Here's a lemma that I think clarifies the key step in the proof.

Lemma 1. *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring, and let* $f_1, \dots, f_r \in \mathfrak{m}$ *with* $V(f_1, \dots, f_r) = \{\mathfrak{m}\}$. *Let* $\mathfrak{p} \subsetneq \mathfrak{m}$ *be a prime with no prime strictly contained between* $\mathfrak{p}$ *and* $\mathfrak{m}$. *Then there exist* $g_1, \dots, g_r \in \mathfrak{m}$ *for which*

(1) $V(g_1, \dots, g_r) = \{\mathfrak{m}\}$ *and*
(2) $\mathfrak{p}$ *contains and is a minimal prime of* $(g_1, \dots, g_{r-1})$.

In "geometric" terms, let Z *be a closed irreducible subset of* $\operatorname{Spec}(A)$ *that is minimal among the closed irreducible sets that properly contain* $\{\mathfrak{m}\}$. *Then we may find* $g_1, \dots, g_r$ *for which* $V(g_1, \dots, g_r) = \{\mathfrak{m}\}$ *and for which* Z *is an irreducible component of* $V(g_1, \dots, g_{r-1})$.

Proof. Since $\mathfrak{m}$ is the unique prime ideal containing $(f_1, \dots, f_r)$, we may assume after reindexing $f_1, \dots, f_r$ as necessary that $f_r \notin \mathfrak{p}$. Then the ideal $\mathfrak{p} + (f_r)$ strictly contains $\mathfrak{p}$ and is contained in $\mathfrak{m}$; our hypotheses on $\mathfrak{p}$ imply that $\mathfrak{m}$ is the only prime ideal containing $\mathfrak{p} + (f_r)$, i.e., that $V(\mathfrak{p} + (f_r)) = \{\mathfrak{m}\}$, or that $\operatorname{rad}(\mathfrak{p} + (f_r)) = \mathfrak{m}$. In particular, for each $1 \le i \le r-1$ we may find n_i for which $f_i^{n_i} \in \mathfrak{p} + (f_r)$, say

$$f_i^{n_i} = g_i + z_i f_r \text{ with } g_i \in \mathfrak{p}, z_i \in A.$$

We claim that the conclusion of the lemma is now satisfied with $g_1, \dots, g_{r-1}$ as above and $g_r := f_r$:

(1) The above equation shows that any prime $\mathfrak{q}$ that contains $g_1, \dots, g_{r-1}, f_r$ also contains $f_i^{n_i}$ and hence f_i for $1 \le i \le r$, hence $\mathfrak{q} = \mathfrak{m}$. Thus $V(g_1, \dots, g_r) = \{\mathfrak{m}\}$.
(2) It's clear by construction that $\mathfrak{p}$ contains $(g_1, \dots, g_{r-1})$. There is thus a minimal prime $\mathfrak{p}'$ of $(g_1, \dots, g_{r-1})$ contained in $\mathfrak{p}$; we must verify that $\mathfrak{p} = \mathfrak{p}'$. (Geometrically, $\mathfrak{p}'$ corresponds to an irreducible component Z' of $V(g_1, \dots, g_{r-1})$ containing Z.) To see this, consider the quotient ring $\overline{A} := A/(g_1, \dots, g_{r-1})$. Let
$$\overline{\mathfrak{m}} \supsetneq \overline{\mathfrak{p}} \supseteq \overline{\mathfrak{p}'} \tag{1}$$
denote the chain of primes in $\overline{A}$ given by the image of $\mathfrak{m} \supsetneq \mathfrak{p} \supseteq \mathfrak{p}' \supseteq (g_1, \dots, g_{r-1})$. Then $(\overline{A}, \overline{\mathfrak{m}})$ is a Noetherian local ring, and our task is equivalent to showing that $\overline{\mathfrak{p}} = \overline{\mathfrak{p}'}$. Let $f \in \overline{A}$ denote the image of f_r. The primes of $\overline{A}$ containing f are in bijection with the primes of A containing $g_1, \dots, g_{r-1}, f_r$, so $V_{\overline{A}}(f) = \{\overline{\mathfrak{m}}\}$. By Krull's principal ideal theorem (in the form of Corollary **??**), it follows that $\operatorname{height}(\overline{\mathfrak{m}}) \le 1$. From (1) we then deduce that $\overline{\mathfrak{p}} = \overline{\mathfrak{p}'}$, as required. (Intuitively, by choosing f_r not to vanish on any irreducible component of $V(f_1, \dots, f_{r-1})$, we guarantee that appending it to the set of generators has the effect of knocking down the dimension of each such component by 1.)

□

We now deduce Theorem **??**. We must show that if $\mathfrak{p}$ is a minimal prime of $(f_1, \dots, f_r)$, then $\operatorname{height}(\mathfrak{p}) \le r$. We may assume without loss of generality (replacing A with $A_{\mathfrak{p}}$ and $\mathfrak{p}$ with $\mathfrak{p}_{\mathfrak{p}}$, which doesn't change the height of or minimality assumption on the latter) that $(A, \mathfrak{p})$ is a Noetherian local ring with $V(f_1, \dots, f_r) = \{\mathfrak{p}\}$; we must show then that $\operatorname{height}(\mathfrak{p}) \le r$. We do this by induction on r. The case $r = 1$ is given by Krull's principal ideal theorem, so suppose $r > 1$. Let $\mathfrak{q} \subsetneq \mathfrak{p}$ be a maximal element of the set of primes strictly contained in $\mathfrak{p}$;

it will suffice then to show that height($\mathfrak{q}$) $\leq r - 1$. By Lemma 1, we may assume without loss of generality that $\mathfrak{q}$ is a minimal prime of $(f_1, \ldots, f_{r-1})$; the required inequality then follows from our inductive hypothesis.

References