

SOME NOTES ON COHERENT STATES

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ABSTRACT. We summarize a paper of Sugita (“Proof of the generalized Lieb-Wehrl conjecture for integer indices larger than one”) and Delbourgo–Fox (“Maximum weight vectors possess minimal uncertainty”).

1. THE GENERALIZED LIEB-WEHRL CONJECTURE FOR INTEGER INDICES LARGER THAN ONE

We summarize Sugita [2].

Let G be a compact connected Lie group. Let π be an irreducible unitary (complex) representation of G .

Definition 1. We say that a unit vector $v \in \pi$ is a *coherent state* if it is a highest weight vector with respect to some maximal torus and ordering.

For convenience, let us now fix a maximal torus and ordering, so that we may speak of the highest weight λ of π and the line of highest weight vectors. Coherent states are then the G -translates of unit vectors in that line.

We equip G with its probability Haar dg .

Theorem 2. Let $q \in \mathbb{Z}_{\geq 2}$. For a unit vector $v \in \pi$, set

$$I(v) := (\dim \pi) \int_G (|\langle gv, v_\lambda \rangle|^2)^q dg.$$

(The normalizing factor is for later convenience.) Then $I(v)$ achieves its maximum precisely when v is a coherent state.

Remark 3. Using the convexity of the q th power function $x \mapsto x^q$, one deduces an ostensibly more general result concerning higher rank tensors in place of $v \otimes \bar{v}$.

We turn to the proof of Theorem 2.

Let Π be an irreducible unitary representation of G with highest weight $q\lambda$. Then Π embeds as a subrepresentation of $\pi^{\otimes q}$, with multiplicity one. We identify Π with a subrepresentation of $\pi^{\otimes q}$:

$$\Pi \subseteq \pi^{\otimes q}.$$

Lemma 4. For unit vectors $v \in \pi$, the quantity $I(v)$ is maximized precisely when $v^{\otimes q} \in \Pi$.

Proof. It is enough to show that

$$I(v) = \frac{\dim \pi}{\dim \Pi} \langle Pv^{\otimes q}, v^{\otimes q} \rangle, \tag{1}$$

where $P : \pi^{\otimes q} \rightarrow \Pi$ denotes the orthogonal projection, because the right hand side of this formula is clearly maximal precisely when $v^{\otimes q}$ lies in the image Π of P . To

establish (1), we first apply the Schur orthogonality relations to see that

$$P = (\dim \Pi) \int_G gv_\lambda^{\otimes q} \otimes \overline{gv_\lambda^{\otimes q}} dg.$$

We conclude by inserting this formula for P into the right hand side of (1). \square

To complete the proof of Theorem 2, we reduce to establishing the following equivalence:

$$v : \text{coherent} \iff v^{\otimes q} \in \Pi. \quad (2)$$

The forward implication follows from the fact that $v_\lambda^{\otimes q} \in \Pi$. It remains only to establish the converse.

To that end, we introduce the Casimir operator Ω for G , given by $\sum_{x \in \mathcal{B}(\mathfrak{g})} x^2$ for an orthonormal basis $\mathcal{B}(\mathfrak{g})$ of $\mathfrak{g} := \text{Lie}(G)$ taken with respect to a G -invariant inner product. This operator lies in the center of the universal enveloping algebra, hence acts on any given irreducible representation as multiplication by some scalar.

Lemma 5. For unit vectors $v \in \pi$, the inner product

$$\langle \Omega v^{\otimes q}, v^{\otimes q} \rangle$$

is maximized precisely when v is coherent.

Proof. We first compute that

$$\langle \Omega v^{\otimes q}, v^{\otimes q} \rangle = q\Omega_\pi + q(q-1)E(v), \quad (3)$$

where Ω_π denotes the scalar by which Ω acts on π and

$$E(v) := \sum_{x \in \mathcal{B}(\mathfrak{g})} |\langle xv, v \rangle|^2.$$

Consider for instance the case $q = 2$. For $x \in \mathfrak{g}$, we have

$$\begin{aligned} x^2 v^{\otimes 2} &= x(xv \otimes v + v \otimes xv) \\ &= x^2 v \otimes v + 2xv \otimes xv + v \otimes x^2 v. \end{aligned}$$

Summing over $x \in \mathcal{B}(\mathfrak{g})$ gives

$$\begin{aligned} \Omega v^{\otimes 2} &= \Omega v \otimes v + 2 \sum_{x \in \mathcal{B}(\mathfrak{g})} xv \otimes xv + v \otimes \Omega v \\ &= 2\Omega_\pi v \otimes v + 2 \sum_{x \in \mathcal{B}(\mathfrak{g})} xv \otimes xv. \end{aligned}$$

When $q = 3$, we obtain instead

$$\Omega v^{\otimes 3} = 3\Omega_\pi v^{\otimes 3} + 2 \sum_{x \in \mathcal{B}} (xv \otimes xv \otimes v + xv \otimes v \otimes xv + v \otimes xv \otimes xv).$$

We get a similar expression in general, which, after pairing with $v^{\otimes q}$, gives the claimed formula (3).

We conclude via appeal to the following lemma. \square

Lemma 6. For unit vectors v , the quantity $E(v)$ is maximized precisely when v is coherent.

Proof. See Section 2. \square

We may now complete the proof of the backwards implication in (2), hence of Theorem 2. Suppose $v^{\otimes q} \in \Pi$. Then, writing Ω_Π for the scalar by which Ω acts on the irreducible representation Π , we have $\Omega v^{\otimes q} = \Omega_\Pi v^{\otimes q}$ and $\Omega v_\lambda^{\otimes q} = \Omega_\Pi v_\lambda^{\otimes q}$, hence

$$\langle \Omega v^{\otimes q}, v^{\otimes q} \rangle = \langle \Omega v_\lambda^{\otimes q}, v_\lambda^{\otimes q} \rangle.$$

By two applications of Lemma 5, we deduce that v is coherent, as required.

2. COHERENT STATES MINIMIZE UNCERTAINTY

We summarize Delbourgo–Fox [1], and in particular, prove Lemma 6.

Let G be a compact connected Lie group. Let π be an irreducible unitary representation of G . Equip the Lie algebra \mathfrak{g} with a G -invariant inner product, and let $\mathcal{B}(\mathfrak{g})$ be an orthonormal basis.

Definition 7. For each unit vector $v \in \pi$, we define

$$\Delta(v) := \sum_{x \in \mathcal{B}(\mathfrak{g})} \|\bar{x}v\|^2$$

where $\bar{x} := x - \langle xv, v \rangle$.

Theorem 8. For unit vectors v , the quantity $\Delta(v)$ is minimized precisely when v is a coherent state.

That Theorem 8 implies Lemma 6 is immediate from the following:

Lemma 9. We have $\Delta(v) = -\Omega_\pi - E(v)$, where Ω_π denotes the Casimir eigenvalue for π .

Proof. We note that $\bar{x}v = xv - \langle xv, v \rangle v$, thus $\langle \bar{x}v, v \rangle = 0$ and

$$\|\bar{x}v\|^2 = \langle \bar{x}v, xv \rangle.$$

By expanding the definition of \bar{x} , we obtain

$$\langle xv, xv \rangle - \langle xv, v \rangle \langle v, xv \rangle.$$

Using here that x acts via a skew-symmetric operator, we see that the first term equals $-\langle x^2v, v \rangle$, while the second term may be abbreviated to $|\langle xv, v \rangle|^2$. Summing over x leads to the required identity. \square

We turn to the proof of Theorem 8.

Lemma 10. Let π be a unitary representation of G , let $v \in \pi$ be a unit vector, let $x \in i\mathfrak{g}$. Then the minimum over all (real) scalars c of the quantity

$$\|(x - c)v\|$$

is achieved by taking $c = \langle xv, v \rangle$.

Proof. Since x lies in $i\mathfrak{g}$, it acts via a self-adjoint operator, so we readily obtain

$$\|(x - c)v\|^2 = \langle xv, xv \rangle - 2c\langle xv, v \rangle + c^2. \quad (4)$$

This is a quadratic polynomial whose minimum is attained at the critical point, which we solve for by taking a derivative. \square

The definition of $\Delta(v)$ remains unchanged upon replacing \mathfrak{g} with its imaginary multiple $i\mathfrak{g}$. By Lemma 10, we see that the minimum of $\Delta(v)$ taken over unit vectors v coincides with the minimum of

$$\sum_{x \in \mathcal{B}(i\mathfrak{g})} \|(x - c_x)v\|^2 \quad (5)$$

taken over unit vectors v and tuples of scalars c_x . Such a minimum exists by continuity and compactness. Let us consider one such minimum. The above expression expands, as in (4), to

$$-\Omega_\pi + \sum_x c_x^2 - 2 \sum_x c_x \langle xv, v \rangle.$$

With the notation $y := \sum_x c_x x \in \mathfrak{g}$, the above may be written

$$-\Omega_\pi + \|y\|^2 - 2\langle yv, v \rangle.$$

We choose a maximal torus whose Lie algebra contains y , and an ordering with respect to y which is dominant. Our assumptions imply that $\langle yv, v \rangle$ cannot be made larger by changing v , so v must lie in the eigenspace for y with largest eigenvalue. We claim that this eigenspace is one-dimensional. If the claim holds, then y is a highest weight vector, so we are done. Assume the claim fails. Let us then modify v , if necessary, so that it is a highest weight vector for the given torus and ordering; we may do so without changing $\langle yv, v \rangle$, hence without changing our assumption that v and c_x realize the minimum. We see then that by modifying y without changing $\|y\|^2$, we may increase the size of $\langle yv, v \rangle$, contradicting the supposed minimality. This completes the proof of Theorem 8.

3. DETERMINING VECTORS BY THEIR MATRIX COEFFICIENTS

It follows from Lemma 10 that for a given unit vector v , the quantity (5) is minimized by taking $c_x = \langle xv, v \rangle$. We record here that *the latter quantities determine v up to multiplication by a unit scalar*. Indeed, let G be a compact connected Lie group, π an irreducible representation, and $u, v \in \pi$ unit vectors with $\langle xu, u \rangle = \langle xv, v \rangle$ for all $x \in \mathfrak{g}$. By expanding the exponential series, we see that $\langle \exp(x)u, u \rangle = \langle \exp(x)v, v \rangle$, so that the $\langle gu, u \rangle = \langle gv, v \rangle$ holds near the identity on G . Since matrix coefficients of finite-dimensional representations define analytic functions, we deduce the equality on all of G . We conclude that u and v are proportional by appeal to the following:

Proposition 11. *Let G be a compact group, let π be an irreducible representation, and let u and v be nonzero vectors in π with the property that*

$$\langle gu, u \rangle = \langle gv, v \rangle \quad (6)$$

for all $g \in G$. Then u and v are proportional.

We record a proof after some lemmas.

Lemma 12 (Schur's lemma). *Let π be an irreducible representation of G . Then $\text{End}_G(\pi) = \mathbb{C}$, that is to say, any linear operator on π that commutes with the action of G is a scalar multiple of the identity.*

Lemma 13. *Let π and σ be unitary representations of G , with π irreducible. Let $u \in \pi$ and $v \in \sigma$ be nonzero vectors such that*

$$\langle gu, u \rangle = \langle gv, v \rangle.$$

Then there is a unique G -equivariant map $T : \pi \rightarrow \sigma$ that sends u to v .

Proof. Any vector in π may be written $\sum_{g \in G} c_g g u$ for some finitely-supported coefficients c_g . We have no choice but to attempt to define

$$T \left(\sum_{g \in G} c_g g u \right) := \sum_{g \in G} c_g g v.$$

We need to check that this is well-defined. To see this, we use that the vanishing of the argument of T may be detected via its inner product with itself, and then use that u and v have the same inner products to see that the right hand side must likewise vanish. \square

Proof of Proposition 11. By Lemma 13, there is a unique $T \in \text{End}_G(\pi)$ that maps u to v . By Schur's lemma, T is a multiple of the identity. [thinking-face] \square

REFERENCES

- [1] R. Delbourgo and J. R. Fox. Maximum weight vectors possess minimal uncertainty. *J. Phys. A*, 10(12):233–235, 1977.
- [2] Ayumu Sugita. Proof of the generalized Lieb-Wehrl conjecture for integer indices larger than one. *J. Phys. A*, 35(42):L621–L626, 2002.