}} \left| \sum_{A \in \text{Cl}_K} \chi(A) f(z_A) \right|^2.$$

For $D > 0$,

$$L(f \times \theta_\chi, \tfrac{1}{2}) = \frac{C_f}{D^{1/2}} \left| \sum_{A \in \text{Cl}_K^+} \chi(A) \int_{C_A} f(z) \frac{|dz|}{y} \right|^2.$$

Theorem 5 (Michel–Venkatesh '05). *Let $\delta = 1/2700$. For $D < 0$:*

$$\left| \left\{ \chi \in \widehat{\text{Cl}_K} : L(f \times \theta_\chi, \tfrac{1}{2}) \neq 0 \right\} \right| \gg D^\delta.$$

Sketch of proof. By orthogonality of characters,

$$\frac{1}{h(D)} \sum_{\chi \in \widehat{\text{Cl}_K}} L(f \times \theta_\chi, \tfrac{1}{2}) = \frac{c_f}{D^{1/2}} \sum_{A \in \text{Cl}_K} |\dots|^2.$$

This converges by Duke's equidistribution theorem. The conclusion then follows from subconvexity for Rankin–Selberg L -functions, due to Harcos–Michel. \square

Same proof does not work for geodesics, cannot apply equidistribution and relate to subconvexity (square is outside integral).

Question 6 (Michel, Oberwolfach 2020). Let K be a real quadratic field of discriminant $D > 0$, and assume that $h_K \gg D^\delta$. Does there exist $A \in \text{Cl}_K^+$ such that

$$\int_{C_A} f \frac{|dz|}{y} \neq 0?$$

Equivalently, does there exist $\chi \in \widehat{\text{Cl}_K^+}$ such that $L(f \times \theta_\chi, \frac{1}{2}) \neq 0$?

The prime geodesic theorem. Let $D > 0, A \in \text{Cl}_K^+$. This gives rise to a closed geodesic C_A , with $I(C_A) = 2 \log \varepsilon_D$. Sound-Young have the best estimate for their count.

Theorem 7 (C-Nordentoft 2024). *Let f be a nonzero Maass form for $\text{SL}_2(\mathbb{Z})$. Then*

$$\# \left\{ C \in \mathcal{C}(X) : \int_C f(z) \frac{|dz|}{y} = 0 \right\} \ll \frac{X}{(\log X)^{5/4}}.$$

Remark 8. We also obtain 100% non-vanishing for periods of weight k holomorphic cusp forms, for any Fuchsian group Γ .

Theorem 9 (C-Nordentoft 2024). *For a positive proportion of positive discriminants $D > 0$ with $\varepsilon_D \leq X$, we get that there exists $\chi \in \widehat{\text{Cl}_{\mathbb{Q}(\sqrt{D})}}$ with $L(f \times \theta_\chi, \frac{1}{2}) \neq 0$.*

We construct a bipartite graph on X_N (double cosets in $\Gamma_\infty \backslash \Gamma / \Gamma_\infty$ with $c \leq N$) times Y_N (conjugacy classes with trace bounded in magnitude by N). This graph relates closed geodesic and vertical geodesics.

4. AN EXCISED ORTHOGONAL MODEL FOR FAMILIES OF CUSP FORMS

Talk by Zoe Batterman (Abstract), Akash Narayanan, Christopher Yao. Joint with Owen Barrett, Aditya Jambhale, and Kishan Sharma. Preprint: [?].

Conjecture (Montgomery–Dyson, 1970’s): zeros of zeta vs. GUE.

2005: S.J. Miller noticed a repulsion of the lowest-lying zeros near the central point of a family of even twists of a fixed elliptic curve L -function with finite conductor.

2011: Duenez, Huynh, Keating, Miller, Snaith: proposed an excised orthogonal model to capture the behavior of this repulsion.

Question 10. How accurately do eigenvalues of random matrices from classical compact groups model the lowest-lying zeros of families of L -functions associated to a cuspidal newform?

Let

$$S_k^{\text{new}}(M, \chi_f) \ni f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z},$$

$$\lambda_f(n) = a_f(n) / n^{(k-1)/2}.$$

$$L(s, f) = \sum_{n \geq 1} \lambda_f(n) n^{-s},$$

Various specific families of twists $L(f \otimes \psi_d, s)$, match with classical compact groups:

- principal character, even twists vs. $\mathrm{SO}(\text{even})$
- principal character, odd twists vs. $\mathrm{SO}(\text{odd})$
- non-principal character, self-dual vs. Sp
- generic vs. U

Pictures.

5. ON AN EXTENSION OF THE ROHRlich-JENSEN FORMULA, *Lejla Smajlović*

Joint work with James Cogdell, Jay Jorgensen. Preprint: [?].

What is a Poisson–Jensen formula? We will view it as a way to characterize meromorphic functions in terms of their divisors. Some notation:

- $D_R = \{z = x + iy \in \mathbb{C} : |z| < R\}$
- F : a non-constant meromorphic function on $\overline{D_R}$,

$$F(z) = c_F z^m + O(z^{m+1}), \quad z \rightarrow 0.$$

Then

$$\int_0^{2\pi} \log|F(Re^{i\theta})| \frac{d\theta}{2\pi} + \sum_{D_R} \dots = F(0).$$

Rohrlich, 1980's: a *modular* generalization, characterizing modular forms via divisors. Given f , a meromorphic function on \mathbb{H} that is invariant by $\mathrm{PSL}(2, \mathbb{Z})$. Assume f is holomorphic at the cusp and that the Fourier expansion of f at ∞ has constant term equal to one. Then

$$\int_{\mathrm{PSL}(2, \mathbb{Z}) \backslash \mathbb{H}} \log|f(z)| \frac{d\mu(z)}{2\pi} + \sum_{w \in \mathcal{F}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} P(w) = 0.$$

Here

- $\mathrm{ord}_w(f)$ is the order of f at w as a meromorphic function,
- $\mathrm{ord}(w)$ denotes the order of the *point* w with respect to the action of $\mathrm{PSL}(2, \mathbb{Z})$ on \mathbb{H} , and
- $P(w) = \log(|\eta(w)|^4 \cdot \Im(w))$ is the Kronecker limit function associated to the parabolic Eisenstein series on $\mathrm{PSL}(2, \mathbb{Z})$. This is the function appearing as the next-order term in the expansion of the Eisenstein series as $s \rightarrow 1$.

Another way to interpret this formula is as follows. We have

$$\langle 1, \log|f(z)| \rangle = \lim_{Y \rightarrow \infty} \int_{\mathcal{F}(Y)} 1 \cdot \log|f(z)| d\mu(z),$$

hence the formula reads

$$\langle 1, \log|f(z)| \rangle = -2\pi \sum_{w \in \mathcal{F}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} P(w).$$

Here $\log|f(z)|$ can be replaced by

$$\log\|f(z)\| = \log(\Im z^k |f(z)|)$$

for weight $2k$ meromorphic modular forms. Generalizations:

- to other Fuchsian groups of the first kind, by Rohrlich.
- to hyperbolic 3-space, by Herrero, Imamoglu, von Pippich, Toth.

Further modular generalization by Bringman and Kane. Keep the modular group setting, but evaluate more general inner products.

- $j(z) = q^{-1} + 744 + O(q)$: Hauptmodul

- $j_1(z) := j(z) - 744$
- $j_n(z) := j_1|T_n(z)$, for $n \geq 2$
- f : weight $2k$ meromorphic modular form with respect to $\mathrm{PSL}(2, \mathbb{Z})$, normalized so that $f(z) = 1 + O(q)$

They evaluated the regularized scalar product in terms of the divisor of f , proving that

$$\langle j_n(z), \log((\Im(z))^k |f(z)|) \rangle = -2\pi \sum_{w \in \mathcal{F}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} \mathbf{j}_n(w) + \frac{k}{6} c_n,$$

where \mathbf{j}_n is characterized in terms of differential operators by Bringman and Kane; application of our results yields a different expression for the same function, namely

$$\mathbf{j}_n(w) = 2\pi\sqrt{n}\partial_s F_{-n}^{\mathrm{PSL}(2, \mathbb{Z})}(w, s)|_{s=1} - 24\sigma(n)P(w).$$

What is our main goal?

- To extend the point of view that the Rohrlich–Jensen formula is the evaluation of a particular type of inner product.
- To prove the extension of this formula in the setting of an arbitrary, not necessarily arithmetic, Fuchsian group of the first kind with one cusp.

We start with $j_n(z) = j_1|T_n(z)$, which is the unique (up to constants) holomorphic function that is $\mathrm{PSL}(2, \mathbb{Z})$ -invariant on \mathbb{H} and whose expansion near ∞ is $q^{-n} + o(q^{-1})$.

These properties hold for the special value $s = 1$ of the Niebur–Poincaré series $F_{-n}^\Gamma(z, s)$, defined for any Fuchsian group Γ of the first kind with one cusp by

$$F_m^\Gamma(z, s) = \sum_{\gamma_\infty \setminus \Gamma} e(m\Re(\gamma z)) (\Im(\gamma z))^{1/2} I_{s-1/2}(2\pi|m\Im(\gamma z)),$$

for $m \neq 0$.

It is an eigenfunction of the hyperbolic Laplacian, and may be expressed in terms of j_m .

The term $\log(\|f(z)\|)$ is a bit complicated, involving findings by Jorgensen, von Pippich and the speaker, plus some additional work done in our paper.

Proposition 11. *Let Γ be a cofinite Fuchsian group with one cusp at ∞ with identity as its scaling matrix. Let $2k \geq 0$ be an even integer, and let f be a weight $2k$ meromorphic form which is Γ -invariant and with q -expansion at ∞ normalized so its constant term is equal to one. Then, we can express $\log(\|f\|(z))$ in terms of parabolic Eisenstein series and Green's functions, as*

$$-2k + 2\pi \sum_{w \in \mathcal{F}_\Gamma} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} \lim_{s \rightarrow 1} \left(G_s^\Gamma(z, w) + \mathcal{E}_{\Gamma, \infty}^{\mathrm{par}(z, s)} \right) + \dots$$

Here

$$\mathcal{E}_{\infty}^{\mathrm{par}}(z, s) = \sum_{\Gamma_\infty \setminus \Gamma} \Im(\gamma z)^s.$$

The Kronecker limit formula says that

$$\mathcal{E}_{\infty}^{\mathrm{par}}(z, s) = \frac{1}{(s-1)\mathrm{vol}(\Gamma \setminus \mathbb{H})} + \beta - \frac{1}{\mathrm{vol}(\Gamma \setminus \mathbb{H})} \log(|\eta_\infty^4(z)| \Im(z)) + O(s-1).$$

Then we need to define β , and the Green's function $G_s(z, w)$, which is obtained by averaging the kernel $k_s(z, w)$; both functions are eigenfunctions of the Laplacian with eigenvalue $s(1-s)$, and the Green's function has a specified singularity on the

diagonal. The Green's function also has a Laurent series expansion as $s \rightarrow 1$ that involves the parabolic Eisenstein series. In particular,

$$\lim_{s \rightarrow 1} \left(G_s^\Gamma(z, w) + \mathcal{E}_{\Gamma, \infty}^{\text{par}}(z, w) \right)$$

exists, with a logarithmic singularity on the diagonal.

The Rohrlich–Jensen formula can be understood through the study of the regularized inner product of this last limit with $F_{-n}(\cdot, 1)$. Regularization is needed only in the cusp (because the logarithmic singularity is integrable). We can thus write

$$\begin{aligned} & \left\langle F_{-n}(\cdot, 1), \overline{\lim_{s \rightarrow 1} \left(G_s^\Gamma(z, w) + \mathcal{E}_{\Gamma, \infty}^{\text{par}}(z, w) \right)} \right\rangle \\ &= \lim_{Y \rightarrow \infty} \int_{\mathcal{F}(Y)} F_{-n}(z, 1) \lim_{s \rightarrow 1} (G_s(z, w) + \mathcal{E}_\infty^{\text{par}}(z, s)) d\mu(z). \end{aligned}$$

A key observation is that all terms are eigenfunctions of the Laplacian, hence one can seek to compute the inner product in a manner similar to that which yields the Maass–Selberg formula. The key identity is that

$$\begin{aligned} & \int_{\mathcal{F}(Y)} F_{-n}(z, 1) \lim_{s \rightarrow 1} (G_s(z, w) + \mathcal{E}_\infty^{\text{par}}(z, s)) d\mu(z) \\ &= \partial_s \left(-s(1-s) \int_{\mathcal{F}(Y)} F_{-n}(z, 1) (G_s(z, w) + \mathcal{E}_\infty^{\text{par}}(w, s) d\mu(z)) \right) \Big|_{s=1}. \end{aligned}$$

We can now absorb $-s(1-s)$ into the integrals after applying the hyperbolic Laplacian Δ to each of the two factors in the integrand.

Main results:

Theorem 12. *For any positive integer n and any point $w \in \mathcal{F}$, we have*

$$\left\langle F_{-n}(\cdot, 1), \overline{\lim_{s \rightarrow 1} (G_s(\cdot, w) + \mathcal{E}_\infty^{\text{par}}(\cdot, s))} \right\rangle = -\partial_s F_{-n}(w, s) \Big|_{s=1}.$$

Combined with the previous proposition (describing $\log \|f(z)\|$), we get the Rohrlich–Jensen formula:

$$\langle F_{-n}(\cdot, 1), \log \|f\| \rangle = -2\pi \sum_{w \in \mathcal{F}} \frac{\text{ord}_w(f)}{\text{ord}(w)} \partial_s F_{-n}(w, s) \Big|_{s=1}.$$

We have further results. For instance, if g is any Γ -invariant analytic function with a pole at ∞ , then (Niebur)

$$g(z) = \sum_{n=1}^K 2\pi \sqrt{n} a_n F_{-n}(z, 1) + c(g)$$

for some constants K , a_n , and $c(g)$ depending only upon g . Then, we have the identity

$$\langle g, \log \|f\| \rangle = -2\pi \sum_{w \in \mathcal{F}} \frac{\text{ord}_w(f)}{\text{ord}(w)} \left(2\pi \sum_{n=1}^K \sqrt{n} a_n \partial_s F_{-n}(w, s) \Big|_{s=1} - \cdots \right).$$

Further found that the generating series for the Niebur Poincaré series at $s = 1$ is, in the z -variable (where we sum over q_z), the holomorphic part of the weight two biharmonic Maass form given by differentiating our linear combination of Eisenstein series and Green's function with respect to z .

6. SHENGHAO HUA, *Joint Value Distribution of Hecke–Maass Forms*

Preprint: [?]

Motivation: semiclassical limit of solutions to Schroedinger equation. Berry’s random wave conjecture. QUE, L^p -norms.

Joint Gaussian moments conjecture: statistical independence of values of large eigenfunctions (multiplied together and integrated against some test function).

Theorem 13. *Let f and g be normalized Hecke–MAass cusp forms. Then if $2t_f \leq t_g - t_g^\varepsilon$, we have*

$$\int_{\Gamma \backslash \mathbb{H}} \psi f^2 g \ll t_g^{-A},$$

while if $2t_f > t_g - t_g^\varepsilon$, then, assuming GLH, we have

$$\int_{\Gamma \backslash \mathbb{H}} \psi f^2 g = \int_{\Gamma \backslash \mathbb{H}} \psi g + O(t_f^{-1/4+\varepsilon}(1 + |2t_f - t_g|^{-1/4})).$$

Proof uses spectral expansion of ψ into eigenfunctions u , then spectral expansion of $\int (ug)(f^2)$, reducing to the case $g = u$, then further spectral expansion of $\int (g^2)(f^2)$.

Theorem 14. *Assuming GRH and GRC, we have as $\max(t_f, t_g) \rightarrow \infty$*

$$\mathbb{E} f^2 g^2 = 1 + O((\log(t_f + t_g))^{-1/4+\varepsilon}).$$

Proof eventually applies ultimately Soundararajan’s method of moments (which requires GRH).

Work in progress: asymptotic formulas for certain averages of L -functions over toroidal families, going beyond [?].

7. ALGEBRAIC PROOF OF MODULAR FORM INEQUALITIES FOR OPTIMAL SPHERE PACKINGS, *Seewoo Lee*

Preprint: [?].

Motivating question: finding densest sphere packings.

- $d = 1$
- $d = 2$: Thue 1910, $\Delta_2 = \frac{\pi}{2\sqrt{3}}$
- $d = 3$: Kepler conjecture 1611, Hales 2005, $\Delta_3 = \frac{\pi}{3\sqrt{2}}$, formally verified 2014 in Isabelle/HOL
- (Korkine–Zolotareff, Blichfeldt, Cohn–Kumar): D_4, D_5, D_6, E_7, E_8 and Leech lattice
- Viazovska, 2016 π -day on arXiv: E_8 lattice packing optimal, with $\Delta_8 = \frac{\pi^4}{388}$
- Cohn–Kumar–Miller–Radchenko–Viazovska, March 21st 2016 on arXiv: $\Delta_{24} = \frac{\pi^{12}}{12!}$ Leech lattice (unique even unimodular with nonzero minimal length 2)

Viazovska et al use:

Theorem 15 (Cohn–Elkies, 2013). *Let $r > 0$. Assume there exists a nice function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying*

- $f(0) = \hat{f}(0) > 0$,
- $f(x) \leq 0$ for all $\|x\| \geq r$,
- $\hat{f}(y) \geq 0$ for all $y \in \mathbb{R}^d$,

then

$$\Delta_d \leq \text{vol}(B_{r/2}^d) = \left(\frac{r}{2}\right)^d \frac{\pi^{d/2}}{(d/2)!}.$$

Proof is not hard (fits in one page).

This leads to the hunt for a “magic function” f satisfying the indicated conditions. Based on their numerical experiments, Cohn–Elkies conjectured that the optimal sphere packing can be achieved by a magic function when $d = 2, 8, 24$. The magic function f and its Fourier transform \hat{f} should vanish at the lattice points.

Viazovska et al constructed the magic functions for $d = 8, 24$ using modular forms. Decompose into Fourier eigenfunctions $f = f_+ + f_-$ (parity under Fourier transform). Viazovska writes them as

$$f_{\pm}(x) = \sin^2\left(\frac{\pi\|x\|^2}{2}\right) \int_0^\infty \varphi_{\pm}(t) e^{-\|x\|^2 t} dt,$$

where the \sin^2 factor enforces desired roots.

For $d = 8$, we have $\varphi_{\pm}(t) = t^2 \psi_{\pm}(i/t)$, where ψ_{\pm} are defined in terms of standard modular forms. We also need (for both $d = 8, 24$) a few non-obvious inequalities involving some of these modular forms. One reason these inequalities are difficult is because they’re inhomogeneous with respect to the weights of the forms involved.

Original proofs use bounds of Fourier coefficients of the form

$$|c(n)| \leq C_1 e^{C_2 \pi \sqrt{n}},$$

following from the Hardy–Ramanujan formula, reducing the question to finite calculations and interval arithmetic.

Ran Romik (2023) gave an alternative and much simpler proof for $d = 8$ that does not use any interval arithmetic, but still requires a “calculator” to check inequalities like

$$e^{3\pi} \frac{9\Gamma(1/4)^{16}}{8192\pi^{12}} < 20480.$$

Question 16. Can we prove such inequalities *algebraically*?

The answer is yes, as we now explain. To that end, we’ll develop a theory of (*completely*) *positive quasimodular forms*.

Definition 17. Let $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$. We say that a quasimodular form F is *positive* if

$$F(it) \geq 0$$

for all $t > 0$.

We call it *completely positive* if it has nonnegative q -coefficients at ∞ .

Note that “completely positive” implies “positive”, but the inclusion is strict in general. For instance, the discriminant form has a product formula that tells you that it is positive, but you can check that it’s not completely positive.

Easy facts:

Theorem 18. • *Anti-derivative preserves positivity.*

- *Derivative preserves complete positivity.*
- *Serre derivative preserves complete positivity.*

“Nontrivial” fact (that we won’t use, and which also follows directly from Bernstein’s theorem):

Theorem 19. *F is completely positive if and only if all its derivatives are positive.*

“Interesting” fact that we will use:

Theorem 20. *Let $F = \sum_{n \geq n_0} a_n q^n$ with $a_{n_0} > 0$. If some Serre derivative of F is positive, then so is F .*

Examples?

Definition 21 (Kaneko–Koike). For given weight w and depth s , we can speak of *extremal quasimodular forms of weight w and depth s* , which are those whose order of zeros at ∞ is as large as possible.

It’s been shown that these have almost all coefficients positive, and conjectured that they’re all positive.

Theorem 22. *That conjecture is true for depth 1 extremal forms.*

We also have similar identities proving complete positivity of the depth 2 extremal forms of weight $w \leq 14$, but not yet for general w .

This reduces some of the inequalities in the arguments of Viazovska et al to checking some limiting properties of ratios of modular functions. For monotonicity, we check that derivatives are positive, and for this it suffices to check the same for Serre derivatives of derivatives.

Possible future directions:

- Classify the completely positive forms?
- Possible applications in other LP problems?
 - Dual LP, uncertainty principle
 - Any results that are “uniform” in dimensions
- Make a formalization of the proof (e.g., in Lean) easier?

REFERENCES