HW3

1. $E[(x-c)^2] = E[x^2 + c^2 - 2xc]$

$= E[x^2] + E[c^2] - 2E[xc]$

$= E[x^2] + c^2 - 2cE[x] + (E[x])^2 - (E[x])^2$

$= \underline{(E[x])^2 - 2cE[x] + c^2}$

$+ \underline{E[x^2] - (E[x])^2}$

$= (c - E[x])^2 + Var[x]$

2. Check the solutions in HW1 and python notebook.

3. Bernoulli:

$$P(r \mid f(s)) = (f(s))^r (1 - f(s))^{1-r}$$

$$J_s = -E_r\left[\frac{\partial^2}{\partial s^2} \log\left((f(s))^r (1 - f(s))^{1-r}\right)\right]$$

$$= -E_r\left[\frac{\partial^2}{\partial s^2}\left(r \log(f(s)) - (1-r)\log(1 - f(s))\right)\right]$$

$$= -E_r\left[\frac{\partial}{\partial s}\left(r \frac{f'(s)}{f(s)} + (1-r)\frac{-f'(s)}{1 - f(s)}\right)\right.$$

$$= -E_r\left[r\left(\frac{f''(s)}{f(s)} - \frac{f'^2(s)}{f^2(s)}\right) - (1-r)\left(\frac{f''(s)}{1-f(s)} + \frac{f'^2(s)}{(1-f(s))^2}\right)\right]$$

$$E_r[r] = f(s)$$

from Bernoulli;

$$= -\left[f''(s) - \frac{f'^2(s)}{f(s)}\right] + \left[f''(s) + \frac{f'^2(s)}{1-f(s)}\right]$$

$$= \frac{f'^2(s)}{f(s)\,(1-f(s))}$$

4. $$P(k) = \int_0^\infty P_{Pois}(k|\mu)\, \rho(\mu)\, d\mu$$

$$= \int_0^\infty \frac{\mu^k}{k!} e^{-\mu} \cdot \frac{\mu^{\alpha-1}\, e^{-\beta\mu}}{\beta^{-\alpha}\,\Gamma(\alpha)}\, d\mu$$

$$= \frac{1}{k!\,\beta^{-\alpha}\,\Gamma(\alpha)} \int_0^\infty \mu^{k+\alpha-1}\, e^{-(\beta+1)\mu}\, d\mu$$

For a Gamma distribution we know that:

$$\int_0^\infty \theta^{a-1} e^{-b\theta} \, d\theta = \frac{\Gamma(a)}{b^a}$$

$$= \frac{\beta^\alpha}{k!\,\Gamma(\alpha)} \int_0^\infty \mu^{\alpha+k-1} e^{-(\beta+1)\mu} \, d\mu$$

$$= \frac{\beta^\alpha}{k!\,\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+k)}{(\beta+1)^{\alpha+k}}$$

From the definition of factorial we also know that: $k! = \Gamma(k+1)$

$$\frac{1}{\Gamma(k+1)\Gamma(\alpha)} \cdot \beta^\alpha \cdot \frac{\Gamma(\alpha+k)}{(\beta+1)^{\alpha+k}}$$

$$= \frac{\Gamma(\alpha+k)}{\Gamma(k+1)\,\Gamma(\alpha)} \frac{1}{(\beta+1)^k} \cdot \left(\frac{\beta}{\beta+1}\right)^\alpha$$

from the definition of factorial:

$$= \frac{(\alpha + k - 1)!}{k!\,(\alpha-1)!} \left(\frac{1}{\beta+1}\right)^k \left(\frac{\beta}{\beta+1}\right)^\alpha$$

combination formula

$$= \binom{\alpha + k - 1}{k} \left(\frac{1}{\beta+1}\right)^k \left(\frac{\beta}{\beta+1}\right)^\alpha$$

Definition of negative binomial:

$$P(r \mid q, p) = \binom{r+q-1}{r} (1-p)^q \, p^r$$

from above : $r = k$

$$p = \frac{1}{\beta+1}$$

$$1-p = \frac{\beta}{\beta+1} = 1 - \frac{1}{\beta+1}$$

$$q = \alpha$$

$$= NB(q, p)$$