MA 274: Solution to 7.3.10

Theorem: Every countable union of countable sets is countable.

We begin by proving a lemma;

Lemma 1. A set X is countable if and only if there exists a surjection $f\colon \mathbb{N} \to X$.

Proof. If such a surjection exists, then X is countable by 7.3.5. Assume, therefore, that X is countable. We will prove that $\mathbb{N}$ surjects X.

X is countable if and only if it is finite or denumerable. If X is denumerable, there is a bijection $f\colon \mathbb{N} \to X$. This bijection is also a surjection, so if X is denumerable we are done.

Suppose that X is finite. By the definition of finite, there exists a bijection

$$g\colon \{1, \ldots, n\} \to X$$

for some $n \in \mathbb{N}$. Define $f\colon \mathbb{N} \to X$ by

$$f(m) = \begin{cases} g(x) & \text{if } x \in \{1, \ldots, n\} \\ g(n) & \text{if } x \notin \{1, \ldots, n\} \end{cases}$$

Since the range of g is a subset of the range of f, f is surjective. $\square$

Proof of 7.3.10. Let $\mathscr{A}$ be a countable set of sets, each of which is countable. We will prove that $B = \bigcup\limits_{A \in \mathscr{A}} A$ is countable.

Since $\mathscr{A}$ is countable, by the lemma there exists a surjection $f\colon \mathbb{N} \to \mathscr{A}$. Let $A_i = f(i)$. Since each A_i is countable, by the lemma, for each i, there exists a surjection $g_i\colon \mathbb{N} \to A_i$. Define the function $h\colon \mathbb{N} \times \mathbb{N} \to B$ by

$$h(i, j) = g_i(j).$$

(In other words, h takes the pair $(i, j) \in \mathbb{N} \times \mathbb{N}$ to the jth element of the ith set.)

Claim: h is a surjection.

Suppose that $x \in B$. By the definition of union, there exists $A \in \mathscr{A}$ such that $x \in A$. Since f is onto there exists $i \in \mathbb{N}$ such that $A = A_i = f(i)$. Since g_i is onto, there exists $j \in \mathbb{N}$ such that $g_i(j) = x$. By the definition of h, $h(i, j) = x$. $\square$(Claim)

In class we proved that there is a bijection $\phi\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Since bijections are also surjections and since the composition of surjections is a surjection, $h \circ \phi\colon \mathbb{N} \to B$ is a surjection. By the lemma, this implies that B is countable. $\square$