DEFINITION OF SOME CHARACTER SUMS

Abstract. We define some character sums relevant for subconvexity on $\mathrm{GL}_{n+1} \times \mathrm{GL}_n$ at "prime depth."

Let F be a finite field of order q. Set

$$(G, H) := (\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F)),$$

with H embedded as the upper-left block. Let (B, B_H) denote the upper-triangular Borel subgroups.

Quasi-invariants for the two-sided action of $B \times B_H$ on G (by left and right translation) are given by the following determinental minors:

- For $m = 1, \dots, n+1$, the "left-and-bottom-anchored" minors
$$A_j(g) := \det(g_{ij})_{n+2-m \le i \le n+1}^{1 \le j \le i}.$$
- For $m = 1, \dots, n$, the "left-and-second-to-bottom-anchored" minors
$$B_j(g) := \det(g_{ij})_{n+1-m \le i \le n}^{1 \le j \le i}.$$

We also adopt the convention $A_0(g) := B_0(g) := 1$.

The action of $B \times B_H$ on G has an open orbit, given by the nonvanishing of each of the invariants $A_1, \dots, A_{n+1}, B_1, \dots, B_n$. Let α be a representative for this orbit.

Example 1. We could take, e.g., for $n + 1 = 4$,

$$\alpha = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

Indeed, we then have $A_i(\alpha) = B_j(\alpha) = 1$ for all i and j.

Example 2. One could instead take

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\chi : B \to \mathrm{U}(1)$ and $\eta : B_H \to \mathrm{U}(1)$ be characters. We define the character sum

$$S(\gamma) := \sum_{\substack{x, y \in B_H : \\ \alpha^{-1} y^{-1} \gamma x \alpha \in B}} \chi(\alpha^{-1} y^{-1} \gamma x \alpha) \eta(x^{-1} y).$$

We would like to understand the magnitude of this sum, together with averaged variants such as

$$\max_{x \in B_H} \sum_{y \in B_H} |S(x \gamma y)|.$$

Remark 3. The sum $S(\gamma)$ is η-equivariant under the action of $B_H \times B_H$, so in studying that sum, we can assume that γ lies in a set of representatives for that action. Generic representatives are given by, e.g., when $n+1=4$,

$$\gamma = \begin{pmatrix} 0 & 0 & 1 & \gamma_1 \\ 0 & 1 & 1 & \gamma_2 \\ 1 & 1 & 0 & \gamma_3 \\ 1 & 0 & 0 & \gamma_4 \end{pmatrix}. \tag{2}$$

Let $\Theta : G \to \mathbb{C}$ denote the function supported on $B_H \alpha B$ and given there by

$$\Theta(y\alpha z) = \eta(y)\chi(z).$$

Then

$$S(\gamma) = \sum_{x \in B_H} \eta^{-1}(x)\Theta(\gamma x \alpha).$$

Lemma 4. Define the components χ_j and η_j of χ and η by writing

$$\begin{aligned} y &= \operatorname{diag}(y_n, \dots, y_1, 1), \qquad & z &= \operatorname{diag}(z_1, \dots, z_{n+1}), \\ \eta(y) &= \prod_j \eta_j(y_j), \qquad & \chi(z) &= \prod_j \chi_j(z_j). \end{aligned} \tag{3}$$

Let α be as in Example 1. Then on $B_H \alpha B$, we have

$$\Theta = \chi_1\left(\frac{A_1}{B_0}\right) \cdots \chi_{n+1}\left(\frac{A_{n+1}}{B_n}\right) \eta_1\left(x^{-1}\frac{B_1}{A_1}\right) \cdots \eta_n\left(x^{-1}\frac{B_n}{A_n}\right).$$

Example 5. Suppose that the η_j are trivial and the χ_j are all equal. Call their common value χ. Then

$$\Theta = \chi\left(\frac{A_1 \cdots A_{n+1}}{B_1 \cdots B_n}\right).$$

Example 6. For $n+1=2$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $x = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ and $\alpha = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, the invariants of

$$\gamma x \alpha = \begin{pmatrix} ax+b & ax \\ cx+d & cx \end{pmatrix}$$

are given by

$$A_1 = cx+d, \quad B_1 = ax+b, \quad A_2 = (ad-bc)x,$$

so the character sum in question is a unit scalar multiple of

$$\sum_{x \in F^\times}^{*} \chi_1(cx+d)\chi_2\left(\frac{x}{ax+b}\right)\eta\left(x^{-1}\frac{ax+b}{cx+d}\right).$$

By Weil, this exhibits square-root cancellation except when

- γ is diagonal,
- γ is lower-triangular and $\chi_1 = \eta$, or
- γ is upper-triangular and $\chi_2 = \eta$.

In those cases, we don't get any cancellation. For example, suppose that $\chi_2 = \eta = 1$. Then the character sum is

$$\sum_{x \in F^\times} \chi_1(cx+d).$$

If γ is upper-triangular, i.e., $c = 0$, then we get no cancellation.

Thus we see that "conductor-dropping" manifests in an expanded degeneracy locus for the character sum.

Example 7. One can check that for $n+1=3$, η trivial, and γ "generic" as in (2), we have, with the notation

$$x = \begin{pmatrix} x_1 & x_3 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{aligned}\Theta(\gamma x \alpha) = \chi_1\left(\gamma_3 + x_3\right) \\ \cdot \chi_2\left(\frac{\gamma_2 x_1 + \gamma_2 x_3 - \gamma_3 x_1 - \gamma_3 x_2 - \gamma_3 x_3 + x_1 x_2}{\gamma_2 + x_2 + x_3}\right) \\ \cdot \chi_3\left(\frac{x_1 x_2(\gamma_1 - \gamma_2 + \gamma_3)}{\gamma_1 x_1 + \gamma_1 x_2 + \gamma_1 x_3 - \gamma_2 x_2 + x_1 x_2}\right).\end{aligned}$$

Here's a more explicit example, obtained by specializing the γ_j at random:

$$\chi_1\left(x_3 - 5\right)\chi_2\left(\frac{x_1 x_2 + 7x_1 + 5x_2 + 7x_3}{x_2 + x_3 + 2}\right)\chi_3\left(-\frac{11 x_1 x_2}{x_1 x_2 - 4x_1 - 6x_2 - 4x_3}\right).$$

Question 8. In the special case $\chi_1 = \chi_2 = \chi_3$, is the character sum obtained in 7 equivalent to that considered by Sharma, namely

$$\sum_{t_1, t_2, t_3} \chi\left(\frac{t_1 t_3 (t_2 + 1)(c_2 k_1 t_3 + c_2 k_2 + t_2)}{t_2 (t_1 + 1)(c_1 t_3 + t_1)(k_1 t_3 + k_2)}\right)?$$

Maybe there is some clever change of variables relating (t_1, t_2, t_3) to (x_1, x_2, x_3).