DECIDABILITY OF SECOND-ORDER THEORIES AND AUTOMATA ON INFINITE TREES(1)

BY

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Introduction. In this paper we solve the decision problem of a certain second-order mathematical theory and apply it to obtain a large number of decidability results. The method of solution involves the development of a theory of automata on infinite trees—a chapter in combinatorial mathematics which may be of independent interest.

Let $\Sigma=\{0, 1\}$, and denote by T the set of all words (finite sequences) on Σ. Let $r_0: T \to T$ and $r_1: T \to T$ be, respectively, the *successor functions* $r_0(x)=x0$ and $r_1(x)=x1$, $x \in T$. Our main result is that *the (monadic) second-order theory of the structure* $\langle T, r_0, r_1\rangle$ *of two successor functions is decidable*. This answers a question raised by Büchi [1].

It turns out that this result is very powerful and many difficult decidability results follow from it by simple reductions. The decision procedures obtained by this method are elementary recursive (in the sense of Kalmar). The applications include the following. (Whenever we refer, in this paper, to second-order theories, we mean monadic second-order; weak second-order means quantification restricted to finite subsets of the domain.)

The second-order theory of countable linearly ordered sets is proved decidable. As a corollary we get that the weak second-order theory of arbitrary linearly ordered sets is decidable; a result due to Läuchli [9] which improves on a result of Ehrenfeucht [5].

In [4] Ehrenfeucht announced the decidability of the first-order theory of a unary function. We prove that the second-order theory of a unary function with a countable domain is decidable. Also, the weak second-order theory of a unary function with an arbitrary domain is decidable.

There are also applications to point set topology. Let CD be Cantor's discontinuum (i.e., $\{0, 1\}^\omega$ with the product topology). Let F_σ be the lattice of all subsets of CD which are denumerable unions of closed sets, and let L_c be the sublattice of all closed subsets of CD. The first-order theory of the lattice F_σ, with L_c as a distinguished sublattice, is decidable. Similar results hold for the real line with the usual topology. This answers in the affirmative Grzegorczyk's question [8] whether

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the first-order theory of the lattice of all closed subsets of the real line is decidable.

Through Stone's representation theorm, the results concerning Cantor's discontinuum lead to results about boolean algebras. Thus the theory of countable boolean algebras with quantification over ideals, is decidable. The first-order theory of arbitrary boolean algebras with a sequence of distinguished ideals is decidable. This last result is an improvement of Tarski's result [15], and of Ershov's [6, Theorem 9].

Finally, we give an application to the theory of games. We show that the statement, proved by Wolfe [17], that every Gale-Stewart game (see §2.5 for terminology) with a set in F_σ is determinate, is expressible in the second-order theory of two successor functions. Thus Wolfe's theorem could be proved by applying the decision procedure.

Due to the fact that we use reductions to a second-order theory, our decidability proofs are very direct. Through appropriate interpretations, the set variables allow us to talk about all structures in a certain class. Thus, for example, for every sentence F of the second-order theory of linear ordering, we write a sentence $\bar{F}$ of the second-order theory of $\langle T, r_0, r_1\rangle$ which asserts that F holds in all countable linearly ordered sets. Since we can decide whether $\bar{F}$ is true in $\langle T, r_0, r_1\rangle$, we can also decide whether F is in fact true in all countable linearly ordered sets.

It would be interesting to see whether this direct approach, involving some powerful decidable second-order theory, would yield a similar unified approach to other classes of solvable decision problems, e.g., in the theory of various fields.

Let us briefly explain the connection with automata theory. The set T can be viewed as the full binary tree with root Λ (the empty sequence), and where $x0$ and $x1$ are the nodes branching out of the node $x \in T$. For a finite set Σ, a Σ-(valued) tree is a mapping $v: T \to \Sigma$. The set of all Σ-trees is denoted by V_Σ. A Σ-automaton is a system $\mathfrak{A} = \langle S, M, S_0, F\rangle$, where S is a finite set, $M: S \times \Sigma \to P(S \times S)$, $S_0 \subseteq S$, and $F \subseteq P(S)$. We define the notion of a finite automaton $\mathfrak{A}$ *accepting* a Σ-tree v. The set of all Σ-trees accepted by $\mathfrak{A}$ is denoted by $T(\mathfrak{A})$. A set $A \subseteq V_\Sigma$ is called *finite automaton* (f.a.) definable if for some $\mathfrak{A}$, $T(\mathfrak{A}) = A$.

For a $\Sigma_1 \times \Sigma_2$-tree v the projection on Σ_1 is the Σ_1-tree $p_1 v$, where $p_1(x, y) = x$. The basic properties of f.a. definable sets are as follows. If $A \subseteq V_\Sigma$, $B \subseteq V_\Sigma$, and $C \subseteq V_{\Sigma_1 \times \Sigma_2}$ are f.a. definable, then so are $A \cup B$, $V_\Sigma - A$, and $p_1(C)$. Automata defining the latter sets can be effectively constructed from automata defining the sets A, B, and C.

The *emptiness problem*, whether for a given automaton $\mathfrak{A}$ we have $T(\mathfrak{A}) = \varnothing$, is effectively solvable.

Now let Σ^n be $\{0, 1\}^n$. We set up a one-to-one correspondence τ between n-tuples $\tilde{A} = (A_1, \ldots, A_n) \in P(T)^n$ of subsets of T, and Σ^n-trees. Namely, $\tau(\tilde{A}) = v_{\tilde{A}}$ where $v_{\tilde{A}}(x) = (\chi_{A_1}(x), \ldots, \chi_{A_n}(x))$, $x \in T$, where χ_A denotes the characteristic function of A. For every formula $F(A_1, \ldots, A_n)$ of the second-order theory of two successor functions, we can effectively construct a Σ^n-automaton $\mathfrak{A}_F$ so that for $\tilde{A} \in P(T)^n$, $v_{\tilde{A}} \in T(\mathfrak{A}_F)$ if and only if $\langle T, r_0, r_1\rangle \models F(\tilde{A})$. This result, coupled with the solvability of the emptiness problem, leads immediately to the decidability of the second-order theory of two successor functions.

As indicated above, our method also yields a complete survey of the relations definable in $\langle T, r_0, r_1\rangle$ using (monadic) second-order language. Via the interpretations used in the decidability proofs, we are in a position to get complete information about definability in all the other theories proved decidable by our method. This question, however, is not explored in the present article.

The paper is organized in three parts. Chapter I contains the basic definitions and the proof, using automata on infinite trees, of the main result concerning the decidability of the second-order theory of $\langle T, r_0, r_1\rangle$. Chapter II contains the various applications of the main decidability result. In Chapter III we develop in detail the theory of automata on infinite trees. We prove the two difficult theorems used in Chapter I; namely, that the class of f.a. definable sets is closed with respect to complementations, and that the emptiness problem is effectively solvable. The treatment of automata theory is self-contained and the relevant results concerning sequential automata and automata on finite trees, are fully explained. Anyone looking for further background information may consult, in addition to the original papers quoted, also the survey article [12].

The reader who is mainly interested in automata, may get a complete picture of the theory of automata on infinite trees by reading §1.4 and Chapter III.

Chapter I. The theory of two successor functions

1.1. **Notations and terminology.** We shall use the usual set theoretic notation throughout this paper. Thus, a function $f: A \to B$ is a subset $f \subseteq A \times B$ satisfying certain conditions. Sometimes we shall describe a mapping by the notation $x \mapsto f(x)$, $x \in A$, which indicates that for $x \in A$, x is mapped into $f(x)$. For example, $x \mapsto x^2$, $x \in [0, 1]$, denotes the squaring function $f(x) = x^2$. If $f: A \to B$ then A and $f(A) = \{f(a) \mid a \in A\}$ are called, respectively, the *domain* D (f) and the *range* R (f) of f. If $f: A \to B$ and $C \subseteq A$, then $f|C$ will denote the restriction $f \cap (C \times B)$ of f to C.

We adopt the convention that every ordinal number α is the set $\alpha = \{\beta \mid \beta < \alpha\}$ of all smaller ordinals. Thus $0 = \varnothing$ (the empty set), $n = \{0, 1, \dots, n-1\}$, and $\omega = \{0, 1, \dots\}$. We shall use $[n]$ to denote the set $\{1, \dots, n\} = n - \{\varnothing\}$. The cardinality of a set A will be denoted by c (A).

The set of all subsets of a set A will be denoted by $P(A)$.

For A a set and α an ordinal, A^α is the set of all α-termed sequences of elements of A; i.e., $A^\alpha = \{\phi \mid \phi: \alpha \to A\}$.

Let A be a set, n an integer, and $1 \leqq i \leqq n$. The *projection onto the ith coordinate* is the mapping $p_i: A^n \to A$ such that $p_i((x_1, \dots, x_n)) = x_i$. Strictly speaking, projections such as $(x, y) \mapsto y$ and $(x, y, z) \mapsto y$ are different mappings, but we shall denote both by p_2.

In writing logical formulas, we shall employ boldface type to denote the predicate or function constants, and the various variables (these will be either set variables or individual variables).

Let $\mathfrak{A}=\langle A, R\rangle$ be a structure and $F(\mathbf{B}, \mathbf{x})$ be a formula of a language appropriate for $\mathfrak{A}$ (here $\mathbf{B}$ is a set variable and $\mathbf{x}$ is an individual variable). For $B \subseteq A$, $x \in A$, the notation $\mathfrak{A} \models F(B, x)$ is used to indicate that the formula $F(\mathbf{B}, \mathbf{x})$ is satisfied in $\mathfrak{A}$ by B and x.

The conjunction of the formulas F_i, $1 \leqq i \leqq n$, is denoted by $\bigwedge_{1 \leqq i \leqq n} F_i$; similarly for disjunction.

1.2. **Structures and theories.** We shall prove the decidability of some first-order and monadic second-order theories which are defined semantically as the set of all sentences true in a certain structure or class of structures. Let $\mathscr{K}$ be a class of similar structures $\mathfrak{M}=\langle A, P_\alpha\rangle_{\alpha<\lambda}$, where λ is an ordinal and P_α is an $n(\alpha)$-ary relation or function on A. With $\mathscr{K}$ we associate a language L appropriate to it. L may be a first-order or a second-order language. L has the usual logical connectives and quantifiers, equality, a sequence $\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots,$ of individual variables, and an $n(\alpha)$-ary predicate or function constant $\mathbf{P}_\alpha$ for each $\alpha<\lambda$.

In the case that L is the (monadic) second-order language appropriate to $\mathscr{K}$, it has, in addition to the above, a sequence of set variables, $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots,$ and the membership symbol $\in$. The atomic formulas of L include formulas of the form $t \in V$, where t is a *term* of L, and V is a set variable. Quantification is possible over both set variables and individual variables. In defining a language L we shall sometimes impose restrictions on some of its set variables. For example, we may confine some or all of the variables to range over finite subsets of the domain. Or else we may confine the variables to range over ideals of an algebra, or over subsets of the domain closed in a given topology, etc.

With a given structure $\mathfrak{M}$ and an appropriate language L, we associate the *theory* $T=\mathrm{Th}\,(\mathfrak{M}, L)$ which is, by definition, the set of all sentences of L true in $\mathfrak{M}$. The theory T constructed in this manner will be referred to as the *theory of the structure* $\mathfrak{M}$, or, sometimes, as the *theory of the relations and functions* P_α, $\alpha<\lambda$. The adjective second-order, with appropriate qualifications, will be added when the language L is of that kind. The theory $\mathrm{Th}\,(\mathscr{K}, L)$ of a class $\mathscr{K}$ of similar structures is, by definition, $\mathrm{Th}\,(\mathscr{K}, L)=\bigcap_{\mathfrak{M}\in\mathscr{K}} \mathrm{Th}\,(\mathfrak{M}, L)$. The first-order theory of $\mathscr{K}$ will be denoted by $\mathrm{Th}_1\,(\mathscr{K})$; the (monadic) second-order theory will be denoted by $\mathrm{Th}_2\,(\mathscr{K})$.

As usual, a theory T is called *decidable* if the set T (more precisely, the set of Gödel numbers of the sentences in T) is recursive. T is called *primitive recursive* or *elementary recursive* if (as a set of integers) it is primitive recursive or elementary recursive (in the sense of Kalmar). We shall not trace this fact explicitly, but all the decidability proofs given in this paper will actually show that the theory in question is elementary recursive.

1.3. **Theory of n successor functions.** For an arbitrary set A, let A^* denote the set of all finite sequences of elements of A. The elements $x \in A$ are also called *words* on A. For $x \in A^*$, $l(x)$ denotes the *length* of the sequence x. The unique word $x \in A^*$ for which $l(x)=0$ is called the *empty word* and is denoted by Λ. If $x \in A^*$, $y \in A^*$, then xy will denote the result of concatenating x with y.

On A^* we define a partial ordering by $x \leqq y$ (x is an *initial* of y) if $y = xz$ for some $z \in A^*$. If $x \leqq y$ and $x \neq y$, then we shall write $x < y$.

If A is linearly ordered by $\leqq_A$, then we can introduce the ordering $\preccurlyeq$ which is the lexicographical ordering of A^* induced by $\leqq_A$. Thus $x \preccurlyeq y$ if and only if $x \leqq y$, or $x = zau$ and $y = zbv$, where $a \in A$, $b \in A$, and $a <_A b$. Note that $\langle A^*, \preccurlyeq \rangle$ is a totally-ordered set.

For $a \in A$, define the (immediate) *successor function* $r_a \colon A^* \to A^*$ by $r_a(x) = xa$. We also define a *predecessor function* pd: $A^* \to A^*$ by pd $(x) = y$ if either $x = y = \Lambda$ or $x = r_a(y)$ for some $a \in A$.

DEFINITION 1.1. For $0 < n \leqq \omega$, let $T_n = n^* = \{i \mid i < n\}^*$. The structure $\mathfrak{N}_n = \langle T_n, r_i, \leqq, \preccurlyeq \rangle_{i<n}$, where $\preccurlyeq$ is the lexicographic ordering induced by the usual ordering on n, is the *structure of n successor functions*. Th_2 $(\mathfrak{N}_n)$ is called the *second-order theory of n successor functions* and will be denoted by SnS.

The structure $\mathfrak{N}_1 = \langle \{0\}^*, r_0, \leqq, \preccurlyeq \rangle$ is (up to isomorphism) the set of integers with the usual successor function $x \mapsto x+1$ and ordering $\leqq$; the relation $\preccurlyeq$ is the same as $\leqq$ in this case. Thus SnS, $1 < n$, is a natural generalization of the ordinary theory of a single successor function.

For every finite $n < \omega$, the relations $\leqq$ and $\preccurlyeq$ on T_n are definable (in second-order language) from $r_0, \ldots, r_{n-1}$. This is not true for T_ω and we, therefore, include these relations in the definition of $\mathfrak{N}_n$, $n \leqq \omega$.

Our main decidability result is

THEOREM 1.1. *The* (*monadic*) *second-order theory of two successor functions* (S2S) *is decidable.*

This result, which will be proved later on, has a large number of consequences. In particular, it trivially implies that SnS for every integer n, as well as SωS, are decidable (§1.9). For this reason we prove directly only the decidability of S2S, even though the very same method would apply to every SnS, $n < \omega$.

1.4. **Trees and automata.** As explained in the Introduction, the solution of the decision problem of S2S will require a theory of automata on infinite trees. In this section we give the basic definitions and results concerning automata.

The *infinite binary tree* is the set $T = T_2 = \{0, 1\}^*$ of all finite words on $\{0, 1\}$. The elements $x \in T$ are the *nodes* of T. For $x \in T$, the nodes $x0$, $x1$ are called the *immediate successors* of x. The empty word Λ is called the *root* of T. Our language is suggested by the following picture. The lowest node of T is the root Λ. The root branches up to the (say) left into the node 0 and to the right into the node 1. The node 0 branches into 00 and 01; the node 1 branches into 10 and 11. And so on ad infinitum.

For $x \in T$, the *subtree* T_x with *root* x is defined by $T_x = \{y \mid y \in T, x \leqq y\}$. Thus, $T_\Lambda = T$.

A *path* π of a tree T_x is a set $\pi \subset T_x$ satisfying: (1) $x \in \pi$; (2) for $y \in \pi$, either $y0 \in \pi$ or $y1 \in \pi$, but not both; (3) if $y \in \pi$ and $x \neq y$ then $\operatorname{pd}(y) \in \pi$. It can be verified that if $y, z \in \pi$, then $y \leqq z$ or $z \leqq y$.

For a mapping $\phi \colon A \to B$, define

$$\operatorname{In}(\phi) = \{b \mid b \in B, c(\phi^{-1}(b)) \geqq \omega\}.$$

In the following, Σ denotes a finite set called the *alphabet*.

DEFINITION 1.2. A Σ-(*valued*) *tree* is a pair (v, T_x) such that $v \colon T_x \to \Sigma$. If (v, T) is a valued tree, then (v, T_x) will denote the induced value subtree $(v \mid T_x, T_x)$. The set of all Σ-trees (v, T_x), for a fixed $x \in T$, will be denoted by $V_{\Sigma,x}$. The set $\bigcup_{x \in T} V_{\Sigma,x}$ of all Σ-trees, will be denoted by V_Σ.

DEFINITION 1.3. A *table* $\mathfrak{A}'$ over Σ-trees is a pair $\mathfrak{A}' = \langle S, M \rangle$ where S is a finite set, the *set of states*, and M is a function $M \colon S \times \Sigma \to P(S \times S)$, the (nondeterministic) *table of moves* ($P(A)$ denotes the set of all subsets of A).

A *finite automaton* (f.a.) over Σ-trees (a Σ-automaton) is a system $\mathfrak{A} = \langle S, M, S_0, F \rangle$ where $\langle S, M \rangle$ is as above, $S_0 \subseteq S$ is the set of *initial states*, $F \subseteq P(S)$ is the set of *designated subsets* of S.

DEFINITION 1.4. A *run* of $\mathfrak{A}' = \langle S, M \rangle$ on the Σ-tree $t = (v, T_x)$ is a mapping $r \colon T_x \to S$ such that for $y \in T_x$, $(r(y0), r(y1)) \in M(r(y), v(y))$. We also talk about a run of an automaton $\mathfrak{A}$ on a tree, meaning a run of the associated table $\mathfrak{A}'$. The set of all $\mathfrak{A}$-runs on t is denoted by $\operatorname{Rn}(\mathfrak{A}, t)$.

DEFINITION 1.5. The automaton $\mathfrak{A} = \langle S, M, S_0, F \rangle$ *accepts* (v, T_x) if there exists an $\mathfrak{A}$-run r on (v, T_x) such that $r(x) \in S_0$ and for every path π of T_x, $\operatorname{In}(r \mid \pi) \in F$.

The set $T(\mathfrak{A})$ of Σ-trees *defined* by $\mathfrak{A}$ is

$$T(\mathfrak{A}) = \{(v, T_x) \mid x \in T, (v, T_x) \textit{ is accepted by } \mathfrak{A}\}.$$

A set $A \subseteq V_\Sigma$ is *f.a. definable* if for some f.a. $\mathfrak{A}$, $A = T(\mathfrak{A})$.

REMARK. A set $A \subseteq V_\Sigma$ is called *invariant* if for every Σ-tree $t = (v, T)$ and every $x \in T$, $t \in A$ if and only if the tree $t' = (v', T_x)$ defined by $v'(xy) = v(y)$, $y \in T$, is in A. The invariant subsets of V_Σ are a boolean algebra. It is clear from Definition 1.5 that every set $T(\mathfrak{A})$ is invariant. To prove that an invariant set A is f.a. definable, it suffices to construct an automaton $\mathfrak{A}$ such that $(v, T) \in T(\mathfrak{A})$ if and only if $(v, T) \in A$.

The following results are immediate.

LEMMA 1.2. *If $A \subseteq V_\Sigma$ is f.a. definable, then there exists an automaton $\mathfrak{A} = \langle S, M, S_0, F \rangle$ such that $S_0 = \{s_0\}$, $s_0 \in S$, and $T(\mathfrak{A}) = A$.*

THEOREM 1.3. *If $A, B \subseteq V_\Sigma$ are f.a. definable, then so are $A \cup B$ and $A \cap B$.*

Proof. Let $A=T(\mathfrak{A})$, $B=T(\mathfrak{B})$ where $\mathfrak{A}=\langle S, M, s_0, F\rangle$, $\mathfrak{B}=\langle S', M', s_0', F'\rangle$. We assume that $S \cap S'=\varnothing$. Construct the automaton

$$\mathfrak{A} \cup \mathfrak{B} = \langle S \cup S', M \cup M', \{s_0, s_0'\}, F \cup F'\rangle.$$

Clearly, $T(\mathfrak{A} \cup \mathfrak{B})=A \cup B$.

With the above notations, define $\mathfrak{A}\times\mathfrak{B}=\langle S\times S', \overline{M}, (s_0, s_0'), \overline{F}\rangle$ as follows. $((s_1, s_1'), (s_2, s_2')) \in \overline{M}((s, s'), \sigma)$ if and only if $(s_1, s_2) \in M(s, \sigma)$ and $(s_1', s_2') \in M'(s', \sigma)$. Let p_1 and p_2 be the projection functions $(x, y)\mapsto x$ and $(x, y)\mapsto y$. Define $\overline{F} =\{G \mid G\subseteq S\times S', p_1(G) \in F, p_2(G) \in F'\}$. We have $T(\mathfrak{A}\times\mathfrak{B})=A \cap B$.

Definition 1.6. Let $t=(v, T)$ be a $\Sigma_1\times\Sigma_2$-tree and let p_1 again be the projection $(x, y)\mapsto x$. The *projection* $p_1(t)$, by definition, is the Σ_1-tree $(p_1 v, T)$.

The *projection* $p_1(A)$ of a set $A\subseteq V_{\Sigma_1\times\Sigma_2}$, is $p_1(A)=\{p_1(t) \mid t \in A\}$. The Σ_2-*cylindrification* of a set $B\subseteq V_{\Sigma_1}$ is the largest set $A\subseteq V_{\Sigma_1\times\Sigma_2}$ such that $p_1(A)=B$.

Theorem 1.4. *If $A\subseteq V_{\Sigma_1\times\Sigma_2}$ is a f.a. definable set, then $p_1(A)\subseteq V_{\Sigma_1}$ is a f.a. definable set. If $B\subseteq V_{\Sigma_1}$ is f.a. definable, so is its Σ_2-cylindrification $A\subseteq V_{\Sigma_1\times\Sigma_2}$.*

Proof. Let $\mathfrak{A}=\langle S, M, s_0, F\rangle$ be a $\Sigma_1\times\Sigma_2$-automaton with $T(\mathfrak{A})=A$. Define a Σ_1-automaton by $\mathfrak{A}_1=\langle S, M_1, s_0, F\rangle$, where $M_1(s, \sigma_1)=\bigcup_{\sigma_2\in\Sigma_2} M(s, (\sigma_1, \sigma_2))$, $\sigma_1 \in\Sigma_1$, $s \in S$. One can check that $T(\mathfrak{A}_1)=p_1(A)$.

The proof concerning cylindrification is left to the reader.

Theorem 1.5. *The complement $V_\Sigma-T(\mathfrak{A})$ of a f.a. definable set is a f.a. definable set.*

Theorem 1.6. *There exists an effective (even elementary-recursive) procedure for deciding for every automaton $\mathfrak{A}$ whether $T(\mathfrak{A})=\varnothing$.*

For a proof of these two difficult theorems, see Chapter III.

1.5. **Definability in $\mathfrak{N}_2$.** The basic facts concerning automata on infinite trees lead, in a natural way, to a proof of Theorem 1.1. The proof proceeds by setting up a correspondence between n-tuples $(A_1, \ldots, A_n)$ of subsets of $T=T_2$ and valued trees.

Definition 1.7. For a set $A\subseteq T$, let $\chi_A\colon T\to\{0, 1\}$ be the characteristic function of A. Denote $\{0, 1\}^n$ by Σ^n, $n<\omega$. With $\tilde{A}=(A_1, \ldots, A_n)$, associate the Σ^n-tree $(v_{\tilde{A}}, T)$ defined by $v_{\tilde{A}}(x)=(\chi_{A_1}(x), \ldots, \chi_{A_n}(x))$, $x \in T$. The mapping $\tau\colon \tilde{A}\mapsto (v_{\tilde{A}}, T)$ sets up a one-to-one correspondence between $P(T)^n$ and $V_{\Sigma^n,\Lambda}=\{(v, T) \mid v : T\to\Sigma^n\}$.

Theorem 1.7. *There exists an (elementary-recursive) effective procedure for assigning to every formula $F(A_1, \ldots, A_n)$ of* S2S *a Σ^n-automaton $\mathfrak{A}_F$ so that*

$$T(\mathfrak{A}_F) \cap V_{\Sigma^n,\Lambda} = \tau(\{(A_1, \ldots, A_n) \mid \mathfrak{N}_2 \models F(A_1, \ldots, A_n)\}). \tag{1.1}$$

If (1.1) *holds for an automaton $\mathfrak{A}$, then we shall say that $\mathfrak{A}$ represents the formula $F(A_1, \ldots, A_n)$.*

Proof. Call two formulas $F(A_1, \ldots, A_n)$ and $G(A_1, \ldots, A_n)$ *equivalent* (in $\mathfrak{N}_2$) if for all $A_1, \ldots, A_n$, $\mathfrak{N}_2 \models F(A_1, \ldots, A_n)$ if and only if $\mathfrak{N}_2 \models G(A_1, \ldots, A_n)$. We start by showing how to assign (effectively) to every formula F an equivalent formula G which is in a special *normal form.*

We introduce the following abbreviation for terms of S2S. A variable x will be abbreviated by x. Inductively, if t is abbreviated by xw, where $w \in T$, then $r_\delta(t)$, $\delta \in \{0, 1\}$, will be abbreviated by $xw\delta$.

A formula $P(A_1, \ldots, A_m)$ of S2S is called *principal* if it has the form

$$\exists x[xw_1\eta_1 A_{i_1} \wedge \cdots \wedge xw_k\eta_k A_{i_k}], \tag{1.2}$$

where each i_j satisfies $1 \leqq i_j \leqq m$ and each η_j is either $\in$ or $\notin$. Note that it is not required that every A_i, $1 \leqq i \leqq m$, actually appear in (1.2).

Every formula $F(A_1, \ldots, A_n)$ is equivalent to a formula G of the normal form $Q_{n+1} \cdots Q_m M(A_1, \ldots, A_m)$, where M is a boolean combination of some principal formulas $P_1, \ldots, P_r$, and each quantifier Q_i is either $\exists A_i$ or $\forall A_i$.

The formula G is obtained from F by a sequence of simple steps as follows. In F, replace every occurrence of $t_1 \preccurlyeq t_2$, where t_1 and t_2 are terms, by

$$t_1 \leqq t_2 \vee \exists z[r_0(z) \leqq t_1 \wedge r_1(z) \leqq t_2].$$

In the resulting formula, replace every occurrence of $t_1 = t_2$ by $\forall A[t_1 \in A \to t_2 \in A]$, and every occurrence of $t_1 \leqq t_2$ by

$$\forall A[\forall x[x \in A \to r_0(x) \in A \wedge r_1(x) \in A] \wedge t_1 \in A \to t_2 \in A].$$

We obtain a formula F' equivalent with F, in which the only atomic subformulas are of the form $t \in V$ where t is a term and V is a set variable. By a well-known procedure of second-order logic, F' is transformed into an equivalent formula F_1 of the form $Q_{n+1} \cdots Q_m M_1(A_1, \ldots, A_m)$ where each Q_i is $\forall A_i$ or $\exists A_i$, and M_1 is in prenex form with quantification only over *individual* variables. By pushing the quantifiers of M_1 one by one into M_1, M_1 is transformed into an equivalent boolean combination $M(A_1, \ldots, A_m)$ of principal formulas. Thus, F is equivalent to the formula $Q_{n+1} \cdots Q_m M$ of the desired normal form.

Returning to the proof of the assertion in our theorem, we may now assume that F itself is in the normal form $Q_{n+1} \cdots Q_m M(A_1, \ldots, A_m)$(2). That the assertion is true for a principal formula $P(A_1, \ldots, A_m)$ of the form (1.2), can be seen by an explicit construction of an automaton $\mathfrak{A}_P$ so that (1.1) holds. Alternatively, this will be obtained as Corollary 3.13. The existence of an automaton $\mathfrak{A}_M$ representing the boolean combination M of principal formulas now follows from Theorems 1.3 and 1.5.

(2) M. Megidor has suggested that a slight modification of the argument which follows will establish our result directly by induction on formulas, without having to pass to normal form In this approach individual variables are treated as special set variables ranging over singleton sets.

The proof of our theorem will be completed if we can establish the following proposition. If $G(A_1, \ldots, A_k)$ is a formula represented by the Σ^k-automaton $\mathfrak{A}$, and Q_k is $\exists A_k$ or $\forall A_k$, then $Q_k G(A_1, \ldots, A_k)$ is representable by a Σ^{k-1}-automaton $\mathfrak{B}$. Since $\forall A_k G$ is equivalent with $\sim \exists A_k \sim G$ and the class of f.a. definable sets is closed under complementation (this is the crucial application of Theorem 1.5!), it suffices to consider the case $\exists A_k G$.

Let p be the projection $p: (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_{k-1})$. Since $\Sigma^k = \Sigma^{k-1} \times \Sigma^1$ (where $\Sigma^1 = \{0, 1\}$), p induces a projection from Σ^k-trees to Σ^{k-1}-trees (Definition 1.6). This projection commutes with the mapping $\tau: \tilde{A} \mapsto (v_{\tilde{A}}, T)$, i.e., $\tau(p(\tilde{A})) = p(\tau(\tilde{A}))$ for $\tilde{A} \in P(T)^k$. Now

$$\begin{aligned} H &= \{(A_1, \ldots, A_{k-1}) \mid \mathfrak{N}_2 \models \exists A_k G(A_1, \ldots, A_{k-1}, A_k)\} \\ &= p(\{(A_1, \ldots, A_k) \mid \mathfrak{N}_2 \models G(A_1, \ldots, A_k)\}). \end{aligned}$$

Applying τ to both sides and interchanging τ and p, we get

$$\tau(H) = p(\tau(\{(A_1, \ldots, A_k) \mid \mathfrak{N}_2 \models G(A_1, \ldots, A_k)\})) = p(T(\mathfrak{A})) \cap V_{\Sigma^{k-1}, \Lambda},$$

the last equation being our assumption about G and $\mathfrak{A}$. By Theorem 1.4, there exists a Σ^{k-1}-automaton $\mathfrak{B}$ such that $T(\mathfrak{B}) = p(T(\mathfrak{A}))$. This $\mathfrak{B}$ represents $\exists A_k G$.

1.6. **Proof of Theorem 1.1.** Let G be a sentence of S2S. We wish to determine effectively whether $\mathfrak{N}_2 \models G$. Without loss of generality, assume that G is of the form $\exists A_1 F(A_1)$. Construct the automaton $\mathfrak{A}_F$. Now $\mathfrak{N}_2 \models \exists A_1 F(A_1)$ if and only if $T(\mathfrak{A}_F) \neq \varnothing$. The question whether $T(\mathfrak{A}_F) \neq \varnothing$ can be effectively decided by Theorem 1.6.

1.7. **Addition of finite-set variables.** Theorem 1.1 can be strengthened to show that S2S remains decidable upon adding to the language set variables $\boldsymbol{a}, \boldsymbol{b}, \ldots$ ranging over finite subsets of T_2. This will be done by proving that the finiteness of a set $A \subseteq T$ is a property definable in S2S.

To do this, and also for later applications, let us briefly recall the notion of a relation R being *definable* in a structure $\mathfrak{M} = \langle A, P_\alpha \rangle_{\alpha < \lambda}$, using a language L (less precisely, R being definable in Th $(\mathfrak{M}, L)$). Thus, for example, $R \subseteq A^2 \times P(A)$ is definable in Th_2 $(\mathfrak{M})$ if there exists a formula $F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{B})$ of L such that for $x, y \in A$, $B \subseteq A$, we have $\langle x, y, B \rangle \in R$ if and only if $\mathfrak{M} \models F(x, y, B)$. Our proofs that a given relation is definable in a given structure will often be informal. We shall give a verbal description of the relation, leaving it to the reader to check that this verbal description is expressible by a formula of the language in question.

LEMMA 1.8. *The predicate* Fn (A), *true for* $A \subseteq T_2$ *if and only if* A *is finite, is definable in* S2S.

Proof. We recall that $A \subseteq T$ is totally-ordered by $\preccurlyeq$. Thus A is finite if and only if every $B \subseteq A$ has both a largest and a smallest element with respect to $\preccurlyeq$.

The same definition of finiteness applies to every SnS, $n \leqq \omega$.

COROLLARY 1.9. *Let* L' *be the second-order language* L *appropriate to* $\mathfrak{N}_2$, *augmented by the addition of variables* $\boldsymbol{a}, \boldsymbol{b}, \ldots$, *ranging over finite subsets of* T_2. Th $(\mathfrak{N}_2, L')$ *is decidable.*

1.8. **Definability of $T(\mathfrak{A})$ in S2S.** Theorem 1.7 asserts that if $R \subseteq P(T)^n$ is definable in S2S, then $\tau(R)$ is finite-automaton definable. The converse of this statement is also true.

THEOREM 1.10. *Let $R \subseteq P(T)^n$ be an n-ary relation between subsets of T and let $\mathfrak{A}$ be a Σ^n-automaton such that $T(\mathfrak{A}) \cap V_{\Sigma^n, \Lambda} = \tau(R)$. There exists a formula $F_{\mathfrak{A}}(A_1, \ldots, A_n)$ of* S2S *so that $\langle A_1, \ldots, A_n \rangle \in R$ if and only if $\mathfrak{N}_2 \models F_{\mathfrak{A}}(A_1, \ldots, A_n)$.*

Proof. We shall use the following notational abbreviation. If $\tilde{A} = (A_1, \ldots, A_n)$ and $(\delta_1, \ldots, \delta_n) = \sigma \in \Sigma^n$ $(= \{0, 1\}^n)$, then $v_{\tilde{A}}(x) = \sigma$ will abbreviate the formula $x\eta_1 A_1 \wedge \cdots \wedge x\eta_n A_n$ where η_i is $\in$ if $\delta_i = 1$ and η_i is $\notin$ if $\delta_i = 0$, $1 \leqq i \leqq n$. This notation captures the intention of Definition 1.7. If $H \subseteq \Sigma^n$, then $v_{\tilde{A}}(x) \in H$ will abbreviate $\bigvee_{\sigma \in H} v_{\tilde{A}}(x) = \sigma$.

Let $\mathfrak{A} = \langle S, M, s_0, F \rangle$. We may assume that $S \subseteq \Sigma^m$ for an appropriate m. Thus to each S-valuation $r: T \to S$ there corresponds a $\tilde{B} = (B_1, \ldots, B_m)$ such that $v_{\tilde{B}}(x) = r(x)$ holds for $\tilde{B}$ and for every $x \in T$.

With r and $\tilde{B}$ as above, $C \subseteq T$ a path, and $s \in S$, $s \in \operatorname{In}(r \mid C)$ holds if and only if $\forall x \exists y [x \in C \to y \in C \wedge x < y \wedge r_{\tilde{B}}(y) = s]$ is true in $\mathfrak{N}_2$ for C and $\tilde{B}$. This implies that for $F \subseteq P(S)$ the statement $\operatorname{In}(r \mid C) \in F$ is expressible by an appropriate formula $\operatorname{In}_F(\tilde{B}, C)$ of S2S. Finally, there exists a formula Path (C) of S2S which is true if and only if $C \subseteq T$ is indeed a path. Note that all the above mentioned formulas can be constructed so as to contain no set quantifiers.

Putting the previous remarks together, we see that for $\tilde{A} = (A_1, \ldots, A_n) \in P(T)^n$, $\tilde{A} = (A_1, \ldots, A_n)$ and $\tilde{B} = (B_1, \ldots, B_m)$ (where the latter two are sequences of set variables), $(v_{\tilde{A}}, T) \in T(\mathfrak{A})$ if and only if $\tilde{A}$ satisfies the formula $F_{\mathfrak{A}}(A_1, \ldots, A_n)$ which reads

$$
\begin{aligned}
&\exists B_1 \cdots \exists B_m \forall C \Big[\forall x \bigwedge_{\sigma \in \Sigma^n} \bigwedge_{s \in S} \Big[v_{\tilde{A}}(x) = \sigma \wedge v_{\tilde{B}}(x) = s \\
&\qquad\qquad \to \bigvee_{(s_1, s_2) \in M(s, \sigma)} v_{\tilde{B}}(x0) = s_1 \wedge v_{\tilde{B}}(x1) = s_2\Big] \\
&\qquad\qquad\qquad \wedge\, v_{\tilde{B}}(\Lambda) = s_0 \wedge [\operatorname{Path}(C) \to \operatorname{In}_F(\tilde{B}, C)]\Big].
\end{aligned}
\tag{1.3}
$$

This formula $F_{\mathfrak{A}}$ is the desired one.

COROLLARY 1.11. *Every formula $F(A_1, \ldots, A_n)$ of* S2S *is equivalent to a formula G of the form $\exists \tilde{B} \forall C M(\tilde{A}, \tilde{B}, C)$ where M is a formula with no set-quantifiers.*

Proof. This follows at once by combining Theorems 1.7 and 1.10 and noting that formula (1.3) has the required form.

The dual form of the previous corollary is, of course, also true.

1.9. **Decidability of SωS.** The binary tree $T_2 = T$, in a certain sense, contains as subtrees all trees with countable branching. For this reason, the decidability of S2S implies decidability of second-order theory of more complicated particular trees and classes of trees with countable branching. Here we shall treat only the case T_ω.

THEOREM 1.12. *The second-order theory* $\mathrm{S}\omega\mathrm{S}=\mathrm{Th}_2(\mathfrak{N}_\omega)$ *of* ω *successor functions is decidable.*

Proof. Let $A \subseteq T$ be a set with a unique element $\Lambda_A \in A$ smallest with respect to $\leqq$. Define a relation $S(A) \subseteq A^2$ by $(x, y) \in S(A)$ if and only if

$$x \in A \ \wedge\ y \in A \ \wedge\ x < y \ \wedge\ \forall z[z \in A \rightarrow \sim[x < z < y]].$$

If $(x, y) \in S(A)$, then we shall say that y is an (immediate) *successor* of x in A. Thus $T(A)=\langle A, S(A)\rangle$ is a tree with root Λ_A.

Note that for $x, y \in A$, $x \leqq y$ (where $\leqq$ is the partial ordering of T) if and only if x precedes y in the tree $T(A)$.

Let $A=\{\Lambda\} \cup \{1^{n_1}01^{n_2}0\cdots 1^{n_k}0 \mid 1 \leqq k<\omega, 0<n_i, 1 \leqq i \leqq k\}$. In $T(A)$, the set of immediate successors of an $x \in A$ is well-ordered in an ω-sequence by $\preccurlyeq$ (the lexicographic ordering of T). Thus we can *define* $r_0^A(x)=y$ by $(x, y) \in S(A) \wedge \forall z[(x, z) \in S(A) \rightarrow y \preccurlyeq z]$ and, inductively on $n<\omega$, $r_{n+1}^A(x)=y$ by $(x, y) \in S(A) \wedge \bigwedge_{i<n} r_i^A(x) \neq y \wedge \forall z[(x, z) \in S(A) \wedge \bigwedge_{i<n} r_i^A(x) \neq z \rightarrow y \preccurlyeq z]$. With this definition of the successor functions r_n^A, $n<\omega$, the structure $\langle A, r_n^A, \leqq|A, \preccurlyeq|A\rangle_{n<\omega}$ is isomorphic to $\mathfrak{N}_\omega$.

Now the set A and the relations $r_n^A(x)=y$, $n<\omega$, are definable in S2S. Combined with the previous remark, this implies decidability of SωS.

CHAPTER II. APPLICATIONS OF THE DECIDABILITY OF S2S

2.1. **Linearly ordered sets.** Let $\mathscr{K}_{\leqq}^{\omega}$ be the class of all linearly ordered sets $\langle \bar{A}, \leqq\rangle$ such that $\mathrm{c}(\bar{A}) \leqq \omega$.

THEOREM 2.1. $\mathrm{Th}_2(\mathscr{K}_{\leqq}^{\omega})$, *the second-order theory of countable linearly ordered sets, is decidable.*

Proof. This is an almost trivial consequence of the decidability of S2S.

Let $B \subset T$ be the set of all sequences $x101$ such that $x101$ has *no* (consecutive) subsequence 101 except the one at the end. Thus if $x, y \in B$ and $x \leqq y$, then $x=y$. It can be easily verified that $\langle B, \preccurlyeq|B\rangle$ has the order type η of the set of rationals. This implies that for every $\langle \bar{A}, \leqq\rangle \in \mathscr{K}_{\leqq}^{\omega}$ there exists a set $A \subset T$ so that $\langle \bar{A}, \leqq\rangle \simeq \langle A, \preccurlyeq|A\rangle$.

Let F be any sentence of the second-order theory of linear ordering. Let F^A be the sentence obtained by replacing in F all occurrences of $\leqq$ by $\preccurlyeq$, relativizing individual quantifiers to A and relativizing set quantifiers to subsets of A. By the above, $F \in \mathrm{Th}_2(\mathscr{K}_{\leqq}^{\omega})$ if and only if $\forall A F^A \in$ S2S.

It has been observed (Corollary 1.9) that S2S remains decidable upon inclusion of set variables ranging over finite sets. The previous theorem may be strengthened in the same way. Combining this with the fact that the downward Skolem-Lowenheim theorem is valid for weak second-order logic, we get as a corollary the following result of Läuchli [9], which strengthens Ehrenfeucht's result [5].

COROLLARY 2.2. *The weak second-order theory of linearly ordered sets is decidable.*

In contrast with the treatment in [5], [9], we get here elementary recursive decision procedures.

Since the notion of well-ordering is obviously definable in $\mathrm{Th}_2\,(\mathscr{K}^{\omega}_{\leqq})$, we have the following result which is related to Büchi's Theorem 1′ of [2].

COROLLARY 2.3. *The second-order theory of countable well-ordered sets is decidable.*

It is not known whether the second-order theory of arbitrary well-ordered sets is decidable. By the same token, it is not known whether $\mathrm{Th}_2\,(\mathscr{K}_{\leqq})$, where $\mathscr{K}_{\leqq}$ is the class of all linearly ordered sets, is decidable. It may be that some sentence of second-order theory of linear ordering is independent of set theory. In this case, it will be impossible to produce a decision procedure for $\mathrm{Th}_2\,(\mathscr{K}_{\leqq})$ by means of arguments formulated within set theory.

Closely related to this is the following question. Does there exist a sentence F of second-order theory of linear ordering so that $\langle A, \leqq\rangle \models F$ if and only if $\mathrm{c}\,(A) \leqq \omega$? The existence of such a sentence would imply that Souslin's Hypothesis is expressible in this theory. Souslin's Hypothesis is known to be independent of set theory.

2.2. **Second-order theory of a unary function.** Let $\mathscr{K}_f$ be the class of all structures $\mathfrak{A}=\langle A,f\rangle$ where $f\colon A \to A$ is a (unary) function from A to A. By $\mathscr{K}_f^{\omega}$ we shall denote the class of all structures $\langle A,f\rangle \in \mathscr{K}_f$ with $\mathrm{c}\,(A)\leqq\omega$. The structures in $\mathscr{K}_f^{\omega}$ will be referred to, throughout this section, as *algebras*. Thus, the term "algebra" always implies countability. We shall list without proofs some simple observations about the structure of algebras.

Two elements $x, y \in A$ of an algebra $\mathfrak{A}=\langle A,f\rangle$ are called *connected* ($x \sim y$) if for some $n<\omega$, $m<\omega$, $f^n(x)=f^m(y)$. The relation $\sim$ is an equivalence relation. For $x \in A$, the equivalence class $\{y \mid y\sim x, y \in A\}$ is a subalgebra of $\mathfrak{A}$. An algebra $\mathfrak{A}$ is called *connected* if every two $x, y \in A$ are connected. Every algebra $\mathfrak{A}=\langle A,f\rangle$ is the cardinal sum of a countable collection of connected algebras; i.e., $A=\bigcup_{n<m\leqq\omega} A_n$, where each $\langle A_n, f|A_n\rangle$ is a connected subalgebra of $\mathfrak{A}$, and $A_n \cap A_k=\varnothing$ for $n<k<m$.

An algebra is called a *prime algebra* if it is one of the following:

$$\mathfrak{A}_n = \langle\{a_i \mid 0 \leqq i < n\}, f\rangle,$$

$1\leqq n\leqq\omega$, where, for $n<\omega$, $f(a_i)=a_{i+1}$, $0\leqq i<n-1$, $f(a_{n-1})=a_0$; and for $n=\omega$, $f(a_i)=a_{i+1}$. Every algebra contains at least one prime algebra. A connected algebra $\mathfrak{A}$ is said to be of *type n*, $1\leqq n\leqq\omega$, if $\mathfrak{A}$ contains an algebra $\mathfrak{A}_n$. The type of a connected algebra $\mathfrak{A}$ is uniquely determined by $\mathfrak{A}$.

Let $\mathfrak{A}=\langle A,f\rangle$ be a prime algebra. The *enveloping algebra* $\mathfrak{B}=\langle B, g\rangle\supset\mathfrak{A}$ of $\mathfrak{A}$ is defined as follows. Let $N=\omega-\{0\}$ be the set of positive integers. Set $B=AN^*$ and define $g(ai_1\cdots i_{k+1})=ai_1\cdots i_k$, $g(a)=f(a)$, for $a\in A$, $i_1\cdots i_{k+1}\in N^*$.

The following basic property of the enveloping algebra is easily verifiable. Let $\mathfrak{A}_n$ be a prime algebra of type n, and $\mathfrak{B} \supset \mathfrak{A}_n$ be its enveloping algebra. If $\mathfrak{C} \supseteq \bar{\mathfrak{A}}_n$ is a connected algebra of type n, where $\bar{\mathfrak{A}}_n$ is a prime algebra, then any isomorphism $\phi: \bar{\mathfrak{A}}_n \to \mathfrak{A}_n$ can be extended to a monomorphism $\bar{\phi}: \mathfrak{C} \to \mathfrak{B}$ of $\mathfrak{C}$ into $\mathfrak{B}$. Thus, every connected algebra is a subalgebra of an enveloping algebra. This implies that every algebra is embeddable in a (countable) cardinal sum of enveloping algebras.

THEOREM 2.4. $\mathrm{Th}_2(\mathscr{K}_f^{\omega})$, *the second-order theory of a unary function with a countable domain, is decidable.*

Proof. We shall interpret $\mathrm{Th}_2(\mathscr{K}_f^{\omega})$ in $\mathrm{S}\omega\mathrm{S} = \mathrm{Th}_2(\mathfrak{N}_\omega)$, which is decidable (Theorem 1.12). This will be done by constructing a relation $F(x, y, C)$ definable in $\mathfrak{N}_\omega$, so that for a fixed $C \subset T_\omega$, the set of pairs $\langle x, y\rangle$ for which $F(x, y, C)$ holds is a unary function f, and the algebra $\mathfrak{A}_C = \langle D(f), f\rangle$ is the cardinal sum of denumerably many enveloping algebras. Conversely, for every cardinal sum $\mathfrak{A}$ of denumerably many enveloping algebras, there will exist a C so that $\mathfrak{A} \simeq \mathfrak{A}_C$.

Assume for the moment the existence of such a relation and let $F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{C})$ be the formula of SωS defining it. The formula $Al(A, C)$ which is

$$\forall \boldsymbol{x} \exists \boldsymbol{y}[\boldsymbol{x} \in \boldsymbol{A} \to F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{C}) \wedge \boldsymbol{y} \in \boldsymbol{A}],$$

is true for $A \subseteq T_\omega$, $C \subset T_\omega$, if and only if $\langle A, f|A\rangle$ is a subalgebra of $\mathfrak{A}_C = \langle D(f), f\rangle$. By the remarks concerning algebras, for every $\mathfrak{A} \in \mathscr{K}_f^{\omega}$ there are sets $A, C \subseteq T_\omega$ so that $Al(A, C)$ holds, and $\mathfrak{A} \simeq \langle A, f|A\rangle$. If S is a sentence of second-order theory of a unary function, let $\bar{S}$ be the formula of SωS obtained from S by replacing $f(\boldsymbol{x}) = \boldsymbol{y}$ by $F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{C})$, relativizing all individual quantifiers to $\boldsymbol{A}$ and relativizing all set quantifiers to subsets of $\boldsymbol{A}$. We have $S \in \mathrm{Th}_2(\mathscr{K}_f^{\omega})$ if and only if

$$\forall A \forall C[Al(A, C) \to \bar{S}] \in \mathrm{S}\omega\mathrm{S}.$$

Thus, all that remains is to construct $F(x, y, C)$. We shall do this informally, leaving verification that the relation in question is definable in $\mathfrak{N}_\omega$, to the reader. Let $C \subset T_\omega$ be a set so that $C \subset \{0^n 1 0^m \mid n < \omega, m < \omega\}$, and for every $n < \omega$, C contains at most one word of the form $0^n 1 0^m$. For $n < \omega$, let $A_n = \{0^n 1 0^i \mid i \leqq m\}$ if $0^n 1 0^m \in C$, and $A_n = \{0^n 1 0^i \mid i < \omega\}$ if no $0^n 1 0^m$ is in C. Let $B_n = A_n N^*$ where $N = \omega - \{0\}$, and let $B = \bigcup_{n<\omega} B_n$.

The idea is to define $F(x, y, C)$ so that under the corresponding unary f, each subset $B_n \subset B$ will be an enveloping algebra. f will be the predecessor function for $x \neq 0^n 1 0^i$, and for $x = 0^n 1 0^i$ the definition will be different. The detailed definition follows.

Let pd (x) be the usual predecessor function on T_ω (see §1.3). Note that $\mathrm{pd}(x) = y$ is definable in $\mathfrak{N}_\omega$ by $\boldsymbol{x} = \boldsymbol{y} = \Lambda \vee \boldsymbol{y} < \boldsymbol{x} \wedge \forall \boldsymbol{z} \sim [\boldsymbol{y} < \boldsymbol{z} < \boldsymbol{x}]$. Let $F(x, y, C)$ hold if and only if C has the above mentioned property, $x \in B_n$ for some n, and y satisfies the following. If $x = 0^n 1 0^i z$, $\Lambda \neq z \in N^*$ then $y = \mathrm{pd}(x)$; if $x = 0^n 1 0^i \notin C$, then $y = r_0(x) = 0^n 1 0^{i+1}$; if $x = 0^n 1 0^m \in C$ then $y = 0^n 1$. Let $f: B \to B$ be the mapping such that $f(x) = y$ if and only if $F(x, y, C)$.

For a fixed $n<\omega$, if $0^n10^m \in C$, then $\langle A_n, f|A_n\rangle$ is the prime algebra of type $m+1$, and if no 0^n10^m is in C then $\langle A_n, f|A_n\rangle$ is the prime algebra of type ω. In either case, $\mathfrak{B}_n=\langle B_n, f|B_n\rangle$ is the enveloping algebra of $\langle A_n, f|A_n\rangle$. Thus $\langle B, f\rangle$ is the cardinal sum of the enveloping algebras $\mathfrak{B}_n$, $n<\omega$, and every denumerable cardinal sum is obtained in this way by an appropriate choice of $C\subset T_\omega$. This concludes the proof(3).

We may again strengthen our result by including in the language set variables $\boldsymbol{a}, \boldsymbol{b}, \ldots$, ranging over finite sets and still retaining decidability (see Corollary 1.9). As in the case of linearly ordered sets, we get

COROLLARY 2.5. *The weak second-order theory of a unary function is decidable.*

This is a strengthened version of Ehrenfeucht's result [4] where he announced the decidability of the first-order theory of a unary function (Th_1 ($\mathscr{K}_f$) in our notation). We again get, both for Theorem 2.4 and Corollary 2.5, that the theories in question are elementary recursive.

2.3. **Subsets of** $\{0, 1\}^\omega$. Let $\mathrm{CD}=\{0, 1\}^\omega$, and introduce on CD the usual product topology. As is well known, CD is essentially the same as Cantor's discontinuum (ternary set).

There is a natural one-to-one correspondence between CD and the set of paths $\pi\subset T$ of the binary tree $T=T_2$. Namely, each path π is simply the set of all (finite) initials of a unique element $\phi\colon \omega\to\{0, 1\}$ of CD. Thus, we shall view the paths $\pi\subset T$ as elements of CD, and sets of paths as subsets of CD.

We wish to define in S2S subsets of CD. This is not directly possible because the paths $\pi\subset T$ are already sets and, therefore, sets of paths are third-order objects (sets of subsets of T). An indirect way for defining subsets of CD is to consider a formula $F(\boldsymbol{B}, A)$ of S2S of the form $G(\boldsymbol{B}, A) \wedge \mathrm{Path}\ (\boldsymbol{B})$, where $\mathfrak{N}_2 \models \mathrm{Path}\ (B)$ if and only if $B\subset T$ is a path. Such a formula gives a mapping $f\colon P(T)\to P(\mathrm{CD})$ defined by $f(A)=\{\pi \mid \pi\subset T, \mathfrak{N}_2 \models F(\pi, A)\}$. When A ranges over subsets of T, $f(A)$ ranges over a class of subsets of CD. Appropriate choices of F will produce interesting classes $\{f(A) \mid A\subseteq T\}\subseteq P(\mathrm{CD})$.

THEOREM 2.6. *Let* Cl $(\boldsymbol{B}, A)$ *be* $[\boldsymbol{B}\subseteq A]\wedge \mathrm{Path}\ (\boldsymbol{B})$ *and* $F_\sigma(\boldsymbol{B}, A)$ *be* $\mathrm{Fn}\ (A\cap \boldsymbol{B}) \wedge \mathrm{Path}\ (\boldsymbol{B})$ (*see Lemma* 1.8). cl (A) *ranges over all closed subsets of* CD, *and* $f_\sigma(A)$ *ranges over all* F_σ (*countable unions of closed sets*) *subsets of* CD. *Here* cl *and* f_σ *are the mappings corresponding to* Cl *and* F_σ *in the above explained manner.*

Proof. That for every $A\subseteq T$ the set cl $(A)=\{\pi \mid \pi\subset T, \pi\subseteq A\}$ is a closed subset of CD is trivial. Conversely, let $S\subseteq \mathrm{CD}$ be a closed set. Let $A=\bigcup_{\pi\in S}\pi$. We have cl $(A)=S$. Note that for this last A, if $\pi\not\subseteq A$, then c $(\pi\cap A)<\omega$.

(3) The interpretation of the second-order theory of unary functions with a countable domain in S2S, was noted independently by J. J. Le Tourneau in his thesis.

Let $A \subseteq T$; we have $f_\sigma(A) = \{\pi \mid \mathrm{c}(\pi \cap A) < \omega\} = \bigcup_{n<\omega} S_n$, where

$$S_n = \{\pi \mid \mathrm{c}(\pi \cap A) \leqq n\}.$$

Each set S_n is closed; hence, $f_\sigma(A) \in F_\sigma$.

Conversely, let $S \subseteq \mathrm{CD}$ be an F_σ set. We may assume $S = \bigcup_{n<\omega} S_n$, where S_n is closed and $S_n \subseteq S_{n+1}$, for $n<\omega$. Let $A_n = \bigcup_{\pi \in S_n} \pi$; then $S_n = \mathrm{cl}(A_n)$ and $A_n \subseteq A_{n+1}$, for $n<\omega$.

If $\pi \notin S$, then for every $n<\omega$, $\mathrm{c}(\pi \cap A_n) < \omega$. This enables us to define a sequence $x_n(\pi)$, $n<\omega$, as follows. $x_0(\pi) = \min_{x \in \pi - A_0} x$ (i.e., $x_0(\pi)$ is the smallest node, with respect to $\leqq$, in π but not in A_0). Inductively, $x_{n+1}(\pi) = \min_{x_n(\pi) < x \in \pi - A_n} x$. Note that if $\pi', \pi'' \notin S$ and $x_n(\pi') \in \pi''$, then $x_n(\pi') = x_n(\pi'')$; this is proved by induction on n.

Let $A = \{x_n(\pi) \mid \pi \notin S, n<\omega\}$. We claim that $f_\sigma(A) = S$. If $\pi \notin S$ then $x_n(\pi) \in \pi \cap A$ for $n<\omega$, so that $\mathrm{c}(\pi \cap A) = \omega$ and $\pi \notin f_\sigma(A)$. Hence, $f_\sigma(A) \subseteq S$. Assume $\pi \in S$; then $\pi \subseteq A_n$ for some $n<\omega$ and hence $\pi \subseteq A_{n+k}$, $k<\omega$. This implies that if $x_m(\pi') \in \pi$ for a $\pi' \notin S$, then $0 \leqq m < n$. We claim $\mathrm{c}(\pi \cap A) < n+1$ and hence $\pi \in f_\sigma(A)$, which will establish $S \subseteq f_\sigma(A)$. Otherwise, there exist $x_{m_0}(\pi_0), \ldots, x_{m_n}(\pi_n)$, which are pairwise different and elements of π. From $m_i < n$, $0 \leqq i \leqq n$, it follows that for some $0 \leqq i < j \leqq n$, $m_i = m_j = m$. Now $x_m(\pi_i) \in \pi$ and $x_m(\pi_j) \in \pi$. Therefore, they are comparable and, say, $x_m(\pi_i) < x_m(\pi_j)$. Hence, $x_m(\pi_i) \in \pi_j$; but this contradicts $x_m(\pi_i) \neq x_m(\pi_j)$.

THEOREM 2.7. *Let* $\mathfrak{C} = \langle \mathrm{CD}, \leqq \rangle$ *be Cantor's discontinuum with the usual ordering; i.e.,* $\pi < \pi'$ *if for some* $z \in T$, $z0 \in \pi$ *and* $z1 \in \pi'$. *Let L be a language appropriate to* $\mathfrak{C}$ *which has (besides the individual variables) set variables* $\boldsymbol{C}_1, \boldsymbol{C}_2, \ldots$ *ranging over closed subsets of* CD, *and set variables* $\boldsymbol{D}_1, \boldsymbol{D}_2, \ldots$ *ranging over* F_σ *subsets of* CD. Th $(\mathfrak{C}, L)$ *is decidable.*

Proof. Let S be a sentence of L. Let $\bar{S}$ be the sentence of S2S obtained from S by replacing $\boldsymbol{x}_i \leqq \boldsymbol{x}_j$ with $\boldsymbol{B}_i = \boldsymbol{B}_j \vee \exists \boldsymbol{x}[\boldsymbol{x}0 \in \boldsymbol{B}_i \wedge \boldsymbol{x}1 \in \boldsymbol{B}_j]$, replacing $\boldsymbol{x}_i \in \boldsymbol{C}_j$ with $\mathrm{Cl}(\boldsymbol{B}_i, \boldsymbol{C}_j)$, replacing $\boldsymbol{x}_i \in \boldsymbol{D}_j$ with $F_\sigma(\boldsymbol{B}_i, \boldsymbol{D}_j)$, and replacing all quantifiers $\exists \boldsymbol{x}_i$ or $\forall \boldsymbol{x}_i$ by $\exists \boldsymbol{B}_i$ or $\forall \boldsymbol{B}_i$ relativized to Path $(\boldsymbol{B}_i)$. We have that $S \in \mathrm{Th}(\mathfrak{C}, L)$ if and only if $\bar{S} \in \mathrm{S2S}$.

THEOREM 2.8. *Let* $\mathfrak{B} = \langle F_\sigma, \cup, \cap, L_c \rangle$ *be the lattice of* F_σ*-subsets of* CD, *with the lattice of all closed subsets of* CD *as a distinguished sublattice (i.e.,* $x \in L_c$ *if and only if x is a closed subset of* CD). $\mathrm{Th}_1(\mathfrak{B})$ *is decidable.*

This is a trivial consequence of Theorem 2.7.

The above results carry over from CD to the segment [0, 1] with the usual topology and order. For $\pi_1, \pi_2 \in \mathrm{CD}$, define an equivalence $\sim$ by $\pi_1 \sim \pi_2$ if and only if $\pi_1 = \pi_2$, or for some $x \in T$, $x10^n \in \pi_1$, $n<\omega$, and $x01^n \in \pi_2$, $n<\omega$, or vice versa with π_1 and π_2 interchanged. The quotient space $\mathrm{CD}/\sim$ is homeomorphic with [0, 1].

THEOREM 2.9. *Let* $I = \langle [0, 1], \leqq \rangle$ *be the unit interval with the usual ordering, and let the language L be the same as in Theorem* 2.7. Th (I, L) *is decidable.*

Proof. The relation $\sim$ between paths of T is definable in S2S. Let S be a sentence of L and let $\bar{S}$ be as in the proof of Theorem 2.7. Replace in $\bar{S}$ all subformulas

$\boldsymbol{B}_i=\boldsymbol{B}_j$ by $\boldsymbol{B}_i\sim\boldsymbol{B}_j$ (recall that the variables $\boldsymbol{B}_i$ in $\bar{S}$ are all relativized to Path $(\boldsymbol{B}_i)$), Cl $(\boldsymbol{B}_i, \boldsymbol{C}_j)$ by $\exists \boldsymbol{B}_{i+1}[\boldsymbol{B}_i\sim\boldsymbol{B}_{i+1}\wedge \mathrm{Cl}(\boldsymbol{B}_{i+1}, \boldsymbol{C}_j)]$, and similarly for $F_\sigma(\boldsymbol{B}_i, \boldsymbol{D}_j)$. The resulting sentence S' is true in $\mathfrak{N}_2$ if and only if $S\in \mathrm{Th}\,(I, L)$.

This last result answers in the affirmative Grzegorczyk's question [8] whether the first-order theory of the lattice of all closed subsets of the real line is decidable.

2.4. **Boolean algebras.** Denote the class of all boolean algebras by $\mathscr{K}_{\mathrm{B}}$, and the class of countable boolean algebras by $\mathscr{K}_{\mathrm{B}}^{\omega}$. Let L_I be the language appropriate for $\mathscr{K}_{\mathrm{B}}$, which has set variables $\boldsymbol{I}, \boldsymbol{J}, \ldots$, ranging over *ideals* of the boolean algebras.

THEOREM 2.10. Th $(\mathscr{K}_{\mathrm{B}}^{\omega}, L_I)$, *the theory of countable boolean algebras with quantification over ideals, is decidable.*

Proof. Let $\mathfrak{B}_\omega=\langle B, \cup, \cap, '\rangle$ be the free boolean algebra on a denumerable number of generators. By Stone's representation theorm (see [14, Sections 8, 14]), $\mathfrak{B}_\omega$ is isomorphic by a mapping $\phi\colon \mathfrak{B}_\omega\to P(\mathrm{CD})$ with the algebra of all closed-open (clopen) subsets of CD([4]). The ideals $I\subseteq\mathfrak{B}_\omega$ stand on a one-to-one correspondence with the *open* subsets of CD by the mapping $U(I)=\bigcup_{b\in I}\phi(b)\subseteq \mathrm{CD}$. Thus, we have $b\in I$ if and only if $\phi(b)\subseteq U(I)$. The notions of open subset of CD, closed-open, and the boolean operations on clopen sets, are all definable in Th $(\mathfrak{C}, L)$. Thus the decidability of Th $(\mathfrak{B}_\omega, L_I)$ follows from Theorem 2.7.

Now, the arbitrary algebra $\mathfrak{B}\in\mathscr{K}_{\mathrm{B}}^{\omega}$ is isomorphic with $\mathfrak{B}_\omega/I$ for an appropriate ideal $I\subseteq\mathfrak{B}_\omega$, and the ideals $J'\subseteq\mathfrak{B}$ stand in a one-to-one correspondence with the ideals J, $I\subseteq J\subseteq\mathfrak{B}_\omega$. This implies the decidability of Th $(\mathscr{K}_{\mathrm{B}}^{\omega}, L_I)$.

As a corollary we get the following improvement of Tarski's result [15] to the effect that the first-order theory of boolean algebras is decidable.

THEOREM 2.11. *The first-order theory of boolean algebras with a sequence of distinguished ideals is decidable.*

Proof. The class, call it $\mathscr{K}_{\mathrm{B}I}$, of structures in question consists of all boolean algebras $\mathfrak{B}=\langle B, \cup, \cap, ', I_n\rangle_{n<\omega}$, where I_n is an ideal of $\mathfrak{B}$ for $n<\omega$. Let F be a sentence of the first-order language of $\mathscr{K}_{\mathrm{B}I}$ and let the list $\boldsymbol{I}_0, \ldots, \boldsymbol{I}_m$ include all the ideal-constants appearing in F. It follows from the Skolem-Lowenheim theorem that $F\in \mathrm{Th}_1\,(\mathscr{K}_{\mathrm{B}I})$ if and only if $\forall \boldsymbol{I}_0\cdots\forall \boldsymbol{I}_m F\in \mathrm{Th}\,(\mathscr{K}_{\mathrm{B}}^{\omega}, L_I)$.

Theorem 2.11 also implies a result of Ershov [6, Theorem 9] to the effect that the first-order theory of Boolean algebras with a distinguished *maximal* ideal, is decidable.

2.5. **Games on** $\{0, 1\}^\omega$. Let $D\subseteq \mathrm{CD}=\{0, 1\}^\omega$. With D we associate the Gale-Stewart [7] game, which is played by two players 1 and 2. Player 1 picks an $\varepsilon_1\in\{0, 1\}$, player 2 then picks an $\varepsilon_2\in\{0, 1\}$, and so they alternate ad infinitum. If the result $\pi=\varepsilon_1\varepsilon_2\varepsilon_3\cdots$ of the play satisfies $\pi\in D$, then player 1 won that play; if $\pi\notin D$, then player 2 won the play.

([4]) This connection was pointed out to me by D. Scott.

A *strategy* is a mapping $f: \{0, 1\}^* \to \{0, 1\}$. A sequence $\pi = \varepsilon_1\varepsilon_2 \cdots$ has been played by player 1 according to the strategy f, if $f(\Lambda) = \varepsilon_1$ and $\varepsilon_{2n+1} = f(\varepsilon_1 \cdots \varepsilon_{2n})$, $1 \leqq n < \omega$. The strategy f is *winning* for player 1 if every π played by 1 according to f satisfies $\pi \in D$. The notion of a winning strategy for player 2 is similar.

The set $D \subseteq \mathrm{CD}$ is called *determinate* if one of the two players has a winning strategy. With the aid of the axiom of choice, it can be shown that not every $D \subseteq \mathrm{CD}$ is determinate. Gale and Stewart [7] have shown that if D is open or closed, then it is determinate. Wolfe [17] proved that if $D \in F_\sigma$, then it is determinate. The best result thus far is due to M. Davis who has proved that $D \in F_{\sigma\delta}$ implies determinacy.

We shall show that the statement that every $D \in F_\sigma$ is determinate, is expressible in the second-order language of $\mathfrak{N}_2$. Thus, Wolfe's theorem is among those decided by the decision procedure for S2S.

A strategy $f: \{0, 1\}^* \to \{0, 1\}$ of the Gale-Stewart game can be viewed as the characteristic function of a set $A \subseteq T$; namely, $A = \{x \mid f(x) = 1\}$. Note that the sets $E_1 = \{x \mid l(x) = 2n, n < \omega\}$, $E_2 = \{x \mid l(x) = 2n+1, n < \omega\}$, are definable in S2S by appropriate formulas $E_1(\boldsymbol{x})$ and $E_2(\boldsymbol{x})$.

Let $W_i(\boldsymbol{A}, \boldsymbol{D})$, $i = 1, 2$, be the formula

$$\forall \boldsymbol{B}[\mathrm{Path}\,(\boldsymbol{B}) \land \forall \boldsymbol{x}[E_i(\boldsymbol{x}) \land \boldsymbol{x} \in \boldsymbol{B} \to [\boldsymbol{x} \in \boldsymbol{A} \leftrightarrow \boldsymbol{x}1 \in \boldsymbol{B}]] \to F_\sigma^i(\boldsymbol{B}, \boldsymbol{D})]$$

where F_σ^1 is just F_σ, and F_σ^2 is $\sim F_\sigma$. $\mathfrak{N}_2 \models W_i(A, D)$ if and only if χ_A is a winning strategy for player i in the game associated with $f_\sigma(D)$.

The sentence of S2S asserting that every set in F_σ is determinate reads

$$\forall \boldsymbol{D} \exists \boldsymbol{A}[W_1(\boldsymbol{A}, \boldsymbol{D}) \lor W_2(\boldsymbol{A}, \boldsymbol{D})].$$

Chapter III. Automata on infinite trees

The basic definitions and some preliminary results concerning f.a. on infinite trees were given in §1.4. This chapter will be devoted to the proofs of Theorems 1.5 and 1.6 which were stated in that section. This will involve a number of auxiliary concepts and results.

3.1. **Sequential automata.** We shall briefly recall some notions and results concerning automata on infinite sequences.

Let Σ be a finite set; Σ^ω is the set of all ω-sequences on Σ.

A (sequential) Σ-table is a system $\mathfrak{A}' = \langle S, M \rangle$ where S is a finite set and $M: S \times \Sigma \to P(S)$. If $\mathrm{c}\,(M(s, \sigma)) = 1$ for all $s \in S$, $\sigma \in \Sigma$, then $\mathfrak{A}'$ is called *deterministic*.

An $\mathfrak{A}'$-run on $v \in \Sigma^\omega$ is a mapping $r: \omega \to S$ such that $r(n+1) \in M(r(n), v(n))$, $n < \omega$.

A (nondeterministic) finite automaton (see [13]) is a system $\mathfrak{A} = \langle S, M, S_0, F \rangle$, where $\langle S, M \rangle$ is as above, $S_0 \subseteq S$, $F \subseteq S$. Following Büchi [1], we say that $\mathfrak{A}$ *accepts* $v \in \Sigma^\omega$ if for some $\mathfrak{A}$-run r on v, $r(0) \in S_0$ and $\mathrm{In}\,(r) \cap F \neq \varnothing$. The set of all $v \in \Sigma^\omega$ which are accepted by $\mathfrak{A}$ is denoted by $T_\omega(\mathfrak{A})$. A set $A \subseteq \Sigma^\omega$ is called *f.a. definable* (in Büchi's sense), if for some f.a. $\mathfrak{A}$, $A = T_\omega(\mathfrak{A})$.

D. Muller [11] formulated a different notion of acceptance of a sequence by an automaton. Let $\mathfrak{A}=\langle S, M, s_0, F\rangle$, where $\langle S, M\rangle$ is a *deterministic* table, $s_0 \in S$, and $F \subseteq P(S)$. Following Muller, we say that $\mathfrak{A}$ *accepts* $v \in \Sigma^\omega$ if for the (unique) $\mathfrak{A}$-run r on v which satisfies $r(0)=s_0$, $\mathrm{In}\,(r) \in F$ holds. Again, we denote by $T_\omega(\mathfrak{A})$ the set of all $v \in \Sigma^\omega$ accepted by the (Muller) automaton $\mathfrak{A}$.

McNaughton [10] proved the following fundamental result. *Every set $A \subseteq \Sigma^\omega$ f.a. definable in Büchi's sense is f.a. definable in Muller's sense, and vice versa.*

We may further generalize the notion of automaton by combining the approaches of Büchi and Muller. Thus in Muller's definition, allow $\langle S, M\rangle$ to be nondeterministic and say that $\mathfrak{A}$ accepts $v \in \Sigma^\omega$ if for *some* $\mathfrak{A}$-run r on v, $r(0)=s_0$ and $\mathrm{In}\,(r) \in F$. Again we do not obtain any new definable sets. This can be seen, for example, by showing that every generalized Muller automaton is equivalent with an appropriate Büchi automaton.

The class of f.a. definable sets $A \subseteq \Sigma^\omega$ is a boolean algebra (Büchi [1]).

3.2. **Generalization of acceptance by automata.** Let (r, T) be an S-tree $(r\colon T \to S)$, and $\pi \subset T$ be a path. With $r|\pi$ we associate an ω-sequence $(r|\pi)_\omega$ of elements of S as follows. Let $\pi=\{x_n \mid n<\omega\}$ where $x_0<x_1<\cdots$. Define $(r|\pi)_\omega(n)=r(x_n)$. This makes $(r|\pi)_\omega$ a mapping from the set ω into S; i.e., $(r|\pi)_\omega \in S^\omega$.

The definition (1.5) of the notion of an automaton $\mathfrak{A}$ accepting a tree $t=(v, T_x)$ involves the condition $\mathrm{In}\,(r|\pi) \in F$, i.e., $\mathrm{In}\,((r|\pi)_\omega) \in F$. This condition on the ω-sequence $(r|\pi)_\omega \in S^\omega$ is recognizable by a sequential S-automaton. The following theorem states that using any (sequential) automaton-definable condition on the sequences $(r|\pi)_\omega$, we still get just f.a. definable sets of Σ-trees.

THEOREM 3.1. *Let $\mathfrak{A}=\langle S, M\rangle$ be a table over Σ, and $B \subseteq S^\omega$ be a f.a. definable set of ω-sequences on S. Define $C \subseteq V_\Sigma$ to be the set of Σ-trees $t=(v, T_x)$ such that for some run $r \in \mathrm{Rn}\,(\mathfrak{A}, t)$, and every path $\pi \subset T_x$, we have $(r|\pi)_\omega \in B$. The set C is f.a. definable.*

Proof. Let $\mathfrak{B}=\langle U, K, u_0, H\rangle$ be a (sequential) deterministic automaton over S such that $T_\omega(\mathfrak{B})=B$. Note that $H \subseteq P(U)$. Define a Σ-automaton $\mathfrak{C}=\langle U\times S, \bar{M}, \{u_0\}\times S, \bar{F}\rangle$ as follows.

For $(u, s) \in U\times S$, $\sigma \in \Sigma$, define

$$\bar{M}((u, s), \sigma) = \{\langle (K(u, s), s_1), (K(u, s), s_2)\rangle \mid (s_1, s_2) \in M(s, \sigma)\}.$$

Let $p_1\colon (x, y) \mapsto x$, $p_2\colon (x, y) \mapsto y$ be the projection functions. For every run r of $\langle U\times S, \bar{M}\rangle$ on a Σ-tree $t=(v, T)$, $p_2 r\colon T \to S$ is an $\mathfrak{A}$-run on t, and every $\mathfrak{A}$-run is obtained in this way. Also, along any path $\pi \subset T$, $(p_1 r|\pi)_\omega$ is a $\mathfrak{B}$-run on the sequence $(p_2 r|\pi)_\omega$.

Define now $\bar{F}=\{A \mid A \subseteq U\times S, p_1(A) \in H\}$. We have $T(\mathfrak{C})=C$.

Note that in the previous proof, we assumed that $B \subseteq S^\omega$ is defined by a deterministic (Muller) automaton. In applying Theorem 3.1, we shall usually prove the f.a. definability of the $B \subseteq S^\omega$ in question by exhibiting a nondeterministic $\mathfrak{B}$

such that $B=T_\omega(\mathfrak{B})$ (in the generalized Muller sense). McNaughton's theorem (see §3.1) assures us of the equivalence of the two notions of f.a. definability.

3.3. **Marked Σ-trees.**

DEFINITION 3.1. Let Q be a finite set, $H\subseteq T_x\times Q$, and $(v, T_x)=t$ be a Σ-tree. The *tree t marked at H*, is the $\Sigma\times P(Q)$-tree $\bar{t}=(\bar{v}, T_x)$ such that for $y\in T_x$, $\bar{v}(y)=(v(y), \{q \mid (y, q)\in H\})$.

Let Q be a finite set and let C_q, for $q\in Q$, be a f.a. definable set of $\Sigma\times P(Q)$-trees. In the sequel we shall consider sets A of Σ-trees (v, T) satisfying an iterative condition which is, roughly, as follows. $(v, T)\in A$ if and only if there exist a $q_0\in Q$ and a set $H^{\Lambda,q_0}\subseteq(T-\{\Lambda\})\times Q$ so that the tree (v, T) marked at H^{Λ,q_0} is in C_{q_0}. Furthermore, for every $(x, q)\in H^{\Lambda,q_0}$, there exists a set $H^{x,q}\subseteq(T_x-\{x\})\times Q$ so that (v, T_x) marked at $H^{x,q}$ is in C_q. And so on. We shall need the fact that such an $A\subseteq V_\Sigma$ is f.a. definable. This situation is made precise in the following

LEMMA 3.2. *Let Q be a finite set, $q_0\in Q$, and let $\mathfrak{A}_q$, $q\in Q$, be a $\Sigma\times P(Q)$-automaton. Define an invariant (see §1.4) set $A\subseteq V_\Sigma$ by the condition: $(v, T)\in A$ if and only if there exist a set $H\subseteq T\times Q$ and a mapping $(x, q)\mapsto H^{x,q}$, $(x, q)\in H$, such that (1) $(\Lambda, q_0)\in H$; (2) $H^{x,q}\subseteq(T_x-\{x\})\times Q$, and $H^{x,q}\subseteq H$; (3) for $(x, q)\in H$, the tree (v, T_x) marked at $H^{x,q}$ is accepted by $\mathfrak{A}_q$.*

The set is f.a. definable.

The proof of this lemma will be given in §3.6.

3.4. **Well-founded mappings.** We shall also require a version of Lemma 3.2 with the additional assumption on $(x, q)\mapsto H^{x,q}$ that it is well-founded.

DEFINITION 3.2. Let $H\neq\varnothing$ be a set and let $x\mapsto H^x$, $x\in H$, $H^x\subseteq H$, be a mapping. We shall say that this mapping is *well-founded* if every sequence $x_1, x_2, \ldots\in H$ such that $x_{k+1}\in H^{x_k}$, $k=1, 2, \ldots$, is finite.

LEMMA 3.3. *The mapping $x\mapsto H^x$, $x\in H$, $H^x\subseteq H$, is well-founded if and only if there exists a decomposition $H=\bigcup_{\alpha<\mu} H_\alpha$ (μ is an ordinal) such that $\alpha<\beta$ implies $H_\alpha\cap H_\beta=\varnothing$, and for every $x\in H_\alpha$, $\alpha<\mu$, $H^x\subseteq\bigcup_{\lambda<\alpha} H_\lambda$.*

Proof. That from the existence of such a decomposition follows the well-foundedness of $x\mapsto H^x$ is clear.

Assume now that the mapping is well-founded. Define $H_0=\{x \mid x\in H, H^x=\varnothing\}$. Note that $H_0\neq\varnothing$. Define by transfinite induction

$$H_\alpha = \Big\{x \mid x\in H, x\notin \bigcup_{\lambda<\alpha} H_\lambda, H^x\subseteq \bigcup_{\lambda<\alpha} H_\lambda\Big\}.$$

The sets H_α are mutually disjoint. Let μ be the smallest ordinal such that $H_\mu=\varnothing$. Assume that $D=H-\bigcup_{\alpha<\mu} H_\alpha\neq\varnothing$. For every $x\in D$ there exists a $y\in H^x$ such that $y\in D$. This entails the existence of an ω-sequence $(x_k)_{k<\omega}$ such that $x_{k+1}\in H^{x_k}$, $k<\omega$, a contradiction.

LEMMA 3.4. *With the same notations as in Lemma 3.2, define an invariant subset $W\subseteq V_\Sigma$ by the condition: $(v, T)\in W$ if and only if there exist a set $H\subseteq T\times Q$ and* *a mapping* $(x, q) \mapsto H^{x,q}$, $(x, q) \in H$, *such that conditions* 1–3 *of Lemma* 3.2 *hold, and in addition*: (4) *the mapping* $(x, q) \mapsto H^{x,q}$ *is well-founded. The set W is f.a. definable.*

The proof of this lemma will be given in §3.7.

3.5. **Simultaneous runs.** Trying to recognize whether a Σ-tree $t=(v, T)$ is in the set A of Lemma 3.2, we may proceed as follows. For each $q \in Q$ we construct a Σ-automaton $\mathfrak{B}_q$ which can move on a Σ-tree $t=(v, T)$ and accept it only if for some $H^{\Lambda,q} \subseteq (T-\{\Lambda\}) \times Q$, the tree t marked at $H^{\Lambda,q}$ is accepted by $\mathfrak{A}_q$. The construction of $\mathfrak{B}_q$ is essentially the one used in the proof of Theorem 1.4. Intuitively speaking, what $\mathfrak{B}_q$ does is to "guess" at each $x \in T$ how $v(x)$ can be supplemented by a $Q' \in P(Q)$ so that on the pair $(v(x), Q') \in \Sigma \times P(Q)$, $\mathfrak{A}_q$ will make the "correct" transition. Now, if at $x \in T$ the set Q' was used, and $q' \in Q'$, then the tree (v, T_x) should be accepted by $\mathfrak{B}_{q'}$; to check this, we must run a copy of $\mathfrak{B}_{q'}$ on (v, T_x). In this way, more and more copies of the automata $\mathfrak{B}_q$, $q \in Q$ are activated, and this process cannot be directly described by a finite automaton.

The crucial observation is that for any $y \in T$, even though many copies of a $\mathfrak{B}_q$ may have been activated at various $x<y$, at y the number of *different* states of $\mathfrak{B}_q$ which appear is still bounded by the cardinality of the set of all states of $\mathfrak{B}_q$. Thus, all the copies of $\mathfrak{B}_q$ reaching y in the same state s can be replaced by just one of these copies. In this way, we have, at any node y, just a bounded number of copies of each $\mathfrak{B}_q$, and this can be described by a finite Σ-table. In addition to having copies of $\mathfrak{B}_q$ move on (v, T), we will also need to record which copies merged when reaching the same state. The above considerations motivate the following formal definition of a Σ-table $\mathfrak{B}$.

In order to simplify notations, we shall formulate Definition 3.3 and prove Lemmas 3.2 and 3.4 for the case $\mathrm{c}(Q)=1$. The treatment of the general case, however, will be essentially the same.

Let $Q=\{q\}$ and let $\mathfrak{A}_q=\mathfrak{A}=\langle S, M, s_0, F\rangle$ be a $\Sigma \times \{\varnothing, \{q\}\}$-automaton. Assume $\mathrm{c}(S)=n$, and denote $[n+1]=\{1, \ldots, n+1\}$.

DEFINITION 3.3. For $\mathfrak{A}$ as above, define the Σ-table $\mathfrak{B}=\langle S^{\mathfrak{B}}, M^{\mathfrak{B}}\rangle$ as follows: Set $U=\{u\colon E \to S \mid u \text{ is } 1-1,\ E \subseteq [n+1]\}$; $D=\{d=(d_1, d_2, d_3) \mid d_1 \subseteq [n+1];\ d_1 \neq \varnothing$ *implies* $d_2=\{m\} \subseteq [n+1];\ d_1=\varnothing$ *implies* $d_2=\varnothing;\ d_3\colon E \to E \subseteq [n+1]\}$. Define now $S^{\mathfrak{B}}$ and $M^{\mathfrak{B}}$ by $S^{\mathfrak{B}}=\{(u, d) \mid u \in U,\ d \in D,\ d_1 \subseteq \mathrm{D}(u),\ d_2 \neq \varnothing$ *implies* $d_2 \nsubseteq \mathrm{D}(u)$, $\mathrm{R}(d_3)=\mathrm{D}(u)\}$, where $\mathrm{R}(\phi)$ and $\mathrm{D}(\phi)$ denote the range and domain of ϕ;

$$((u', d'), (u'', d'')) \in M^{\mathfrak{B}}((u, d), \sigma)$$

if and only if $\mathrm{D}(d_3')=\mathrm{D}(d_3'')=\mathrm{D}(u) \cup d_2$ and

$$\begin{aligned}(u'(d_3'(m)), u''(d_3''(m))) &\in M(s_0, (\sigma, \varnothing)), && \text{for } m \in d_2,\\ &\in M(u(m), (\sigma, \{q\})), && \text{for } m \in d_1,\\ &\in M(u(m), (\sigma, \varnothing)), && \text{for } m \in \mathrm{D}(u)-d_1.\end{aligned}$$

For later use, we also introduce an initial state $s_0^{\mathfrak{B}} \in S^{\mathfrak{B}}$ defined by

$$s_0^{\mathfrak{B}} = (\{\langle 1, s_0\rangle\}, (\varnothing, \varnothing, \{\langle 1, 1\rangle\})).$$

The interpretation is that if $(u, d) \in S^{\mathfrak{B}}$, $\mathrm{D}(u)=\{m_1, \ldots, m_k\}$, and $d=(d_1, d_2, d_3)$, then the "copies" $m_1, \ldots, m_k$, of $\mathfrak{A}$ are active and in the states $u(m_1), \ldots, u(m_k)$; the copies $m' \in d_1 \subseteq \mathrm{D}(u)$ behave as if they see $(\sigma, \{q\})$; the new copy $m \in d_2$ is activated in the state s_0; and, finally, the copies $m_1, \ldots, m_k$, are the replacements, by the mapping d_3, of the copies $m \in \mathrm{D}(d_3)$ of $\mathfrak{A}$ active at the *predecessor node*.

DEFINITION 3.4. Let $r \in \mathrm{Rn}(\mathfrak{B}, (v, T))$ and $r(\Lambda)=s_0^{\mathfrak{B}}$. Denote $r(x)=(u^x, d^x)$, $d^x=(d_1^x, d_2^x, d_3^x)$. We shall say that $m \in [n+1]$ is *active at* $x \in T$ if $m \in \mathrm{D}(u^x) \cup d_2^x$. Also, $m \in [n+1]$ is *activated* at $x \in T$ if $m=1$ and $x=\Lambda$, or $m \in d_2^x$.

Let m be active at x and let $y=x0$ or $y=x1$. We say that *m at x is replaced by m_1 at y* $((m, x) \to (m_1, y))$ if $m_1=d_3^y(m)$. The notion of replacement is extended by passing to the transitive closure. Thus, assume $x<y$, $x=x_0$, $x_{i+1}=x_i\varepsilon_i$, $\varepsilon_i \in \{0, 1\}$, $0 \leqq i \leqq k-1$, and $y=x_k$. We shall say that *m at x is replaced by m' at y* $((m, x) \to (m', y))$, if for a sequence $m_i \in [n+1]$, $0 \leqq i \leqq k$, $m_0=m$, $m_k=m'$, and $(m_i, x_i) \to (m_{i+1}, x_{i+1})$, $0 \leqq i \leqq k-1$.

LEMMA 3.5. *Let $r \in \mathrm{Rn}(\mathfrak{B}, (v, T))$, $(v, T) \in V_\Sigma$, and $r(\Lambda)=s_0^{\mathfrak{B}}$. Denote again $r(x)$ by (u^x, d^x). Define $H=(\{\Lambda\} \cup \{x \mid d_2^x \neq \varnothing\}) \times \{q\}$.*

For $(x, q) \in H$, let m be activated at x; define

$$H^{x,q} = \{y \mid x \neq y \in T_x, \exists m'[(m, x) \to (m', y) \text{ and } m' \in d_1^y]\} \times \{q\}.$$

For $(x, q) \in H$ and m as above, define $r_{x,q}\colon T_x \to S$ by $r_{x,q}(x)=s_0$; $r_{x,q}(y)=u^y(m')$, if $x<y \in T_x$ and $(m, x) \to (m', y)$.

For $(x, q) \in H$, $r_{x,q}$ is an $\mathfrak{A}$-run on the tree (v, T_x) marked at $H^{x,q}$.

The proof is clear from the definitions.

3.6. **Proof of Lemma 3.2.** We still restrict ourselves to the case $Q=\{q\}$, and retain the notations of §3.5. Let $A \subseteq V_\Sigma$ be as in Lemma 3.2. We claim that $(v, T) \in A$ if and only if there exists a $\mathfrak{B}$-run $r\colon T \to S^{\mathfrak{B}}$ on (v, T) with $r(\Lambda)=s_0^{\mathfrak{B}}$, such that for every path $\pi \subset T$, and every $x \in \pi$, at which some $m \in [n+1]$ is activated, $\mathrm{In}(r_{x,q}|\pi) \in F$.

Assume the existence of such an r. Define H, and a mapping $(x, q) \mapsto H^{x,q}$, $(x, q) \in H$, as in Lemma 3.5. Now $r_{x,q}\colon T_x \to S$ is an $\mathfrak{A}$-run on the tree (v, T_x) marked at $H^{x,q}$. Our condition implies that this tree is accepted by $\mathfrak{A}$. Thus, 1–3 of Lemma 3.2 hold and $(v, T) \in A$.

Assume $(v, T) \in A$ and let $\bar{H}$ and $(x, q) \mapsto \bar{H}^{x,q}$ satisfy 1–3 of Lemma 3.2. Thus for every $(x, q) \in \bar{H}$ there exists an $\mathfrak{A}$-run r_x on the tree (v, T_x) marked at $\bar{H}^{x,q}$ with $r_x(x)=s_0$, such that for every path $x \in \pi \subset T$, $\mathrm{In}(r_x|\pi) \in F$.

We shall define a $\mathfrak{B}$-run $r\colon T \to S^{\mathfrak{B}}$ and show that it satisfies the above condition. The run r will have some additional properties. If $m \in \mathrm{D}(u^x) \cup d_2^x$, $y=x0$ or $y=x1$, and $m \in \mathrm{D}(u^y)$, then $d_3^y(m)=m$ will hold. Thus we shall have a well-defined function $\rho^x\colon \mathrm{D}(u^x) \cup d_2^x \to \{y \mid y \leqq x\}$, such that $\rho^x(m)=y$ if m was last activated at y. In particular, if $m \in d_2^x$, then $\rho^x(m)=x$. We shall use this function in the sequel, writing $\rho(x, m)$ instead of $\rho^x(m)$.

Also, if $m \in \mathrm{D}(u^x)$, then $(\rho(x, m), q) \in \bar{H}$, $u^x(m) = r_{\rho(x,m)}(x)$; and $m \in d_1^x$ only if $(x, q) \in \bar{H}^{\rho(x,m),q}$. Thus, if the copy m of $\mathfrak{A}$ was activated at y, then $(y, q) \in \bar{H}$, and m goes through the run r_y until replaced.

The values $r(x)$ will be defined by induction on $l(x)$. Set $r(\Lambda) = s_0^{\mathfrak{B}}$; thus $\rho(\Lambda, 1) = \Lambda$ (recall that 1 is activated at Λ). Assume that $r: \{x \mid l(x) \leqq n\} \to S^{\mathfrak{B}}$ is defined, is the restriction of a $\mathfrak{B}$-run, and has the special properties mentioned above. For x with $l(x) = n$, we shall define $r(x0)$; the definition of $r(x1)$ is completely analogous. For $m \in \mathrm{D}(u^x) \cup d_2^x$, where $l(x) = n$, denote $r_{\rho(x,m)}(x0)$ by $s(m)$.

The idea is that in going from x to $x0$, all the $m' \in \mathrm{D}(u^x) \cup d_2^x$ for which $s(m') = s$ are replaced by that m for which $s(m) = s$ and $\rho(x, m)$ is minimal. This leads to the following formal definitions.

Put $\mathrm{D}(u^{x0}) = \{m \mid m \in \mathrm{D}(u^x) \cup d_2^x,\ \rho(x, m) \leqq \rho(x, m')$ *for all* m' *such that* $s(m) = s(m')\}$; $u^{x0}(m) = s(m)$, for $m \in \mathrm{D}(u^{x0})$. Note that u^{x0} is 1-1. $d_1^{x0} = \{m \mid m \in \mathrm{D}(u^{x0}),\ (x0, q) \in \bar{H}^{\rho(x,m),q}\}$. If $d_1^{x0} \neq \varnothing$, then set $d_2^{x0} = \{m\}$ for some $m \in [n+1] - \mathrm{D}(u^{x0})$ (this set is nonempty); otherwise, put $d_2^{x0} = \varnothing$. Define $d_3^{x0}(m') = m$ for $m' \in \mathrm{D}(u^x) \cup d_2^x$, $m \in \mathrm{D}(u^{x0})$, $s(m') = s(m)$.

It is readily checked that these definitions extend r to a mapping

$$r: \{x \mid l(x) \leqq n+1\} \to S^{\mathfrak{B}}$$

which has the same properties as the original mapping.

The above inductive definition yields a well-defined function $r: T \to S^{\mathfrak{B}}$. It is easily verified that r is a $\mathfrak{B}$-run on (v, T). Now, using the notion of replacement (Definition 3.4), $(m, x) \to (m', y)$ and $m \neq m'$ imply that $\rho(y, m') < \rho(x, m)$. This implies that along every path π, for every $x \in \pi$ and m which is activated at x, there exists a $y > x$ and a m' such that $(m, x) \to (m', y)$ and, along that path π, m' at $\rho(y, m') = w$ is always replaced by itself; i.e., for $w < z \in \pi$, $(m', w) \to (m', z)$. Hence, for $z \geqq y$, $z \in \pi$, $r_{x,q}(z) = r_{w,q}(z)$ (see Lemma 3.5). But $\mathrm{In}(r_{w,q}|\pi) = \mathrm{In}(r_w|\pi) \in F$ which proves our assertion.

The proof that A is f.a. definable will be completed by showing that the property of the paths $\pi \subset T$ that for every $x \in \pi$ and $m \in [n+1]$ activated at x, $\mathrm{In}((r_{x,q}|\pi)) \in F$, is recognizable by a (sequential) automaton, and appealing to Theorem 3.1.

Let $D \subset (S^{\mathfrak{B}})^{\omega}$ be the set of all sequences $\delta = ((u^k, d^k))_{k<\omega}$ such that $(u^0, d^0) = s_0^{\mathfrak{B}}$ and $\mathrm{D}(d_3^{k+1}) = \mathrm{D}(u^k) \cup d_2^k$. Note that for every $\mathfrak{B}$-run r on any $t \in V_{\Sigma}$, and every path π, $(r|\pi)_{\omega} \in D$; thus, it suffices to restrict ourselves to sequences in D. The set D is obviously f.a. definable.

The notions of $m \in [n+1]$ being activated at k, $0 \leqq k < \omega$, and of m at k being replaced by m' at $k' > k$ ($(m, k) \to (m', k')$), are defined for sequences $\delta \in D$ in the same way as the corresponding notions were defined for runs $r \in \mathrm{Rn}(\mathfrak{B}, t)$ (Definition 3.4). Similarly, we have for $k < \omega$ and $m \in [n+1]$ activated at k, the sequence $\delta_{k,q} \in S^{\omega}$ of states of $\mathfrak{A}$.

What we must show is that the set $B \subset (S^{\mathfrak{B}})^{\omega}$, $B = \{\delta \mid \delta \in D$, *for every* $k < \omega$ *and* m *activated at* k, $\mathrm{In}(\delta_{k,q}) \in F\}$, is f.a. definable. This will be done by constructing a nondeterministic (sequential) automaton $\mathfrak{C}=\langle S^{\mathfrak{C}}, M^{\mathfrak{C}}, S_0^{\mathfrak{C}}, F^{\mathfrak{C}}\rangle$ such that $D \cap T_\omega(\mathfrak{C}) = D-B$. (We do not care about what $\mathfrak{C}$ accepts outside of D.) Define $S^{\mathfrak{C}}=\{e\} \cup ([n+1]\times S)$, where e is a "new" element, and S is the set of states of $\mathfrak{A}$. Put $M^{\mathfrak{C}}(e, (u, d))=\{e, (m, s_0)\}$ for $m \in d_2$, $(u, d) \in S^{\mathfrak{B}}$. Define

$$\begin{aligned} M^{\mathfrak{C}}((m, s), (u, d)) &= e && \text{if } m \notin \mathrm{D}\,(d_3), \\ &= (d_3(m)), u(d_3(m)), && \text{for } m \in \mathrm{D}\,(d_3). \end{aligned}$$

Finally, define $S_0^{\mathfrak{C}}=\{e, (1, s_0)\}$, and $F^{\mathfrak{C}}=\{H \mid H\subset[n+1]\times S, p_2(H) \in P(S)-(F \cup \{\varnothing\})\}$. We have $D \cap T_\omega(\mathfrak{C})=D-B$ and this proves that B is f.a. definable.

3.7. **Proof of Lemma 3.4.** We use again the automaton $\mathfrak{B}$ (Definition 3.3) and the concepts and notations of Definition 3.4, Lemma 3.5, and §3.6. Note that we again assume $Q=\{q\}$.

Let $W\subset V_\Sigma$ be as in Lemma 3.4. We claim that $(v, T) \in W$ if and only if there exists an $r \in \mathrm{Rn}\,(\mathfrak{B}, (v, T))$ with $r(\Lambda)=s_0^{\mathfrak{B}}$, so that for every path $\pi\subset T$: (i) for every $x \in \pi$ and $m \in [n+1]$ activated at x, $\mathrm{In}\,(r_{x,q}|\pi) \in F$; and (ii) every sequence $(x_k, m_k, \overline{m}_k)$, $k=1, 2, \ldots$, such that $x_k \in \pi$, $\overline{m}_k$ is activated at x_k, $m_{k+1} \in d_1^{x_{k+1}}$, and $(\overline{m}_k, x_k) \to (m_{k+1}, x_{k+1})$, is finite.

Assume the existence of such a run r. Define the set $H\subseteq T\times\{q\}$, the mapping $(x, q)\mapsto H^{x,q}$ and the $\mathfrak{A}$-runs $r_{x,q}\colon T_x \to S$, $(x, q) \in H$, corresponding to this run r as in Lemma 3.5 and the proof of Lemma 3.2. By the proof of Lemma 3.2, condition (i) on r implies that H and $(x, q)\mapsto H^{x,q}$ satisfy conditions 1–3 of Lemma 3.2.

Assume now that $(x_k)_{k<\omega}$ is a sequence such that $(x_{k+1}, q) \in H^{x_k,q}$, $k<\omega$. This relationship implies $x_k<x_{k+1}$, $k<\omega$. Therefore, there exists a unique path $\pi\subset T$ such that $x_k \in \pi$, $k<\omega$. Let $\overline{m}_k \in [n+1]$ be activated at x_k, and $m_{k+1} \in [n+1]$ be such that $(\overline{m}_k, x_k) \to (m_{k+1}, x_{k+1})$. Now $(x_{k+1}, q) \in H^{x_k,q}$. This implies, by our definitions, that $m_{k+1} \in d_1^{x_{k+1}}$. Thus the sequence $(x_k, m_k, \overline{m}_k)_{k<\omega}$ satisfies the conditions of (ii), but is infinite, a contradiction. Hence $(x, q)\mapsto H^{x,q}$ is well-founded and thus $(v, T) \in W$.

To prove the asserted implication in the other direction, assume the existence of $\overline{H}\subseteq T\times\{q\}$, and a mapping $(x, q)\mapsto \overline{H}^{x,q}$, $(x, q) \in \overline{H}$, satisfying conditions 1–4 of Lemma 3.4. Let $r_x\colon T_x \to S$, for $(x, q) \in \overline{H}$, be an $\mathfrak{A}$-run on the tree (v, T_x) marked at $\overline{H}^{x,q}$ such that $r_x(x)=s_0$ and for every path $\pi\subset T_x$, $\mathrm{In}\,(r_x|\pi) \in F$.

Since $(x, q)\mapsto \overline{H}^{x,q}$ is well-founded, $\overline{H}$ can be decomposed as in Lemma 3.3. Define a mapping $\mathrm{Od}\colon \overline{H} \to \{\alpha \mid \alpha<\mu\}$ by $\mathrm{Od}\,(x, q)=\alpha$ if and only if $(x, q) \in \overline{H}_\alpha$. We have that $(y, q) \in \overline{H}^{x,q}$ implies $\mathrm{Od}\,(y, q)<\mathrm{Od}\,(x, q)$.

As in the proof of Lemma 3.2, define by induction on $l(x)$ $x \in T$, a $\mathfrak{B}$-run $r\colon T\to S^{\mathfrak{B}}$, and a function $\rho^x\colon \mathrm{D}\,(u^x) \cup d_2^x \to T$. The only difference will be in the definition of $\mathrm{D}\,(u^{x0})$; here we define it as follows:

$\mathrm{D}\,(u^{x0})=\{m \mid m \in \mathrm{D}\,(u^x) \cup d_2^x$; *for all* m' *such that* $s(m)=s(m')$ *either* $\mathrm{Od}\,(\rho(x, m), q)<\mathrm{Od}\,(\rho(x, m'), q)$, *or* $\mathrm{Od}\,(\rho(x, m), q)=\mathrm{Od}\,(\rho(x, m'), q)$ *and* $\rho(x, m) < \rho(x, m')\}$.

The idea behind this definition is that in going from x to $x0$, we replace all the $m' \in \mathrm{D}(u^x) \cup d_2^x$ such that $s(m')=s$ by the m with $s(m)=s$ for which $\rho(x, m)$ is minimal with respect to the lexicographic ordering: $(\mathrm{Od}(y, q), y) < (\mathrm{Od}(z, q), z)$ if $\mathrm{Od}(y, q) < \mathrm{Od}(z, q)$, or $\mathrm{Od}(y, q) = \mathrm{Od}(z, q)$ and $y < z$. Note that for a fixed path π, this is a well-ordering of $\{(y, q) \mid y \in \pi, (y, q) \in \bar{H}\}$.

If $(m, x) \to (m', y)$ and $m \neq m'$, then $(\mathrm{Od}(\rho(y, m'), q), \rho(y, m')) < (\mathrm{Od}(\rho(x, m), q), \rho(x, m))$. This entails, in exactly the same manner as in the proof of Lemma 3.2, that for every path $\pi \subset T$, $\mathrm{In}(r_{x,q}|\pi) \in F$. Thus (i) holds for r.

To show that r satisfies condition (ii), let $(x_k, m_k, \bar{m}_k)$, $k=1, 2, \ldots$, be a sequence such that $m_{k+1} \in d_1^{x_{k+1}}$, $\bar{m}_k$ is activated at x_k, and $(\bar{m}_k, x_k) \to (m_{k+1}, x_{k+1})$. The last two conditions imply $\mathrm{Od}(x_k, q) = \mathrm{Od}(\rho(x_k, \bar{m}_k), q) \geqq \mathrm{Od}(\rho(x_{k+1}, m_{k+1}), q)$. Denote $\rho(x_{k+1}, m_{k+1})$ by y. From the definition of r it follows that $m_{k+1} \in d_1^{x_{k+1}}$ implies $(x_{k+1}, q) \in \bar{H}^{y,q}$. Thus, $\mathrm{Od}(y, q) > \mathrm{Od}(x_{k+1}, q)$. Combining with previous relations, we have $\mathrm{Od}(x_k, q) > \mathrm{Od}(x_{k+1}, q)$; this forces the sequence to be finite and establishes (ii).

To complete the proof that W is f.a. definable, we must again show that the properties (i) and (ii) of a sequence $r|\pi$ are recognizable by a (sequential) automaton. Again we consider $\delta = (r|\pi)_\omega$ and restrict ourselves to $\delta \in D$ (see end of §3.6). For property (i) this has already been done in the proof of Lemma 3.2. Since the class of definable subsets of $(S^{\mathfrak{B}})^\omega$ is closed with respect to intersections, it suffices to show that the set $B \subseteq D$ of all sequences δ with property (ii) is f.a. definable.

It is again easier to show that the *complement* of the set of sequences with property (ii) is f.a. definable. Let $C \subseteq D$ be the set of all sequences $\delta = ((u^k, d^k))_{k<\omega}$, $\delta \in D$, such that there exist an increasing ω-sequence $(k_i)_{i<\omega}$ of integers and sequences $(m_i)_{i<\omega}$, $(\bar{m}_i)_{i<\omega}$, such that $\bar{m}_i$ is activated at k_i, $(\bar{m}_i, k_i) \to (m_{i+1}, k_{i+1})$ and $m_{i+1} \in d_1^{k_{i+1}}$ for $i < \omega$. We shall show that C is f.a. definable.

Construct an $S^{\mathfrak{B}}$-automaton $\mathfrak{C} = \langle S^{\mathfrak{C}}, M^{\mathfrak{C}}, g, F^{\mathfrak{C}} \rangle$ as follows. Define $S^{\mathfrak{C}} = \{g\} \cup ([n+1] \times \{e, f\})$, where g, e, f are any new elements. $M^{\mathfrak{C}}(g, (u, d)) = \{g\} \cup \{(m, e) \mid m \in d_2\}$, $(u, d) \in S^{\mathfrak{B}}$. Let $s = (m, e)$ or $s = (m, f)$, $m \in [n+1]$, define

$$\begin{aligned} M^{\mathfrak{C}}(s, (u, d)) &= g && \text{if } m \notin \mathrm{D}(d_3), \\ &= \{(d_3(m), e)\} && \text{if } d_3(m) \notin d_1, \\ &= \{(d_3(m), e), (m', f)\} && \text{if } d_3(m) \in d_1, \{m'\} = d_2. \end{aligned}$$

Set $F^{\mathfrak{C}} = \{G \mid G \subseteq S^{\mathfrak{C}}, G \cap ([n+1] \times \{f\}) \neq \varnothing\}$. It is readily seen that $T_\omega(\mathfrak{C}) \cap D = C = D - B$. This implies that B is f.a. definable.

3.8. **Dual acceptance by automata.** In order to analyze the structure of the complement $V_\Sigma - T(\mathfrak{A})$ of a f.a. definable set, we need certain auxiliary concepts.

DEFINITION 3.5. Let $\Omega = ((L_i, U_i))_{1 \leqq i \leqq k}$ be a sequence of pairs of sets. For a mapping $\phi: A \to S$ we shall say that ϕ *is of type* Ω (notation: $\phi \in [\Omega]$) if for some i, $1 \leqq i \leqq k$, $\mathrm{In}(\phi) \cap U \neq \varnothing$ and $\mathrm{In}(\phi) \cap L_i = \varnothing$. Furthermore, we shall say that ϕ is *of type Ω-empty* ($\phi \in [\Omega, e]$), if $\phi \in [\Omega]$ and for some j, $1 \leqq j \leqq k$, $\phi(A) \cap L_j = \varnothing$.

The following concept is, in a sense, dual to the notion of the set defined by an automaton. It will turn out, in fact, that every *complement* of a f.a. definable set can be described in this way.

DEFINITION 3.6. Let $\mathfrak{A}=\langle S, M\rangle$ be a Σ-table, $s \in S$, and let Ω be as above. We shall say that $\mathfrak{A}$, with Ω and s, *(dually) accepts* (v, T_x) (notation: $(v, T_x) \in \mathrm{D}(\mathfrak{A}, \Omega, s)$) if for *every* $r \in \mathrm{Rn}(\mathfrak{A}, (v, T_x))$ satisfying $r(x)=s$, there exists a path $\pi \subset T_x$ such that $r|\pi \in [\Omega]$. Using formal notation:

$$(v, T_x) \in \mathrm{D}(\mathfrak{A}, \Omega, s) \equiv \forall r \exists \pi[r \in \mathrm{Rn}(\mathfrak{A}, (v, T_x)) \wedge r(x) = s \rightarrow r \mid \pi \in [\Omega]].$$

Notice that the prefix $\forall r \exists \pi$ in the above definition is dual to the prefix $\exists r \forall \pi$ in the definition of acceptance by an automaton (Definition 1.5).

LEMMA 3.6. *For every Σ-automaton $\mathfrak{A}=\langle S, M, s_0, F\rangle$ there exists a Σ-table $\mathfrak{B}=\langle S^{\mathfrak{B}}, M^{\mathfrak{B}}\rangle$, a sequence Ω of pairs of sets, and an element $s_0^{\mathfrak{B}} \in S^{\mathfrak{B}}$ such that $(v, T) \in V_\Sigma - T(\mathfrak{A})$ if and only if $(v, T) \in \mathrm{D}(\mathfrak{B}, \Omega, s_0^{\mathfrak{B}})$.*

Proof. We have $(v, T) \in V_\Sigma - T(\mathfrak{A})$ if and only if

$$\forall r \exists \pi[r \in \mathrm{Rn}(\mathfrak{A}, (v, T)) \wedge r(\Lambda) = s_0 \rightarrow \mathrm{In}(r|\pi) \in P(S) - F]. \tag{3.1}$$

Arrange $P(S)-F-\{\varnothing\}$ in a sequence $(S_1, S_2, \ldots, S_k)$. Set $Q=\bigtimes_{1 \leqq i \leqq k} P(S_i)$, and $S^{\mathfrak{B}}=Q \times S$. The table $M^{\mathfrak{B}}$ is defined by

$$((q_1, s_1), (q_2, s_2)) \in M^{\mathfrak{B}}((q, s), \sigma), \qquad (q, s) \in Q \times S,$$

if and only if $(s_1, s_2) \in M(s, \sigma)$ and, for $1 \leqq i \leqq k, j=1, 2$,

$$\begin{aligned} q_j(i) &= q(i) \cup \{s\}, && \text{if } s \in S_i, q(i) \neq S_i, \\ &= q(i), && \text{if } s \notin S_i, q(i) \neq S_i, \\ &= \varnothing, && \text{if } q(i) = S_i. \end{aligned}$$

Finally, set $s_0^{\mathfrak{B}}=(q_0, s_0)$ where $q_0(i)=\varnothing$ for $1 \leqq i \leqq k$.

Let $r \in \mathrm{Rn}(\mathfrak{B}, (v, T))$. The mapping $p_2 r: T \rightarrow S$ is an $\mathfrak{A}$-run on (v, T) with $p_2 r(\Lambda)=s_0$, and every $\mathfrak{A}$-run is obtained in this way.

Define now, for $1 \leqq i \leqq k$, $L_i=\{(q, s) \mid q \in Q, s \in S-S_i\}$ $U_i=\{(q, s) \mid q \in Q, s \in S, q(i)=S_i\}$; let $\Omega=((L_i, U_i))_{1 \leqq i \leqq k}$.

Let $r \in \mathrm{Rn}(\mathfrak{B}, (v, T))$, and let $\pi \subset T$ be a path. For a fixed i, $1 \leqq i \leqq k$, $\mathrm{In}(p_2 r|\pi)=S_i$, if and only if $\mathrm{In}(r|\pi) \cap U_i \neq \varnothing$ and $\mathrm{In}(r|\pi) \cap L_i=\varnothing$; i.e., $\mathrm{In}(p_2 r|\pi) \notin F$ if and only if $r|\pi \in \Omega$. Thus (3.1) holds if and only if $(v, T) \in \mathrm{D}(\mathfrak{B}, \Omega, s_0^{\mathfrak{B}})$.

LEMMA 3.7. *Let $\mathfrak{A}=\langle S, M\rangle$ be a Σ-table, $\bar{s} \in S$, and $\Omega=((L_i, U_i))_{1 \leqq i \leqq k}$. Let A be the set of all $\bar{\Sigma}=\Sigma \times P(S)$-trees $\bar{t}$ such that $\bar{t}=(\bar{v}, T_x) \in A$ if and only if*

$$\begin{aligned} &\forall r[r \in \mathrm{Rn}(\mathfrak{A}, p_1(\bar{t})) \wedge r(x) = \bar{s} \\ &\qquad \rightarrow \exists \pi[r|\pi \in [\Omega]] \vee \exists y[y \in T_x \wedge \bar{v}(y) = (\sigma, q) \wedge r(y) \in q]]. \end{aligned} \tag{3.2}$$

There exist a $\bar{\Sigma}$-table $\mathfrak{B}=\langle S^{\mathfrak{B}}, M^{\mathfrak{B}}\rangle$, a state $s_0 \in S^{\mathfrak{B}}$, and a sequence $\bar{\Omega}=((\bar{L}_i, \bar{U}_i))_{1 \leqq i \leqq k}$ such that $A=\mathrm{D}(\mathfrak{B}, \bar{\Omega}, s_0)$. Note that here the length of $\bar{\Omega}$ equals that of Ω.

Proof. Let $u \notin S$. Set $S^{\mathfrak{B}} = S \cup \{u\}$; $s_0 = \bar{s}$. Define $M^{\mathfrak{B}}(u, \bar{\sigma}) = \{(u, u)\}$ for $\bar{\sigma} \in \bar{\Sigma}$. For $s \in S$, $\sigma \in \Sigma$, $q \in P(S)$, define

$$\begin{aligned} M^{\mathfrak{B}}(s, (\sigma, q)) &= M(s, \sigma), \quad \text{for } s \notin q, \\ &= \{(u, u)\}, \quad \text{for } s \in q. \end{aligned}$$

We see that for a $\mathfrak{B}$-run r on a $\bar{\Sigma}$-tree $(\bar{v}, T_x)$, either there exists a $y \in T_x$ such that $\bar{v}(y) = (\sigma, q)$ and $r(y) \in q$, in which case $r(z) = u$ for all $z > y$, or r is also an $\mathfrak{A}$-run on the Σ-tree $p_1((\bar{v}, T_x))$.

Let $\bar{U}_1 = U_1 \cup \{u\}$, $\bar{L}_i = L_i$, $1 \leqq i \leqq k$, $\bar{U}_i = U_i$, $2 \leqq i \leqq k$, $\bar{\Omega} = ((\bar{L}_i, \bar{U}_i))_{1 \leqq i \leqq k}$. With these definitions, $A = \mathrm{D}(\mathfrak{B}, \bar{\Omega}, s_0)$.

LEMMA 3.8. *With the same assumptions and notations as in Lemma* 3.7, *let B be the set of $\bar{\Sigma}$-trees such that $\bar{t} = (\bar{v}, T_x) \in B$ if and only if*

$$\forall r[r \in \mathrm{Rn}(\mathfrak{A}, p_1(\bar{t})) \wedge r(x) = \bar{s}$$
$$\rightarrow \exists \pi[\pi \subset T_x \wedge r|\pi \in [\Omega, e]] \vee \exists y[y \in T_x \wedge \bar{v}(y) = (\sigma, q) \wedge r(y) \in q]].$$

There exists a table $\mathfrak{B} = \langle S^{\mathfrak{B}}, M^{\mathfrak{B}} \rangle$, a state $s_0 \in S^{\mathfrak{B}}$, and a sequence

$$\bar{\Omega} = ((\bar{L}_i, \bar{U}_i))_{1 \leqq i \leqq k},$$

such that $\bar{L}_k = \varnothing$ and $B = \mathrm{D}(\mathfrak{B}, \bar{\Omega}, s_0)$. Again the length of $\bar{\Omega}$ equals that of Ω.

Proof. Set $[i] = \{1, \ldots, i\}$, $0 \leqq i \leqq k$. Define Φ by $\Phi = \{\phi \mid \phi\colon [i] \rightarrow [k],\ \phi \text{ is } 1\text{-}1,\ 0 \leqq i \leqq k\}$. Note that Φ contains the function $\phi\colon \varnothing \rightarrow [k]$; i.e., $\varnothing \in \Phi$. Define $S^{\mathfrak{B}} = (\Phi \times S) \cup \{u\}$, and $s_0 = (\varnothing, \bar{s})$.

We shall define $M^{\mathfrak{B}}$ so that $\mathfrak{B}$-runs will have the following features. If $r^{\mathfrak{B}}$ is a $\mathfrak{B}$-run on $\bar{t} = (\bar{v}, T)$ and $r^{\mathfrak{B}}(x) \neq u$ for all $x \in T$, then $r = p_2 r^{\mathfrak{B}}$ is an $\mathfrak{A}$-run on $p_1(\bar{t})$. For $r^{\mathfrak{B}}(x) = (\phi, s)$, where $\phi\colon [i] \rightarrow [k]$, $L_{\phi(1)}, \ldots, L_{\phi(i)}$, will be, in order of appearance along $\{y | y \leqq x\}$, the L_j for which $L_j \cap r(\{y \mid y \leqq x\}) \neq \varnothing$. The state u will appear if and only if for some $y \in T$, $\bar{v}(y) = (\sigma, q)$ and $r(y) \in q$. This motivates the following formal definitions.

Let $(\phi, s) \in S^{\mathfrak{B}}$, where $\phi\colon [i] \rightarrow [k]$; $\bar{\sigma} = (\sigma, q) \in \Sigma \times P(S)$. Define $M^{\mathfrak{B}}((\phi, s), \bar{\sigma}) = \{(u, u)\}$ if $s \in q$; also, $M^{\mathfrak{B}}(u, \bar{\sigma}) = \{(u, u)\}$. In all other cases, let $b = \{j \mid s \in L_j\}$, denote $b - \mathrm{R}(\phi) = \{j_1, \ldots, j_n\}$ where $j_1 < j_2 < \cdots < j_n$, and define $\bar{\phi}\colon [i+n] \rightarrow [k]$ by $\bar{\phi} \mid [i] = \phi$, $\bar{\phi}(i+m) = j_m$, $1 \leqq m \leqq n$ (if $b - \mathrm{R}(\phi) = \varnothing$ then $\bar{\phi} = \phi$). With these notations, define $M^{\mathfrak{B}}((\phi, s), \bar{\sigma}) = \{((\bar{\phi}, s_1), (\bar{\phi}, s_2)) \mid (s_1, s_2) \in M(s, \sigma)\}$.

For $1 \leqq i < k$ define $\bar{U}_i = \{(\phi, s) \mid i \in \mathrm{D}(\phi) \neq [k],\ \phi(i) = j$ *implies* $s \in U_j\}$; $\bar{L}_i = \{(\phi, s) \mid i \in \mathrm{D}(\phi) \neq [k],\ \phi(i) = j$ *implies* $s \in L_j\}$. Finally, define $\bar{U}_k = \{u\} \cup \{(\phi, s) \mid$ *for some* i, $1 \leqq i \leqq k$, $i \notin \mathrm{R}(\phi)$ *and* $s \in U_i\}$; $\bar{L}_k = \varnothing$. Set $\bar{\Omega} = ((\bar{L}_i, \bar{U}_i))_{1 \leqq i \leqq k}$. It can now be verified that $B = \mathrm{D}(\mathfrak{B}, \bar{\Omega}, s_0)$.

3.9. **Results concerning** $\mathrm{D}(\mathfrak{A}, \Omega, s)$. To prove that the class of f.a. definable sets is closed under complementation, it suffices, by Lemma 3.6, to show that sets of the form $\mathrm{D}(\mathfrak{A}, \Omega, s_0)$ are f.a. definable. The proof will be accomplished by induction on the length $l(\Omega)$ of Ω. Assume that the statement has been established for all sets

D $(\mathfrak{A}', \Omega', s)$ where $\mathfrak{A}'$ is a table over any alphabet Σ' and $l(\Omega')=k-1$. We shall prove that for every Σ-table $\mathfrak{A}$, every $s_0 \in S^{\mathfrak{A}}$, and every sequence $\Omega=((L_i, U_i))_{1\leq i\leq k}$, the set D $(\mathfrak{A}, \Omega, s_0)$ is f.a. definable. The induction from $k-1$ to k will actually proceed in two "half-steps" (§3.11).

This plan calls for an analysis of $(v, T) \in$ D $(\mathfrak{A}, \Omega, s_0)$ by dual acceptance involving shorter sequences Ω'.

LEMMA 3.9. *Let* $\mathfrak{A}=\langle S, M\rangle$, $\Omega=((L_i, U_i))_{1\leq i\leq k}$, *and* $\Omega_{k-1}=((L_i, U_i))_{1\leq i\leq k-1}$. *If* $t=(v, T)\in$ D $(\mathfrak{A}, \Omega, s_0)$, *then there exists a set* $H_0\subseteq(T-\{\Lambda\})\times U_k$ *such that* (1) $(x, s)\in H_0$ *implies* $(v, T_x)\in$ D $(\mathfrak{A}, \Omega, s)$; (2) *for every* $\mathfrak{A}$*-run* r *on* t *satisfying* $r(\Lambda)=s_0$, *if* $r|\pi\notin[\Omega_{k-1}]$ *for every path* $\pi\subset T$, *then there exists a* $(x, s)\in H_0$ *with* $r(x)=s$.

Proof. Define

$$(3.3)\qquad H_0 = \{(x, s) \mid x\in T-\{\Lambda\}, s\in U_k, (v, T_x)\in \mathrm{D}\,(\mathfrak{A}, \Omega, s)\}.$$

Condition 1 holds for this H_0. Assume now that r is an $\mathfrak{A}$-run on (v, T) with $r(\Lambda)=s_0$, such that $r|\pi\notin[\Omega_{k-1}]$ for every path $\pi\subset T$, and $r(x)\neq s$ for every $(x, s)\in H_0$.

Let $\Pi=\{\pi \mid \pi\subset T, r(\pi-\{\Lambda\})\cap U_k\neq\varnothing\}$. Define a mapping $\pi\mapsto x(\pi)$, $\pi\in\Pi$, by $x(\pi)=\min\{y \mid \Lambda\neq y\in\pi, r(y)\in U_k\}$. If $\pi, \pi'\in\Pi$, then either $T_{x(\pi)}=T_{x(\pi')}$ or $T_{x(\pi)}\cap T_{x(\pi')}=\varnothing$.

Note that the set $D=T-\bigcup_{\pi\in\Pi} T_{x(\pi)}$ contains no $y\neq\Lambda$ with $r(y)\in U_k$. Consequently, for every path $\pi\subset D$, $r|\pi\notin[\Omega]$.

Since $(x(\pi), r(x(\pi)))\notin H_0$ for $\pi\in\Pi$, there exists a run $r_\pi\colon T_{x(\pi)}\to S$ with $r_\pi(x(\pi))=r(x(\pi))$ so that for every path $\pi'\subset T_{x(\pi)}$, $r_\pi|\pi'\notin[\Omega]$. Define an $\mathfrak{A}$-run $r'\colon T\to S$ by

$$\begin{aligned} r'(y) &= r(y), \quad \text{if } y\notin \bigcup_{\pi\in\Pi} T_{x(\pi)},\\ &= r_\pi(y), \quad \text{if } y\in T_{x(\pi)} \text{ for some } \pi\in\Pi. \end{aligned}$$

We have $r'(\Lambda)=r(\Lambda)=s_0$. Let $\pi'\subset T$ be a path. Either $\pi'\subset D$ which implies $r|\pi'=r'|\pi'$, and hence $r'|\pi'\notin[\Omega]$; or for some $\pi\in\Pi$ we have that $\pi'-T_{x(\pi)}$ is finite and hence again $r'|\pi'\notin[\Omega]$. Thus $(v, T)\notin$ D $(\mathfrak{A}, \Omega, s_0)$, a contradiction. This establishes condition 2 for H_0.

REMARK. The above lemma, and its proof, remain valid for $k=1$. In this case Ω_{k-1} is empty and we stipulate that $r|\pi\notin[\Omega_{k-1}]$ always holds. The same remark applies to the following

LEMMA 3.10. *Let* $\mathfrak{A}=\langle S, M\rangle$ *be a* Σ*-table*; $\Omega=((L_i, U_i))_{1\leq i\leq k}$ *with* $L_k=\varnothing$; $\Omega_{k-1}=((L_i, U_i))_{1\leq i\leq k-1}$; *and* $s_0\in S$. *A* Σ*-tree* (v, T) *is in* D $(\mathfrak{A}, \Omega, s_0)$ *if and only if there exist a set* $H\subseteq(T-\{\Lambda\})\times U_k\cup\{(\Lambda, s_0)\}$ *and a mapping* $(x, s)\mapsto H^{x,s}$, $(x, s)\in H$ *such that* (1) $(\Lambda, s_0)\in H$; (2) *for* $(x, s)\in H$, $H^{x,s}\subseteq((T_x-\{x\})\times U_k)\cap H$; (3) *for* $(x, s)\in H$,

$$\forall r[r\in \mathrm{Rn}\,(\mathfrak{A}, (v, T_x)) \wedge r(x) = s \to \exists\pi[\pi\subset T_x \wedge r|\pi\in[\Omega_{k-1}]] \\ \vee \exists y[x\neq y\in T_x \wedge (y, r(y))\in H^{x,s}]].$$

Proof. Assume existence of H and $(x, s) \mapsto H^{x,s}$ satisfying 1–3. Let r be an $\mathfrak{A}$-run on (v, T) with $r(\Lambda)=s_0$. We must prove the existence of a path $\pi \subset T$ with $r|\pi \in [\Omega]$. If for some $\pi \subset T$, $r|\pi \in [\Omega_{k-1}]$ then we are done. Otherwise, for every subtree (v, T_x) and every $\pi \subset T_x$, $(r|T_x) \mid \pi = r|\pi \notin [\Omega_{k-1}]$. By 1–3 there exists a $x_0 > \Lambda$ such that $(x_0, r(x_0)) \in H$ and, hence, $r(x_0) \in U_k$. By 2–3 there exists an $x_1 > x_0$ such that $(x_1, r(x_1)) \in H$ and again $r(x_1) \in U_k$. Continuing in this way, we get an ω sequence $(x_i)_{i<\omega}$ such that $x_i < x_{i+1}$, $r(x_i) \in U_k$, $i<\omega$. For the unique path $\pi \subset T$ such that $x_i \in \pi$, $i<\omega$, we have $\mathrm{In}\,(r|\pi) \cap U_k \neq \varnothing$ and $\mathrm{In}\,(r|\pi) \cap L_k = \mathrm{In}\,(r|\pi) \cap \varnothing = \varnothing$; thus $r|\pi \in [\Omega]$.

To prove the other direction, assume $(v, T) \in \mathrm{D}\,(\mathfrak{A}, \Omega, s_0)$. Define $H=\{(\Lambda, s_0)\} \cup H_0$, where H_0 is as in (3.3); thus, $(\Lambda, s_0) \in H$. Applying Lemma 3.9 to each tree (v, T_x) and $s \in S$ such that $(x, s) \in H$, we obtain the existence of a set $H^{x,s} \subset H$ such that 2–3 hold. The mapping $(x, s) \mapsto H^{x,s}$ is the desired one.

REMARK. For the case $k=1$, Lemma 3.10 reads: Let $\mathfrak{A}=\langle S, M\rangle$, $s_0 \in S$, and $U \subset S$. A Σ-tree $t=(v, T)$ satisfies

$$\forall r[r \in \mathrm{Rn}\,(\mathfrak{A}, t) \wedge r(\Lambda) = s_0 \rightarrow \exists \pi[\mathrm{In}\,(r|\pi) \cap U \neq \varnothing]]$$

if and only if there exist a set H and a mapping $(x, s) \mapsto H^{x,s}$ satisfying 1–2 of Lemma 3.10 and: (3′) for $(x, s) \in H$,

$$\forall r[r \in \mathrm{Rn}\,(\mathfrak{A}, (v, T_x)) \wedge r(x) = s \rightarrow \exists y[x \neq y \in T_x \wedge (y, r(y)) \in H^{x,s}]].$$

LEMMA 3.11. *Let* $\mathfrak{A}=\langle S, M\rangle$ *be a* Σ*-table*, $\Omega=((L_i, U_i))_{1\le i\le k}$, *and* $s_0 \in S$. $(v, T) \in \mathrm{D}\,(\mathfrak{A}, \Omega, s_0)$ *if and only if there exist a set* $H \subseteq T \times S$ *and a mapping* $(x, s) \mapsto H^{x,s}$, $(x, s) \in H$, *such that*

(1) $(\Lambda, s_0) \in H$;

(2) $H^{x,s} \subseteq (T_x - \{x\}) \times S$ *and* $H^{x,s} \subseteq H$;

(3) *if* $(x, s) \in H$ *then*

$$\forall r[r \in \mathrm{Rn}\,(\mathfrak{A}, (v, T_x)) \wedge r(x) = s \rightarrow \exists \pi[\pi \subset T_x \wedge r|\pi \in [\Omega, e]] \vee \exists y[x \neq y \in T_x \wedge (y, r(y)) \in H^{x,s}]];$$

(4) *the mapping* $(x, s) \mapsto H^{x,s}$ *is well-founded.*

Proof. Assume that $t=(v, T)$ satisfies the conditions concerning the existence of H and $(x, s) \mapsto H^{x,s}$. Let $r: T \rightarrow S$ be an $\mathfrak{A}$-run with $r(\Lambda)=s_0$. We must prove that for some path $\pi \subset T$, $r|\pi \in [\Omega]$. If for some path $\pi \subset T$ we have $r|\pi \in [\Omega, e]$, then we are finished. Otherwise, by conditions 1 and 3, there exists a $(x_1, s_1) \in H^{\Lambda, s_0}$ such that $r(x_1)=s_1$. If for some path $\pi \subset T_{x_1}$ we have $r|\pi \in [\Omega, e]$, then again we are finished. Otherwise, since $(x_1, s_1) \in H$, there exists a $(x_2, s_2) \in H^{x_1, s_1}$ with $r(x_2)=s_2$; and so on. Since the mapping $(x, s) \mapsto H^{x,s}$ is well-founded, we must arrive at a pair $(x_n, s_n) \in H$ such that $r(x_n)=s_n$ and for some path $\pi \subset T_{x_n}$, $r|\pi \in [\Omega, e]$. Let $\pi' = \{y \mid y \leqq x_n\} \cup \pi$. Then π' is a path of T and $r|\pi' \in [\Omega]$.

Assume, conversely, that $t \in \mathrm{D}(\mathfrak{A}, \Omega, s_0)$. Let $H_0 \subseteq T \times S$ be defined by $(x, s) \in H_0$ if and only if

$$\forall r[r \in \mathrm{Rn}(\mathfrak{A}, (v, T_x)) \wedge r(x) = s \rightarrow \exists \pi[\pi \subset T_x \wedge r|\pi \in [\Omega, e]]].$$

A sequence of sets $H_\alpha \subseteq T \times S$ will be defined by transfinite induction on α. Set $G_\alpha = \bigcup_{\lambda < \alpha} H_\lambda$. Define H_α by $(x, s) \in H_\alpha$ if and only if $(x, s) \notin G_\alpha$ and

$$\forall r[r \in \mathrm{Rn}(\mathfrak{A}, (v, T_x)) \wedge r(x) = s \rightarrow \exists \pi[\pi \subset T_x \wedge r|\pi \in [\Omega, e]] \vee \exists y \exists s_1[(y, s_1) \in G_\alpha \wedge r(y) = s_1]].$$

There exists an ordinal μ such that $H_\mu = \varnothing$. Set $H = G_\mu$ $(= \bigcup_{\lambda < \mu} H_\lambda)$. For $(x, s) \in H$ let $\alpha < \mu$ be such that $(x, s) \in H_\alpha$. Define $H^{x,s} = ((T_x - \{x\}) \times S) \cap G_\alpha$. Our definitions insure that H and the mapping $(x, s) \mapsto H^{x,s}$ satisfy conditions 2–4. It remains to show that $(\Lambda, s_0) \in H$.

Assume that $(x, s) \notin H$. There exists an $\mathfrak{A}$-run $r_{x,s}$ on (v, T_x) such that $r_{x,s}(x) = s$, for every $y \in T_x$ we have $(y, r_{x,s}(y)) \notin H$, and for every path $\pi \subset T_x$ we have $r_{x,s}|\pi \notin [\Omega, e]$.

Thus, if $\pi \subset T_x$ and $r_{x,s}|\pi \in [\Omega]$, then there exists a minimal $y(\pi)$, $x < y(\pi) \in \pi$ such that $r_{x,s}(\{z \mid x \leqq z < y(\pi)\}) \cap L_i \neq \varnothing$ for all $1 \leqq i \leqq k$. Define $F(x, s) = \{y(\pi) \mid \pi \subset T_x, r_{x,s}|\pi \in [\Omega]\}$. The set $F(x, s)$ consists of pairwise incomparable elements. Notice the $T_x - \bigcup_{y \in F(x,s)} T_y$ contains no path π with $r_{x,s}|\pi \in [\Omega]$.

Assume by way of contradiction that $(\Lambda, s_0) \notin H$. Define by (ordinary) induction a sequence $(E_n)_{n<\omega}$, $E_n \subseteq T$, and a mapping $\phi: \bigcup_{n<\omega} E_n \rightarrow S$. Let $E_0 = \{\Lambda\}$, $\phi(\Lambda) = s_0$. Assume that E_n and $\phi: E_n \rightarrow S$ are already defined, that $(y, \phi(y)) \notin H$ for $y \in E_n$, and that the elements of E_n are pairwise incomparable. Define $E_{n+1} = \bigcup_{y \in E_n} F(y, \phi(y))$. Since $F(y, \phi(y)) \subseteq T_y - \{y\}$ and the elements of $F(y, \phi(y))$ are pairwise incomparable, it follows that the elements of E_{n+1} are pairwise incomparable.

For every $x \in E_{n+1}$, there is a unique $y \in E_n$ such that $x \in F(y, \phi(y))$. Define $\phi: E_{n+1} \rightarrow S$ by

$$\phi(x) = r_{y,\phi(y)}(x), \qquad x \in F(y, \phi(y)), y \in E_n.$$

Again, $(x, \phi(x)) \notin H$ for $x \in E_{n+1}$.

For $x \in T$ there exists a maximal n such that for some (unique) $y \in E_n$ we have $x \in T_y$. Define a mapping $r: T \rightarrow S$ by $r(x) = r_{y,\phi(y)}(x)$ where x and y retain their above meanings. It can be verified that r is an $\mathfrak{A}$-run on (v, T).

We wish to show that for every path $\pi \subset T$, $r|\pi \notin [\Omega]$. This will contradict $(v, T) \in \mathrm{D}(\mathfrak{A}, \Omega, s_0)$ and, therefore, imply $(\Lambda, s_0) \in H$, thus finishing the proof. Observe that the construction of $(E_n)_{n<\omega}$ and the fact that $F(y, \phi(y)) \subseteq T_y - \{y\}$, imply that if $\pi \cap E_n \neq \varnothing$ then $\mathrm{c}(\pi \cap E_n) = \mathrm{c}(\pi \cap E_m) = 1$ for $0 \leqq m \leqq n$. Thus there either exists a maximal n such that $\pi \cap E_n \neq \varnothing$, or else $\pi \cap E_n \neq \varnothing$ for $n < \omega$.

In the first case, let $\pi \cap E_n = \{x\}$. We have

$$\pi' = \{y \mid y \in \pi, x \leqq y\} \subseteq T_x - \bigcup_{y \in F(x,\phi(x))} T_y = D.$$

For $y \in D$ we have $r(y) = r_{x,\phi(x)}(y)$. Hence $r|\pi' = r_{x,\phi(x)}|\pi' \notin [\Omega]$, so that $r|\pi \notin [\Omega]$.

In the second case, let $\pi \cap E_n = \{x_n\}$, $n < \omega$. Now $x_{n+1} \in F(x_n, \phi(x_n))$ for $n < \omega$. Hence

$$r(\{y \mid x_n \leqq y < x_{n+1}\}) \cap L_i = r_{x_n,\phi(x_n)}(\{y \mid x_n \leqq y < x_{n+1}\}) \cap L_i \neq \varnothing,$$

for $1 \leqq i \leqq k$. Thus In $(r|\pi) \cap L_i \neq \varnothing$, $1 \leqq i \leqq k$; this again implies $r|\pi \notin [\Omega]$.

3.10. **Automata on finite trees.** We recall some facts concerning automata on finite trees. Our formulation differs from the one employed by Doner, Thatcher, and Wright (see [3], [16], [12]). It is, however, not hard to prove the equivalence of the various definitions.

A *finite binary tree* is a finite subset $E \subset T$, closed with respect to the predecessor function pd. The nodes $x \in E$ for which $x0 \notin E$ and $x1 \notin E$, are called the *terminal* nodes of E. The set of terminal nodes of E is called the *frontier* of E and is denoted by Ft (E). A finite binary tree E is called *frontiered* if $x \in E$ and $x \notin \mathrm{Ft}(E)$ imply $x0 \in E$ and $x1 \in E$. The term finite tree will, henceforth, always mean finite frontiered tree.

Let Σ be a finite set. A *finite Σ-tree* is a pair (v, E) where $v: E - \mathrm{Ft}(E) \to \Sigma$.

A Σ-automaton on finite Σ-trees is a system $\mathfrak{A} = \langle S, M, s_0, f\rangle$ where S is a finite set, $M: S \times \Sigma \to P(S \times S \cup \{(f,f)\})$, $s_0 \in S$, and $f \notin S$.

The notion of an $\mathfrak{A}$-run on (v, E) is completely analogous to Definition 1.4.

We shall say that $\mathfrak{A}$ *accepts* $t = (v, E)$ if for some $\mathfrak{A}$-run r on t, $r(\Lambda) = s_0$ and $r(x) = f$ for $x \in \mathrm{Ft}(E)$. The set of all (v, E) accepted by $\mathfrak{A}$ is denoted, as usual, by $T(\mathfrak{A})$. Whether this notation refers to the finite or infinite case will be clear from the context. A set B of finite Σ-trees will be called *finite automaton definable* if for some $\mathfrak{A}$, $B = T(\mathfrak{A})$.

The f.a. definable sets of finite Σ-trees form a boolean algebra. There exists a natural one-to-one correspondence between the f.a. definable sets of finite $\{0, 1\}^n$-trees and the n-ary relations between finite subsets of $\{0, 1\}^*$ which are definable in the weak second-order theory of two successor functions.

If (v, T) is a Σ-tree and $E \subset T$ is a finite tree, then we shall denote the finite Σ-tree $(v \mid (E - \mathrm{Ft}(E)), E)$ by (v, E).

LEMMA 3.12. *Let A be a f.a. definable set of finite Σ-trees. Let $B \subseteq V_\Sigma$ be the invariant set defined by the condition*: $(v, T) \in B$ *if and only if there exists a finite tree $E \subset T$ such that* $(v, E) \in A$. *The set B is f.a. definable.*

Proof. Let $A = T(\mathfrak{A})$ where $\mathfrak{A} = \langle S, M, s_0, f\rangle$. Set $\bar{S} = S \cup \{f\}$. Define $\bar{M}: \bar{S} \times \Sigma \to P(\bar{S} \times \bar{S})$ by $\bar{M}(s, \sigma) = M(s, \sigma)$ for $s \in S$, $\bar{M}(f, \sigma) = \{(f, f)\}$. Define $\mathfrak{B} = \langle \bar{S}, \bar{M}, s_0, \{\{f\}\}\rangle$. We claim $T(\mathfrak{B}) = B$.

That $B \subseteq T(\mathfrak{B})$ is quite obvious. To prove $T(\mathfrak{B}) \subseteq B$, assume $t=(v, T) \in T(\mathfrak{B})$. Thus, for some $r \in \mathrm{Rn}\,(\mathfrak{B}, t)$, $r(\Lambda)=s_0$ and $\mathrm{In}\,(r|\pi)=\{f\}$ for every path π. Define $x(\pi)=\min\{x|x \in \pi, r(x)=f\}$. The set $E=\{y|y \leqq x(\pi)$ *for some* $\pi \subset T\}$ is a finite (frontiered) tree. Let $\bar{t}=(v, E)$. $r|E$ is an $\mathfrak{A}$-run on $\bar{t}$ and $r(x)=f$ for $x \in \mathrm{Ft}\,(E)$. Hence, $\bar{t} \in A$ and $t \in B$.

We shall now fill up the small gap remaining in the proof of Theorem 1.7.

COROLLARY 3.13. *If P is a principal formula* (§1.5), *then there exists an automaton representing it.*

Proof. Assume that $P(A_1, \ldots, A_m)$ is the formula (1.2). Let

$$R = \tau(\{\tilde{A} \mid \tilde{A} \in P(T)^m, \mathfrak{N}_2 \models P(\tilde{A})\}).$$

A $\{0, 1\}^m$-tree $t=(v, T)$ is in R if and only if for some $x \in T$, and for $1 \leqq j \leqq k$, $p_{i_j}(v(xw_j))=\varepsilon_j$, where $\varepsilon_j=1$ if η_j is $\in$, and $\varepsilon_j=0$ if η_j is $\notin$.

Now, a tree t satisfies this condition if and only if for some finite subtree $E \subset T$, (v, E) satisfies it. Since the set of finite $\{0, 1\}^m$-trees satisfying the above condition (which, for finite trees, is expressible in weak second-order theory of $\mathfrak{N}_2$) is f.a. definable, our assertion follows from Lemma 3.12.

LEMMA 3.14. *Let $\mathfrak{A}=\langle S, M\rangle$, $s_0 \in S$ and $t=(v, T)$. If $H_0 \subseteq T \times S$ is such that for every $r \in \mathrm{Rn}\,(\mathfrak{A}, t)$ with $r(\Lambda)=s_0$ there exists an $x \in T$ so that $(x, r(x)) \in H_0$, then there exists a finite $H_1 \subseteq H_0$ with the same property.*

Proof. Assume the conclusion not to hold. Denote $T^n=\{x \mid x \in T, l(x) \leqq n\}$. For every $n<\omega$ there exists an $\mathfrak{A}$-run r_n on t satisfying $r_n(\Lambda)=s_0$ and, for $x \in T_n$, $(x, r_n(x)) \notin H_0$. By König's Infinity Lemma there exists an increasing sequence $(n(i))_{i<\omega}$ such that for $i \leqq j<\omega$, $r_{n(i)}|T^i=r_{n(j)}|T^i$. Let $r: T \to S$ be the limiting function; i.e., $r|T^i=r_{n(i)}|T^i$, $i<\omega$. r is an $\mathfrak{A}$-run on t with $r(\Lambda)=s_0$. Furthermore, $(x, r(x)) \notin H_0$ for all $x \in T$, a contradiction.

COROLLARY 3.15. *Let $\mathfrak{A}=\langle S, M\rangle$ be a Σ-table, $s_0 \in S$, $P=P(S)$. Let B be the invariant set of $\bar{\Sigma}=\Sigma \times P$-trees defined by the condition: $\bar{t}=(\bar{v}, T) \in B$ if and only if for every $\mathfrak{A}$-run r on $p_1(\bar{t})$, if $r(\Lambda)=s_0$ then for some $x \in T$, $\bar{v}(x)=(\sigma, q)$ and $r(x) \in q$. The set B is f.a. definable.*

Proof. Let A be the set of all finite (frontiered) $\bar{\Sigma}$-trees $(\bar{v}, E)$ satisfying the above condition. The set A is f.a. definable.

By Lemma 3.14, $\bar{t} \in B$ if and only if for some finite tree $E \subset T$, $(\bar{v}, E) \in A$. The set B is now f.a. definable by Lemma 3.12.

3.11. **Proof of Theorem 1.5.** To show that the class of f.a. definable subsets of V_Σ is closed under complements, it suffices, by Lemma 3.6, to show that sets of the form $\mathrm{D}\,(\mathfrak{A}, \Omega, s_0)$ are f.a. definable. This will be done by induction on $l(\Omega)$.

We assume as our induction hypothesis that for all $\mathfrak{A}=\langle S, M\rangle$ (over any Σ), all sequences $\Omega_{k-1}=((L_i, U_i))_{1 \leqq i \leqq k-1}$ of length $k-1$ and all $s \in S$, the set

$\mathrm{D}(\mathfrak{A}, \Omega_{k-1}, s)$ is f.a. definable (if $k=1$ then our assumption is vacuous). We wish to show that the same statement holds for all sequences $\Omega=((L_i, U_i))_{1\leq i\leq k}$ of length k with $L_k=\varnothing$.

Proof. Let $\mathfrak{A}$ and Ω be as above (thus $L_k=\varnothing$), $s_0\in S$, and $t=(v, T)\in V_\Sigma$. By Lemma 3.10, $t\in \mathrm{D}(\mathfrak{A}, \Omega, s_0)$ if and only if there exist a set $H\subseteq(T-\{\Lambda\})\times U_k \cup \{(\Lambda, s_0)\}$ and a mapping $(x, s)\mapsto H^{x,s}$, $(x, s)\in H$, such that for $k>1$ conditions 1–3 hold, and for $k=1$ conditions 1–3′ (of the remark following Lemma 3.10) hold.

Consider first the case $k=1$. By Corollary 3.15 there exists for every $s\in S$ an automaton $\mathfrak{A}_s$ over $\Sigma\times P(S)$-trees such that for $(x, s)\in H$ the tree (v, T_x) marked at $H^{x,s}$ (Definition 3.1) is accepted by $\mathfrak{A}_s$ if and only if 3′ holds.

Consider next the case $k>1$. Let $s\in S$. By Lemma 3.7 there exists a $\Sigma\times P(S)$-table $\overline{\mathfrak{A}}_s$, a state $\bar{s}$ of $\overline{\mathfrak{A}}_s$, and a sequence $\overline{\Omega}_{k-1}$ of length $k-1$ such that for $(x, s)\in H$, the tree (v, T_x) marked at $H^{x,s}$ is in $\mathrm{D}(\overline{\mathfrak{A}}_s, \overline{\Omega}_{k-1}, \bar{s})$ if and only if condition 3 of Lemma 3.10 holds. By the induction hypothesis, there exists a finite automaton $\mathfrak{A}_s$ such that $T(\mathfrak{A}_s)=\mathrm{D}(\overline{\mathfrak{A}}_s, \overline{\Omega}_{k-1}, \bar{s})$.

In either case, $\mathrm{D}(\mathfrak{A}, \Omega, s_0)$ is f.a. definable by Lemma 3.2.

We have thus far established that if $\mathfrak{A}=\langle S, M\rangle$ is a $\overline{\Sigma}$-table, $s\in S$ and $\overline{\Omega}=((L_i, U_i))_{1\leq i\leq k}$ with $L_k=\varnothing$. Then $\mathrm{D}(\mathfrak{A}, \overline{\Omega}, s)$ is f.a. definable. We now wish to show that $\mathrm{D}(\mathfrak{A}, \Omega, s_0)$ is f.a. definable for an arbitrary sequence Ω of length k.

Proof. By Lemma 3.11, $(v, T)\in \mathrm{D}(\mathfrak{A}, \Omega, s_0)$ if and only if there exist a set $H\subseteq T\times S$ and a mapping $(x, s)\mapsto H^{x,s}$ such that conditions 1–4 of that lemma hold. By Lemma 3.4 the proof that $\mathrm{D}(\mathfrak{A}, \Omega, s_0)$ is f.a. definable will be finished if we can show that for $s\in S$ there exists an automaton $\mathfrak{A}_s$ over the alphabet $\overline{\Sigma}=\Sigma\times P(S)$, such that for a Σ-tree $t=(v, T_x)$ and a set $H^{x,s}\subset T_x\times S$, $\mathfrak{A}_s$ accepts the tree t marked at $H^{x,s}$ if and only if 3 of Lemma 3.11 holds.

According to Lemma 3.8 there exists a $\overline{\Sigma}$-table $\mathfrak{B}_s$, a sequence $\overline{\Omega}_s=((\bar{L}_i, \bar{U}_i))_{1\leq i\leq k}$, with $\bar{L}_k=\varnothing$, and a state $\bar{s}\in S^{\mathfrak{B}_s}$ such that for a Σ-tree $t=(v, T_x)$ and a set $H^{x,s}\subseteq T_x\times S$, condition 3 of Lemma 3.11 holds if and only if the tree t marked at $H^{x,s}$ is in $\mathrm{D}(\mathfrak{B}_s, \overline{\Omega}, \bar{s})$. But $\mathrm{D}(\mathfrak{B}_s, \overline{\Omega}_s, \bar{s})$ (which is a set of $\overline{\Sigma}$-trees) is f.a. definable by a $\overline{\Sigma}$-automaton $\mathfrak{A}_s$ according to our inductive assumption. This completes the proof.

3.12. **Solution of the emptiness problem.** We wish to give an effective procedure which will enable us to determine for every f.a. $\mathfrak{A}=\langle S, M, s_0, F\rangle$, over any Σ, whether $T(\mathfrak{A})=\varnothing$. Consider the automaton $\overline{\mathfrak{A}}=\langle S, \overline{M}, s_0, F\rangle$ over the single letter alphabet $\overline{\Sigma}=\{\alpha\}$ defined by $\overline{M}(s, \alpha)=\bigcup_{\sigma\in\Sigma} M(s, \sigma)$, $s\in S$. Instead of $\overline{M}(s, \alpha)$, we shall simply write $\overline{M}(s)$. Since there exists for every subtree T_x just one $\{\alpha\}$-valued tree $(\bar{v}, T_x)$ (namely, $\bar{v}(y)=\alpha$, $y\in T_x$) we shall omit mention of the valuation and talk about $\overline{\mathfrak{A}}$-runs on T_x, $\overline{\mathfrak{A}}$ accepting T, etc.

Notice that every $\mathfrak{A}$-run $r\colon T\to S$ on a Σ-tree $t=(v, T)$ is also an $\overline{\mathfrak{A}}$-run on T. Conversely, if r is an $\overline{\mathfrak{A}}$-run on T, then there exists a Σ-tree t so that r is an $\mathfrak{A}$-run on t. Coupled with the definition of $T(\mathfrak{A})$, this implies that $T(\mathfrak{A})\neq\varnothing$ if and only if

$T(\bar{\mathfrak{A}}) \neq \varnothing$; i.e., if and only if $T(\bar{\mathfrak{A}}) = \{T_x \mid x \in T\}$. Thus, the emptiness problem is reduced to the case of automata over a single letter alphabet $\bar{\Sigma}$ and, henceforth, we shall restrict ourselves exclusively to this case.

The set $V_{\bar{\Sigma}} - T(\mathfrak{A})$ is f.a. definable and, given $\mathfrak{A}$, we can effectively construct a $\mathfrak{B}$ such that $T(\mathfrak{B}) = V_{\bar{\Sigma}} - T(\mathfrak{A})$. Now, $T(\mathfrak{A}) = V_{\bar{\Sigma}} - T(\mathfrak{B})$. According to Lemma 3.6 we can effectively construct a table $\mathfrak{C}$, a sequence $\Omega = ((L_i, U_i))_{1 \leqq i \leqq k}$, and a state s_0 of $\mathfrak{C}$ so that $T(\mathfrak{A}) = V_{\bar{\Sigma}} - T(\mathfrak{B}) = \mathrm{D}(\mathfrak{C}, \Omega, s_0)$.

Thus the emptiness problem will be solved if we exhibit an effective procedure for deciding for every $\bar{\Sigma}$-table $\mathfrak{A} = \langle S, M \rangle$, sequence $\Omega = ((L_i, U_i))_{1 \leqq i \leqq k}$, and $s_0 \in S$, whether $T \in \mathrm{D}(\mathfrak{A}, \Omega, s_0)$. This will be done by reducing the length k of Ω.

In fact, the reduction will proceed in the same "half steps" used in proving (for an arbitrary Σ) that the sets $\mathrm{D}(\mathfrak{A}, \Omega, s_0)$ are f.a. definable.

We shall show how to reduce the question whether $T \in \mathrm{D}(\mathfrak{A}, \Omega, s_0)$ to a finite number of questions whether $T \in \mathrm{D}(\mathfrak{A}, \Omega', s)$, where $\Omega' = ((L_i', U_i'))_{1 \leqq i \leqq k}$, and $L_k' = \varnothing$. Next we shall show that a question whether $T \in \mathrm{D}(\mathfrak{A}, \Omega', s)$, with Ω' as above and $k > 1$, reduces to a finite number of questions whether $T \in \mathrm{D}(\mathfrak{A}, \Omega_{k-1}, s)$, where Ω_{k-1} is a sequence of length $k-1$. Finally, we show how to solve effectively the problem $T \in \mathrm{D}(\mathfrak{A}, \Omega, s)$ where $\Omega = ((\varnothing, U))$. We shall need the following

Definition 3.7. Let $\mathfrak{A} = \langle S, M \rangle$ be a table, $S' \subseteq S$. The table $\mathfrak{A}$ *restricted to* S', denote it by $\mathfrak{A}|S'$, is $\langle S, M' \rangle$ where $M'(s) = M(s) \cap S'$, $s \in S$.

Remark Note that $\mathfrak{A}|S'$-runs r on T_x are precisely those $\mathfrak{A}$-runs for which $r(y) \in S'$ for $y > x$.

In the case $1 \leqq k$ and arbitrary Ω, let $\mathfrak{A} = \langle S, M \rangle$, $\Omega = ((L_i, U_i))_{1 \leqq i \leqq k}$, $s_0 \in S$. In the proof of Lemma 3.11 we constructed a sequence $(H_\alpha)_{\alpha < \mu}$ of sets and showed that $(v, T) \in \mathrm{D}(\mathfrak{A}, \Omega, s_0)$ if and only if $(\Lambda, s_0) \in \bigcup_{\alpha < \mu} H_\alpha$. In the case of a single-letter alphabet, the construction specializes as follows. Define $H_0 \subseteq S$ by $s \in H_0$ if and only if $\forall r[r \in \mathrm{Rn}(\mathfrak{A}, T) \wedge r(\Lambda) = s \Rightarrow \exists \pi[r|\pi \in [\Omega, e]]]$.

Set $G_m = \bigcup_{l \leqq m} H_l$ and define $H_{m+1} \subseteq S$ by $s \in H_{m+1}$ if and only if $s \notin G_m$ and

$$\forall r[r \in \mathrm{Rn}(\mathfrak{A}, T) \wedge r(\Lambda) = s \rightarrow \exists \pi[r|\pi \in [\Omega, e]] \vee \exists y[r(y) \in G_m]]. \tag{3.4}$$

Let $\mathrm{c}(S) = n$; then $H_m = \varnothing$ for $m > n$. Now $T \in \mathrm{D}(\mathfrak{A}, \Omega, s_0)$ if and only if $s_0 \in \bigcup_{m \leqq n} H_m$.

By Lemma 3.8 there exists for any $s \in S$ and $\varnothing \subseteq G_m \subseteq S$, a table $\mathfrak{B}$, a sequence $\bar{\Omega} = ((\bar{L}_i, \bar{U}_i))_{1 \leqq i \leqq k}$ with $\bar{L}_k = \varnothing$, and a state $\bar{s} \in S^{\mathfrak{B}}$ such that (3.4) holds if and only if $T \in \mathrm{D}(\mathfrak{B}, \bar{\Omega}, \bar{s})$. Thus the computation of $(H_m)_{0 \leqq m \leqq n}$ and the question whether $T \in \mathrm{D}(\mathfrak{A}, \Omega, s_0)$, are reduced to deciding a finite number of questions whether $T \in \mathrm{D}(\mathfrak{B}, \bar{\Omega}, \bar{s})$ where each $\bar{\Omega}$ is a sequence of length k with $\bar{L}_k = \varnothing$.

The case $1 < k$, $L_k = \varnothing$. Let $\mathfrak{A} = \langle S, M \rangle$, $s_0 \in S$, $\Omega = ((L_i, U_i))_{1 \leqq i \leqq k}$, $L_k = \varnothing$, and $\Omega_{k-1} = ((L_i, U_i))_{1 \leqq i \leqq k-1}$.

Specializing Lemma 3.9 to our case of a single letter alphabet, we see that

$T \in \mathrm{D}(\mathfrak{A}, \Omega, s_0)$ if and only if there exists a nonempty subset $H \subseteq U_k$ such that for $s \in H \cup \{s_0\}$

(3.5) $\forall r[r \in \mathrm{Rn}(\mathfrak{A}, T) \wedge r(\Lambda) = s \rightarrow \exists \pi[r|\pi \in [\Omega_{k-1}]] \vee \exists x[\Lambda < x \wedge r(x) \in H]]$.

In view of Definition 3.7 and the remark following it, (3.5) holds if and only if

$$\forall r'[r' \in \mathrm{Rn}(\mathfrak{A}', T) \wedge r'(\Lambda) = s \Rightarrow \exists \pi[r'|\pi \in [\Omega_{k-1}]]], \tag{3.6}$$

where $\mathfrak{A}' = \mathfrak{A}|(S-H)$. But (3.6) is equivalent to $T \in \mathrm{D}(\mathfrak{A}', \Omega_{k-1}, s)$. Thus the question whether $T \in \mathrm{D}(\mathfrak{A}, \Omega, s_0)$ effectively reduces to a finite number of questions whether $T \in \mathrm{D}(\mathfrak{A}', \Omega_{k-1}, s)$ where $\mathfrak{A}'$ ranges over automata $\mathfrak{A}|(S-H)$, $H \in P(S)$, $s \in S$, and Ω_{k-1} is of length $k-1$.

The case $k=1$, $L_1 = \varnothing$. Let $\mathfrak{A} = \langle S, M \rangle$, $\Omega = ((\varnothing, U))$, $s_0 \in S_0$. It follows at once from Lemmas 3.9 and 3.14, that $T \in \mathrm{D}(\mathfrak{A}, \Omega, s_0)$ if and only if there exist a nonempty set $U_0 = \{s_1, \ldots, s_m\} \subseteq U$ and *finite* trees E_i, $0 \leqq i \leqq m$, such that for $0 \leqq i \leqq m$, if r is an $\mathfrak{A}$-run on E_i with $r(\Lambda) = s_i$ then $r(E_i - \{\Lambda\}) \cap U_0 \neq \varnothing$.

Now the question whether for a set $U_0 \subseteq U$ and an element $s \in S$ there exists a tree E such that for every $\mathfrak{A}$-run r on E, $r(\Lambda) = s$ implies $r(E - \{\Lambda\}) \cap U_0 \neq \varnothing$, is expressible in the weak second-order theory of $\mathfrak{N}_2$, and hence, decidable by [3], [16].

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