LARGE VALUES OF WHITTAKER FUNCTIONS FOR GL_2

ABSTRACT. Note written around 2016 where, following Templier [3] and Saha [1], we exhibit large values of p-adic Whittaker functions for principal series representations of GL_2.

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1. GENERAL NOTATION

Let k be a local field. Fix a nontrivial additive character ψ of k. When k is non-archimedean, we assume ψ is unramified and adopt the accompanying notation $\mathfrak{o}, \mathfrak{p}, q$ for its maximal order, maximal ideal and residue field cardinality. Let χ_1, χ_2 be unitary characters of $k^\times$ with analytic conductors $C_1 := C(\chi_1)$, $C_2 := C(\chi_2)$. Set $G := \mathrm{GL}_2(k)$. We use the notation

$$n(x) := \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \quad a(y) := \begin{pmatrix} y & \\ & 1 \end{pmatrix}.$$

Let $\pi := \mathcal{I}(\chi_1, \chi_2)$ be the induced principal series representation of conductor $C := C_1 C_2$.

We write the character group of $k^\times$ additively. We write α for the character given by the normalized modulus $|.|$. We denote by $\omega := \chi_1 + \chi_2$ the central character of π. We realize π in its induced model as a space of functions $f : G \to \mathbf{C}$ satisfying

$$f(n(x)a(y)zg) = y^{\alpha/2+\chi_1} z^\omega f(g).$$

Here $z \in k^\times$ identifies with an element of the center of G, and we use notation such as

$$y^{\alpha/2+\chi_1} := |y|^{1/2}\chi_1(y).$$

We denote in what follows by ι_k a constant such as $\zeta_k(1)$, $\zeta_k(2)$, or a product of (inverses of) similar quantities; we allow the precise value of ι_k to vary from one occurrence to another.

2. Whittaker functions in principal series representations

2.1. Schwartz space parametrization. For $\Phi \in \mathcal{S}(k^2)$, define the element $f_\Phi \in \pi$ by the formula

$$f_\Phi(g) := \int_{z \in k^\times} \Phi(e_2 z g) z^{-\omega} \det(zg)^{\alpha/2+\chi_1}.$$

Thus if

$$g = \begin{pmatrix} * & * \\ c & d \end{pmatrix},$$

then

$$f_\Phi(g) = \det(g)^{\alpha/2+\chi_1} \int_{z \in k^\times} \Phi(cz, dz) z^{\alpha+\chi_1-\chi_2}.$$

This association defines a surjective map $\Phi \mapsto f_\Phi$ onto π.

2.2. The Whittaker intertwiner. Embed $f \in \pi$ into its ψ-Whittaker model via the map $f \mapsto W_f$, where

$$W_f(g) := \int_{u \in k} f(wn(u)g)\psi(-u).$$

For $\Phi \in \mathcal{S}(k^2)$, set $W_\Phi := W_{f_\Phi}$. For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$, we compute that

$$W_\Phi(g) = \det(g)^{\alpha/2+\chi_1} \int_{z \in k^\times} \int_{u \in k} \Phi(z(a+cu), z(b+du)) z^{\alpha+\chi_1-\chi_2} \psi(-u).$$

After the change of variables $u \mapsto u/z$ followed by $z \mapsto uz$, the above becomes

$$W_\Phi(g) = \det(g)^{\alpha/2+\chi_1} \int_{z \in k^\times, u \in k} \Phi(u(az+c), u(bz+d))(uz)^{\chi_1-\chi_2} \psi(-1/z). \quad (1)$$

2.3. Specialization to diagonal matrices. In the special case $g = a(y) := \left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right)$, the formula (1) specializes to

$$W_\Phi(a(y)) = y^{\alpha/2+\chi_1} \int_{z \in k^\times, u \in k} \Phi(uyz, u)(uz)^{\chi_1-\chi_2} \psi(-1/z),$$

which after the substitutions $u \mapsto u/yz$, $z = 1/yt$ gives

$$W_\Phi(a(y)) = y^{\alpha/2+\chi_2} \int_{u,t \in k} \Phi(u, ut) u^{\chi_1-\chi_2} \psi(-yt). \quad (2)$$

2.4. **Specialization to matrices in the open Bruhat cell.** The most relevant group element is

$$g = a(y)wn(x) = \begin{pmatrix} & -y \\ 1 & x \end{pmatrix},$$

for which (1) gives

$$W_\Phi(a(y)wn(x)) = y^{\alpha/2+\chi_1} \int_{z \in k^\times} \int_{u \in k} \Phi(u, u(-yz + x))(uz)^{\chi_1-\chi_2} \psi(-1/z).$$

We substitute $z \mapsto z/y$ to rewrite this as

$$W_\Phi(a(y)wn(x)) = y^{\alpha/2+\chi_2} \int_{z \in k^\times} \int_{u \in k} \Phi(u, u(x - z))(uz)^{\chi_1-\chi_2} \psi(-y/z)$$

and then substitute $z = x - t$ to obtain

$$W_\Phi(a(y)wn(x)) = y^{\alpha/2+\chi_2} \int_{u,t \in k} \Phi(u, ut) u^{\chi_1-\chi_2} (x - t)^{\chi_1-\chi_2-\alpha} \psi(-y(x - t)^{-1})$$

and finally $t \mapsto xt$ to conclude that

$$W_\Phi(a(y)wn(x)) = y^{\alpha/2+\chi_2} \int_{u,t \in k} \Phi(u, utx)(ux)^{\chi_1-\chi_2} (1 - t)^{\chi_1-\chi_2-\alpha} \psi\left(\frac{-y/x}{1 - t}\right). \tag{3}$$

3. The normalized newvector in a principal series representation

We suppose in this section that k is non-archimedean.

3.1. **Choice of Schwartz function.** Define $\Phi \in \mathcal{S}(k^2)$ by the formula

$$\Phi(c, d) := C_2^{1/2} c^{-\chi_1} d^{\chi_2} \cdot \begin{cases} 1_{C_2|c|=1} & C_1 > 1 \\ 1_{C_2|c| \leq 1} & C_1 = 1 \end{cases} \cdot \begin{cases} 1_{|d|=1} & C_2 > 1 \\ 1_{|d| \leq 1} & C_2 = 1. \end{cases}$$

The restriction of f_Φ to $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K := \mathrm{GL}_2(\mathfrak{o})$ is

$$f_\Phi(g) = \det(g)^{\alpha/2+\chi_1} \int_{z \in k^\times} \Phi(cz, dz) z^{\alpha+\chi_1-\chi_2}.$$

This defines a newvector in π with L^2-norm $\asymp 1$. One can see this directly in the induced model (e.g., observe that it is supported on the same $B \times K_1(C)$-double coset modulo $\mathfrak{p}^{n_1+n_2}$ as the one given in Schmidt's note [2], or just verify directly that it satisfies the needed invariance properties and has norm $\asymp 1$). We give another verification below.

We henceforth assume for simplicity that C_1, C_2 are both strictly greater than 1, so that we may simplify the above formula to

$$\Phi(c, d) = C_2^{1/2} c^{-\chi_1} d^{\chi_2} 1_{C_2|c|=1} 1_{|d|=1}.$$

Put another way, we have

$$\Phi(u, ut) u^{\chi_1-\chi_2} = C_2^{1/2} t^{\chi_2} 1_{C_2|u|=1} 1_{|t|=C_2}.$$

3.2. **Sanity check: the Whittaker function at diagonal matrices.** The general formula (2) for the Schwartz function-parametrized Whittaker function at a diagonal element specialized to the choice of Φ given above gives

$$W(a(y)) = C_2^{1/2} y^{\alpha/2+\chi_2} \int_{u,t\in k} t^{\chi_2} 1_{C_2|u|=1} 1_{|t|=C_2} \psi(-yt).$$

The integrand has support of volume $\asymp 1$; the u integrand is constant, while the t integrand is a Gauss sum vanishing unless $|y|=1$, in which case it has magnitude $\iota_k C_2^{-1/2}$. The change of variables $t \mapsto t/y$ and $z := 1/t$ gives

$$W(a(y)) = \iota_k C_2^{1/2} y^{\alpha/2} \int_{z\in k^\times} 1_{C_2|yz|=1} z^{-\chi_2} \psi(-1/z),$$

which by the stated properties of Gauss sums implies that

$$W(a(y)) = \iota_k 1_{\mathfrak{o}^\times}(y),$$

as expected.

3.3. **The Whittaker function at elements of the open Bruhat cell.** As a variant of the formula for $\Phi(u, ut)$ recorded above, we have

$$\Phi(u, uxt)(ux)^{\chi_1-\chi_2} = C_2^{1/2} x^{\chi_1} t^{\chi_2} 1_{C_2|u|=1} 1_{|xt|=C_2}.$$

From the general formula for $W(a(y)wn(x))$ recorded above in (3), it follows that

$$W(a(y)wn(x)) = C_2^{1/2} x^{\chi_1} y^{\alpha/2+\chi_2} \int_{u,t\in k} t^{\chi_2} (1-t)^{\chi_1-\chi_2-\alpha} 1_{C_2|u|=1} 1_{|xt|=C_2} \psi\left(\frac{-y/x}{1-t}\right).$$

The \$u\$-integral is constant with support of volume $\iota_k C_2^{-1}$, while the support of the t integral has volume $\iota_k C_2/|x|$, so executing the u integral while renaming $t \in k$ to $z \in k^\times$ gives

$$W(a(y)wn(x)) = \iota_k \left(\frac{C_2|y|}{|x|^2}\right)^{1/2} x^{\chi_1} y^{\chi_2} \int_{z\in k^\times} z^{\chi_2} 1_{|xz|=C_2} (1-z)^{\chi_1-\chi_2-\alpha} \psi\left(\frac{-y/x}{1-z}\right)$$

Note that all calculations up until now basically work over an archimedean field, too; we just have to replace conditions like $1_{|z|=1}$ with a smooth cutoff.

3.4. **Final formula for the magnitude.** In particular, taking absolute values in the final formula from section 3.3 gives

$$W(a(y)wn(x)) \asymp \left(\frac{C_2|y|}{|x|^2}\right)^{1/2} \int_{z\in k^\times} 1_{|z|=C_2/|x|} z^{\chi_2} (1-z)^{\chi_1-\chi_2-\alpha} \psi\left(-\frac{y/x}{1-z}\right).$$

It will also be convenient (for later purposes unrelated to the lower bound) to record the result of the substitution $z \mapsto z^{-1}$:

$$W(a(y)wn(x)) \asymp \left(\frac{|y|}{C_2}\right)^{1/2} \int_{z\in k^\times} 1_{|z|=|x|/C_2} z^{-\chi_1} (1-z)^{\chi_1-\chi_2-\alpha} \psi\left(\frac{zy/x}{1-z}\right).$$

4. The lower bound

4.1. **Assumptions.** We consider in this section characters whose conductors satisfy

- $C_1 > C_2 > 1$,
- C_1 is a square,
- $C_2 < C_1^{1/2}$,

and x, y satisfying

- $C_1(C_2/|x|)^2 \leq 1$,
- $|y/x|(C_2/|x|)^2 \leq 1$,
- $C_2 < |x| < C_1$.

4.2. **The model case.** It is clear that x, y satisfying the above assumptions exist; for instance, it suffices to require that

$$|x| = C_2 C_1^{1/2}, |y| = C_2 C_1^{3/2}. \tag{4}$$

4.3. **Calculation.** Under the above assmptions, for any $z \in k^\times$ with $|z| = C_2/|x|$ one has $|1 - z| = 1$ and

$$(1-z)^{\chi_1 - \chi_2} = \psi(\xi z)$$

for some $\xi \in k$ satisfying

$$|\xi| = C(\chi_1 - \chi_2) = C_1.$$

Moveover, by a quadratic Taylor expansion $(1-z)^{-1} = 1 + z + z^2(1-z)^{-1}$ (using our assumptions to kill off the contribution of the remainder),

$$\psi\left(-\frac{y/x}{1-z}\right) = \psi(-y/x)\psi(-yz/x)$$

and so by the first of the above formulas,

$$W(a(y)wn(x)) \asymp \left(\frac{C_2|y|}{|x|^2}\right)^{1/2} \int_{z \in k^\times} 1_{|z| = C_2/|x|} z^{\chi_2} \psi((\xi - y/x)z).$$

The inner integral is a Gauss sum of size

$$\int_{z \in k^\times} 1_{|z|=C_2/|x|} z^{\chi_2} \psi((\xi - y/x)z) \asymp 1_{|\xi - y/x| \cdot C_2/|x| = C_2} C_2^{-1/2} = 1_{|x\xi - y| = |x|^2} C_2^{-1/2}.$$

From our assumption $|\xi| = C_1 > |x|$ we see that $|x\xi - y| = |x|^2$ implies $|y| = C_1|x|$ and thus

$$W(a(y)wn(x)) \asymp \left(\frac{C_1}{|x|}\right)^{1/2} 1_{|x\xi - y| = |x|^2}.$$

Choosing x, y as in the model case above which satisfy $|x\xi - y| = |x|^2$ (which is clearly possible), we obtain

$$W(a(y)wn(x)) \asymp \left(\frac{C_1^{1/2}}{C_2}\right)^{1/2}.$$

When is this big? Well, for instance, if $C_2 = 1$ and C_1 is large. But note that if C_1 and C_2 are the same size (e.g., if the central character is trivial), then the above is actually small, so we don't get values that are obviously large.

References

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