hand, it suffices to show that the line

$$L: \; y - b = (1-n)(x-b)$$

through $(b, b)$ has no other intersections with the graph of $g$ in the first quadrant. Thus, we have the algebraic problem of showing that

$$\frac{b^n}{x^{n-1}} - \frac{b^n}{b^{n-1}} = (1-n)(x-b) \tag{4}$$

has no *positive* solutions other than $x = b$. Clearing the fractions, cancelling a factor of $x - b$, and collecting all the terms of the polynomial on one side looks like the way to start. For $n = 3$, this leads to the equation

$$(x-b)(2x+b) = 0,$$

from which the desired result is obvious. The appearance of $b$ as a *double* root of the polynomial obtained from (4) is a consequence of the tangency of $L$ to the graph of $g$, but an explanation of that won't help the students. However, this case will help them see what has to be done in general. Follow the same steps: Clear the fractions, cancel the first factor of $x - b$, and collect all the terms of the polynomial to one side. The result is the equation

$$(n-1)x^{n-1} - bx^{n-2} - b^2x^{n-3} - \cdots - b^{n-2}x - b^{n-1} = 0.$$

The factorization of the left-hand side is not obvious for $n > 3$, but long division by $x - b$ comes out even, and the quotient is

$$(n-1)x^{n-2} + (n-2)bx^{n-3} + \cdots + b^{n-2},$$

which clearly has no positive roots.

Precalculus students who are both persistent and intrigued by mathematical challenges can follow the whole argument. Such students will surely appreciate the much simpler and more compelling arguments that calculus provides.

*Remark.* The details of this argument for $n = 2$ and the calculus-based argument for larger $n$ were given by R. G. Kuller in [*Proceedings of a Conference on Computers in the Undergraduate Curricula*, University of Iowa, 1970, pp. 4.1–4.11]. The $n = 2$ case is also developed in my book [*Interface*: *Calculus and the Computer*, 2nd ed., Saunders College Publishing, 1984, pp. 61–63].

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## Two Simple Recursive Formulas for Summing $1^k + 2^k + 3^k + \cdots + n^k$

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Most students of mathematics encounter the series

$$S_k(n) = 1^k + 2^k + 3^k + \cdots + n^k \qquad (k = 0, 1, 2, 3, \ldots) \tag{1}$$

in a calculus course, where they are asked to evaluate the definite integral $\int_0^1 x^k \, dx$ via the limit of inscribed rectangles, $\lim_{n\to\infty}(1/n^{k+1})\sum_{j=1}^n j^k$. Although the general formula for $S_k(n)$ involves the Bernoulli Polynomials and the corresponding Bernoulli numbers, which most beginning calculus students cannot appreciate, the students' knowledge is advanced enough to understand a method for determining $S_k(n)$ that we describe below.

Our approach uses elementary algebra and calculus to obtain a simple recursive formula for $S_k(n)$. Similar results were previously obtained by L. S. Levy [*Amer.*

*Math. Monthly* 77 (October 1970) 840–847]. However, our approach is more direct and more accessible to first-year calculus students.

We begin by noting from the definition of $S_k(n)$ that for each fixed $k (k = 0, 1, 2, \dots)$:

$$S_k(n) - S_k(n-1) = n^k \qquad (n = 1, 2, 3, \dots). \tag{2}$$

If, in addition, we define $S_k(0) = 0$, then $S_k(n)$ is uniquely determined for all nonnegative integers $n$. Since the form of (2) does not require that $n$ be an integer, we are motivated to define functions $f_k(x)$ $(k = 0, 1, 2, \dots)$ of the *continuous* variable $x$ by the difference relations

$$f_k(0) = 0 \quad \text{and} \quad f_k(x) - f_k(x-1) = x^k. \tag{3}$$

Note that an immediate consequence of (3) is the fact that

$$f_k(-1) = 0 \qquad (k = 1, 2, 3, \dots). \tag{4}$$

Note also that any function $f_k(x)$ which satisfies (3) will determine $S_k(n)$ since $S_k(n) = f_k(n)$.

Since $S_0(n) = n$, $S_1(n) = n(n+1)/2$, and $S_2(n) = n(n+1)(2n+1)/6$ suggest that $S_k(n)$ is a polynomial in $n$, we are further motivated to find a polynomial solution of (3). To show that a polynomial solution exists, replace $k$ in (3) by $k+1$ and differentiate with respect to the continuous variable $x$, obtaining

$$f'_{k+1}(x) - f'_{k+1}(x-1) = (k+1)x^k.$$

This can be recast, using (3), as

$$f'_{k+1}(x) - (k+1)f_k(x) = f'_{k+1}(x-1) - (k+1)f_k(x-1). \tag{5}$$

Since the expressions on both sides of (5) have the same functional form, suppose we define

$$\phi_k(x) = f'_{k+1}(x) - (k+1)f_k(x) \qquad (k = 0, 1, 2, \dots). \tag{6}$$

Since we are requiring that $f_k(x)$ be a polynomial in $x$, it follows that $\phi_k(x)$ is a polynomial in $x$ that satisfies

$$\phi_k(x) = \phi_k(x-1)$$

for all $x$. In particular, if $r$ is a root of $\phi_k(x)$, then so is $r - m$ for every integer $m$. Since a nonconstant polynomial has only a finite number of roots, $\phi_k(x)$ must be a constant $C_k$. We may thus write (6) as

$$f'_{k+1}(x) = (k+1)f_k(x) + C_k \qquad (k = 0, 1, 2, \dots), \tag{7}$$

where the constant $C_k$ could (and in fact does) depend on $k$. Since we defined $f_k(0) = 0$, integrating (7) from 0 to $x$ yields

$$f_{k+1}(x) = (k+1)\int_0^x f_k(t)\,dt + C_k x \qquad (k = 0, 1, 2, \dots). \tag{8}$$

From (4) and (8), we obtain

$$C_k = (k+1)\int_0^{-1} f_k(t)\,dt \qquad (k = 0, 1, 2, \dots).$$

Substituting for $C_k$ in (8) gives

$$f_{k+1}(x) = (k+1)\left\{\int_0^x f_k(t)\,dt + x\int_0^{-1} f_k(t)\,dt\right\} \qquad (k = 0, 1, 2, \dots).$$

This can be put in the more convenient form

$$f_{k+1}(x) = F_k(x) + xF_k(-1), \tag{9}$$

where

$$F_k(x) = (k+1)\int_0^x f_k(t)\,dt. \tag{10}$$

Thus, (9) and (10) provide a simple recursive formula for obtaining $f_k(x)$, and hence $S_k(n)$ (which equals $f_k(n)$). We need only start with $f_0(x) = x$, which satisfies (3), and successively generate $f_1(x)$, $f_2(x)$, etc. For example:

$$f_0(x) = x$$

$$F_0(x) = \int_0^x t\,dt = \left(\tfrac{1}{2}\right)x^2$$

$$f_1(x) = F_0(x) + xF_0(-1) = \left(\tfrac{1}{2}\right)x^2 + \left(\tfrac{1}{2}\right)x$$

$$F_1(x)I = 2\int_0^x \left(\tfrac{1}{2}\right)t^2 + \left(\tfrac{1}{2}\right)t\,dt = \left(\tfrac{1}{3}\right)x^3 + \left(\tfrac{1}{2}\right)x^2$$

$$f_2(x) = F_1(x) + xF_1(-1) = \left(\tfrac{1}{3}\right)x^3 + \left(\tfrac{1}{2}\right)x^2 + \left(\tfrac{1}{6}\right)x$$

$$F_2(x) = 3\int_0^x \left(\tfrac{1}{3}\right)t^3 + \left(\tfrac{1}{2}\right)t^2 + \left(\tfrac{1}{6}\right)t\,dt = \left(\tfrac{1}{4}\right)x^4 + \left(\tfrac{1}{2}\right)x^3 + \left(\tfrac{1}{4}\right)x^2$$

$$f_3(x) = F_2(x) + xF_2(-1) = \left(\tfrac{1}{4}\right)x^4 + \left(\tfrac{1}{2}\right)x^3 + \left(\tfrac{1}{4}\right)x^2$$

$$F_3(x) = 4\int_0^x \left(\tfrac{1}{4}\right)t^4 + \left(\tfrac{1}{2}\right)t^3 + \left(\tfrac{1}{4}\right)t^2\,dt = \left(\tfrac{1}{5}\right)x^5 + \left(\tfrac{1}{2}\right)x^4 + \left(\tfrac{1}{3}\right)x^3$$

$$f_4(x) = F_3(x) + xF_3(-1) = \left(\tfrac{1}{5}\right)x^5 + \left(\tfrac{1}{2}\right)x^4 + \left(\tfrac{1}{3}\right)x^3 - \left(\tfrac{1}{30}\right)x$$

$$F_4(x) = 5\int_0^x \left(\tfrac{1}{5}\right)t^5 + \left(\tfrac{1}{2}\right)t^4 + \left(\tfrac{1}{3}\right)t^3 - \left(\tfrac{1}{30}\right)t\,dt$$

$$= \left(\tfrac{1}{6}\right)x^6 + \left(\tfrac{1}{2}\right)x^5 + \left(\tfrac{5}{12}\right)x^4 - \left(\tfrac{1}{12}\right)x^2$$

$$f_5(x) = F_4(x) + xF_4(-1) = \left(\tfrac{1}{6}\right)x^6 + \left(\tfrac{1}{2}\right)x^5 + \left(\tfrac{5}{12}\right)x^4 - \left(\tfrac{1}{12}\right)x^2,$$

and so on. This gives the corresponding sums

$$S_0(n) = n$$

$$S_1(n) = \left(\tfrac{1}{2}\right)n^2 + \left(\tfrac{1}{2}\right)n = n(n+1)/2$$

$$S_2(n) = \left(\tfrac{1}{3}\right)n^3 + \left(\tfrac{1}{2}\right)n^2 + \left(\tfrac{1}{6}\right)n = n(n+1)(2n+1)/6$$

$$S_3(n) = \left(\tfrac{1}{4}\right)n^4 + \left(\tfrac{1}{2}\right)n^3 + \left(\tfrac{1}{4}\right)n^2 = [n(n+1)/2]^2$$

$$S_4(n) = \left(\tfrac{1}{5}\right)n^5 + \left(\tfrac{1}{2}\right)n^4 + \left(\tfrac{1}{3}\right)n^3 - \left(\tfrac{1}{30}\right)n = n(n+1)(2n+1)(3n^2+3n-1)/30$$

$$S_5(n) = \left(\tfrac{1}{6}\right)n^6 + \left(\tfrac{1}{2}\right)n^5 + \left(\tfrac{5}{12}\right)n^4 - \left(\tfrac{1}{12}\right)n^2 = n^2(n+1)^2(2n^2+2n-1)/12,$$

etc.

For a more efficient computer-assisted method to carry out the integrations in (8), write the polynomials $f_k(x)$ as power series and solve for a recursive relation between the coefficients of the powers of $x$. Thus, for

$$f_k(x)=\sum_{j=1}^{k+1} a_{k,j}x^j \quad \left(\text{or } S_k(n)=\sum_{j=1}^{k+1} a_{k,j}n^j\right), \tag{11}$$

equation (8) becomes

$$\sum_{j=1}^{k+2} a_{k+1,j}x^j=\sum_{j=1}^{k+1} a_{k,j}\left(\frac{k+1}{j+1}\right)x^{j+1}+C_k x$$
$$=\sum_{j=2}^{k+2} a_{k,j-1}\left(\frac{k+1}{j}\right)x^j+C_k x,$$

and equating powers of $x$ yields for each $k$ $(k=0,1,2,\dots)$:

$$a_{k+1,j}=\left(\frac{k+1}{j}\right)a_{k,j-1} \qquad (j=2,3,4,\dots,k+2),$$

or, alternately for each $k$ $(k=1,2,3,\dots)$:

$$a_{k,j}=\left(\frac{k}{j}\right)a_{k-1,j-1} \qquad (j=2,3,4,\dots,k+1). \tag{12}$$

Note, furthermore, that $a_{0,1}=1$ follows from (3) and (11). To obtain $a_{k,1}$ for $k\geq 1$, we can use (4) and (11) yielding

$$a_{k,1}=\sum_{j=2}^{k+1} a_{k,j}(-1)^j, \tag{13a}$$

or we can use (11) and the fact that $f_k(1)=S_k(1)=1^k=1$ to obtain

$$a_{k,1}=1-\sum_{j=2}^{k+1} a_{k,j}. \tag{13b}$$

The advantage of (13b) over (13a) is that it is not necessary to compute $(-1)^j$, which could be tedious on a small computer or programmable calculator.

A sample computer code using (12) and (13b) can be written to evaluate the coefficients $a_{k,j}$ in $S_k(n)=\sum_{j=1}^{k+1}a_{k,j}n^j$ for fixed $n$. The table below presents the results of such a code. Note that for $n>1$, the column under $j=1$ yields the Bernoulli numbers mentioned in this capsule's introduction.

| $k\backslash j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | |
| 1 | 1/2 | 1/2 | | | | | | |
| 2 | 1/6 | 1/2 | 1/3 | | | | | |
| 3 | 0 | 1/4 | 1/2 | 1/4 | | | | |
| 4 | −1/30 | 0 | 1/3 | 1/2 | 1/5 | | | |
| 5 | 0 | −1/12 | 0 | 5/12 | 1/2 | 1/6 | | |
| 6 | 1/42 | 0 | −1/6 | 0 | 1/2 | 1/2 | 1/7 | |
| 7 | 0 | 1/12 | 0 | −7/24 | 0 | 7/12 | 1/2 | 1/8 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |

*Acknowledgment.* I wish to thank C. Rorres for reviewing this manuscript and making me aware of the Levy article, and I would like to thank the Editor for his useful suggestions.

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