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Constructing Projections on Sums and Intersections

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Abstract—In this note, we list several formulas for computing orthogonal projections onto the linear sum and intersection of two subspaces of $\mathbb{C}^n$ in terms of the projections on the individual subspaces.

Keywords—Projection, Pseudoinverse.

In the first edition of his problem book [1], Halmos poses the following problem.

PROBLEM 96[1]. If E and F are projections with ranges $\mathbf{M}$ and $\mathbf{N}$, find the projection $E \wedge F$ with range $\mathbf{M} \cap \mathbf{N}$.

He goes on to explain the kind of solution he would prefer. In this expository note, we show how to compute the projections onto the linear sum and intersection of two linear subspaces of $\mathbb{C}^n$ in terms of the projections onto the individual subspaces. In particular, we exhibit several formulas for $E \wedge F$ and $E \vee F$, the projection on $\mathbf{M} + \mathbf{N}$.

The key to these results is the concept of the pseudoinverse of a matrix. For an m-by-n matrix A over the complex numbers $\mathbb{C}$, the pseudoinverse of A is the unique solution A^+ to the equations

(1) $AA^+A = A$,
(2) $A^+AA^+ = A^+$,
(3) $(AA^+)^* = AA^+$,
(4) $(A^+A)^* = A^+A$.

These four defining equations of the pseudoinverse plus uniqueness imply many other relationships including

(5) $(A^*)^+ = (A^+)^*$,
(6) $(AA^*)^+ = (A^*)^+A^+$,
(7) $A^+ = A^*(AA^*)^+$, and
(8) $A^+ = (A^*A)^+A^*$.

Of course, $*$ means conjugate transpose. In addition, equations (1)–(4) imply that AA^+ and A^+A are self-adjoint and idempotent; that is, they are orthogonal projections. Recall that a matrix P (or operator P if you prefer) is a projection iff $P^* = P = P^2$. It is known that AA^+

[1]This has been Problem 122 since the 1984 edition.

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is the projection onto the range of A, A^+A is the projection on the range of A^*, (equivalently on the orthogonal complement of the null space of A), $I - AA^+$ is the projection on the null space of A^* (equivalently on the orthogonal complement of the range of A) and $I - A^+A$ is the projection on the null space of A. I denotes the identity matrix of appropriate size. While our focus is primarily finite dimensional, there are direct extensions to bounded linear operators on Hilbert space when the operators have a closed range. These are precisely the bounded linear operators that possess a pseudoinverse that is also a bounded linear operator (see [2]).

We take advantage of the one-to-one correspondence between linear subspaces of $\mathbb{C}^n$ and projections (see [3,4]); that is, if $\mathcal{M}$ is a subspace of $\mathbb{C}^n$, we write $P_{\mathcal{M}}$ for the unique projection onto $\mathcal{M}$. More specifically $\mathcal{M} = \mathrm{Ran}(P_{\mathcal{M}})$ and $\mathcal{M}^\perp = \mathrm{Ker}(P_{\mathcal{M}})$. Recall that the inclusion of subspaces can be captured in the algebra of their projections: $\mathcal{N} \subseteq \mathcal{M}$ iff $P_{\mathcal{N}} = P_{\mathcal{N}}P_{\mathcal{M}}$ iff $P_{\mathcal{N}} = P_{\mathcal{M}}P_{\mathcal{N}}$. We write $P_{\mathcal{N}} \leq P_{\mathcal{M}}$ when $\mathcal{N} \subseteq \mathcal{M}$.

We are mainly interested in projections on $\mathbb{C}^n$ but we give a more general result in Theorem 1 that aids in our discussion. The result in the preliminary lemma is well known [4,5]. However, we give a simple proof here using the pseudoinverse.

LEMMA 1. *For any matrix* $M, \mathrm{Ran}(M) = \mathrm{Ran}(MM^*)$.

PROOF. If $x \in \mathrm{Ran}(MM^*)$, then $MM^*y = x$ for some vector y. But then $x = M(M^*y)$, so plainly $x \in \mathrm{Ran}(M)$. On the other hand, if $x \in \mathrm{Ran}(M)$, then $x = My$ for some y. Note that $MM^+x = MM^+My = My = x$ so by (8) above substituting for M^+, we see $M(M^*(MM^*)^+)$ $x = x$. In other words, $MM^*(z) = x$ where $z = (MM^*)^+x$. This puts z in $\mathrm{Ran}(MM^*)$ and the lemma is established.

THEOREM 1. *Let A be an $m \times n$ matrix, and B an $m \times k$ matrix both defined on the complex field, then*

$$\mathrm{Ran}\,[AA^* + BB^*] = \mathrm{Ran}[A] + \mathrm{Ran}[B] \qquad \text{and} \tag{1.1}$$

$$\mathrm{Ker}\,[AA^* + BB^*] = \mathrm{Ker}\,[AA^*] \cap \mathrm{Ker}[BB^*]. \tag{1.2}$$

PROOFS.

PROOF OF (1.1). Let $M = [A \;\vdots\; B]$ be the m–by–$(n+k)$ augmented matrix. Then $\mathrm{Ran}(M)$ $= \mathrm{Ran}(A) + \mathrm{Ran}(B)$. By the lemma, $\mathrm{Ran}(M) = \mathrm{Ran}(MM^*)$, but

$$MM^* = \begin{bmatrix} A & \vdots & B \end{bmatrix} \begin{bmatrix} A & \vdots & B \end{bmatrix}^* = \begin{bmatrix} A & \vdots & B \end{bmatrix} \begin{bmatrix} A^* \\ \cdots \\ B^* \end{bmatrix} = AA^* + BB^*.$$

Hence, $\mathrm{Ran}[AA^* + BB^*] = \mathrm{Ran}(MM^*) = \mathrm{Ran}(M) = \mathrm{Ran}(A) + \mathrm{Ran}(B)$.

PROOF OF (1.2). Let $x \in \mathrm{Ker}[AA^*] \cap \mathrm{Ker}[BB^*]$. Then $AA^*x = 0$, $BB^*x = 0$, and $[AA^* + BB^*]$ $x = 0$, so $x \in \mathrm{Ker}[AA^* + BB^*]$. Now let $x \in \mathrm{Ker}[AA^* + BB^*]$. Then $[AA^* + BB^*]x = 0$. Thus, $x^*[AA^* + BB^*]x = x^*AA^*x + x^*BB^*x = 0$ which implies $x^*AA^*x = 0$ and $x^*BB^*x = 0$. If $x^*AA^*x = 0$, then $A^*x = 0$ and $AA^*x = 0$ so $x \in \mathrm{Ker}[AA^*]$. The same argument gives $x \in \mathrm{Ker}[BB^*]$, which in turn implies $x \in \mathrm{Ker}[AA^*] \cap \mathrm{Ker}[BB^*]$.

Next, we note that for a projection P, $P = P^*$, and $P = P^2 = PP^*$. Hence, if Q is also a projection $P + Q = PP^* + QQ^*$.

COROLLARY 2. *If P and Q are projections on $\mathbb{C}^n$*

$$\mathrm{Ran}(P + Q) = \mathrm{Ran}(P) + \mathrm{Ran}(Q), \tag{2.1}$$

$$\mathrm{Ker}(P + Q) = \mathrm{Ker}(P) \cap \mathrm{Ker}(Q). \tag{2.2}$$

Next suppose A is a matrix whose column space is $\mathcal{M}$ and B is a matrix whose column space is $\mathcal{N}$. Then $P_{\mathcal{M}} = P_{\mathrm{Ran}(A)} = AA^+$ and $P_{\mathcal{N}} = P_{\mathrm{Ran}(B)} = BB^+$. We have noted that $[P_{\mathcal{M}} + P_{\mathcal{N}}][P_{\mathcal{M}} + P_{\mathcal{N}}]^+$ is the projection on the range of $P_{\mathcal{M}} + P_{\mathcal{N}}$ which, by the argument above, is $P_{\mathcal{M}+\mathcal{N}}$.

As a shorthand notation, let $P^\perp = I - P$ for a projection P.

THEOREM 3. *Let $\mathcal{M}$ and $\mathcal{N}$ be subspaces of $\mathbb{C}^n$. Let $P = P_{\mathcal{M}}$ and $Q = P_{\mathcal{N}}$. Then all the following expressions are equal to the orthogonal projection on $\mathcal{M} + \mathcal{N}$.*

$$P_{\mathcal{M}+\mathcal{N}}, \tag{3.1}$$

$$[P+Q][P+Q]^+, \tag{3.2}$$

$$[P+Q]^+[P+Q], \tag{3.3}$$

$$Q + \left[(PQ^\perp)^+(PQ^\perp)\right], \tag{3.4}$$

$$P + \left[(QP^\perp)^+(QP^\perp)\right], \tag{3.5}$$

$$P + P^\perp\left[P^\perp Q\right]^+, \tag{3.6}$$

$$Q + Q^\perp\left[Q^\perp P\right]^+. \tag{3.7}$$

PROOF. That (3.2) equals (3.3) follows from the fact that $[P + Q][P + Q]^+$ is self-adjoint. That (3.1) equals (3.2) follows from Corollary 2 and the discussion above. We give an order theoretic argument that (3.1) and (3.4) are equal. The equality of (3.5) will follow by symmetry. Let $H = Q + (PQ^\perp)^+(PQ^\perp)$. First note that $(PQ^\perp)^+(PQ^\perp)Q = 0$ so H is a projection. Also $HQ = Q + 0 = Q$ so $Q \leq H$. Next

$$\begin{aligned} PH &= P[Q + (PQ^\perp)^+(PQ^\perp)] = PQ + P(PQ^\perp)^+(PQ^\perp) = PQ + P[I - (I - (PQ^\perp)^+(PQ^\perp))] \\ &= PQ + P - P(I - (PQ^\perp)^+(PQ^\perp))] = P + [PQ - P(I - (PQ^\perp)^+(PQ^\perp))]. \end{aligned}$$

But

$$\begin{aligned} O &= (PQ^\perp)[I - (PQ^\perp)^+(PQ^\perp)] = P(I-Q)[I - (PQ^\perp)^+(PQ^\perp)] \\ &= [P - PQ][I - (PQ^\perp)^+(PQ^\perp)] = P(I - (PQ^\perp)^+(PQ^\perp)) - PQ, \end{aligned}$$

since $Q \leq I - (PQ^\perp)^+(PQ^\perp)$. Thus, $PQ = P(I - (PQ^\perp)^+(PQ^\perp))$ and consequently $PH = PQ + P - PQ = P$. Thus, $P \leq H$. Now let K be any projection with $K \geq P$ and $K \geq Q$. Then

$$KH = K[Q + (PQ^\perp)^+(PQ^\perp)] = KQ + K(PQ^\perp)^+(PQ^\perp) = Q + K(PQ^\perp)^+(PQ^\perp).$$

But $Q \leq K$ so $QK^\perp = O$, whence $PQ^\perp K^\perp = PK^\perp = O$ so $K^\perp = [I - (PQ^\perp)^+(PQ^\perp)]K^\perp$. This says $K^\perp \leq I - (PQ^\perp)^+(PQ^\perp)$ or equivalently $(PQ^\perp)^+(PQ^\perp) \leq K$. Thus, $K(PQ^\perp)^+(PQ^\perp) = (PQ^\perp)^+(PQ^\perp)$ and so $KH = H$ putting $H \leq K$. Therefore, H is the least upper bound of P and Q. Now (3.5) follows by symmetry.

(3.6) Let $U = P^\perp Q$. Then $UQ = U$ so $U^+UQ = U^+U$. This says $U^+UQP^\perp = U^+UP^\perp$. By taking the $*$ of both sides, we get

$$P^\perp QU^+U = P^\perp U^+U, \qquad \text{so } U = UU^+U = P^\perp U^+U.$$

Thus, $UU^+ = P^+U^+UU^+ = P^\perp U^+$. Therefore, $UU^+ = P^\perp U^+$; that is $(P^\perp Q)(P^\perp Q)^+ = P^\perp(P^\perp Q)^+$.

But

$$\begin{array}{c} \left[(P^\perp Q)\left(P^\perp Q\right)^+\right]^* = (P^\perp Q)\left(P^\perp Q\right)^+ \\ \| \\ \left(P^\perp Q\right)^{+*}\left(P^\perp Q\right)^* = \left(QP^\perp\right)^+\left(QP^\perp\right), \end{array}$$

so

$$\left(QP^{\perp}\right)^{+}\left(QP^{\perp}\right)=P^{\perp}\left[P^{\perp}Q\right]^{+}$$

(see also [6]).

Equation (3.7) follows by symmetry.

So, our first goal is accomplished; namely, in finite dimensions the projection on the linear sum of two subspaces is computable in many ways in terms of the individual projections. We now turn to our second goal, representing the projection on the intersection of two subspaces in terms of the individual projections. The first representation (4.2) seems to have first appeared in [7], but does not seem to be very well known. We are indebted to D. Foulis for pointing out formula (4.6).

THEOREM 4. *Let $\mathcal{M}$ and $\mathcal{N}$ be subspaces of $\mathbb{C}^n$. Let $P = P_{\mathcal{M}}$ and $Q = P_{\mathcal{N}}$. Then all the following expressions are equal to the orthogonal projection onto $\mathcal{M}\cap\mathcal{N}$.*

$$P_{\mathcal{M}\cap\mathcal{N}}, \tag{4.1}$$

$$2Q(Q+P)^{+}P, \tag{4.2}$$

$$2P(Q+P)^{+}Q, \tag{4.3}$$

$$2\left[P-P\left(Q+P\right)^{+}P\right], \tag{4.4}$$

$$2\left[Q-Q\left(Q+P\right)^{+}Q\right], \tag{4.5}$$

$$P-(P-QP)^{+}(P-QP)=P-\left(Q^{\perp}P\right)^{+}\left(Q^{\perp}P\right), \tag{4.6}$$

$$Q-(Q-PQ)^{+}(Q-PQ)=Q-\left(P^{\perp}Q\right)^{+}\left(P^{\perp}Q\right), \tag{4.7}$$

$$P-P\left[PQ^{\perp}\right]^{+}, \tag{4.8}$$

$$Q-Q\left[QP^{\perp}\right]^{+}. \tag{4.9}$$

PROOF. We begin by showing (4.2) and (4.3) are equal. To do this, we first show $Q(Q+P)^{+}P-P(Q+P)^{+}Q=0$. But

$$\begin{aligned}Q(Q+P)^{+}P-P(Q+P)^{+}Q&=Q(Q+P)^{+}P+Q(Q+P)^{+}Q-Q(Q+P)^{+}Q+P(Q+P)^{+}Q\\&=Q(Q+P)^{+}(Q+P)-(Q+P)(Q+P)^{+}Q.\end{aligned}$$

By Corollary 2, $\mathrm{Ran}(Q)\subseteq\mathrm{Ran}(Q)+\mathrm{Ran}(P)=\mathrm{Ran}(Q+P)$ so $Q(Q+P)^{+}(Q+P)=Q=(Q+P)(Q+P)^{+}Q$. Thus, $Q(Q+P)^{+}P-P(Q+P)^{+}Q=Q-Q=0$ and so $Q(Q+P)^{+}P=P(Q+P)^{+}Q$ and it follows that (4.2) equals (4.3).

Next, we argue that (4.2) and (4.3) equal (4.1). For this let $H=Q(Q+P)^{+}P+P(Q+P)^{+}Q=2Q(Q+P)^{+}P=2P(Q+P)^{+}Q$. Now $HP=[2Q(Q+P)^{+}P]P=2Q(Q+P)^{+}P^{2}=2Q(Q+P)^{+}P=H$ and similarly $HQ=H$. Thus, $\mathrm{Ran}(H)\subseteq\mathcal{M}\cap\mathcal{N}$. But also

$$\begin{aligned}H&=HP_{\mathcal{M}\cap\mathcal{N}}=[Q(Q+P)^{+}P+P(Q+P)^{+}Q]P_{\mathcal{M}\cap\mathcal{N}}=Q(Q+P)^{+}PP_{\mathcal{M}\cap\mathcal{N}}+P(Q+P)^{+}QP_{\mathcal{M}\cap\mathcal{N}}\\&=Q(Q+P)^{+}P_{\mathcal{M}\cap\mathcal{N}}+P(Q+P)^{+}P_{\mathcal{M}\cap\mathcal{N}}=[Q(Q+P)^{+}+P(Q+P)^{+}]P_{\mathcal{M}\cap\mathcal{N}}\\&=(Q+P)(Q+P)^{+}P_{\mathcal{M}\cap\mathcal{N}}=P_{\mathcal{M}\cap\mathcal{N}}.\end{aligned}$$

This last equality follows because $\mathcal{M}\cap\mathcal{N}\subseteq\mathrm{Ran}(Q+P)$. Thus, $H=P_{\mathcal{M}\cap\mathcal{N}}$.

Next we show (4.4) and (4.5) equal (4.1) by showing $P(Q+P)^{+}Q=P-P(Q+P)^{+}P$. The argument that $Q(Q+P)^{+}P=Q-Q(Q+P)^{+}Q$ is similar and will be omitted. Now

$$\begin{aligned}P(Q+P)^{+}Q-(P-P(Q+P)^{+}P)&=P(Q+P)^{+}Q-P+P(Q+P)^{+}P\\&=P(Q+P)^{+}Q+P(Q+P)^{+}P-P=P(Q+P)^{+}(Q+P)-P=P-P=0.\end{aligned}$$

To see that (4.6) and (4.7) equal (4.1), note first that $\mathcal{M} \cap \mathcal{N} \subseteq \mathcal{M}$ and $\mathcal{M} \cap \mathcal{N} \subseteq \mathcal{N}$ so $PP_{\mathcal{M}\cap\mathcal{N}} = P_{\mathcal{M}\cap\mathcal{N}}$, $QP_{\mathcal{M}\cap\mathcal{N}} = P_{\mathcal{M}\cap\mathcal{N}}$, and $QPP_{\mathcal{M}\cap\mathcal{N}} = P_{\mathcal{M}\cap\mathcal{N}}$. Now let $S = P - (P - QP)^+(P - QP)$. We claim S is a projection. Clearly $S^* = S$ and $SP = S$. It follows $PS = S$. In particular $\mathrm{Ran}(S) \subseteq \mathrm{Ran}(P) = \mathcal{M}$. Next

$$\begin{aligned} S^2 &= P^2 - P(P - QP)^+(P - QP) - (P - QP)^+(P - QP)P + ((P - QP)^+(P - QP))^2 \\ &= P[P - (P - QP)^+(P - QP)] = PS = S. \end{aligned}$$

Now

$$P - QP = (P - QP)(P - QP)^+(P - QP) = P(P - QP)^+(P - QP) - QP(P - QP)^+(P - QP),$$

so

$$\begin{aligned} S = PS &= P[P - (P - QP)^+(P - QP)] = P - P(P - QP)^+(P - QP) \\ &= QP - QP(P - QP)^+(P - QP) = Q[P - P(P - QP)^+(P - QP)] = QPS = QS. \end{aligned}$$

Thus, $QS = S$ and $SQ = S$ so $\mathrm{Ran}(S) \subseteq \mathrm{Ran}(Q) = \mathcal{N}$. Therefore, $\mathrm{Ran}(S) \subseteq \mathcal{M} \cap \mathcal{N}$ and so $P_{\mathcal{M}\cap\mathcal{N}}S = S$. But $SP_{\mathcal{M}\cap\mathcal{N}} = [P - (P - QP)^+(P - QP)]P_{\mathcal{M}\cap\mathcal{N}} = PP_{\mathcal{M}\cap\mathcal{N}} - (P - QP)^+(P - QP)P_{\mathcal{M}\cap\mathcal{N}} = P_{\mathcal{M}\cap\mathcal{N}} - 0 = P_{\mathcal{M}\cap\mathcal{N}}$. Thus, $S = P_{\mathcal{M}\cap\mathcal{N}}$. The argument for (4.7) is similar with the role of P and Q interchanged. The arguments for (4.8) and (4.9) are similar to the ones given for (3.6). This completes the proof of Theorem 4.

It is an easy corollary from our formulas that if P and Q commute, i.e., $PQ = QP$, then $P_{\mathcal{M}\cap\mathcal{N}} = PQ$, since

$$P_{\mathcal{M}\cap\mathcal{N}} = P_{\mathcal{M}\cap\mathcal{N}}Q = [P-(P-QP)^+(P-QP)]Q = PQ-(P-QP)^+(PQ-QPQ) = PQ-0 = PQ.$$

Also in this case, $P_{\mathcal{M}+\mathcal{N}} = P + Q - PQ$. First, note since $PQ = QP, P + Q - PQ$ is a projection. Then since $(Q + P)(Q + P)^+PQ = PQ$ we see

$$(P+Q)-PQ = (P+Q)(P+Q)^+(P+Q)-(Q+P)(Q+P)^+PQ = (P+Q)(P+Q)^+[P+Q-PQ].$$

Also

$$[P+Q-PQ](P+Q)(P+Q)^+ = ((P+Q)(P+Q)-(PQ)(P+Q))(P+Q)^+ = (P+Q)(P+Q)^+.$$

Thus, $(P + Q)(P + Q)^+ = P + Q - PQ$.

There is a well-known formula in Hilbert space that says

$$P_{\mathcal{M}\cap\mathcal{N}} = \lim_{n\to\infty} P(QP)^n.$$

With this, we can now complete a summary table.

SUMMARY TABLE

Let A and B be complex m-by-n and m-by-k matrices, respectively, with $n \leq m,\ k \leq m$. Then

(1) $AA^+ = P_{\mathrm{Ran}(A)}$,
(2) $A^+A = P_{\mathrm{Ran}(A^*)} = P_{\mathrm{Ker}(A)^\perp}$,
(3) $I - AA^+ = P_{\mathrm{Ker}(A^*)} = P_{\mathrm{Ran}(A)^\perp}$,
(4) $I - A^+A = P_{\mathrm{Ker}(A)} = P_{\mathrm{Ran}(A^*)^\perp}$,
(5) $[A \ \vdots\ B][A \ \vdots\ B]^+ = P_{\mathrm{Ran}(A)+\mathrm{Ran}(B)}$,
(6) $2AA^+[AA^+ + BB^+]^+BB^+ = P_{\mathrm{Ran}(A)\cap\mathrm{Ran}(B)}$.

If $P = P_{\mathcal{M}}$ and $Q = P_{\mathcal{N}}$ then

(7) $P_{\mathcal{M}+\mathcal{N}} = [P+Q][P+Q]^{+} = [P+Q]^{+}[P+Q] = Q + [(PQ^{\perp})^{+}(PQ^{\perp})] = P + [(QP^{\perp})^{+}(QP^{\perp})] = P + P^{\perp}[P^{\perp}Q]^{+} = Q + Q^{\perp}[Q^{\perp}P]^{+}$,

(8) $P_{\mathcal{M}\cap\mathcal{N}} = 2Q(Q+P)^{+}P = 2P(Q+P)^{+}Q$
$= 2[P - P(Q+P)^{+}P] = 2[Q - Q(Q+P)^{+}Q]$
$= P - (P - QP)^{+}(P - QP) = Q - (Q - PQ)^{+}(Q - PQ) = P - P[PQ^{\perp}]^{+}$
$= Q - Q[QP^{\perp}]^{+} = \lim_{n\to\infty} P(QP)^{n}$.

If moreover, $PQ = QP$

(9) $P_{\mathcal{M}\cap\mathcal{N}} = PQ = QP$,

(10) $P_{\mathcal{M}+\mathcal{N}} = P + Q - PQ$,

If $P \perp Q$ (i.e., $PQ = 0$)

(11) $P_{\mathcal{M}\cap\mathcal{N}} = 0$,

(12) $P_{\mathcal{M}+\mathcal{N}} = P + Q$.

CONCLUDING REMARKS

Recently, Yang [8] gave a method for finding a basis for the intersection of two subspaces given a basis of each. In fact, his method can be extended to the case where only spanning sets are given for each subspace. We can produce the projection onto the intersection from this approach. Let $\mathcal{M} = \operatorname{span}\{A_1, A_2, \ldots, A_r\}$ and $\mathcal{N} = \operatorname{span}\{B_1, B_2, \ldots, B_s\}$ be subspaces of $\mathbb{C}^n$. Thinking of the A_i's and B_j's as column vectors, we form the matrices

$$A = \begin{bmatrix} A_1 & \vdots & A_2 & \vdots & \cdots & \vdots & A_r \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_1 & \vdots & B_2 & \vdots & \cdots & \vdots & B_s \end{bmatrix}$$

which are n-by-r and n-by-s, respectively. Of course, $AA^{+} = P_{\mathrm{Ran}(A)} = P_{\mathcal{M}}$ and $BB^{+} = P_{\mathrm{Ran}(B)} = P_{\mathcal{N}}$. Now we form the augmented matrix $M = [A \;\vdots\; B]$ which is n-by-$(r+s)$ and with a left multiplication by a suitable invertible matrix R, we produce the row reduced Echelon form of M; say

$$r\mathrm{ref}[M] = RM = R\begin{bmatrix} A & \vdots & B \end{bmatrix} = \begin{bmatrix} RA & \vdots & RB \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} \\ \mathbb{O} & E_{22} \\ \mathbb{O} & \mathbb{O} \end{bmatrix}.$$

Now

$$B = R^{-1}\begin{bmatrix} E_{12} \\ E_{22} \\ \mathbb{O} \end{bmatrix} = R^{-1}\begin{bmatrix} E_{12} \\ \mathbb{O} \\ \mathbb{O} \end{bmatrix} + R^{-1}\begin{bmatrix} \mathbb{O} \\ E_{22} \\ \mathbb{O} \end{bmatrix}$$

and

$$\begin{bmatrix} E_{11} \\ \mathbb{O} \\ \mathbb{O} \end{bmatrix}\begin{bmatrix} E_{11} \\ \mathbb{O} \\ \mathbb{O} \end{bmatrix}^{+}\begin{bmatrix} E_{12} \\ \mathbb{O} \\ \mathbb{O} \end{bmatrix} = \begin{bmatrix} E_{11} \\ \mathbb{O} \\ \mathbb{O} \end{bmatrix}\begin{bmatrix} E_{11}^{+} & \mathbb{O} & \mathbb{O} \end{bmatrix}\begin{bmatrix} E_{12} \\ \mathbb{O} \\ \mathbb{O} \end{bmatrix} = \begin{bmatrix} E_{12} \\ \mathbb{O} \\ \mathbb{O} \end{bmatrix}.$$

Thus, $R^{-1}\begin{bmatrix} E_{12} \\ \mathbb{O} \\ \mathbb{O} \end{bmatrix}$ has columns in $\mathcal{M}\cap\mathcal{N}$. If we let $W = R^{-1}\begin{bmatrix} E_{12} \\ \mathbb{O} \\ \mathbb{O} \end{bmatrix}$, then WW^{+} is the projection on $\mathcal{M}\cap\mathcal{N}$. For example, suppose $\mathcal{M} = \operatorname{span}\{(1,1,1,1,0), (1,2,3,4,0), (3,5,7,9,0), (0,1,2,2,0)\}$

and $\mathcal{N} = \text{span}\{(2,3,4,7,0),\ (1,0,1,0,0),\ (3,3,5,7,0),\ (0,1,0,3,0)\}$. Then

$$RM = \begin{bmatrix} 0 & 2 & 0 & -1 & 0 \\ \frac{1}{2} & -1 & -\frac{1}{2} & 1 & 0 \\ -1 & 1 & 1 & -1 & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{cccc|cccc} 1 & 1 & 3 & 0 & 2 & 1 & 3 & 0 \\ 1 & 2 & 5 & 1 & 3 & 0 & 3 & 1 \\ 1 & 3 & 7 & 2 & 4 & 1 & 5 & 0 \\ 1 & 4 & 9 & 2 & 7 & 0 & 7 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 0 & 3 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 & -2 & 0 & -2 & -2 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] = r\text{ref}[M].$$

Now

$$W = R^{-1}\begin{bmatrix} E_{12} \\ \mathbb{O} \\ \mathbb{O} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 2 & 1 & 0 \\ 1 & 4 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 & -1 \\ 3 & 0 & 3 & 2 \\ -2 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 & 1 \\ 3 & 0 & 3 & 1 \\ 4 & 0 & 4 & 1 \\ 7 & 0 & 7 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$W^{+} = \begin{bmatrix} -\frac{1}{12} & \frac{1}{12} & \frac{1}{4} & -\frac{1}{12} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{12} & \frac{1}{12} & \frac{1}{4} & -\frac{1}{12} & 0 \\ \frac{1}{2} & -\frac{1}{3} & -\frac{7}{6} & \frac{2}{3} & 0 \end{bmatrix}.$$

The projection onto $\mathcal{M} \cap \mathcal{N}$ is thus

$$WW^{+} = \begin{bmatrix} \frac{1}{6} & 0 & -\frac{1}{6} & \frac{1}{3} & 0 \\ 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & 0 \\ -\frac{1}{6} & \frac{1}{3} & \frac{5}{6} & 0 & 0 \\ \frac{1}{3} & \frac{1}{6} & 0 & \frac{5}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The trace of this projection is 2 which gives the dimension of $\mathcal{M} \cap \mathcal{N}$. A basis for $\mathcal{M} \cap \mathcal{N}$ is $\{(2,3,4,7,0),\ (1,1,1,3,0)\}$.

REFERENCES

1. P.R. Halmos, *A Hilbert Space Problem Book*, Van Nostrand, Princeton, NJ, (1967).
2. C.W. Groetsch, *Generalized Inverses of Linear Operators*, Marcel Dekker, New York, (1977).
3. S.K. Berberian, *Introduction to Hilbert Space*, Oxford University, New York, (1961).
4. J. Weidmann, *Linear Operators in Hilbert Spaces*, Springer-Verlag, New York, (1980).
5. C.R. Rao and S.K. Mitra, *Generalized Inverse of Matrices and its Applications*, John Wiley & Sons, (1971).
6. I. Kaplansky, Projections in Banach algebras, *Ann. of Math.* **53**, 235–249, (1951).
7. W.N. Anderson, Jr. and R.J. Duffin, Series and parallel addition of matrices, *J. Math. Anal. Appl.* **26**, 576–595, (1969).
8. K. Yang, A basis for the intersection of subspaces, *Mathematics Magazine* **70** (4), 297, (1997).
9. W.N. Anderson, Jr. and M. Schreiber, The infinum of two projections, *Acta Sci. Math.* (Szeged) **33**, 165–168, (1972).
10. A. Ben-Israel and T.N.E. Greville, *Generalized Inverses: Theory and Applications*, John Wiley & Sons, (1974).
11. R.E. Hartwig and M.P. Drazin, Lattice properties of the *-order for complex matrices, *J. Math. Anal. and Appl.* **86** (2), 359–378, (April 1982).
12. R. Piziak, Involution rings and projections—*I*, *Journal of Natural Sciences and Mathematics* **X** (2), 215–227, (October 1970).
13. T. Yester, The reduced row Echelon form of a matrix is unique, *Mathematics Magazine* **57** (2), 93–94, (1984).