INSTRUCTIONS:

(a) You may use your brain, and pen/pencil on this exam – and No Other Resources!

(b) You are required to show all your work and provide the necessary explanations to get full credit.

(c) This exam has 8 problems for a total of 100 possible points.

Problem 1:

Problem 2:

Problem 3:

Problem 4:

Problem 5:

Problem 6:

Problem 7:

Problem 8:

Extra:

Total:

Problem 1. [10 points]
For which value(s) of k is the following matrix invertible?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{bmatrix}$$

Problem 2. [10 points]
Let A, B, C be $n \times n$ invertible matrices. When you simplify the expression

$$C^{-1}(AB^{-1})^{-1}(CA^{-1})^{-1}C^2,$$

which matrix do you get?

(a) A

(b) $C^{-1}A^{-1}BC^{-1}AC^2$

(c) B

(d) C^2

(e) $C^{-1}BC$

(f) C

Problem 3. [15 points]

Find all 2×2 symmetric matrices $A = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}$ satisfying

$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

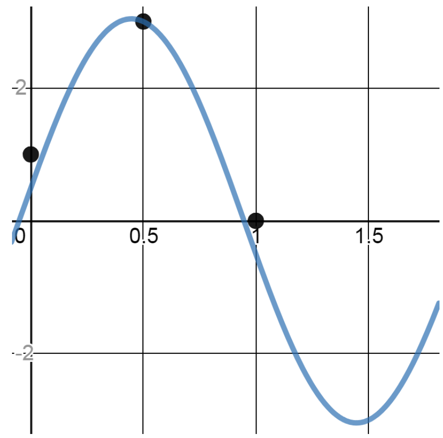
(blank page for your work if you need it)

Problem 4. [15 points]
Solve a least squares problem to find the trigonometric polynomial

$$p(x) = a\sin(\pi x) + b\cos(\pi x)$$

that best fits the points:

$(0, 1)$
$(1/2, 3)$
$(1, 0)$



(blank page for your work if you need it)

Problem 5.[15 points]
The QR factorization of the matrix A (via Gram-Schmidt) yields $A = QR$, where

$$Q = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/2 \\ 0 & 1/\sqrt{2} & 1/2 \\ -1/\sqrt{2} & 0 & 1/2 \\ 0 & -1/\sqrt{2} & 1/2 \end{bmatrix} \qquad \text{and} \qquad R = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

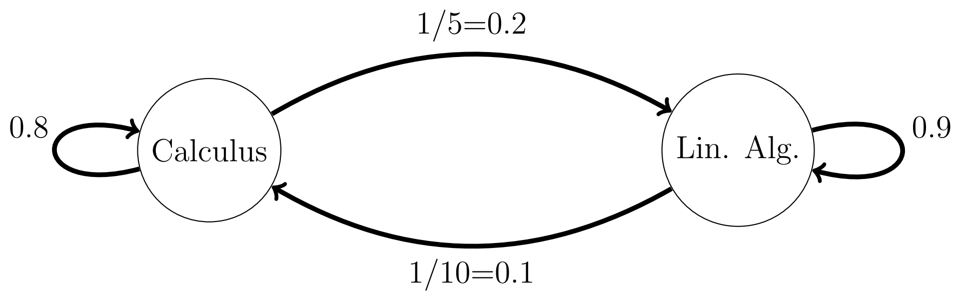
Solve the least squares problem for the matrix A and the vector $b = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Hint: the problem can be answered without multiplying QR to get A. You should be able to *quickly* get an upper-triangular system of equations for the least squares solution.

(blank page for your work if you need it)

Problem 6. [15 points]
Suppose x_k is the fraction of UM students who prefer calculus to linear algebra at year k. The remaining fraction $y_k = 1 - x_k$ prefers linear algebra.

At year $k+1$, 1/5 of those who prefer calculus change their mind (possibly after taking M221). Also at year $k+1$, 1/10 of those who prefer linear algebra change their mind (possibly because of this exam).



Create the matrix A to give $\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = A \begin{bmatrix} x_k \\ y_k \end{bmatrix}$, and find the limit of $A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as $n \to \infty$.

(blank page for your work if you need it)

Problem 7. [15 points]
Let

$$A = \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}$$

Determine the values of t such that the matrix A is diagonalizable.

(blank page for your work if you need it)

Problem 8. [5 points]
Show that if x is an eigenvector of the matrix A with eigenvalue λ, then x is an eigenvector of the matrix $A^2 + A$ with eigenvalue $\lambda^2 + \lambda$.

Problem for 5 extra credits
Suppose u is a unit vector, so $u^T u = 1$. This problem is about the $n \times n$ matrix $H = I_n - 2uu^T$.

(a) Show that H is symmetric.

(b) Show that $H^2 = H$.

(c) One eigenvector of H is u itself. Find the corresponding eigenvalue.