Monoidal functors (aka idioms!)

[Symmetric] monoidal functors

- A *lax monoidal functor* between monoidal categories $(\mathcal{C}, I, \otimes)$ and $(\mathcal{C}, I', \otimes')$ is
  - a functor F from $\mathcal{C}$ to $\mathcal{C}'$
  - with natural transformations $\mathrm{e} : I' \to FI$ and $\mathrm{m}_{A,B} : FA \otimes' FB \to F(A \otimes B)$

  such that

$$
\begin{tikzcd}
I' \otimes' FA \arrow[r, "\mathrm{e} \otimes' FA"] \arrow[d, "\lambda'_{FA}"'] & FI \otimes' FA \arrow[r, "\mathrm{m}_{I,A}"] & F(I \otimes A) \arrow[d, "F\lambda_A"] \\
FA \arrow[rr, equal] & & FA
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \arrow[rr, equal] \arrow[d, "\rho'_{FA}"'] & & FA \arrow[d, "F\rho_A"] \\
FA \otimes' I' \arrow[r, "FA \otimes' \mathrm{e}"] & FA \otimes' FI \arrow[r, "\mathrm{m}_{A,I}"] & F(A \otimes I)
\end{tikzcd}
$$

$$
\begin{tikzcd}
(FA \otimes' FB) \otimes' FC \arrow[r, "\mathrm{m}_{A,B} \otimes' FC"] \arrow[d, "\alpha'_{FA,FB,FC}"'] & F(A \otimes B) \otimes' FC \arrow[r, "\mathrm{m}_{A \otimes B, C}"] & F((A \otimes B) \otimes' C) \arrow[d, "F\alpha_{A,B,C}"] \\
FA \otimes' (FB \otimes' FC) \arrow[r, "FA \otimes' \mathrm{m}_{B,C}"'] & FA \otimes' F(B \otimes C) \arrow[r, "\mathrm{m}_{A, B \otimes C}"'] & F(A \otimes (B \otimes C))
\end{tikzcd}
$$

- A lax monoidal functors between symmetric monoidal categories is *lax symmetric monoidal*, if also

$$
\begin{tikzcd}
FA \otimes' FB \arrow[r, "\mathsf{m}_{A,B}"] \arrow[d, "\sigma'_{FA,FB}"'] & F(A \otimes B) \arrow[d, "F\sigma_{A,B}"] \\
FB \otimes' FA \arrow[r, "\mathsf{m}_{B,A}"] & F(B \otimes A)
\end{tikzcd}
$$

- An *oplax [symmetric] monoidal functor* is like a lax [symmetric] monoidal functor, but e, m go in the opposite direction.
- A *monoidal [symmetric] functor* is like a lax [symmetric] monoidal functor, but e, m are required to be natural isomorphisms.

- A *lax [symmetric] monoidal natural transformation* between two lax [symmetric] monoidal functors $(F, \mathrm{e}, \mathrm{m})$, $(G, \mathrm{e}', \mathrm{m}')$ is a natural transformation $\tau : F \dot{\to} G$ satisfying

$$
\begin{tikzcd}
I' \arrow[r, "\mathrm{e}"] \arrow[d, equal] & FI \arrow[d, "\tau_I"] \\
I' \arrow[r, "\mathrm{e}'"] & GI
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes' FB \arrow[r, "\mathrm{m}_{A,B}"] \arrow[d, "\tau_A \otimes' \tau_B"'] & F(A \otimes B) \arrow[d, "\tau_{A \otimes B}"] \\
GA \otimes' GB \arrow[r, "\mathrm{m}'_{A,B}"] & G(A \otimes B)
\end{tikzcd}
$$

- *Oplax [symmetric] monoidal* and *[symmetric] monoidal natural transformations* are defined similarly.

- Any functor F between Cartesian categories is canonically oplax symmetric monoidal via
  - $\mathsf{e} = F1 \xrightarrow{!} 1$,
  - $\mathsf{m}_{A,B} = F(A \times B) \xrightarrow{\langle F\mathsf{fst}, F\mathsf{snd}\rangle} FA \times FB$.
- Any natural transformation between functors F, G between Cartesian categories is oplax symmetric monoidal for the canonical oplax symmetric monoidalities on F and G.

Lax monoidal functors $\cap$ containers

- Containers whose interpretation carries a lax monoidal functor are given by

  $(S, \mathrm{e}, \bullet) : \mathrm{Monoid}$
  $P : S \to \mathrm{Set}$
  $\mathrm{e} : S$
  $\bullet : S \times S \to S$
  $\nwarrow\, : \Pi(\{s_0\}, s_1) : (\{S\} \times S).\, P\,(s_0 \bullet s_1) \to P\, s_0$
  $\nearrow\, : \Pi(s_0, \{s_1\}) : (S \times \{S\}).\, P\,(s_0 \bullet s_1) \to P\, s_1$

  such that

  $(\{s\}, \mathrm{e}) \nwarrow p = p$
  $(\mathrm{e}, \{s\}) \nearrow p = p$
  $(\{s\}, s') \nwarrow ((\{s \bullet s'\}, s'') \nwarrow p) = (\{s\}, s' \bullet s'') \nwarrow p$
  $(s, \{s'\}) \nearrow ((\{s \bullet s'\}, s'') \nwarrow p) = (\{s'\}, s'') \nwarrow$
  $((s, \{s' \bullet s''\}) \nearrow p)$
  $(s \bullet s', \{s''\}) \nearrow p = (s', \{s''\}) \nearrow ((s, \{s' \bullet s''\}) \nearrow p)$
- $\nwarrow$ resembles a left action, $\nearrow$ a right action of $(S, \mathrm{e}, \bullet)$.

Monoidal monads

[Symmetric] monoidal monads

- A *lax [symmetric] monoidal monad* on a [symmetric] monoidal category $(\mathcal{C}, I, \otimes)$ is a monad (T, η, μ) with a lax [symmetric] monoidality (e, m) of T for which η and μ are lax [symmetric] monoidal, i.e., satisfy

$$
\begin{tikzcd}
I \arrow[r, equal] \arrow[d, equal] & I \arrow[d, "\eta_I"] \\
I \arrow[r, "\mathrm{e}"'] & TI
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "\mathrm{e}"] \arrow[d, equal] & TI \arrow[r, "T\mathrm{e}"] & T(TI) \arrow[d, "\mu_I"] \\
I \arrow[rr, "\mathrm{e}"'] & & TI
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[r, equal] \arrow[d, "\eta_A \otimes \eta_B"'] & A \otimes B \arrow[d, "\eta_{A \otimes B}"] \\
TA \otimes TB \arrow[r, "\mathrm{m}_{A,B}"'] & T(A \otimes B)
\end{tikzcd}
$$

$$
\begin{tikzcd}
T(TA) \otimes T(TB) \arrow[r, "\mathrm{m}_{TA,TB}"] \arrow[d, "\mu_A \otimes \mu_B"'] & T(TA \otimes TB) \arrow[r, "T\mathrm{m}_{A,B}"] & T(T(A \otimes B)) \arrow[d, "\mu_{A \otimes B}"] \\
TA \otimes TB \arrow[rr, "\mathrm{m}_{A,B}"'] & & T(A \otimes B)
\end{tikzcd}
$$

(Note that Id is lax [symmetric] monoidal and, if F, G are lax [symmetric] monoidal, then so is $G \cdot F$.)

- The 1st law forces that $\mathsf{e} = \eta_I$ and the 2nd law follows from one of the monad laws, so we only need m and the 3rd and 4th laws.

- On a Cartesian category, every monad is canonically oplax symmetric monoidal.

Lax monoidal monads = Comm. bistrong monads

- There is a bijection of lax [symmetric] monoidalities m on a monad (T, η, μ) on a [symmetric] monoidal category $(\mathcal{C}, I, \otimes)$ and commutative [symmetric] bistrengths (θ, ϑ).
- It is defined by
  - $\mathrm{m}_{A,B} = \mathrm{m}^{lr}_{A,B} = \mathrm{m}^{rl}_{A,B}$

  and
  - $\theta_{A,B} = A \otimes TB \xrightarrow{\eta_A \otimes TB} TA \otimes TB \xrightarrow{\mathrm{m}_{A,B}} T(A \otimes B)$,
  - $\vartheta_{A,B} = TA \otimes B \xrightarrow{TA \otimes \eta_B} TA \otimes TB \xrightarrow{\mathrm{m}_{A,B}} T(A \otimes B)$.

- On $(\mathbf{Set}, 1, \times)$, as any monad has a unique left strength and [symmetric] bistrength, it is lax [symmetric] monoidal in at most one way.

Exception idioms

- Lax [symmetric] monoidalities (e, m) on the exception functor for E
  - $TA = E + A$

  are in a bijection with [commutative] semigroup structures $\otimes$ on E via
  - $\mathrm{e}\,* = \mathrm{inr}\,*$,
    $\mathrm{m}_{A,B}\,(\mathrm{inl}\,e_0, \mathrm{inl}\,e_1) = \mathrm{inl}\,(e_0 \otimes e_1)$,
    $\mathrm{m}_{A,B}\,(\mathrm{inl}\,e, \mathrm{inr}\,b) = \mathrm{inl}\,e$
    $\mathrm{m}_{A,B}\,(\mathrm{inr}\,a, \mathrm{inl}\,e) = \mathrm{inl}\,e$
    $\mathrm{m}_{A,B}\,(\mathrm{inr}\,a, \mathrm{inr}\,b) = \mathrm{inr}\,(a, b)$;
  - $e_0 \otimes e_1 = \mathrm{case}\ \mathrm{m}_{0,0}\ \mathrm{of}\ \mathrm{inl}\,e \mapsto e$.
- Two special cases are $e_0 \otimes e_1 = e_0$ (the left zero semigroup) and $e_0 \otimes e_1 = e_1$ (the right zero semigroup).
- The exception monad for E is not lax [symmetric] monoidal except for the special case $E = 1$.

Writer idioms

- Lax [symmetric] monoidalities (e, m) on the writer functor for a set P
  - $TA = P \times A$

  are in a bijection with [commutative] monoid structures $(\mathsf{i}, \otimes)$ on P.

- Lax [symmetric] monoidalities m on the writer monad for a monoid $(P, \mathsf{o}, \oplus)$ are in a bijection with those [commutative] monoid structures $(\mathsf{i}, \otimes)$ on P that satisfy
  - $\mathsf{i} = \mathsf{o}$
  - $(e_0 \oplus e_1) \otimes (e_2 \oplus e_3) = (e_0 \otimes e_2) \oplus (e_1 \otimes e_3)$ (middle-four interchange)

- Under the 1st condition, the 2nd condition implies

  $$e_0 \otimes e_1 = (e_0 \oplus \mathsf{o}) \otimes (\mathsf{o} \oplus e_1) = (e_0 \otimes \mathsf{i}) \oplus (\mathsf{i} \otimes e_1) = e_0 \oplus e_1$$

  and further

  $$e_0 \oplus e_1 = (\mathsf{o} \oplus e_0) \oplus (e_1 \oplus \mathsf{o}) = (\mathsf{o} \oplus e_1) \oplus (e_0 \oplus \mathsf{o}) = e_1 \oplus e_0$$

  as well as follows from these conditions.
- Hence the writer monad is lax [symmetric] monoidal if and only if $\oplus$ is commutative.