A Hands-On Introduction to Computational Fluid Dynamics

Mohammad Umair[†], Gerardo Zampino[†] and Ricardo Vinuesa[†]

[†]FLOW, Engineering Mechanics, KTH Royal Institute of Technology, Sweden

VETENSKAP OCH KONST

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SERC Swedish e-Science Research Centre

Computational Fluid Dynamics

17th Century
France and England

PURE
EXPERIMENTS

Computational Fluid Dynamics

17th Century
France and England

PURE
EXPERIMENTS

PURE
THEORY

18th – 19th Century
Europe

Computational Fluid Dynamics

17th Century
France and England

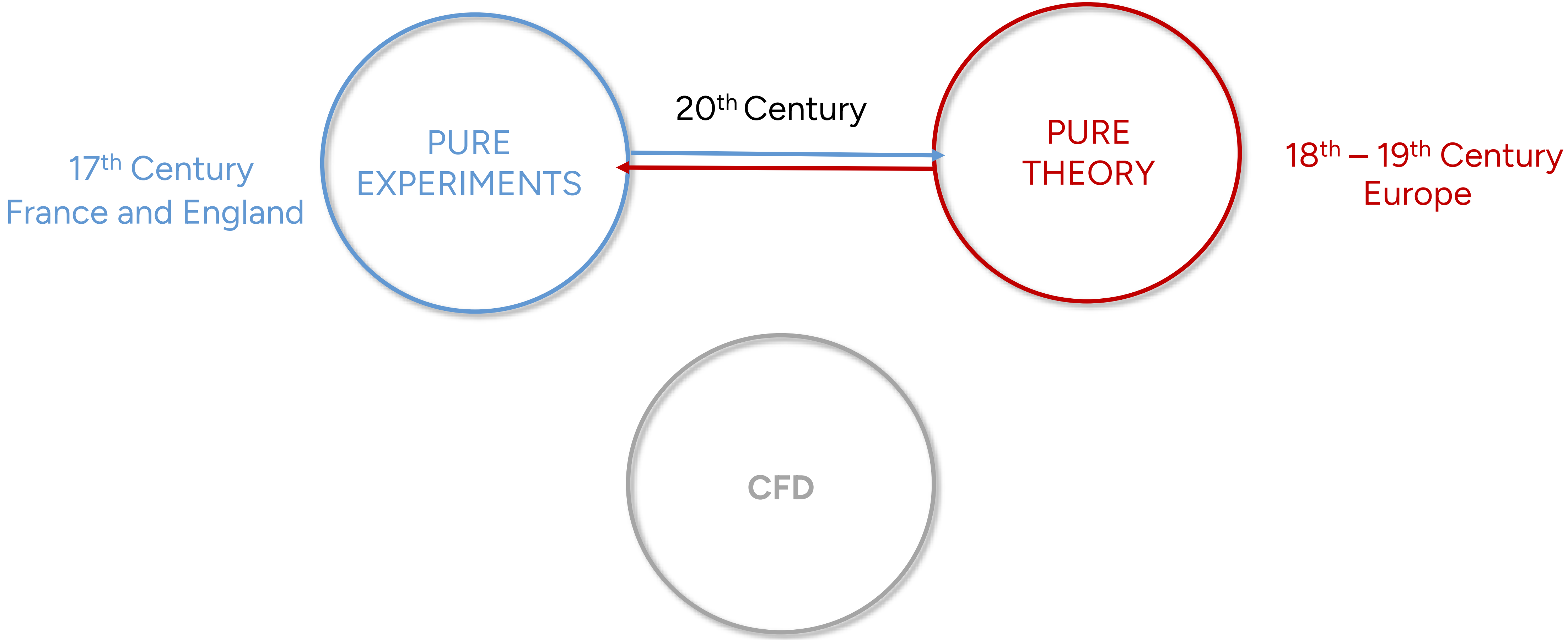
PURE
EXPERIMENTS

20th Century

PURE
THEORY

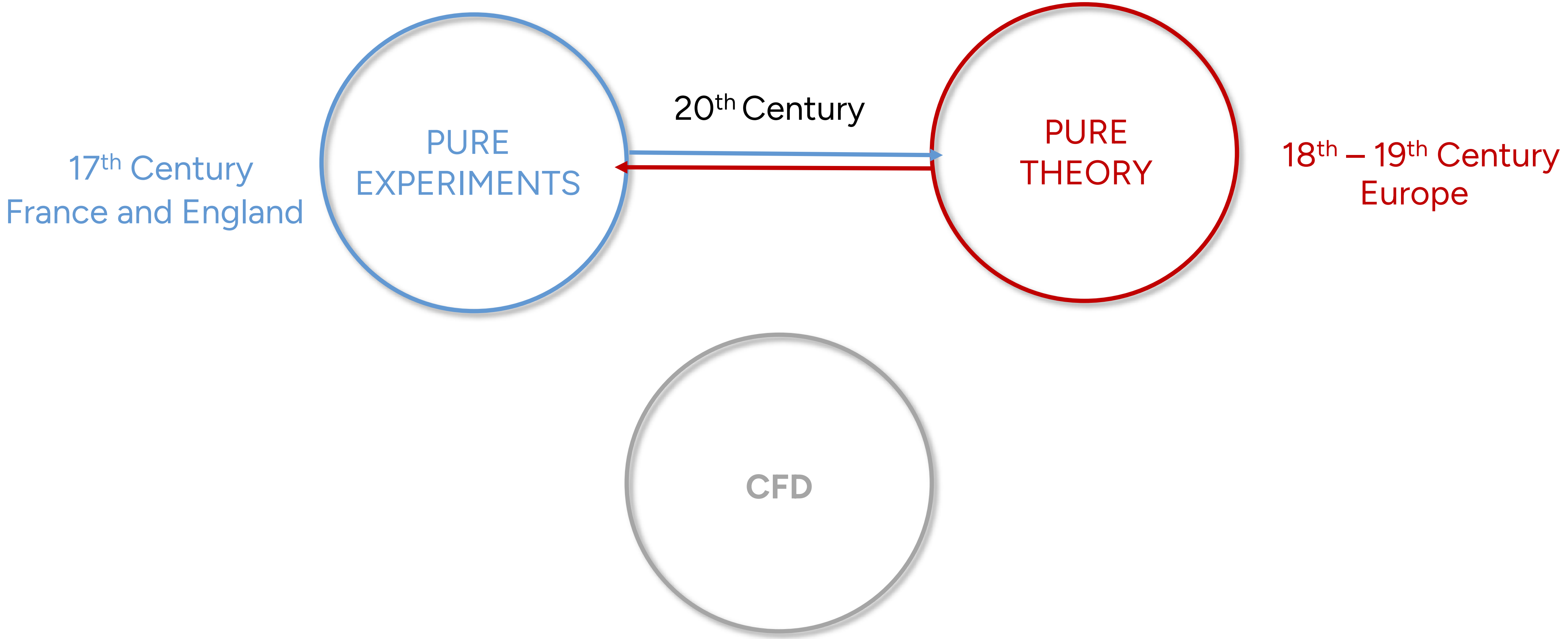
18th – 19th Century
Europe

Computational Fluid Dynamics



A new *"third approach"* in the philosophical study of fluid dynamics

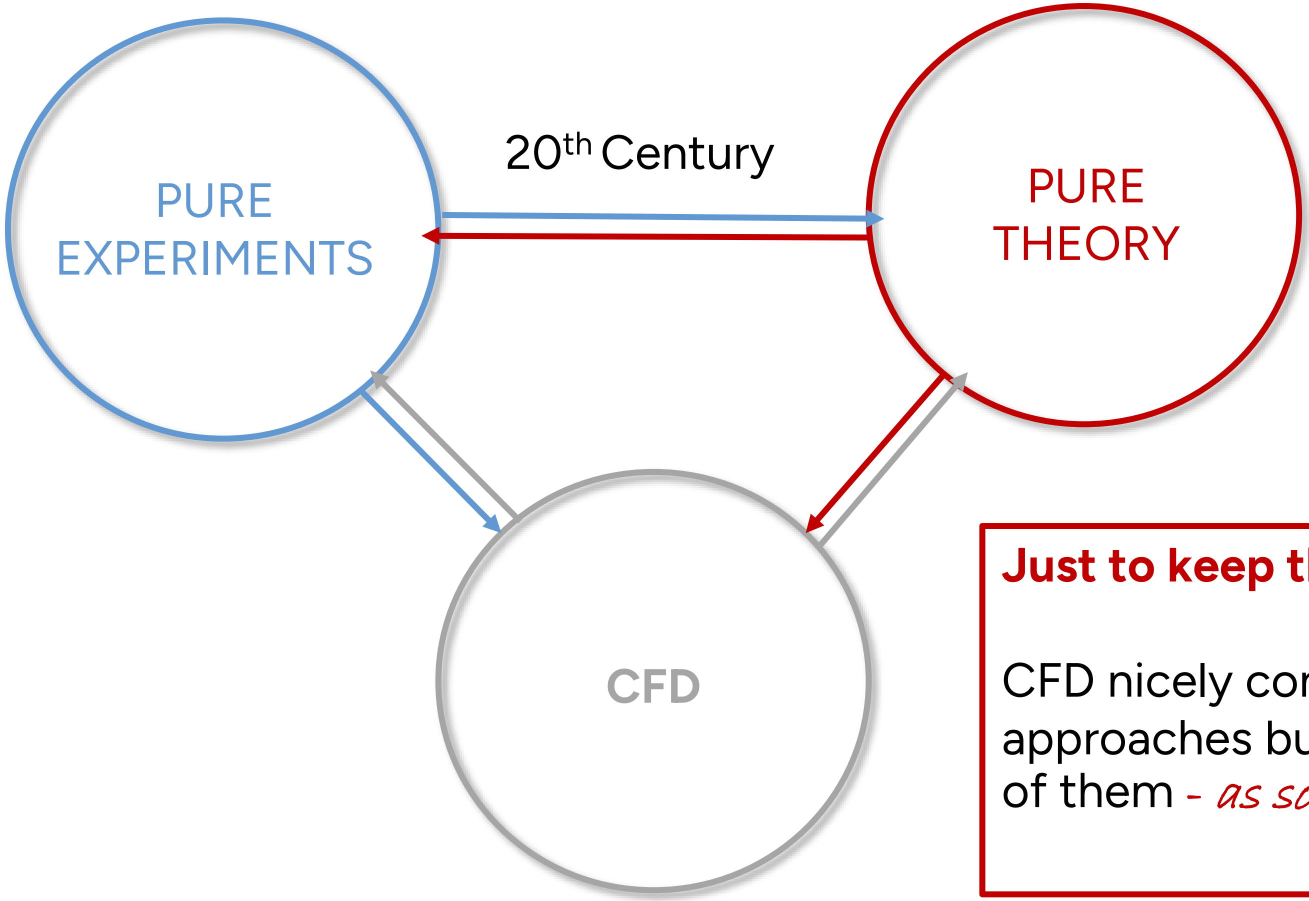
Computational Fluid Dynamics



A new *"third approach"* in the philosophical study of fluid dynamics

(Nothing more than that!)

Computational Fluid Dynamics



Just to keep things in perspective:

CFD nicely compliments the other two approaches but will ***never*** replace either of them - *as sometimes suggested!*

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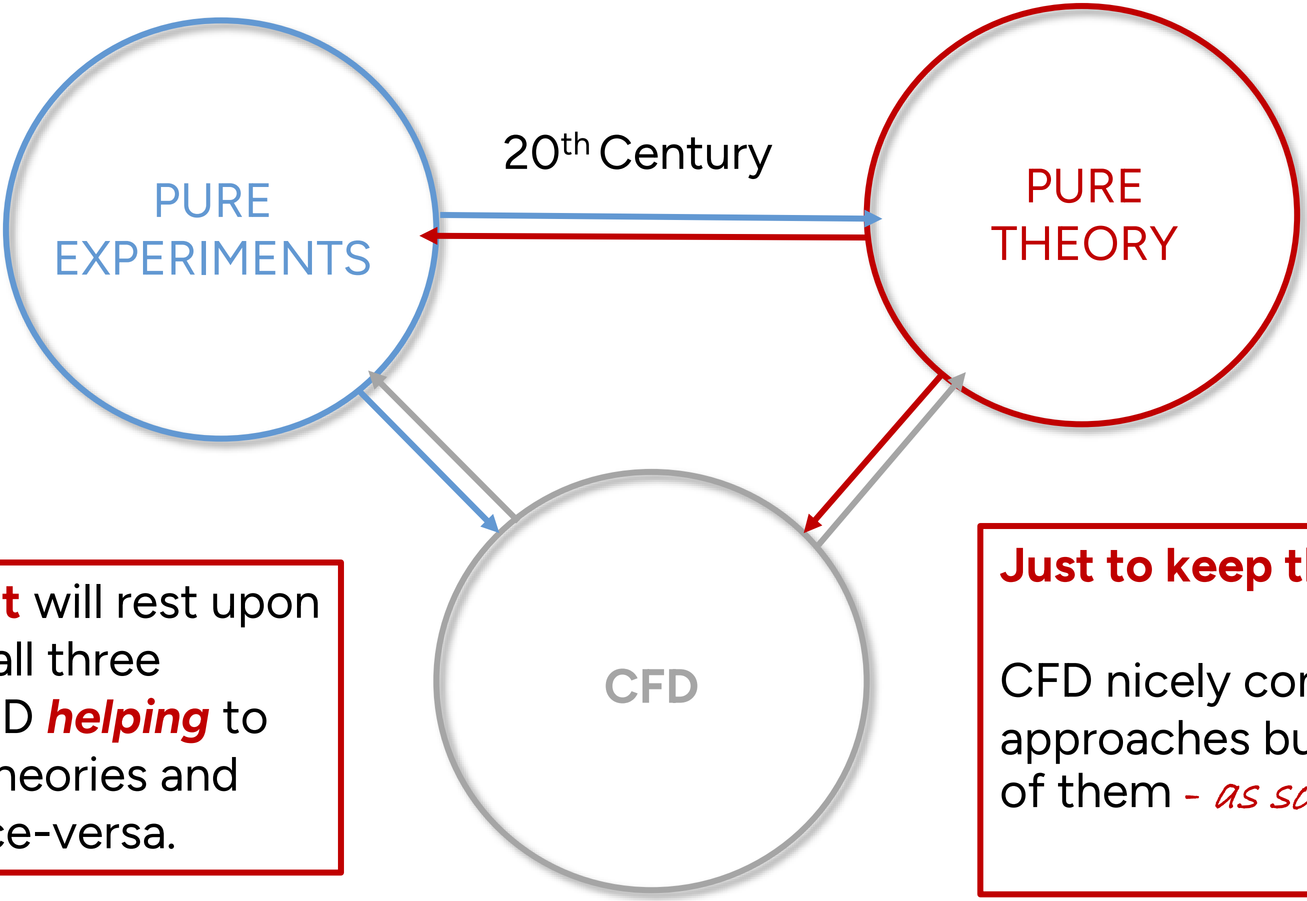

Future advancement will rest upon a proper balance of all three approaches, with CFD ***helping*** to interpret results of theories and experiments, and vice-versa.

Just to keep things in perspective:

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CFD as a research tool

CFD results are directly analogous to Wind Tunnel results.

However, unlike a wind tunnel a computer program can be carried everywhere or can be accessed sitting *1000 miles away!*

A CFD program is, therefore, a *"readily transportable wind tunnel"*, where you can carry out *"numerical experiments".*

CFD as a research tool: Who cares?

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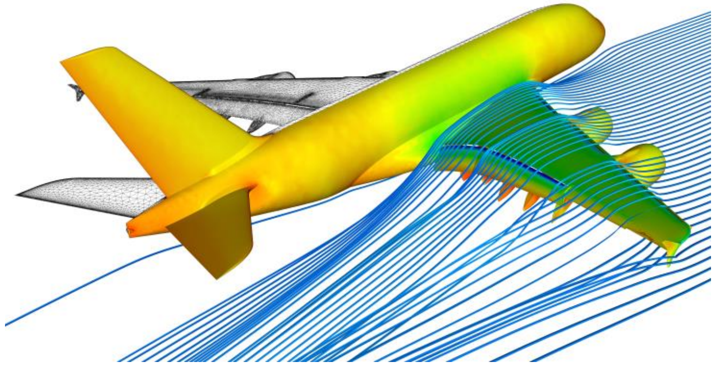
Aviation



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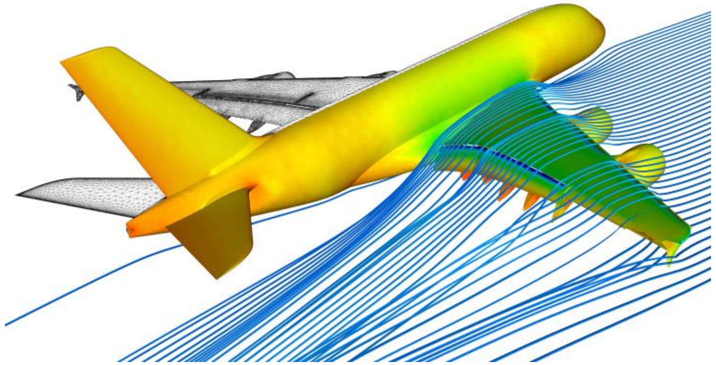
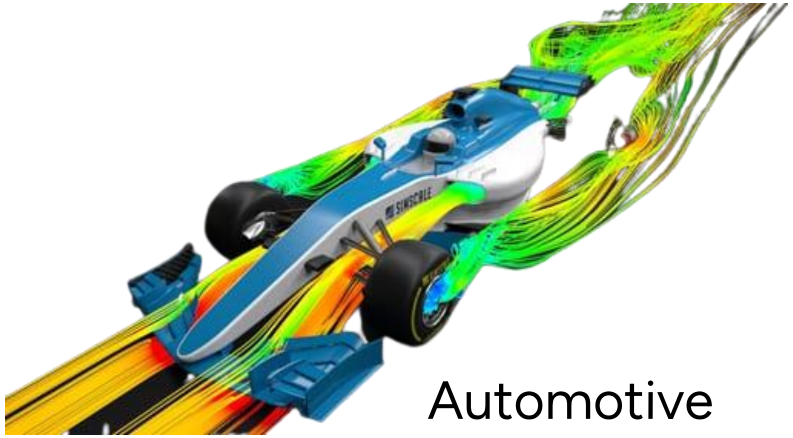
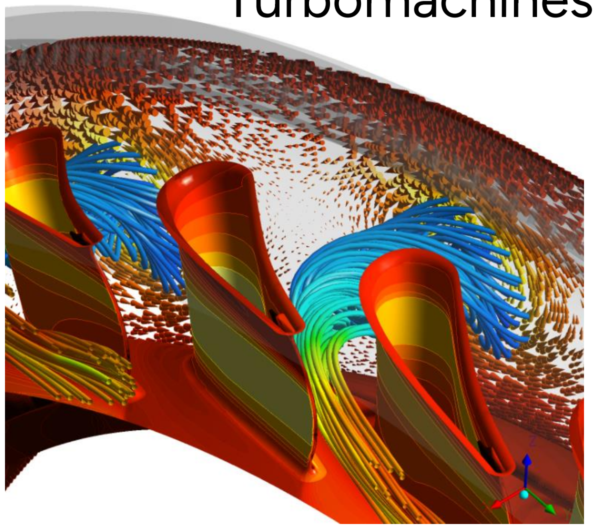




Automotive

CFD as a research tool: Who cares?

Turbomachines



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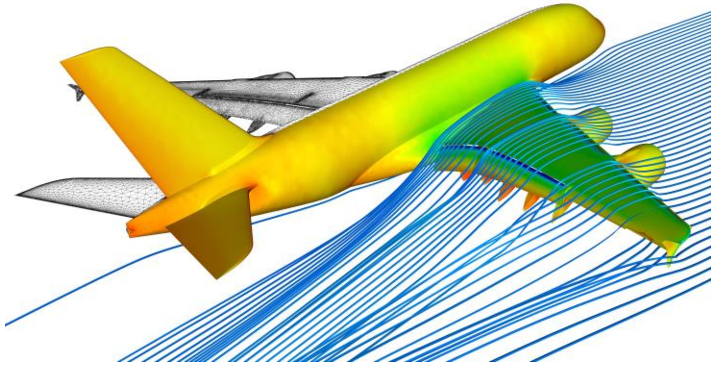

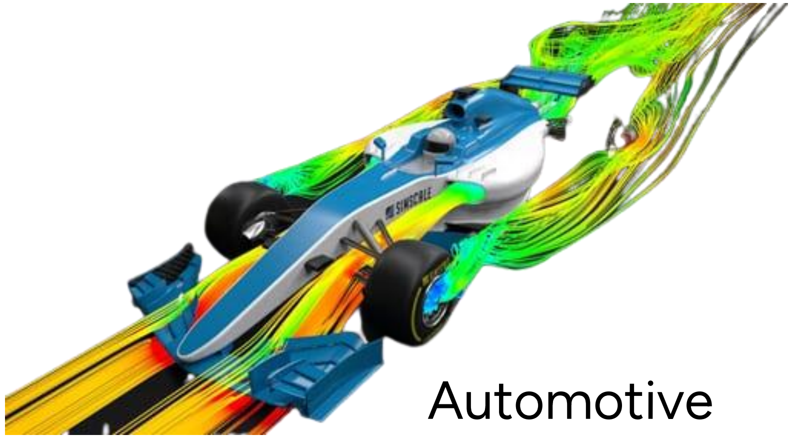

Automotive

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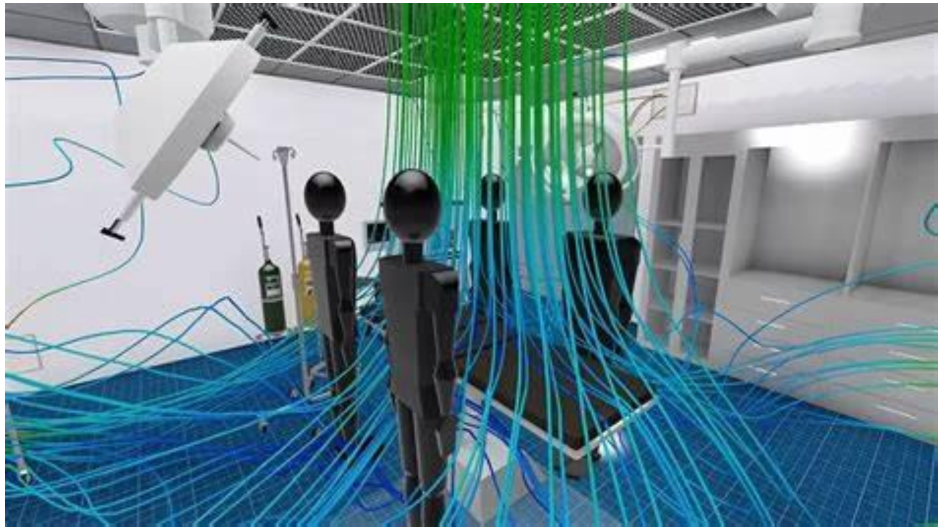
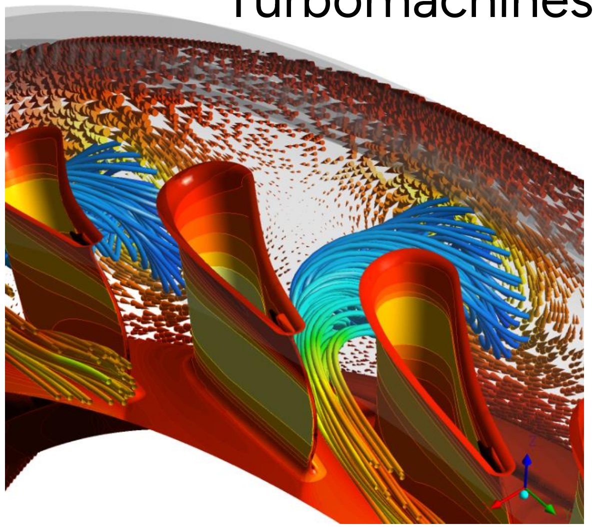
HVAC



Turbomachines



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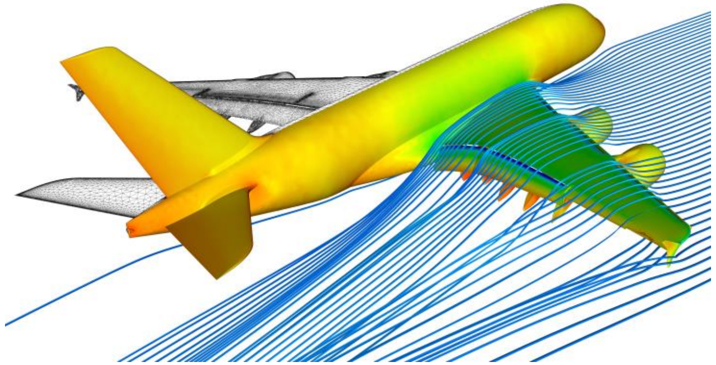
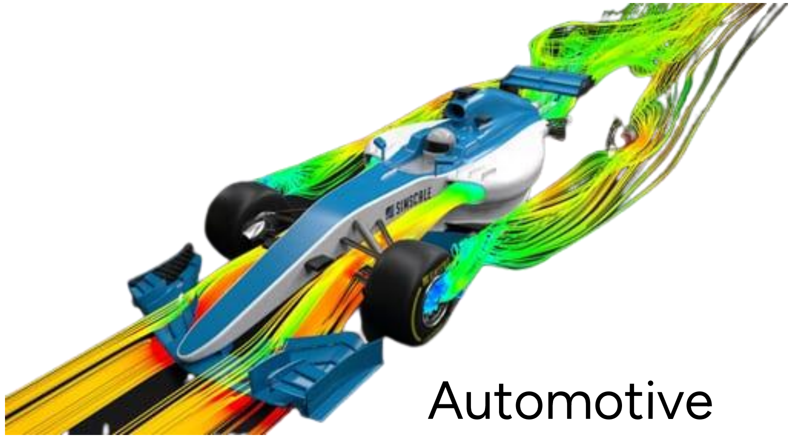
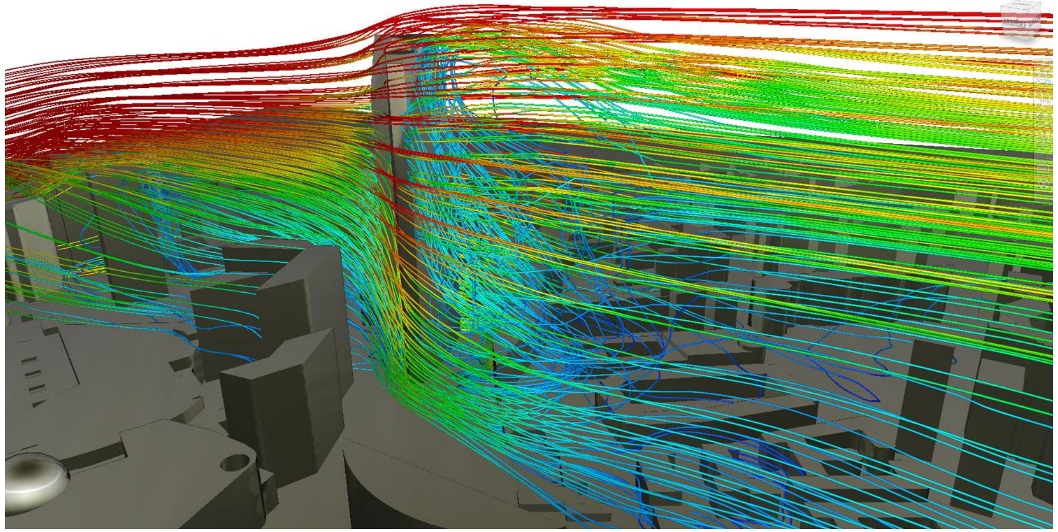




Automotive

CFD as a research tool: Who cares?

Cities



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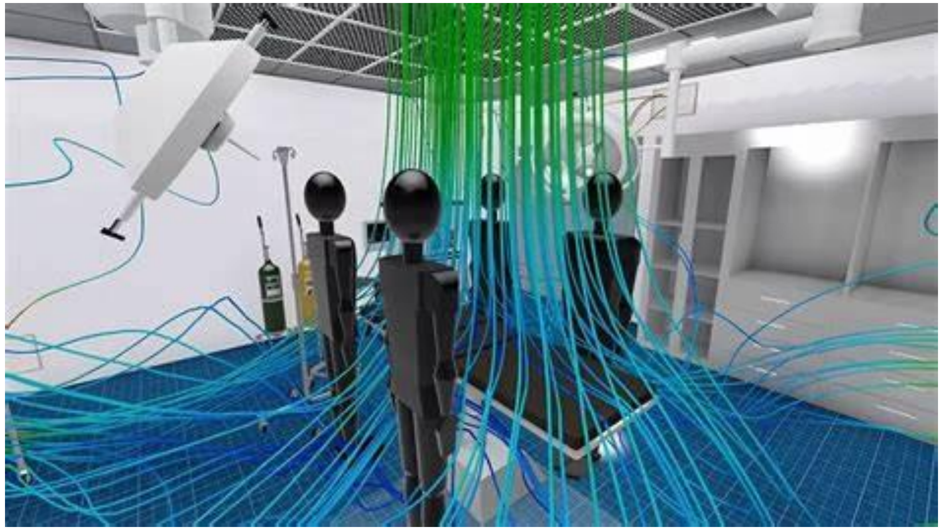
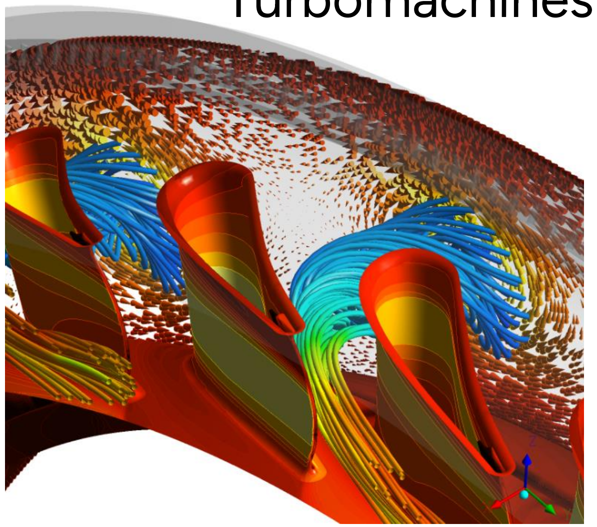
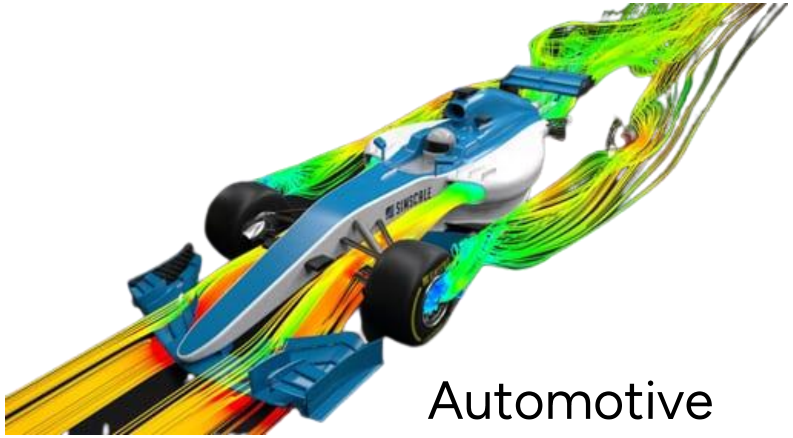


Turbomachines





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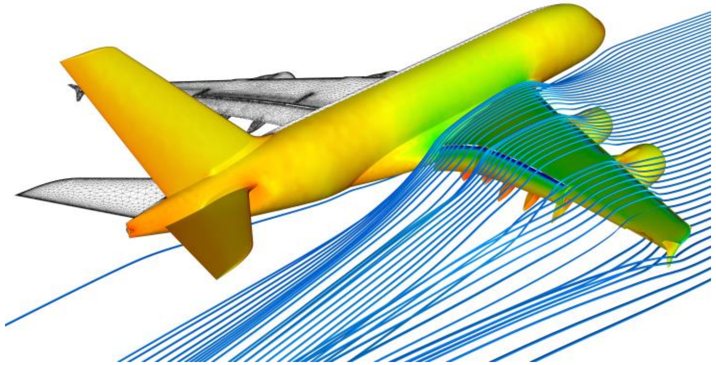

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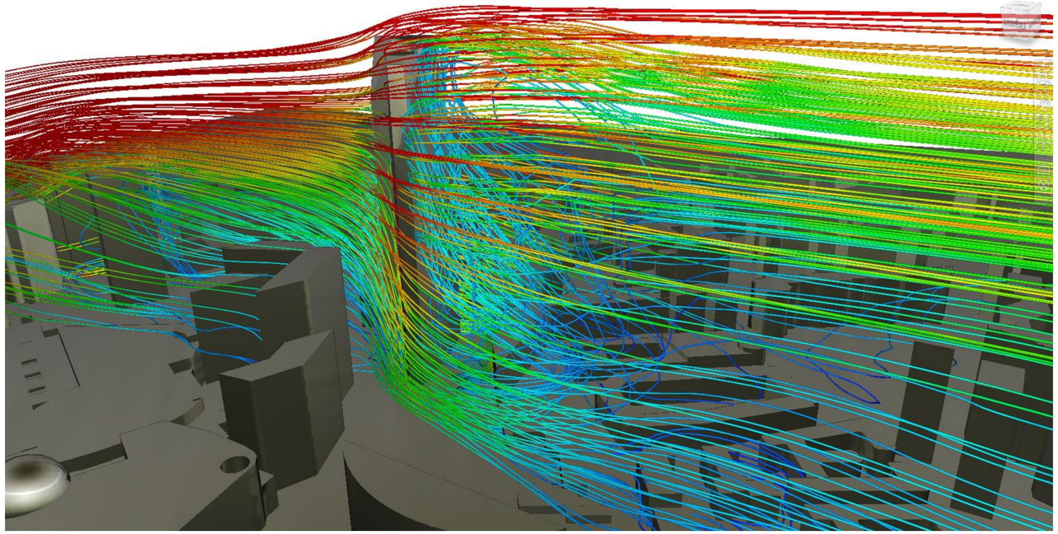

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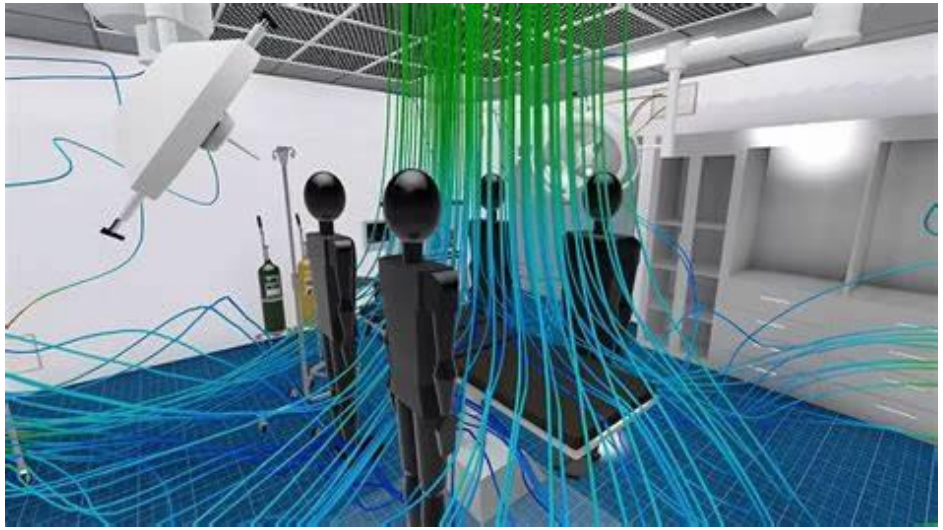

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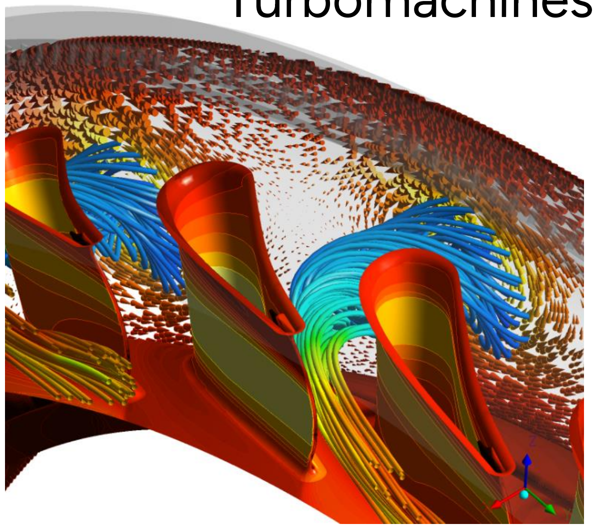

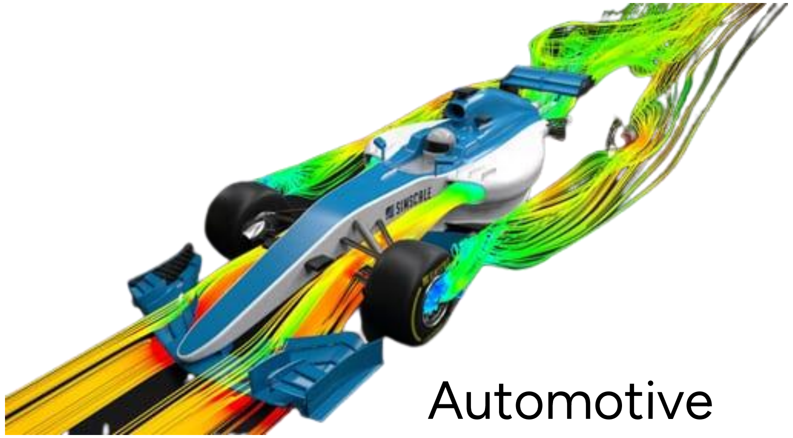

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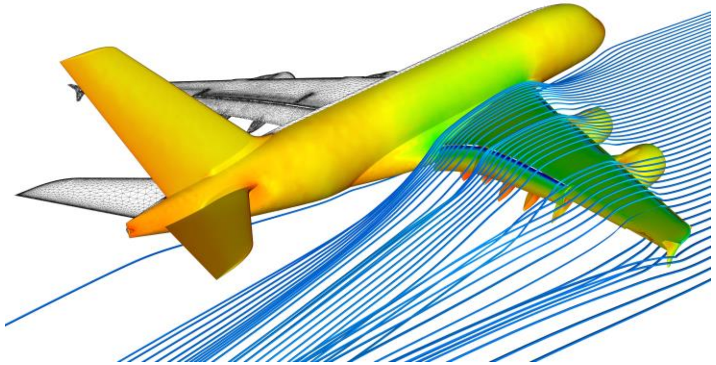
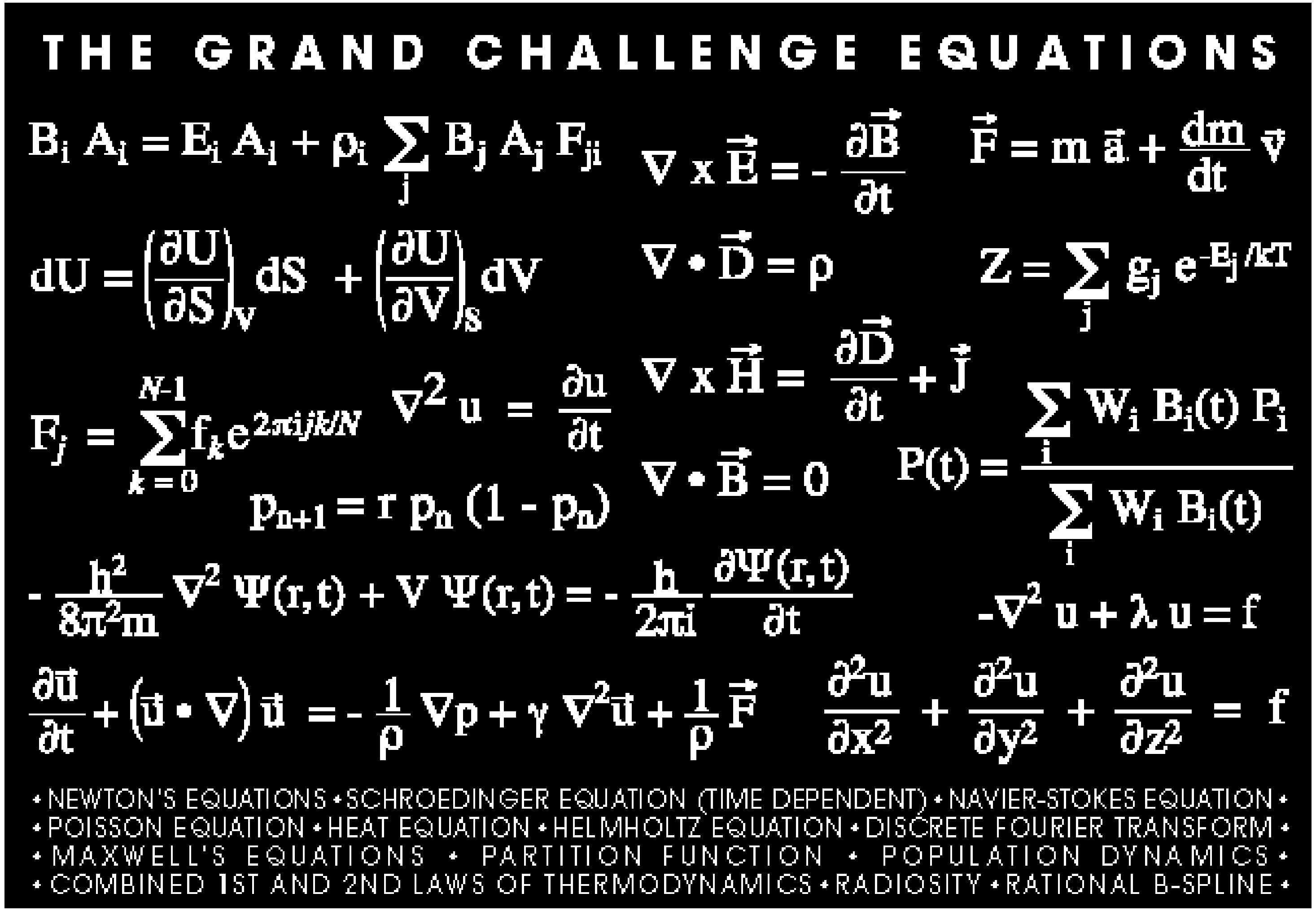


+ many more !

So How to CFD?

CFD involves:

- Identifying the physical phenomenon, that includes Governing equations, Initial & Boundary conditions.
- Breaking down the continuous problem to a discrete representation.
- Solving the discrete set of equations using adequate numerical methods.
- Post processing the results.



What are we solving?

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Navier Stokes equations

Continuity equation: $$\partial_t \rho + \nabla.(\rho \boldsymbol{u}) = 0$$

Momentum equation: $$\partial_t \rho \boldsymbol{u} + \nabla.(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p - \nabla.\boldsymbol{\tau} = \boldsymbol{f}$$

Energy equation: $$\partial_t E + \nabla.[(E + p)\boldsymbol{u}] - \nabla.(\boldsymbol{\tau u}) - \nabla.(\kappa \nabla T) = S$$

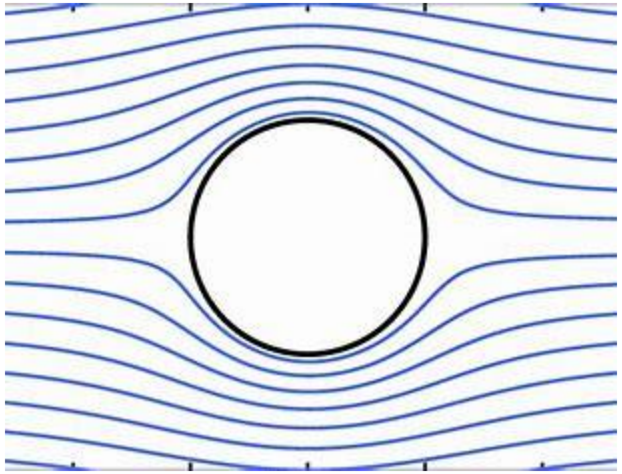
Depending on the nature of physics governing the fluid motion one or more terms might be negligible.

Presence of each term and their combinations determines the appropriate solution algorithm and the numerical procedure.

Classification of PDEs

Elliptic

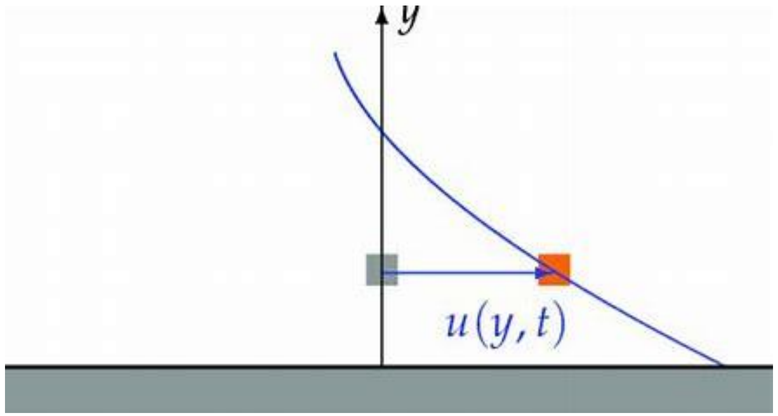
$$\nabla^2 u = 0$$



Potential Flow

Parabolic

$$\frac{\partial u}{\partial t} = \nu \nabla^2 u$$



Flow over an oscillating plate
(Stokes 2^{nd} Problem)

Hyperbolic

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

Wave motion

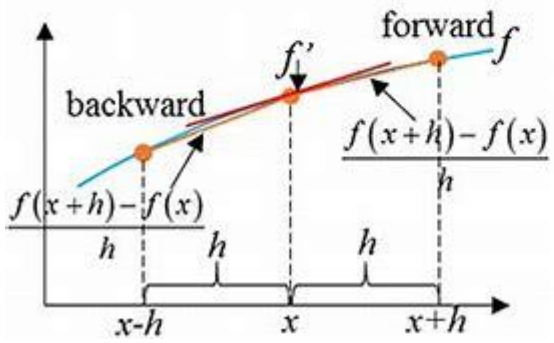
Equations belonging to each of these classifications behave in different ways both *physically* and *numerically*.

Techniques for Numerical Discretization

Governing Equations $\xrightarrow{\text{Discretization}}$ Numerical Analogue

Commonly used discretization methods

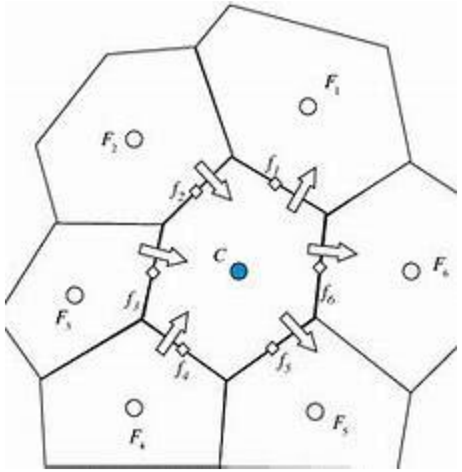
Finite Difference Methods



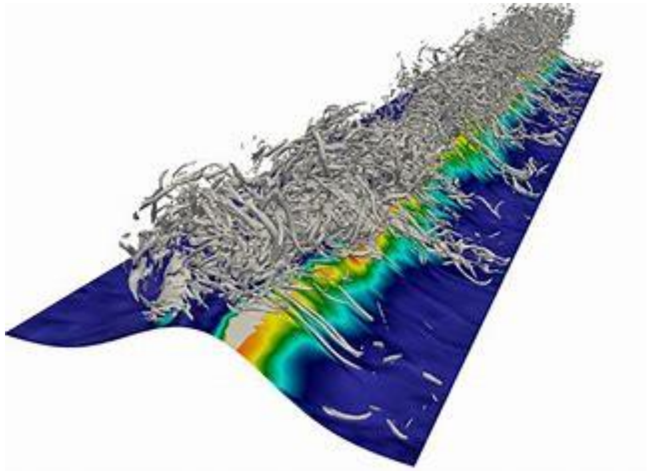
Spectral Methods

$$
\begin{aligned}
G(f) &= \Im\{\cos(2\pi A t)\} = \int_{-\infty}^{\infty} \frac{e^{i2\pi At} + e^{-i2\pi At}}{2} e^{-i2\pi ft}\,dt \\
&= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{i2\pi At} e^{-i2\pi ft}\,dt + \int_{-\infty}^{\infty} e^{-i2\pi At} e^{-i2\pi ft}\,dt\right] \\
&= \frac{1}{2}\left[\delta(f-A) + \delta(f+A)\right]
\end{aligned}
$$

Finite Volume Methods



Finite/Spectral Element Methods



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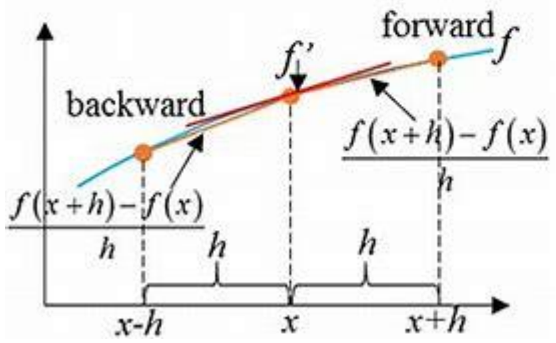
TODAY

TOMORROW

Finite Difference Methods



Spectral Methods

$$G(f)=\Im\{\cos(2\pi At)\}=\int_{-\infty}^{\infty}\frac{e^{i2\pi At}+e^{-i2\pi At}}{2}e^{-i2\pi ft}\,dt$$
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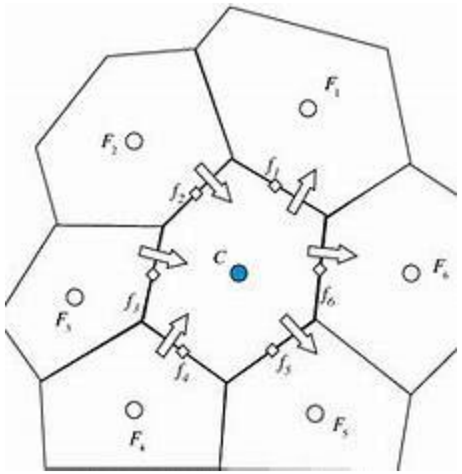
$$=\frac{1}{2}\left[\delta(f-A)+\delta(f+A)\right]$$

Finite Volume Methods



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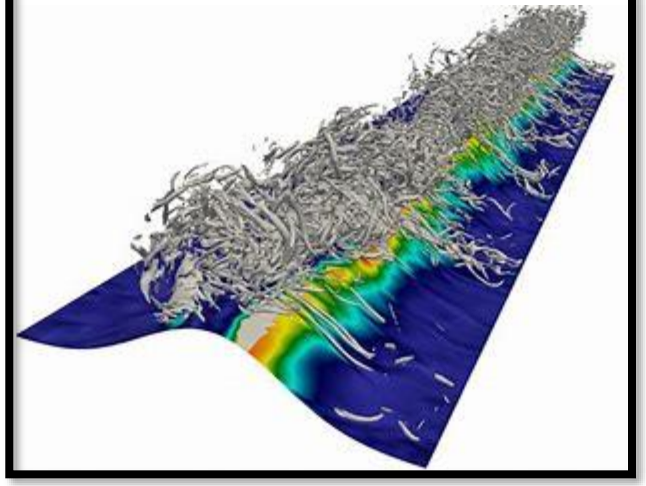

Finite Difference Method

Definition of a derivative

Exact

$$\frac{df}{dx} = \lim_{dx \to 0} \frac{f(x + dx) - f(x)}{dx}$$

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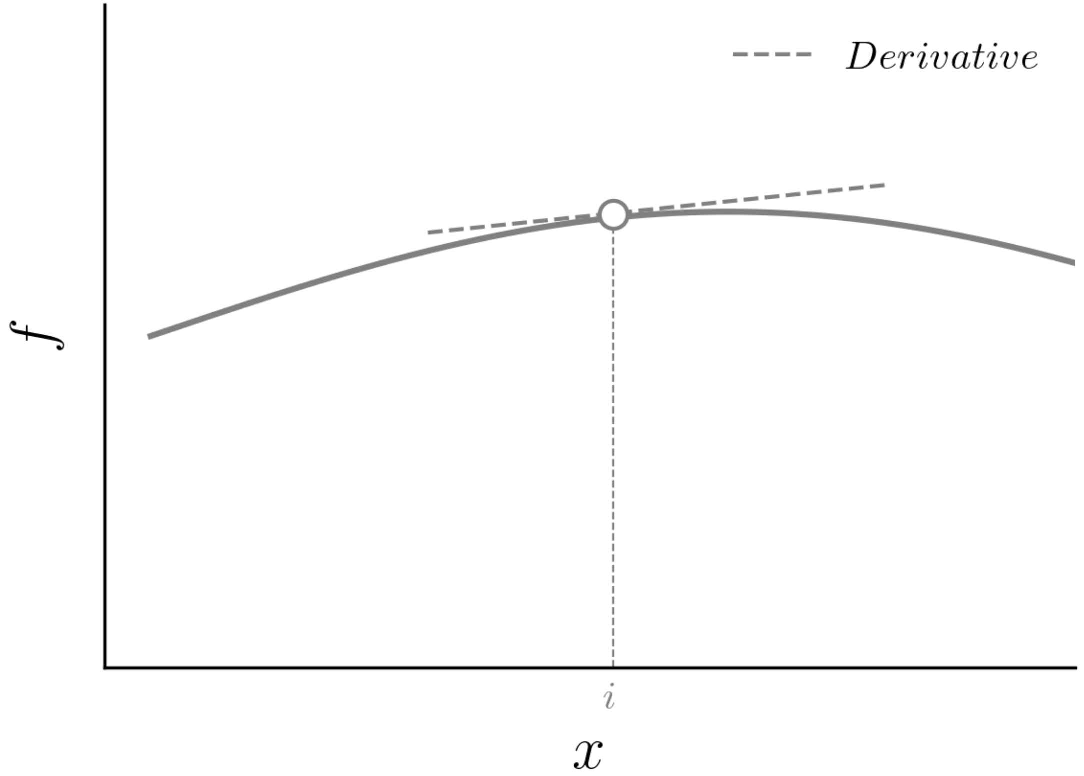

Finite Difference Method

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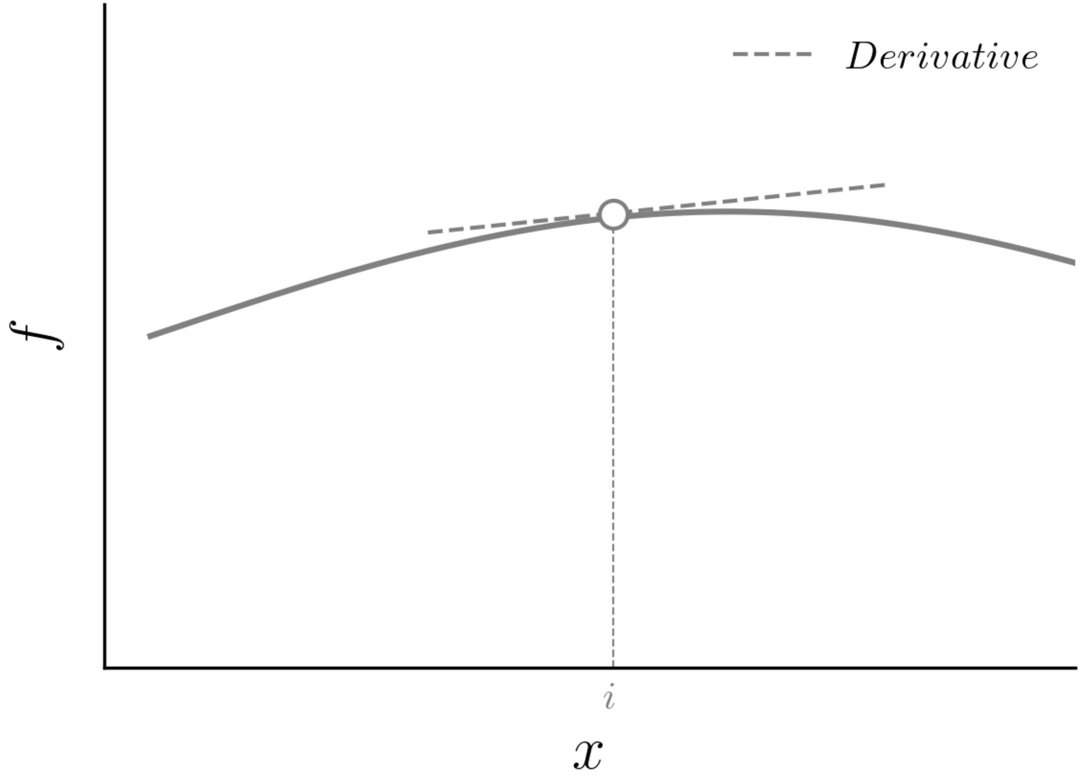
$$\frac{df}{dx} = \lim_{dx \to 0} \frac{f(x + dx) - f(x)}{dx}$$

$$\frac{df}{dx} = \lim_{dx \to 0} \frac{f(x) - f(x - dx)}{dx}$$

$$\frac{df}{dx} = \lim_{dx \to 0} \frac{f(x + dx) - f(x - dx)}{2dx}$$



Finite Difference Method

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$$\frac{df}{dx} \approx \frac{f(x+dx)-f(x)}{dx}$$

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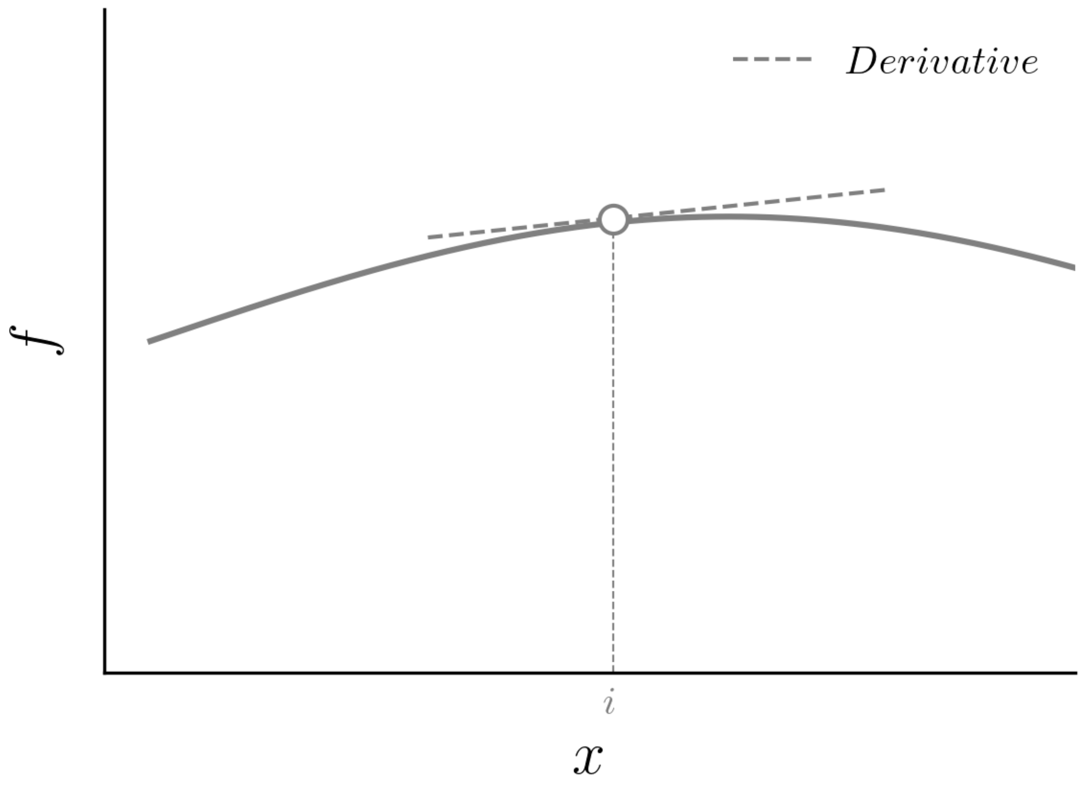

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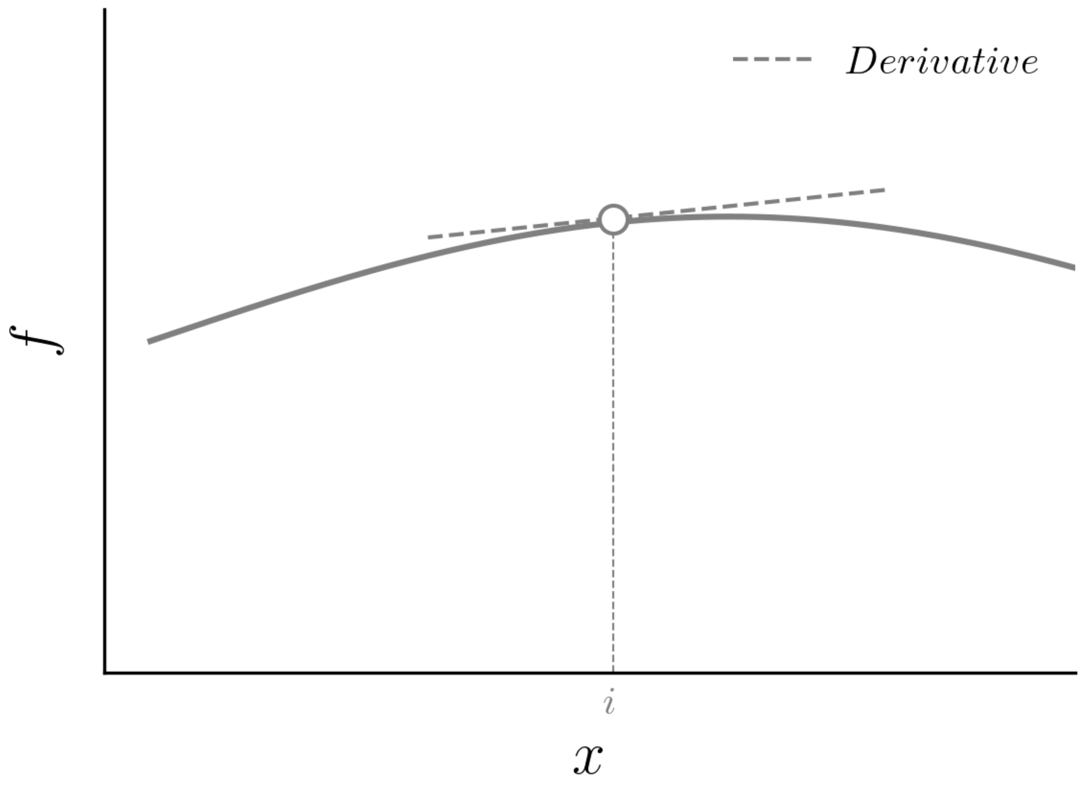

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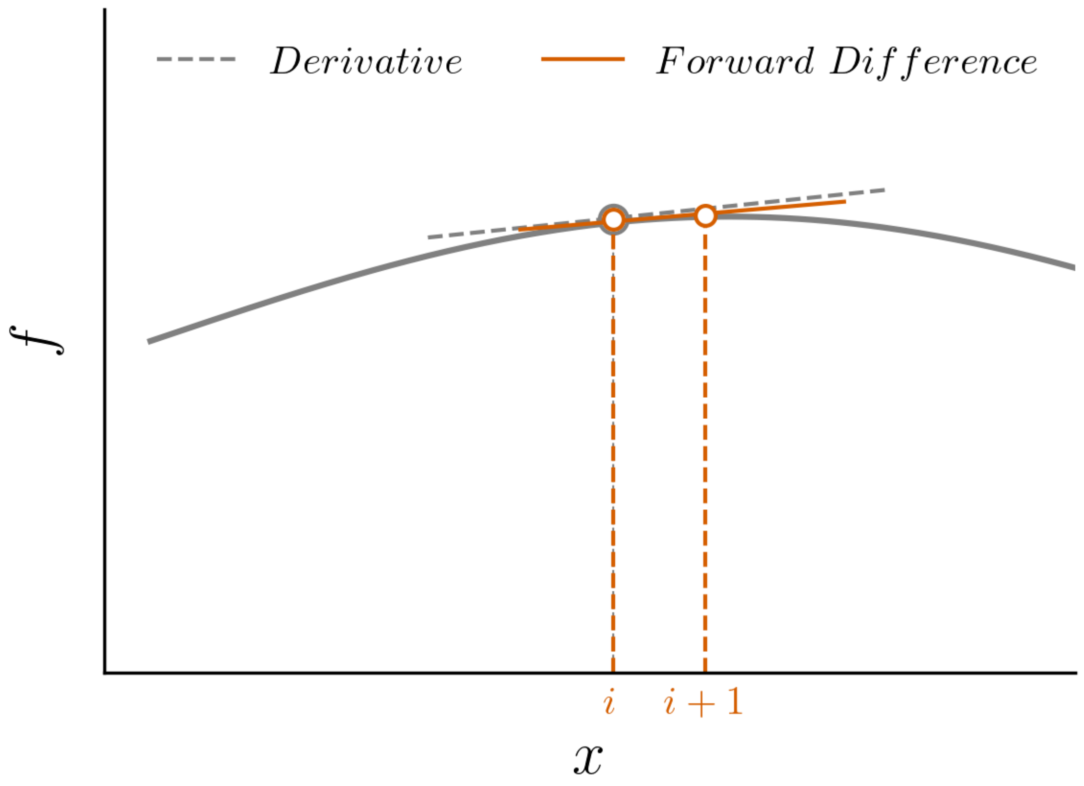

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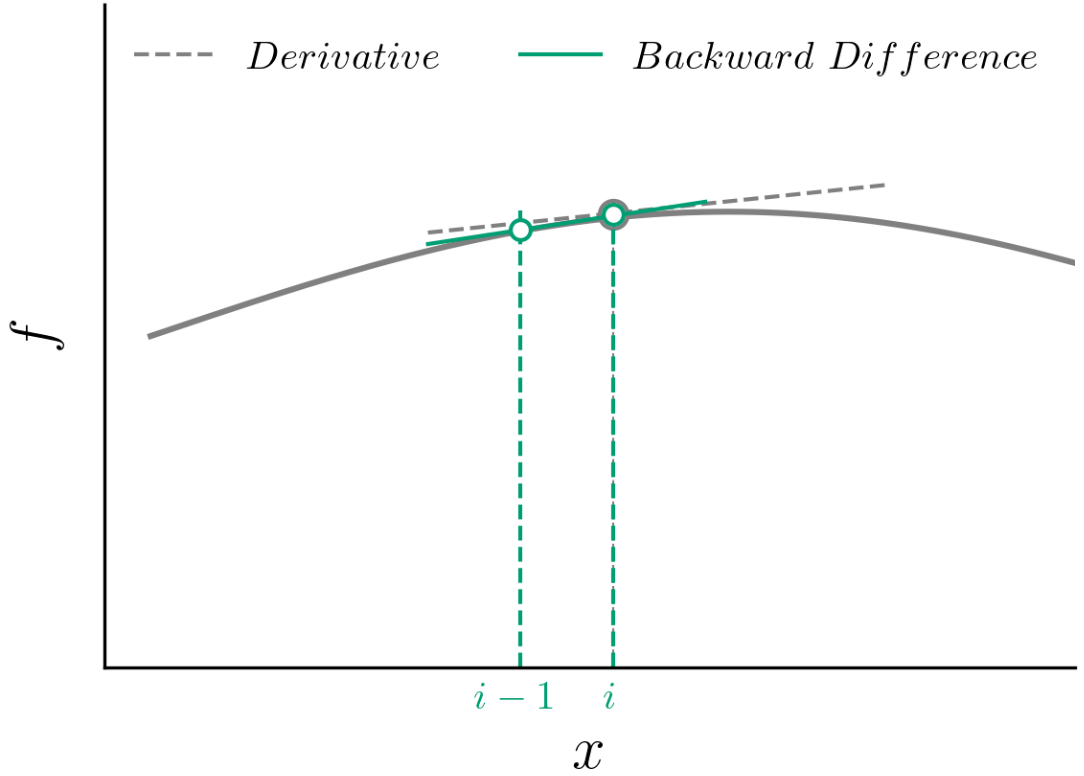

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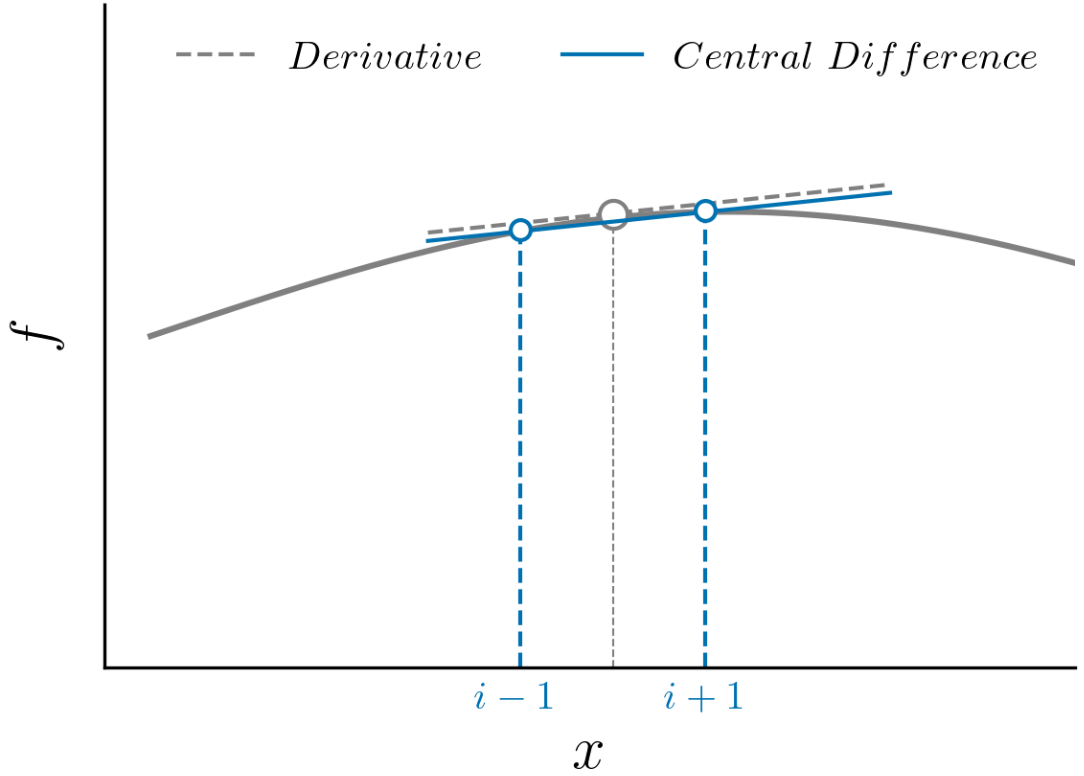
Centered $$\frac{df}{dx} \approx \frac{f(x+dx) - f(x-dx)}{2dx}$$



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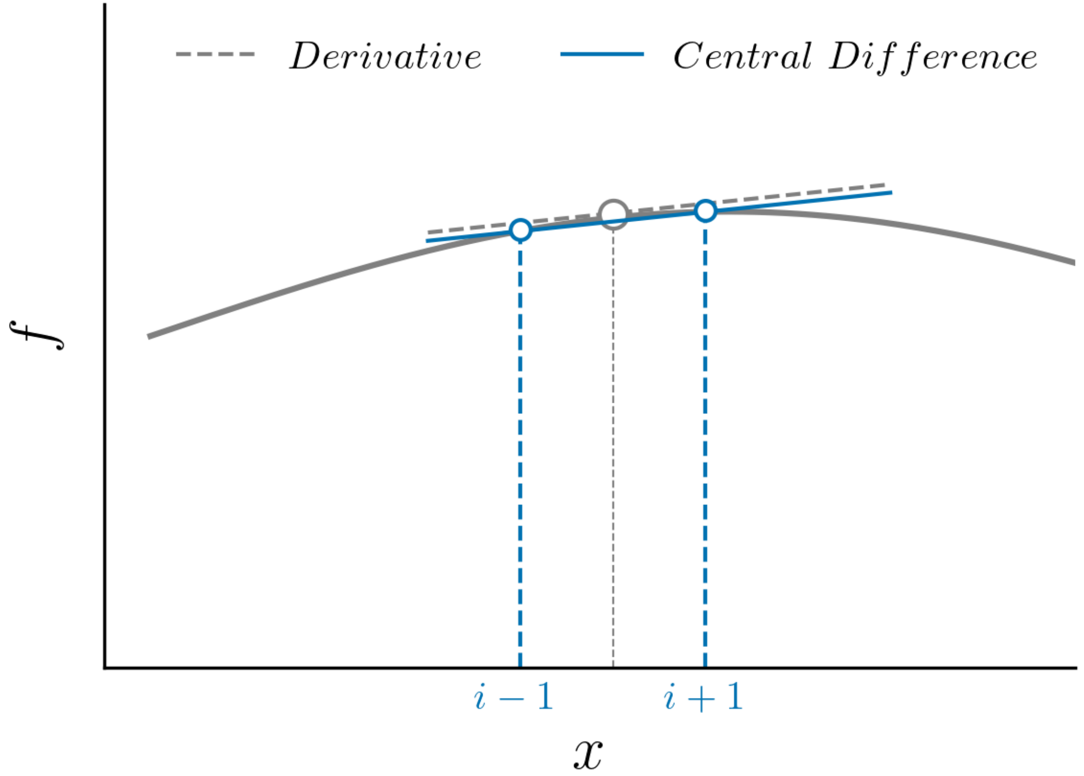

How good is the approximation?

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Taylor Series

$$f(x_0 + dx) = f(x_0) + \frac{df}{dx}\Big|_{x_0} dx + \frac{1}{2!}\frac{d^2 f}{dx^2}\Big|_{x_0} dx^2 + \frac{1}{3!}\frac{d^3 f}{dx^3}\Big|_{x_0} dx^3 + \cdots$$

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$$\frac{f(x_0+dx)-f(x_0)}{dx}=\frac{df}{dx}\bigg|_{x_0}+\frac{1}{2!}\frac{d^2f}{dx^2}\bigg|_{x_0}dx+\frac{1}{3!}\frac{d^3f}{dx^3}\bigg|_{x_0}dx^2+\cdots$$

$$\frac{f(x_0+dx)-f(x_0)}{dx}=\frac{df}{dx}\bigg|_{x_0}+O(dx)$$

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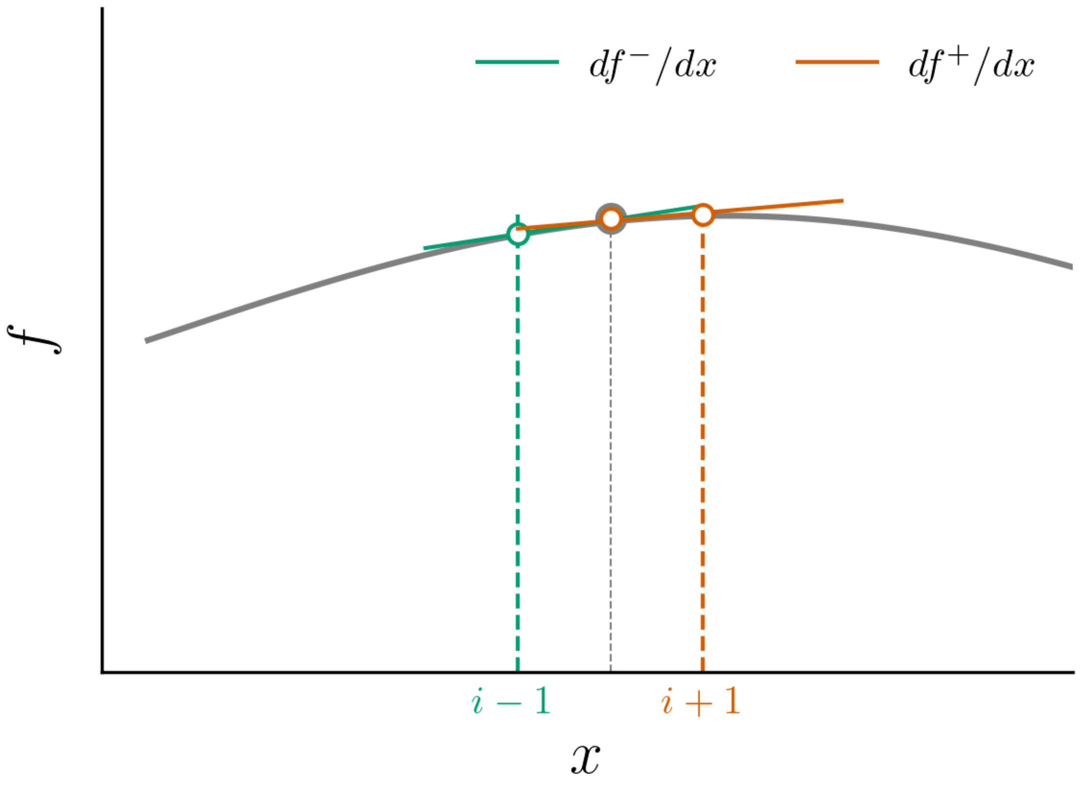
$$\frac{f(x_0+dx) - f(x_0)}{dx} = \left.\frac{df}{dx}\right|_{x_0} + O(dx)$$

1st order accurate

Second derivative

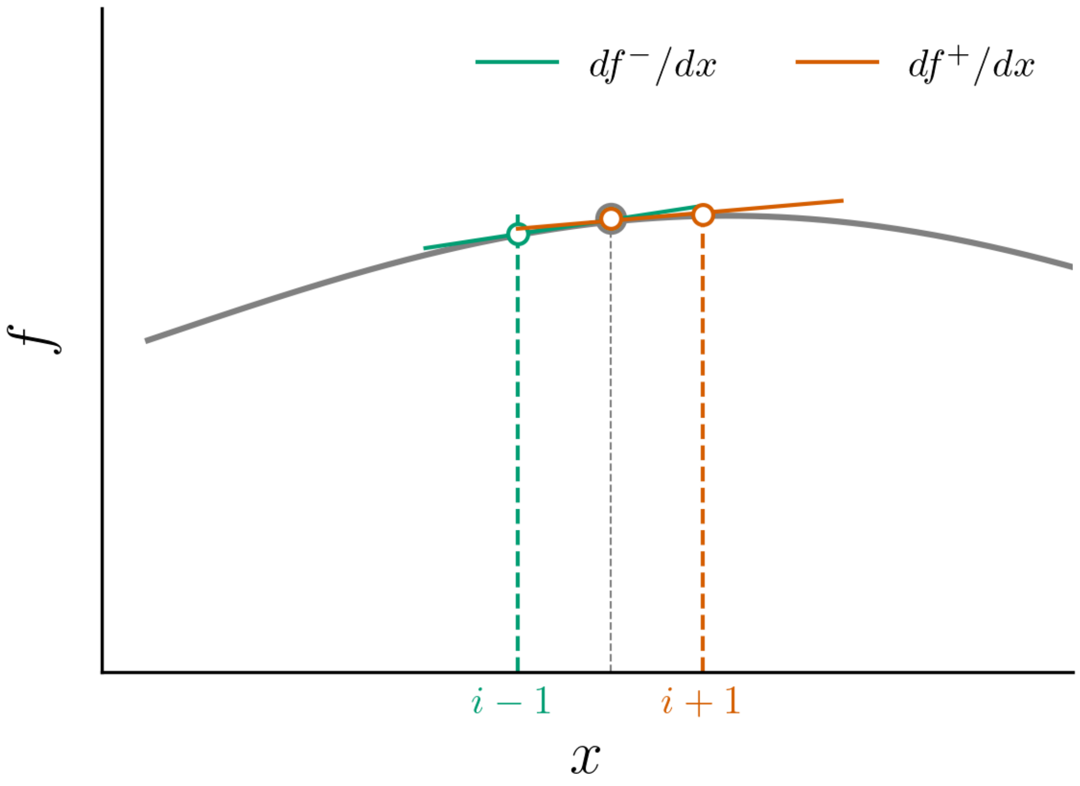
$$\frac{\left.\frac{df}{dx}\right|^{+} - \left.\frac{df}{dx}\right|^{-}}{dx} \approx \frac{d^2 f}{dx^2}$$



Second derivative

$$\frac{\left.\frac{df}{dx}\right|^{+} - \left.\frac{df}{dx}\right|^{-}}{dx} \approx \frac{d^2 f}{dx^2}$$

$$\frac{\frac{f(x+dx)-f(x)}{dx} - \frac{f(x+dx)-f(x)}{dx}}{dx} \approx \frac{d^2 f}{dx^2}$$



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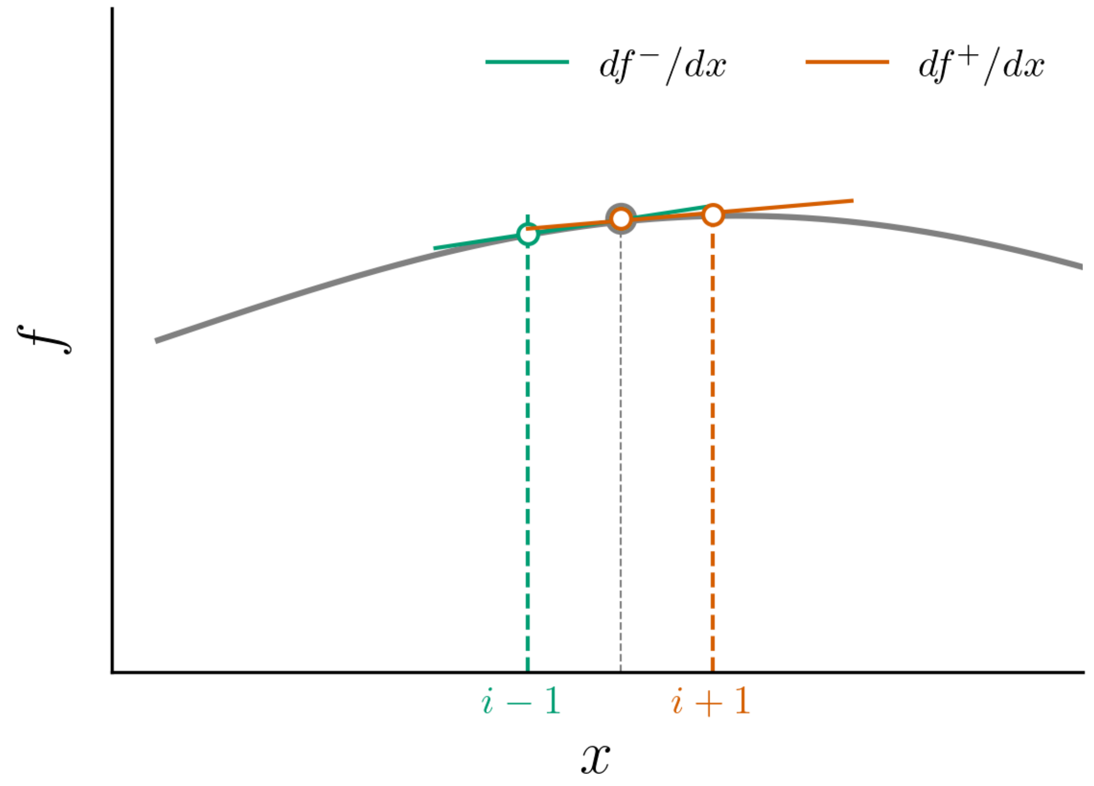
$$\frac{\frac{f(x+dx)-f(x)}{dx} - \frac{f(x+dx)-f(x)}{dx}}{dx} \approx \frac{d^2 f}{dx^2}$$

$$\frac{f(x+dx) - 2f(x) + f(x-dx)}{dx^2} \approx \frac{d^2 f}{dx^2}$$



Second derivative

$$\frac{\left.\frac{df}{dx}\right|^{+} - \left.\frac{df}{dx}\right|^{-}}{dx} \approx \frac{d^2 f}{dx^2}$$

$$\frac{\frac{f(x+dx)-f(x)}{dx} - \frac{f(x+dx)-f(x)}{dx}}{dx} \approx \frac{d^2 f}{dx^2}$$

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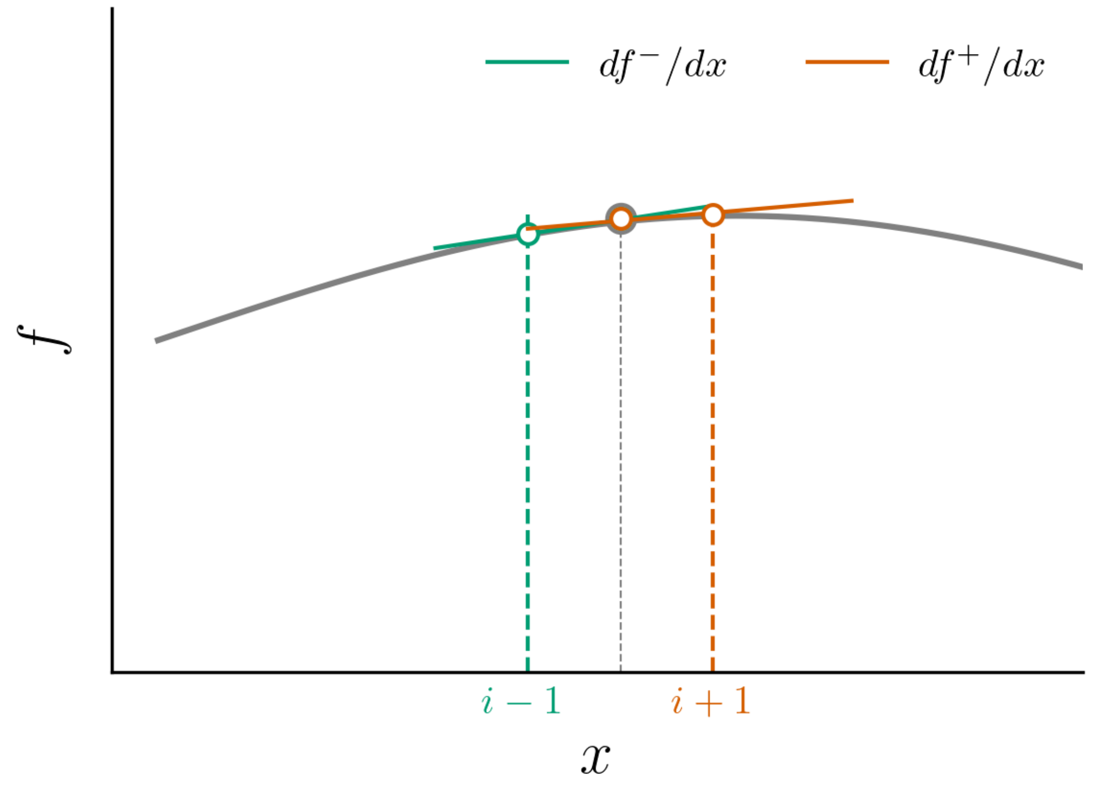

Central Differencing formula for 2nd derivative

Derivative Operators

Taylor Series

$$[f(x+dx)] = \left[f(x) + f'(x)dx + \frac{1}{2!}f''(x)dx^2 + \cdots\right]$$

$$[f(x)] = [f(x)]$$

$$[f(x-dx)] = \left[f(x) - f'(x)dx + \frac{1}{2!}f''(x)dx^2 + \cdots\right]$$

Derivative Operators

MODELAIR

Taylor Series

$$a[f(x+dx)] = a\left[f(x) + f'(x)dx + \frac{1}{2!}f''(x)dx^2 + \cdots\right]$$

$$b[f(x)] = b[f(x)]$$

$$c[f(x-dx)] = c\left[f(x) - f'(x)dx + \frac{1}{2!}f''(x)dx^2 + \cdots\right]$$

Derivative Operators

Taylor Series

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$$af(x+dx) + bf(x) + cf(x-dx) \approx (a+b+c)f(x) + (a-c)f'(x)dx + \frac{(a+c)}{2}f''(x)dx^2$$

Derivative Operators

$$af(x+dx)+bf(x)+cf(x-dx)\approx(a+b+c)f(x)+(a-c)f'(x)dx+\frac{(a+c)}{2}f''(x)dx^2$$

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1^{st} Derivative

$$(a+b+c) = 0$$

$$(a \quad -c) = 1/dx$$

$$(a \quad +c) = 0$$

Derivative Operators

$$af(x+dx) + bf(x) + cf(x-dx) \approx (a+b+c)f(x) + (a-c)f'(x)dx + \frac{(a+c)}{2}f''(x)dx^2$$

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$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}} \underset{w}{\begin{pmatrix} a \\ b \\ c \end{pmatrix}} = \underset{s}{\begin{pmatrix} 0 \\ 1/dx \\ 0 \end{pmatrix}}$$

Derivative Operators

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$$a = \frac{1}{2dx}$$
$$b = 0$$
$$c = \frac{-1}{2dx}$$

Derivative Operators

$$af(x+dx)+bf(x)+cf(x-dx)\approx(a+b+c)f(x)+(a-c)f'(x)dx+\frac{(a+c)}{2}f''(x)dx^2$$

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2nd Derivative

$$(a+b+c)=0$$
$$(a\quad -c)=0$$
$$(a\quad +c)=2!/dx^2$$

$$\begin{array}{ccccc} A & & w & & s \\ \begin{bmatrix}1 & 1 & 1\\ 1 & 0 & -1\\ 1 & 0 & 1\end{bmatrix} & & \begin{pmatrix}a\\ b\\ c\end{pmatrix} & = & \begin{pmatrix}0\\ 0\\ 2/dx^2\end{pmatrix}\end{array}\xrightarrow{\text{Inversion}}\quad a=\frac{1}{dx^2},\quad b=\frac{-2}{dx^2},\quad c=\frac{1}{dx^2}$$

Finite Difference in action

1D Linear Convection

$$\frac{du}{dt} + c\frac{du}{dx} = 0$$

Finite Difference in action

1D Linear Convection $$\frac{du}{dt} + c\frac{du}{dx} = 0$$

Initial Condition: $$u(x, 0) = u_0(x)$$

Exact Solution: $$u(x, t) = u_0(x - ct)$$

Finite Difference in action

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With an Initial Condition (given as a wave), the equation represents propagation of a wave with a speed *c*, *without change in its shape*!

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Backward Difference

$$\frac{du}{dx} \approx \frac{u(x) - u(x - \Delta x)}{\Delta x}$$

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1D Linear Convection

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$$\frac{du}{dx} \approx \frac{u(x) - u(x - \Delta x)}{\Delta x}$$

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$$\frac{du}{dt} \approx \frac{u^{n+1}(x) - u^{n}(x)}{\Delta t}$$

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Finite Difference in action

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Let's write our 1st CFD code !

Barba et al., (2018). CFD Python: the 12 steps to Navier-Stokes equations. Journal of Open-Source Education, 1(9), 21. https://doi.org/10. 21105/jose.00021

1st Code: 1D Linear Convection

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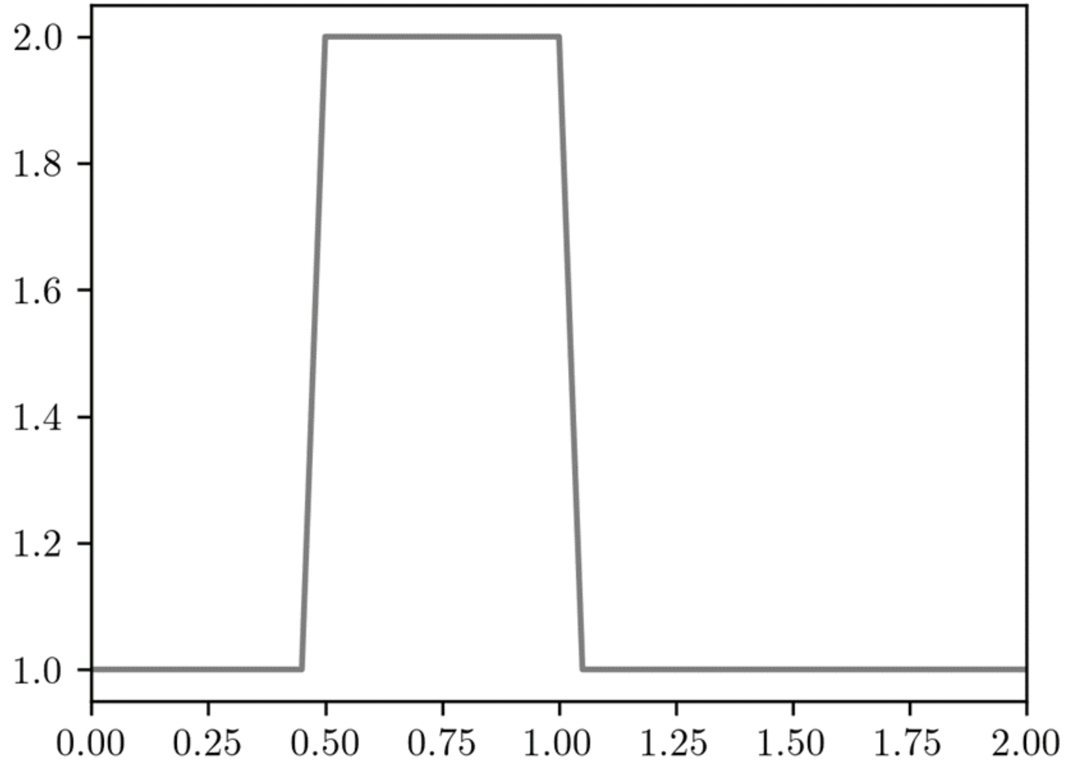
The initial velocity u_0 is given as 2 in the interval $0.5 \leq x \leq 1$ and 1 elsewhere in $(0,2) \Longrightarrow$ Hat function



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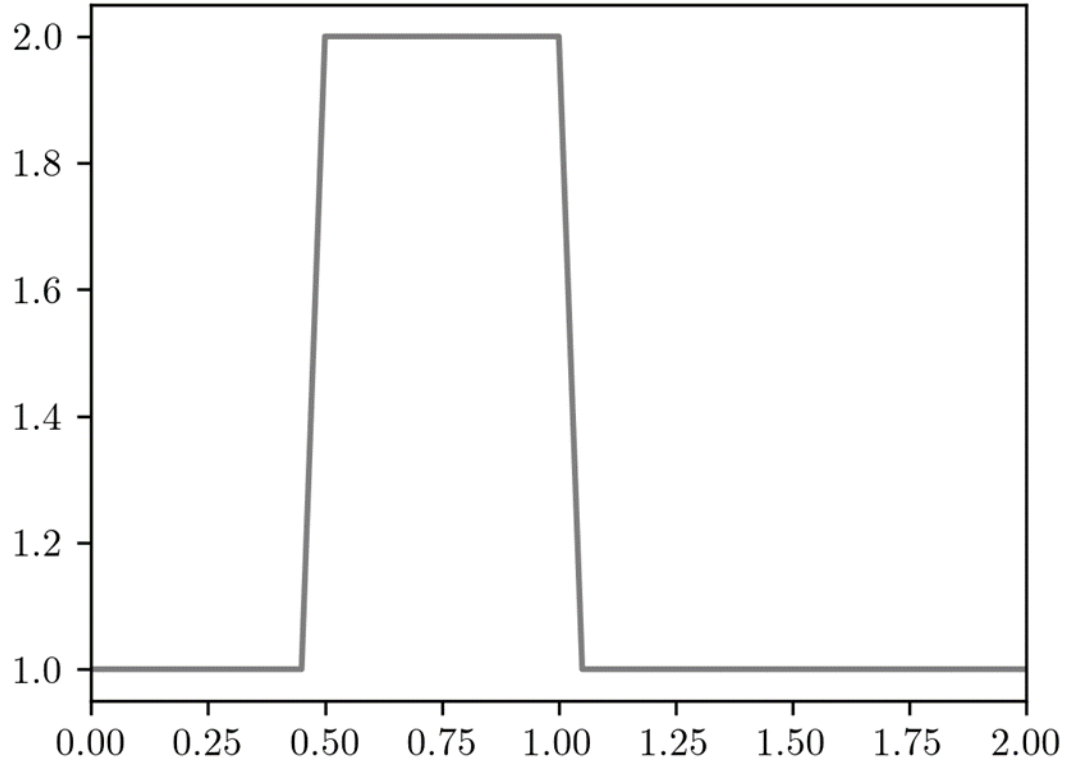

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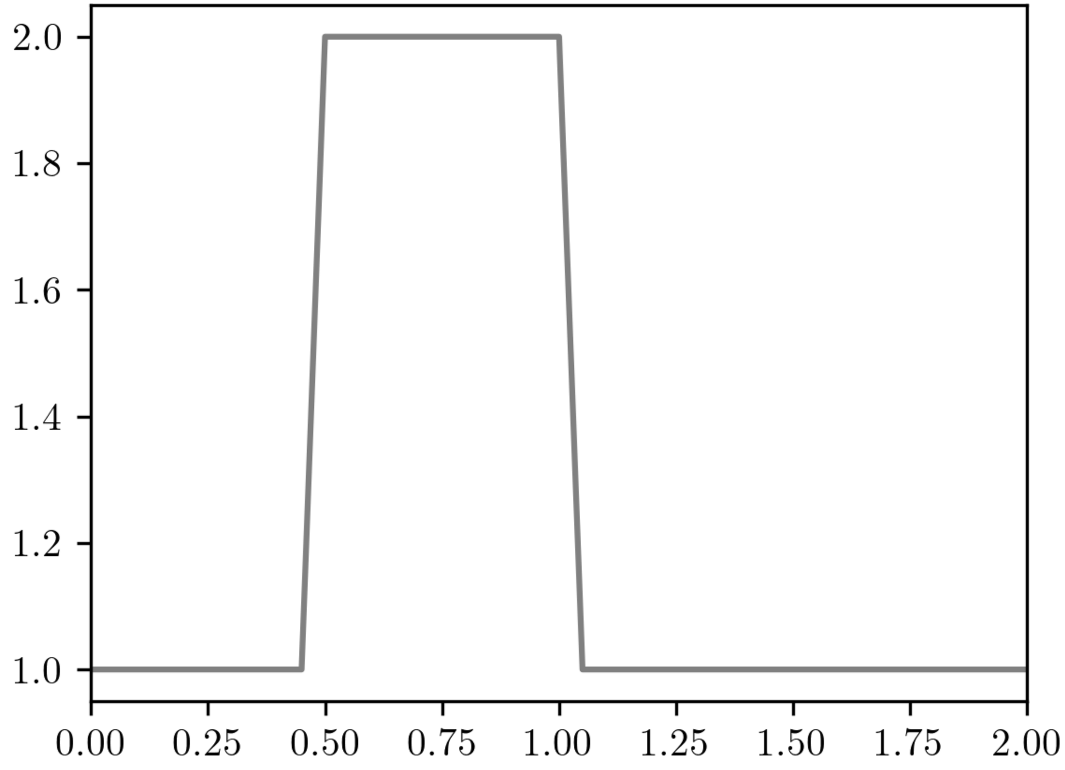

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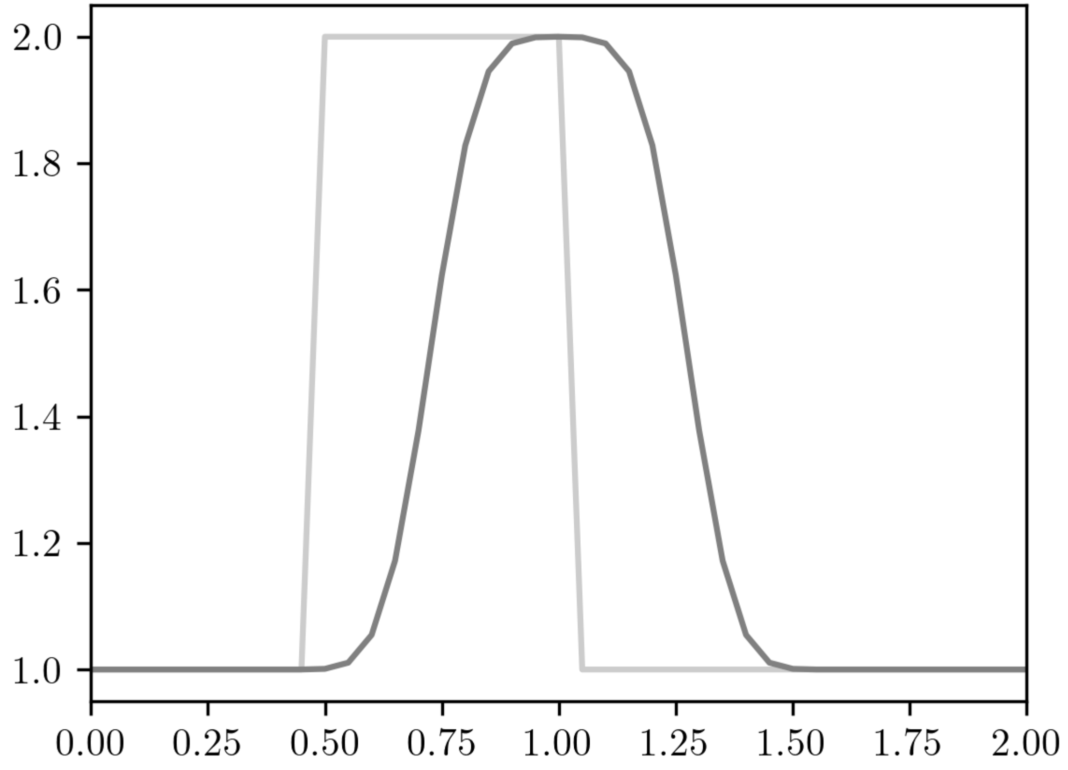

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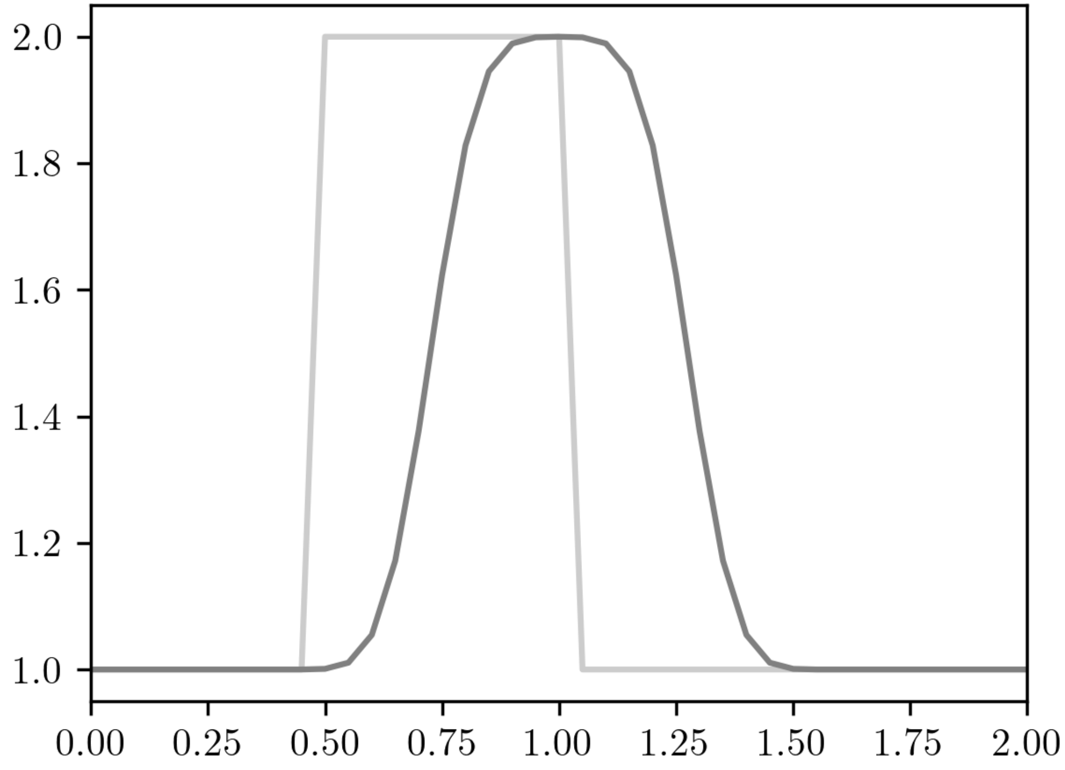

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Let's try with central differencing !

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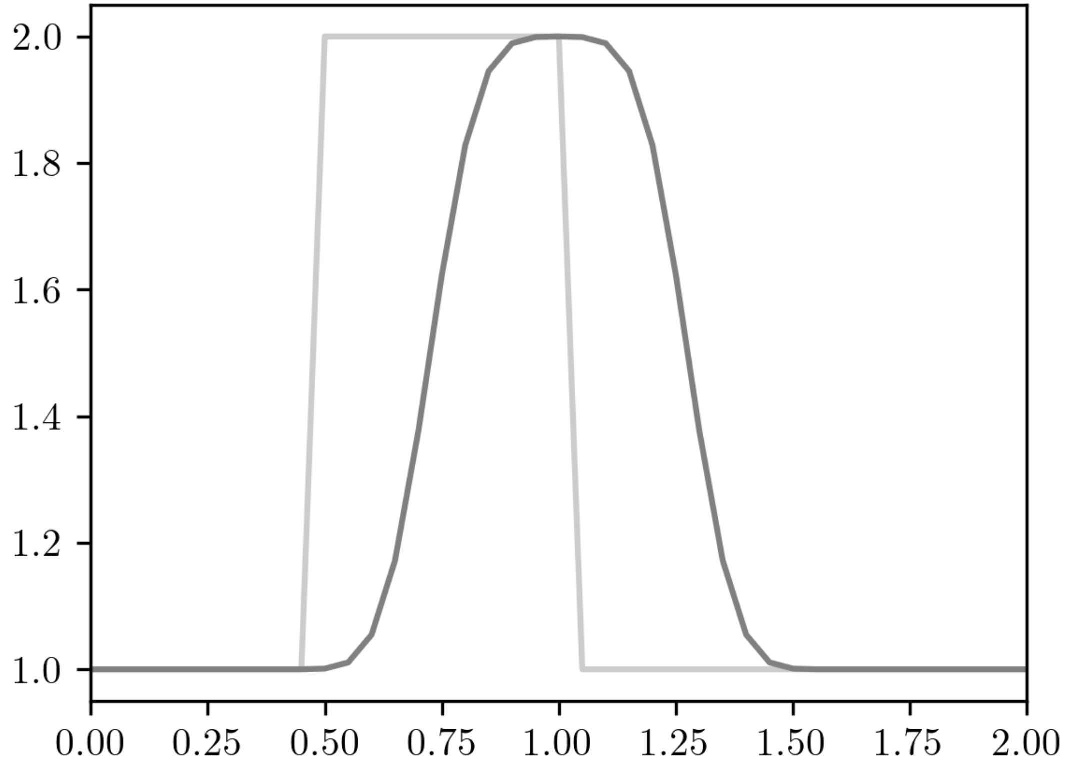

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Change these !

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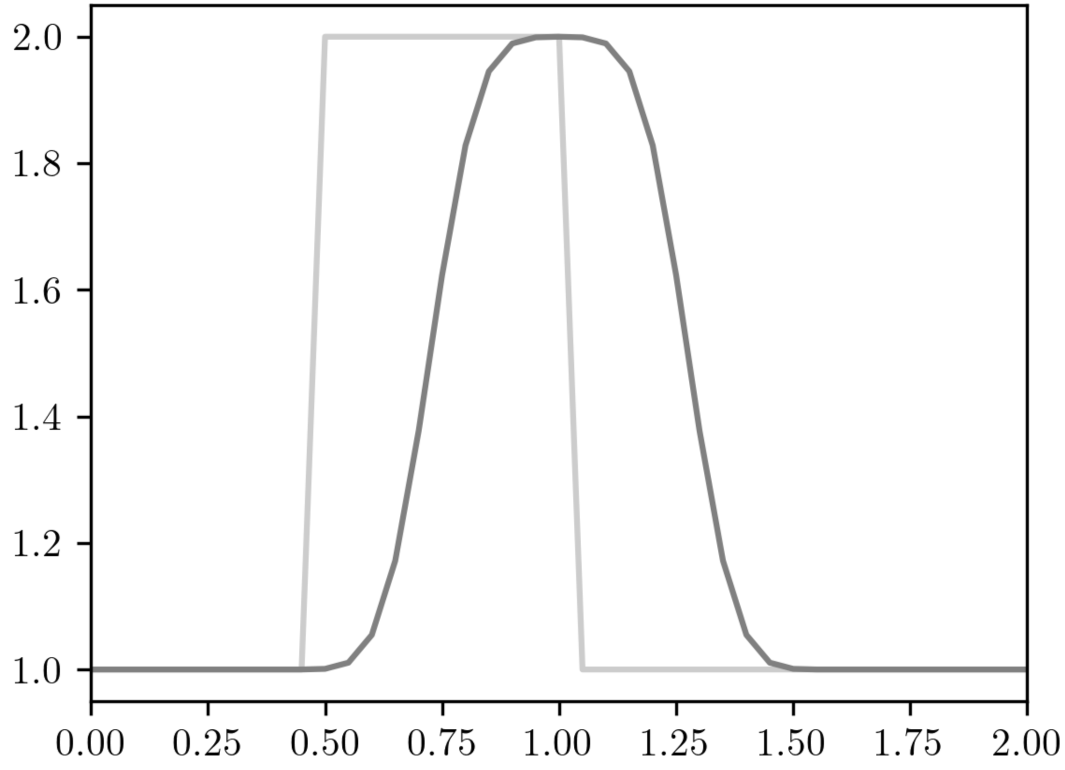

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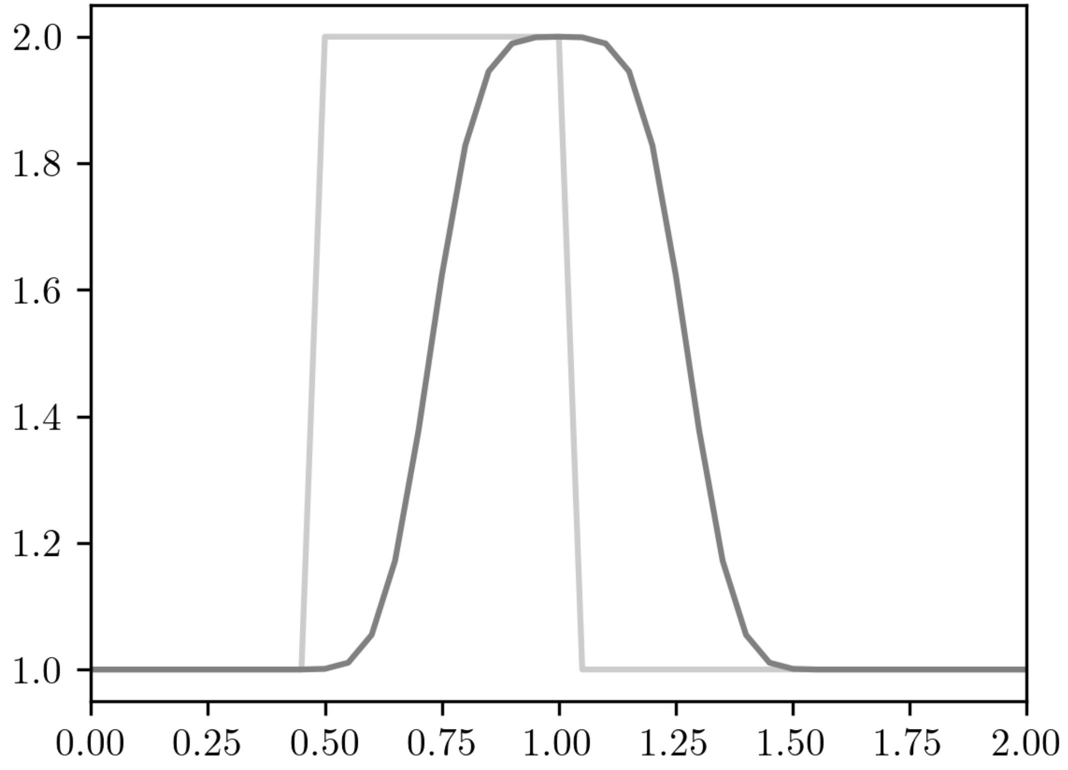

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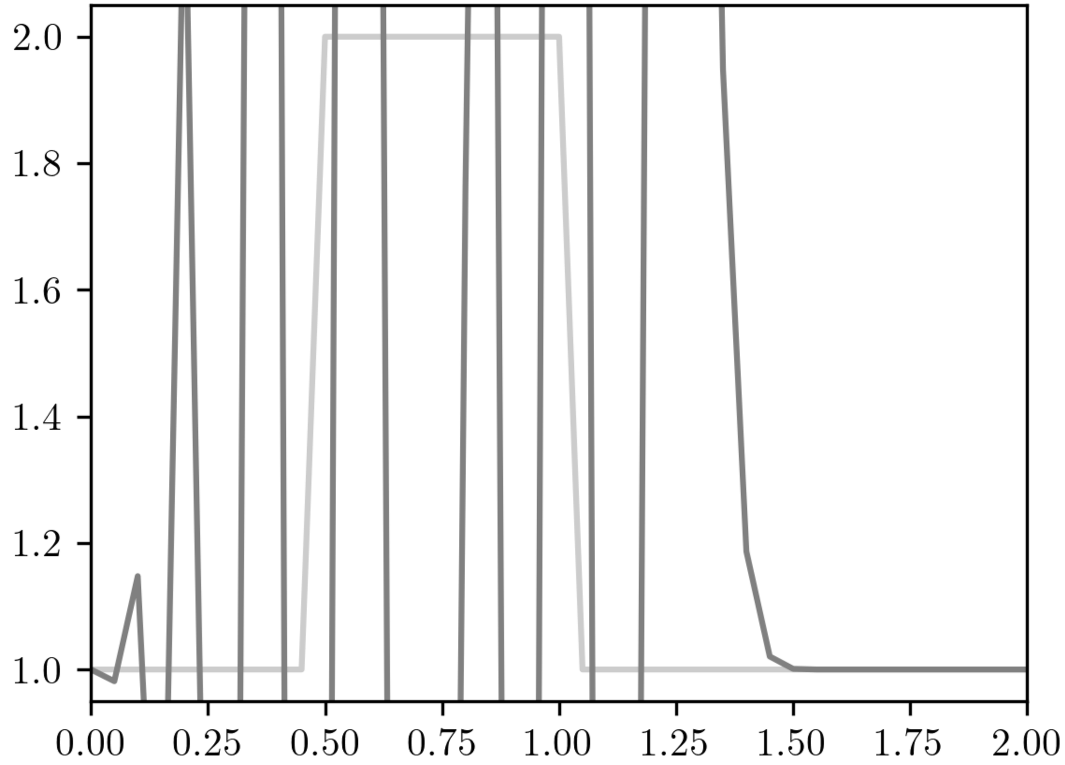

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What's that !

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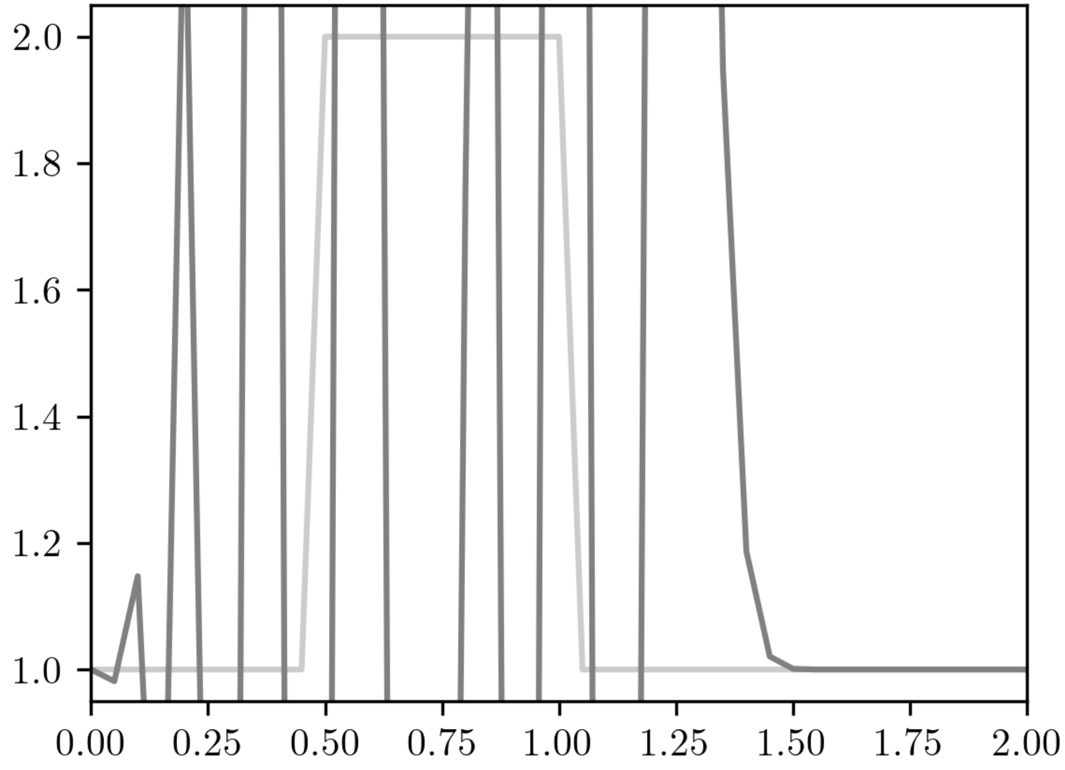

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1D Linear Convection with Central Differencing in Space

$$\frac{du}{dt} + c\frac{du}{dx} = 0 \longrightarrow \frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{2\Delta \mathrm{x}}(u_{i+1}^n - u_{i-1}^n) = 0$$

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1D Linear Convection with Central Differencing in Space

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$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) - \left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right)_i^n = \frac{\Delta t}{2!}\left.\frac{\partial^2 u}{\partial t^2}\right|_i^n + \frac{c\Delta x^3}{3!}\left.\frac{\partial^3 u}{\partial t^3}\right|_i^n + O(\Delta t^2, \Delta x^4)$$

1D Linear Convection with Central Differencing in Space

$$\frac{du}{dt} + c\frac{du}{dx} = 0 \longrightarrow \frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) = 0$$

$$u_i^{n+1} = u_i^n + \Delta t \frac{\partial u}{\partial t}\bigg|_i^n + \frac{\Delta t^2}{2!}\frac{\partial^2 u}{\partial t^2}\bigg|_i^n + \frac{\Delta t^3}{3!}\frac{\partial^3 u}{\partial t^3}\bigg|_i^n + \cdots$$

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Truncation Error (ϵ_t)

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Truncation Error (ϵ_t)

As Δt & $\Delta x \longrightarrow 0$: $\epsilon_T \longrightarrow 0 \Rightarrow$ Numerical Scheme is Consistent

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Modified Differential Equation !

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Modified Differential Equation !

NOT a convection equation, It is a Convection-Diffusion equation !!

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-ve Diffusion Coefficient !!

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Explosion : Unstable Scheme !!

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1D Linear Convection with Backward Differencing in Space

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Diffusion Coefficient must be +ve !!

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For stability, we need: $0 \leq \frac{c\Delta t}{\Delta x} \leq 1$

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Courant-Friedrich-Lewy (CFL) Number!!

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Poor Accuracy !!

2^{nd} Code: 1D non-Linear Convection

```
import numpy as np
import matplotlib.pyplot as plt
import time, sys

nx = 41
dx = 2/(nx-1)
nt = 25
dt = 0.025
c = 1

u = np.ones(nx)
u[int(0.5/dx):int(1/dx+1)] = 2
print(u)
plt.plot(np.linspace(0,2,nx), u)

un = np.ones(nx)
for n in range(nt):
    un=u.copy()
    for i in range(1,nx):
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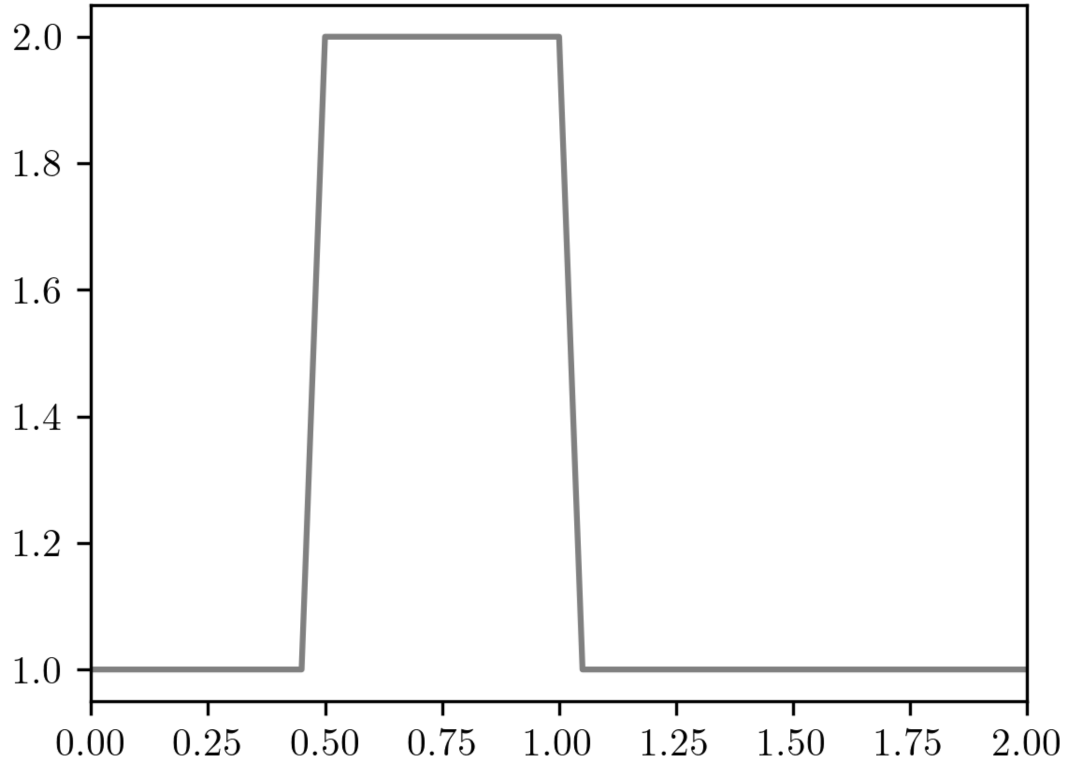
```

The initial velocity u_0 is given as 2 in the interval $0.5 \leq x \leq 1$ and 1 elsewhere in $(0,2)$ $\Longrightarrow$ Hat function



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Change this only !

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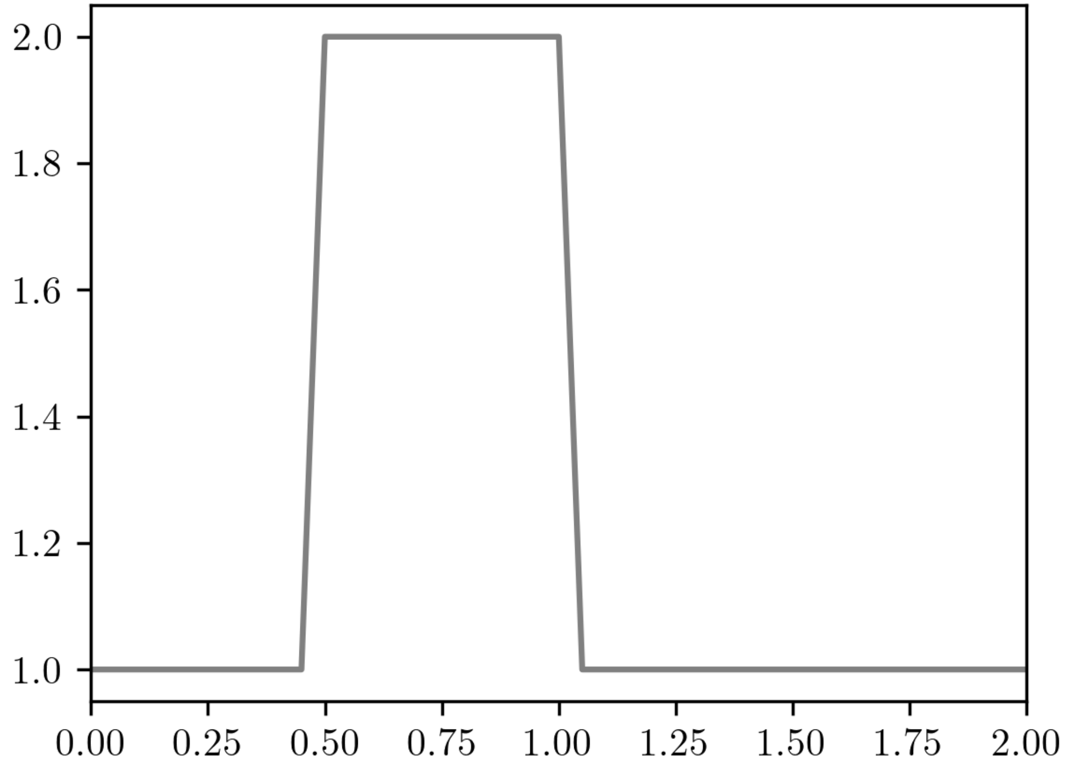

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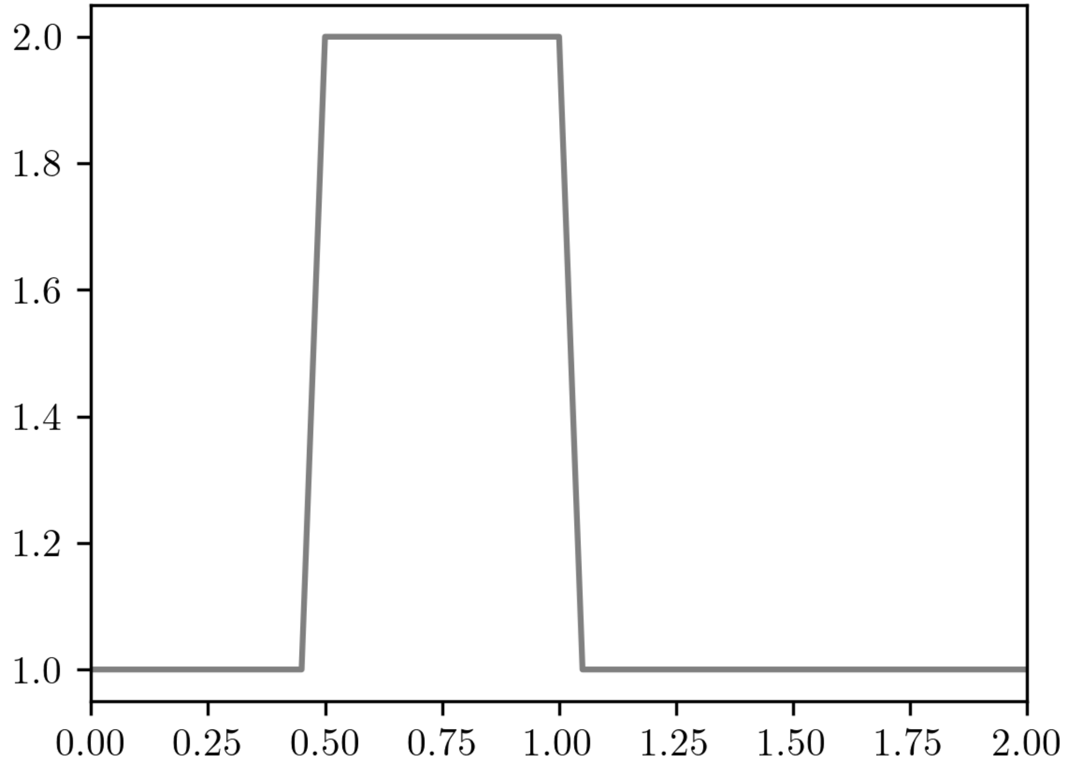

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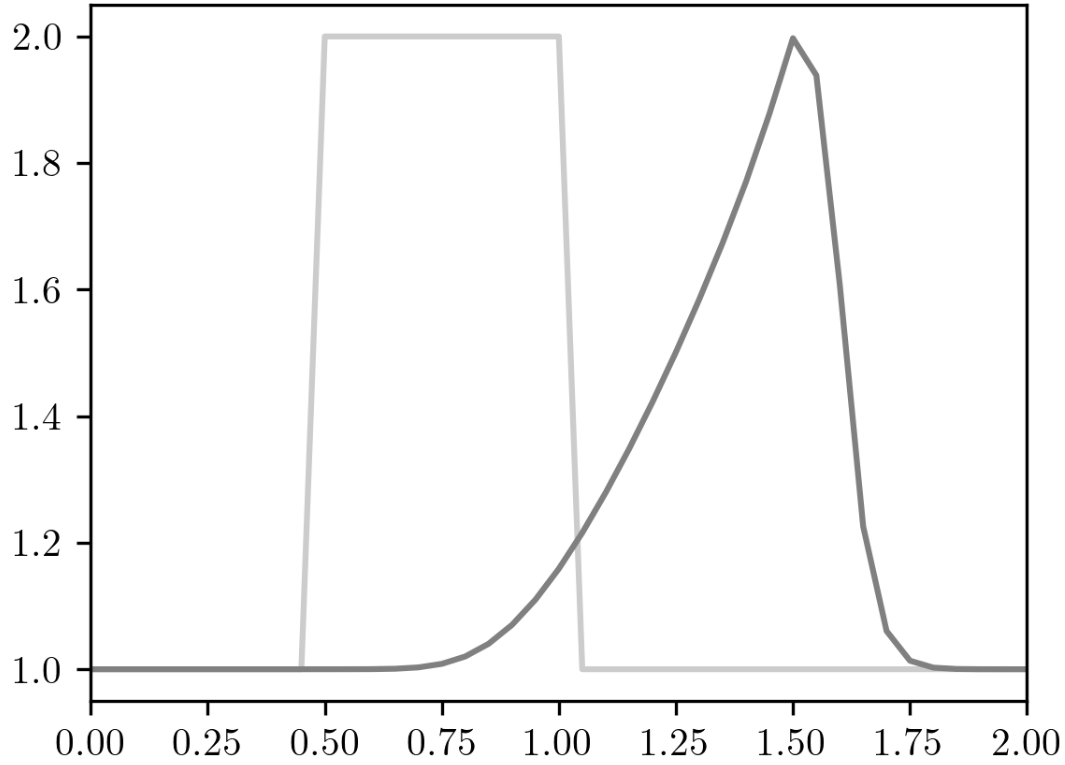
The initial velocity u_0 is given as 2 in the interval $0.5 \leq x \leq 1$ and 1 elsewhere in $(0,2)$ $\Longrightarrow$ Hat function



2^{nd} Code: 1D non-Linear Convection

MODELAIR

```
import numpy as np
import matplotlib.pyplot as plt
import time, sys

nx = 41
dx = 2/(nx-1)
nt = 25
dt = 0.025
c = 1

u = np.ones(nx)
u[int(0.5/dx):int(1/dx+1)] = 2
print(u)
plt.plot(np.linspace(0,2,nx), u)

un = np.ones(nx)
for n in range(nt):
    un=u.copy()
    for i in range(1,nx):
        u[i] = un[i] – un[i] * dt/dx * (un[i]-un[i-1])
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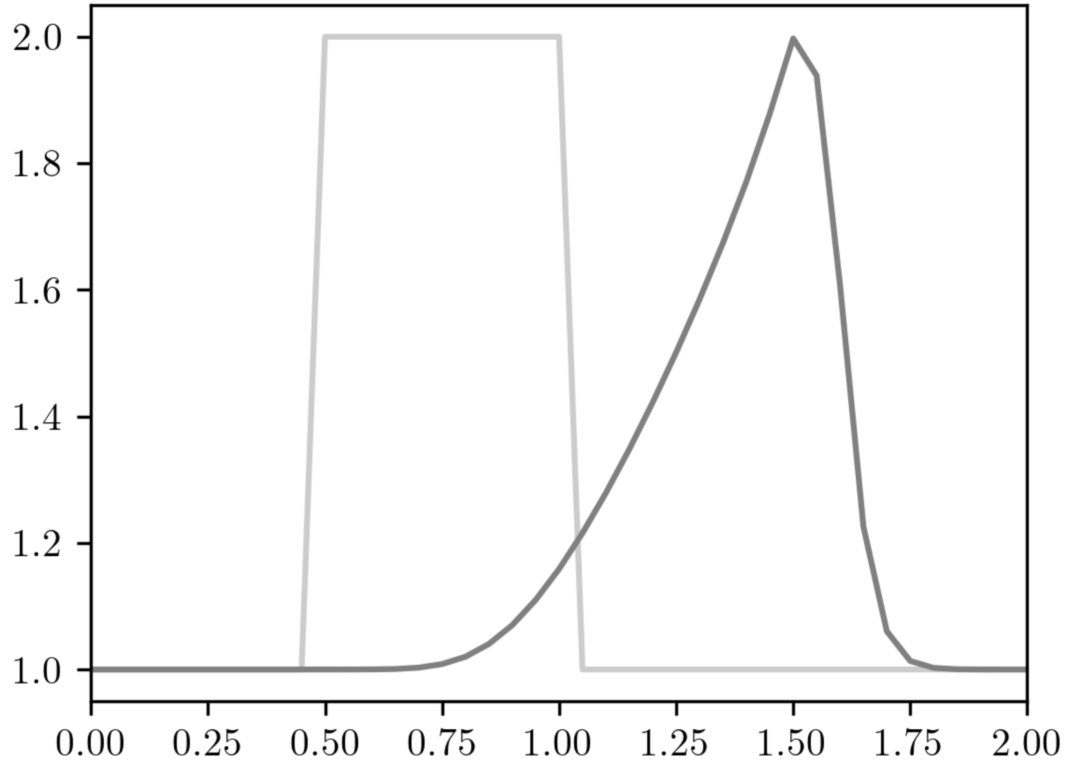
```

Let's introduce CFL

The initial velocity u_0 is given as 2 in the interval $0.5 \leq x \leq 1$ and 1 elsewhere in $(0,2)$ $\Longrightarrow$ Hat function



2^{nd} Code: 1D non-Linear Convection

```
import numpy as np
import matplotlib.pyplot as plt
import time, sys

nx = 41
dx = 2/(nx-1)
nt = 25
CFL = 0.9

u = np.ones(nx)
u[int(0.5/dx):int(1/dx+1)] = 2
print(u)
plt.plot(np.linspace(0,2,nx), u)

dt = CFL*dx/max(abs(u))

un = np.ones(nx)
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    for i in range(1,nx):
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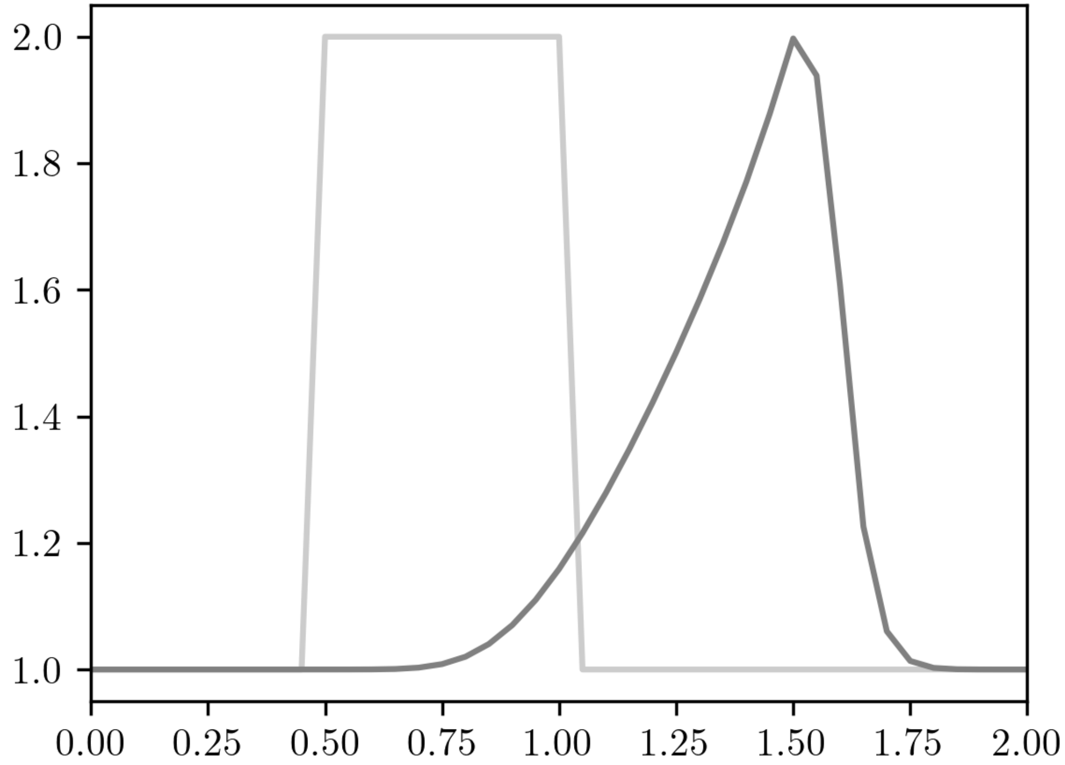

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Let's introduce CFL

Play a bit & see what happens!

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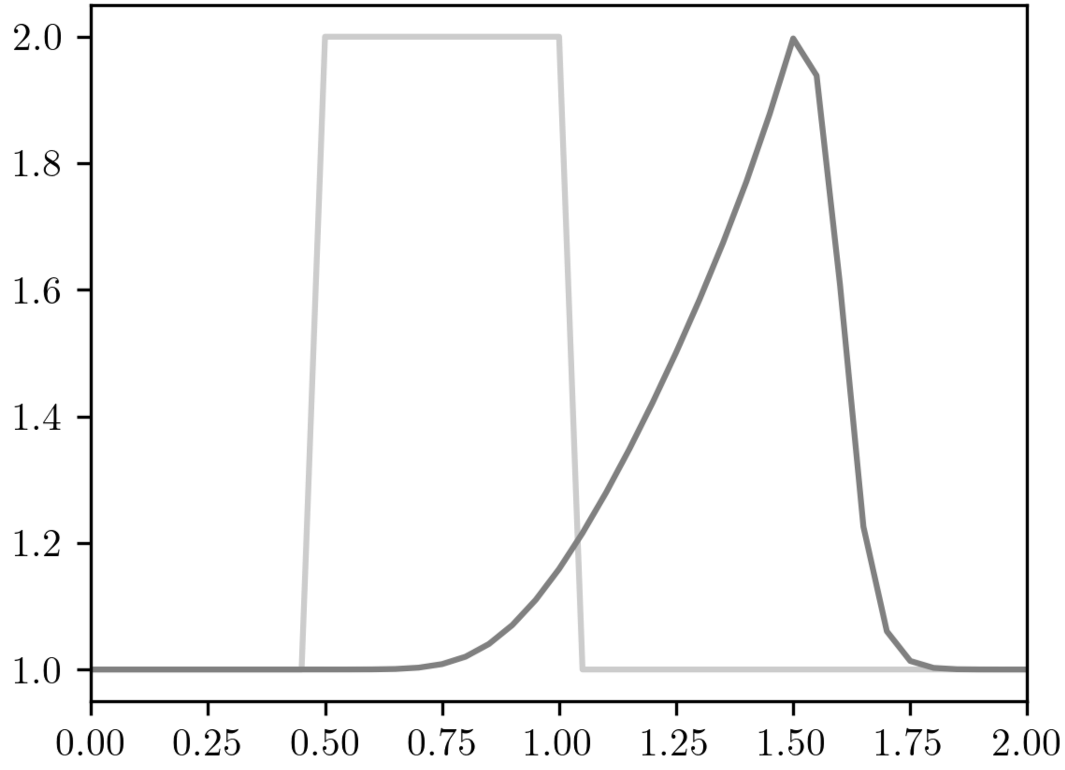

3rd Code: 1D Diffusion

$$\frac{du}{dt} = \nu \frac{d^2u}{dx^2}$$

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Exponential Damping for $\nu > 0$

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Diffusion is isotropic in nature

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Central-Differencing

3rd Code: 1D Diffusion

$$\frac{du}{dt} = \nu \frac{d^2u}{dx^2}$$

Central Difference $$\frac{d^2u}{dx^2} \approx \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{\Delta x^2}$$

Forward Euler $$\frac{du}{dt} \approx \frac{u^{n+1}(x) - u^n(x)}{\Delta t}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

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Let's Code It !

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dt = sigma * dx**2 / nu

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u[int(0.5/dx):int(1/dx+1)] = 2
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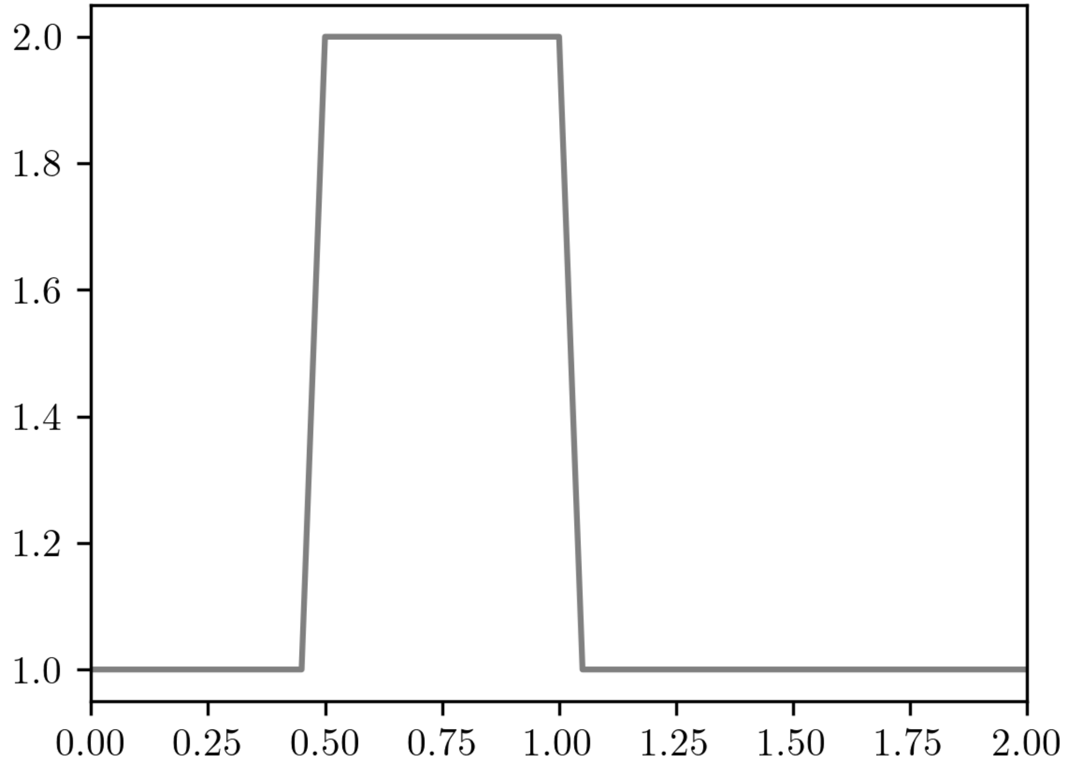

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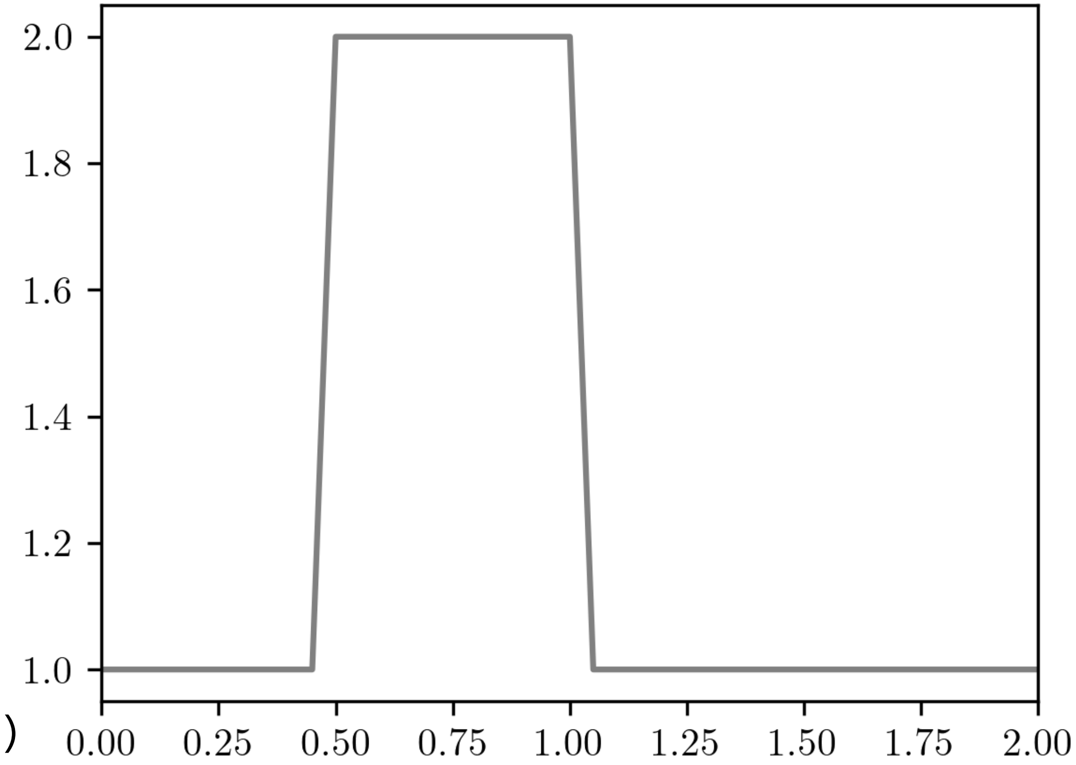

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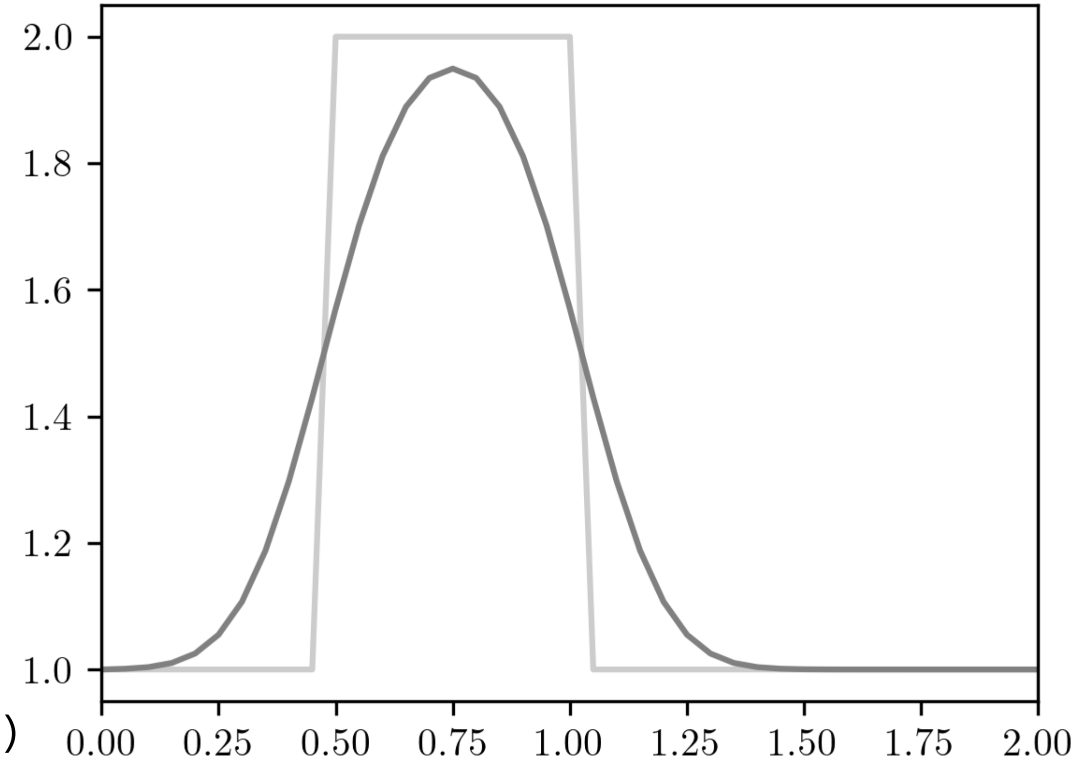

1D Diffusion (Caution!)

$$\frac{du}{dt} = \nu \frac{d^2u}{dx^2}$$

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Parabolic PDE

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Characteristic lines are lines of constant 't'

All information at a given time should affect the solution

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Our Numerical Formulation:

$$u_i^{n+1} = u_i^n + \frac{\nu \Delta t}{\Delta x^2}(u_{i-1}^n - 2u_i^n + u_{i-1}^n)$$

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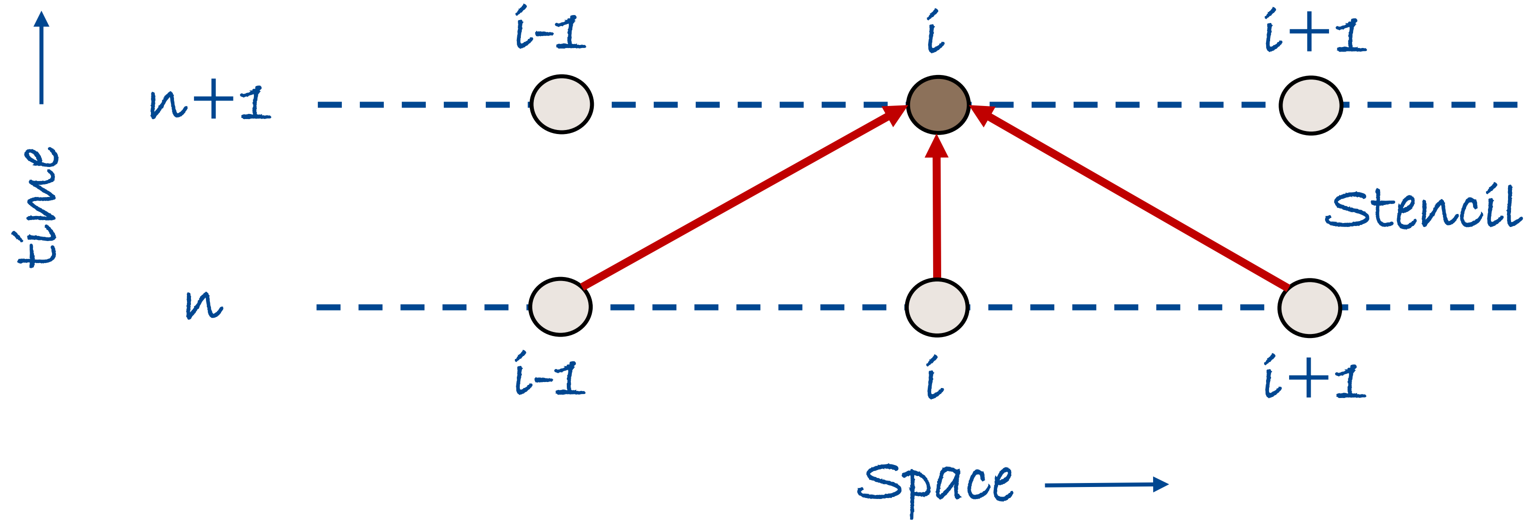

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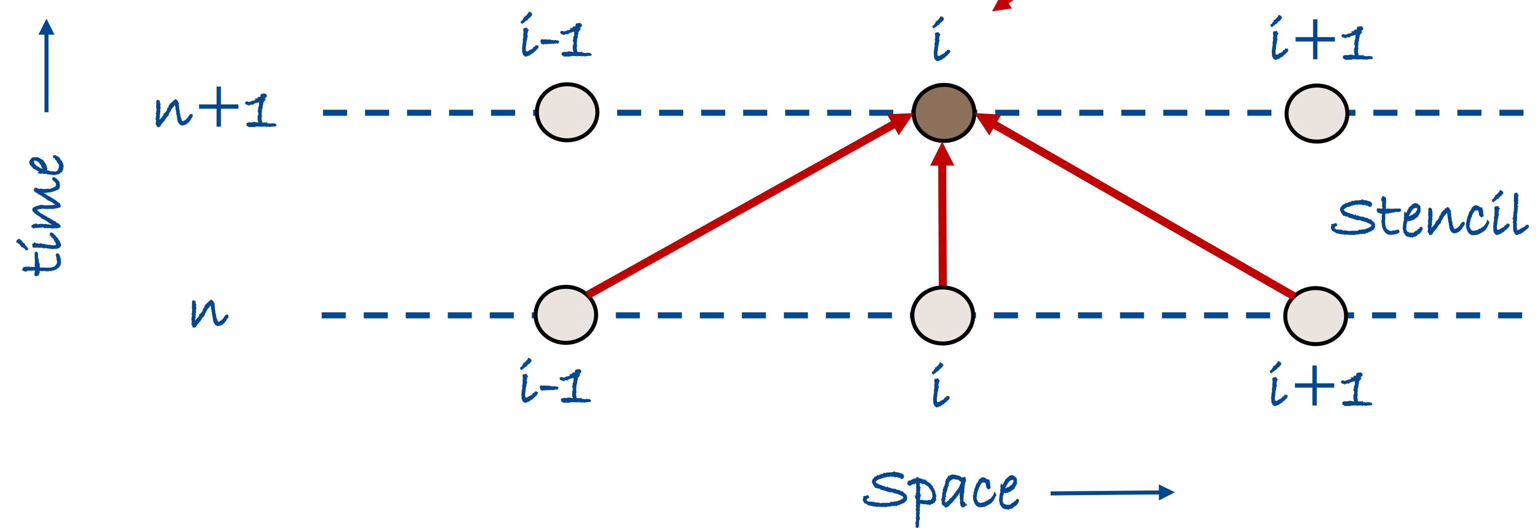

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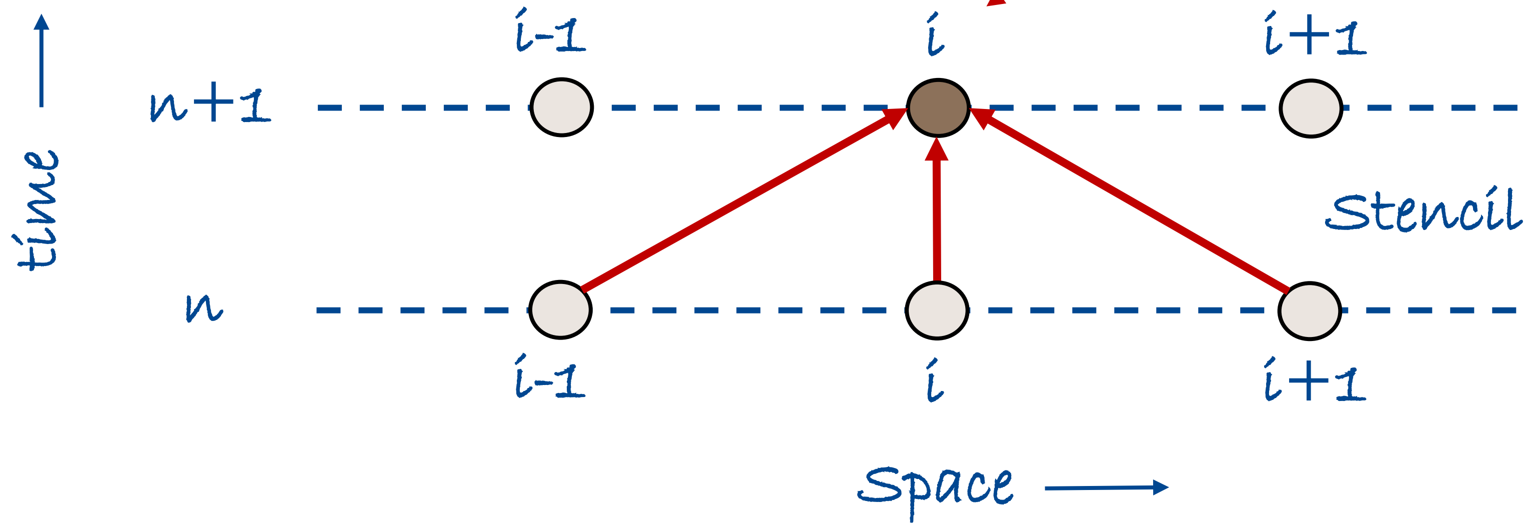
"Explicit Formulation"



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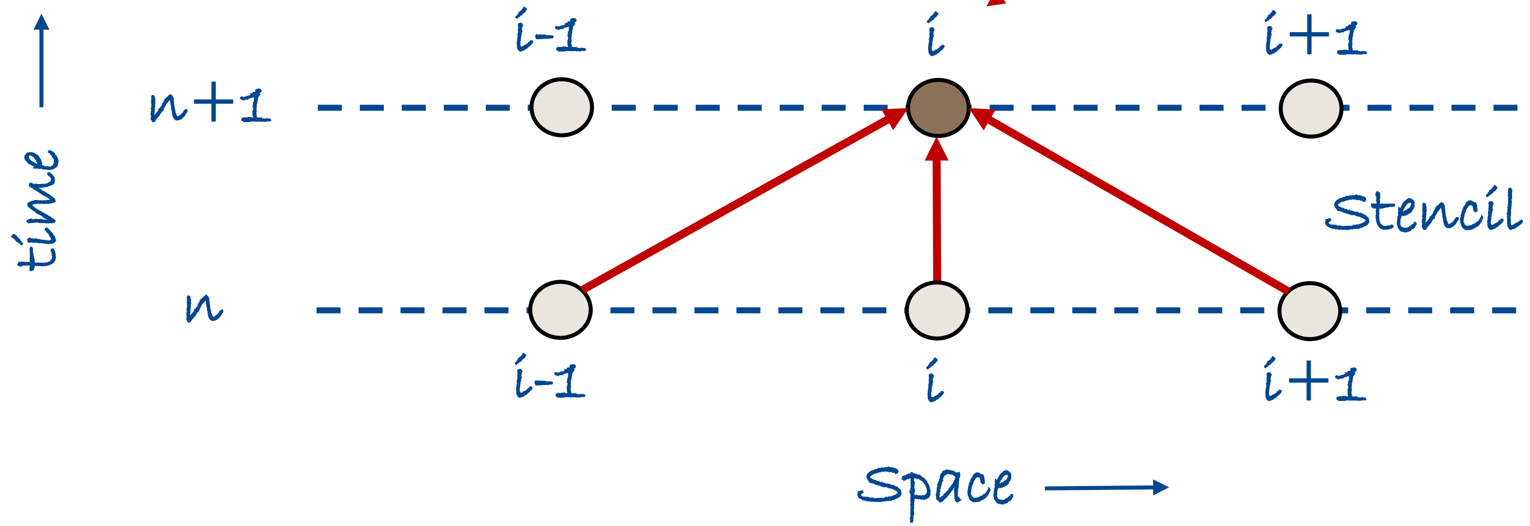

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Problem if Boundary Conditions are time-dependent!

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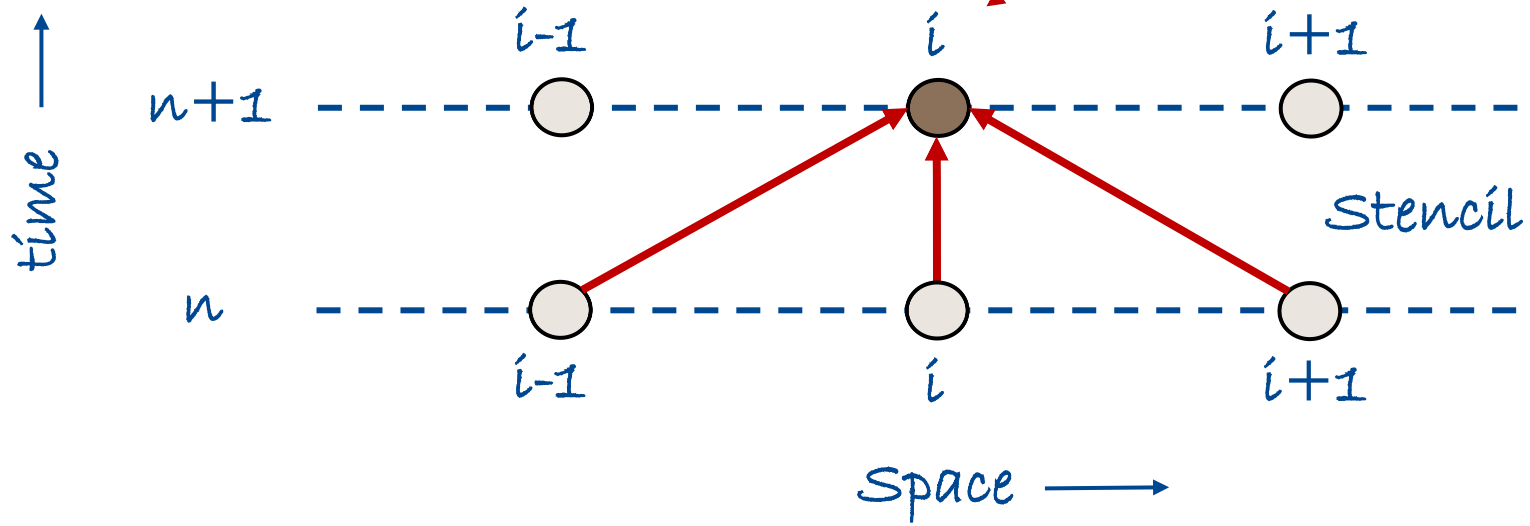

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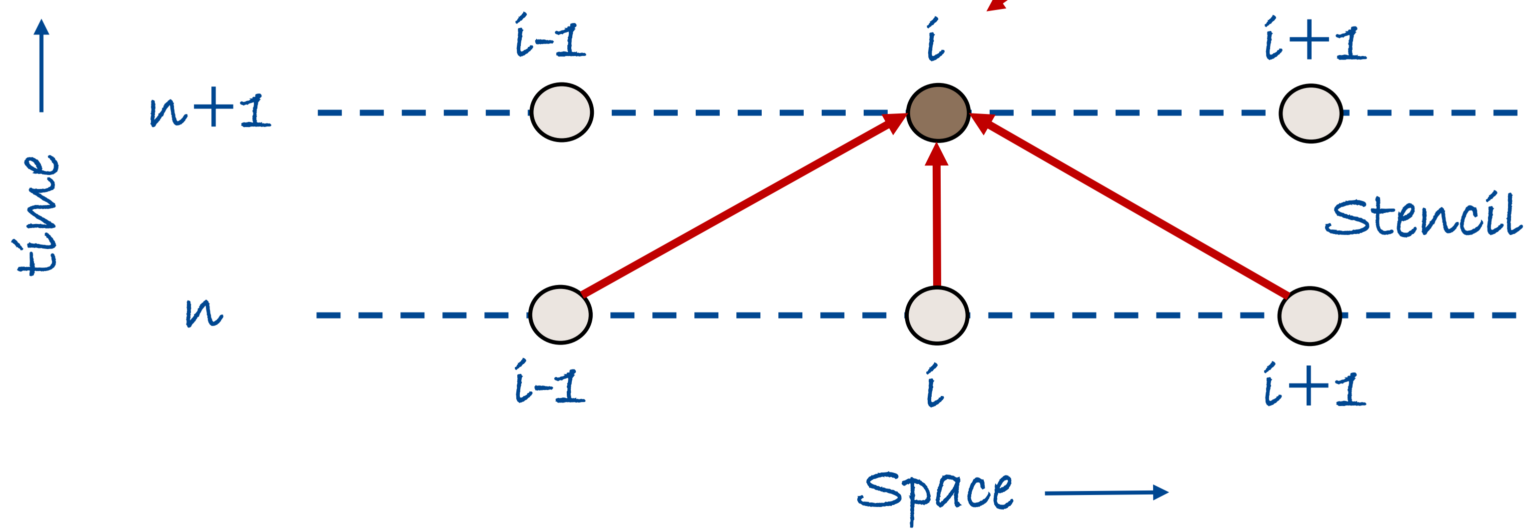

How can we include BCs at current time?

1D Diffusion (Caution!)

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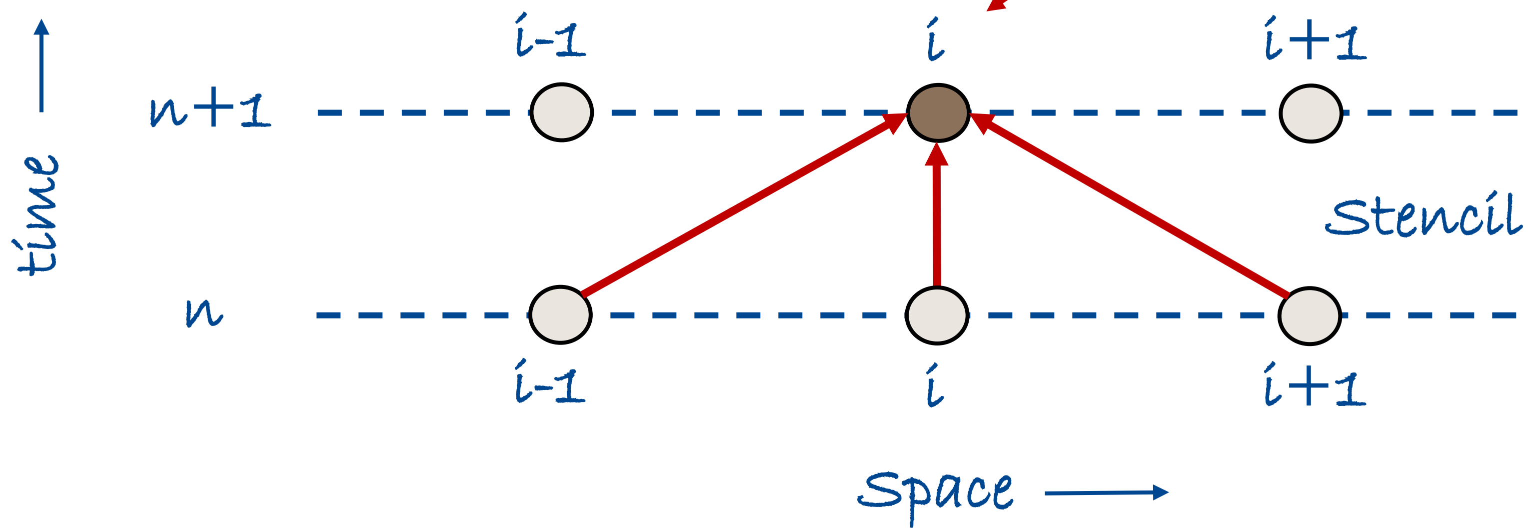

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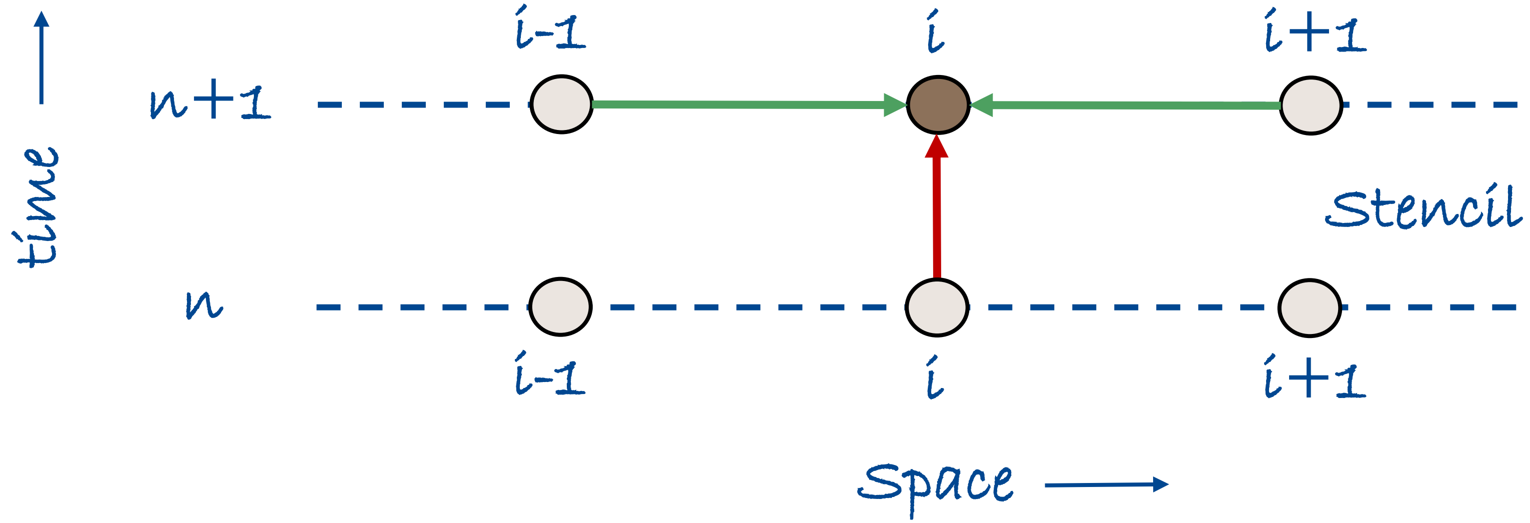

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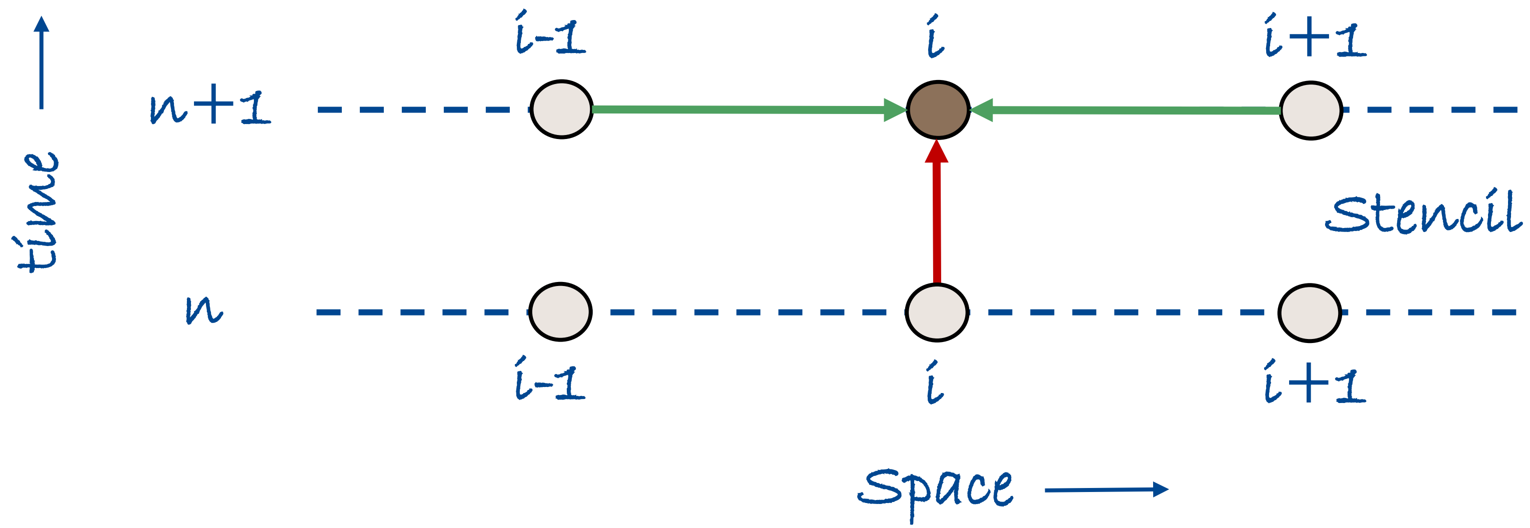

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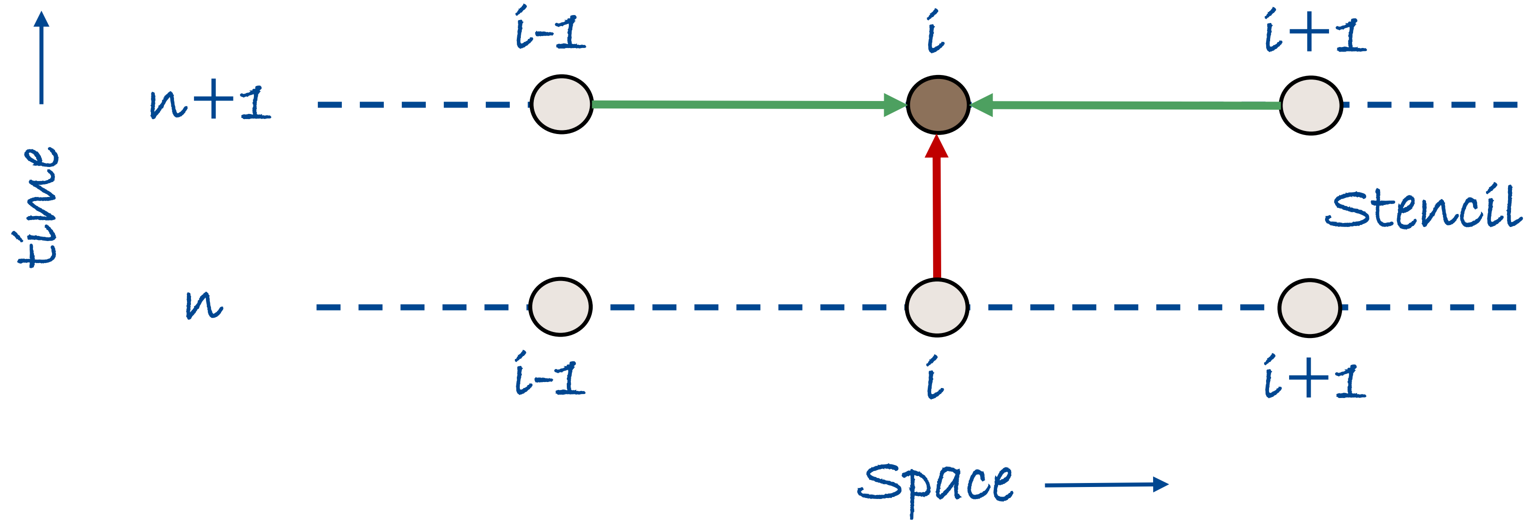
Need Algorithms to solve such a sparse matrix system!



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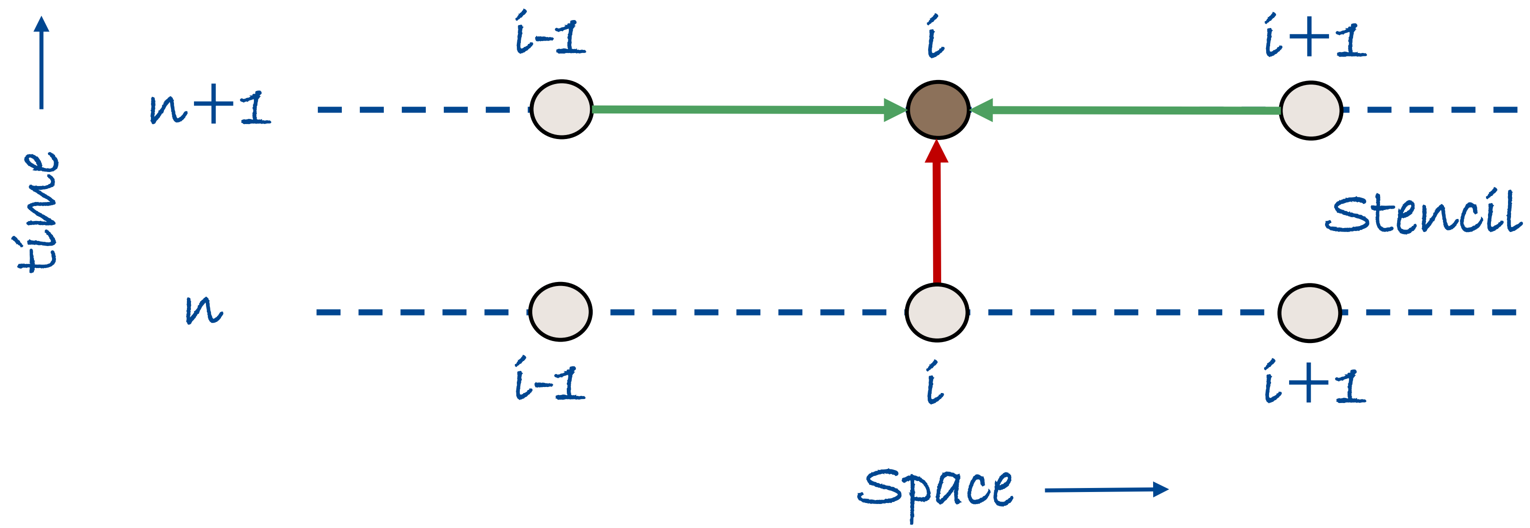

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Direct or Iterative Methods

1D Diffusion (Implicit Formulation)

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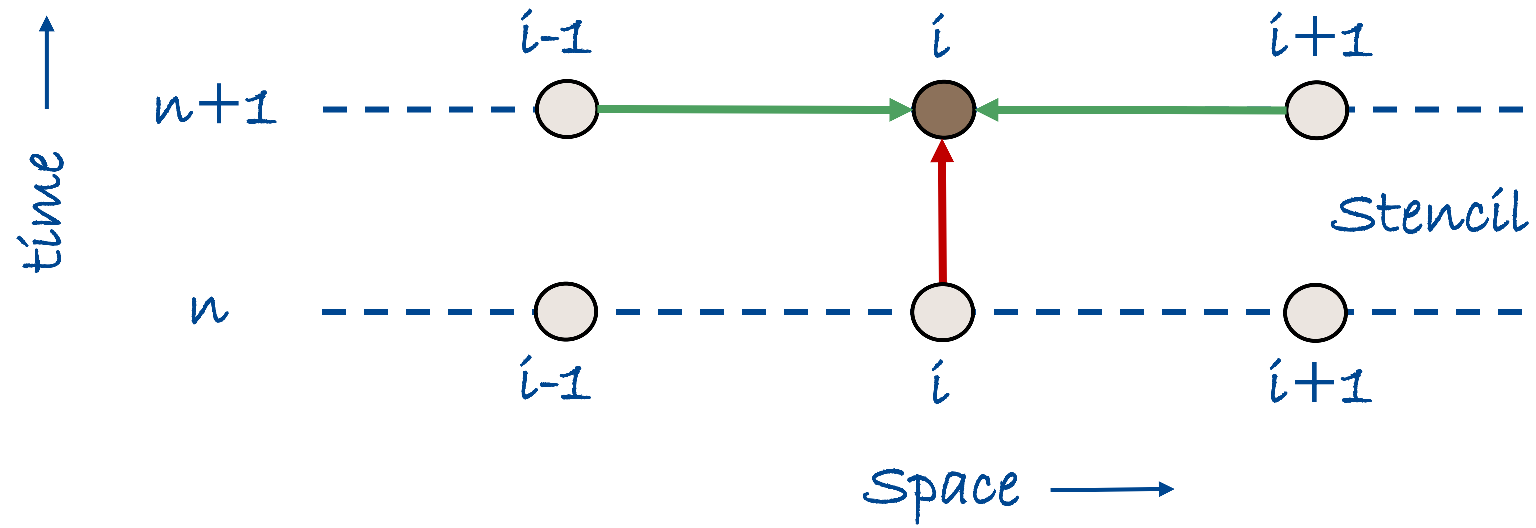

Need Algorithms to solve such a sparse matrix system!

Direct or Iterative Methods

Can be Expensive!

Crank-Nicholson Method

Well-known for Parabolic PDEs

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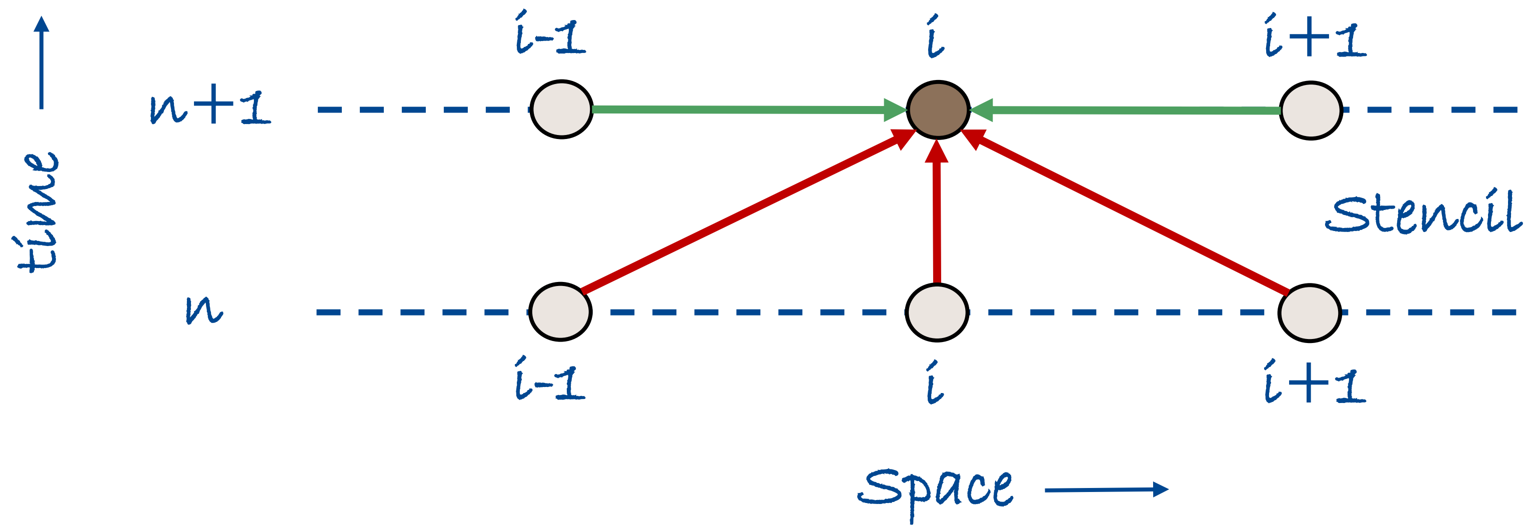

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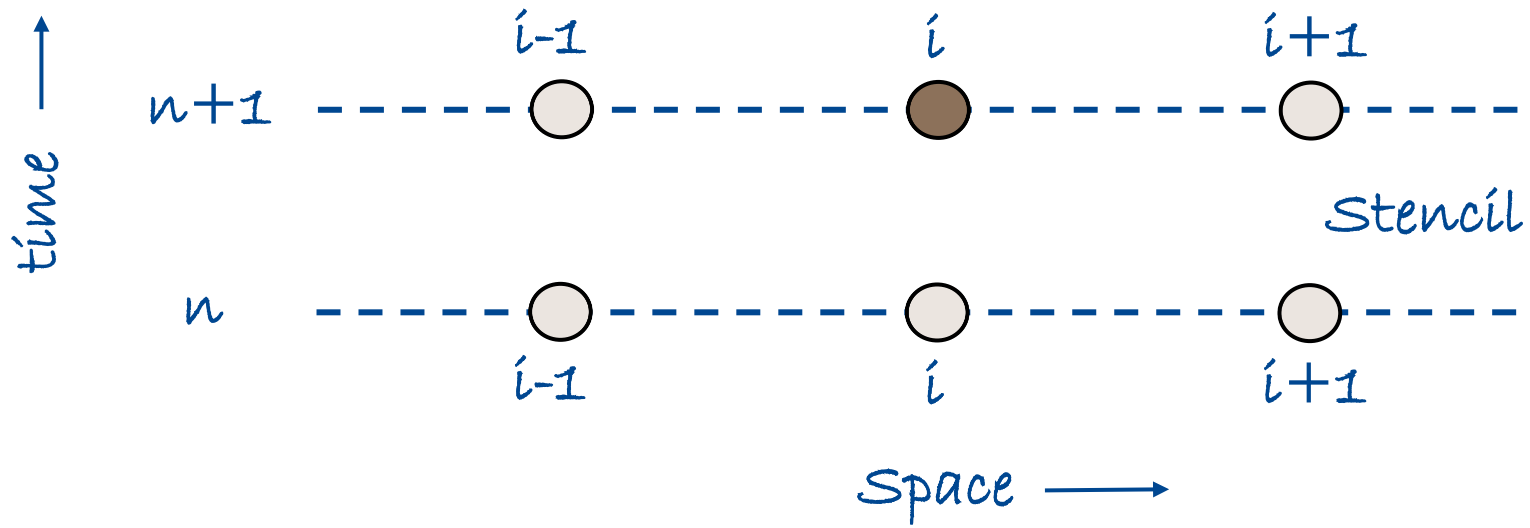

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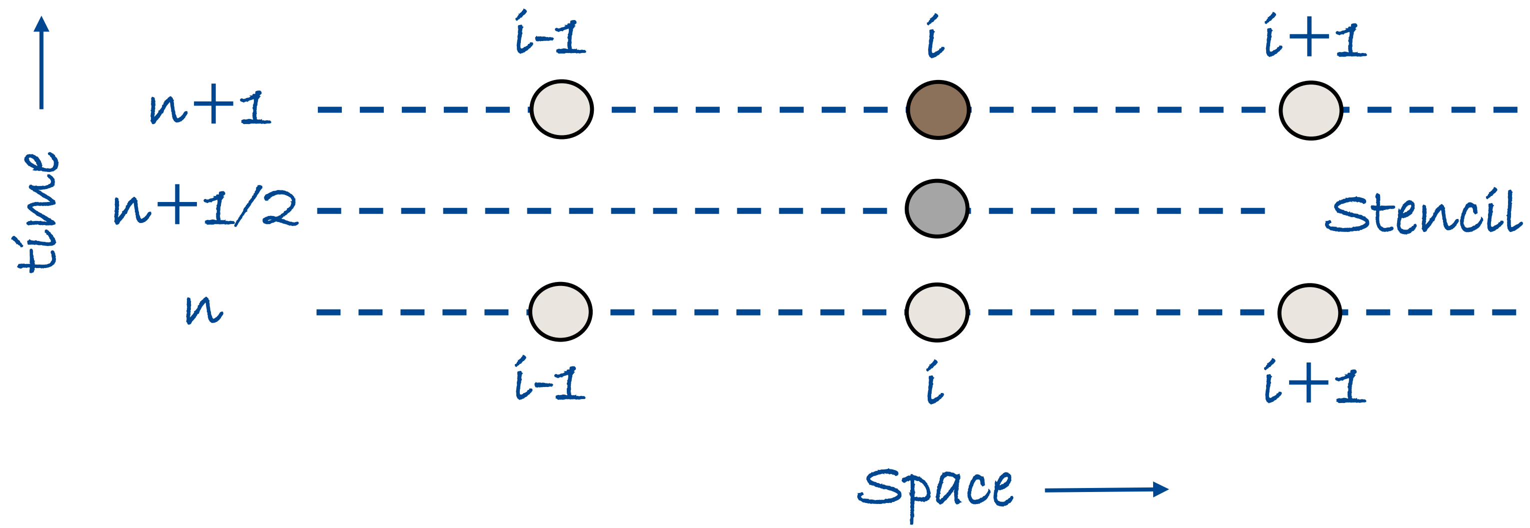

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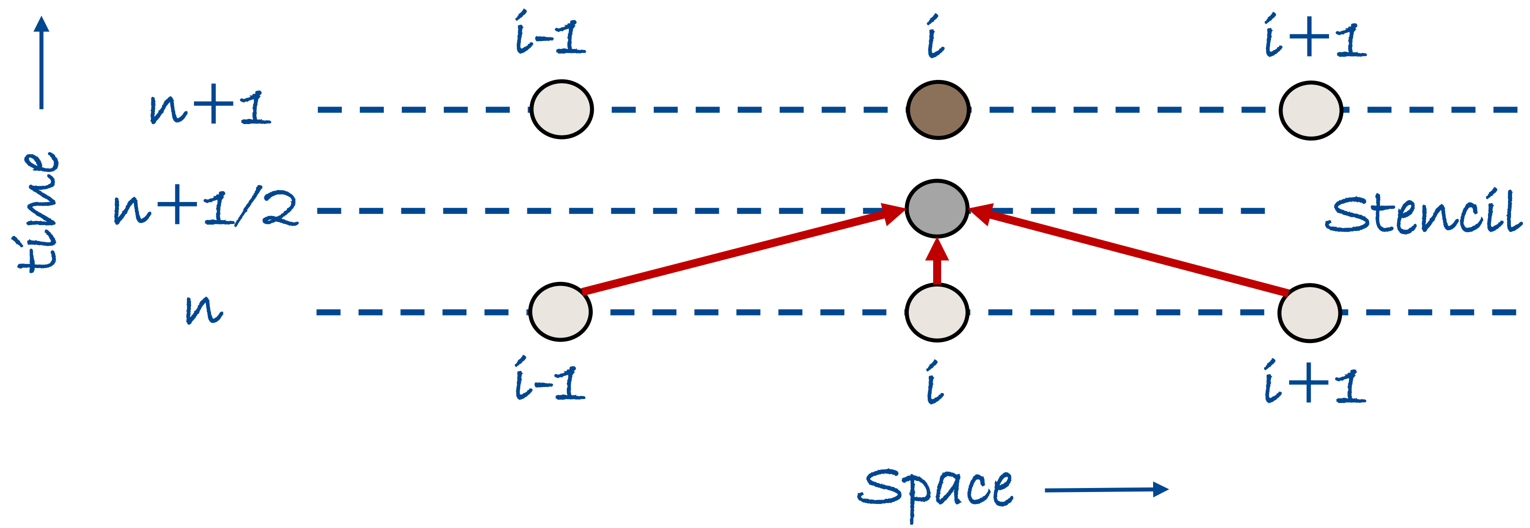

Explicit:

$$\frac{u_i^{n+1/2} - u_i^{n}}{\Delta t/2} = \frac{\nu}{\Delta x^2}\left(u_{i-1}^{n} - 2u_i^{n} + u_{i-1}^{n}\right)$$

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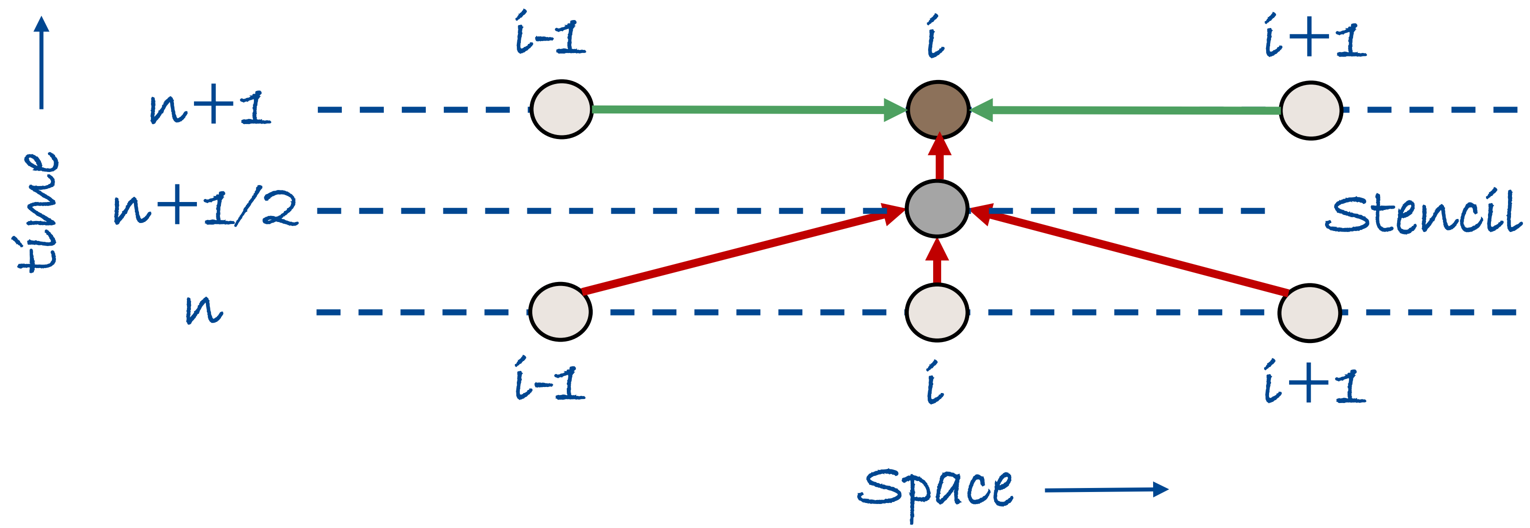

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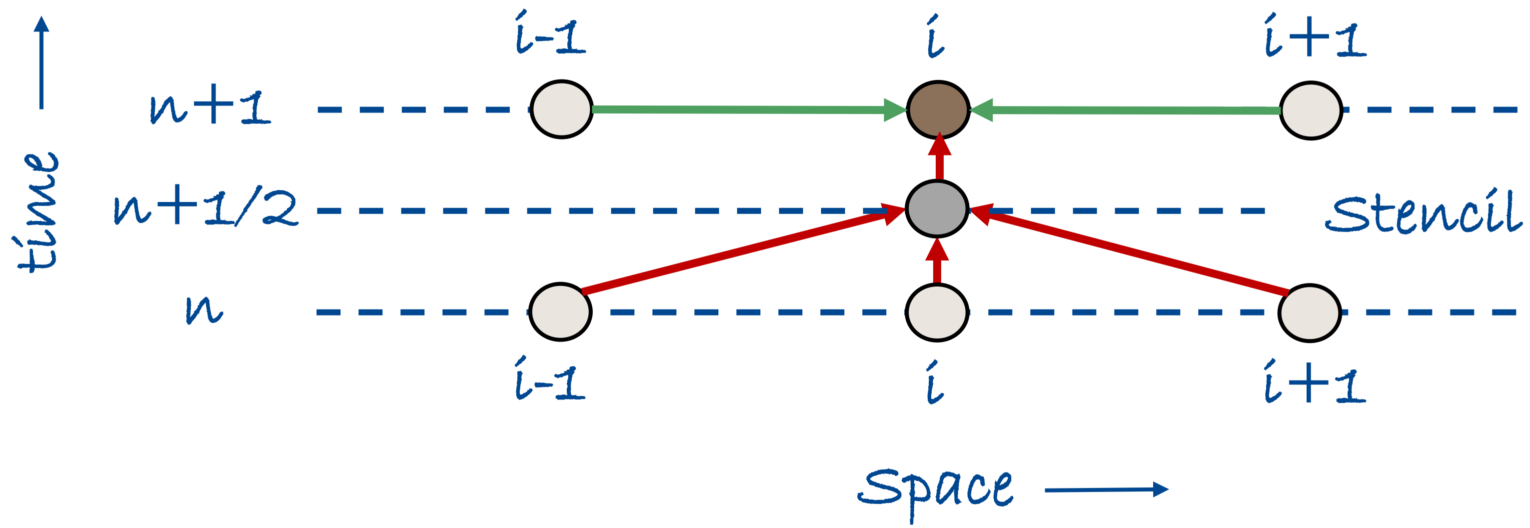

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✓ 2nd order in time & space !

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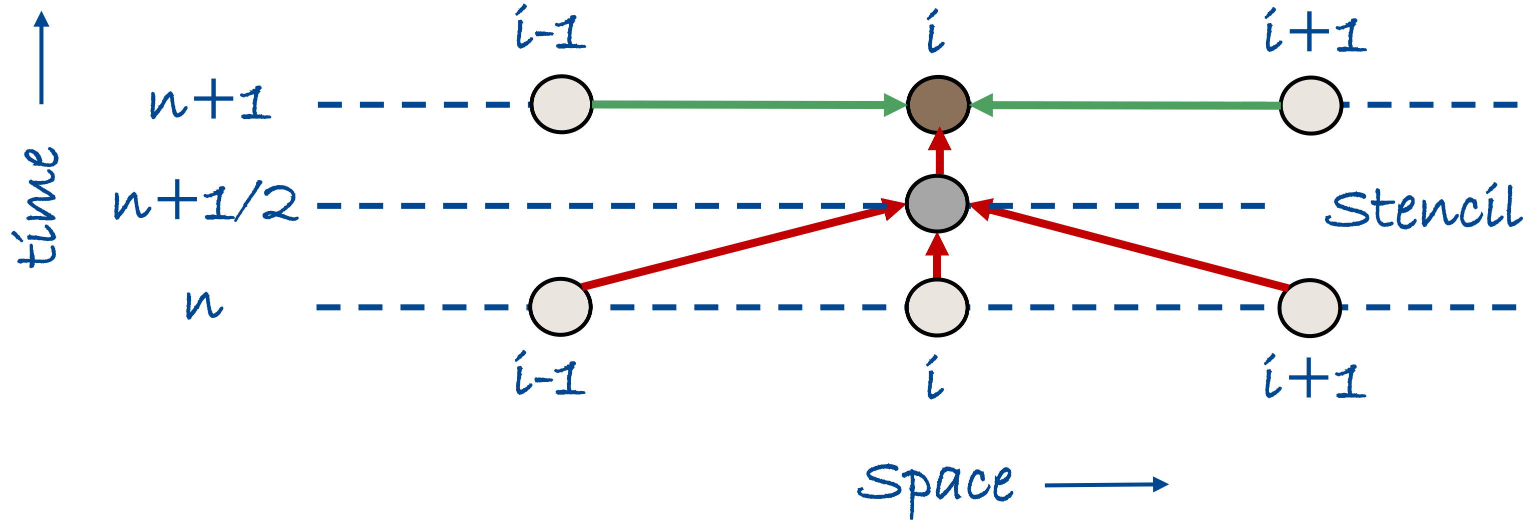

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✔ 2nd order in time & space !

⚠ Tridiagonal system to solve!

4th Code: 1D Burger's Equation

$$\frac{du}{dt} + u\frac{du}{dx} = \nu\frac{d^2u}{dx^2}$$

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Initial Condition: $u(x,0) = -\frac{2\nu}{\phi}\frac{\partial\phi}{\partial x} + 4 \quad \phi = \exp\left(-\frac{x^2}{4\nu}\right) + \exp\left(-\frac{(x-2\pi)^2}{4\nu}\right)$ Boundary Condition: $u(0) = u(2\pi)$

Periodic BC!

Exact Solution: $u = -\frac{2\nu}{\phi}\frac{\partial\phi}{\partial x} + 4 \qquad \phi = \exp\left(-\frac{(x-4t)^2}{4\nu(t+1)}\right) + \exp\left(-\frac{(x-4t-2\pi)^2}{4\nu(t+1)}\right)$

4^{th} Code: 1D Burger's Equation

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Let's Code It!

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import sympy as sp
import pylab as pl
pl.ion()
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x, nu, t = sp.symbols('x nu t')
phi = sp.exp(-(x-4*t)**2/(4*nu*(t+1))) + sp.exp(-(x-4*t-2*sp.pi)**2/(4*nu*(t+1)))

phiprime = phi.diff(x)
u = -2*nu*(phiprime/phi)+4

from sympy.utilities.lambdify import lambdify
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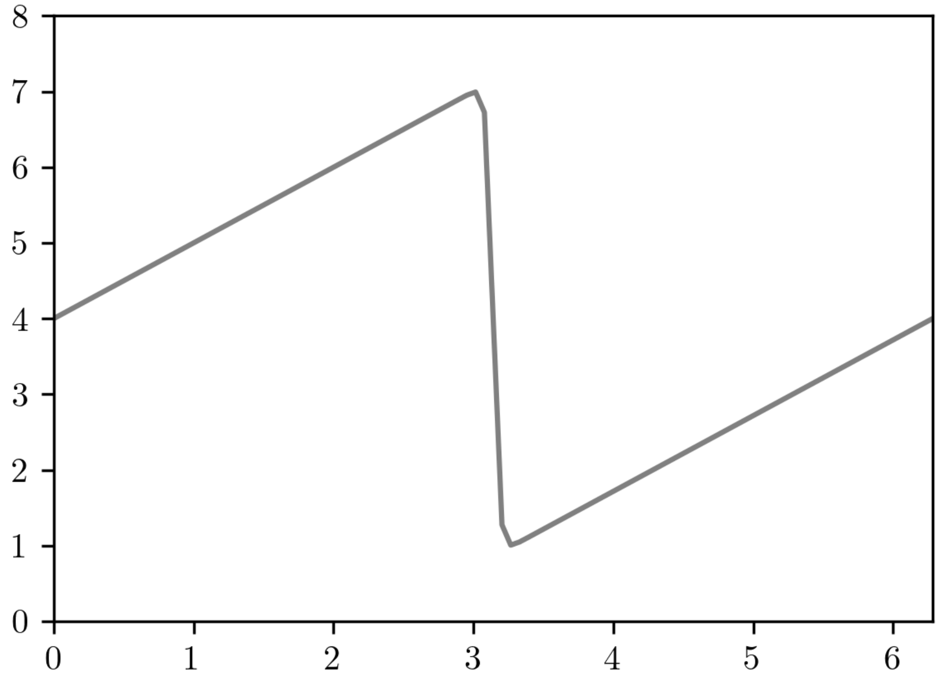
nx = 101
dx = 2*np.pi/(nx-1)
nt = 100
nu = 0.07
dt = dx*nu
T = nt*dt

grid = np.linspace(0, 2*np.pi, nx)
u = np.empty(nx)
t = 0
u = np.asarray([ufunc(t, x, nu) for x in grid])


pl.figure(figsize=(11,7), dpi=100)
pl.plot(grid,u, marker='o', lw=2)
pl.xlim([0,2*np.pi])
pl.ylim([0,10])
pl.xlabel('X')
pl.ylabel('Velocity')
pl.title('1D Burgers Equation - Initial condition')
```



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for n in range(nt):
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        u[i] = un[i] - un[i] * dt/dx * (un[i]-un[i-1]) + \
            nu * dt/(dx**2) * (un[i+1] - 2*un[i] + un[i-1])
    # infer the periodicity
    u[-1] = un[-1] - un[-1] * dt/dx * (un[-1]-un[-2]) + \
            nu * dt/(dx**2) * (un[0] - 2*un[-1] + un[-2])

u_analytical = np.asarray([ufunc(T, xi, nu) for xi in grid])

pl.figure(figsize=(11,7), dpi=100)
pl.plot(grid, u, marker='o', lw=2, label='Computational')
pl.plot(grid, u_analytical, label='Analytical')
pl.xlim([0, 2*np.pi])
pl.ylim([0,10])
pl.legend()
pl.xlabel('X')
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pl.title('1D Burgers Equation - Solutions')
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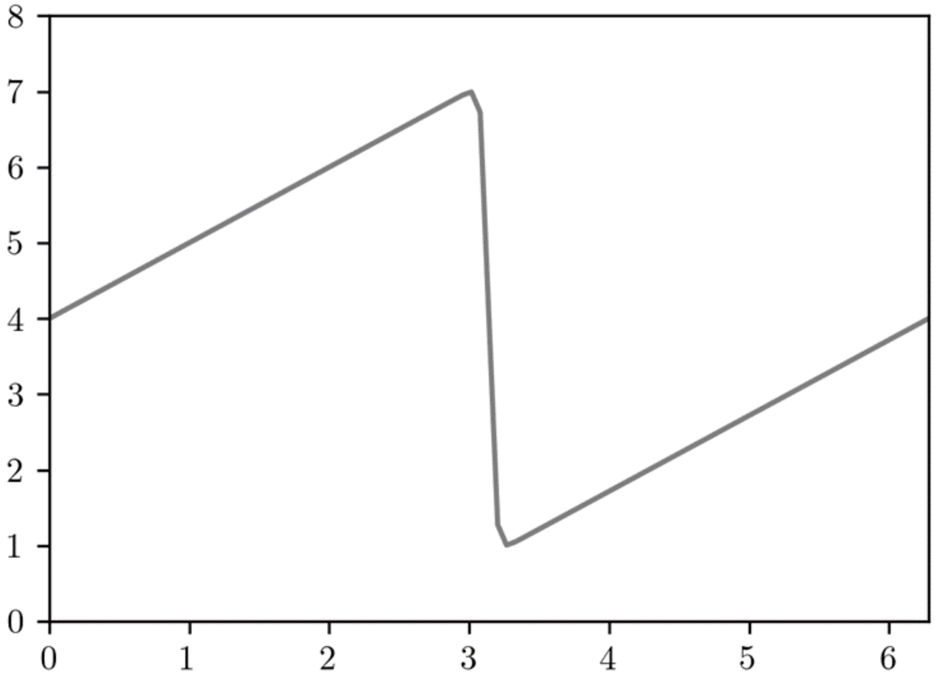

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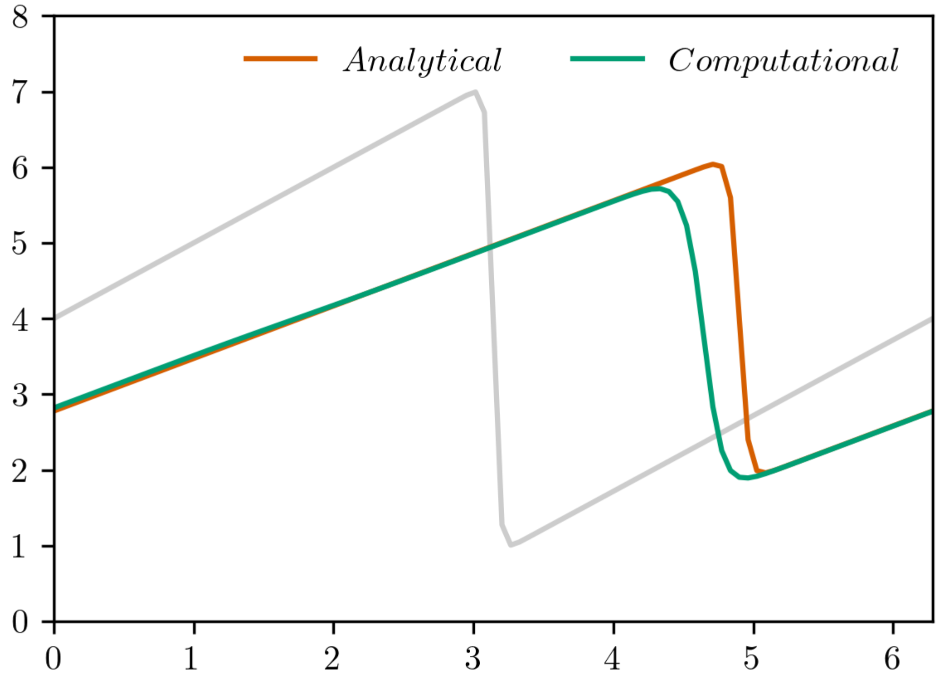

5^{th} Code: 2D Laplace Equation

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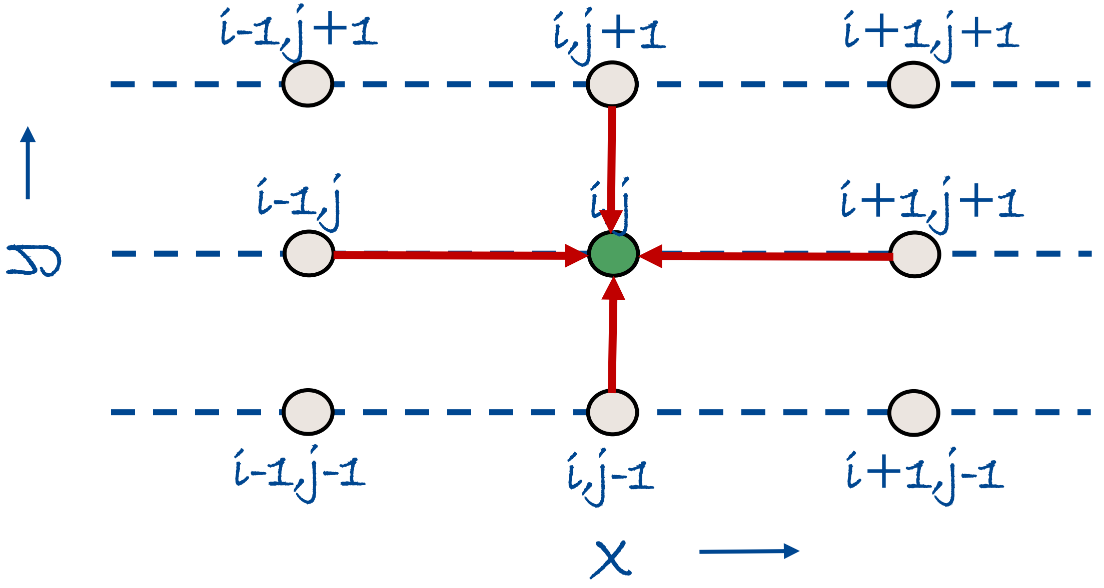
5 points stencil



5^{th} Code: 2D Laplace Equation

No time dependence !

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0$$

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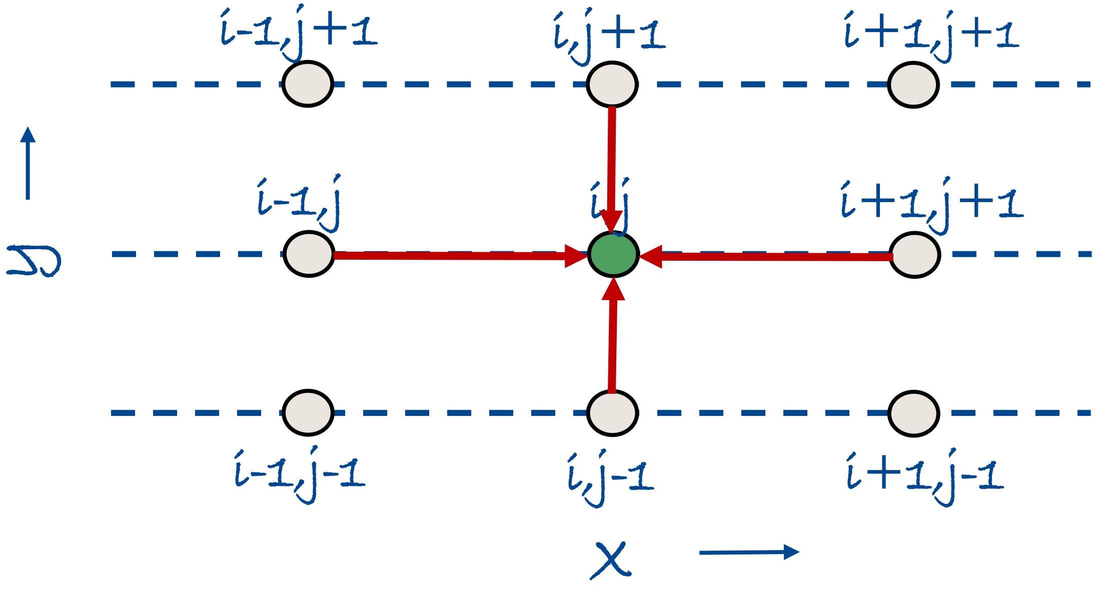

5th Code: 2D Laplace Equation

No time dependence !

Calculates an equilibrium state given the specified BCs

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0$$

$$\frac{p^n_{i+1,j} - 2p^n_{i,j} + p^n_{i-1,j}}{\Delta x^2} + \frac{p^n_{i,j+1} - 2p^n_{i,j} + p^n_{i,j-1}}{\Delta y^2} = 0$$

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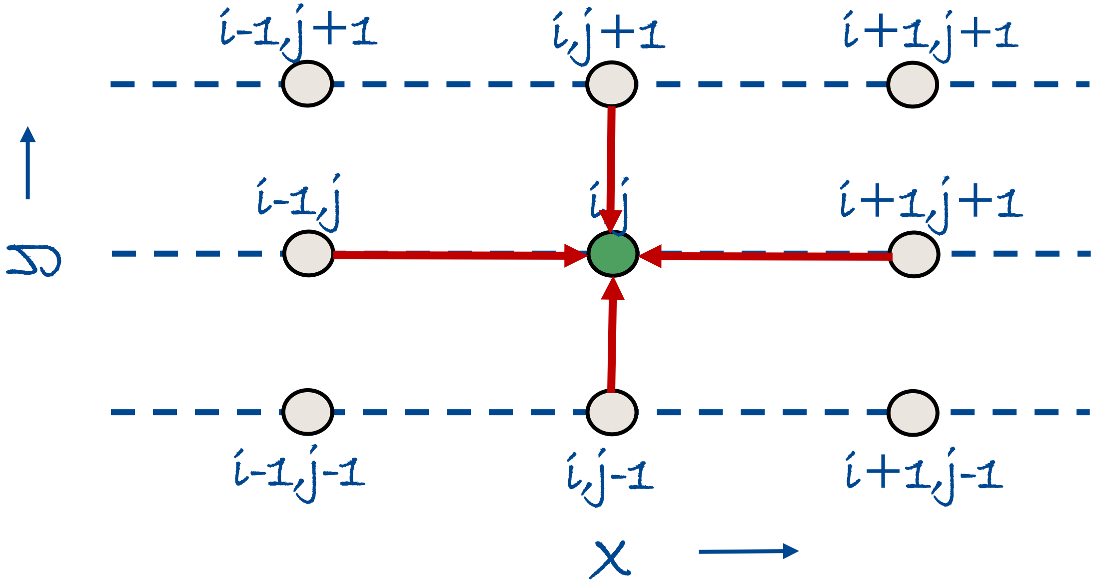

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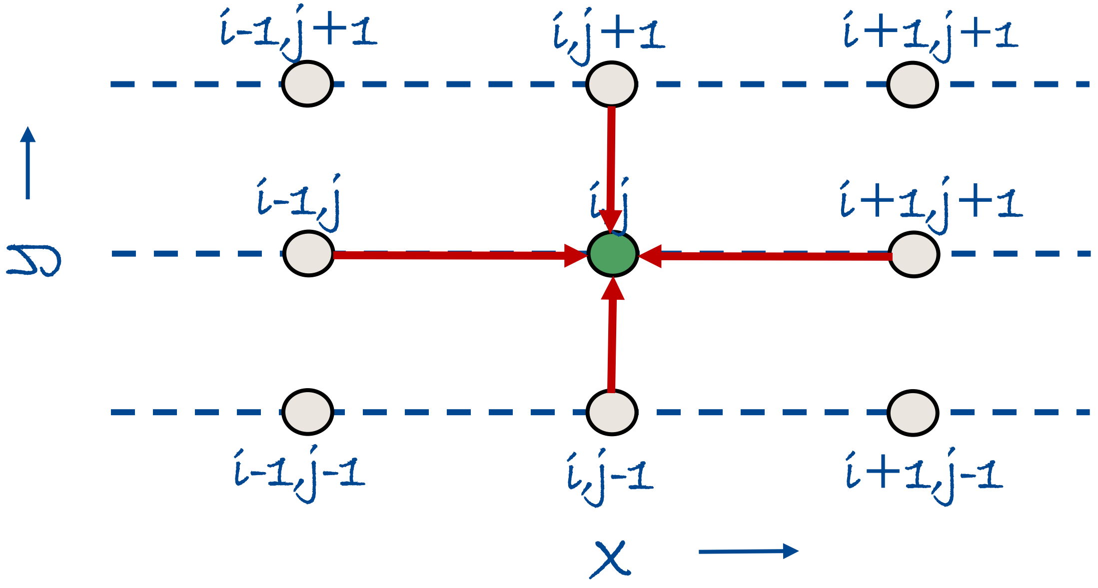

Need to solve iteratively

Will reach equilibrium as iterations $\rightarrow \infty$

Need to specify a threshold!

KTH VETENSKAP OCH KONST

5^{th} Code: 2D Laplace Equation

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No time dependence !

Calculates an equilibrium state given the specified BCs

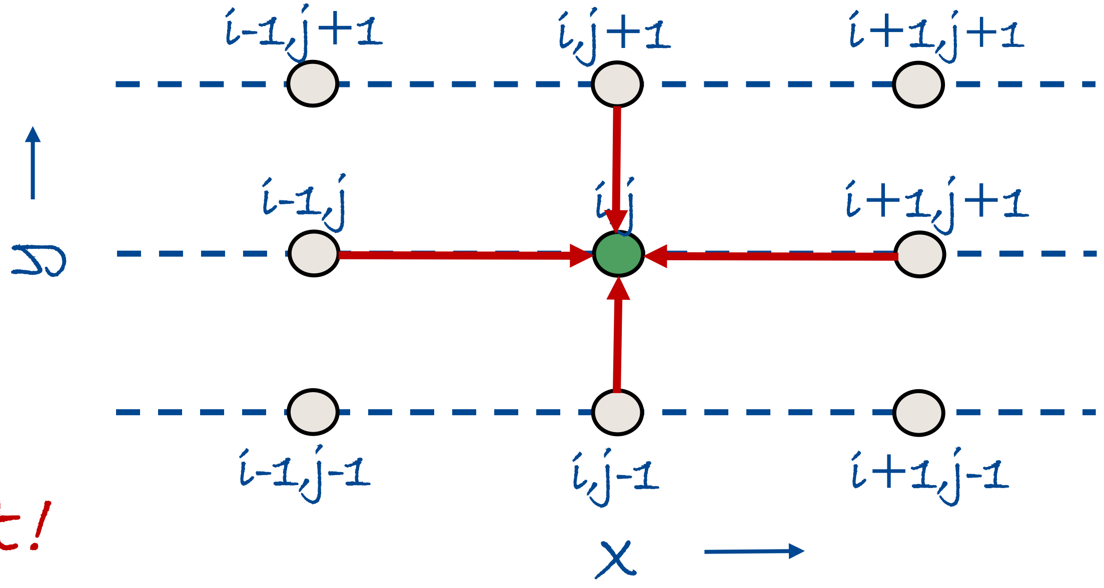
$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0$$

BCs: $p = 0 \; at \; x = 0$, $p = y \; at \; x = 2$, $\partial p/\partial y = 0 \; at \; y = 0{,}1$

$$\frac{p_{i+1,j}^n - 2p_{i,j}^n + p_{i-1,j}^n}{\Delta x^2} + \frac{p_{i,j+1}^n - 2p_{i,j}^n + p_{i,j-1}^n}{\Delta y^2} = 0$$

$$p_{i,j}^n = \frac{\Delta y^2 (p_{i+1,j}^n + p_{i-1,j}^n) + \Delta x^2 (p_{i,j+1}^n + p_{i,j-1}^n)}{2(\Delta x^2 + \Delta y^2)}$$

5 points stencil



Need to solve iteratively

Will reach equilibrium as iterations → ∞

Need to specify a threshold!

Let's Code It!

5^{th} Code: 2D Laplace Equation

No time dependence !

Calculates an equilibrium state given the specified BCs

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0$$

BCs: $p = 0 \; at \; x = 0$, $p = y \; at \; x = 2$, $\partial p / \partial y = 0 \; at \; y = 0{,}1$

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$$p_{i,j}^{n} = \frac{\Delta y^2 (p_{i+1,j}^{n} + p_{i-1,j}^{n}) + \Delta x^2 (p_{i,j+1}^{n} + p_{i,j-1}^{n})}{2(\Delta x^2 + \Delta y^2)}$$

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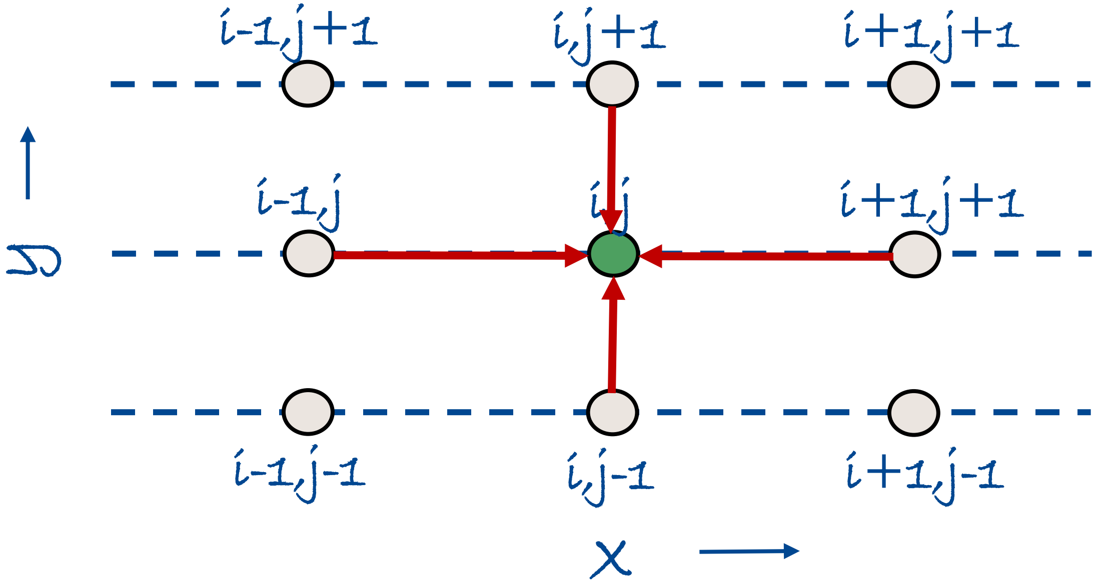

5th Code: 2D Laplace Equation

```
import numpy as np
from matplotlib import pyplot, cm
from mpl_toolkits.mplot3d import Axes3D
```

5th Code: 2D Laplace Equation

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def plot2D(x,y,p):
    fig = pyplot.figure( figsize=(11,7), dpi=100 )
    ax  = fig.gca(projection='3d')
    X, Y = np.meshgrid(x,y)
    surf = ax.plot_surface( X, Y, p[:], rstride=1, cstride=1, cmap=cm.viridis,
                           linewidth=0, antialiased=False )
    ax.set_xlabel("$x$");  ax.set_xlim(0,2)
    ax.set_ylabel("$y$");  ax.set_ylim(0,1)
    ax.view_init(30,225)
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def laplace2D(p,y,dx,dy,target_norm):
   norm=1
   pn = np.empty_like(p)
   while norm > target_norm:
      pn = p.copy()
      p[1:-1,1:-1] = (
                        (dy**2 * (pn[1:-1,2:] – pn[1:-1,:-2]))   +
                        (dx**2 * (pn[2:,1:-1] – pn[:-2,1:-1]))
                     ) / (2* (dx**2 + dy**2))
      p[:,0] = 0                # p=0 at x=0
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      p[0,:] = p[1,:]           # dp/dy=0 at y=0
      p[-1,:] = p[-2,:]         # dp/dy=0 at y=1
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nx = 31; ny = 31; dx = 2/(nx-1); dy =1/(ny-1)
x = np.linspace(0, 2, nx); y = np.linspace(0, 1, ny)
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5th Code: 2D Laplace Equation

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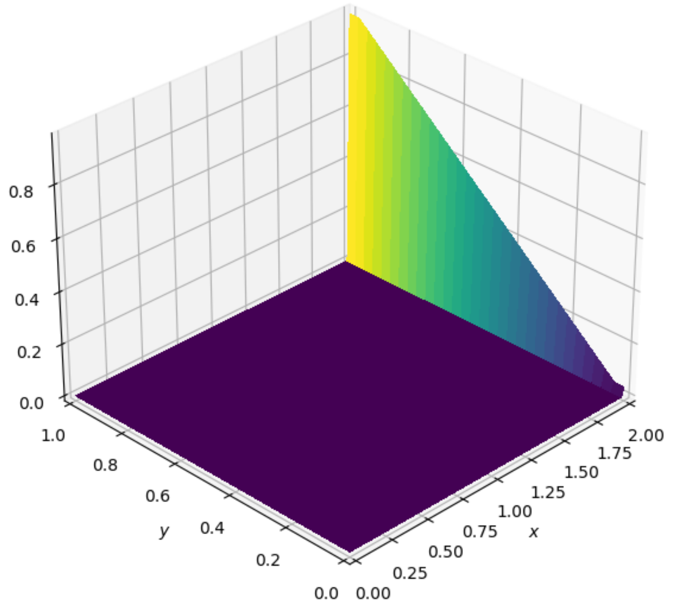
plot2D(x,y,p)
```



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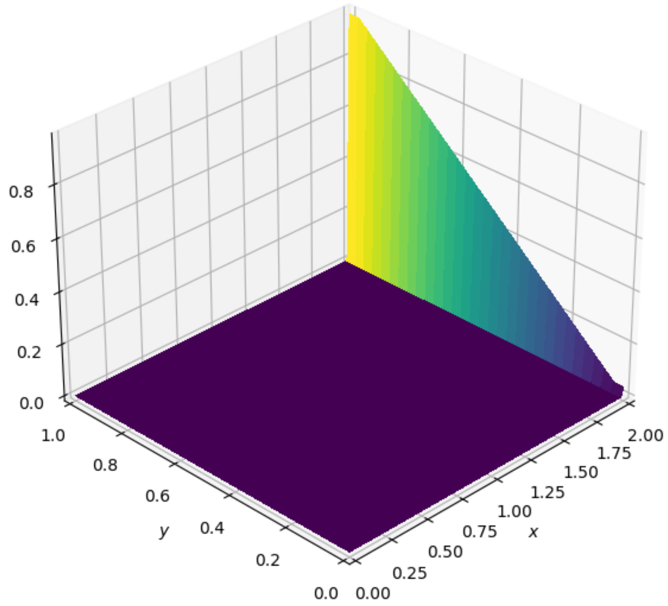
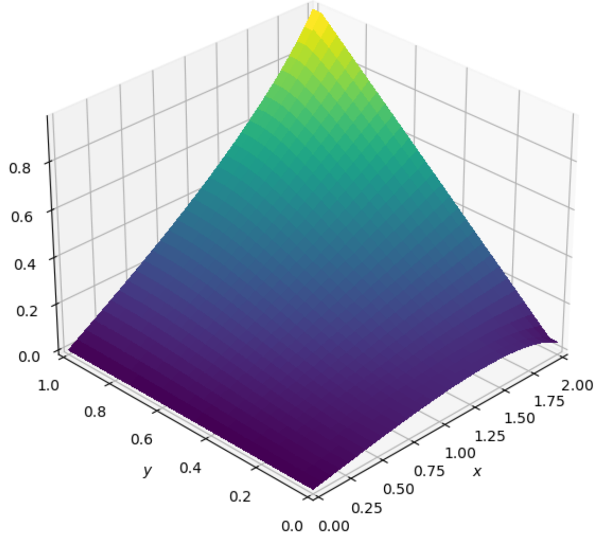
plot2D(x,y,p)
p = laplace2D(p,y,dx,dy,1e-4)
plot2D(x,y,p)
```





Navier-Stokes equations for incompressible flows

Continuity equation: $$\nabla . \vec{v} = 0$$

Momentum equation: $$\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v}$$

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No equation for pressure !

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There is no obvious way of coupling these equations !

Navier-Stokes equations for incompressible flows

Continuity equation: $$\nabla . \vec{v} = 0$$

Provides a kinematic constraint that requires pressure field to evolve such that the velocity field remains solenoidal !

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Such a relation can be obtained by taking the divergence of the momentum equation.

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$$\nabla^2 p = -\nabla .(\vec{v}.\nabla)\vec{v}$$

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Pressure Poisson Equation

$$\boxed{\nabla^2 p = -\nabla.(\vec{v}.\nabla)\vec{v}}$$

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Pressure Poisson Equation

$$\boxed{\nabla^2 p = -\nabla.(\vec{v}.\nabla)\vec{v}}$$

Let's look at the 6th code to learn how to solve a Poisson's equation.

6^{th} Code: 2D Poisson's Equation

Obtained by adding a source term on the right-hand side of the Laplace's equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = b$$

6th Code: 2D Poisson's Equation

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Poisson's equation act to "relax" the initial sources in the field

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$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = b$$

Poisson's equation act to "relax" the initial sources in the field

Discretized form:

$$\frac{p_{i+1,j}^{n} - 2p_{i,j}^{n} + p_{i-1,j}^{n}}{\Delta x^2} + \frac{p_{i,j+1}^{n} - 2p_{i,j}^{n} + p_{i,j-1}^{n}}{\Delta y^2} = b_{i,j}^{n}$$

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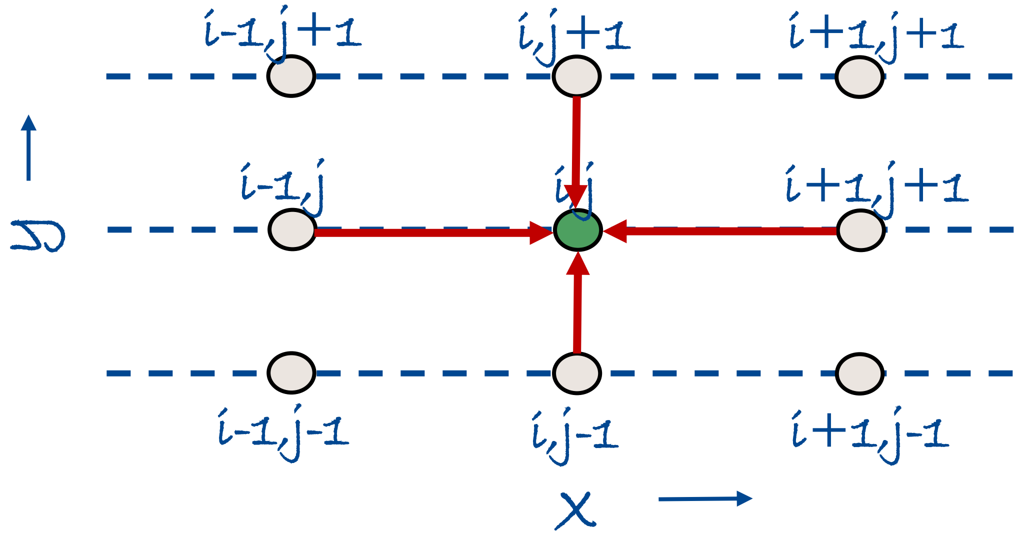
Poisson's equation act to "relax" the initial sources in the field

Discretized form:

$$\frac{p^n_{i+1,j} - 2p^n_{i,j} + p^n_{i-1,j}}{\Delta x^2} + \frac{p^n_{i,j+1} - 2p^n_{i,j} + p^n_{i,j-1}}{\Delta y^2} = b^n_{i,j}$$

$$p^n_{i,j} = \frac{\Delta y^2 (p^n_{i+1,j} + p^n_{i-1,j}) + \Delta x^2 (p^n_{i,j+1} + p^n_{i,j-1}) - b^n_{i,j} \Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}$$

5 points stencil



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Poisson's equation act to "relax" the initial sources in the field

Discretized form:

$$\frac{p_{i+1,j}^{n} - 2p_{i,j}^{n} + p_{i-1,j}^{n}}{\Delta x^2} + \frac{p_{i,j+1}^{n} - 2p_{i,j}^{n} + p_{i,j-1}^{n}}{\Delta y^2} = b_{i,j}^{n}$$

$$p_{i,j}^{n} = \frac{\Delta y^2 (p_{i+1,j}^{n} + p_{i-1,j}^{n}) + \Delta x^2 (p_{i,j+1}^{n} + p_{i,j-1}^{n}) - b_{i,j}^{n} \Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}$$

Source term:

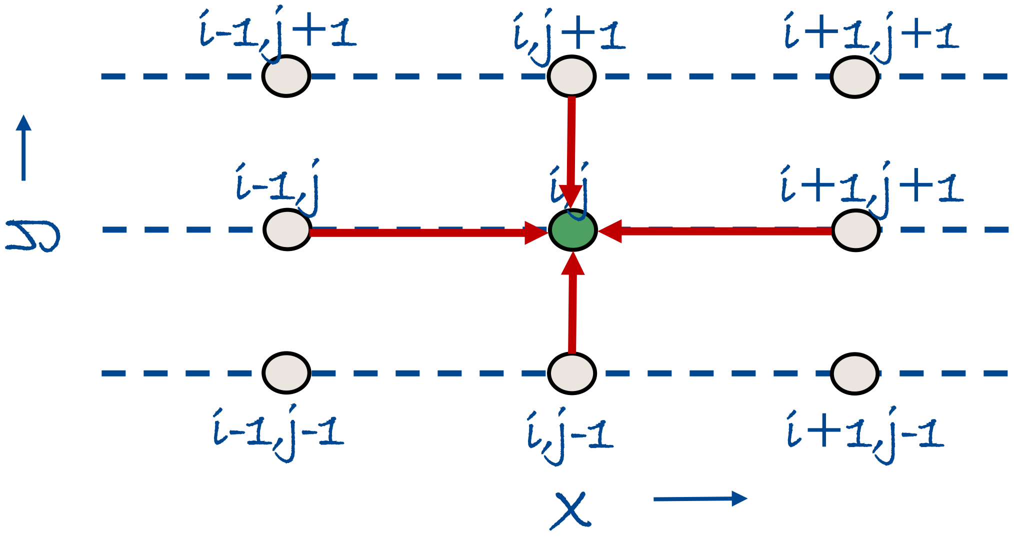
2 Spikes

$$b_{i,j} = -100 \ at \ i = \frac{nx}{4}, j = \frac{3ny}{4}$$

$$b_{i,j} = 100 \ at \ i = \frac{3nx}{4}, j = \frac{ny}{4}$$

$$b_{i,j} = 0 \ \ elsewhere$$

5 points stencil



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Poisson's equation act to "relax" the initial sources in the field

Discretized form:

$$\frac{p^n_{i+1,j} - 2p^n_{i,j} + p^n_{i-1,j}}{\Delta x^2} + \frac{p^n_{i,j+1} - 2p^n_{i,j} + p^n_{i,j-1}}{\Delta y^2} = b^n_{i,j}$$

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Source term:

$$b_{i,j} = -100 \ at \ i = \frac{nx}{4}, j = \frac{3ny}{4}$$

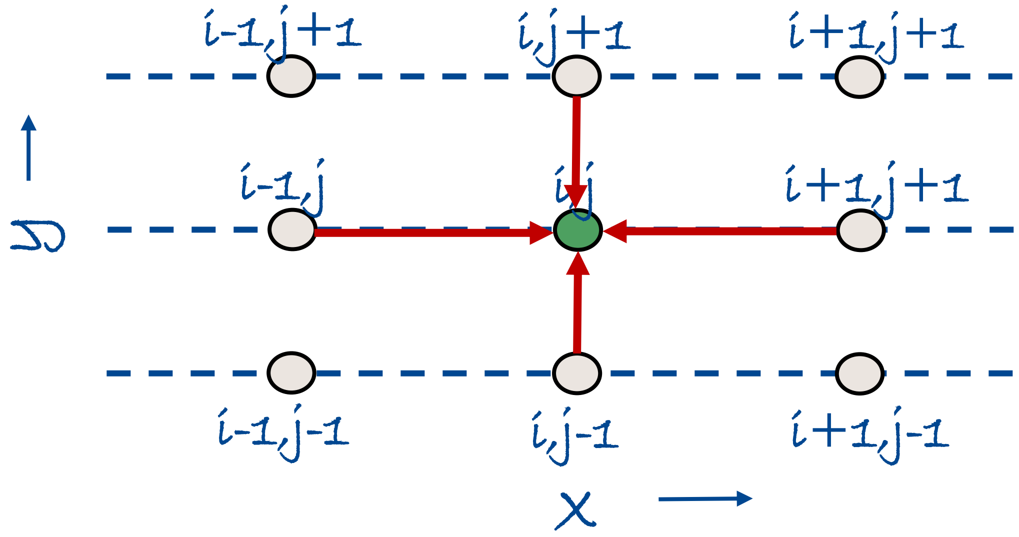
2 Spikes

$$b_{i,j} = 100 \ at \ i = \frac{3nx}{4}, j = \frac{ny}{4}$$

$$b_{i,j} = 0 \ elsewhere$$

The iteration will relax the initial spikes!

5 points stencil



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$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = b$$

Poisson's equation act to "relax" the initial sources in the field

Discretized form:

$$\frac{p^n_{i+1,j} - 2p^n_{i,j} + p^n_{i-1,j}}{\Delta x^2} + \frac{p^n_{i,j+1} - 2p^n_{i,j} + p^n_{i,j-1}}{\Delta y^2} = b^n_{i,j}$$

BCs:

$$p = 0 \; at \; x = 0,2$$

$$p = 0 \; at \; y = 0,2$$

$$p^n_{i,j} = \frac{\Delta y^2 (p^n_{i+1,j} + p^n_{i-1,j}) + \Delta x^2 (p^n_{i,j+1} + p^n_{i,j-1}) - b^n_{i,j} \Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}$$

Source term:

2 Spikes

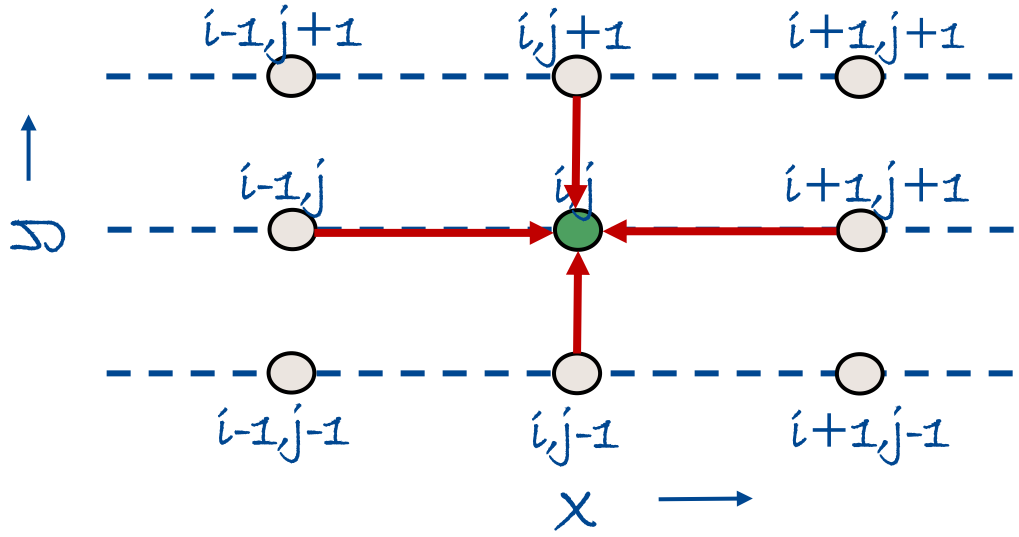
$$b_{i,j} = -100 \; at \; i = \frac{nx}{4}, j = \frac{3ny}{4}$$

$$b_{i,j} = 100 \; at \; i = \frac{3nx}{4}, j = \frac{ny}{4}$$

$$b_{i,j} = 0 \; elsewhere$$

The iteration will relax the initial spikes!

5 points stencil



6^{th} Code: 2D Poisson's Equation

Obtained by adding a source term on the right-hand side of the Laplace's equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = b$$

Poisson's equation act to "relax" the initial sources in the field

Discretized form:

$$\frac{p_{i+1,j}^n - 2p_{i,j}^n + p_{i-1,j}^n}{\Delta x^2} + \frac{p_{i,j+1}^n - 2p_{i,j}^n + p_{i,j-1}^n}{\Delta y^2} = b_{i,j}^n$$

BCs:

$$p = 0 \ at \ x = 0,2$$

$$p = 0 \ at \ y = 0,2$$

$$p_{i,j}^n = \frac{\Delta y^2 (p_{i+1,j}^n + p_{i-1,j}^n) + \Delta x^2 (p_{i,j+1}^n + p_{i,j-1}^n) - b_{i,j}^n \Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}$$

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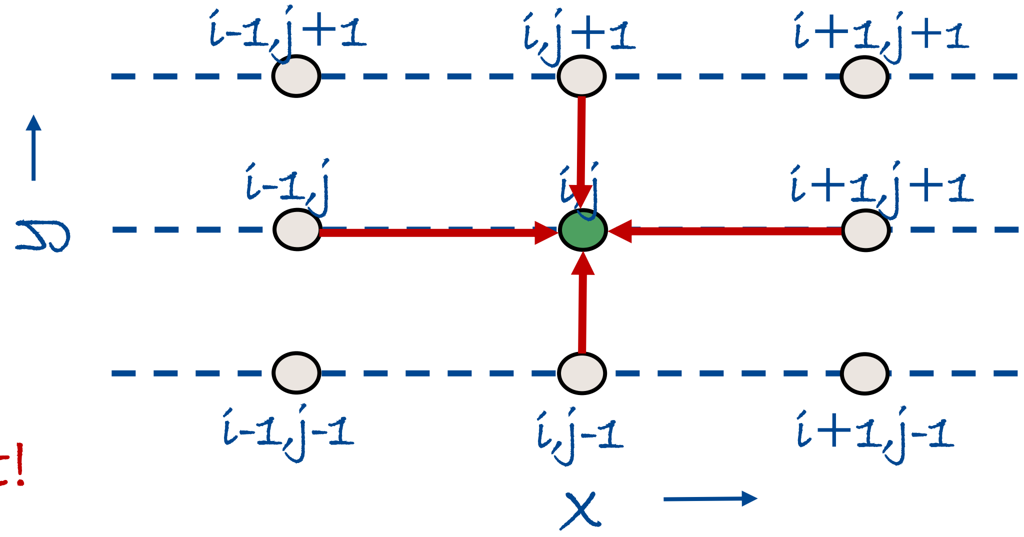

Source term:

2 Spikes

$$b_{i,j} = -100 \ at \ i = \frac{nx}{4}, j = \frac{3ny}{4}$$

$$b_{i,j} = 100 \ at \ i = \frac{3nx}{4}, j = \frac{ny}{4}$$

$$b_{i,j} = 0 \ elsewhere$$

The iteration will relax the initial spikes!

Let's Code it!

6th Code: 2D Poisson's Equation

```
import numpy as np
from matplotlib import pyplot, cm
from mpl_toolkits.mplot3d import Axes3D
```

6^{th} Code: 2D Poisson's Equation

```
import numpy as np
from matplotlib import pyplot, cm
from mpl_toolkits.mplot3d import Axes3D

def plot2D(x,y,p):
    fig = pyplot.figure( figsize=(11,7), dpi=100 )
    ax  = fig.gca(projection='3d')
    X, Y = np.meshgrid(x,y)
    surf = ax.plot_surface( X, Y, p[:], rstride=1, cstride=1, cmap=cm.viridis,
                            linewidth=0, antialiased=False )
    ax.set_xlabel("$x$");  ax.set_xlim(0,2)
    ax.set_ylabel("$y$");  ax.set_ylim(0,1)
    ax.view_init(30,225)
```

6th Code: 2D Poisson's Equation

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def  poisson2D(p,b,dx,dy,target_norm):
    norm=1; small=1e-8; niter=0
    pn = np.zeros_like(p)
    while norm > target_norm:
       pn = p.copy(); niter+=1
       p[1:-1,1:-1] = ( ( dy**2 * (pn[2:,1:-1] – pn[:-2,1:-1])    +
                          dx**2 * (pn[1:-1,2:] – pn[1:-1,:-2])    -
                          dx**2 * dy**2 * b[1:-1,1:-1] ) /
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       p[0,:]  = 0            # p=0 at x=0
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       norm = (np.sum(np.abs(p)) – np.sum(np.abs(pn)))/(np.sum(np.abs(pn))+small)
    return p
```

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   return p


nx = 51; ny = 51; dx = 2/(nx-1); dy = 2/(ny-1); nt = 100;
x = np.linspace(0, 2, nx); y = np.linspace(0, 2, ny)
```

6th Code: 2D Poisson's Equation

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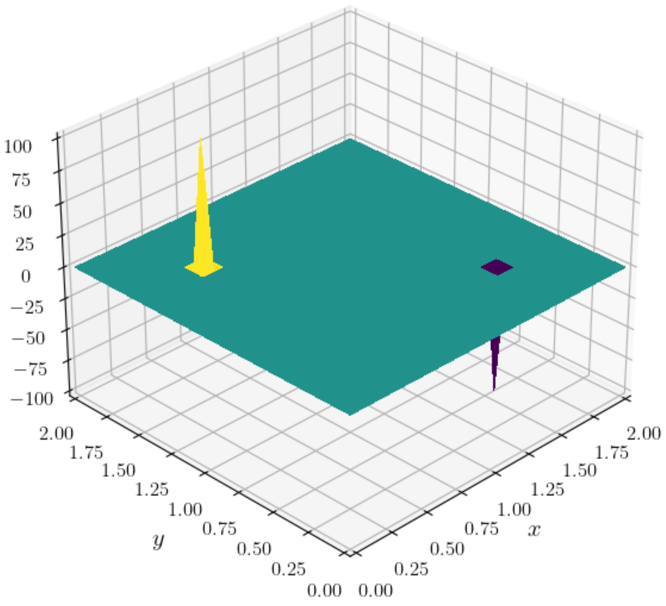
nx = 51; ny = 51; dx = 2/(nx-1); dy = 2/(ny-1); nt = 100;
x = np.linspace(0, 2, nx); y = np.linspace(0, 1, ny)
```

```python
p  = np.zeros((nx,ny))
b  = np.zeros((nx,ny))

b[int(nx/4), int(3*ny/4)] = -100
b[int(3*nx/4), int(ny/4)] = 100

plot2D(x,y,p)
plot2D(x,y,b)
```



6th Code: 2D Poisson's Equation

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p  = np.zeros((nx,ny))
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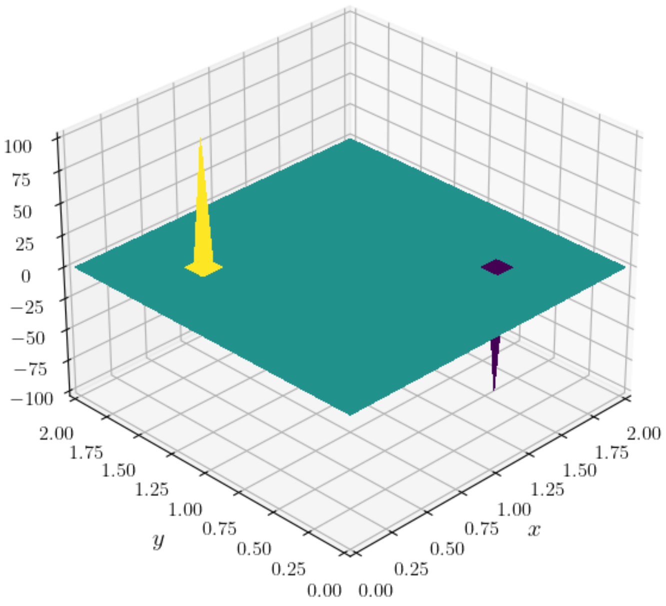
b[int(nx/4), int(3*ny/4)] = -100
b[int(3*nx/4), int(ny/4)] = 100

plot2D(x,y,p)
plot2D(x,y,b)

p, niter = poisson2D(p,b,dx,dy,1e-4)
```



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plot2D(x,y,p)
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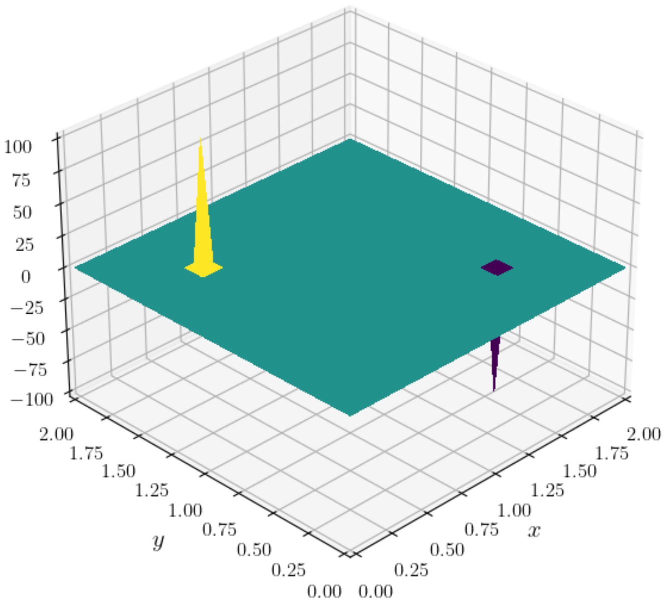
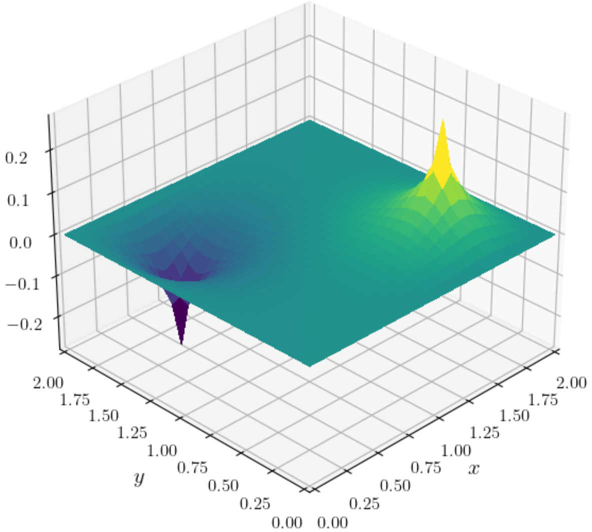
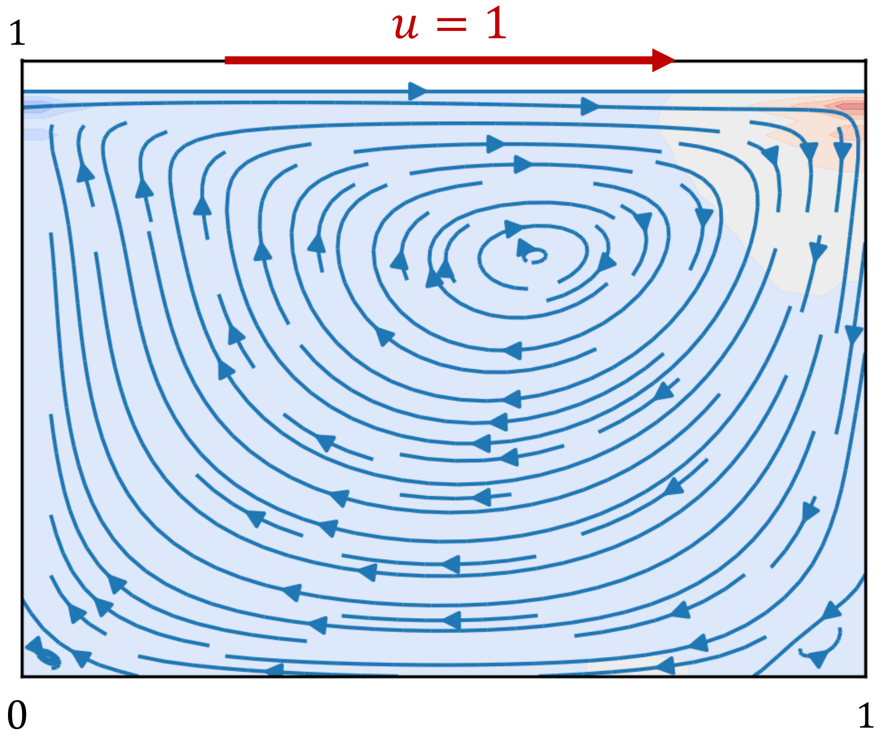




Practical Session: 2D Lid Driven Cavity Flow

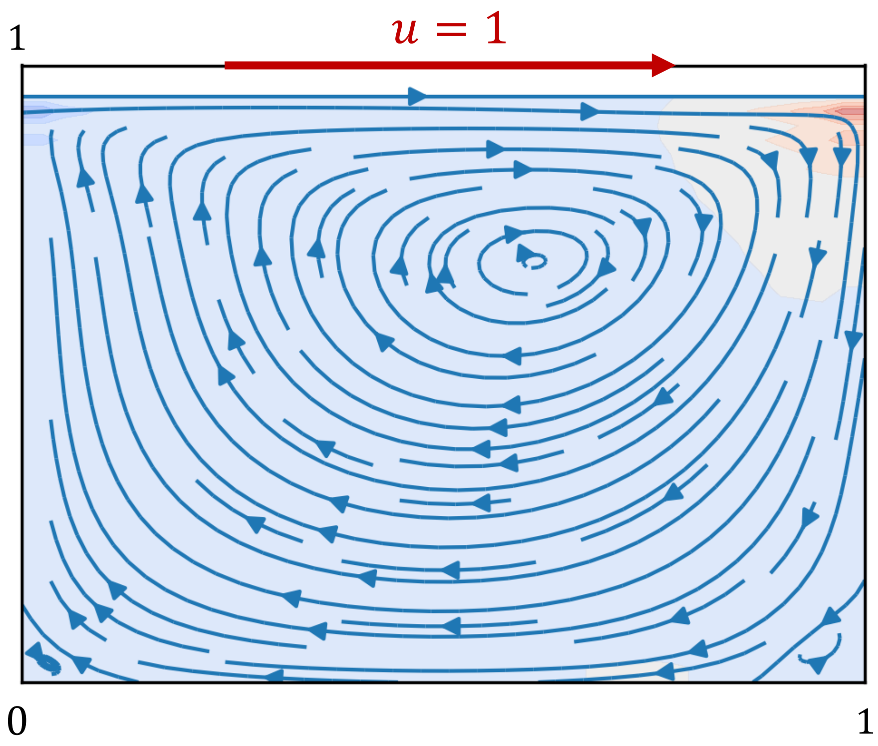


Practical Session: 2D Lid Driven Cavity Flow

MODELAIR

Continuity: $\nabla . \vec{v} = 0$

Momentum: $\dfrac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v} = -\dfrac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v}$



Practical Session: 2D Lid Driven Cavity Flow

KTH VETENSKAP OCH KONST

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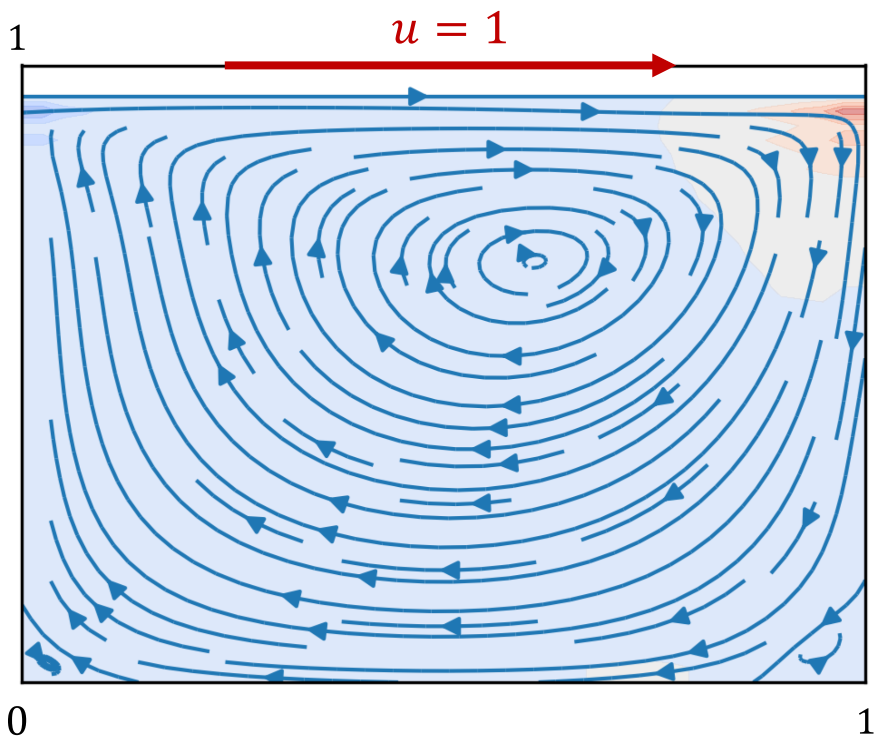
X-velocity: $\dfrac{\partial u}{\partial t} + u\left(\dfrac{\partial u}{\partial x}\right) + v\left(\dfrac{\partial u}{\partial y}\right) = -\dfrac{1}{\rho}\left(\dfrac{\partial p}{\partial x}\right) + \nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right)$

Y-velocity: $\dfrac{\partial v}{\partial t} + u\left(\dfrac{\partial v}{\partial x}\right) + v\left(\dfrac{\partial v}{\partial y}\right) = -\dfrac{1}{\rho}\left(\dfrac{\partial p}{\partial y}\right) + \nu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right)$



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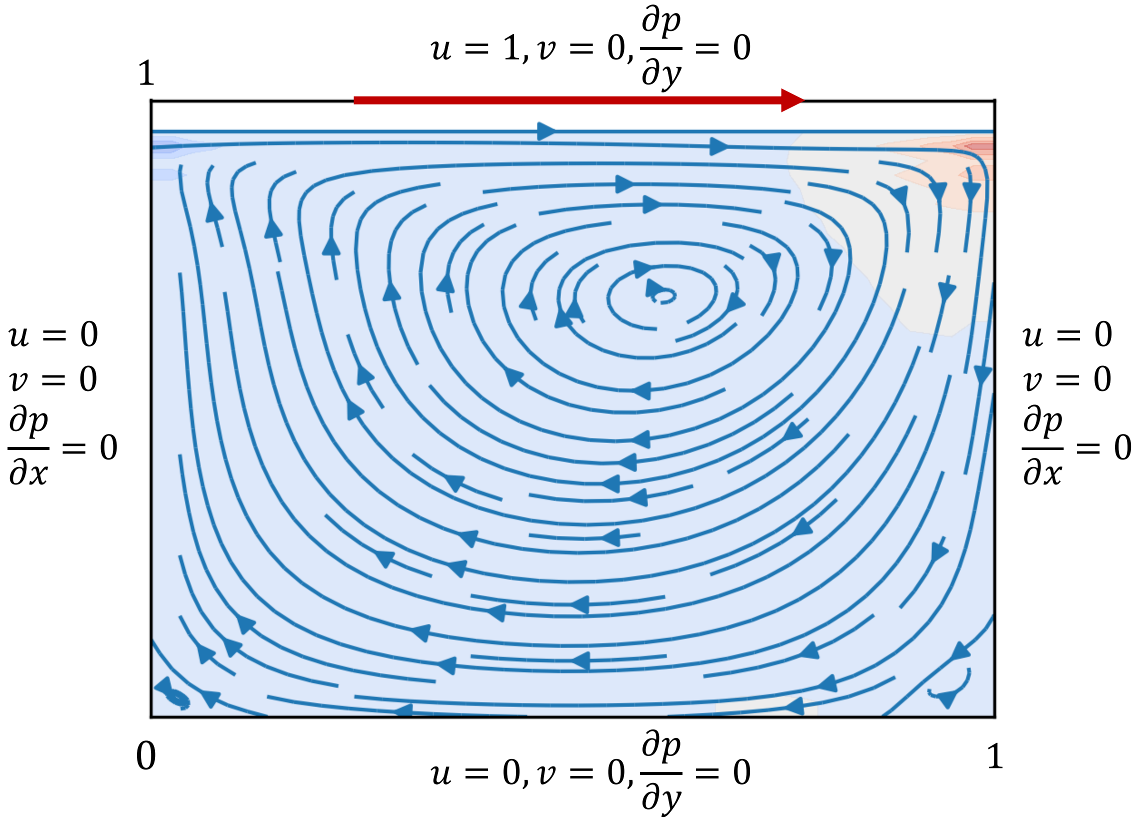

Practical Session: 2D Lid Driven Cavity Flow

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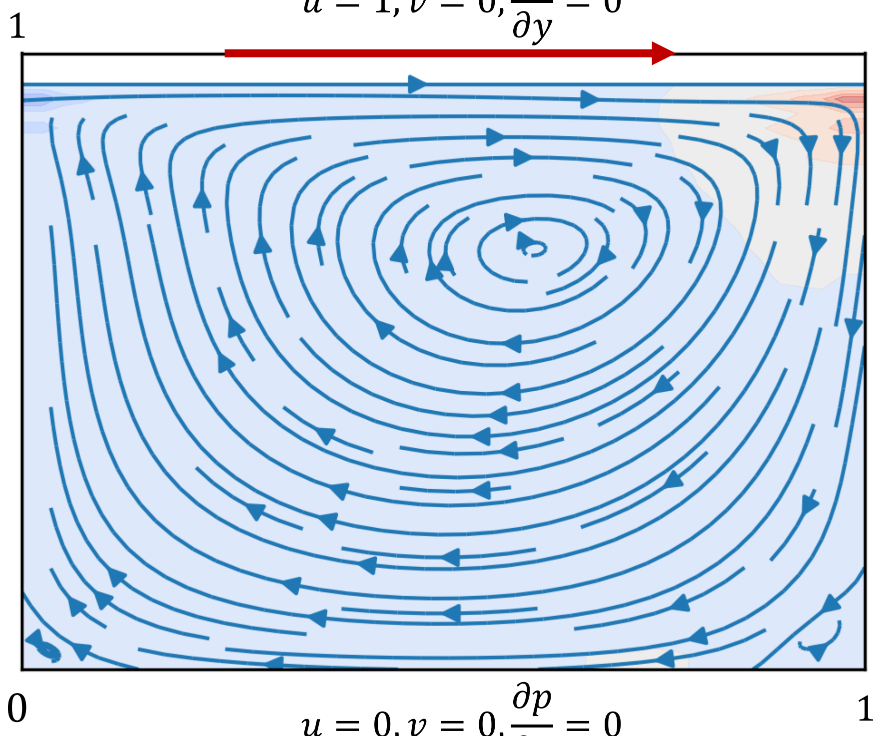
What about the pressure?

$u = 1, v = 0, \dfrac{\partial p}{\partial y} = 0$



$u = 0$, $v = 0$, $\dfrac{\partial p}{\partial x} = 0$ (left wall)

$u = 0$, $v = 0$, $\dfrac{\partial p}{\partial x} = 0$ (right wall)

$u = 0, v = 0, \dfrac{\partial p}{\partial y} = 0$

Practical Session: 2D Lid Driven Cavity Flow

Continuity: $\nabla.\vec{v} = 0$

Momentum: $$\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{v}$$

X-velocity: $$\frac{\partial u}{\partial t} + u\left(\frac{\partial u}{\partial x}\right) + v\left(\frac{\partial u}{\partial y}\right) = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right) + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

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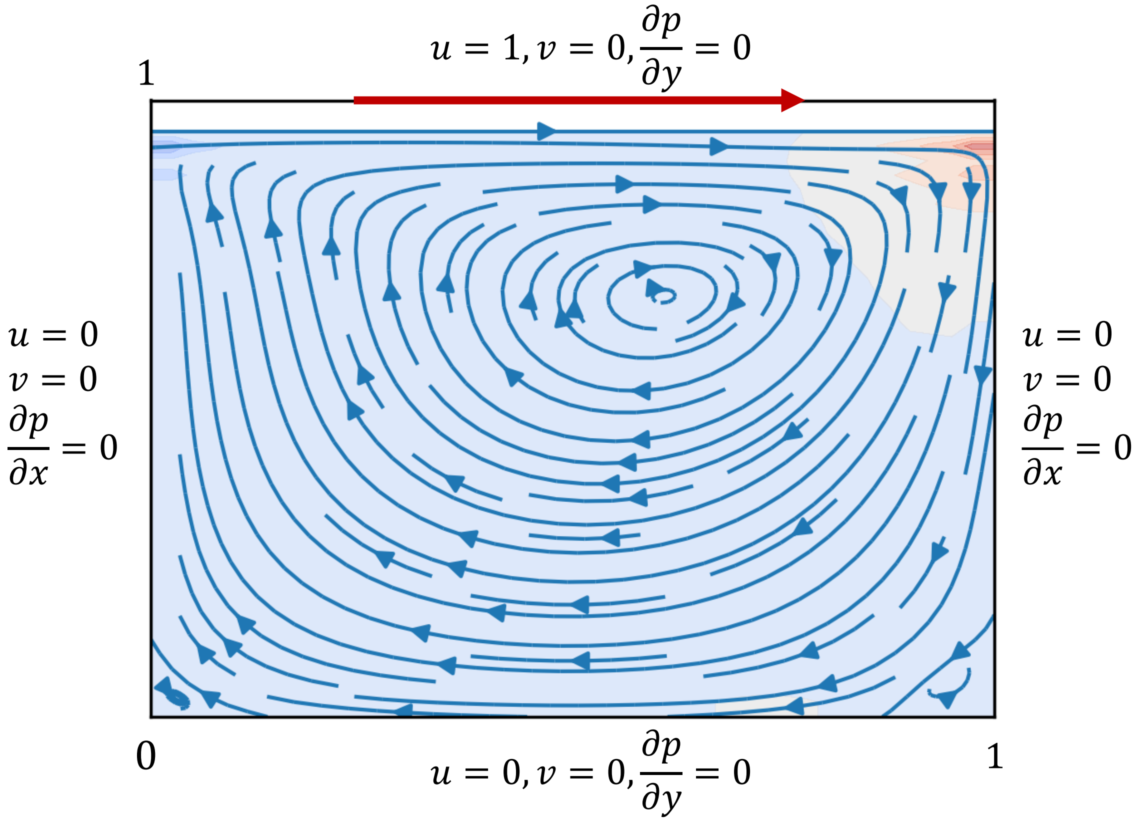
What about the pressure?

Recall: The pressure Poisson's Equation

$$\nabla^2 p = -\nabla.(\vec{v}.\nabla)\vec{v}$$



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What about the pressure?

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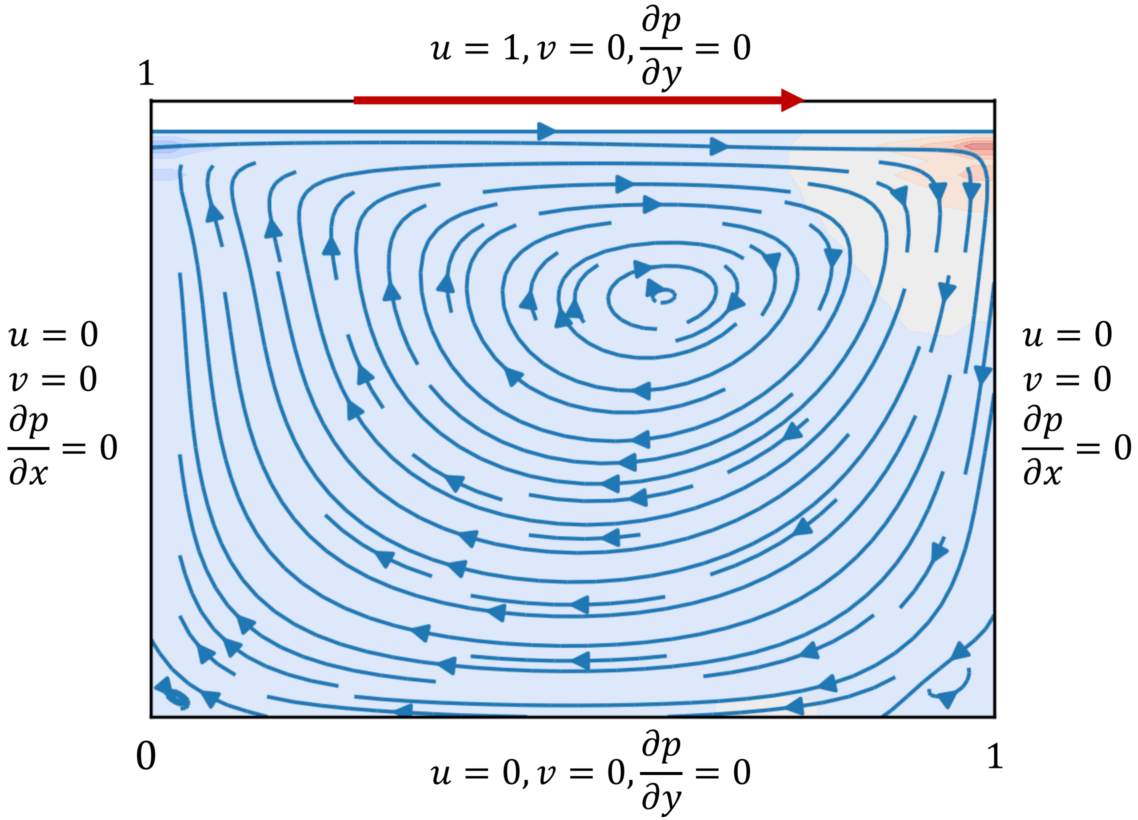
$$\nabla^2 p = -\nabla . (\vec{v}.\nabla)\vec{v}$$

$$\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) = -\rho\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\right)$$



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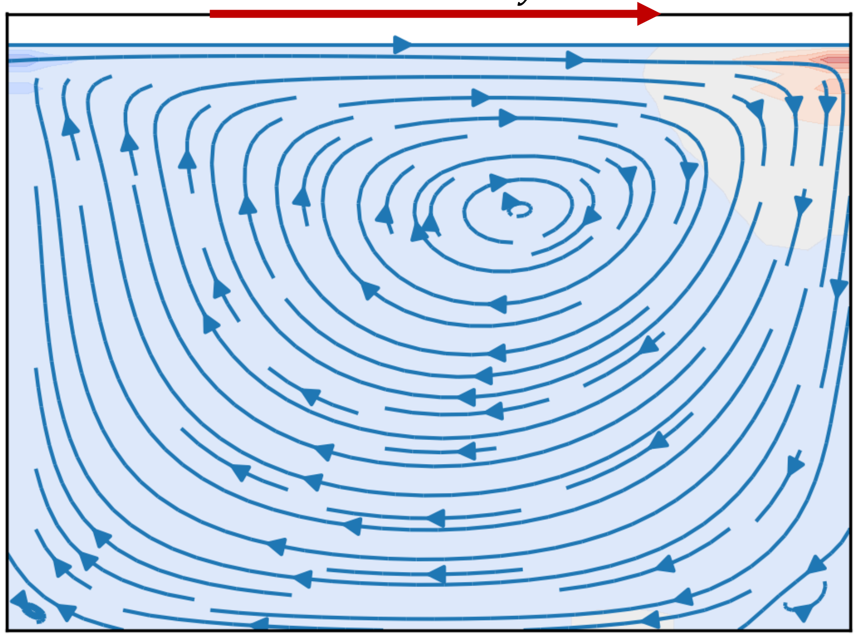


Now it's your turn to discretize the equations, code it & visualize the results!

Practical Session: 2D Lid Driven Cavity Flow

Continuity: $\nabla . \vec{v} = 0$

Momentum: $\dfrac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v} = -\dfrac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v}$

X-velocity: $\dfrac{\partial u}{\partial t} + u\left(\dfrac{\partial u}{\partial x}\right) + v\left(\dfrac{\partial u}{\partial y}\right) = -\dfrac{1}{\rho}\left(\dfrac{\partial p}{\partial x}\right) + \nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right)$

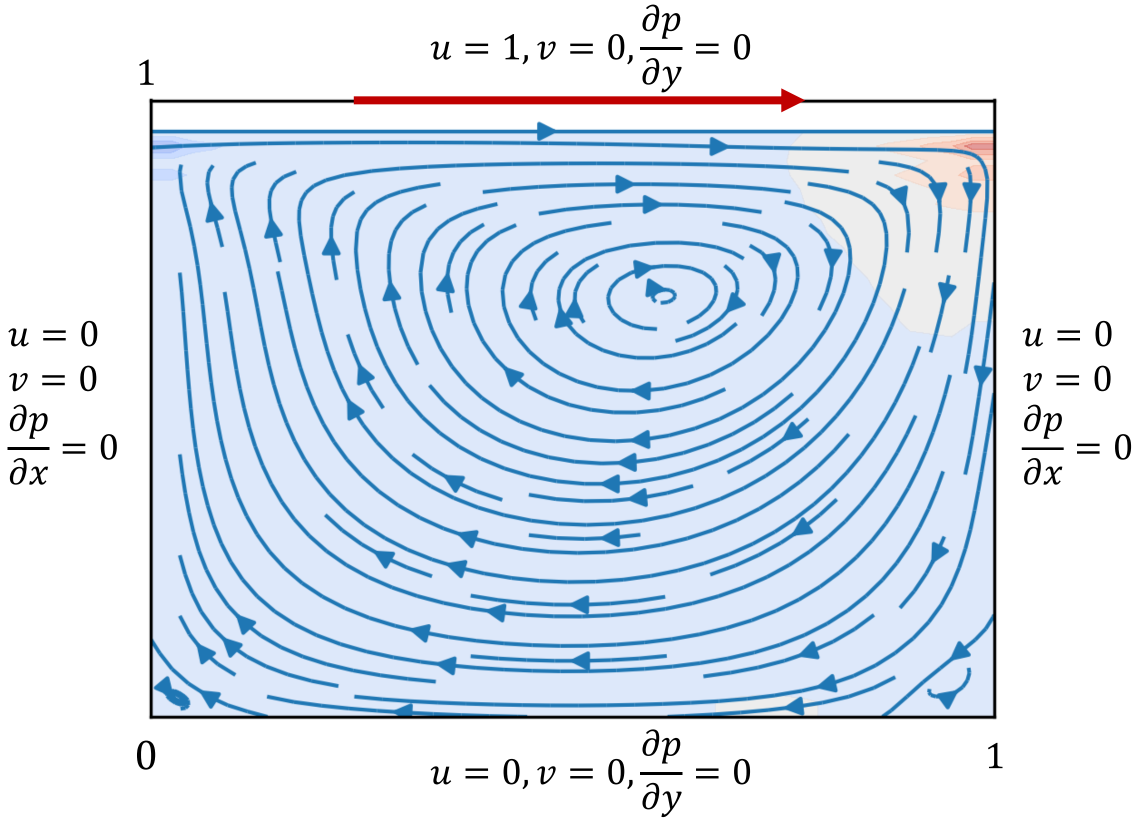
Y-velocity: $\dfrac{\partial v}{\partial t} + u\left(\dfrac{\partial v}{\partial x}\right) + v\left(\dfrac{\partial v}{\partial y}\right) = -\dfrac{1}{\rho}\left(\dfrac{\partial p}{\partial y}\right) + \nu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right)$

What about the pressure?

Recall: The pressure Poisson's Equation

$$\nabla^2 p = -\nabla . (\vec{v}.\nabla)\vec{v}$$

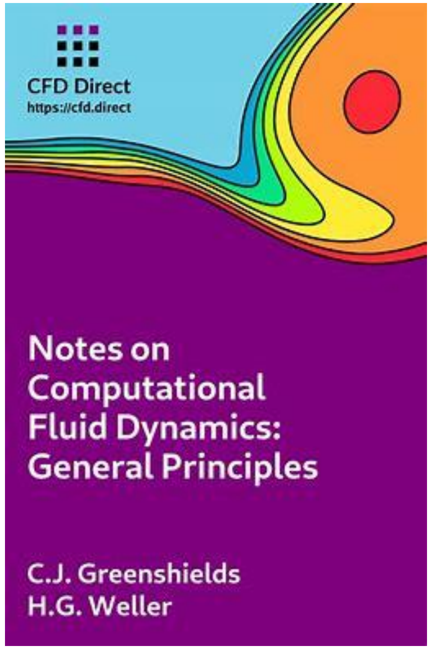
$$\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) = -\rho\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\right)$$

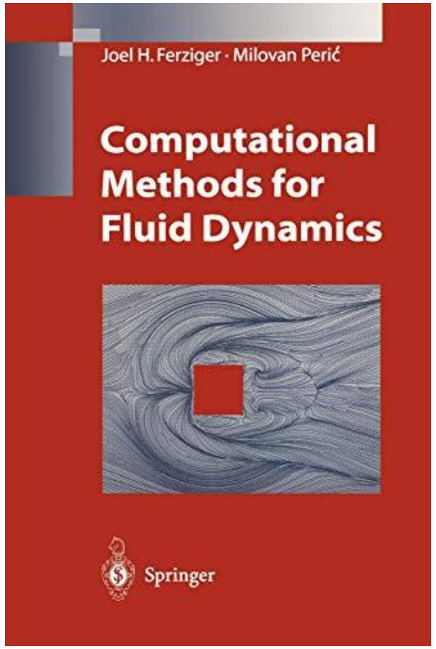


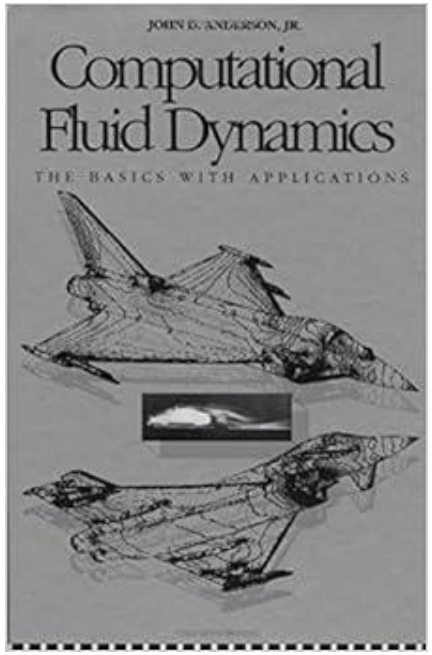
Now it's your turn to discretize the equations, code it & visualize the results!

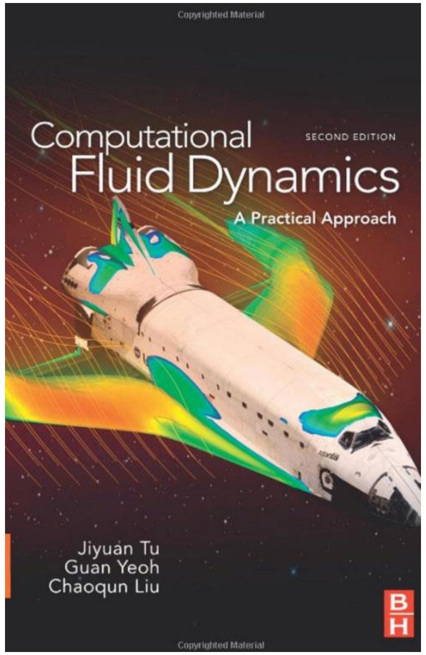
Good Luck!

References









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