$\boxed{\text{Ex 1.1}}$ 1) $f_1(t)$

a) • $f_1(t)$ est continue par morceaux et dérivable par morceaux car $f_1(t)$ est une fct constante par morceaux

• $f_1(t)$ discontinue en 0 et π mais la discontinuité est de 1ère espèce

car $\lim\limits_{t \to 0^+} f_1(t) = 1$ et $\lim\limits_{t \to 0^-} f_1(t) = -1$

$\lim\limits_{t \to \pi^+} f_1(t) = -1$ et $\lim\limits_{t \to \pi^-} f_1(t) = 1$

limites finies !

• $f_1'(t) = 0 \quad \forall t$ donc $f_1'(t)$ continue

Conclusion $f_1(t)$ est bien de classe $\mathcal{C}^1$ par morceaux.

3). a_0 et a_n sont nuls car f_1 fct impaire 2/7

$$b_n = \frac{\cancel{2}}{\cancel{2}\pi}\int_{-\pi}^{\pi} f_1(t)\sin(nt)\,dt = \frac{2}{\pi}\int_0^{\pi} 1.\sin(nt)\,dt$$

$$= \frac{2}{\pi n}\Big[-\cos(nt)\Big]_0^{\pi} = \frac{2}{\pi n}\Big(1-\underbrace{\cos(n\pi)}_{(-1)^n}\Big)$$

$$\boxed{b_n = \frac{2}{\pi n}\left(1-(-1)^n\right)}$$

$n=2k \Rightarrow b_{2k} = 0$

$n=2k+1 \Rightarrow b_{2k+1} = \frac{4}{\pi(2k+1)}$

Ex1.1

f_1

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$$a_n = 0 \quad \forall n \geq 0$$

$$b_n = \frac{2}{n\pi}\left(1-(-1)^n\right) \quad \forall n \geq 1$$

$$\begin{cases} b_{2k} = 0 & k \geq 0 \\ b_{2k+1} = \dfrac{4}{\pi(2k+1)} & k \geq 0 \end{cases}$$

Ex 1.1 (Suite)

~~3~~ 4) $S_1(t) = \frac{4}{\pi} \sum_{p=0}^{+\infty} \frac{1}{2p+1} \sin[(2p+1)t]$

4) pour tout $t \neq k\pi$ où $k \in \mathbb{Z}$ $\quad f(t) = S_1(t)$

5) Si $t = t_k = k\pi$, $k \in \mathbb{Z}$ $\quad \frac{1}{2}\left[f(t_k^+) + f(t_k^-)\right] = S_1(t_k)$

or $\forall\, t_k$ $\quad f(t_k^+) + f(t_k^-) = 0$ pour $k \in \mathbb{Z}$

donc $\frac{4}{\pi} \sum_{p=1}^{+\infty} \frac{1}{2p+1} \sin[(2p+1)k\pi] = 0$ Vrai pour tout $k \in \mathbb{Z}$

car $\sin(k\pi) = 0$

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$$S(t) = \frac{4}{\pi} \sum \frac{1}{2k+1} \cdot \sin\left[(2k+1)t\right]$$

$S(t) = f(t)$ si f continue en t càd
$t \neq t_k = k\pi \quad k \in \mathbb{Z}$

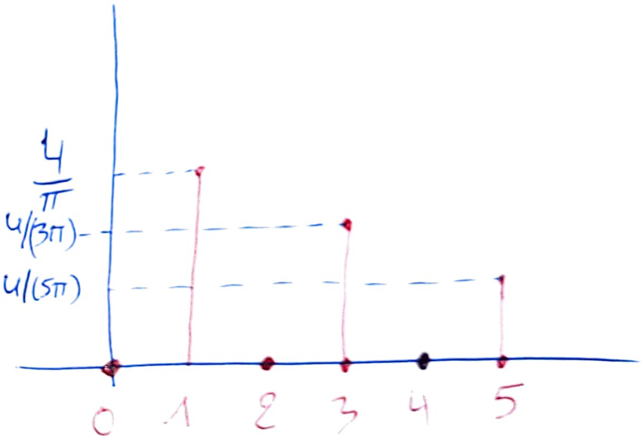
si $t_k = k\pi \quad f(t_k^-) = -f(t_k^+)$

$$\frac{1}{2}\left(f(t_k^-) + f(t_k^+)\right) = 0 = \frac{4}{\pi} \sum \frac{1}{2k+1} \underbrace{\sin\left[(2k+1)k\pi\right]}_{=0}$$

Relation vraie

6) $A_n = \sqrt{a_n^2 + b_n^2} \qquad A_{2p} = 0 \quad \forall p \in \mathbb{N}$

$$A_{2p+1} = |b_{2p+1}| = \frac{4}{\pi(2p+1)} \quad \forall p \in \mathbb{N}$$



7) Parseval :

$$\frac{1}{2\pi}\int_0^{2\pi} [f(t)]^2 \, dt = a_0^2 + \sum_{n \geq 1} \frac{a_n^2 + b_n^2}{2}$$

$$\frac{1}{2\pi}\int_0^{\pi} (f(t))^2 \, dt + \frac{1}{2\pi}\int_{\pi}^{2\pi} (f(t))^2 \, dt$$

$$= \frac{1}{2\pi}\left(\int_0^{\pi} dt + \int_{\pi}^{2\pi} dt\right) = \frac{1}{2\pi}\left[t\right]_0^{2\pi} = \frac{2\pi}{2\pi} = 1$$

donc

$$\frac{1}{2}\sum_{p \geq 0} \frac{1}{(2p+1)^2} \cdot \left(\frac{4}{\pi}\right)^2 = 1 \Rightarrow \boxed{\sum_{p \geq 0} \frac{1}{(2p+1)^2} = \frac{\pi^2}{8}}$$

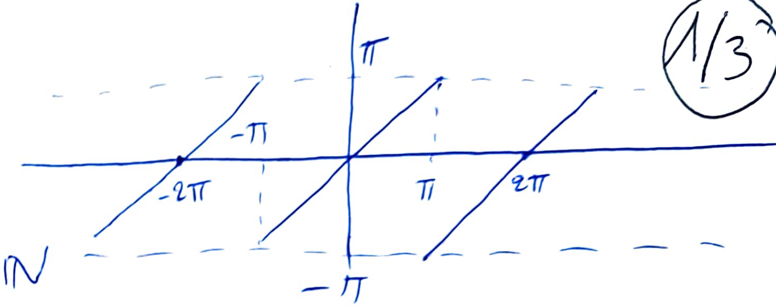
Ex 1.1 Suite

$$\sum_{n\geq 1}\frac{1}{n^2}=\sum_{p\geq 1}\frac{1}{(2p)^2}+\sum_{p\geq 0}\frac{1}{(2p+1)^2}$$

$$=\frac{1}{4}\sum_{p\geq 1}\frac{1}{p^2}+\frac{\pi^2}{8} \quad \text{or} \quad \sum_{p\geq 1}\frac{1}{p^2}=\sum_{n\geq 1}\frac{1}{n^2}$$

donc $\displaystyle \frac{3}{4}\sum_{n\geq 1}\frac{1}{n^2}=\frac{\pi^2}{8} \Longrightarrow \boxed{\sum_{n\geq 1}\frac{1}{n^2}=\frac{\pi^2}{6}}$

Ex 1.1 (2) $f_2(x) = t$ si $t \in [-\pi, \pi[$, 2π



f_2 impaire donc $a_n = 0$ pour $\forall n \in \mathbb{N}$

$n \neq 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t \sin(nt)\,dt = \frac{2}{\pi}\cdot\int_0^{\pi} t \sin(nt)\,dt \quad \text{car } \boxed{t\sin(nt) \text{ paire}}$$

$u = t \quad v' = \sin(nt)$

$u' = 1 \quad v = -\frac{1}{n}\cos(nt)$

$$\frac{\pi}{2} b_n = \left[-\frac{t}{n}\cos(nt)\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos(nt)\,dt$$

$$= \frac{-\pi\cos(n\pi)}{n} + \frac{1}{n^2}\underbrace{\left[\sin(nt)\right]_0^{\pi}}_{=0} \quad \text{donc } \frac{\pi}{2} b_n = \frac{-\pi(-1)^n}{n}$$

$$\boxed{b_n = (-1)^{n+1}\frac{2}{n}} \quad \forall\, n \geq 1$$

4) donc $\boxed{S_2(t) = 2\sum_{n\geq 1} \frac{(-1)^{n+1}}{n}\sin(nt)}$

d'où $\forall\ t \neq (2k+1)\pi = t_k$

$f_2(t) = S_2(t)$

si $t = t_k = (2k+1)\pi$

$\frac{1}{2}\left[f_2(t_k^+) + f_2(t_k^-)\right] = S_2(t_k)$

(2/3)

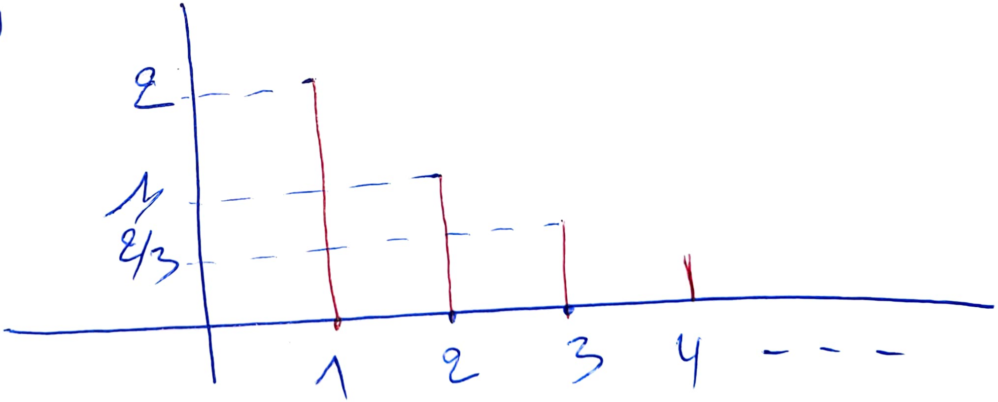
5) Si $t_k = (2k+1)\pi$ $\frac{1}{2}\left[f_2(t_k^+) + f_2(t_k^-)\right] = 0$ $(-1+1)$ ou $(1-1) = 0$

Si $t_k = 2k\pi$ $f_2(t_k) = 0$ continue en $t_k = 2k\pi$

donc on a $2 \sum_{n \geqslant 1} \frac{(-1)^{n+1}}{n} \sin(n t_k) = 0 \quad \forall k$

Vrai.

6) $A_n = |b_n| = \frac{2}{n} \quad \forall n \geq 1$



7) Parseval :

$$I = \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(t)]^2 dt = \frac{1}{2}\sum_{n\geq 1} b_n^2 = \frac{1}{2}\sum_{n\geq 1}\frac{4}{n^2} = 2\sum_{n\geq 1}\frac{1}{n^2}$$

$$I = \frac{1}{2\pi}\left(\int_0^{\pi} t^2 dt + \int_{-\pi}^{0} t^2 dt\right) = \frac{\cancel{2}}{\cancel{2}\pi}\int_0^{\pi} t^2 dt \quad \text{car } t^2 \text{ fct paire}$$

$$= \frac{1}{\pi}\left[\frac{1}{3}t^3\right]_0^{\pi} = \frac{\pi^3}{3\pi} = \frac{\pi^2}{3} \quad \text{donc } 2\sum\frac{1}{n^2} = \frac{\pi^2}{3}$$

$$\Longrightarrow \boxed{\sum\frac{1}{n^2} = \frac{\pi^2}{6}}$$

Ex 1.1 (3) $f_3(t) = t^2$ si $t \in [-\pi, \pi[$ $\quad$ f paire

3) $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{\cancel{2}}{\cancel{2}\pi}\int_0^{\pi} t^2\,dt = \frac{1}{\pi}\left[\frac{1}{3}t^3\right]_0^{\pi} = \frac{\pi^3}{3\pi} = \frac{\pi^2}{3}$

$n \geq 1$

$a_n = \frac{\cancel{2}}{\cancel{2}\pi}\int_{-\pi}^{\pi} t^2\cos(nt)\,dt = \frac{2}{\pi}\int_0^{\pi} t^2\cos(nt)\,dt$ (car $t^2\cos(nt)$ fct paire)

$\begin{cases} u = t^2 & v' = \cos(nt) \\ u' = 2t & v = \frac{1}{n}\sin(nt) \end{cases}$

$\frac{\pi}{2}a_n = \cancel{\left[\frac{1}{n}t^2\sin(nt)\right]_0^{\pi}} - \frac{2}{n}\int_0^{\pi} t\sin(nt)\,dt$

$\begin{cases} u = t & v' = \sin(nt) \\ u' = 1 & v = -\frac{1}{n}\cos(nt) \end{cases}$

$= \frac{-2}{n}\left\{\left[-\frac{1}{n}t\cos(nt)\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nt)\,dt\right\}$

$= \frac{2}{n^2}\cdot\pi\cos(n\pi) - \frac{2}{n^2}\cancel{\left[\frac{1}{n}\sin(nt)\right]_0^{\pi}} = \frac{2\pi(-1)^n}{n^2}$

donc $\boxed{a_n = (-1)^n\frac{4}{n^2}}$ donc $\boxed{S_3(t) = \frac{\pi^2}{3} + 4\sum\frac{(-1)^n}{n^2}\cos(nt)}$

4) f_3 continue donc $f_3(t) = t^2 = \frac{\pi^2}{3} + 4\sum_{n\geq 1} \frac{(-1)^n}{n^2} \cos(nt) \quad \forall t \in [-\pi, \pi]$

5) Si $t = \pi$ on a $\pi^2 = \frac{\pi^2}{3} + 4\sum_{n\geq 1} \frac{(-1)^n}{n^2} \cos(n\pi)$ or $\cos(n\pi) = (-1)^n$

donc $4\sum_{n\geq 1} \frac{1}{n^2} = \pi^2 - \frac{\pi^2}{3} = \frac{2\pi^2}{3} \Rightarrow \boxed{\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}}$

Si $t = 0$ on a $0 = \frac{\pi^2}{3} + 4\sum_{n\geq 1} \frac{(-1)^n}{n^2}$

$\Rightarrow \sum_{n\geq 1} \frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}$ ou encore $\boxed{\sum_{n\geq 1} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}}$

<u>Càd</u> : $\boxed{1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} \cdots + \frac{(-1)^{n-1}}{n^2} + \cdots = \frac{\pi^2}{12}}$

<u>Parseval :</u> $\frac{1}{2\pi}\int_{-\pi}^{\pi} t^4 dt = \frac{\pi^4}{9} + \frac{1}{2}\sum_{n\geq 1}\frac{16}{n^4}$

$$\frac{2}{2\pi}\int_0^{\pi} t^4 dt = \frac{\pi^4}{9} + \frac{16}{2}\sum \frac{1}{n^4}$$

$$\frac{1}{5\pi}\left[t^5\right]_0^{\pi} = \frac{\pi^4}{9} + 8\sum\frac{1}{n^4} = \frac{1}{5}\pi^4$$

donc $8\sum_{n\geq 1}\frac{1}{n^4} = \frac{\pi^4}{5} - \frac{\pi^4}{9} = \frac{9\pi^4 - 5\pi^4}{45} = \frac{4\pi^4}{45}$

Conclusion $\boxed{\sum_{n\geq 1}\frac{1}{n^4} = \frac{\pi^4}{90}}$.