We consider the family of all π on $\mathrm{PGL}_2(\mathbb{Z})\backslash \mathrm{PGL}_2(\mathbb{R})$ with $C(\pi) \leq Q$, and aim to write down the shape of the Kuznetsov formula for

$$\frac{1}{Q} \sum_{C(\pi)\leq Q} \lambda(m)\lambda(n)$$

in the range

$$m, n \asymp T,$$

where T and Q are parameters at our disposal.

We pick off the family using f_0, the normalized characteristic function of the archimedean variant $K_0(Q)$ of the standard congruence subgroup, like [1]:

$$K_0(Q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a = 1 + o(1), \quad b = o(1), \quad c \lll 1/Q, \quad d = 1 + o(1) \right\}.$$

Then we have the pretrace formula

$$\sum_{\gamma \in \Gamma} f_0(x^{-1}\gamma y) = \sum_{C(\pi)\leq Q} \sum_{\substack{\varphi \in \mathcal{B}(\pi): \\ \text{analytic newvector}}} \varphi(x)\overline{\varphi(y)}.$$

Each such φ will have roughly the Fourier expansion

$$\varphi(n(x)a(y)) = \sum_{n \geq 1} \frac{\lambda(n)}{\sqrt{n}} W(ny) e(nx).$$

Here the test function $W(ny)$ roughly detects $ny \asymp 1$, i.e., $n \asymp 1/y$. So if we integrate the above against $e(-mx)$ over $x \in \mathbb{R}/\mathbb{Z}$, we get

$$\int_{x\in\mathbb{R}/\mathbb{Z}} \varphi(n(x)a(y)) e(-nx)\, dx = 1_{n \asymp 1/y} \frac{\lambda(n)}{\sqrt{n}}.$$

We choose $y := 1/T$. Then we have the following:

$$\int_{u,v\in\Gamma_N\backslash N} \varphi(ua(1/T))e(-mu)\overline{\varphi(va(1/T))e(-nv)} \approx 1_{m,n\asymp T} \frac{1}{\sqrt{mn}} \lambda(m)\lambda(n).$$

Thus for $m, n \asymp T$,

$$\sum_{C(\pi)\leq Q} \frac{\lambda(m)\lambda(n)}{\sqrt{mn}} \approx \int_{u,v\in\Gamma_N\backslash N} e(-mu+nv) \sum_{\gamma\in\Gamma} f_0(a(T)u^{-1}\gamma v a(1/T))\, du\, dv.$$

Here we think

$$u \leftrightarrow \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad v \leftrightarrow \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}.$$

Now we do the Bruhat decomposition. First, let's consider the diagonal contribution from $\gamma \in \Gamma_N$. This unfolds to

$$1_{m=n} \int_{u\in N} f_0(a(T)ua(1/T)) e(mu)\, du,$$

which for our f_0, which we recall has the shape

$$f_0 \approx Q 1_{K_0(Q)}, \quad K_0(Q) = G \cap \left(1 + \begin{pmatrix} o(1) & o(1) \\ o(1/Q) & o(1) \end{pmatrix}\right).$$

should have size $\asymp Q/T$, because the upper-right entry of the following is $o(1/T)$:

$$J := a(1/T)K_0(Q)a(T) \subseteq \left(1 + \begin{pmatrix} o(1) & o(1/T) \\ o(T/Q) & o(1) \end{pmatrix}\right).$$

This is as expected, because for $n \asymp T$,

$$\sum_{C(\pi)\leq Q} \frac{|\lambda(n)|^2}{n} \approx \frac{Q}{T}.$$

Next we consider the off-diagonal. So we write, for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } c \neq 0,$$

$$\gamma = \begin{pmatrix} 1 & a/c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix} =: n(a/c)w(c)n(d/c).$$

The off-diagonal contribution is

$$Q \sum_{c\neq 0} \sum_{a,d\in\mathbb{Z}:ad\equiv 1(c)} \int_{u,v\in\mathbb{R}/\mathbb{Z}} e(-mu+nv) 1_{n(a/c+u)w(c)n(d/c+v)\in J}\, du\, dv.$$

We want to recognize Kloosterman sums and then evaluate everything else. To that end, we split the sum over a and d into arithmetic progressions modulo c:

- $a = a_0 + ca_1$,
- $d = d_0 + cd_1$,

where $a_0, d_0 \in (\mathbb{Z}/c)^\times$ and $a_1, d_1 \in \mathbb{Z}$. Then the above is rewritten as

$$Q \sum_{c\neq 0} \sum_{\substack{a_0,d_0\in(\mathbb{Z}/c)^\times: \\ a_0d_0\equiv 1(c)}} \sum_{a_1,d_1\in\mathbb{Z}} \int_{u,v\in\mathbb{R}/\mathbb{Z}} e(-mu+nv) 1_{n(a_0/c+a_1-u)w(c)n(d_0/c+d_1+v)\in J}\, du\, dv.$$

Now we substitute $u \mapsto u - a_0/c$ and $v \mapsto v - d_0/c$:

$$Q \sum_{c\neq 0} \sum_{\substack{a_0,d_0\in(\mathbb{Z}/c)^\times: \\ a_0d_0\equiv 1(c)}} e_c(-ma_0-nd_0) \sum_{a_1,d_1\in\mathbb{Z}} \int_{u,v\in\mathbb{R}/\mathbb{Z}} e(-mu+nv) 1_{n(a_1+u)w(c)n(d_1+v)\in J}\, du\, dv.$$

Now for this last integral, we unfold (a_1, u) and (d_1, v):

$$Q \sum_{c\neq 0} \sum_{\substack{a_0,d_0\in(\mathbb{Z}/c)^\times: \\ a_0d_0\equiv 1(c)}} e_c(-ma_0-nd_0) \int_{u,v\in\mathbb{R}} e(-mu+nv) 1_{n(u)w(c)n(v)\in J}\, du\, dv.$$

Now we rewrite this as

$$Q \sum_{c\neq 0} S(m,n,c) I(m,n,c),$$

where

$$I(m,n,c) := \int_{u,v\in\mathbb{R}} e(-mu+nv) 1_{n(u)w(c)n(v)\in J}\, du\, dv. \tag{1}$$

So in summary, we have shown thus far that for $m, n \asymp T$,

$$\sum_{C(\pi)\leq Q} \frac{\lambda(m)\lambda(n)}{\sqrt{mn}} \approx 1_{m=n}\frac{Q}{T} + Q\sum_{c\neq 0} S(m,n,c)I(m,n,c).$$

In other words,

$$\frac{1}{Q} \sum_{C(\pi)\leq Q} \lambda(m)\lambda(n) \approx 1_{m=n} + T\sum_{c\neq 0} S(m,n,c)I(m,n,c). \tag{2}$$

To study (1), we apply the matrix multiplication identity

$$\begin{pmatrix} 1 & u/c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} \begin{pmatrix} 1 & v/c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} u & \frac{uv}{c} - \frac{1}{c} \\ c & v \end{pmatrix}.$$

For this to lie in J, we should have

$$u = o(1), \quad v = o(1), \quad c \lll T/Q,$$

say $c \asymp T/Q$, and then

$$uv - 1 \ll 1/Q.$$

This essentially detects $v = 1/u$ but we save a factor of $1/Q$ from the volume of the set of relevant v. We arrive at, for $c \asymp T/Q$,

$$I(m,n,c) = \frac{1}{c^2 Q} \int_{u=1+o(1)}^{\text{smooth}} e\left(\frac{mu + n/u}{c}\right) du\, dv.$$

We have, as functions of u,

$$(mu + nu^{-1})' = m - nu^{-2}.$$

The stationary points are

$$u_0 = \pm\sqrt{n/m}.$$

Near that point, we can approximate the phase using its second degree Taylor expansion. The second derivative is

$$(mu + nu^{-1})'' = \frac{1}{2} nu^{-3}.$$

The value of the phase is

$$\frac{mu_0 + nu_0^{-1}}{c} = \pm \frac{m\sqrt{n/m} + n\sqrt{m/n}}{c} = \pm 2 \frac{\sqrt{mn}}{c}.$$

The second degree Taylor expansion then looks like

$$\frac{mu + nu^{-1}}{c} \approx \pm 2 \frac{\sqrt{mn}}{c} \pm \frac{1}{2} \frac{nu_0^{-3}}{2c} (u - u_0)^2.$$

So we're reduced to the Fresnel integral

$$\int_u e\left(\frac{1}{2} \frac{nu_0^{-3}}{2c} u^2\right) du.$$

Doing $u \mapsto u(c/n)^{1/2}$ gets rid of the part of the phase that is not $\asymp 1$, so the above has size

$$\asymp (c/n)^{1/2}.$$

So we arrive at

$$I(m,n,c) \approx \frac{1}{c^2 Q} \frac{c^{1/2}}{n^{1/2}} e\left(\pm 2 \frac{\sqrt{mn}}{c}\right).$$

For $c \asymp T/Q$ and $n \asymp T$, the above has magnitude

$$\frac{1}{(T/Q)^2 Q^{3/2}} = \frac{Q^{1/2}}{T^2}.$$

Substituting back into (2) gives now

$$\frac{1}{Q} \sum_{C(\pi) \leq Q} \lambda(m)\lambda(n) \approx 1_{m=n} + \frac{Q^{1/2}}{T} \sum_{c \asymp T/Q} S(m,n,c) e\left(\pm 2 \frac{\sqrt{mn}}{c}\right).$$

References

[1] Subhajit Jana and Paul D. Nelson. Analytic newvectors for $\mathrm{GL}_n(\mathbb{R})$. *arXiv e-prints*, page arXiv:1911.01880, Nov 2019.