CSCI 6339 Theoretical Foundations of Computer Science
Homework 2
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Questions and Answers:

Problem 1. Let $N = \{1, 2, 3, \ldots\}$ be the set of all natural numbers. Prove $N \times N \times N = \{(x, y, z) \mid x, y, \text{ and } z \text{ are all in } N\}$ is countable.

Answer 1.

The easiest way to prove the countability is to find the one-to-one function. Here, the possible solution for $N \times N \times N \longrightarrow N$:

$$f(x, y, z) = 2^x 3^y 5^z$$

Since that for each tuple (x, y, z) the answer is unique prime factorisation, we get the one-to-one representation, **which proves the countability**

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Problem 2. Prove that the set of irrational numbers in $[0, 1]$ is not countable.

Answer 2.

To understand if the set of irrational numbers in $[0, 1]$ is not countable, we should know that the set of real numbers between $[0, 1]$ is not countable. Knowing that the real numbers contains both rational and irrational numbers, let's start to prove that the set of rational numbers in $[0, 1]$ is countable. It might be done by mapping, for example, $\frac{a}{b} \longrightarrow 2^a 3^b$ where $0 \leq \frac{a}{b} \leq 1$ and $b \neq 0$.

Now, we can prove that if the set of irrational numbers in $[0, 1]$ is not countable. If it were countable, the set of real numbers between $[0, 1]$ would be countable, because the union of countable sets is the countable set. Since it is not, **the set of irrational numbers in $[0, 1]$ is not countable.**

Problem 3. Show that there is a correspondence between the two intervals $[0, 1)$ and $[0, 1]$.

Answer 3.

Let's create one-to-one function $R \longrightarrow (0, 1)$ with the function $f(x) = \frac{e^x}{1+e^x}$ where $x \in R$. In this case, $|R| = |(0, 1)|$. Let's create the inverse map $(0, 1) \longrightarrow R$ with the function $-\log\left(-\frac{x-1}{x}\right)$ where $x \neq 0$ and $x \neq 1$. According to the Schröder–Bernstein theorem, if there exist two one-to-one function $f : A \longrightarrow B$ and $g : B \longrightarrow A$ between the sets A and B, there is a correspondence function $h : A \longrightarrow B$. Since that $(0, 1) \subset [0, 1) \subset [0, 1] \subset R$, **we can define that there is the correspondence function between $[0, 1)$ and $[0, 1]$, since all sets have the same cardinality.**

Problem 4. Show that the language $\{M : M$ is a Turing machine with $L(M)$ to be a finite set } is undecidable. You need to establish its connection to

$$A_{TM} = \{< M, w > \mid \text{Turing machine } M \text{ accepts input } w\}$$

Answer 4.

Let's define that M is decidable. In this case, there is the Turing machine M_2 such that

- accepts $\{0^n 1^n, n \geq 0\}$ if M does not accept some string w;
- accepts w if M results accept on w;

Then, define the Turing machine R such checks if $L(M_2)$ is regular or not; it is done if M accepts w. Finally, design the Turing machine A_{TM} such that accepts $< M, w >$ and

- gives "M accept w" if R accepts M_2;
- gives "M reject w" if R rejects M_2;

In this case, R will say if M_2 is a regular language if M accepts w and A_{TM} says "M accept w" if R decides M_2 is regular, which is contradiction.