ω-CONTINUOUS SEMIRINGS, ALGEBRAIC SYSTEMS AND PUSHDOWN AUTOMATA

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ABSTRACT. In Kuich [4] we generalized the Kleene and the Parikh Theorem to l-complete semirings whose natural limit function is compatible with their partial order. In this paper we generalize in the same manner the following language theoretic result: A language is context-free iff it is accepted by a pushdown automaton.

1. CONTINUOUS MONOIDS AND SEMIRINGS

The reader is assumed to be familiar with the theory of semirings, automata and algebraic systems as developed in Kuich, Salomaa [5].

A monoid $\langle A,+,0\rangle$ is called *partially ordered* iff

(i) A is partially ordered as a set,

(ii) A is commutative,

(iii) $0 \le a$ holds for all $a \in A$,

(iv) $a_1 \le a_2$ implies $a_1 + a \le a_2 + a$ for all $a, a_1, a_2 \in A$.

A semiring $\langle A,+,\cdot,0,1\rangle$ is called *partially ordered* iff the monoid $\langle A,+,0\rangle$ is partially ordered and, additionally, for all $a, a_1, a_2 \in A$,

$$a_1 \le a_2 \text{ implies } a_1 \cdot a \le a_2 \cdot a \text{ and } a \cdot a_1 \le a \cdot a_2 .$$

A monoid (resp. semiring) A is called *naturally ordered* iff A is partially ordered by the following relation $\le$:

$$a \le b \textit{ iff there exists a } c \in A \textit{ such that } a + c = b.$$

A sequence $(a_i \mid i \in \mathbb{N})$ of elements of a partially ordered set A is called a *chain* (in A) iff, for all $i \in \mathbb{N}$, $a_i \le a_{i+1}$. The *supremum* (least upper bound) $\sup S$ of a set $S = \{a_i \mid i \in I\}$ or a sequence $S = (a_i \mid i \in I)$ of elements of a partially ordered set A is defined by

(i) $a_i \le \sup S$, for all $i \in I$,

(ii) $a_i \le c$ for all $i \in I$ implies $\sup S \le c$.

If the supremum exists, it is unique.

A *completely partially ordered* set is a partially ordered set with a least element, such that for each chain the supremum exists. A partially ordered monoid (resp. partially ordered semiring) A is called *completely partially ordered* iff for each chain in A the supremum exists.

A commutative monoid A is called *complete* iff infinite sums of elements of A can be defined such that the order of summation is irrelevant. A semiring $\langle A,+,\cdot,0,1\rangle$ is called *complete* iff $\langle A,+,0\rangle$ is a complete monoid and the distribution laws hold also for infinite sums (see Hebisch [2]). We use the term *partially ordered complete monoid* (resp. *partially ordered complete semiring*) to denote a partially ordered monoid (resp. partially ordered semiring) that is complete.

The next definition is from Goldstern [1]. A partially ordered complete monoid (resp. partially ordered complete semiring) A is called *finitary* iff, for all index sets I, all sequences $(a_i \mid i\in I)$ in A and all $c\in A$, the following condition is satisfied:

$$\text{If } \sum_{i\in E} a_i \le c \text{ for all finite } E\subseteq I \text{ then } \sum_{i\in I} a_i \le c\,.$$

If this condition is satisfied for a naturally ordered complete monoid (resp. naturally ordered complete semiring), this monoid (resp. semiring) is called *continuous* (see Sakarovitch [8]). For example, the power set of any monoid is a continuous semiring.

A partially ordered complete monoid (resp. partially ordered complete semiring) A is called *ω-finitary* iff, for all sequences $(a_i \mid i\in\mathbb{N})$ in A and all $c\in A$, the following condition is satisfied:

$$\text{If } \sum_{0\le i\le n} a_i \le c \text{ for all } n\in\mathbb{N} \text{ then } \sum_{i\in\mathbb{N}} a_i \le c\,.$$

If this condition is satisfied for a naturally ordered complete monoid (resp. naturally ordered complete semiring), this monoid (resp. semiring) is called *ω-continuous*. The ω-continuous semirings are exactly the l-complete semirings of Kuich [4] whose natural limit function is compatible with their partial order (see Theorem 2 (iii)). Clearly, each continuous semiring is ω-continuous. The converse is not true.

The next two theorems are due to Karner [3].

THEOREM 1. *Let A be a naturally ordered complete monoid (resp. naturally ordered complete semiring). Then the following statements are equivalent:*

(i) *A is continuous;*

(ii) $\sup\{\sum_{i\in E} a_i \mid E\subseteq I \text{ finite}\}$ *exists for all sequences $(a_i\mid i\in I)$ of elements of A and is equal to* $\sum_{i\in I} a_i$. □

THEOREM 2. *Let A be a naturally ordered complete monoid (resp. naturally ordered complete semiring). Then the following statements are equivalent:*

(i) *A is ω-continuous;*

(ii) $\sup(\sum_{0\le i\le n} a_i \mid n\in\mathbb{N})$ *exists for all sequences $(a_i\mid i\in\mathbb{N})$ of elements of A and is equal to* $\sum_{i\in\mathbb{N}} a_i$;

(iii) *if* $\sum_{0\le i\le n} a_i \le \sum_{0\le i\le n} b_i$ *for all $n\in\mathbb{N}$ then* $\sum_{i\in\mathbb{N}} a_i \le \sum_{i\in\mathbb{N}} b_i$, *for all sequences $(a_i\mid i\in\mathbb{N})$ and $(b_i\mid i\in\mathbb{N})$ of elements of A.*

Each of these statements implies

(iv) *if* $\sum_{0\le i\le n} a_i = \sum_{0\le i\le n} b_i$ *for all $n\in\mathbb{N}$ then* $\sum_{i\in\mathbb{N}} a_i = \sum_{i\in\mathbb{N}} b_i$, *for all sequences $(a_i\mid i\in\mathbb{N})$ and $(b_i\mid i\in\mathbb{N})$ of elements of A.*

Proof. We have only to prove that statements (i) and (iii) are equivalent.

Assume (i) and $\sum_{0\le i\le n} a_i \le \sum_{0\le i\le n} b_i$. This implies $\sum_{0\le i\le n} a_i \le \sum_{i\in\mathbb{N}} b_i$. Since A is ω-continuous, we obtain $\sum_{i\in\mathbb{N}} a_i \le \sum_{i\in\mathbb{N}} b_i$.

Assume (iii) and $\sum_{0\le i\le n} a_i \le c$ for all $n\in\mathbb{N}$ and some $c\in A$. Consider the sequence $(b_i\mid i\in\mathbb{N})$, where $b_0 = c$ and $b_i=0$, $i>0$. Then, by (iii), $\sum_{i\in\mathbb{N}} a_i \le c$ and A ist ω-continuous. □

Semirings that satisfy condition (iv) of Theorem 2 are exactly the l-complete semirings of Kuich [4].

EXAMPLE. We consider the semiring $\langle \mathbb{R}_+^{(\infty)},+,\cdot,0,1\rangle$, where $\mathbb{R}_+^{(\infty)} = \{a \in \mathbb{R} \mid a \geq 0\} \cup \{\infty\}$, with the obvious extension of the operations to ∞ (observe $0\cdot\infty = \infty\cdot 0 = 0$).

The definition of infinite sums is possible in two different ways for $a_i \in \mathbb{R}_+^{(\infty)}$, $i \in I$:

(1) $\sum_{i\in I} a_i = \sup\{\sum_{i\in E} a_i \mid E \subseteq I \text{ finite}\}$. For example $\sum_{i\in\mathbb{N}} \frac{1}{2^i} = 2$.

(2) $\sum_{i\in I} a_i = \infty$ iff infinitely many of the a_i's are unequal to 0 or at least one a_i is ∞. For example, $\sum_{i\in\mathbb{N}} \frac{1}{2^i} = \infty$.

Clearly, the semiring $\mathbb{R}_+^{(\infty)}$ with infinite sums according to (1) (resp. (2)) is continuous and, hence, ω-continuous (resp. neither continuous nor ω-continuous). □

Let $C = (a_i \mid i \in \mathbb{N})$ be a chain in an ω-continuous monoid (resp. ω-continuous semiring) A. Since A is naturally ordered, there exist $b_i \in A$, $i \in \mathbb{N}$, such that

$$a_0 = b_0,\ a_{i+1} = a_i + b_{i+1},\ i \in \mathbb{N}.$$

This implies $a_i = \sum_{0\leq j\leq i} b_j$ for all $i \in \mathbb{N}$ and we obtain, by Theorem 2 (ii), $\sup C = \sum_{i\in\mathbb{N}} b_i$. By Theorem 2 (iv), the choice of the b_i's, $i \in \mathbb{N}$, is irrelevant. Thus we have shown the next theorem.

THEOREM 3. *An ω-continuous monoid (resp. ω-continuous semiring) is a completely naturally ordered monoid (resp. completely naturally ordered semiring).* □

We now consider matrices and formal power series.

THEOREM 4. *Let A be an ω-continuous monoid, J and J' be index sets and Σ be an alphabet. Then $A^{J\times J'}$ and $A\langle\langle\Sigma^*\rangle\rangle$ are again ω-continuous monoids.*

Proof. Firstly, assume $\sum_{0\leq i\leq n} M_i \leq M$ for all $n \in \mathbb{N}$, where $M, M_i \in A^{J\times J'}$, $i \in \mathbb{N}$. This implies, for all $j \in J$ and $j' \in J'$,

$$(\sum_{0\leq i\leq n} M_i)_{j,j'} = \sum_{0\leq i\leq n} (M_i)_{j,j'} \leq M_{j,j'}.$$

Since A is ω-continuous, we get

$$\sum_{i\in\mathbb{N}} (M_i)_{j,j'} \leq M_{j,j'}.$$

Hence, $\sum_{i\in\mathbb{N}} M_i \leq M$ and $A^{J\times J'}$ is ω-continuous.

Secondly, assume $\sum_{0\leq i\leq n} r_i \leq r$ for all $n \in \mathbb{N}$, where $r, r_i \in A\langle\langle\Sigma^*\rangle\rangle$, $i \in I$. This implies, for all $w \in \Sigma^*$,

$$(\sum_{0\leq i\leq n} r_i, w) = \sum_{0\leq i\leq n} (r_i, w) \leq (r, w).$$

Since A is ω-continuous, we get

$$\sum_{i\in\mathbb{N}} (r_i, w) \leq (r, w).$$

Hence, $\sum_{i\in\mathbb{N}} r_i \leq r$ and $A\langle\langle\Sigma^*\rangle\rangle$ is ω-continuous. □

THEOREM 5. *Let A be an ω-continuous monoid, J and J' be index sets and Σ be an alphabet. Let $(M_i \mid i \in \mathbb{N})$ be a chain in $A^{J\times J'}$ and $(r_i \mid i \in \mathbb{N})$ be a chain in $A\langle\langle\Sigma^*\rangle\rangle$. Then*

$$\sup(M_i \mid i \in \mathbb{N})_{j,j'} = \sup((M_i)_{j,j'} \mid i \in \mathbb{N}) \text{ for all } j \in J,\ j' \in J'$$

and

$$(\sup(r_i \mid i \in \mathbb{N}), w) = \sup((r_i, w) \mid i \in \mathbb{N}) \text{ for all } w \in \Sigma^*.$$

Proof. We only prove the first equality. The proof of the second equality is similar.

Since A is naturally ordered, there exist $M_i' \in A^{J \times J'}$, $i \in \mathbb{N}$, such that

$$M_0 = M_0', \ M_{i+1} = M_i + M_{i+1}', \ i \in \mathbb{N}.$$

This implies $M_i = \sum_{0 \le j \le i} M_j'$ for all $i \in \mathbb{N}$. As in the proof of Theorem 3, we get

$$\sup (M_i \mid i \in \mathbb{N}) = \sum_{i \in \mathbb{N}} M_i'$$

and

$$\sup ((M_i)_{j,j'} \mid i \in \mathbb{N}) = \sum_{i \in \mathbb{N}} (M_i')_{j,j'}, \text{ for all } j \in J, \ j' \in J'.$$

Taking the (j,j')-entries of the first equality proves the theorem. ☐

Let A and A' be partially ordered sets. A mapping $f : A \to A'$ is called *monotonic* iff $a \le b$ in A implies $f(a) \le f(b)$ in A'.

Let A and A' be completely partially ordered sets. A mapping $f : A \to A'$ is called *continuous* iff it is monotonic and, for every chain $(a_i \mid i \in \mathbb{N})$ in A,

$$f(\sup (a_i \mid i \in \mathbb{N})) = \sup (f(a_i) \mid i \in \mathbb{N}).$$

For the proof of Theorem 6 in the next section we quote Theorem 4.2 of Kuich [4], which we rephrase to fit to our definitions:

Let A be an ω-continuous semiring and let $(a_i \mid i \in \mathbb{N})$ and $(b_i \mid i \in \mathbb{N})$ be chains in A. Then $(a_i b_i \mid i \in \mathbb{N})$ is a chain and $\sup (a_i b_i \mid i \in \mathbb{N}) = \sup (a_i \mid i \in \mathbb{N}) \sup (b_i \mid i \in \mathbb{N})$. *Hence, multiplication $\cdot : A \times A \to A$ is a continuous mapping. (This is true also for addition.)*

Another theorem which we will need in the next section is the *Fixpoint Theorem* (see Loeckx, Sieber [7], Theorem 4.24):

Let A be a completely partially ordered set with least element 0, *and let $f : A \to A$ be a continuous function. Then f has a least fixpoint $\mu f \in A$, and, in fact,*

$$\mu f = \sup (f^i(0) \mid i \in \mathbb{N}).$$

2. ALGEBRAIC SYSTEMS AND PUSHDOWN AUTOMATA

In the sequel, A denotes a semiring, A' a subset of A containing 0 and 1, and $Y = \{y_1, \ldots, y_n\}$ an alphabet of variables. By $A(Y)$ we denote the polynomial algebra over the semiring A with variables in Y (see Lausch, Nöbauer [6], Chapter 1, 4.1 and 4.4). The elements of $A(Y)$ are called *polynomials*. Each polynomial can be represented by a finite sum of *product terms*. Here, a product term t has the form

$$t(y_1, \ldots, y_n) = a_0 y_{i_1} \ldots y_{i_k} a_k, \ a_j \in A, \ k \ge 0, \tag{1}$$

and a polynomial has the form

$$p(y_1, \ldots, y_n) = \sum_{j=1}^{m} t_j(y_1, \ldots, y_n). \tag{2}$$

The set of all polynomials (2) that are sums of product terms (1) in which the a_j's are in A' is denoted by $A'(Y)$.

In the usual way, a polynomial in $A(Y)$ defines a mapping from A^n into A:

$$t(b_1, \ldots, b_n) = a_0 b_{i_1} \ldots b_{i_k} a_k, \ t \text{ as in (1)}, \ b_j \in A,$$

$$p(b_1, \ldots, b_n) = \sum_{j=1}^{m} t_j(b_1, \ldots, b_n), \ p \text{ as in (2)}, \ b_j \in A.$$

THEOREM 6. *Let A be an ω-continuous semiring and let $p(y_1,\ldots,y_n)$ be a polynomial in $A(Y)$. Then the mapping $p : A^n \to A$ is continuous.*

Proof. Let p be as in (2) and $((b_{1k},\ldots,b_{nk}) \mid k\in\mathbb{N})$ be a chain in A^n. Then $C_i = (b_{ik} \mid k\in\mathbb{N})$, $1\le i\le n$, $(t_j(b_{1k},\ldots,b_{nk}) \mid k\in\mathbb{N})$, $1\le j\le m$, and $(\sum_{j=1}^{m} t_j(b_{1k},\ldots,b_{nk}) \mid k\in\mathbb{N})$ are chains in A and we obtain

$$p(\sup((b_{1k},\ldots,b_{nk}) \mid k\in\mathbb{N})) = p(\sup C_1,\ldots,\sup C_n) = \sum_{j=1}^{m} t_j(\sup C_1,\ldots,\sup C_n) =$$
$$= \sum_{j=1}^{m} \sup(t_j(b_{1k},\ldots,b_{nk}) \mid k\in\mathbb{N}) = \sup(\sum_{j=1}^{m} t_j(b_{1k},\ldots,b_{nk}) \mid k\in\mathbb{N}) = \sup(p(b_{1k},\ldots,b_{nk}) \mid k\in\mathbb{N}).$$

Here, the third equality holds by Theorem 4.2 of Kuich [4]. □

Let be $p(y_1,\ldots,y_n)\in A(Y)^{n\times 1}$. Then p defines a mapping $p : A^n \to A^n$ by

$$(p(b_1,\ldots,b_n))_i = p_i(b_1,\ldots,b_n),\quad 1\le i\le n,$$

i.e., the i-th component of the application of p is given by applying the i-th component of p.

COROLLARY 7. *Let A be an ω-continuous semiring and let $p\in A(Y)^{n\times 1}$. Then the mapping $p : A^n \to A^n$ is continuous.* □

By the Substitution Principle (see Lausch, Nöbauer [6], Chapter 1, 6.31) the mapping $s : A(Y) \to A$, $s=(s_1,\ldots,s_n)\in A^n$, defined by

$$s(p) = p(s_1,\ldots,s_n),$$

is a semiring morphism.

An *A'-algebraic system with variables in Y* is a family of $n\ge 1$ equations of the form

$$y_i = p_i(y_1,\ldots,y_n),\quad 1\le i\le n, \tag{3}$$

where the p_i are polynomials in $A'(Y)$. An element $(b_1,\ldots,b_n)\in A^n$ is called *solution* of (3) iff $b_i=p_i(b_1,\ldots,b_n)$, $1\le i\le n$, i.e., if $(b_1,\ldots,b_n)$ is a fixpoint of p with entries p_i of (3). If A is partially ordered, $(b_1,\ldots,b_n)\in A^n$ is called the *least solution* of (3) iff it is a solution less than all other solutions.

Corollary 7 and the Fixpoint Theorem prove the next theorem.

THEOREM 8. *Let A be an ω-continuous semiring. Let p be in $A'(Y)^{n\times 1}$ with entries p_i of* (3). *Then the least solution of* (3) *exists in A^n and equals μp.* □

Clearly, in case of a power series semiring, μp is nothing else than the strong solution of an algebraic system (see Kuich, Salomaa [5], p. 298).

We now generalize the A'-finite automata of Kuich [4] and the $A\langle\langle\Sigma^*\rangle\rangle$-automata of Kuich, Salomaa [5] to A'-automata, whose set of states may be countable.

An *A'-automaton*

$$\mathfrak{A} = (I,M,S,P)$$

is given by

(i) a countable set I of *states*,

(ii) a matrix $M\in A'^{I\times I}$, called the *transition matrix*,

(iii) $S\in\{0,1\}^{1\times I}$, called the *initial state vector*,

(iv) $P\in\{0,1\}^{I\times 1}$, called the *final state vector*.

If M^* exists in $A^{I\times I}$, the *behavior* $\|\mathfrak{A}\|\in A$ of $\mathfrak{A}$ is defined by

$$\|\mathfrak{A}\| = SM^*P.$$

In case of an ω-continuous semiring A,

$$M^* = \sup\,(\sum_{0\le i\le n} M^i \mid n\in\mathbb{N}).$$

Hence, in this case the behavior $\|\mathfrak{A}\|$ of $\mathfrak{A}$ exists.

A'-pushdown automata are now defined analogous to the $A\langle\langle\Sigma^*\rangle\rangle$-pushdown automata of Kuich, Salomaa [5]. Let Q be a finite set (of states) and Γ be an alphabet (of pushdown symbols). A matrix

$$M\in(A'^{Q\times Q})_J^{\Gamma^*\times\Gamma^*}$$

is termed an *A'-pushdown transition matrix* iff, for all $\pi_1,\pi_2\in\Gamma^*$,

$$M_{\pi_1,\pi_2} = \begin{cases} M_{p,\pi_3} & \text{if there exist } p\in\Gamma,\ \pi_4\in\Gamma^* \text{ with } \pi_1=p\pi_4,\ \pi_2=\pi_3\pi_4 \\ 0 & \text{otherwise} \end{cases}.$$

An *A'-pushdown automaton*

$$\mathfrak{P} = (Q,\Gamma,M,S,p_0,P)$$

is given by

(i) a finite set Q of *states*,
(ii) an alphabet Γ of *pushdown symbols*,
(iii) an *A'-pushdown transition matrix* M,
(iv) $S\in\{0,1\}^{1\times Q}$, called the *initial state vector*,
(v) $p_0\in\Gamma$, called the *initial pushdown symbol*,
(vi) $P\in\{0,1\}^{Q\times 1}$, called the *final state vector*.

If M^* exists in $(A^{Q\times Q})^{\Gamma^*\times\Gamma^*}$, the *behavior* $\|\mathfrak{P}\|\in A$ of $\mathfrak{P}$ is defined by

$$\|\mathfrak{P}\| = S(M^*)_{p_0,\varepsilon}P.$$

Clearly, in case of an ω-continuous semiring A, $\|\mathfrak{P}\|$ exists.

Analogous to Theorem 10.1 of Kuich, Salomaa [5], we have the result that for each A'-pushdown automaton $\mathfrak{P} = (Q,\Gamma,M,S,p_0,P)$, where $\|\mathfrak{P}\|$ exists, there is an equivalent A'-automaton $\mathfrak{A} = (\Gamma^*\times Q,M',S',P')$.

In the sequel we assume that A is an ω-continuous semiring. We now want to show that A'-algebraic systems and A'-pushdown automata are equivalent mechanisms.

Given a pushdown alphabet Γ, define $Y=\{Y_p \mid p\in\Gamma\}$ to be an alphabet of variables and let be

$$Y_\varepsilon = E \text{ (the matrix of unity)},\quad Y_{p\pi} = Y_pY_\pi,\ p\in\Gamma,\ \pi\in\Gamma^*.$$

Given an A'-pushdown transition matrix M, we now consider the $A'^{Q\times Q}$-algebraic system

$$Y_p = \sum_{\pi\in\Gamma^*} M_{p,\pi}Y_\pi,\ p\in\Gamma. \tag{4}$$

Additionally, we consider the $A'^{Q\times Q}$-linear system

$$Y = MY+F, \tag{5}$$

where $F\in(A^{Q\times Q})^{\Gamma^*\times 1}$, $F_\varepsilon = E$, $F_\pi = 0$, $\pi\in\Gamma^+$, and Y is a variable.

Let $T_p\in A^{Q\times Q}$, $p\in\Gamma$. Then we define by help of these matrices a column vector $T\in(A^{Q\times Q})^{\Gamma^*\times 1}$, whose entries are $T_\varepsilon = E$, $T_{p\pi} = T_pT_\pi$, $p\in\Gamma$, $\pi\in\Gamma^*$.

THEOREM 9. *Let A be an ω-continuous semiring. If the matrices $T_p\in A^{Q\times Q}$, $p\in\Gamma$, are the components of a solution of (4) (when substituted for Y_p, $p\in\Gamma$), then $T\in(A^{Q\times Q})^{\Gamma^*\times 1}$ is a solution of (5).*

Proof. Since M is a pushdown transition matrix, we obtain, for all $p\in\Gamma$ and $\pi_1\in\Gamma^*$,

$$(MT)_{p\pi_1} = \sum_{\pi\in\Gamma^*} M_{p\pi_1,\pi\pi_1} T_{\pi\pi_1} = \sum_{\pi\in\Gamma^*} M_{p,\pi} T_\pi T_{\pi_1} = (MT)_p T_{\pi_1} .$$

Since T_p, $p\in\Gamma$, are the components of a solution of (4), we have $T_p = (MT)_p$ for all $p\in\Gamma$. Hence,

$$(MT)_{p\pi_1} = T_p T_{\pi_1} = T_{p\pi_1},\ p\in\Gamma,\ \pi_1\in\Gamma^*.$$

The proof is finished by $T_\varepsilon = E = F_\varepsilon$. □

The next lemma is similar to Theorem 10.5 of Kuich, Salomaa [5].

LEMMA 10. *Let M be an A'-pushdown transition matrix. Then, for all $\pi_1,\pi_2\in\Gamma^*$,*

$$(M^*)_{\pi_1\pi_2,\varepsilon} = (M^*)_{\pi_1,\varepsilon}(M^*)_{\pi_2,\varepsilon} .$$ □

COROLLARY 11. *The components of the least solution of the $A'^{Q\times Q}$-algebraic system (4) are given by*

$$(M^*)_{p,\varepsilon},\ p\in\Gamma.$$

Proof. By Lemma 10 and the fact that M^*F is the least solution of (5). □

Given an A'-pushdown automaton $\mathfrak{P} = (Q,\Gamma,M,S,p_0,P)$, we construct an A'-algebraic system such that $\|\mathfrak{P}\|$ is a component of the least solution of this system.

By definition, Y_p, $p\in\Gamma$, is a $Q\times Q$-matrix of variables $y^p_{q_1,q_2}$, $q_1,q_2\in Q$, such that $y^p_{q_1,q_2}$ is the (q_1,q_2)-entry of Y_p and y_1 is a new variable. Moreover, $Y_\varepsilon = E$, $Y_{p\pi} = Y_p Y_\pi$, $p\in\Gamma$, $\pi\in\Gamma^*$.

Consider the A'-algebraic system (in matrix notation)

$$\begin{cases} y_1 = S\,Y_{p_0}P, \\ Y_p = \sum_{\pi\in\Gamma^*} M_{p,\pi} Y_\pi , \end{cases}$$

constructed from (4). The next theorem is now clear.

THEOREM 12. *Let A be an ω-continuous semiring and let $\mathfrak{P}$ be an A'-pushdown automaton. Then $\|\mathfrak{P}\|$ is a component of the least solution of an A'-algebraic system.* □

Observe that the proof of Theorem 12 is simpler than the proof of Theorem 14.20 of Kuich, Salomaa [5], i.e., the proof of the analogous result for $A' = \mathbb{B}\langle\Sigma\cup\varepsilon\rangle$ and $A = \mathbb{B}\langle\langle\Sigma^*\rangle\rangle$.

We now show the converse.

In the formulation of the next lemma we need the two A'-algebraic systems

$$\begin{cases} y_1 = p_1 + q_1 p_{n+1} q_2, \\ y_i = p_i,\ 2\le i\le n; \end{cases} \qquad (6) \qquad \text{and} \qquad \begin{cases} y_1 = p_1 + q_1 y_{n+1} q_2, \\ y_i = p_i,\ 2\le i\le n+1; \end{cases} \qquad (7)$$

here, p_i, q_1, q_2 and $q_1p_{n+1}q_2$ are polynomials in $A'\langle Y\rangle$ and y_{n+1} is a new variable.

LEMMA 13. *If $(s_1,\ldots,s_n,s_{n+1})$ is a solution of (7) then $s_{n+1} = p_{n+1}(s_1,\ldots,s_n)$. Furthermore, $(s_1,\ldots,s_n)$ is the least solution of (6) iff $(s_1,\ldots,s_n,p_{n+1}(s_1,\ldots,s_n))$ is the least solution of (7).*

Proof. The first sentence of the lemma is obvious. By the Substitution Principle we get the equality

$$p_1(s_1,\ldots,s_n) + q_1(s_1,\ldots,s_n)p_{n+1}(s_1,\ldots,s_n)q_2(s_1,\ldots,s_n) = p_1(s_1,\ldots,s_n) + (q_1p_{n+1}q_2)(s_1,\ldots,s_n).$$

This proves that $(s_1,\ldots,s_n)$ is a solution of (6) iff $(s_1,\ldots,s_n,p_{n+1}(s_1,\ldots,s_n))$ is a solution of (7). Since the least solution of (7) is determined by its first n components, the second sentence of Lemma 13 is proven. □

In analogy to the Chomsky normal form we define the binary normal form. An A'-algebraic system is in *binary normal form* iff the equations have the form

$$y_i = \sum_{k,m=1}^{n} a^i_{km} y_k y_m + a_i, \ 1 \le i \le n, \tag{8}$$

where $a^i_{km} \in \{0,1\}$ and $a_i \in A'$.

THEOREM 14. *Let A be an ω-continuous semiring. If $a \in A$ is a component of the least solution of an A'-algebraic system, then a is also a component of the least solution of an A'-algebraic system in binary normal form.*

Proof. Use a construction similar to that given in the proof of Theorem 14.27 of Kuich, Salomaa [5] by help of Lemma 13. □

Given an A'-algebraic system (8) in binary normal form we define an A'-pushdown transition matrix $M \in A'^{Y^* \times Y^*}$ with $|Q| = 1$ by

$$M_{y_i, y_k y_m} = a^i_{km}, \ M_{y_i, \varepsilon} = a_i, \ 1 \le i,k,m \le n. \tag{9}$$

We write (8) in the form

$$y_i = \sum_{k,m=1}^{n} M_{y_i, y_k y_m} y_k y_m + M_{y_i, \varepsilon}, \ 1 \le i \le n.$$

This A'-algebraic system is of the form (4). Hence, Corollary 11 proves the next theorem.

THEOREM 15. *Let A be an ω-continuous semiring. If the A'-pushdown transition matrix M is defined by* (9) *then $(M^*)_{y_i,\varepsilon}$, $1 \le i \le n$, are the components of the least solution of the A'-algebraic system* (8). □

COROLLARY 16. *Let $(s_1, \ldots, s_n)$ be the least solution of* (8). *Then $s_i = \|\mathfrak{P}_i\|$, $1 \le i \le n$, where*

$$\mathfrak{P}_i = (\{q\}, Y, M, 1, y_i, 1)$$

has its A'-pushdown transition matrix M defined by (9). □

Finally, we state our main theorem.

THEOREM 17. *Let A be an ω-continuous semiring. Then the following statements on $a \in A$ are equivalent:*

(i) *a is a component of the least solution of an A'-algebraic system;*

(ii) *a is a component of the least solution of an A'-algebraic system in binary normal form;*

(iii) *a is the behavior of an A'-pushdown automaton.* □

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