INSTRUCTIONS:

(a) You may use your brain, and pen/pencil on this exam – and No Other Resources!

(b) You are required to show all your work and provide the necessary explanations to get full credit.

(c) This exam has 5 problems for a total of 100 possible points.

(d) You have exactly 50 minutes to answer all **5** questions.

Problem 1:

Problem 2:

Problem 3:

Problem 4:

Problem 5:

Total:

Problem 1. [30 points]

(a) [10 points] Project b onto the column space C(A) by solving $A^T Ax = A^T b$ to obtain the projection $p = Ax$ for

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}.$$

(b) [10 points] Find $e = b - p$, and verify that it is perpendicular to the columns of A.

(c) [10 points] Compute the projection matrix P onto $C(A)$, and verify that Pb gives the same p.

(blank page for your work if you need it)

Problem 2. [20 points] QR factorization of the matrix A yields $A = QR$, where

$$Q = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/2 \\ 0 & 1/\sqrt{2} & 1/2 \\ -1/\sqrt{2} & 0 & 1/2 \\ 0 & -1/\sqrt{2} & 1/2 \end{bmatrix} \qquad \text{and} \qquad R = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solve the least squares problem for the matrix A and the vector $b = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Problem 3. [20 points]

(a) [10 points] Find an orthonormal basis for the column space of the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 0 & 3 \\ 0 & 1 \end{bmatrix}$.

(b) [10 points] Find the QR factorization of A.

Problem 4. [20 points] Consider the following orthonormal basis for $\mathbb{R}^3$:

$$q_1 = \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix} \qquad q_2 = \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix} \qquad \text{and} \qquad q_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Write the vector $\left[\begin{smallmatrix} 2 \\ -3 \\ 1 \end{smallmatrix}\right]$ as a linear combination of the basis vectors.

Problem 5. [10 points] Explain why it is impossible:

(a) [5 points] to construct a matrix A such that its column space contains $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$, and its nullspace contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(b) [5 points] to construct a matrix A such that $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has a solution, and $A^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0$.