Factorization of Nonnegative Matrices—II

Cony M. Lau and Thomas L. Markham
Department of Mathematics and Computer Science
University of South Carolina
Columbia, South Carolina 29208

Submitted by Ky Fan

ABSTRACT

Suppose A is an $n\times n$ nonnegative matrix. Necessary and sufficient conditions are given for A to be factored as LU, where L is a lower triangular nonnegative matrix, and U is an upper triangular nonnegative matrix with $u_{ii}=1$.

I. INTRODUCTION

Suppose A is a matrix of order n over the complex field. Necessary and sufficient conditions are given for A to have an LU-factorization in [3], where other related results may be found. In this paper, we consider the problem of factoring a nonnegative A ($A\geqslant 0$) as LU, where L is a lower triangular nonnegative matrix, and U is an upper triangular nonnegative matrix with main diagonal consisting entirely of ones.

In [4], factorizations of this type were considered with the restriction that all principal minors of A are nonzero. In Theorem 1 of this paper, these restrictions are removed. Throughout the paper, the Schur complement of a nonsingular principal submatrix of A [2] plays an important role in these factorizations, and some of the later results elucidate this role.

II. NONNEGATIVE FACTORIZATIONS

We introduce first the notation that we shall use. Let α and β be increasing sequences on $\{1,\ldots,n\}$. Then $A(\alpha|\beta)$ is the minor of A with rows indexed by α and columns indexed by β, whereas $A[\alpha|\beta]$ denotes the

 0024-3795/78/0020-0051/$1.25

submatrix of A in rows α and columns β. A principal submatrix of A is written $A[\alpha]$. For $1 \leqslant k \leqslant n$, $A_k = A[1,\dots,k]$. Further, $\hat{\alpha}$ denotes the complement of α.

If A is an $n\times n$ matrix, and A_k is nonsingular, the Schur complement of A_k in

$$A = \begin{pmatrix} A_k & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

is defined as $(A|A_k) = A_{22} - A_{21}A_k^{-1}A_{12}$ [2].

THEOREM 1. *An $n\times n$ nonnegative matrix A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$ iff A_{n-1} has an $\tilde{L}\tilde{U}$-factorization with $\tilde{L}\geqslant 0$, $\tilde{U}\geqslant 0$, $\tilde{u}_{ii}=1$, and there exist nonnegative $1\times(n-1)$ vectors v and w such that $\tilde{L}w^T = A[\hat{n}|n]$, $v\tilde{U} = A[n|\hat{n}]$ and $a_{nn} - vw^T \geqslant 0$.*

Proof. Assume A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$. Partition A, L and U conformally so that

$$A = \begin{pmatrix} A_{n-1} & z^T \\ y & a_{nn} \end{pmatrix} = \begin{pmatrix} L_{n-1} & 0 \\ r & l \end{pmatrix}\begin{pmatrix} U_{n-1} & s^T \\ 0 & 1 \end{pmatrix} = LU.$$

Let $\tilde{L} = L_{n-1}$, $\tilde{U} = U_{n-1}$, $v = r$ and $w = s$. Then $\tilde{L}, \tilde{U}, v, w$ are all nonnegative, $\tilde{L}\tilde{U} = L_{n-1}U_{n-1} = A_{n-1}$, $\tilde{L}w^T = L_{n-1}s^T = z^T = A[\hat{n}|n]$, $v\tilde{U} = rU_{n-1} = y = A[n|\hat{n}]$, and $a_{nn} - vw^T = l \geqslant 0$.

Conversely, assume A_{n-1} has an $\tilde{L}\tilde{U}$-factorization and there exist v and w satisfying conditions on the right-hand side of the statement. Partition A as above. Let

$$L = \begin{pmatrix} \tilde{L} & 0 \\ v & a_{nn} - vw^T \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} \tilde{U} & w^T \\ 0 & 1 \end{pmatrix}.$$

Then $L\geqslant 0$, $U\geqslant 0$, and

$$LU = \begin{pmatrix} \tilde{L} & 0 \\ v & a_{nn} - vw^T \end{pmatrix}\begin{pmatrix} \tilde{U} & w^T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \tilde{L}\tilde{U} & \tilde{L}w^T \\ v\tilde{U} & a_{nn} \end{pmatrix} = A. \qquad \blacksquare$$

COROLLARY 1. *Let A be an $n\times n$ nonnegative matrix with* $\det(A_{n-1})\neq 0$. *Then A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$, $u_{ii}=1$ iff A_{n-1} has an $\tilde{L}\tilde{U}$-factorization with $\tilde{L}\geqslant 0$, $\tilde{U}\geqslant 0$, $\tilde{u}_{ii}=1$; and for $y=A[n|\hat{n}]$ and $z^T=A[\hat{n}|n]$, we have $y\tilde{U}^{-1}\geqslant 0$, $\tilde{L}^{-1}z^T\geqslant 0$, and $(A|A_{n-1})=a_{nn}-yA_{n-1}^{-1}z^T\geqslant 0$.*

Proof. Note that since $\tilde{U}$ is always invertible, $\tilde{L}$ is invertible if and only if A_{n-1} is invertible. Thus for $v\tilde{U}=y$ and $\tilde{L}w^T=z^T$, we have $v=y\tilde{U}^{-1}$, $w^T=\tilde{L}^{-1}z^T$ and $a_{nn}-vw^T=a_{nn}-y\tilde{U}^{-1}\tilde{L}^{-1}z^T=a_{nn}-yA_{n-1}^{-1}z^T=(A_n|A_{n-1})$. It is now easy to see that the corollary is a direct consequence of Theorem 1. ■

COROLLARY 2. *Let A be an $n\times n$ nonnegative matrix with nonzero leading principal minors. If A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$, then $(A_{k+1}|A_k)>0$ for $k=1,2,\dots,n-1$. Consequently* $\det(A_k)>0$ *for $k=1,2,\dots,n$.*

Proof. Since A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$, it follows from Corollary 1 that A_{n-1} has an $\tilde{L}\tilde{U}$-factorization with $\tilde{L}\geqslant 0$, $\tilde{U}\geqslant 0$ and $\tilde{u}_{ii}=1$ and $(A|A_{n-1})\geqslant 0$. Now apply the same argument to A_{n-1}. The process can be continued, yielding $(A_{k+1}|A_k)\geqslant 0$ for $k=1,2,\dots,n-1$. Suppose $(A_k|A_{k-1})=0$ for some k. Then it follows from the proofs of Theorem 1 and Corollary 1 that $l_{kk}=0$, contradicting the fact that A is nonsingular. Finally, note that $a_{11}>0$ and $\det(A_{k+1})=\det(A_k)(A_{k+1}|A_k)$. The fact that $\det(A_k)>0$ for $k=1,2,\dots,n$ follows iteratively. ■

COROLLARY 3. *Let A be an $n\times n$ nonnegative matrix with the property that for $k=1,2,\dots,n-1$, there exist nonnegative vectors v_k and w_k such that $A_kw_k^T=A[1,2,\dots,k|k+1]$, $v_kA_k=A[k+1|1,2,\dots,k]$ and $a_{k+1,k+1}-v_kA_kw_k^T\geqslant 0$. Then A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$.*

Proof. We shall use induction on n. For $n=1$, the statement is trivially true. Assume it is true for $n=k$, and consider $n=k+1$. Partition A into $\begin{pmatrix} A_k & z^T \\ y & a_{nn}\end{pmatrix}$, where y,z are $1\times k$ nonnegative vectors. A_k satisfies conditions of the hypothesis and hence has an $\tilde{L}\tilde{U}$-factorization with $\tilde{L}\geqslant 0$, $\tilde{U}\geqslant 0$ and $\tilde{u}_{ii}=1$. Also by hypothesis, there exist nonnegative vectors v_k and w_k such that $A_kw_k^T=A[\hat{n}|n]$, $v_kA_k=A[n|\hat{n}]$ and $a_{nn}-v_kA_kw_k^T\geqslant 0$. Let $v=v_k\tilde{L}$ and $w^T=\tilde{U}w_k^T$. Then $v\geqslant 0$, $w\geqslant 0$, $\tilde{L}w^T=A_kw_k^T=A[\hat{n}|n]$, $v\tilde{U}=v_kA_k=A[n|\hat{n}]$, and $a_{nn}-vw^T=a_{nn}-v_k\tilde{L}\tilde{U}w_k^T=a_{nn}-v_kA_kw_k^T\geqslant 0$. Thus by Theorem 1, A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$. ■

EXAMPLE 1. The converse of Corollary 1 is not true, as can be seen in

$$A=\begin{bmatrix}2&2&0\\2&3&1\\0&2&3\end{bmatrix}=\begin{bmatrix}2&0&0\\2&1&0\\0&2&1\end{bmatrix}\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix},$$

where

$$A[1,2|3]=\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}2&2\\2&3\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix}=A_2w_2^T.$$

w_2^T is unique but not nonnegative, and yet A has a nonnegative factorization.

COROLLARY 4. *Let A be an $n\times n$ nonnegative matrix with nonzero proper leading principal minors such that for $k=1,2,\dots,n-1$, we have $A_k^{-1}s_k^T\geqslant 0$, $r_kA_k^{-1}\geqslant 0$, and $(A_{k+1}|A_k)\geqslant 0$, where $r_k=A[k+1|1,2,\dots,k]$ and $s_k^T=A[1,2,\dots,k|k+1]$. Then A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$.*

Proof. Let $w_k^T=A_k^{-1}s_k^T$ and $v_k=r_kA_k^{-1}$. Then $A_kw_k^T=s_k^T$, $v_kA_k=r_k$, and $a_{k+1,k+1}-v_kA_kw_k^T=a_{k+1,k+1}-r_kA_k^{-1}s_k^T=(A_{k+1}|A_k)\geqslant 0$. It follows directly from Corollary 3 that A has an LU-factorization of the specified type. ■

THEOREM 2. *Let A be an $n\times n$ nonnegative matrix. If A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$, then every almost principal submatrix of the type $A[1,2,\dots,k,i|1,2,\dots,k,j]$ also has an $\tilde{L}\tilde{U}$-factorization with $\tilde{L}\geqslant 0$, $\tilde{U}\geqslant 0$ and $\tilde{u}_{ii}=1$.*

Proof. Let $A=LU$ with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$. Note that $a_{ij}=L[i|\hat{\phi}]U[\hat{\phi}|j]$. Let $v=L[i|1,2,\dots,k]$ and $w^T=U[1,2,\dots,k|j]$. Then we have $a_{ij}-vw^T\geqslant 0$, $L_kw^T=A[1,2,\dots,k|j]$, and $vU_k=A[i|1,2,\dots,k]$. Finally, letting

$$\tilde{L}=\begin{pmatrix}L_k&0\\v&a_{ij}-vw^T\end{pmatrix}\quad\text{and}\quad\tilde{U}=\begin{pmatrix}U_k&w^T\\0&1\end{pmatrix},$$

we obtain

$$\tilde{L}\tilde{U}=\begin{pmatrix}L_kU_k&L_kw^T\\vU_k&a_{ij}\end{pmatrix}=A[1,2,\dots,k,i|1,2,\dots,k,j].$$ ■

COROLLARY 5. *Let A be an $n\times n$ nonnegative matrix with* $\det(A_k)\neq 0$ *for some k. If A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$, then* $(A|A_k)\geqslant 0$.

Proof. Theorem 2 implies that $A[1,2,\dots,k,i|1,2,\dots,k,j]$ has an $\tilde{L}\tilde{U}$-factorization with $\tilde{L}\geqslant 0$, $\tilde{U}\geqslant 0$ and $\tilde{u}_{ii}=1$. Thus $(A[1,2,\dots,k,i|1,2,\dots,k,j]|A_k)\geqslant 0$ by Corollary 1. Partition A into $\begin{pmatrix} A_k & A_{12}\\ A_{21} & A_{22}\end{pmatrix}$. Then $(A|A_k)=A_{22}-A_{21}A_k^{-1}A_{12}=(h_{ij})_{i,j=k+1}^n$. Note that $h_{ij}=(A[1,2,\dots,k,i|1,2,\dots,k,j]|A_k)$. Hence $(A|A_k)\geqslant 0$. ∎

THEOREM 3. *Let A be an $n\times n$ nonnegative matrix with $a_{11}>0$. Then A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$ iff $(A|a_{11})$ has an $\tilde{L}\tilde{U}$-factorization with $\tilde{L}\geqslant 0$, $\tilde{U}\geqslant 0$ and $\tilde{u}_{ii}=1$.*

Proof. Assume A has an LU-factorization with $L\geqslant 0$, $U\geqslant 0$ and $u_{ii}=1$. Partition A, L and U conformally so that

$$A=\begin{pmatrix} a_{11} & z\\ y^T & A_{22}\end{pmatrix}=\begin{pmatrix} a_{11} & 0\\ r^T & L_{22}\end{pmatrix}\begin{pmatrix} 1 & s\\ 0 & U_{22}\end{pmatrix}=LU,$$

where y, z, r and s are $1\times(n-1)$ nonnegative vectors. Necessarily $r=y$, $s=a_{11}^{-1}z$, and $A_{22}=r^Ts+L_{22}U_{22}$. Thus $(A|a_{11})=A_{22}-y^Ta_{11}^{-1}z=A_{22}-r^Ts=L_{22}U_{22}$, which is a factorization of the type specified.

Conversely assume $(A|a_{11})$ has an $\tilde{L}\tilde{U}$-factorization with $\tilde{L}\geqslant 0$, $\tilde{U}\geqslant 0$ and $\tilde{u}_{ii}=1$. Partition A as above. Note that $(A|a_{11})=A_{22}-y^Ta_{11}^{-1}z=\tilde{L}\tilde{U}$. Let

$$L=\begin{pmatrix} a_{11} & 0\\ y^T & \tilde{L}\end{pmatrix}\quad\text{and}\quad U=\begin{pmatrix} 1 & a_{11}^{-1}z\\ 0 & \tilde{U}\end{pmatrix}.$$

Then L is nonnegative lower triangular, U is nonnegative upper triangular, $u_{ii}=1$, and

$$LU=\begin{pmatrix} a_{11} & 0\\ y^T & \tilde{L}\end{pmatrix}\begin{pmatrix} 1 & a_{11}^{-1}z\\ 0 & \tilde{U}\end{pmatrix}=\begin{pmatrix} a_{11} & z\\ y^T & A_{22}\end{pmatrix}=A. \qquad ∎$$

COROLLARY 6. *Let A be an $n\times n$ nonnegative matrix with nonzero proper leading principal minors. If $(A|A_k)\geqslant 0$ for $k=1,2,\dots,n-1$, then A*

has an LU-factorization with $L \geqslant 0$, $U \geqslant 0$ *and* $u_{ii}=1$. *Necessarily* $\det(A_k) > 0$ *for* $k=1,2,\ldots,n-1$.

Proof. Since $(A|A_{n-1}) \geqslant 0$ and consists of a single element, trivially it has a factorization of the specified type. Now using the quotient formula (see [1]) of Schur complements, we have $(A|A_{n-1}) = \big((A|A_{n-2})|(A_{n-1}|A_{n-2})\big)$. Note that $(A_{n-1}|A_{n-2})$ is the element in the first row and first column of $(A|A_{n-2})$ and hence is nonnegative. In fact $(A_{n-1}|A_{n-2}) > 0$, since $(A_{n-1}|A_{n-2}) = 0$ would imply that $\det(A_{n-1}) = \det(A_{n-2})(A_{n-1}|A_{n-2}) = 0$, a contradiction. Now it follows from Theorem 3 that $(A|A_{n-2})$ has a factorization of the specified type. Next apply the same argument to $(A|A_{n-2}) = \big((A|A_{n-3})|(A_{n-2}|A_{n-3})\big)$ to obtain a nonnegative factorization for $(A|A_{n-3})$. The process can be continued until eventually we obtain $A = LU$ with $L \geqslant 0$, $U \geqslant 0$ and $u_{ii}=1$. Also, since $a_{11} > 0$ and $\det(A_{k+1}) = \det(A_k)(A_{k+1}|A_k)$, we obtain iteratively $\det(A_k) > 0$ for $k = 1,2,\ldots,n-1$. ∎

THEOREM 4. *Let A be an* $n \times n$ *nonnegative matrix with nonzero proper leading principal minors. Then A has an LU-factorization with* $L \geqslant 0$, $U \geqslant 0$ *and* $u_{ii}=1$ *iff* $(A|A_k) \geqslant 0$ *for* $k=1,2,\ldots,n-1$.

Proof. This is a direct consequence of Corollaries 5 and 6. ∎

The statement in Theorem 4 is a known result (cf. [4]).

REFERENCES

1 Douglas Crabtree and Emilie Haynsworth, An identity for the Schur complement of a matrix, *Proc. Am. Math. Soc.* **22** (1969), 364–366.
2 Emilie Haynsworth, Determination of the inertia of a partitioned Hermitian matrix, *Linear Algebra Appl.* **1** (1968), 73–81.
3 Cony M. Lau and Thomas L. Markham, *LU* factorizations, submitted for publication.
4 T. L. Markham, Factorizations of nonnegative matrices, *Proc. Am. Math. Soc.* **32** (1972), 45–47.

Received 12 October 1976; revised 22 October 1976