INSTRUCTIONS:

(a) You may use your brain, and pen/pencil on this exam – and No Other Resources!

(b) You are required to show all your work and provide the necessary explanations to get full credit.

(c) This exam has 7 problems for a total of 100 possible points.

(d) You have exactly 2 hours to answer all **7** questions.

Problem 1:

Problem 2:

Problem 3:

Problem 4:

Problem 5:

Problem 6:

Problem 7:

Extra credit:

Total:

Problem 1. [15 points] Consider the matrix $M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

(a) [5 points] Find an orthonormal basis for the nullspace of M.

(b) [5 points] Find the projection matrix P onto the orthogonal complement of the nullspace of M.

(c) [5 points] Verify that $PM^T = M^T$. How could you have explained this before computing P?

Problem 2 [15 points] Consider the matrix A and the vector b

$$A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & -3 & 4 \\ 1 & -2 & 1 & -4 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 3 \\ 9 \\ -4 \end{bmatrix}.$$

(a) [10 points] Find the complete solution to $Ax = b$.

(b) [5 points] Find a solution to $Ax = b$ perpendicular to the vector $v = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

(blank page for your work if you need it)

Problem 3. [10 points] A matrix $A = LU$ has the LU factors

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ -1 & -1 & -2 & 1 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find the dimensions of the four fundamental subspaces of A.

Hint: the question can be answered without multiplying LU to get A.

Problem 4. [15 points] Find the trigonometric polynomial $y(x) = a + b\sin(x) + c\cos(x)$ that best fits the data by solving a least squares problem.

x	$y(x)$
0	7
$\pi/2$	0
π	0
$3\pi/2$	0

(blank page for your work if you need it)

Problem 5. [15 points] Consider the matrix A and the vectors v_1, v_2 and v_3:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \text{and} \qquad v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

(a) [5 points] Check that v_1, v_2 and v_3 are eigenvectors of the matrix A. What are the eigenvalues of A?

(b) [5 points] Check that v_1, v_2 and v_3 are a basis for $\mathbb{R}^3$.

(c) [5 points] Find $A^{100}v_2$.

(blank page for your work if you need it)

Problem 6. [15 points] Explain why the following statements are true.

(a) [5 points] If A is a nonzero square matrix such that $A^2 = 0$ (the zero matrix), then A is not invertible.

Hint: What subspace are the columns of A in?

(b) [5 points] If z is a nonzero vector such that $A^T z = 0$ and $z^T b \neq 0$, then $Ax = b$ is not solvable.

(c) [5 points] If 0 is an eigenvalue of the matrix A, then A is not invertible.

Problem 7. [15 points] Suppose the singular value decomposition $A = U\Sigma V^T$ has

$$U = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad V = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

(a) [5 points] Verify that both U and V are orthogonal matrices.

(b) [5 points] Find the rank of A.

(c) [5 points] Find bases for the four fundamental subspaces of A.

(blank page for your work if you need it)

Problem for 5 extra credits. I was looking for a 5×10 matrix A and vectors b and c such that $Ax = b$ has no solution and $A^T x = c$ has exactly one solution. Why can I not find A, b and c?