Q1a) $A = \{1, 2, 3, 4\}$

$B = \{3k \mid k \in \mathbb{Z} - \{-1, 0, 1\}\}$

$= \{\dots, -9, -6, 0, 6, 9, \dots\}$

$C = \{n \in \mathbb{Z} \mid n^2 + n = 0\}$

$n^2 + n = 0$

$n(n+1) = 0$

$n = 0$ or $n = -1$

$C = \{0, -1\}$

i) $A \cap B = \{1, 2, 3, 4\} \cap \{\dots, -9, -6, 0, 6, 9, \dots\}$

$= \phi$ (True)

ii) $B - C = \{\dots, -9, -6, 0, 6, 9, \dots\} - \{0, -1\}$

$= \{\dots, -9, -6, 6, 9, \dots\}$

$= \{3p \mid p \in \mathbb{Z} - \{-1, 0, 1\}\} \neq \phi$ (False)

iii) $B \cap C = \{\dots, -9, -6, 0, 6, 9, \dots\} \cap \{0, -1\}$

$= \{0\} \neq 0$ (False)

iv) $A \cup C = \{1, 2, 3, 4\} \cup \{0, -1\}$

$= \{-1, 0, 1, 2, 3, 4\} \neq \phi$ (False)

b) let A, B be sets, and $A \subseteq B$

i) Proof: let $x \in A \cup B$ to show $A \cup B \subseteq B$

$\Rightarrow x \in A$ or $x \in B$

since $A \subseteq B$

$\Rightarrow x \in B$

in other case, $x \in B$

$\Rightarrow A \cup B \subseteq B$

~~let $x \in B$ then $x \in A \cup B$~~ to show $B \subseteq A \cup B$

let $x \in B$ then $x \in A \cup B$ by union definition

$\Rightarrow B \subseteq A \cup B$

$\because$ since $A \cup B \subseteq B$ and $B \subseteq A \cup B \Rightarrow A \cup B = B$ (shown)

ii) $A \cap B = A$

to show that $A \cap B \subseteq A$

let $x \in A \cap B$

$\Rightarrow x \in A$ and $x \in B$

~~$x \in A$~~ in particular, $x \in A$

$\therefore A \cap B \subseteq A$

to show that $A \subseteq A \cap B$

let $x \in A$ and since $A \subseteq B \Rightarrow x \in B$

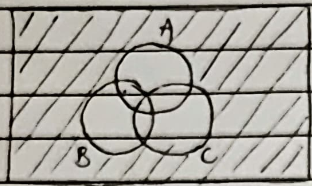
$\therefore x \in A$ and $x \in B \Rightarrow x \in A \cap B$

$\Rightarrow A \subseteq A \cap B$

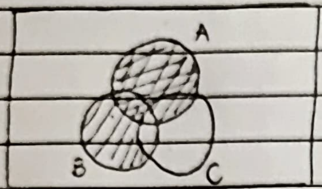
$\because$ since $A \cap B \subseteq A$ and $A \subseteq A \cap B$

$\Rightarrow A \cap B = A$ (shown)

Q1c) i) $\bar{A} \cap \bar{B} \cap \bar{C} = \overline{(A \cup B \cup C)}$ by DML



ii) $(A-B) \cup (A-C) \cup (B-C)$



d) $A \subseteq B \leftrightarrow \bar{B} \subseteq \bar{A}$

to show ~~if~~ $A \subseteq B \Rightarrow \bar{B} \subseteq \bar{A}$

let $x \in \bar{B} \Rightarrow x \notin B \Rightarrow x \notin A$ as $A \subseteq B$

$\Rightarrow x \in \bar{A}$

$\Rightarrow \bar{B} \subseteq \bar{A}$

to show if $\bar{B} \subseteq \bar{A} \Rightarrow A \subseteq B$

let $x \in A \Rightarrow x \notin \bar{A}$

$\Rightarrow x \notin \bar{B}$ $(\because \bar{B} \subseteq \bar{A})$

$\Rightarrow x \in B$

$\Rightarrow A \subseteq B$

Q2a) $f:\mathbb{R} \mapsto \mathbb{R}$

$g:\mathbb{R} \mapsto \mathbb{R}$

$f(x) = x^3+1$, $g(x) = \sqrt[3]{x-1}$

$= (x-1)^{1/3}$

$(g \circ f)(1) = g(f(1))$

$= g(2)$

$= (2-1)^{1/3}$

$= 1$

$(g \circ f)(2) = g(f(2))$

$= g(9)$

$= (9-1)^{1/3}$

$= 2$

$(g \circ f)(x) = g(f(x))$

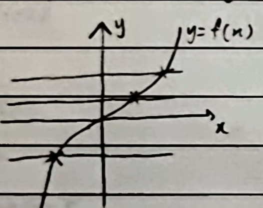
$= ([f(x)]-1)^{1/3}$

$= ((x^3+1)-1)^{1/3}$

$= (x^3)^{1/3}$

$= x$

b) (ii) $f(x) = x^3+x$ is a bijection for $\forall x \in \mathbb{R}$



(i) and (iii) are parabolas ∴ thus there is a horizontal lines that pass through more than once

(iv) has horizontal lines that pass through more than once

∴ (i), (iii) & (iv) are not a bijection

to show that $f(x)$ is one-to-one

if $f(a) = f(b)$

$a^3 + a = b^3 + b$

$a^3 - b^3 + a - b = 0$

$(a-b)(a^2+ab+b^2) + (a-b) = 0$

$(a-b)(a^2+ab+b^2+1) = 0$

$a - b = 0$ or $a^2+ab+b^2+1 = 0$ (rejected)

$a = b$ because $(a+\frac{1}{2}b)^2 + \frac{3}{4}b^2 + 1 \geq 1 \Rightarrow a^2+ab+b^2+1 \geq 1$ i.e. $a^2+ab+b^2+1 \neq 0$

to show that $f(x)$ is onto

let $y \in \mathbb{R}$

$y = f(x)$

$x^3 + x = y$

$x^3 + x - y = 0$

As the equation is a cubic equation, there is always at least 1 real root

ie there exists at least one $x \in \mathbb{R}$ such that $f(x) = y$

$\Rightarrow f(x)$ is onto

∴ $f(x)$ is a bijection

Q2 c) $2\log_9(\sqrt{x}) - \log_9(6x-1) = 0$

$\log_9(\sqrt{x})^2 - \log_9(6x-1) = 0$

$\log_9 x - \log_9(6x-1) = 0$

$\log_9\left(\frac{x}{6x-1}\right) = 0$

$\frac{x}{6x-1} = 9^0$

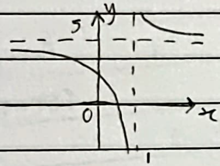
$\frac{x}{6x-1} = 1$

$x = 6x-1$

$5x = 1$

$x = 1/5$

d) $f: \mathbb{R} - \{1\} \longrightarrow \mathbb{R}$ $\quad f(x) = \frac{5x-3}{x-1}$



i) Image of $f: \mathbb{R} - \{5\}$

ii) $f: \mathbb{R} - \{1\} \longrightarrow \mathbb{R} - \{d\}$

$C_f = \mathbb{R} - \{d\} = R_f$

$R_f = \mathbb{R} - \{5\}$

$\Rightarrow d = 5$

iii) let $y = \frac{5x-3}{x-1}$

$y(x-1) = 5x-3$

$xy - y = 5x - 3$

$-5x + xy = y - 3$

$x(y-5) = y-3$

$x = \frac{y-3}{y-5}, \quad y \neq 5$

$\therefore f^{-1}: \mathbb{R} - \{5\} \to \mathbb{R} - \{1\}$ where $f^{-1}(x) = \left(\frac{x-3}{x-5}\right)$

Q3a) i)

p	q	r	$\neg p$	$\neg q$	$p \to q$	$r \to \neg p$	$(p \to q) \wedge (r \to \neg p) \wedge r$	$[(p \to q) \wedge (r \to \neg p) \wedge r] \to \neg q$
0	0	0	1	1	1	1	0	1
1	1	0	0	0	1	1	0	1
0	1	0	1	0	1	1	0	1
1	0	0	0	1	0	1	0	1
0	0	1	1	1	1	1	1	1
1	1	1	0	0	1	0	0	1
0	1	1	1	0	1	1	1	0
1	0	1	0	1	0	0	0	1

ii) it is not a tautology as not every entry is a 1

b) $p = r = F$, $q = s = T$

$((p \vee q) \wedge (q \vee s)) \to ((\neg r \vee p) \wedge (q \vee s))$

$((F \vee T) \wedge (T \vee T)) \to ((\neg F \vee F) \wedge (T \vee T))$

$(T \wedge T) \to (T \wedge T)$

$T \to T = T$

c) p: It's time for bed

q: It's after 9 p.m.

r: I'm tired

i) $p \to (q \wedge r)$

ii) $\neg q \to \neg r$

iii) $p \wedge q \leftrightarrow r$

d) $\forall x \in \mathbb{R}$, if $x(x+2) > 0$ then $x > 0$ or $x < -2$

contrapositive: $\forall x \in \mathbb{R}$, if $-2 \leq x \leq 0$, then $x(x+2) \leq 0$

converse: $\forall x \in \mathbb{R}$, if $x > 0$ or $x < -2$, then $x(x+2) > 0$

inverse: $\forall x \in \mathbb{R}$, if $x(x+2) \leq 0$, then $-2 \leq x \leq 0$

e) $(p \to (q \vee r)) \Leftrightarrow ((p \wedge \neg q) \to r)$

LHS $(p \to (q \vee r)) \equiv \neg p \vee (q \vee r)$ implication

$\equiv (\neg p \vee q) \vee r$ associative laws

$\equiv \neg(p \wedge \neg q) \vee r$ De Morgans Law

$\equiv (p \wedge \neg q) \to r$ implication

$=$ RHS (proven)

Q4 a) i) $(s \to d) \land \neg s \land d$

ii) $(d \lor n) \land (s \to \neg n)$

b) i) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{R} - \{0\}$ such that $xy < 1$

if $x > 0$: then choose any negative real number for y,
$xy < 1$

if $x < 0$: then choose any positive real number for y,
$xy < 1$

if $x = 0$: any $y \in \mathbb{R} - \{0\}$
$xy < 0$ as if $x = 0$, $xy = 0$

$\therefore \forall x \in \mathbb{Z}, \exists y \in \mathbb{R} - \{0\}$ is true such that $xy < 1$

ii) $\forall x \in \mathbb{R} - \{0\}, \forall y \in \mathbb{Z}$ such that $xy < 1$

if $x = 4$, $y = 6$
$xy = 24$
$24 > 1$ $\therefore xy \not< 1$

thus $\forall x \in \mathbb{R} - \{0\}, \forall y \in \mathbb{Z}$ such that $xy < 1$ is false

iii) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $xy = 1$

if $x = 0$, no $y \in \mathbb{R}$ will fulfill $xy = 1$
as $0 \cdot y = 0$

thus $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $xy = 1$ is false

c)

step	reason
1. p	premise
2. $p \lor q$	addition using (1)
3. $(p \lor q) \to \neg r$	premise
4. $\neg r$	Modus Ponens using (2) and (3)
5. $s \to r$	premise
6. $\neg s$	Modus Tollens using (4) and (5)
7. $\neg s \to (\neg q \to r)$	premise
8. $\neg q \to r$	Modus Ponens using (6) and (7)
9. q	Modus Tollens using (4) and (8)

$\therefore$ argument is valid

d) i) $\neg \exists x \in D (P(x) \land Q(x))$
$= \forall x \in D \; \neg (P(x) \land Q(x))$
$= \forall x \in D \; \neg P(x) \lor \neg Q(x)$ by De Morgan's Law

ii) $\neg \forall x \in D (P(x) \to Q(x))$
$= \neg \forall x \in D (\neg P(x) \lor Q(x))$ by Implication
$= \exists x \in D \; \neg (\neg P(x) \lor Q(x))$
$= \exists x \in D \; P(x) \land \neg Q(x)$ by Double ~~complement~~ negation and De Morgan's Law

Q5a) i) $\overline{(\overline{a\cdot b\cdot\bar{c}})+(\overline{\bar{c}\cdot d})}$

$=\overline{\overline{(a\cdot b\cdot\bar{c})}}\cdot\overline{\overline{(\bar{c}\cdot d)}}$ De Morgans Law

$=a\cdot b\cdot\bar{c}\cdot\bar{c}\cdot d$ Double complement

$=a\cdot b\cdot\bar{c}\cdot d$ Idempotent Law

ii) $\overline{\bar{a}+b}\cdot\overline{b+\bar{c}}\cdot\overline{\bar{c}+d}$

$\cancel{=(\bar{\bar{a}}\cdot\bar{b})\cdot(\bar{b}+\bar{c})+}$

$=(\bar{\bar{a}}\cdot\bar{b})\cdot(\bar{b}\cdot\bar{\bar{c}})\cdot(\bar{\bar{c}}\cdot\bar{d})$ De Morgans Law

$=(a\cdot\bar{b})\cdot(\bar{b}\cdot c)\cdot(c\cdot\bar{d})$ Double complement

$=a\cdot\bar{b}\cdot c\cdot\bar{d}$ Idempotent Law

b) i) $Q=\overline{\overline{A\cdot\bar{B}}\cdot\overline{\bar{C}\cdot D}}$

ii) $Q=\overline{\overline{A\cdot\bar{B}}}+\overline{\overline{\bar{C}\cdot D}}$ De Morgans Law

$=(A\cdot\bar{B})+(C\cdot\bar{D})$ De Morgans Law & Double complement

c) $a\cdot b+c\cdot\bar{d}=(a+c)\cdot(a+\bar{d})\cdot(b+c)\cdot(b+\bar{d})$

$(a+b)\cdot(c+\bar{d})=(a\cdot c)+(a\cdot\bar{d})+(b\cdot c)+(b\cdot\bar{d})$

d) $F(a,b,c,d)=(\bar{a}+b\cdot\bar{d})\cdot(c\cdot b\cdot a+\bar{c}\cdot d)$

i)

a	b	c	d	$\bar{a}$	$\bar{d}$	$(b\cdot\bar{d})$	$(c\cdot b\cdot a)$	$\bar{c}$	$(\bar{c}\cdot d)$	$(\bar{a}+b\cdot\bar{d})$	$(c\cdot b\cdot a+\bar{c}\cdot d)$	$(\bar{a}+b\cdot\bar{d})\cdot(c\cdot b\cdot a+\bar{c}\cdot d)$
1	0	0	0	0	1	0	0	1	0	0	0	0
0	1	0	0	1	1	1	0	1	0	1	0	0
1	1	0	0	0	1	1	0	1	0	1	0	0
0	0	0	0	1	1	0	0	1	0	1	0	0
1	0	1	0	0	1	0	0	0	0	0	0	0
0	1	1	0	1	1	1	0	0	0	1	0	0
1	1	1	0	0	1	1	1	0	0	1	1	1
0	0	1	0	1	1	0	0	0	0	1	0	0
1	0	0	1	0	0	0	0	1	1	0	1	0
0	1	0	1	1	0	0	0	1	1	1	1	1
1	1	0	1	0	0	0	0	1	1	0	1	0
0	0	0	1	1	0	0	0	1	1	1	1	1
1	0	1	1	0	0	0	0	0	0	0	0	0
0	1	1	1	1	0	0	0	0	0	1	0	0
1	1	1	1	0	0	0	1	0	0	0	1	0
0	0	1	1	1	0	0	0	0	0	1	0	0

$F(a,b,c,d)=a\cdot b\cdot c\cdot\bar{d}+\bar{a}\cdot b\cdot\bar{c}\cdot d+\bar{a}\cdot\bar{b}\cdot\bar{c}d$ (SOP)

ii)

ab \ cd	00	01	11	10
00	0	1	0	0
01	0	1	0	0
11	0	0	0	$\boxed{1}$
10	0	0	0	0

iii) $F(a,b,c,d)=\bar{a}\cdot\bar{c}\cdot d+a\cdot b\cdot c\cdot\bar{d}$