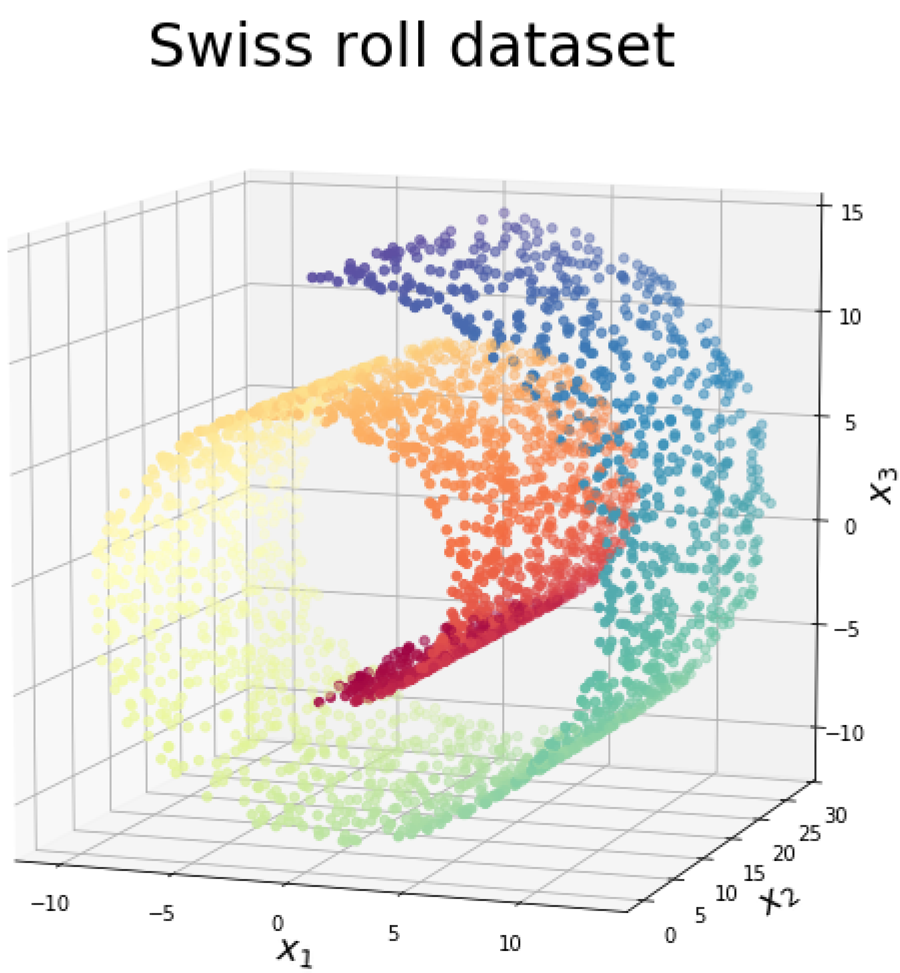
In [1]:
```
import matplotlib.pyplot as plt
import numpy as np
from mpl_toolkits.mplot3d import Axes3D
```

Projection is not always the best approach to dimensionality reduction. For example, consider the famous *Swiss roll* toy dataset

In [171]:
```
'Make a Swiss roll'
m = 3000
t = 1.5 * np.pi * (1 + 2 * np.random.rand(1, m))
x1 = t * np.cos(t)
x2 = 30* np.random.rand(1, m)
x3 = t * np.sin(t)
X = np.concatenate((x1, x2, x3))
X = X.T
t = np.squeeze(t)

'Plot the Swiss roll'
axes = [-11.5, 14, -2, 30, -12, 15]

fig = plt.figure(figsize=(10,10))
ax = plt.axes(projection='3d')
ax.scatter(X[:, 0], X[:, 1], X[:, 2], c=t, cmap=plt.cm.Spectral)
ax.view_init(10, -70)
ax.set_xlabel("$x_1$", fontsize=18)
ax.set_ylabel("$x_2$", fontsize=18)
ax.set_zlabel("$x_3$", fontsize=18)
ax.set_xlim(axes[0:2])
ax.set_ylim(axes[2:4])
ax.set_zlim(axes[4:6])
plt.title('Swiss roll dataset',fontsize=30)
```

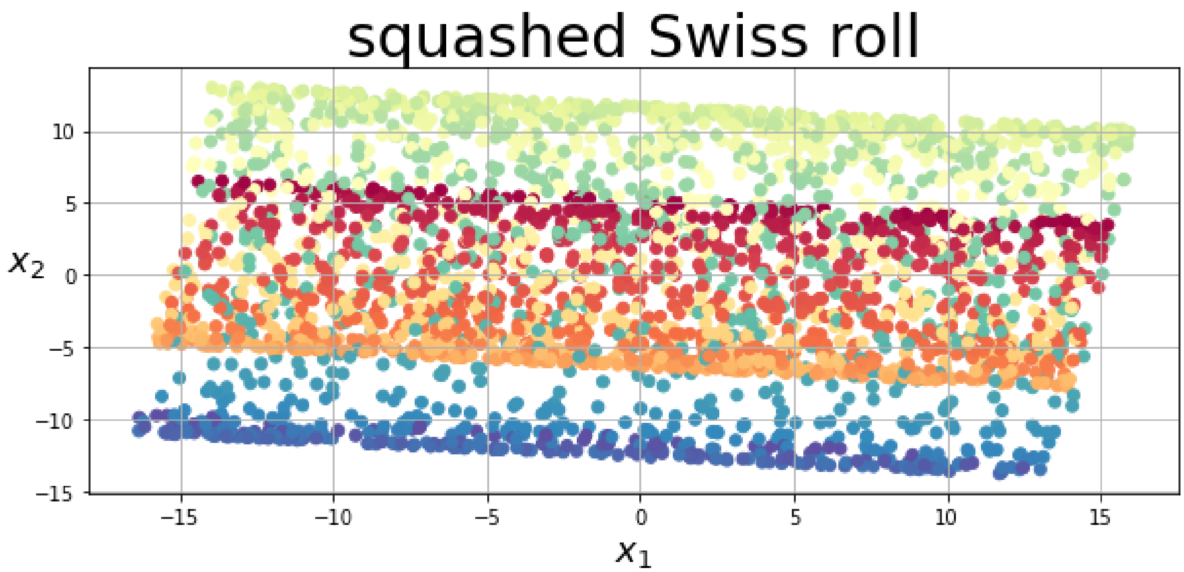
Out[171]: Text(0.5, 0.92, 'Swiss roll dataset')



Let's use PCA to project the *Swiss roll* into a two-dimensional subspace

In [159]:
```
'PCA'
# center the dataset
X_centered = X - np.mean(X,axis=0)

# Singular Value Decomposition
u,s,v = np.linalg.svd(X_centered)

# principal components
k = 2
c = v[:,:k]

# projection
X_reduced = X_centered@v[:k].T
```

In [160]:
```
'plot PCA projection'
plt.figure(figsize=(10,4))
plt.scatter(X_reduced[:,0],X_reduced[:,1], c=t, cmap=plt.cm.Spectral)
plt.xlabel("$x_1$", fontsize=18)
plt.ylabel("$x_2$", fontsize=18, rotation=0)
plt.title('squashed Swiss roll',fontsize=30)
plt.grid(True)
```



PCA squashes different layers of the Swiss roll together. What we really want is to "unroll" the Swiss roll.

11.3 - Locally Linear Embedding (LLE)

The Swiss roll is an example of a two-dimensional **manifold** embedded in a three-dimensional space. Key property of a manifold: locally, a manifold looks flat. **Locally linear embedding (LLE)** is a clever scheme that exploits this property for unrolling twisted manifolds.

Here's how LLE works:

11.3.1 - Locally Linear Embedding Step by Step

Step 1: First, for each dataset point, the algorithm identifies its k closest neighbors (in the preceding code $k = 10$).

In [161]:
```
n_neighbors = 10
m,n = X.shape
# compute all pairwise distances
distance = np.zeros((m,m))
for i in range(m):
    for j in range(m):
        distance[i,j] = np.linalg.norm(X[i]-X[j])
#  find local neighborhoods for each point
index = np.argsort(distance,axis=1)
neighborhoods = index[:,1:n_neighbors+1]
```

The above code is slow. A better approach uses scipy distance function

In [172]:
```
'faster alternative'
from scipy.spatial import distance
n_neighbors = 10
m,n = X.shape
# compute all pairwise distances
distances = distance.squareform(distance.pdist(X))
#  Find local neighborhoods for each point
index = np.argsort(distances,axis=1)
neighborhoods = index[:,1:n_neighbors+1]
```

For example, the closest neighbors of $X[0]$

In [163]:
```
X[0]
```

Out[163]:
```
array([ 9.66042746, 16.7876403 ,  9.17960572])
```

are

In [164]:
```
X[neighborhoods[0]]
```

Out[164]:
```
array([[ 9.56448165, 16.29019057,  9.29570298],
       [10.1177305 , 16.77678891,  8.58641792],
       [ 9.67484816, 17.71733307,  9.16192259],
       [ 9.9906982 , 17.84091668,  8.7582643 ],
       [ 8.75364487, 17.07569776, 10.1812902 ],
       [ 9.10501346, 17.89979871,  9.8170381 ],
       [10.57234096, 17.02986477,  7.91895413],
       [10.68979475, 16.56540252,  7.73115584],
       [10.26008042, 15.28963198,  8.38669504],
       [ 9.49973637, 18.63829849,  9.37255747]])
```

Step 2: Second, the algorithm tries to reconstruct each point as a **linear combination** of its k closest neighbors. More specifically, if we denote by $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k$ the k closest neighbors of x_i, the algorithm finds the weights w_{ij} that minimize

$$\|x_i - \sum_{j=1}^{k} w_{ij}\bar{x}_j\|_2^2 + \lambda \sum_{j=1}^{k} w_{ij}^2$$

subject to the condition

$$\sum_{j=1}^{k} w_{ij} = 1.$$

The regularization term $\lambda \sum_{j=1}^{k} w_{ij}^2$ is there to prevent overfitting issues, and the condition $\sum_{j=1}^{k} w_{ij} = 1$ is imposed to make the algorithm invariant to translations of the dataset.

Using the matrix and the (row) vector

$$\bar{X}_i = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_k \end{bmatrix} \qquad \text{and} \qquad w = \begin{bmatrix} w_{i1} & w_{i2} & \cdots & w_{ik} \end{bmatrix},$$

the problem of finding the weights becomes:

$$\begin{aligned} &\text{minimize } \ \|x_i - w\bar{X}_i\|_2^2 + \lambda\|w\|_2^2 \\ &\text{subject to } \ \sum_{j=1}^{k} w_{ij} = 1 \end{aligned}$$

This is a constrained optimization problem, and its solution can be computed in closed form (hooray for linear algebra!!); see the problem at the end of the Jupyter Notebook.

First, we form the matrix

$$Z_i = \begin{bmatrix} \bar{x}_1 - x_i \\ \bar{x}_2 - x_i \\ \vdots \\ \bar{x}_k - x_i \end{bmatrix}.$$

Second, we compute the matrix

$$C_i = Z_i Z_i^T + \lambda I_k.$$

Third, we solve the linear system

$$C_i\, w^T = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

for the vector w.

Finally, we normalize w so that its entries add up to 1.

In [174]:
```
lam = 1e-3 # regularization parameter
W = np.zeros((m,m)) # store the weights in this matrix
for i in range(m):
    Z = X[neighborhoods[i]]-X[i] # form matrix Z
    C = Z@Z.T # compute matrix C
    C = C + lam*np.identity(n_neighbors)*np.linalg.norm(Z) # add the regularization term
    w = np.linalg.solve(C,np.ones(n_neighbors)) # solve the linear system
    W[i,neighborhoods[i]] = w/np.sum(w) # normalize w
```

After this step, the weight matrix W (containing the weights w_{ij}) encodes the local linear relationships between tha dataset points.

Step 3: The goal of the LLE algorithm is to preserve the local linear structure of the high-dimensional space as accurately as possible in a low-dimensional space. Hence, the matrix W is kept fixed and embedded coordinates y_i are sought by minimizing the following function

$$\sum_{i=1}^{m} \|y_i - \sum_{i=1}^{m} w_{ij} y_j\|_2^2.$$

Using the matrix

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix},$$

the problem of finding the embedded coordinates y_i becomes

$$\text{minimize} \sum_{i=1}^{m} \| \text{ row } i \text{ of } (I_m - W)Y\|_2^2$$

This optimization problem also has a closed-form solution. To solve it, first we form the matrix

$$M = (I_m - W)^T (I_m - W).$$

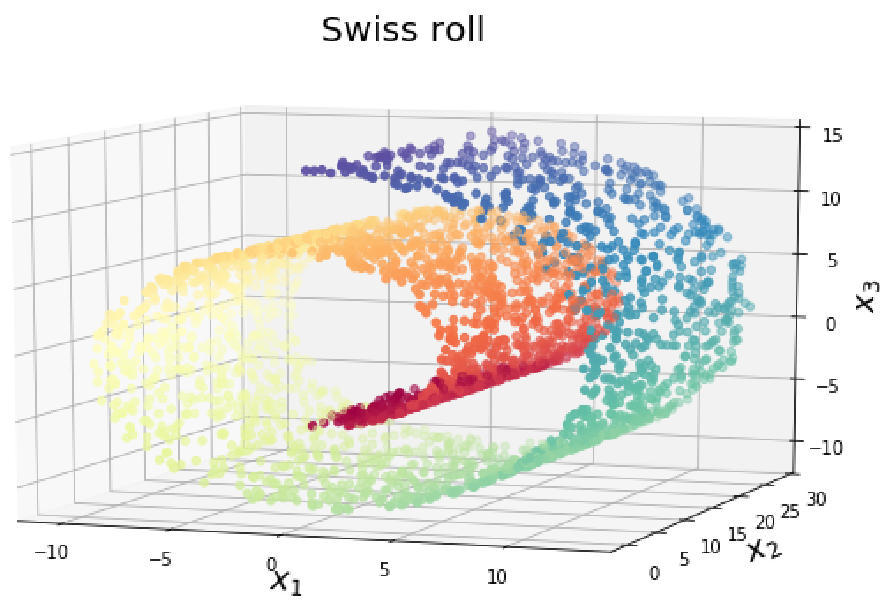
If we are embedding the dataset into a space of dimension d, then the columns of Y are the eigenvectors of M associated with the d smallest nonzero eigenvalues.

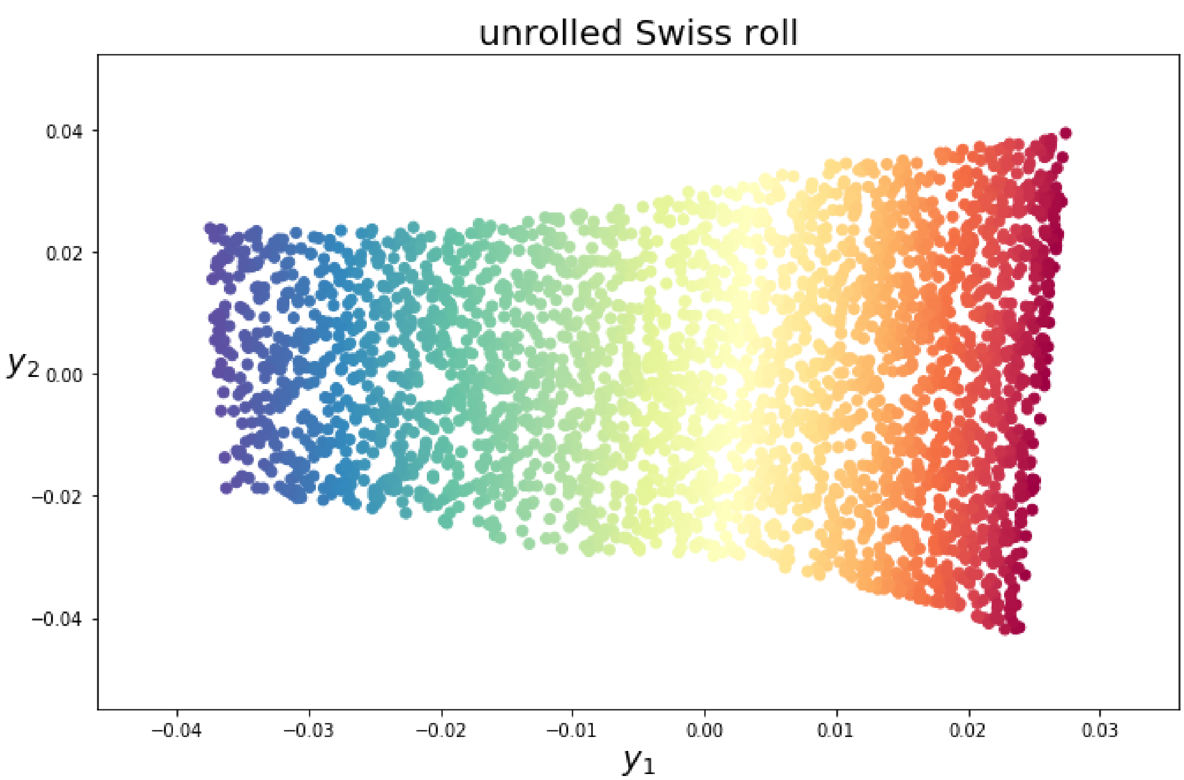
In [175]:
```
d = 2
from scipy.sparse.linalg import eigsh # function for finding eigenvalues and e
igenvectors of symmetric matrices

#M = (np.identity(m)-W).T@(np.identity(m)-W)
M = np.identity(m)-W.T-W+W.T@W
evalues,evectors = eigsh(M,
                         k=d+1,
                         sigma = 0, # compute the eigenvectors with the smalle
st eigenvalues
                         maxiter = 100,
                         tol = 1e-06)
index = np.argsort(evalues) # sort eigenvalues
reduced_X = np.real(evectors[:,index[1:d+1]]) # drop the eigenvector with eige
nvalue equal to 0
```

The resulting 2D dataset is shown in the figure below. As you can see, the Swiss roll is completely unrolled, and the distances between points are locally preserved.

In [176]:
```
'plot the Swiss roll and the unrolled Swiss roll'

fig = plt.figure(figsize=(11, 15))

# Swiss roll
ax = plt.subplot(2,1,1, projection='3d')
axes = [-11.5, 14, -2, 30, -12, 15]
ax.scatter(X[:, 0], X[:, 1], X[:, 2], c=t, cmap=plt.cm.Spectral)
ax.view_init(10, -70)
ax.set_xlabel("$x_1$", fontsize=18)
ax.set_ylabel("$x_2$", fontsize=18)
ax.set_zlabel("$x_3$", fontsize=18)
ax.set_xlim(axes[0:2])
ax.set_ylim(axes[2:4])
ax.set_zlim(axes[4:6])
plt.title('Swiss roll',fontsize=20)

# unrolled Swiss roll
plt.subplot(2,1,2)
plt.scatter(reduced_X[:,0],reduced_X[:,1],c=t,cmap=plt.cm.Spectral)
plt.xlabel("$y_1$", fontsize=18)
plt.ylabel("$y_2$", fontsize=18, rotation=0)
plt.title('unrolled Swiss roll',fontsize=20)
```

Out[176]: Text(0.5, 1.0, 'unrolled Swiss roll')





Amazing, isn't it?

11.3.1 Locally Linear Embedding Implementation

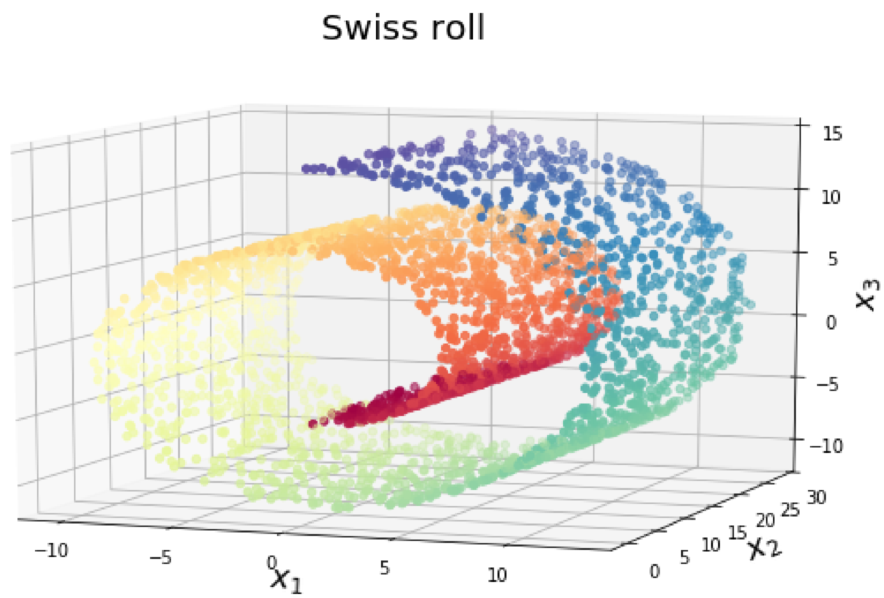
In [2]:
```
def lle(X,d=2,n_neighbors=5):
    from scipy.spatial import distance
    from scipy.sparse.linalg import eigsh
    from scipy.linalg import solve

    m,n = X.shape
    'step 1: compute pairwise distances and find neighbors'
    print('step 1: computing pairwise distances')
    distances = distance.squareform(distance.pdist(X))
    index = np.argsort(distances,axis=1)
    neighbors = index[:,1:n_neighbors+1]

    'step 2: solve for reconstruction weights'
    print('step 2: computing weights')
    if n_neighbors>m:
        lam = 1e-3
    else:
        lam = 0
    reg = 1e-3
    W = np.zeros((m,m))
    for i in range(m):
        Z = X[neighbors[i]]-X[i]
        C = Z@Z.T
        C = C + reg*np.identity(n_neighbors)*np.linalg.norm(Z)
        w = solve(C,np.ones(n_neighbors),sym_pos=True)
        W[i,neighbors[i]] = w/np.sum(w)


    'step 3: compute embedding from eigenvectors of the matrix (I-W)^T*(I-W)'
    print('step 3: computing new points')
    M = np.identity(m)-W.T-W+W.T@W
    tol = 1e-05
    max_iter=100
    evalues,evectors = eigsh(M,
                             k=d+1,
                             sigma = 0,
                             maxiter = max_iter,
                             tol = tol)
    index = np.argsort(evalues)
    reduced_X = np.real(evectors[:,index[1:d+1]])
    return reduced_X
```

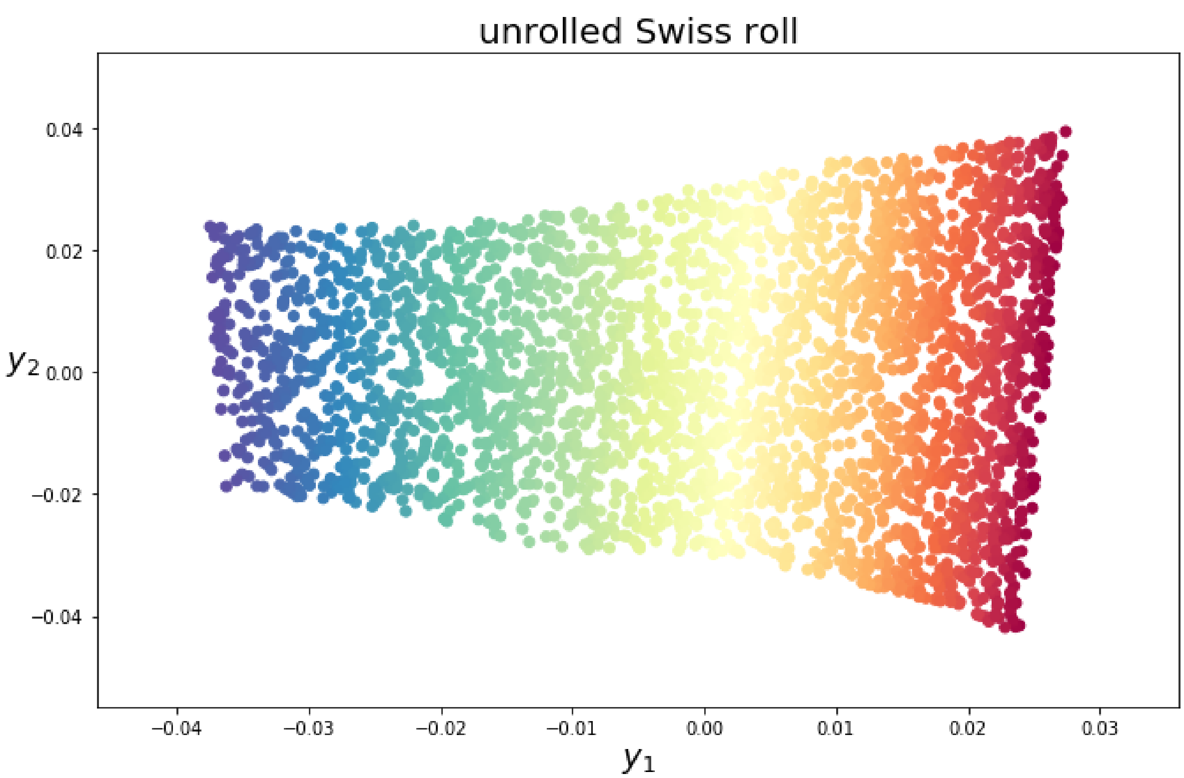
In [178]:
```
'test lle'
reduced_X = lle(X,
                d = 2,
                n_neighbors = 10)
'Plot original and reduced datasets'
fig = plt.figure(figsize=(11, 15))
# Swiss roll
ax = plt.subplot(2,1,1, projection='3d')
axes = [-11.5, 14, -2, 30, -12, 15]
ax.scatter(X[:, 0], X[:, 1], X[:, 2], c=t, cmap=plt.cm.Spectral)
ax.view_init(10, -70)
ax.set_xlabel("$x_1$", fontsize=18)
ax.set_ylabel("$x_2$", fontsize=18)
ax.set_zlabel("$x_3$", fontsize=18)
ax.set_xlim(axes[0:2])
ax.set_ylim(axes[2:4])
ax.set_zlim(axes[4:6])
plt.title('Swiss roll',fontsize=20)

# unrolled Swiss roll
plt.subplot(2,1,2)
plt.scatter(reduced_X[:,0],reduced_X[:,1],c=t,cmap=plt.cm.Spectral)
plt.xlabel("$y_1$", fontsize=18)
plt.ylabel("$y_2$", fontsize=18, rotation=0)
plt.title('unrolled Swiss roll',fontsize=20)
```

```
step 1: computing pairwise distances
step 2: computing weights
step 3: computing new points
```

Out[178]:

```
Text(0.5, 1.0, 'unrolled Swiss roll')
```



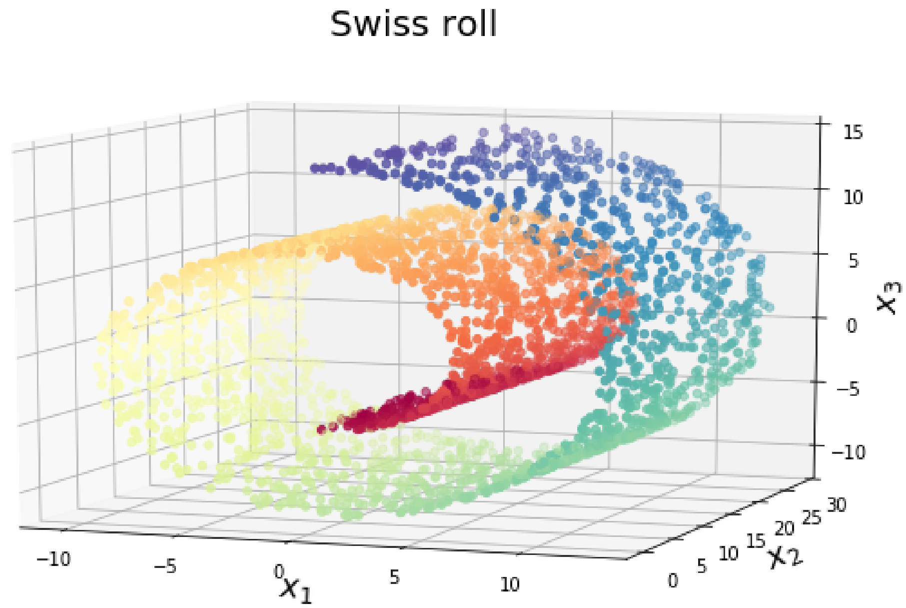


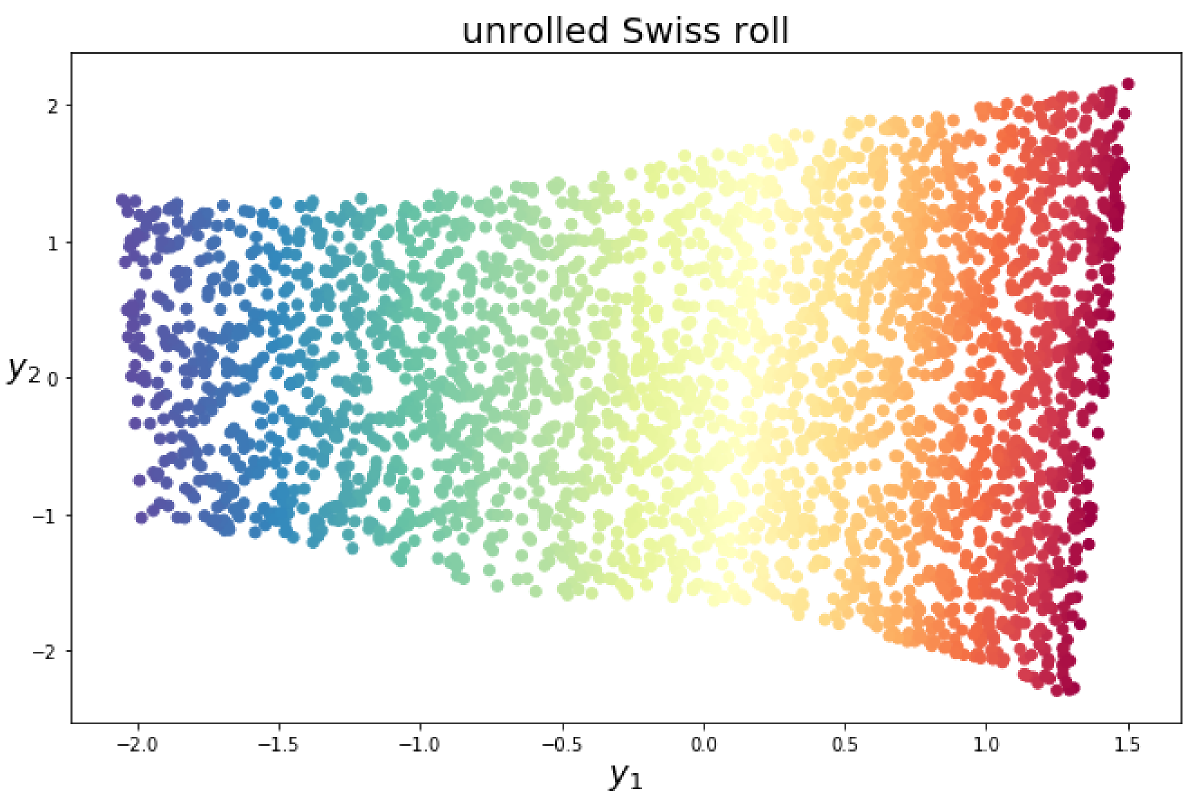
In [179]:
```
def lle_sparse(X,d=2,n_neighbors=5):
    'Use this version if the dataset is big'
    from scipy.spatial import distance
    from scipy.sparse import eye
    from scipy.sparse.linalg import eigsh
    from scipy.linalg import solve
    from sklearn.utils import check_random_state

    m,n = X.shape
    'step 1: compute pairwise distances and find neighbors'
    print('step 1: computing pairwise distances')
    distances = distance.squareform(distance.pdist(X))

    index = np.argsort(distances,axis=1)
    neighbors = index[:,1:n_neighbors+1]

    'step 2: solve for reconstruction weights'
    print('step 2: computing weights')
    reg = 1e-3
    W = np.zeros((m,n_neighbors))
    for i in range(m):
        Z = X[neighbors[i]]-X[i]
        C = Z@Z.T
        C = C + reg*np.identity(n_neighbors)*np.linalg.norm(Z) #np.trace(C)
        w = solve(C,np.ones(n_neighbors),sym_pos=True)
        W[i] = w/np.sum(w)

    'step 3: compute embedding from eigenvectors of the matrix (I-W)^T*(I-W)'
    print('step 3: computing new points')
    'form matrix M'
    M = eye(m,format='lil') # sparse matrix
    for i in range(m):
        j = neighbors[i]
        M[i,j] = M[i,j] - W[i]
    M = (M.T * M).tocsr()
    'get eigenvectors'
    random_state = check_random_state(None)
    v0 = random_state.uniform(-1, 1, M.shape[0])
    tol = 1e-06
    max_iter = 100
    evalues,evectors = eigsh(M,
                             k=d+1,
                             sigma = 0,
                             maxiter = max_iter,
                             tol = tol,
                             v0 = v0)
    reduced_X = np.real(evectors[:,1:])*np.sqrt(m)
    return reduced_X
```

In [180]:
```
'test lle_sparse function'
reduced_X = lle_sparse(X,
                d = 2,
                n_neighbors = 10)
'Plot original and reduced datasets'
fig = plt.figure(figsize=(11, 15))
# Swiss roll
ax = plt.subplot(2,1,1, projection='3d')
axes = [-11.5, 14, -2, 30, -12, 15]
ax.scatter(X[:, 0], X[:, 1], X[:, 2], c=t, cmap=plt.cm.Spectral)
ax.view_init(10, -70)
ax.set_xlabel("$x_1$", fontsize=18)
ax.set_ylabel("$x_2$", fontsize=18)
ax.set_zlabel("$x_3$", fontsize=18)
ax.set_xlim(axes[0:2])
ax.set_ylim(axes[2:4])
ax.set_zlim(axes[4:6])
plt.title('Swiss roll',fontsize=20)

# unrolled Swiss roll
plt.subplot(2,1,2)
plt.scatter(reduced_X[:,0],reduced_X[:,1],c=t,cmap=plt.cm.Spectral)
plt.xlabel("$y_1$", fontsize=18)
plt.ylabel("$y_2$", fontsize=18, rotation=0)
plt.title('unrolled Swiss roll',fontsize=20)
```

```
step 1: computing pairwise distances
step 2: computing weights
step 3: computing new points
```

Out[180]: Text(0.5, 1.0, 'unrolled Swiss roll')





11.3.3 Examples

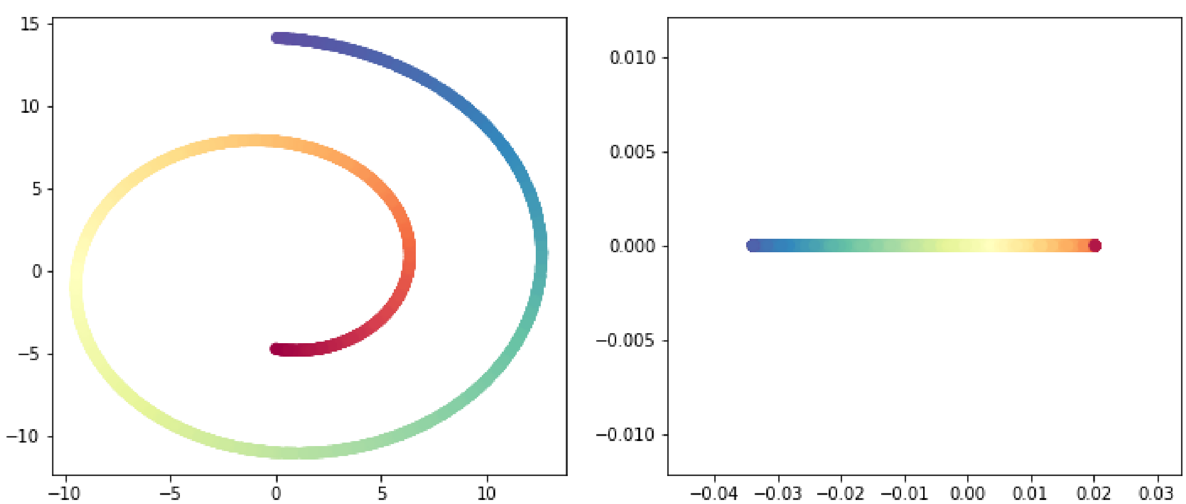
Example 1: a Spiral

In [12]:
```
'Make a spiral'
m = 3000
t = 1.5 * np.pi * (1 + 2 * np.random.rand(1, m))
x = t * np.cos(t)
y = t * np.sin(t)
X = np.concatenate((x, y))
X = X.T
t = np.squeeze(t)

'Unroll the spiral'
reduced_X = lle(X,
                d = 1,
                n_neighbors = 10)
```

```
step 1: computing pairwise distances
step 2: computing weights
step 3: computing new points
```

In [14]:
```
plt.figure(figsize=(12,5))
'Plot the spiral'
plt.subplot(1,2,1)
plt.scatter(X[:,0],X[:,1],c=t,cmap = plt.cm.Spectral)
'Plot the unrolled spiral'
plt.subplot(1,2,2)
plt.scatter(reduced_X,np.zeros(len(reduced_X)), c=t, cmap=plt.cm.Spectral)
```

Out[14]: <matplotlib.collections.PathCollection at 0x23499b8ec88>



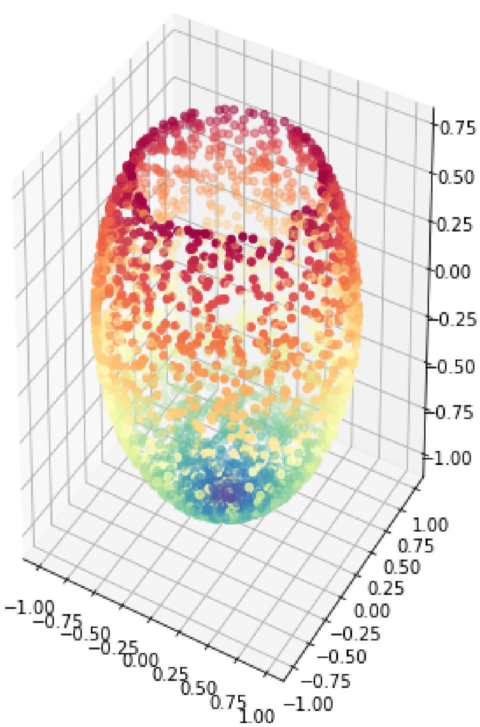
Example 2: a Sphere with a Hole

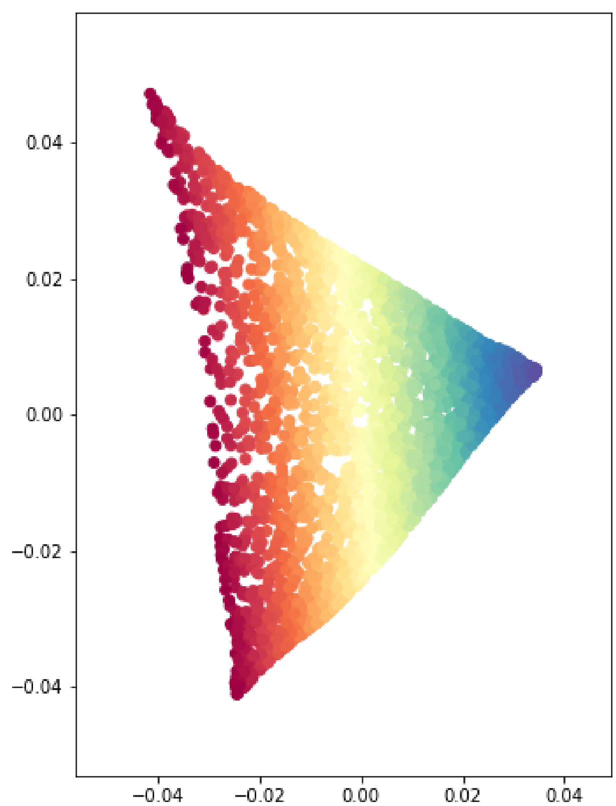
In [50]:
```
'Make a sphere with a hole'
m = 3000
r = 1
phi = 2 * np.pi * np.random.rand(1, m)
t = np.pi/4 + (np.pi-np.pi/4) * np.random.rand(1, m)
x = r*np.sin(t)*np.cos(phi)
y = r*np.sin(t)*np.sin(phi)
z = r*np.cos(t)
X = np.concatenate((x, y,z))
X = X.T
t = np.squeeze(t)

'Unroll the sphere'
reduced_X = lle(X,
                d = 2,
                n_neighbors = 10)
```

```
step 1: computing pairwise distances
step 2: computing weights
step 3: computing new points
```

In [51]:
```
plt.figure(figsize=(12,8))
ax = plt.subplot(1,2,1, projection='3d')
ax.scatter(X[:, 0], X[:, 1], X[:, 2], c=t, cmap=plt.cm.Spectral)


plt.subplot(1,2,2)
plt.scatter(reduced_X[:,0],reduced_X[:,1],c=t,cmap=plt.cm.Spectral)
```

Out[51]: <matplotlib.collections.PathCollection at 0x23497a61908>





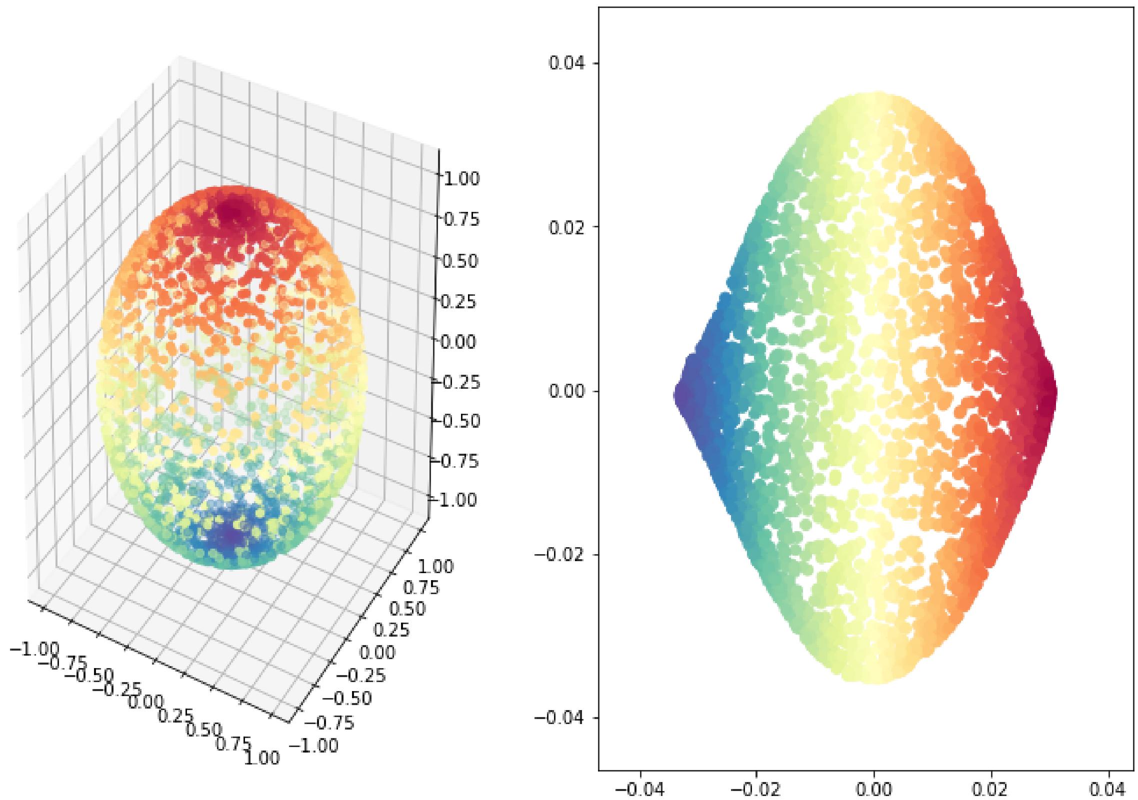
Example 3: a Sphere

In [54]:
```
'Make a sphere'
m = 3000
r = 1
phi = 2 * np.pi * np.random.rand(1, m)
t = np.pi * np.random.rand(1, m)
x = r*np.sin(t)*np.cos(phi)
y = r*np.sin(t)*np.sin(phi)
z = r*np.cos(t)
X = np.concatenate((x, y,z))
X = X.T
t = np.squeeze(t)

'Unroll the sphere'
reduced_X = lle(X,
                d = 2,
                n_neighbors = 15)
```

```
step 1: computing pairwise distances
step 2: computing weights
step 3: computing new points
```

In [55]:
```
plt.figure(figsize=(12,8))
ax = plt.subplot(1,2,1, projection='3d')
ax.scatter(X[:, 0], X[:, 1], X[:, 2], c=t, cmap=plt.cm.Spectral)


plt.subplot(1,2,2)
plt.scatter(reduced_X[:,0],reduced_X[:,1],c=t,cmap=plt.cm.Spectral)
```

Out[55]:
```
<matplotlib.collections.PathCollection at 0x234997f4f08>
```



Example 4: Visualizing 3's and 5's

In [168]:
```
'Load the MNist dataset'
from sklearn.datasets import fetch_openml
mnist = fetch_openml('mnist_784',version=1)
X,y=mnist['data'],mnist['target']
X = X[:5000]
y = y[:5000]
```

In [169]:
```
y = y.astype(np.uint8)
```

We'll focus on digits 3 and 5, which are often misclassified

In [125]:
```
y35 = y[np.logical_or(y == 3, y == 5)]
X35 = X[np.logical_or(y == 3, y == 5)]
len(y35)
```

Out[125]: 927

Now let's use LLE to reduce dimensionality down to 2D so we can plot the dataset:

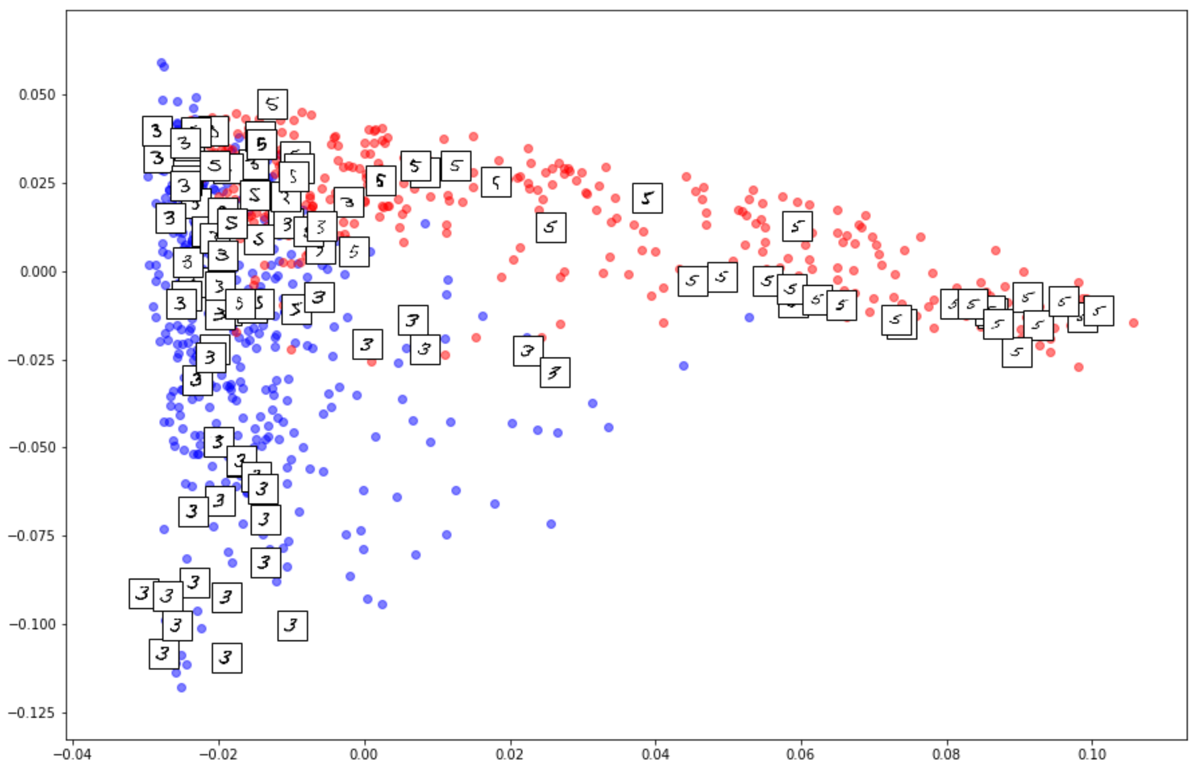
In [126]:
```
reduced_X = lle(X35,
                d = 2,
                n_neighbors = 15)
```

```
step 1: computing pairwise distances
step 2: computing weights
step 3: computing new points
```

Now let's use Matplotlib's scatter() function to plot a scatterplot, using a different color for each digit.

In [127]:

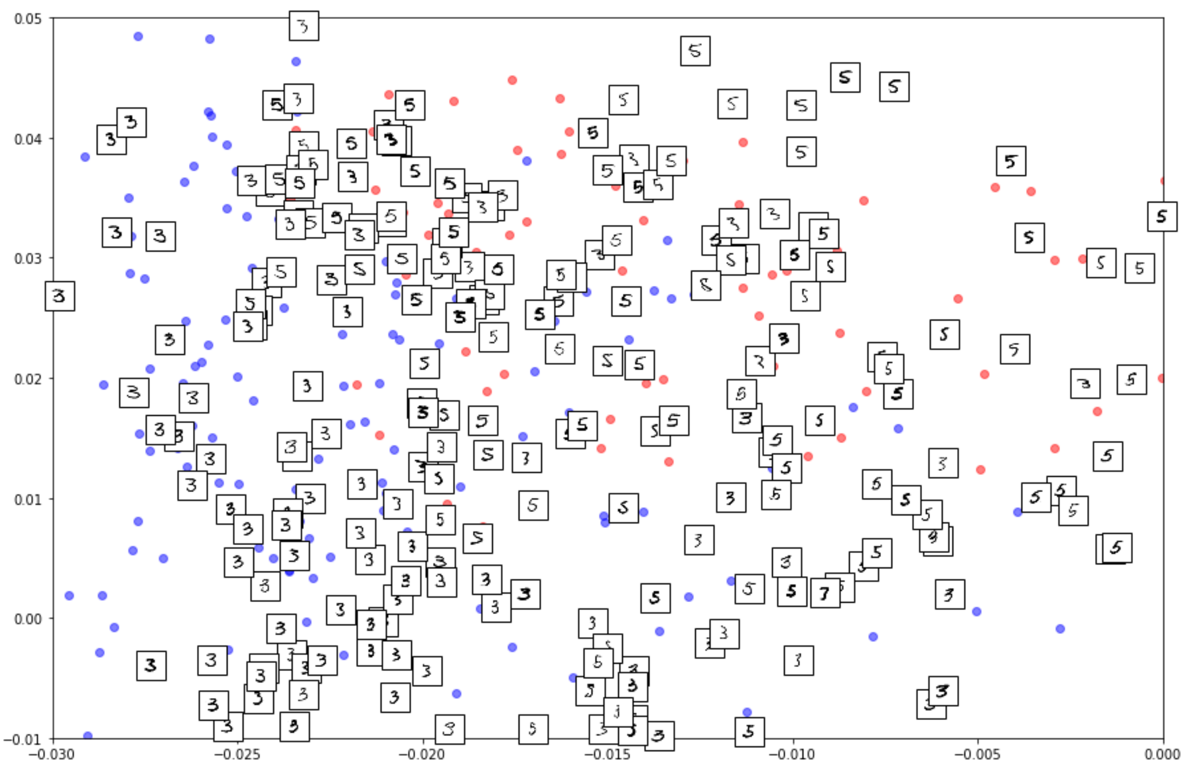
```python
colors = ['blue','red']
digits = [3,5]
plt.figure(figsize=(15,10))
for i in range(2):
    plt.scatter(reduced_X[y35==digits[i],0],reduced_X[y35==digits[i],1],color=
colors[i], alpha = 0.5)
ax = plt.gcf().gca()  # get current axes in current figure

'Plot scaled-down versions of some of digit images'
from matplotlib.offsetbox import AnnotationBbox, OffsetImage
n_digits = 100
for i in range(n_digits):
    image = X35[i].reshape(28, 28)
    image_coord = reduced_X[i]
    imagebox = AnnotationBbox(OffsetImage(image, zoom=.5, cmap="binary"), imag
e_coord)
    ax.add_artist(imagebox)
```



Isn't this just beautiful? :) This plot tells us which 3's are easily distinguishable from 5's , and it also tells us which 3's are often hard to distinguish from 5's.

Let's focus on the region where 3's and 5's seem to overlap a lot.

In [131]:
```
colors = ['blue','red']
digits = [3,5]
plt.figure(figsize=(15,10))
for i in range(2):
    plt.scatter(reduced_X[y35==digits[i],0],reduced_X[y35==digits[i],1],color=
colors[i], alpha = 0.5)
ax = plt.gcf().gca()  # get current axes in current figure
from matplotlib.offsetbox import AnnotationBbox, OffsetImage
n_digits = 500
for i in range(n_digits):
    image = X35[i].reshape(28, 28)
    image_coord = reduced_X[i]
    imagebox = AnnotationBbox(OffsetImage(image, zoom=.5, cmap="binary"), imag
e_coord)
    ax.add_artist(imagebox)
    ax.axis([-0.03,0.00,-0.01,0.05]) # zoom in
```



Example 5: Visualizing all the Digits

In [ ]:
```
'Load the MNist dataset'
from sklearn.datasets import fetch_openml
mnist = fetch_openml('mnist_784',version=1)
X,y=mnist['data'],mnist['target']
X = X[:10000]
y = y[:10000]
```

In [181]:
```
for i in range(10):
    Xi = X[y==i]
    reduced_X = lle(X2,
                d = 2,
                n_neighbors = 15)
    plt.figure(figsize=(15,10))
    plt.scatter(reduced_X[:,0],reduced_X[:,1], alpha = 0.7)
    plt.title('Digit '+str(i), fontsize=30)

    ax = plt.gcf().gca()  # get current axes in current figure
    n_digits = 200
    for i in range(n_digits):
        image = Xi[i].reshape(28, 28)
        image_coord = reduced_X[i]
        imagebox = AnnotationBbox(OffsetImage(image, zoom=.5, cmap="binary"),
image_coord)
        ax.add_artist(imagebox)
```

```
step 1: computing pairwise distances
step 2: computing weights
step 3: computing new points
step 1: computing pairwise distances
step 2: computing weights
step 3: computing new points
step 1: computing pairwise distances
step 2: computing weights
step 3: computing new points
step 1: computing pairwise distances
step 2: computing weights
step 3: computing new points
step 1: computing pairwise distances
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step 1: computing pairwise distances
step 2: computing weights
step 3: computing new points
step 1: computing pairwise distances
step 2: computing weights
step 3: computing new points
step 1: computing pairwise distances
step 2: computing weights
step 3: computing new points
step 1: computing pairwise distances
step 2: computing weights
step 3: computing new points
step 1: computing pairwise distances
step 2: computing weights
step 3: computing new points
```
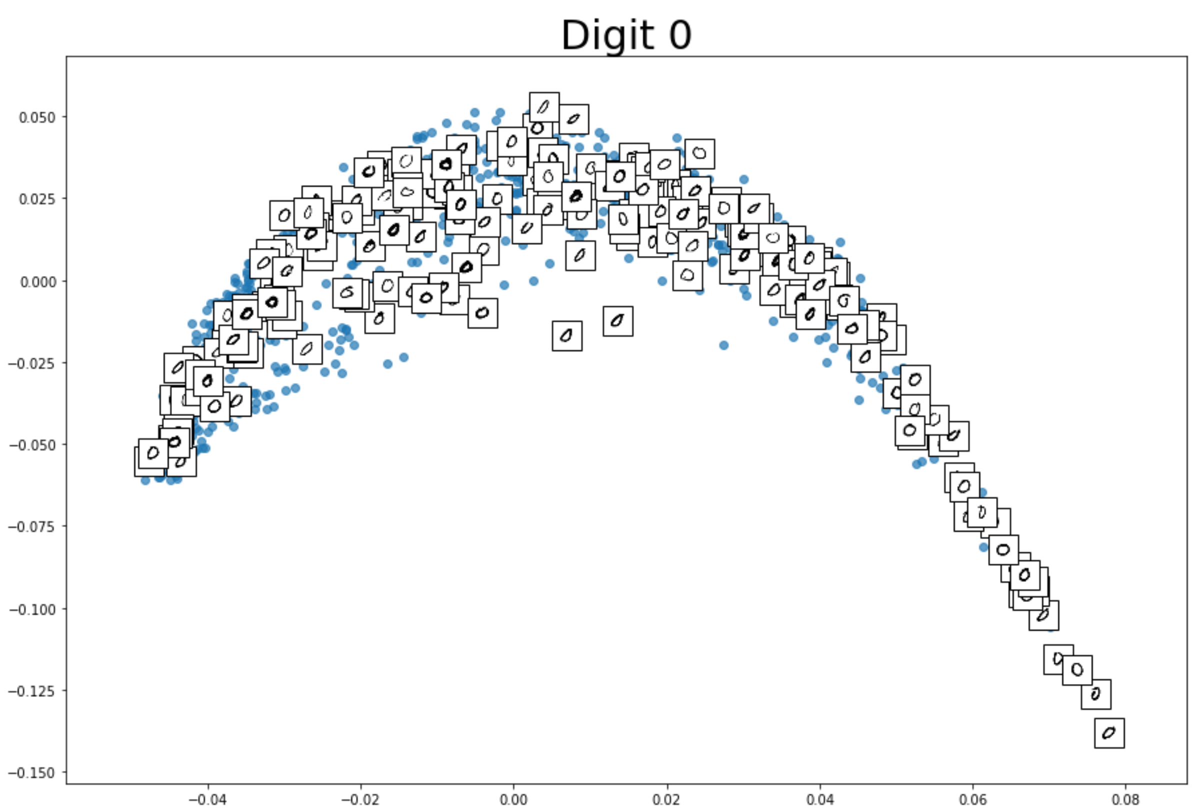
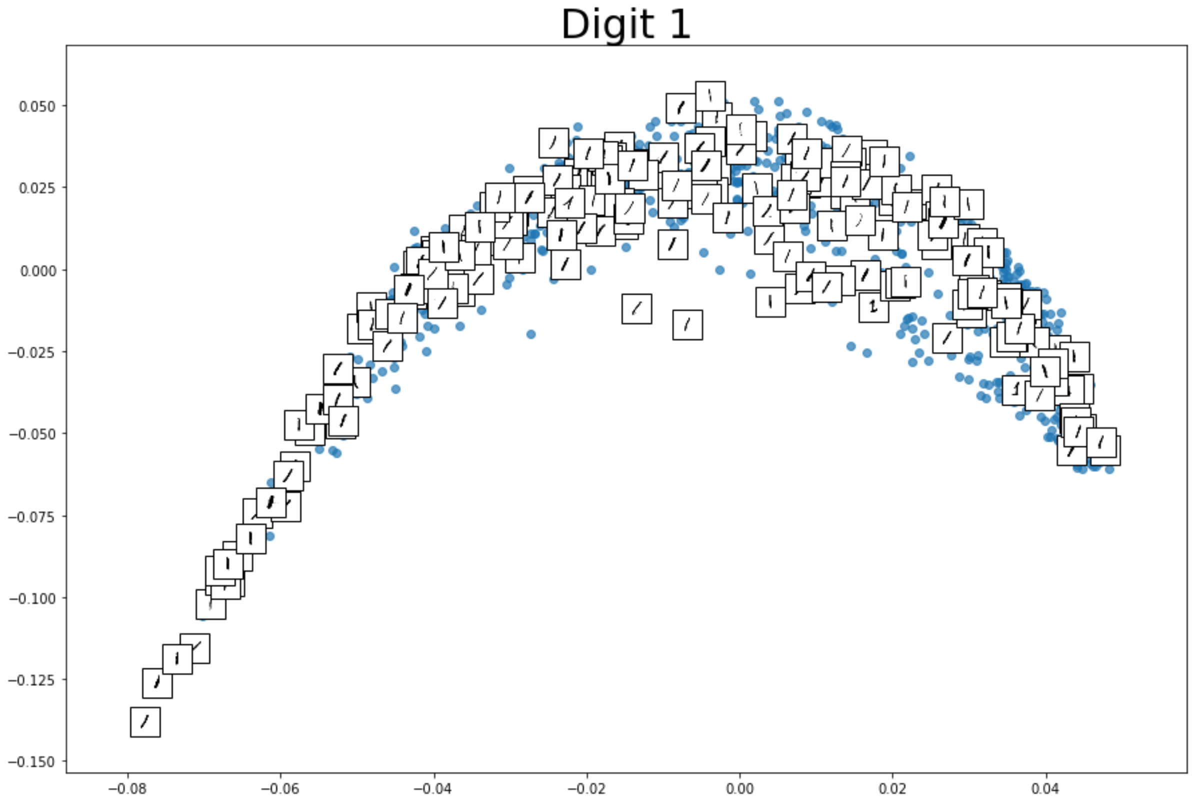
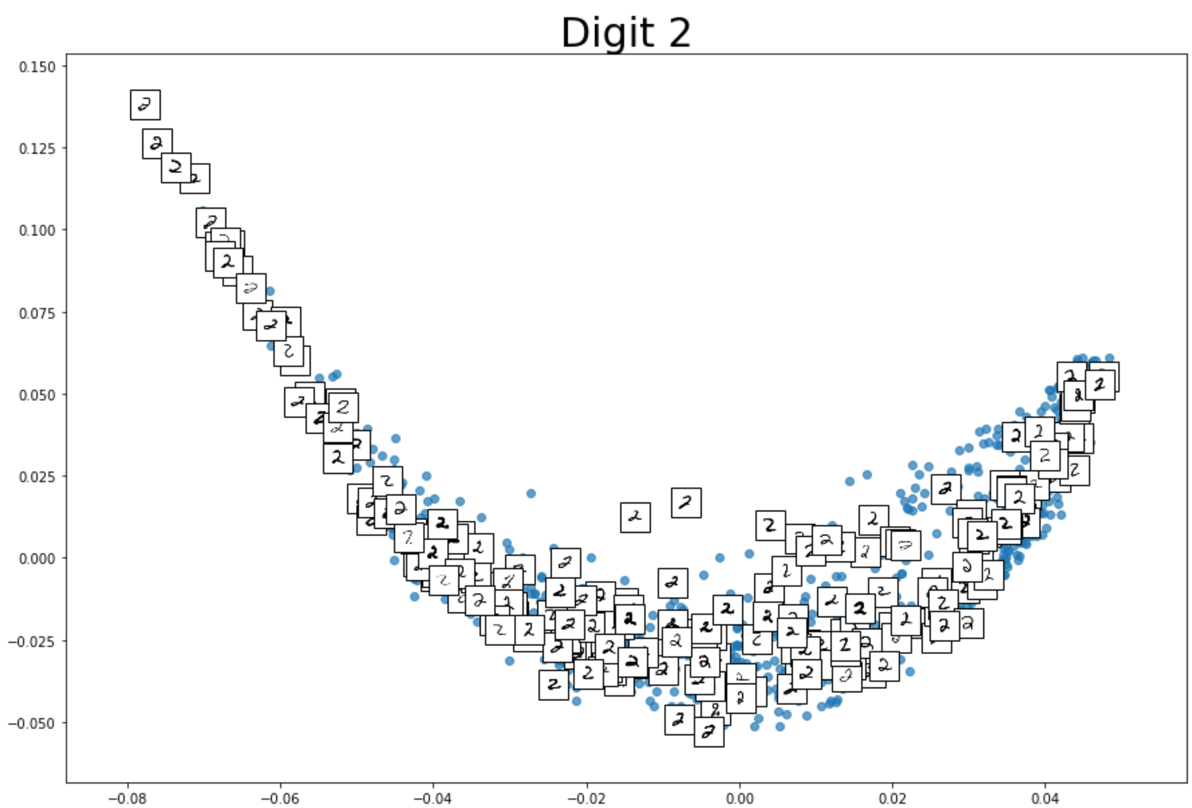
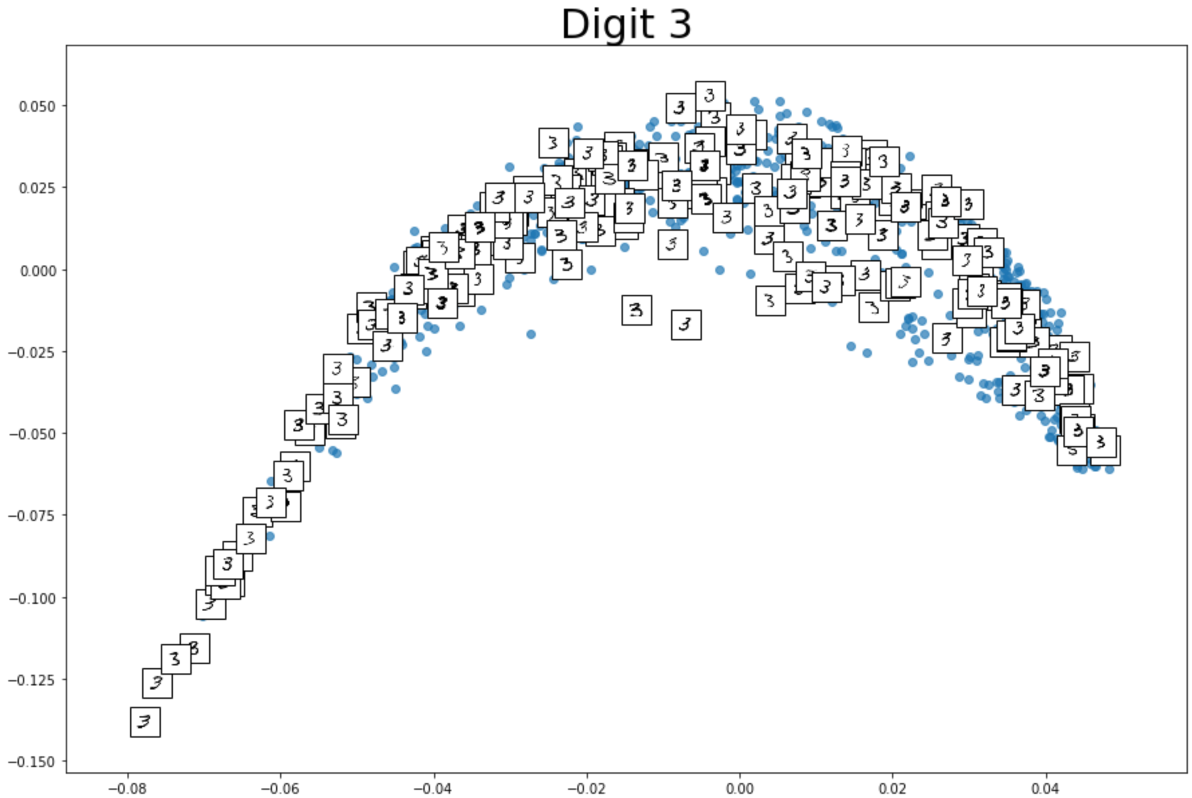
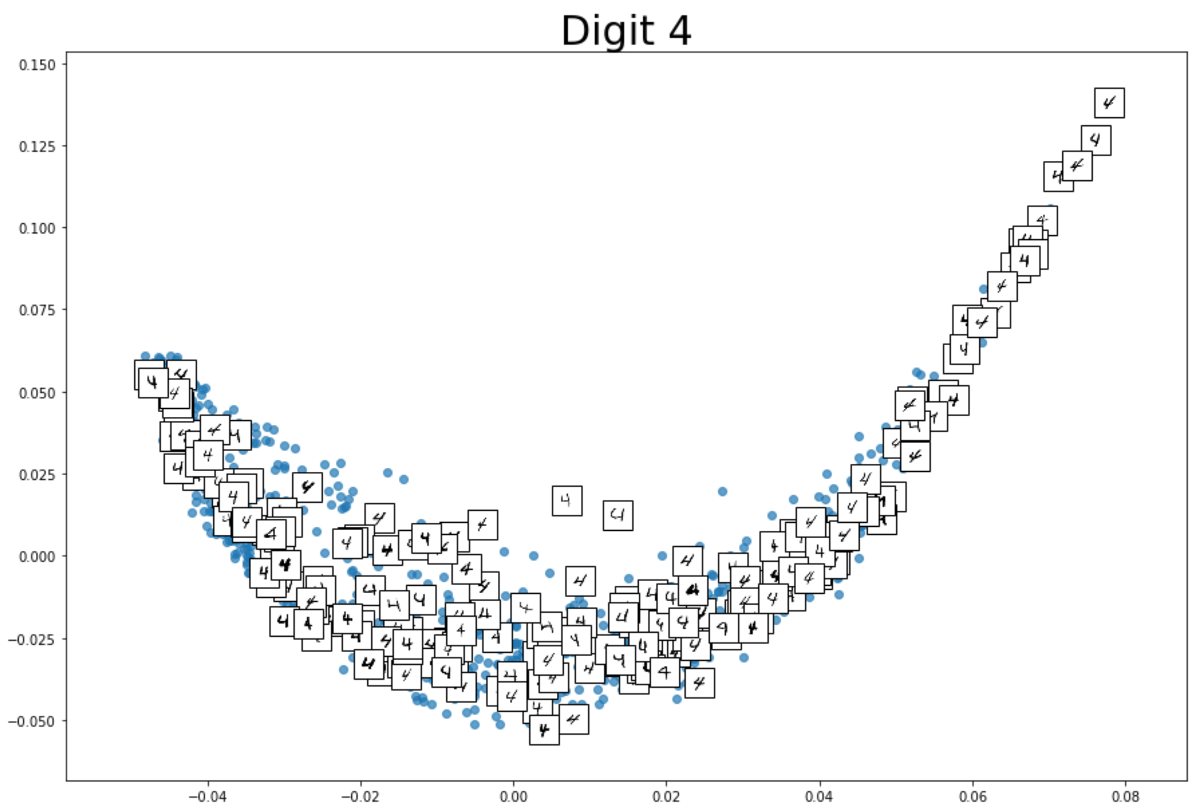



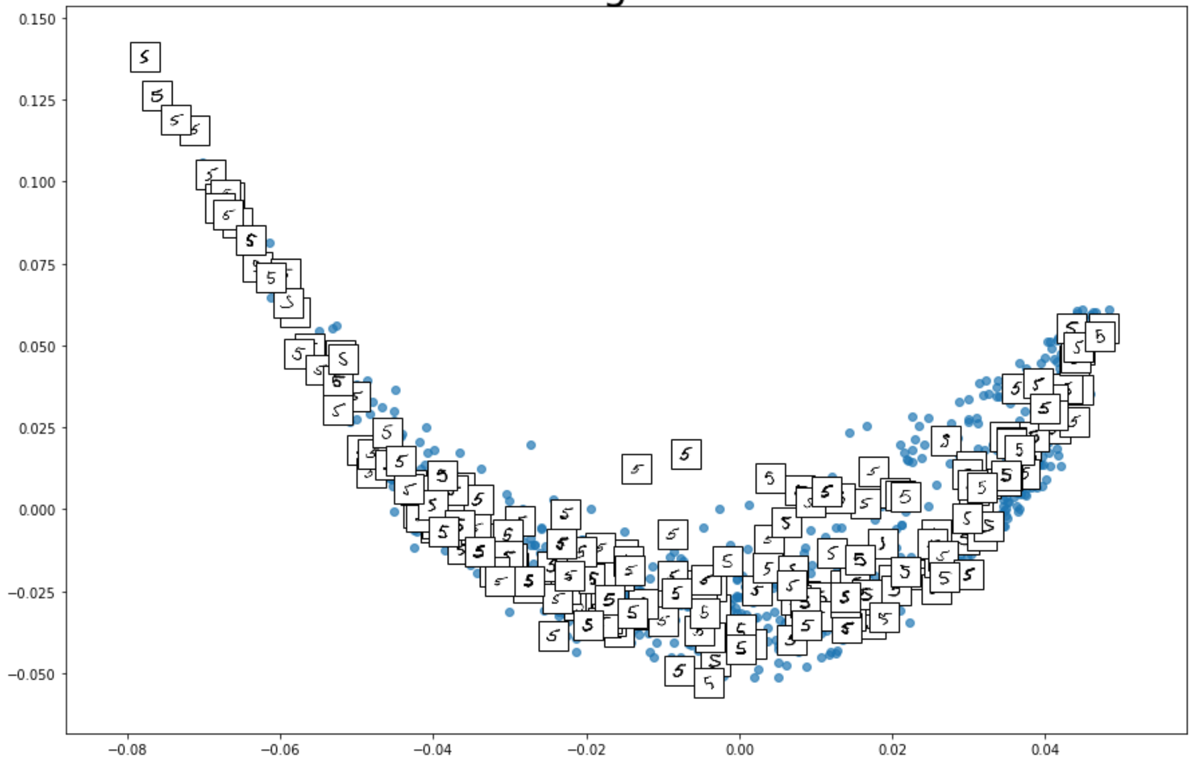

Digit 1
0.050
0.025
0.000
−0.025
−0.050
−0.075
−0.100
−0.125
−0.150
−0.08
−0.06
−0.04
−0.02
0.00
0.02
0.04


Digit 2
0.150
0.125
0.100
0.075
0.050
0.025
0.000
−0.025
−0.050
−0.08
−0.06
−0.04
−0.02
0.00
0.02
0.04

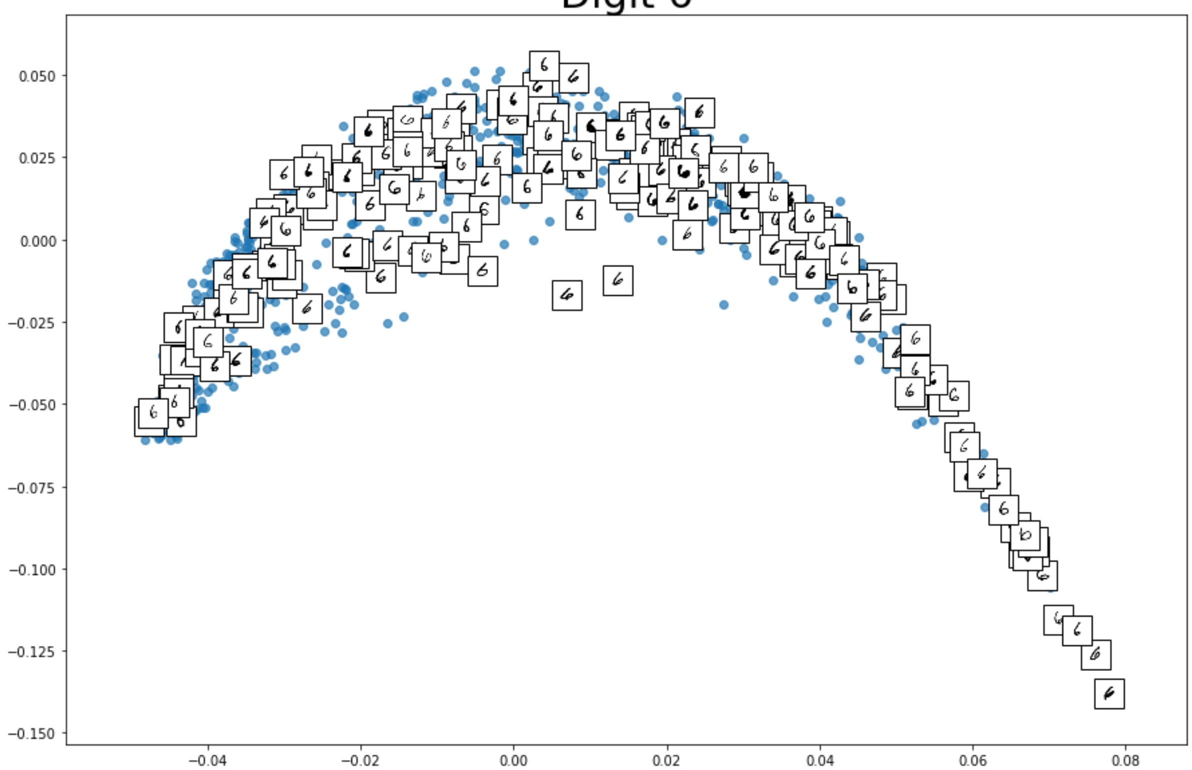

Digit 3
0.050
0.025
0.000
−0.025
−0.050
−0.075
−0.100
−0.125
−0.150
−0.08
−0.06
−0.04
−0.02
0.00
0.02
0.04

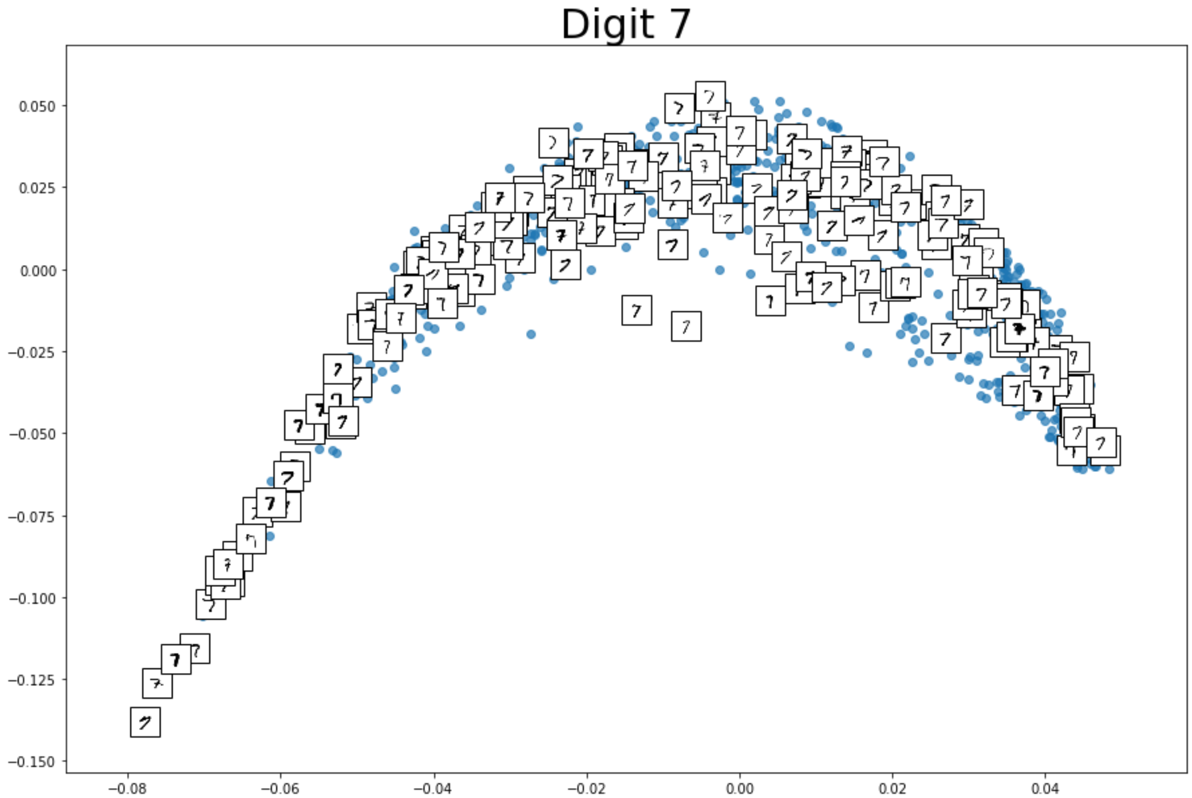

Digit 4
0.150
0.125
0.100
0.075
0.050
0.025
0.000
−0.025
−0.050
−0.04
−0.02
0.00
0.02
0.04
0.06
0.08

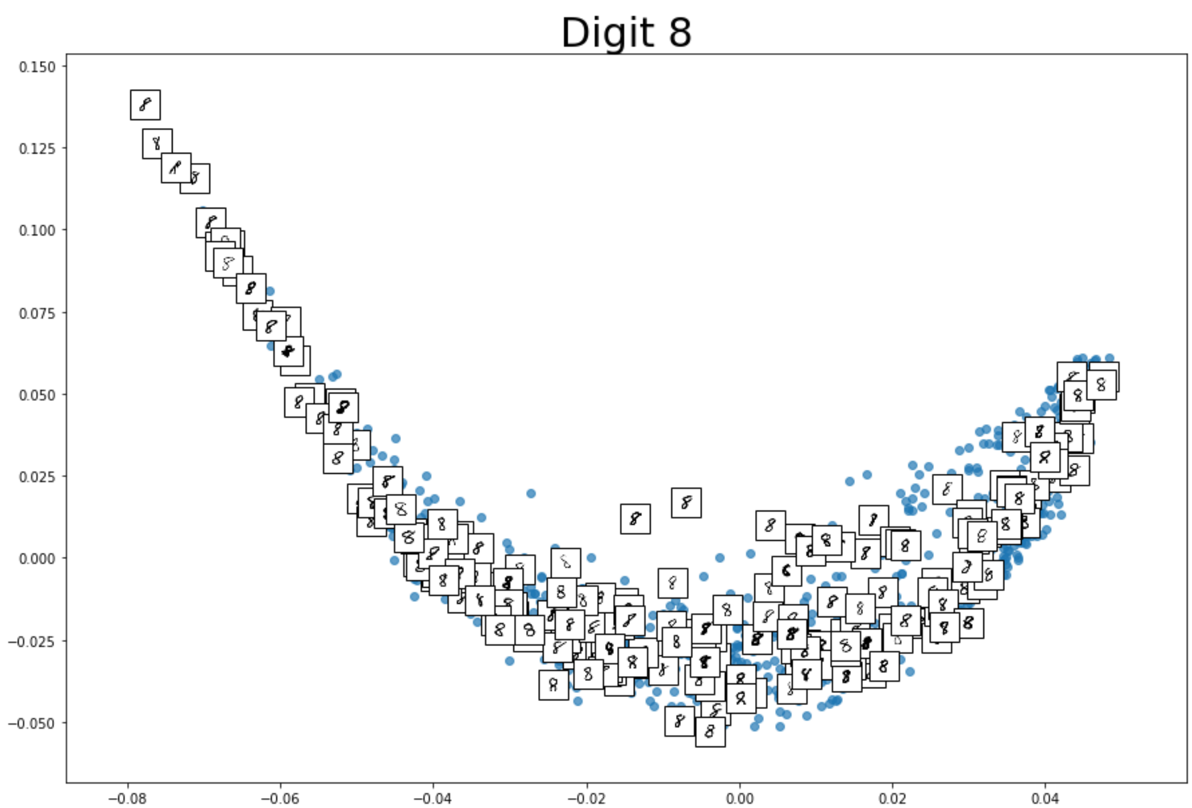
Digit 5

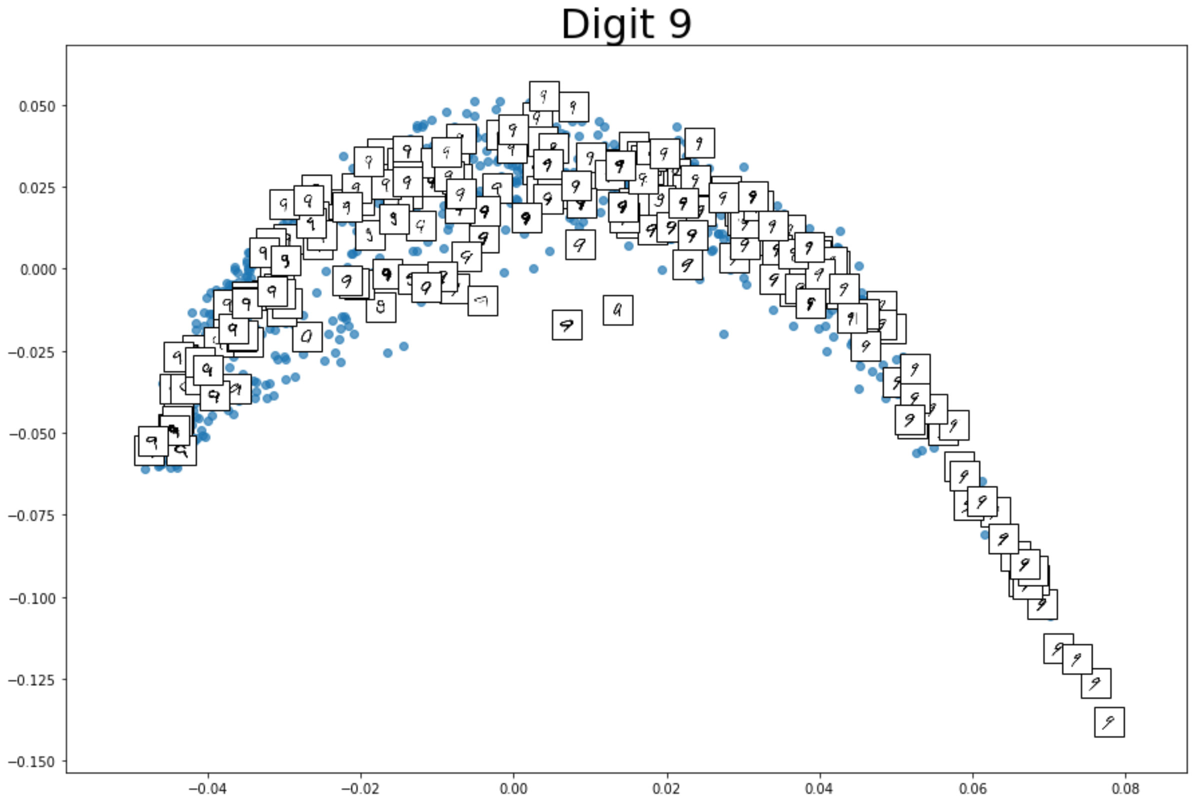


Digit 6




Digit 7
0.050
0.025
0.000
−0.025
−0.050
−0.075
−0.100
−0.125
−0.150
−0.08
−0.06
−0.04
−0.02
0.00
0.02
0.04


Digit 8
0.150
0.125
0.100
0.075
0.050
0.025
0.000
−0.025
−0.050
−0.08
−0.06
−0.04
−0.02
0.00
0.02
0.04



obj74 from coil-100

11.3.4 Problems

Problem 1 (for M562 students and brave M462 students): Let A be an $m \times n$ matrix (with $m > n$), and let b be an m component vector. Consider the following constrained optimization problem:

$$\min_x \|Ax - b\|_2^2$$

$$\text{subject to } \sum_{j=1}^{n} x_i = 1$$

(a) Suppose $\sum_{j=1}^{n} x_i = 1$ and define the $m \times n$ matrix $B = \begin{bmatrix} b & b & \cdots & b \end{bmatrix}$, show $\|Ax - b\|_2^2 = x^T(A-B)^T(A-B)x$.

(b) Using Lagrange multipliers, solve the constrained optimization problem.