Here I'm recording my own notes, closely following the presentation of [1, §4] but filling in the details. Only a small part of this material was covered in lecture.

1. Definitions and statements

The Bernoulli numbers B_n may be defined (up to conventions) by the generating function

$$\frac{1}{e^t - 1} = \frac{1}{t + t^2/2 + \cdots} = \sum_{n \geq 0} \frac{B_n}{n!} t^{n-1}.$$

Exercise 1. Use the methods of these previous notes to determine the leading order asymptotics of B_n as $n \to \infty$.

The Bernoulli polynomials $B_n(x)$ may be defined by the generating function

$$\frac{e^{xt}}{e^t - 1} = \sum_{n \geq 0} \frac{B_n(x)}{n!} t^{n-1}, \tag{1}$$

so that $B_n = B_n(0)$.

Example 1. By direct calculation, we have

$$B_0(x) = 1. \tag{2}$$

$$B_1(x) = x - \frac{1}{2},$$

$$B_2(x) = x^2 - x + \frac{1}{6}.$$

Notation 2. For $x \in \mathbb{R}$, we denote by $\lfloor x \rfloor \in \mathbb{Z}$ its integer part and $\{x\} \in [0,1)$ its fractional part, which are characterized by the identity

$$x = \lfloor x \rfloor + \{x\}.$$

Theorem 3 (Euler–Maclaurin formula)**.** *For integers $a < b$, and a smooth function f on the real line,*

$$\int_a^b f(x)\,dx = \frac{f(a)}{2} + \sum_{n=a+1}^{b-1} f(n) + \frac{f(b)}{2} + \sum_{n=1}^{N-1} \frac{(-1)^n B_{n+1}}{(n+1)!} \left(f^{(n)}(b) - f^{(n)}(a) \right)$$
$$+ (-1)^N \int_a^b f^{(N)}(x) \frac{B_N(\{x\})}{N!}\,dx.$$

Example 4. In the case $N = 1$, this specializes to the "trapezoid rule":

$$\int_a^b f(x)\,dx = \frac{f(a)}{2} + \sum_{n=a+1}^{b-1} f(n) + \frac{f(b)}{2} - \int_a^b \{x - \tfrac{1}{2}\} f'(x)\,dx.$$

Exercise 2. Prove 3. [Reduce to the case $(a,b) = (0,1)$, then integrate by parts.]

2. Applications

Theorem 3 can be useful for asymptotic analysis. The reference [1, §4] contains many interesting examples. Here is a less interesting example, which can be studied in other ways (e.g., using Poisson summation).

2.1. **Asymptotics of Riemann sums.** (Compare with external §**??**.)

Let $f : \mathbb{R} \to \mathbb{C}$ be smooth and compactly-supported, or more generally, an element of the Schwartz space $\mathcal{S}(\mathbb{R})$. Define the function $g : \mathbb{R}^+ \to \mathbb{C}$ by

$$g(y) := \sum_{n \in \mathbb{Z}} f(ny).$$

Then as $y \to 0$, the quantity $yg(y)$ is a Riemann sum, hence converges to $\int f$. We will verify that this convergence is quite rapid:

Lemma 5. *For each fixed $N \geq 0$, we have for $y \in (0,1)$ the estimate*

$$g(y) = y^{-1} \int f + \mathrm{O}(y^N).$$

Proof. We take the limit of Theorem 3 as $(a,b) \to (-\infty, +\infty)$, applied to $f_y(x) := f(xy)$; we can do this because f and all its derivatives vanish at ∞. Since $f_y^{(n)}(x) = y^n f^{(n)}(x)$, we obtain for each $N \geq 1$,

$$\int_{\mathbb{R}} f(xy)\,dx = \sum_{n \in \mathbb{Z}} f(ny) + (-y)^N \int_{\mathbb{R}} f^{(N)}(xy) \frac{B_N(\{x\})}{N!}\,dx.$$

We see that the left hand side is $y^{-1} \int f$ via the substitution $x \mapsto x/y$. We estimate the integral on the right hand side using that $f^{(N)}$ has finite L^1-norm and $B_N(\{x\})$ is bounded, due to the periodicity of $\{x\}$. The required conclusion follows. $\square$

Remark 6. It's more interesting to analyze $\sum_{n \geq 0} f(ny)$ this way. We refer again to [1, §4].

3. Proofs

Here we record the proof of Theorem 3 (again, following the presentation of [1, §4]), after some preliminaries.

Lemma 7. *For $n \geq 1$, we have*

$$B_n'(x) = nB_{n-1}(x).$$

Proof. It suffices to show that

$$\sum_{n \geq 0} \frac{B_n'(x)}{n!} t^{n-1} = \sum_{n \geq 1} \frac{B_{n-1}(x)}{(n-1)!} t^{n-1},$$

or equivalently, that

$$\frac{d}{dx} \sum_{n \geq 0} \frac{B_n(x)}{n!} t^{n-1} = t \sum_{n \geq 0} \frac{B_n(x)}{n!} t^{n-1},$$

which is clear from the definition (1). $\square$

Lemma 8. *The Bernoulli polynomials are given by the formula*

$$B_n(x) = \sum_{r=0}^{n} \binom{n}{r} B_r x^{n-r}.$$

Proof. The claimed formula holds when $x = 0$ because the right hand side simplifies to B_n. It also holds when $n = 0$ in view of (2). We may thus suppose by induction on $n \geq 1$ that the claimed formula holds for smaller values of n.

Since we have checked that the claimed formula holds when $x = 0$, we reduce to verifying that the derivatives of the two sides coincide. The derivative of the right hand side is

$$\sum_{r=0}^{n-1} \frac{n!}{r!(n-r)!}(n-r)B_r x^{n-1-r},$$

which simplifies to

$$n \sum_{r=0}^{n-1} \frac{(n-1)!}{r!(n-1-r)!} B_r x^{n-1-r}.$$

By our inductive hypothesis, this simplifies to $nB_{n-1}(x)$, which, in view of Lemma 7, coincides with the derivative of the left hand side, $B_n'(x)$. □

Lemma 9. *For each $a \geq 0$, we have*

$$\int_a^{a+1} B_n(x)\, dx = a^n.$$

Here we interpret $0^0 = 1$.

Proof. It is enough to check that

$$\int_a^{a+1} \sum_{n \geq 0} \frac{B_n(x)}{n!} t^n \, dx = \sum_{n \geq 0} \frac{a^n}{n!} t^n,$$

or equivalently, in view of (1), that

$$\int_a^{a+1} \frac{te^{xt}}{e^t - 1}\, dx = e^{at}. \tag{3}$$

Indeed, we have

$$\begin{aligned}\int_a^{a+1} e^{xt}\, dx &= \frac{1}{t} e^{xt}|_{x=a}^{a+1} \\ &= \frac{e^{(a+1)t} - e^{at}}{t} \\ &= e^{at} \frac{e^t - 1}{t},\end{aligned}$$

from which the formula (3) follows. □

Lemma 10. *We have*

$$B_n(x+1) - B_n(x) = nx^{n-1}.$$

Proof. Combine Lemmas 7 and 9. □

Lemma 11. *For $n \geq 2$, we have*

$$B_n(1) = B_n.$$

Proof. We need to check that

$$B_n(1) - B_n(0) = 0.$$

By the fundamental theorem of calculus, it is enough to check that

$$\int_0^1 B_n'(x)\,dx = 0.$$

By Lemma 7, it is equivalent to check that

$$\int_0^1 B_{n-1}(x)\,dx = 0.$$

This follows from Lemma 9. We use here that $n-1 \geq 1$, which ensures that $0^{n-1} = 0$. $\square$

Lemma 12. *For a smooth function f on $[0,1]$, we have*

$$\int_0^1 f^{(n)}(x)\frac{B_n(x)}{n!}\,dx = -\int_0^1 f^{(n+1)}(x)\frac{B_{n+1}(x)}{(n+1)!}\,dx + \begin{cases} \frac{f(0)+f(1)}{2} & \text{if } n = 0, \\ \frac{B_{n+1}}{(n+1)!}\left(f^{(n)}(1) - f^{(n)}(0)\right) & \text{if } n \geq 1. \end{cases}$$

Proof. We integrate by parts. Take

$$u = f^{(n)}(x), \qquad v = \frac{B_{n+1}(x)}{(n+1)!}.$$

Then, by Lemma 7,

$$u\,dv = f^{(n)}(x)\frac{B_n(x)}{n!}.$$

The first term on the right hand side of the claimed formula is the integral of $v\,du$. The second term is $uv|_0^1$, using here that $B_m(1) = B_m(0) = B_m$ for $m \geq 2$, while $B_1(0) = 1/2 = -B_1(0)$. $\square$

By inducting on N and using that $B_0(x) = 1$, we obtain:

$$\int_0^1 f(x)\,dx = (-1)^N \int_0^1 f^{(N+1)}(x)\frac{B_{N+1}(x)}{(N+1)!}\,dx + \frac{f(0)+f(1)}{2} + \sum_{n=1}^{N-1} \frac{B_{n+1}}{(n+1)!}\left(f^{(n)}(1) - f^{(n)}(0)\right).$$

By replacing f with its shifts by integer and summing, we obtain Theorem 3.

References

[1] Don Zagier. The mellin transform and other useful analytic techniques. `http://people.mpim-bonn.mpg.de/zagier/files/tex/MellinTransform/fulltext.pdf`, 2006.