

N72-16462

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TR 71-45

October 6, 1971

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Technical Report  
University of Maryland  
College Park, Maryland

This research was partially supported by NSF Grant GY 8987.

Computer time for this research was supported by NASA Grant NsG-398 to the Computer Science Center of the University of Maryland.

# Eigenvalues of the Laplacian of a Graph

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Let  $G$  be a finite undirected graph with no loops or multiple edges. We define the Laplacian matrix of  $G$ ,  $\Delta(G)$ , by  $\Delta_{ii}$  = degree of vertex  $i$  and  $\Delta_{ij}$  = -1 if there is an edge between vertex  $i$  and vertex  $j$ . In this paper we relate the structure of the graph  $G$  to the eigenvalues of  $\Delta(G)$ ; in particular we prove that all the eigenvalues of  $\Delta(G)$  are non-negative, less than or equal to the number of vertices, and less than or equal to twice the maximum vertex degree. Precise conditions for equality are given.

## 1. Introduction

Let  $G$  be a finite undirected graph with no loops or multiple edges. We define the Laplacian matrix of  $G$ ,  $\Delta(G)$ , by  $\Delta_{ii}$  = the degree of vertex  $i$  and  $\Delta_{ij} = -1$  if there is an edge between vertex  $i$  and vertex  $j$ . This matrix is discussed by Harary [5]. Our name for  $\Delta$  is chosen because  $\Delta$  arises in numerical analysis as a discrete analog of the Laplacian operator [3]. In this paper we relate the structure of the graph  $G$  to the eigenvalues of  $\Delta(G)$ ; in particular, we prove that all the eigenvalues of  $\Delta$  are non-negative, less than or equal to the number of vertices, and less than or equal to twice the maximum vertex degree.

There is a considerable body of literature relating the eigenvalues of the adjacency matrix of a graph to its structure [6]; except for Fisher's paper [2], little seems to be known about the Laplacian.

The authors wish to thank Professor Henry P. McKean, Jr. for suggesting this study.

## 2. Preliminaries

Our basic graph theory reference is Harary [5]. The definitions of  $\Delta$  and  $E$ , as well as Lemma 1, are taken from Chapter 13 of Harary. To define  $E$ , the vertex-edge incidence matrix, we first orient  $G$ . Then  $E_{ij} = 1$  if edge  $j$  points toward vertex  $i$ ,  $E_{ij} = -1$  if edge  $j$  points away from vertex  $i$ , and  $E_{ij} = 0$  otherwise. Let  $E^*$  denote the transpose of  $E$ .

Lemma 1.  $\Delta = EE^*$ .

Proof. Two distinct rows of  $E$  will have a non-zero entry in the same column if and only if an edge joins the corresponding vertices; the corresponding entry will be 1 in one row and -1 in the other, giving a product of -1. Q.E.D.

We will also need to consider the matrix  $N$  defined by  $N = E^*E$ . The important property of  $N$  is that if  $\lambda$  is a non-zero eigenvalue of  $\Delta$ , then it is also an eigenvalue of  $N$ , and conversely. In fact, if  $\Delta x = \lambda x$  with  $\lambda \neq 0$ , then  $NE^*x = E^*\Delta x = \lambda E^*x$  so that  $\lambda$  is an eigenvalue of  $N$  with eigenvector  $E^*x$ . The matrix  $N$  of course depends on the choice of orientation; we will vary the orientation as needed. In particular, if  $G$  is a bipartite graph, we may point all edges toward vertices of one class. Then all entries of  $N$  are non-negative; in fact  $N = 2I + A$ , where  $A$  is the adjacency matrix of the line graph of  $G$ . Results about line graphs of bipartite graphs thus translate directly into the present context [1].

If  $M$  is a matrix, let  $\rho(M)$  denote the spectral radius of  $M$ .

Let  $\bar{G}$  denote the graph complementary to  $G$ . That is,  $\bar{G}$  has the same set of vertices as  $G$ , and vertices  $v$  and  $w$  are joined in  $\bar{G}$  if and only if they are not joined in  $G$ .

Let  $K_n$  denote the complete graph on  $n$  vertices.

### 3. The Global Structure of G

In this section we obtain bounds for the eigenvalues of  $\Delta(G)$  in terms of the number of vertices and the number of components of  $G$ .

Lemma 2. The eigenvalues of  $\Delta(K_n)$  are 0, with multiplicity 1, and  $n$ , with multiplicity  $n-1$ .

Proof: Let  $u$  be the vector with all components equal to 1; then  $\Delta(K_n)u = 0$ . If  $x$  is any vector orthogonal to  $u$ , it may be easily verified that  $\Delta(K_n)x = nx$ . Q.E.D.

Theorem 1. If the graph  $G$  has  $n$  vertices, and  $\lambda$  is an eigenvalue of  $\Delta(G)$  then  $0 \leq \lambda \leq n$ . The multiplicity of 0 equals the number of components of  $G$ ; the multiplicity of  $n$  is equal to one less than the number of components of  $\bar{G}$ .

Proof. Suppose  $\lambda$  is an eigenvalue of  $\Delta$ . Then for some vector  $x$ , with  $\|x\| = 1$ ,  $\Delta x = \lambda x$ . Thus  $\lambda = (\lambda x, x) = (\Delta x, x) = (EE^*x, x) = \|E^*x\|^2$ . Therefore  $\lambda$  is real and non-negative.

Let the vertices  $v_1, \dots, v_K$  be the vertices of a connected component of  $G$ ; then the sum of the corresponding rows of  $E$  is 0, and any  $K-1$  of these rows are independent. Therefore the nullity of  $E$ , and thus of  $EE^*$ , is equal to the number of components of  $G$ .

If  $G$  has  $n$  vertices, then  $\Delta(G) + \Delta(\bar{G}) = \Delta(K_n)$ . If  $u$  is the vector with all components 1, then  $\Delta(G)u = \Delta(\bar{G})u = \Delta(K_n)u = 0$ . If  $\Delta(G)x = \lambda x$  for some vector  $x$  orthogonal to  $u$ , then using Lemma 2 we have  $\Delta(\bar{G})x = \Delta(K_n)x - \Delta(G)x = (n-\lambda)x$ . Since the eigenvalues of  $\Delta(\bar{G})$  are also non-negative, we must have  $\lambda \leq n$ . Moreover  $\lambda = n$  if and only

if  $\Delta(\bar{G})x = 0$ , and the dimension of the space of such vectors is one less than the nullity of  $\Delta(\bar{G})$  (since all such  $x$  are orthogonal to  $u$ ).

Q.E.D.

Corollary. If  $G$  has  $n$  vertices, and  $\lambda = n$  is an eigenvalue of  $\Delta(G)$ , then  $G$  is connected.

Proof. If  $G$  were not connected, then  $\bar{G}$  would be, and by the theorem  $n$  could not be an eigenvalue of  $\Delta(G)$ . Q.E.D.

#### 4. The Local Structure of $G$

In this section we obtain an upper bound for the eigenvalues of  $\Delta(G)$  in terms of vertex degrees.

Before proceeding we need to recall a few facts from the theory of non-negative matrices; our basic reference is Chapter XIII of Gantmacher [4]. Briefly, a matrix  $M$  is said to be non-negative if  $M_{ij} \geq 0$  for all  $i$  and  $j$ . If  $M$  is a matrix, denote by  $M^+$  the matrix obtained by replacing each entry of  $M$  by its absolute value. If  $M$  is irreducible, and  $\lambda$  is an eigenvalue of  $M$ , then  $|\lambda| \leq \rho(M^+)$ , with equality if and only if  $M = e^{i\phi} D M^+ D^{-1}$  where  $D^+ = I$ . For an irreducible non-negative matrix  $M$ ,  $\rho(M) \leq$  the maximum row sum with equality if and only if all row sums are equal.

Theorem 2. Let  $G$  be a graph. Then  $\rho(\Delta(G)) \leq \text{Max} (\deg v + \deg w)$  where the maximum is taken over all pairs of vertices  $(v, w)$  joined by an edge of  $G$ . If  $G$  is connected, then equality holds if and only if  $G$  is bipartite and the degree is constant on each class of vertices.

Proof. We will work with the matrix  $N$  rather than  $\Delta$ .

First consider a connected graph  $G$ , then  $N$  is irreducible, and thus  $\rho(N) \leq \rho(N^+) \leq \text{maximum row sum of } N^+$ . But if  $e$  is an edge of  $G$  joining vertices  $v$  and  $w$ , then the row sum in the row corresponding to  $e$  is  $\deg v + \deg w$ . The inequality is thus established for connected graphs.

If  $G$  is bipartite, then we may orient  $G$  with all edges pointing toward the vertices in one of the two classes; thus  $N(G) = N^+(G)$ . Then  $\rho(N) = \text{max row sum}$  if and only if all row sums are equal; i.e., if and only if the condition of the theorem holds. Equivalently, equality holds if and only if the line graph of  $G$  is regular.

If  $G$  is not bipartite, then we will show that  $\rho(N) < \rho(N^+)$ , so that equality cannot hold in the theorem. In fact, suppose  $N = e^{i\phi} D N^+ D^{-1}$ . Then since  $N_{ii} = 2$ , we have  $2 = e^{i\phi} \cdot D_{ii} \cdot 2 \cdot D_{ii}^{-1}$ , so that  $e^{i\phi} = 1$ . Now suppose that the edges  $1, \dots, K$  form an odd cycle (if no odd cycle exists, then  $G$  is bipartite); we may orient  $G$  so that the corresponding entries of  $N$  are  $-1$ . Then  $N_{12} = -1 = D_{11} \cdot 1 \cdot D_{22}^{-1}$ , so that  $D_{22} = -D_{11}$ ; continuing around the cycle we have  $D_{11} = -D_{11}$ , contradicting the requirement that  $D^+ = I$ . Therefore, if  $G$  is not bipartite, equality cannot hold in the theorem.

If  $G$  is not connected, the inequality, and the corresponding equality statement, follow by applying the theorem to each component separately.

Q.E.D.

Corollary. Let  $G$  be a connected graph. Then  $\rho(\Delta(G)) \leq$  twice the maximum vertex degree with equality if and only if  $G$  is a regular bipartite graph.

Proof. This is a special case of the theorem

Q.E.D.

## 5. Explicit Computations

Theorems 1 and 2 were conjectured from explicit computations with eigenvalues; many of these were done on a digital computer. Some of these results are stated below; the reader may verify them without difficulty.

If  $G$  is the complete bipartite graph  $K_{m,n}$ , then the eigenvalues of  $\Delta(G)$  are  $m+n$ ,  $m$ ,  $n$ ,  $0$  with respective multiplicities  $1$ ,  $n-1$ ,  $m-1$ ,  $1$ .

If  $G$  is the cycle with  $n$  vertices, then the eigenvalues of  $\Delta(G)$  are  $4 \sin^2 \frac{\pi K}{n}$ ,  $K = 1, 2, \dots, n$ .

If  $G$  is the path with  $n$  vertices, the eigenvalues of  $\Delta(G)$  are  $4 \sin^2 \frac{\pi K}{2n}$ ,  $K = 0, 1, \dots, n-1$ .

If  $G$  is the wheel with  $n+1$  vertices, the eigenvalues of  $\Delta(G)$  are  $n+1$ ,  $1$ , and  $1 + 4 \sin^2 \frac{\pi K}{n}$ ,  $K = 1, 2, \dots, n-1$ .



### References

- [1] M. Doob, On characterizing certain graphs with four eigenvalues by their spectra, *Linear Algebra and Appl.* 3 (1970), 461-482.
- [2] M. E. Fisher, On hearing the shape of a drum, *J. Comb. Theory* 1 (1966), 105-125.
- [3] G. I. Forsythe and W. R. Wasow, *Finite-difference methods for partial differential equations*, Wiley, New York, 1960.
- [4] F. R. Gantmacher, *The theory of matrices, Volume II*, Chelsea, New York, 1959.
- [5] F. Harary, *Graph theory*, Addison-Wesley, New York, 1969.
- [6] A. J. Hoffman, Some recent results on spectral properties of graphs, in *Beitrage zur Graphentheorie*, Teubner, Leipzig, 1968.

### Footnotes

AMS Subject Classification: 05.40

Key Words: Graph, Eigenvalue, Laplacian

This research was partially supported by NSF Grant GY 8987.

Computer time for this research was supported by NASA Grant Nsg-398 to the Computer Science Center of the University of Maryland.