1. Anne-Maria Ernvall-hytönen, *Lattices and modular forms in coset coding*

Wyner (1975). Wiretap channel (Alice, Bob, Eve). Gaussian noise with variance $\sigma^2$ (between Alice and Bob) and $\sigma_e^2$ (for Eve). We assume $\sigma_e > \sigma$. We would like to do this whole scheme in such a way that we can use the noise so that Bob can still receive the image, but Eve cannot. We aim to do this with lattices $\Lambda \geq \Lambda_e$, where each coset corresponds to a code-letter.

Belfiore and Oggier (2010) (maybe [**?**]?): secrecy gain. Transmit codeword $x \in \mathbb{R}^n$.

$$\frac{1}{(\sigma_\Lambda\sqrt{2\pi})^n}\int_{V_{\Lambda(x)}} e^{-\|y-x\|^2/2\sigma^2}\,dy.$$

$$\frac{1}{(\sigma_\Lambda\sqrt{2\pi})^n}\sum_{t\in\Lambda_e\cap R}\int_{V_{\Lambda(x+t)}} e^{-\|y-x\|^2/2\sigma^2}\,dy.$$

End up trying to minimize

$$\theta_{\Lambda_e}\left(-\frac{1}{2\pi i\sigma_e^2}\right).$$

Secrecy function

$$\Xi(y) := \frac{\theta_{\lambda\mathbb{Z}^n}(yi)}{\theta_\Lambda(yi)}.$$

Belfiore and Sole: For unimodular $\Lambda$, maximum at $y = 1$. Even unimodular: polynomials in

$$E_4 = \frac{1}{2}\left(\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8\right), \qquad \Lambda = \frac{1}{256}\vartheta_2^8\vartheta_3^8\vartheta_4^8.$$

Inverse of $\Xi(y)$ a polynomial in $\frac{\vartheta_4^4\vartheta_2^4}{\vartheta_3^8}$.

(Maybe [**?**] is a reference.)

$\ell$-modular: $1/\sqrt{\ell}$, $\mathbb{Z}\oplus\sqrt{2}\mathbb{Z}\oplus 2\mathbb{Z}$.

The connection to deeper mathematics is, what kinds of representations as polynomials do you have for various theta functions?

2. Jolanta Marzec-ballesteros, *Doubling method for self-dual linear codes*

Garrett, Piatetski-Shapiro–Rallis (1980's). Integral of cusp form against restriction of Siegel-type Eisenstein series equals $L$-function attached to cusp form times cusp form or Eisenstein series attached to cusp form:

$$\left\langle E\left(\begin{pmatrix} g & \\ & g' \end{pmatrix}, s\right), f(g)\right\rangle = L(f,s)f(g').$$

Done for $G$ symplectic, orthogonal, unitary over global field, also for congreunce subgroups.

Let's start with an overview. Let $f$ be a cusp form on $G$, and $H$ a subgroup for which $G\times G \hookrightarrow H$. Then form an Eisenstein series on $H$

$$E(h,s) = \sum_{\gamma\in P\backslash H}\phi(\gamma h, s).$$

Restrict to $h = \mathrm{diag}(g, g')$ and take inner product with $F(g)$. This leads to an unfolding involving a sum over $\gamma \in P\backslash H/(G\times G)$. In favorable cases, only one representative $\gamma_0$ contributes, leaving us with

$$\sum_{(k,k')\in G\times G} \left\langle \phi\left(\gamma_0 \begin{pmatrix} kg & \\ & k'g' \end{pmatrix}, s\right), f(g) \right\rangle$$
$$= \sum_{\beta\in\gamma_0(G\times 1)} \psi(f) f|_\beta(g')$$
$$= L(f,s)f(g').$$

We would like to do something similar, but now over finite fields.

A *linear code of length* $2n$ over a finite field $\mathbb{F}$ is a linear subspace $C \subseteq \mathbb{F}^{2n}$. We denote by $\langle , \rangle : C \times C \to \mathbb{F}$ the Euclidean inner product. We say that $C$ is *self-dual* if

$$C = C^\perp := \left\{ v \in \mathbb{F}^{2n} : \langle v, C\rangle = 0 \right\}.$$

Then (the length $2n$ is even and) $\dim C = n$.

The *weight* of a codeword $c = (c_1, \ldots, c_{2n}) \in C$ is

$$\operatorname{wt} c = \#\left\{ i \in \{1, \ldots, 2n\} : c_i \neq 0 \right\}.$$

*Weight enumerators* are certain homogeneous polynomials of degree $2n$ in variables from the set $V = \{x_\alpha : \alpha \in \mathbb{F}^g\}$, where $g \in \mathbb{N}$ is the genus.

The *genus one weight enumerator* of a code $C \subseteq \mathbb{F}_2^{2n}$ is a polynomial

$$W_1(C, (x_0, x_1)) = \sum_{c\in C} x_0^{2n-\operatorname{wt} c} x_1^{\operatorname{wt} c}.$$

The *genus g weight enumerator* of a code $C \subseteq \mathbb{F}^{2n}$ is a polynomial

$$W_g(C, x) = \sum_{(c^1,\ldots,c^g)\in C^g} \prod_{\alpha\in\mathbb{F}^g} x_\alpha^{w_\alpha(c^1,\ldots,c^g)}$$

of degree $2n$, where $x = (x_\alpha)_{\alpha\in\mathbb{F}^g}$ and

$$w_\alpha(c^1, \ldots, c^g) = \#\left\{ \text{rows } r \text{ in } (c_i^j)_{i=1..2n}^{j=1..g} : r = \alpha \right\}.$$

As an example, we give a basis for a Hamming code $H_8$ 5nof weight 8, the span over $\mathbb{F}_2$ inside $\mathbb{F}_2^8$ of the vectors

$$\begin{pmatrix}1\\0\\0\\1\\0\\1\\1\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\\0\\1\\0\\1\\0\\1\end{pmatrix} \quad \begin{pmatrix}0\\0\\1\\1\\0\\0\\1\\1\end{pmatrix} \quad \begin{pmatrix}0\\0\\0\\0\\1\\1\\1\\1\end{pmatrix}.$$

Then

$$W_2(H_8, (x_{00}, x_{01}, x_{10}, x_{11})) = \sum_{\alpha\in\mathbb{F}_2^2} x_\alpha^8 + 14 \sum_{\substack{\alpha_1,\alpha_2\in\mathbb{F}_2^2\\ \alpha_1<\alpha_2}} x_{\alpha_1}^4 x_{\alpha_2}^4 + 168 x_{00}^2 x_{01}^2 x_{10}^2 x_{11}^2$$
$$= (8) + 14(4,4) + 168(2,2,2,2).$$

In general,

$$W_g(C) = \sum_A b_A \cdot (A),$$

where

$$A \in \left\{ (a_0, \dots, a_{2^g-1}) : \text{admissible tuples}, \sum_{i=0}^{2^g-1} a_i = 2n \right\}.$$

Some analogies with modular forms:

- $W_g(C)$ is like a modular form $f$ of genus $g$,
- $\sum_A b_A \cdot (A)$ is like a Fourier expansion,
- $(2n)$ is like a constant term $a(0)$.

EXamples of cusp forms:

- $W_1(G_{24}) - W_1(H_8 \times H_8 \times H_8) = -42(20,4) + 168(16,8) - 252(12,12)$ is a cusp form of genus one.
- $W_3(E_{16}) - W_3(H_8 \times H_8) = -2688(9,1,1,1,1,1,1,1) + \cdots$ is a cusp form of genus 3.

**Theorem 1** (Runge, 1996; Nebe, Rains, Sloane, 2006)**.** *We have*

$$\langle W_g(C) : C \text{ self-dual, over } \mathbb{F} \rangle = \left(\mathbb{C}[x_\alpha : \alpha \in \mathbb{F}^g]\right)^{\mathcal{C}_g},$$

*where*

$$\mathcal{C}_g := \langle m_r, d_\phi, h_{\iota,u_\iota,v_\iota} : r \in \mathrm{GL}_g(\mathbb{F}), \phi, \iota \rangle$$

*with*

$$\begin{aligned} m_r &: x_\alpha \mapsto x_{r\alpha}, \\ d_\phi &: x_\alpha \mapsto e^{2\pi i \phi(\alpha)} x_\alpha, \\ h_{\iota,u_\iota,v_\iota} &: x_\alpha \mapsto (\#\iota\mathbb{F}^g)^{-1/2} \sum_{w \in \iota\mathbb{F}^g} e^{\frac{2\pi i}{p}\langle w, v_\iota \alpha\rangle} x_w + \cdots . \end{aligned}$$

Consider the mean polynomial ("Siegel-type Eisenstein series")

$$\begin{aligned} M_{2g}((2n)) &= \sum_{\gamma \in P_{2g} \backslash \mathcal{C}_{2g}} (2n)^\gamma \\ &= \sum_{\gamma \in P_{2g} \backslash \mathcal{C}_{2g}} \sum_{\alpha \in \mathbb{F}^{2g}} \left((x_\alpha)^\gamma\right)^{2n} = \text{const} \sum_{\substack{C \subset \mathbb{F}^{2n} \\ \text{fixed type}}} W_{2g}(C, x), \end{aligned}$$

and an inner product defined on monomials.

What we prove with Bourganis in 2024 is the following:

**Theorem 2.** *Let $\mathcal{T}$ be a family of self-dual codes of length $2n$ over a field $\mathbb{F}$. Assume either that $\mathbb{F}$ has odd characteristic or is equal to $\mathbb{F}_2$. Let $C \in \mathcal{T}$ be doubly-even, and fix $g \in \mathbb{N}$. Then there exists an (explicit) constant $C$ such that for a cusp form $f \in \mathcal{T}$ of genus $r$, with $\deg f = 2n$, we have*

$$\langle M_{2g}((2n))(xy), f(x) \rangle = \begin{cases} 0 & \text{if } r < g, \\ C \cdot f(y) & \text{if } r = g \end{cases}.$$

## 3. Petru Constantinescu, *Non-vanishing of geodesic periods of automorphic forms*

Preprint: [?], joint with Asbhjørn Nordentoft.

Class groups:

- $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, $K = \mathbb{Q}(\sqrt{D})$,
- $\mathrm{Cl}_K$ class group, $h(D) = h_K = |\mathrm{Cl}_K|$ class number,

- $\mathcal{Q}_D$: set of primitive integral binary quadratic forms of discriminant $D$,
$$\mathcal{Q}_D = \left\{Q(x,y) = ax^2 + bxy + cy^2 : (a,b,c) = 1,\, b^2 - 4ac = D\right\}.$$
- Gauss: $\Gamma \circlearrowright \mathcal{Q}_D$,
$$(Q.\gamma)\begin{pmatrix} x \\ y \end{pmatrix} = Q\left(\gamma \begin{pmatrix} x \\ y \end{pmatrix}\right),$$
- isomorphism
$$\mathrm{Cl}_K \xrightarrow{\cong} \Gamma\backslash\mathcal{Q}_D$$
$$A \mapsto [a,b,c].$$

Heegner points and closed geodesics:

- $D < 0$: $A \in \mathrm{Cl_K} \rightsquigarrow$ Heegner point $z_A \in \Gamma\backslash\mathbb{H}$,
$$[a,b,c] \rightsquigarrow \frac{-b - i\sqrt{|D|}}{2a}.$$
$$h(D) = |\mathrm{Cl}_K| = |D|^{1/2+o(1)}.$$
- $D > 0$: $A \in \mathrm{Cl}_K^+ \rightsquigarrow$ closed geodesic $C_A \subset \Gamma\backslash\mathbb{H}$, $[a,b,c] \rightsquigarrow$ semicircle with endpoints $\frac{-b\pm\sqrt{D}}{2a}$.
$$h(D)\log\varepsilon_D = D^{1/2+o(1)},$$
$$I(C_A) = 2\log\varepsilon_D.$$

**Theorem 3** (Duke '88)**.** *Fix* $\Omega \subset \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$.

- $D < 0$*: equidistribution of* $z_A$*,* $A \in \mathrm{Cl}_{\mathbb{Q}(\sqrt{D})}$.
- $D > 0$*: similar for closed geodesics.*

**Waldspurger formulas**. Let $f$ be a nonzero Maass form. Our goal is to study closed geodesic periods $\int_{C_A} f(z)\,\frac{|dz|}{y}$.

$\chi \in \widehat{\mathrm{Cl}_K} \rightsquigarrow \theta_\chi$, the associated theta series (weight one on $\Gamma_0(D)$, nebentypus $\chi_D$).

**Theorem 4** (Waldspurger/Zhang/Popa)**.** *Let $D$ be a fundamental discriminant. For $D < 0$,*
$$L(f \times \theta_\chi, \tfrac{1}{2}) = \frac{C_f}{D^{1/2}}\left|\sum_{A\in\mathrm{Cl}_K} \chi(A) f(z_A)\right|^2.$$
*For $D > 0$,*
$$L(f \times \theta_\chi, \tfrac{1}{2}) = \frac{C_f}{D^{1/2}}\left|\sum_{A\in\mathrm{Cl}_K^+} \chi(A)\int_{C_A} f(z)\,\frac{|dz|}{y}\right|^2.$$

**Theorem 5** (Michel–Venkatesh '05)**.** *Let* $\delta = 1/2700$*. For* $D < 0$*:*
$$\left|\left\{\chi \in \widehat{\mathrm{Cl}_K} : L(f\times\theta_\chi, \tfrac{1}{2}) \neq 0\right\}\right| \gg D^\delta.$$

*Sketch of proof.* By orthogonality of characters,
$$\frac{1}{h(D)}\sum_{\chi\in\widehat{\mathrm{Cl}_K}} L(f\times\theta_\chi, \tfrac{1}{2}) = \frac{c_f}{D^{1/2}}\sum_{A\in\mathrm{Cl}_K} |\cdots|^2.$$
This converges by Duke's equidistribution theorem. The conclusion then follows from subconvexity for Rankin–Selberg $L$-functions, due to Harcos–Michel. $\square$

Same proof does not work for geodesics, cannot apply equidistribution and relate to subconvexity (square is outside integral).

**Question 6** (Michel, Oberwolfach 2020)**.** Let $K$ be a real quadratic field of discriminant $D > 0$, and assume that $h_K \gg D^\delta$. Does there exist $A \in \mathrm{Cl}_K^+$ such that

$$\int_{C_A} f \,\frac{|dz|}{y} \neq 0?$$

Equivalently, does there exist $\chi \in \widehat{\mathrm{Cl}_K^+}$ such that $L(f \times \theta_\chi, \frac{1}{2}) \neq 0$?

**The prime geodesic theorem**. Let $D > 0, A \in \mathrm{Cl}_K^+$. This gives rise to a closed geodesic $C_A$, with $I(C_A) = 2\log \varepsilon_D$. Sound–Young have the best estimate for their count.

**Theorem 7** (C–Nordentoft 2024)**.** *Let $f$ be a nonzero Maass form for* $\mathrm{SL}_2(\mathbb{Z})$*. Then*

$$\#\left\{C \in \mathcal{C}(X) : \int_C f(z)\,\frac{|dz|}{y} = 0\right\} \ll \frac{X}{(\log X)^{5/4}}.$$

**Remark 8.** We also obtain 100% non-vanishing for periods of weight $k$ holomorphic cusp forms, for any Fuchsian group $\Gamma$.

**Theorem 9** (C–Nordentoft 2024)**.** *For a positive proportion of positive discriminants $D > 0$ with $\varepsilon_\Delta \leq X$, we get that there exists $\chi \in \widehat{\mathrm{Cl}_{\mathbb{Q}(\sqrt{D})}}$ with $L(f \times \theta_\chi, \frac{1}{2} \neq 0)$.*

We construct a bipartite graph on $X_N$ (double cosets in $\Gamma_\infty\backslash\Gamma/\Gamma_\infty$ with $c \leq N$) times $Y_N$ (conjugacy classes with trace bounded in magnitude by $N$). This graph relates closed geodesic and vertical geodesics.

## 4. An excised orthogonal model for families of cusp forms

Talk by Zoe Batterman (Abstract), Akash Narayanan, Christopher Yao. Joint with Owen Barrett, Aditya Jambhale, and Kishan Sharma. Preprint: [**?**].

Conjecture (Montgomery–Dyson, 1970's): zeros of of zeta vs. GUE.

2005: S.J. Miller noticed a repulsion of the lowest-lying zeros near the central point of a family of even twists of a fixed elliptic curve $L$-function with finite conductor.

2011: Duenez, Huynh, Keating, Miller, Snaith: proposed an excised orthogonal model to capture the behavior of this repulsion.

**Question 10.** How accurately do egienvalues of random matrices from classical compact groups model the lowest-lying zeros of families of $L$-functions associated to a cuspidal newform?

Let

$$S_k^{\mathrm{new}}(M, \chi_f) \ni f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z},$$

$\lambda_f(n) = a_f(n)/n^{(k-1)/2}$.

$$L(s, f) = \sum_{n \geq 1} \lambda_f(n) n^{-s},$$

Various specific families of twists $L(f \otimes \psi_d, s)$, match with classical compact groups:

- principal character, even twists vs. SO(even)
- principal character, odd twists vs. SO(odd)
- non-principal character, self-dual vs. Sp
- generic vs. U

Pictures.

## 5. On an extension of the Rohrlich-Jensen formula, *Lejla Smajlović*

Joint work with James Cogdell, Jay Jorgensen. Preprint: [**?**].

What is a Poisson–Jensen formula? We will view it as a way to characterize meromorphic functions in terms of their divisors. Some notation:

- $D_R = \{z = x + iy \in \mathbb{C} : |z| < R\}$
- $F$: a non-constant meromorphic function on $\overline{D_R}$,
$$F(z) = c_F z^m + \mathrm{O}(z^{m+1}), \quad z \to 0.$$

Then
$$\int_0^{2\pi} \log|F(Re^{i\theta})| \frac{d\theta}{2\pi} + \sum_{D_R} \cdots = F(0).$$

Rohrlich, 1980's: a *modular* generalization, characterizing modular forms via divisors. Given $f$, a meromorphic function on $\mathbb{H}$ that is invariant by $\mathrm{PSL}(2, \mathbb{Z})$. Assume $f$ is holomorphic at the cusp and that the Fourier expansion of $f$ at $\infty$ has constant term equal to one. Then
$$\int_{\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}} \log|f(z)| \frac{d\mu(z)}{2\pi} + \sum_{w \in \mathcal{F}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} P(w) = 0.$$

Here

- $\mathrm{ord}_w(f)$ is the order of $f$ at $w$ as a meromorphic function,
- $\mathrm{ord}(w)$ denotes the order of the *point* $w$ with respect to the action of $\mathrm{PSL}(2, \mathbb{Z})$ on $\mathbb{H}$, and
- $P(w) = \log\left(|\eta(w)|^4 \cdot \Im(w)\right)$ is the Kronecker limit function associated to the parabolic Eisenstein series on $\mathrm{PSL}(2, \mathbb{Z})$. This is the function appearing as the next-order term in the expansion of the Eisenstein series as $s \to 1$.

Another way to interpret this formula is as follows. We have
$$\langle 1, \log|f(z)|\rangle = \lim_{Y\to\infty} \int_{\mathcal{F}(Y)} 1 \cdot \log|f(z)|\, d\mu(z),$$
hence the formula reads
$$\langle 1, \log|f(z)|\rangle = -2\pi \sum_{w\in\mathcal{F}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} P(w).$$

Here $\log|f(z)|$ can be replaced by
$$\log\|f(z)\| = \log\left(\Im z^k |f(z)|\right)$$
for weight $2k$ meromorphic modular forms. Generalizations:

- to other Fuchsian groups of the first kind, by Rohrlich.
- to hyperbolic 3-space, by Herrero, Imamoglu, von Pippich, Toth.

Further modular generalization by Bringman and Kane. Keep the modular group setting, but evaluate more general inner products.

- $j(z) = q^{-1} + 744 + \mathrm{O}(q)$: Hauptmodul

- $j_1(z) := j(z) - 744$
- $j_n(z) := j_1|T_n(z)$, for $n \geq 2$
- $f$: weight $2k$ meromorphic modular form with respect to $\mathrm{PSL}(2,\mathbb{Z})$, normalized so that $f(z) = 1 + \mathrm{O}(q)$

They evaluated the regularized scalar product in terms of the divisor of $f$, proving that

$$\left\langle j_n(z), \log\left((\Im(z))^k |f(z)|\right)\right\rangle = -2\pi \sum_{w\in\mathcal{F}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} \mathbf{j}_n(w) + \frac{k}{6}c_n,$$

where $\mathbf{j}_n$ is characterized in terms of differential operators by Bringman and Kane; application of our results yields a different expression for the same function, namely

$$\mathbf{j}_n(w) = 2\pi\sqrt{n}\partial_s F_{-n}^{\mathrm{PSL}(2,\mathbb{Z})}(w,s)|_{s=1} - 24\sigma(n)P(w).$$

What is our main goal?

- To extend the point of view that the Rohrlich–Jensen formula is the evaluation of a particular type of inner product.
- To prove the extension of this formula in the setting of an arbitrary, not necessarily arithmetic, Fuchsian group of the first kind with one cusp.

We start with $j_n(z) = j_1|T_n(z)$, which is the unique (up to constants) holomorphic function that is $\mathrm{PSL}(2,\mathbb{Z})$-invariant on $\mathbb{H}$ and whose expansion near $\infty$ is $q^{-n} + o(q^{-1})$.

These properties hold for the special value $s = 1$ of the Niebur–Poincaré series $F_{-n}^{\Gamma}(z,s)$, defined for any Fuchsian group $\Gamma$ of the first kind with oen cusp by

$$F_m^{\Gamma}(z,s) = \sum_{\gamma_\infty\backslash\Gamma} e\left(m\Re(\gamma z)\right)\left(\Im(\gamma z)\right)^{1/2} I_{s-1/2}\left(2\pi|m|\Im(\gamma z)\right),$$

for $m \neq 0$.

It is an eigenfucntion of the hyperbolic Laplacian, and may be expressed in terms of $j_m$.

The term $\log\left(\|f(z)\|\right)$ is a bit complicated, involving findings by Jorgensen, von Pippich and the speaker, plus some additional work done in our paper.

**Proposition 11.** *Let $\Gamma$ be a cofinite Fuchsian group with one cusp at $\infty$ with identity as its scaling matrix. Let $2k \geq 0$ be an even integer, and let $f$ be aw eight $2k$ meromorphic form which is $\Gamma$-invariant and with $q$-expansion at $\infty$ normalized so its constant term is equal to one. Then, we can express $\log\left(\|f\|(z)\right)$ in terms of parabolic Eisenstein series and Green's functions, as*

$$-2k + 2\pi \sum_{w\in\mathcal{F}_\Gamma} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} \lim_{s\to 1}\left(G_s^{\Gamma}(z,w) + \mathcal{E}_{\Gamma,\infty}^{\mathrm{par}(z,s)}\right) + \cdots.$$

Here

$$\mathcal{E}_\infty^{\mathrm{par}}(z,s) = \sum_{\Gamma_\infty\backslash\Gamma} \Im(\gamma z)^s.$$

The Kronecker limit formula says that

$$\mathcal{E}_\infty^{\mathrm{par}}(z,s) = \frac{1}{(s-1)\,\mathrm{vol}(\Gamma\backslash\mathbb{H})} + \beta - \frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}\log\left(|\eta_\infty^4(z)|\Im(z)\right) + \mathrm{O}(s-1).$$

Then we need to define $\beta$, and the Green's function $G_s(z,w)$, which is obtained by averaging the kernel $k_s(z,w)$; both functions are eigenfunctions of the Laplacian with eigenvalue $s(1-s)$, and the Green's function has a specified singularity on the diagonal. The Green's function also has a Laurent series expansion as $s \to 1$ that involves the parabolic Eisenstein series. In particular,

$$\lim_{s\to 1}\left(G_s^{\Gamma}(z,w)+\mathcal{E}^{\mathrm{par}}_{\Gamma,\infty}(z,w)\right)$$

exists, with a logarithmic singularity on the diagonal.

The Rohrlich–Jensen formula can be understood through the study of the regularized inner product of this last limit with $F_{-n}(\cdot,1)$. Regularization is needed only in the cusp (because the logarithmic singularity is integrable). We can thus write

$$\begin{aligned}&\left\langle F_{-n}(\cdot,1), \overline{\lim_{s\to 1}\left(G_s^{\Gamma}(z,w)+\mathcal{E}^{\mathrm{par}}_{\Gamma,\infty}(z,w)\right)}\right\rangle\\ &\quad=\lim_{Y\to\infty}\int_{\mathcal{F}(Y)} F_{-n}(z,1)\lim_{s\to 1}\left(G_s(z,w)+\mathcal{E}^{\mathrm{par}}_{\infty}(z,s)\right)\,d\mu(z).\end{aligned}$$

A key observation is that all terms are eigenfunctions of the Laplacian, hence one can seek to compute the inner product in a manner similar to that which yields the Maass–Selberg formula. The key identity is that

$$\begin{aligned}&\int_{\mathcal{F}(Y)} F_{-n}(z,1)\lim_{s\to 1}\left(G_s(z,w)+\mathcal{E}^{\mathrm{par}}_{\infty}(z,s)\right)\,d\mu(z)\\ &\quad=\partial_s\left(-s(1-s)\int_{\mathcal{F}(Y)} F_{-n}(z,1)\left(G_s(z,w)+\mathcal{E}^{\mathrm{par}}_{\infty}(w,s)\,d\mu(z)\right)\right)\Big|_{s=1}.\end{aligned}$$

We can now absorb $-s(1-s)$ into the integrals after applying the hyperbolic Laplacian $\Delta$ to each of the two factors in the integrand.

Main results:

**Theorem 12.** *For any positive integer $n$ and any point $w \in \mathcal{F}$, we have*

$$\left\langle F_{-n}(\cdot,1), \overline{\lim_{s\to 1}\left(G_s(\cdot,w)+\mathcal{E}^{\mathrm{par}}_{\infty}(\cdot,s)\right)}\right\rangle = -\partial_s F_{-n}(w,s)|_{s=1}.$$

Combined with the previous proposition (describing $\log\|f(z)\|$), we get the Rohrlich–Jensen formula:

$$\langle F_{-n}(\cdot,1), \log\|f\|\rangle = -2\pi\sum_{w\in\mathcal{F}}\frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)}\partial_s F_{-n}(w,s)|_{s=1}.$$

We have further results. For instance, if $g$ is any $\Gamma$-invariant analytic function with a pole at $\infty$, then (Niebur)

$$g(z)=\sum_{n=1}^{K}2\pi\sqrt{n}a_nF_{-n}(z,1)+c(g)$$

for some constants $K$, $a_n$, and $c(g)$ depending only upon $g$. Then, we have the identity

$$\langle g, \log|f|\rangle = -2\pi\sum_{w\in\mathcal{F}}\frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)}\left(2\pi\sum_{n=1}^{K}\sqrt{n}a_n\partial_sF_{-n}(w,s)|_{s=1}-\cdots\right).$$

Further found that the generating series for the Niebur Poincaré series at $s=1$ is, in the $z$-variable (where we sum over $q_z$), the holomorphic part of the weight two biharmonic Maass form given by differentiating our linear combination of Eisenstein series and Green's function with respect to $z$.

## References