MODELAIR

KTH VETENSKAP OCH KONST

A Hands-On Introduction to Computational Fluid Dynamics

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FLOW

SERC Swedish e-Science Research Centre

Computational Fluid Dynamics

17th Century
France and England

PURE
EXPERIMENTS

Computational Fluid Dynamics

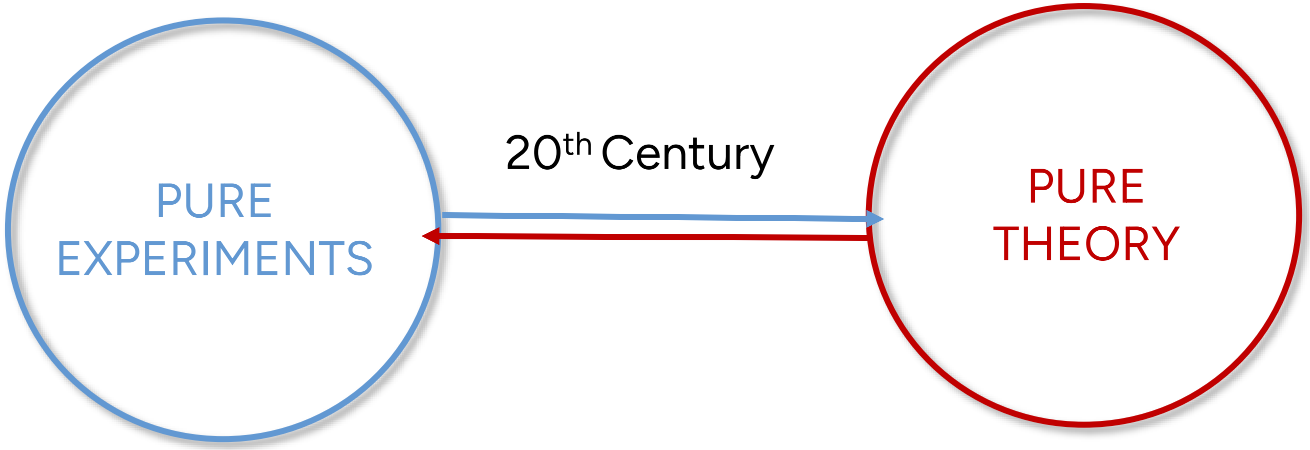
17th Century
France and England

PURE
EXPERIMENTS

PURE
THEORY

18th – 19th Century
Europe

Computational Fluid Dynamics



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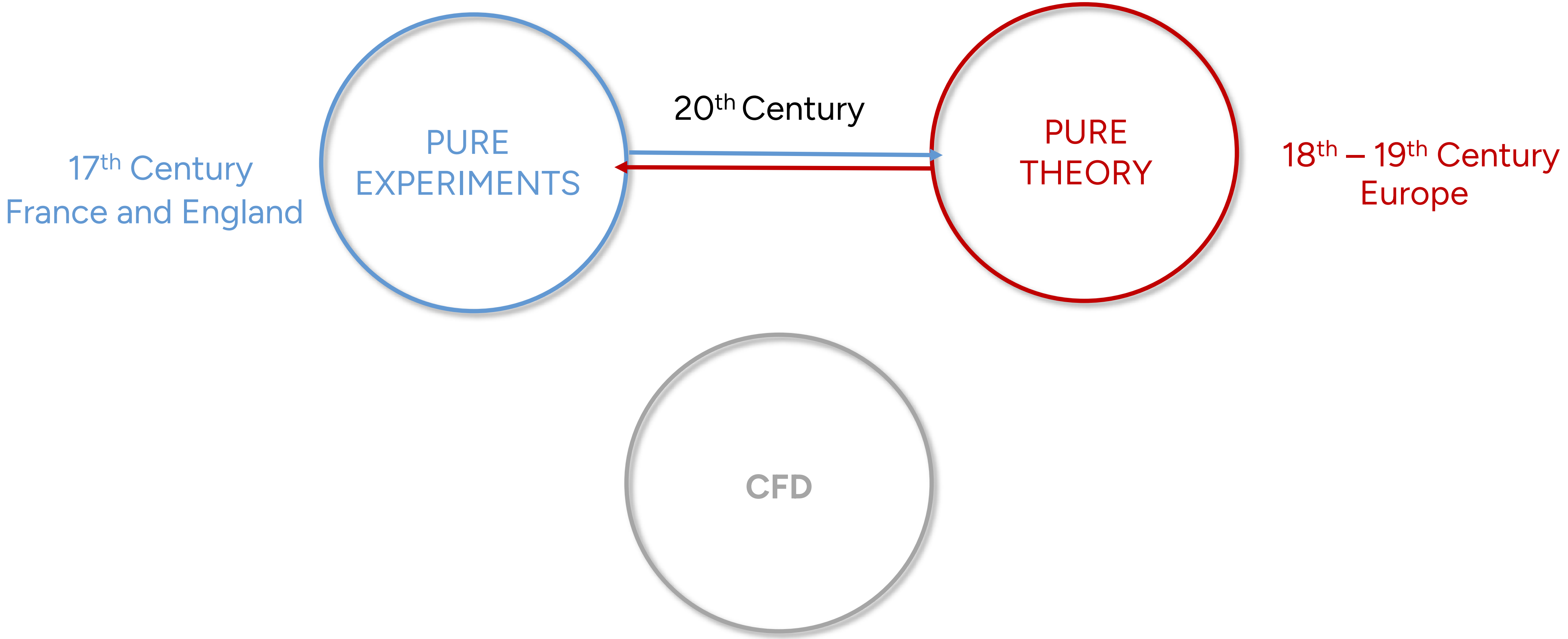
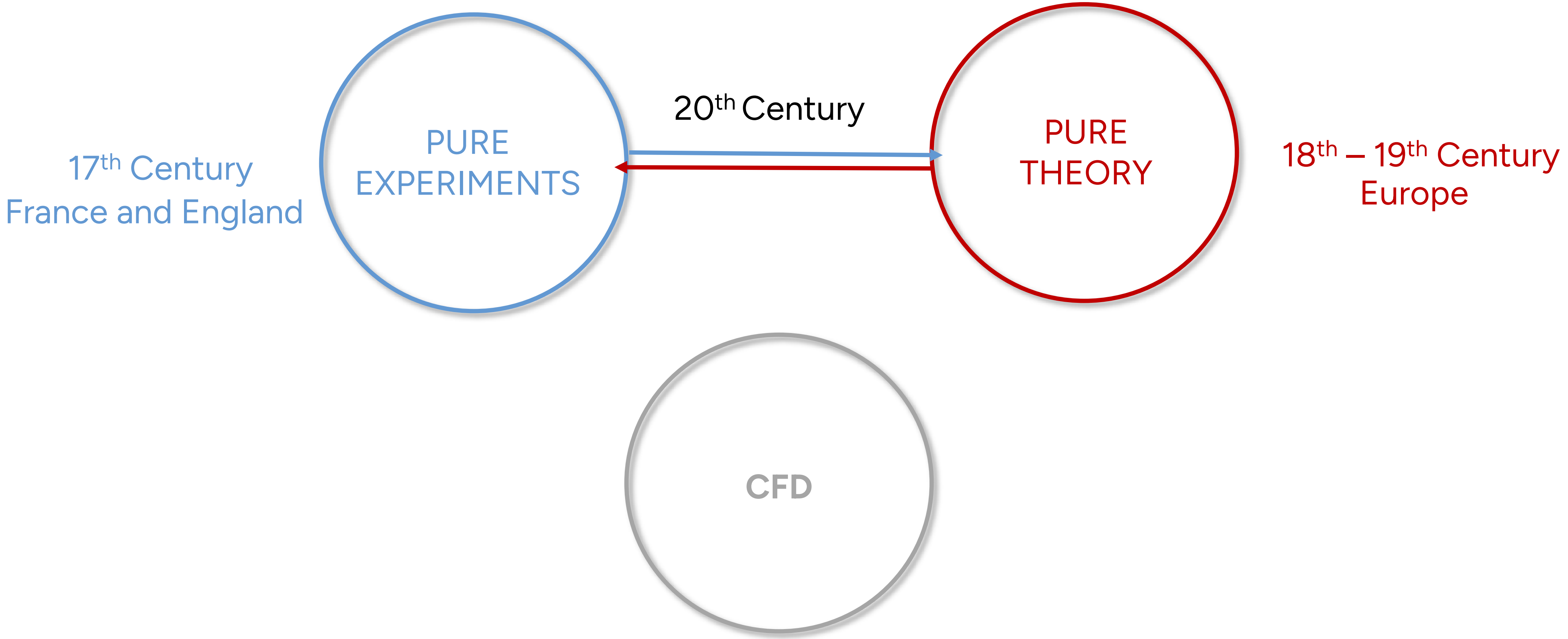


A new *"third approach"* in the philosophical study of fluid dynamics

Computational Fluid Dynamics



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CFD

A new *"third approach"* in the philosophical study of fluid dynamics

(Nothing more than that!)

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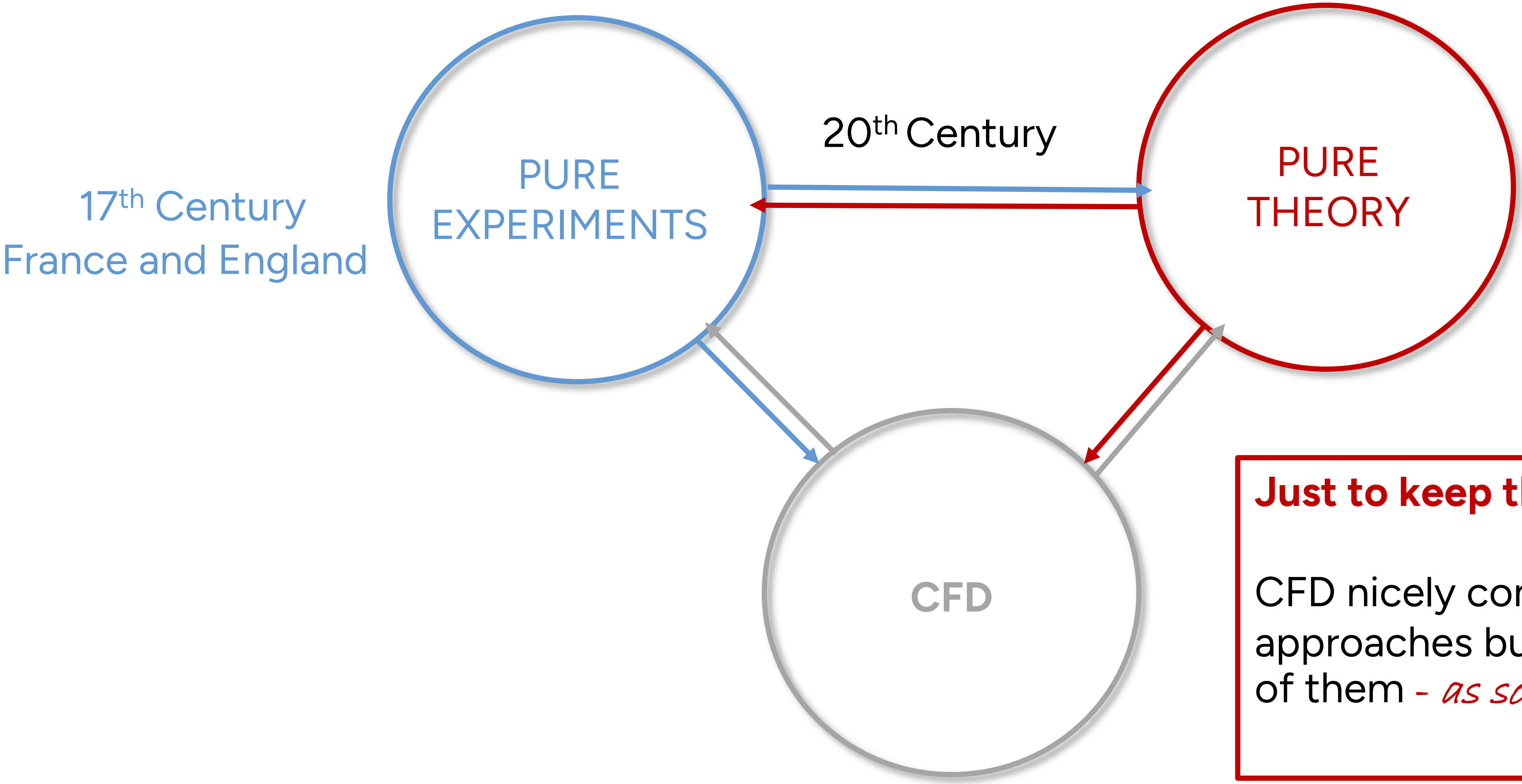

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Just to keep things in perspective:

CFD nicely compliments the other two approaches but will ***never*** replace either of them - *as sometimes suggested!*

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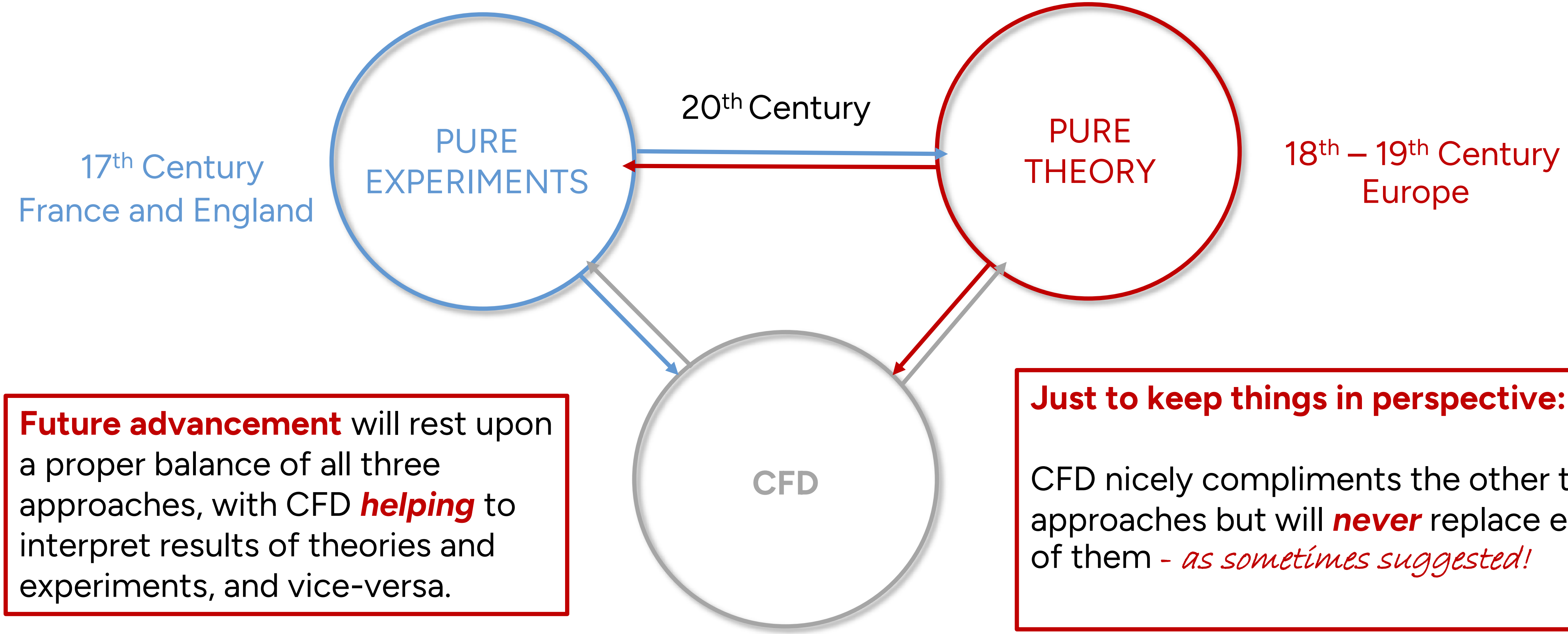
Computational Fluid Dynamics



Future advancement will rest upon a proper balance of all three approaches, with CFD ***helping*** to interpret results of theories and experiments, and vice-versa.

Just to keep things in perspective:

CFD nicely compliments the other two approaches but will ***never*** replace either of them - *as sometimes suggested!*

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CFD as a research tool

CFD results are directly analogous to Wind Tunnel results.

However, unlike a wind tunnel a computer program can be carried everywhere or can be accessed sitting *1000 miles away!*

A CFD program is, therefore, a *“readily transportable wind tunnel”*, where you can carry out *“numerical experiments”.*

CFD as a research tool: Who cares?

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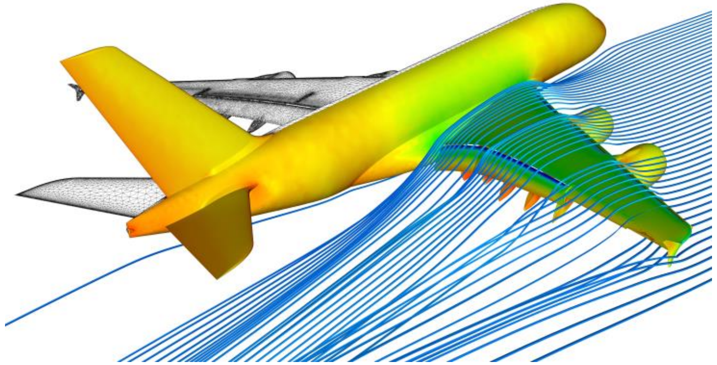
Aviation



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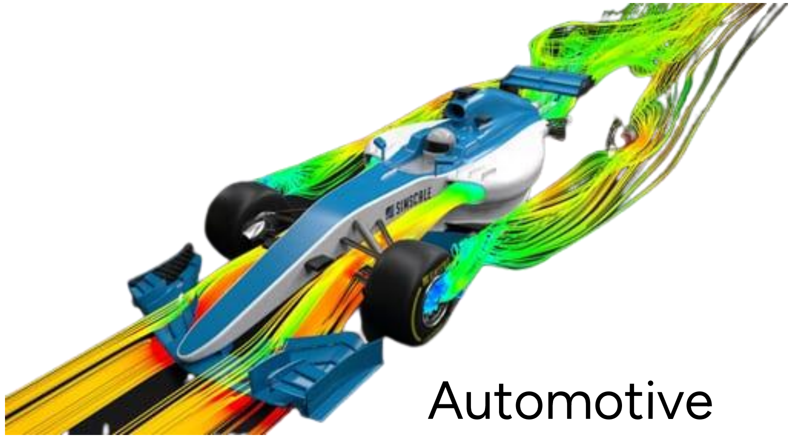
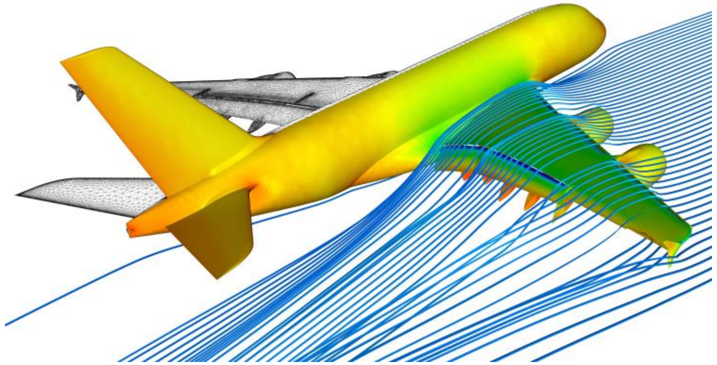
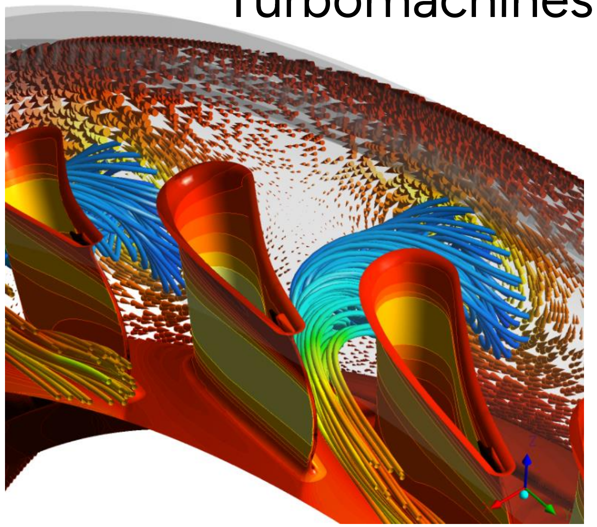




Automotive

CFD as a research tool: Who cares?

Turbomachines



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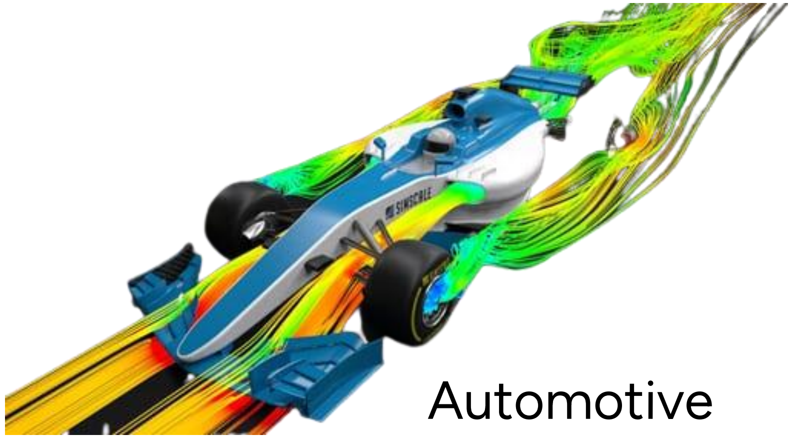

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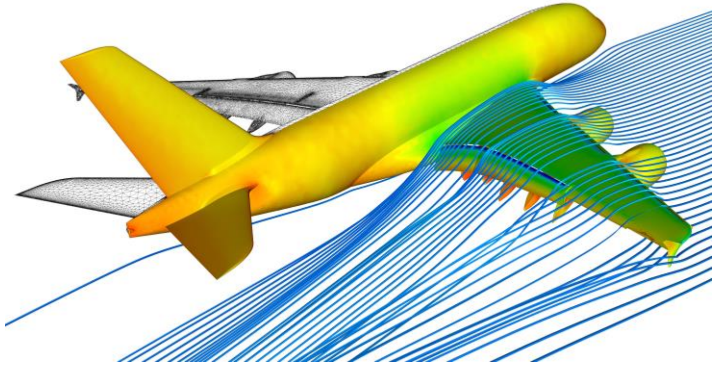
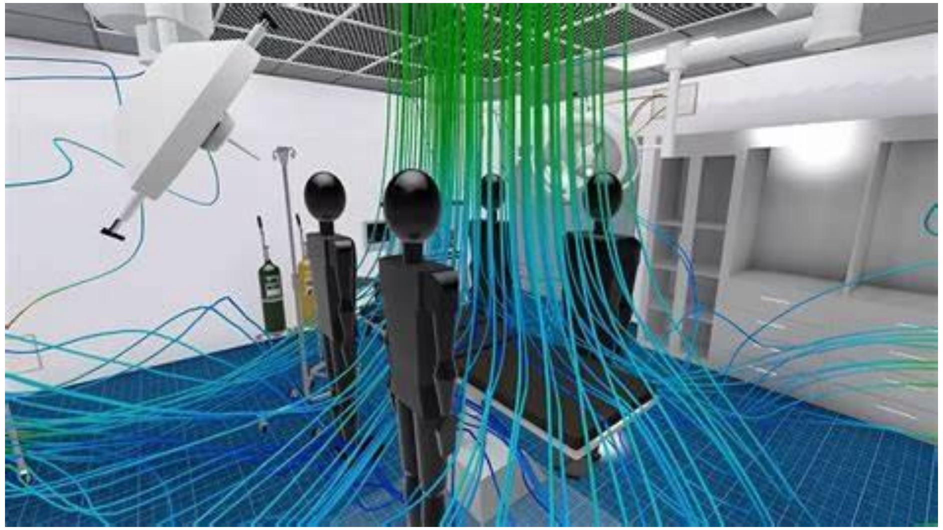
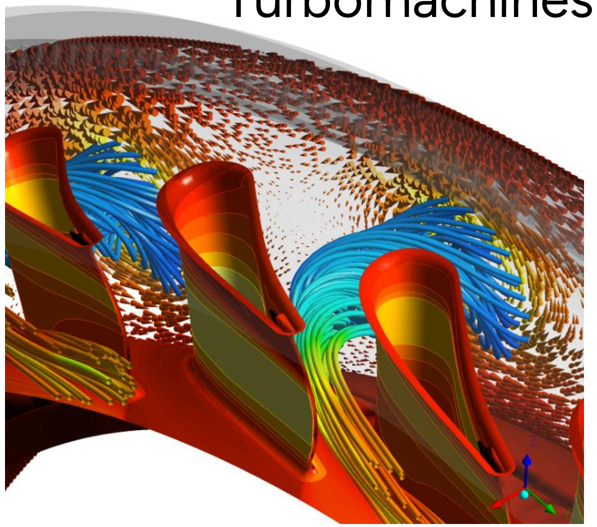


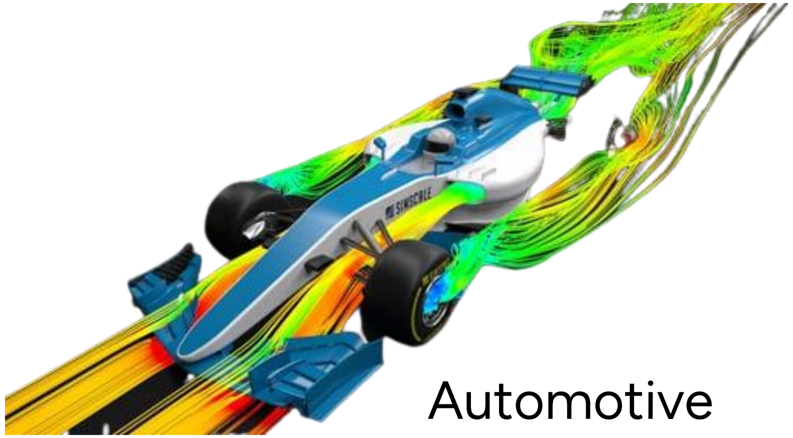
CFD as a research tool: Who cares?

HVAC



Turbomachines





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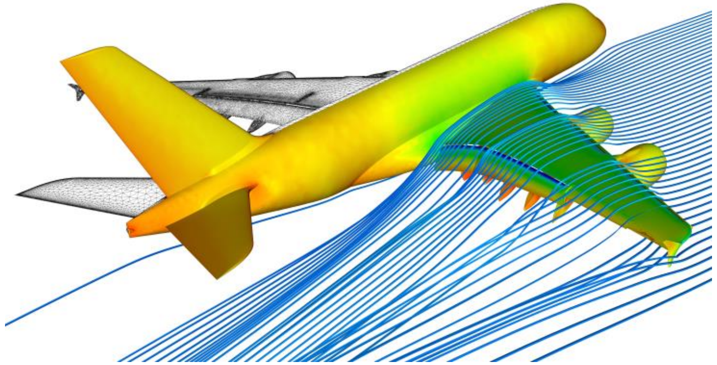
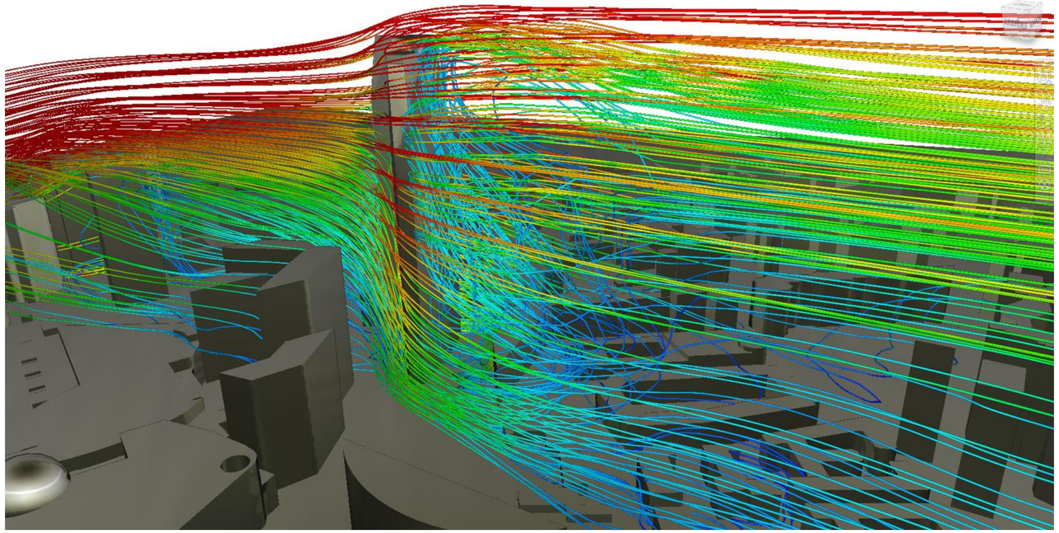


CFD as a research tool: Who cares?

Cities



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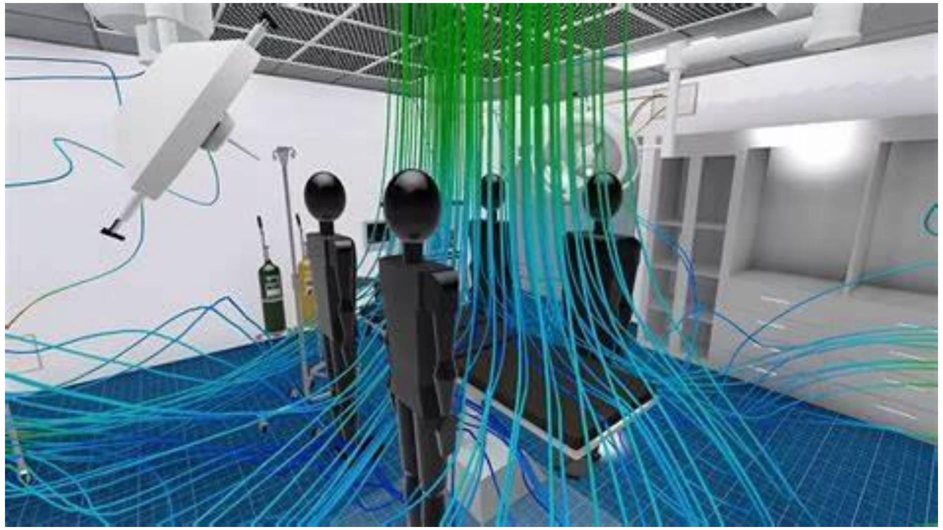

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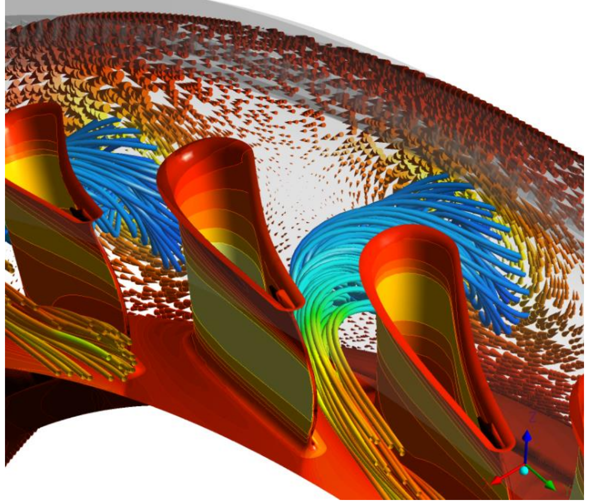

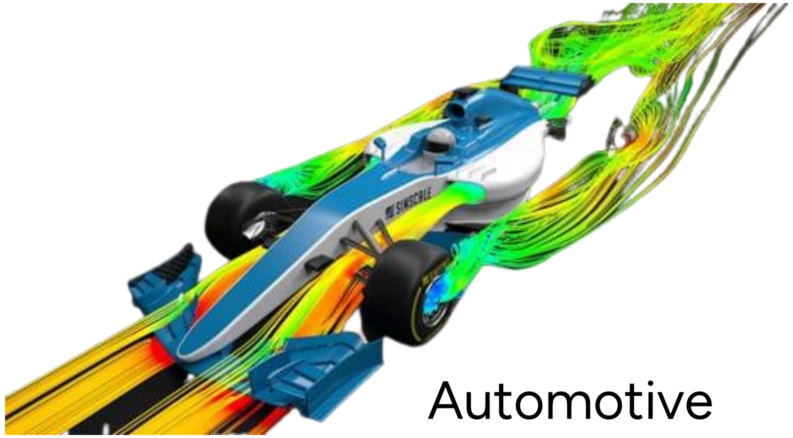

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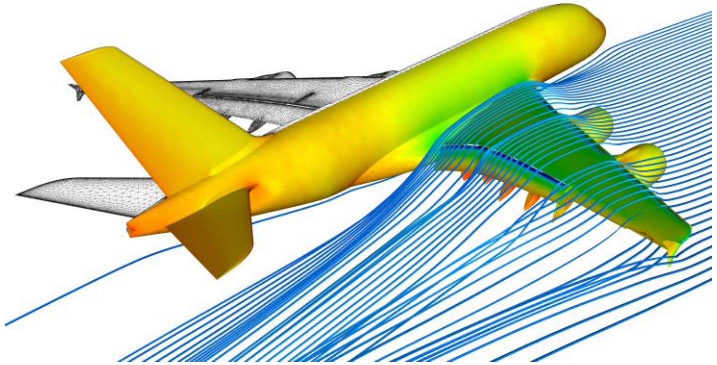

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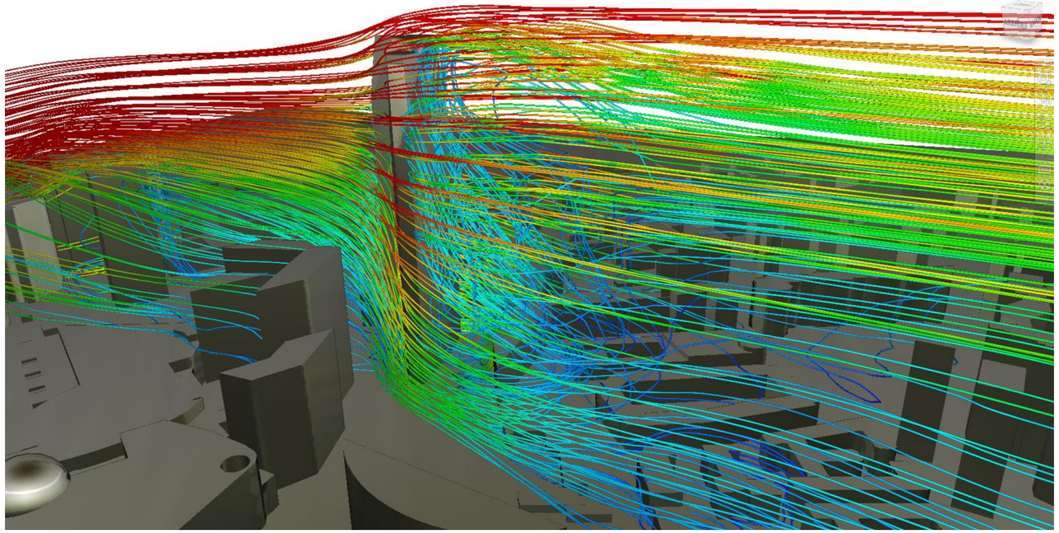

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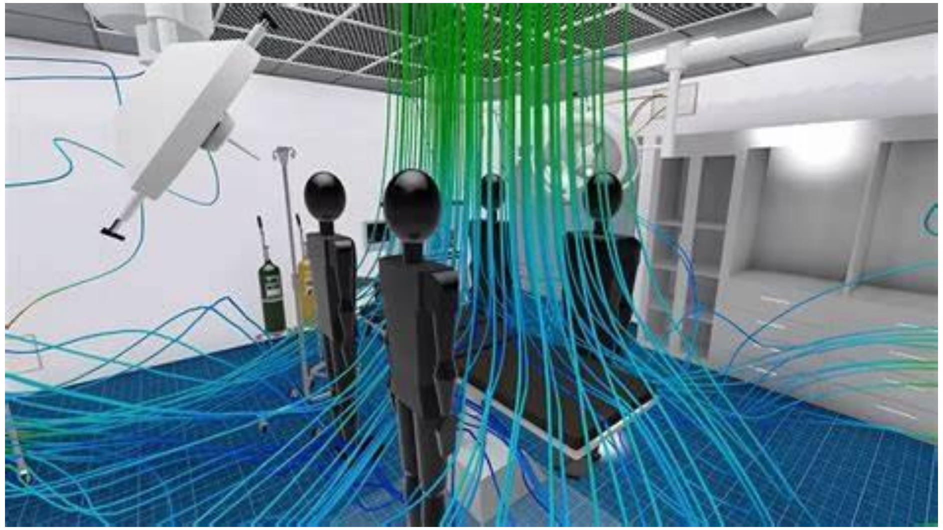

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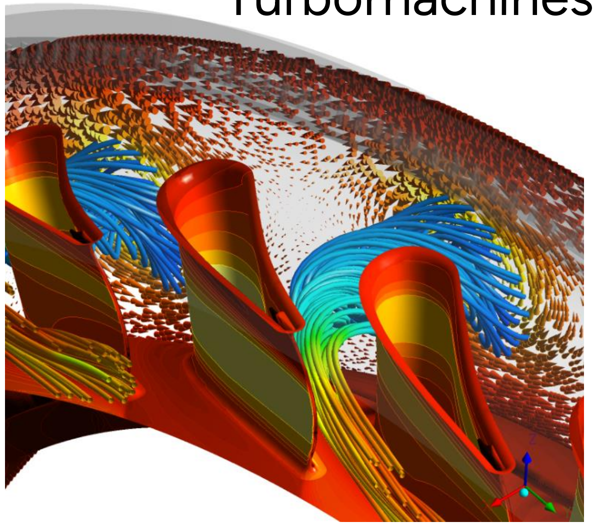

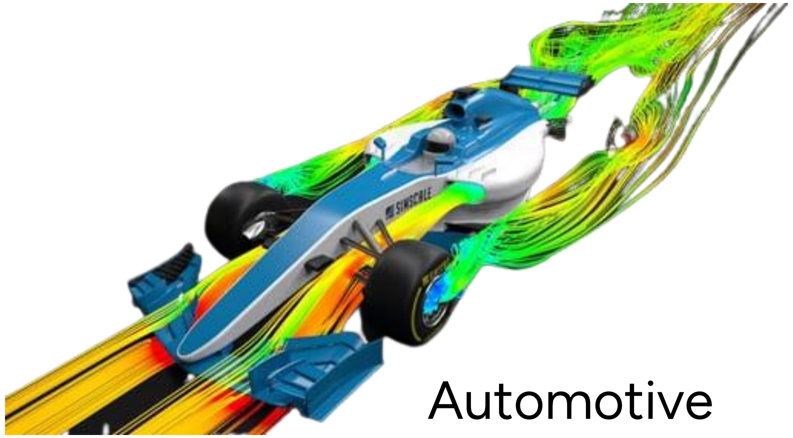

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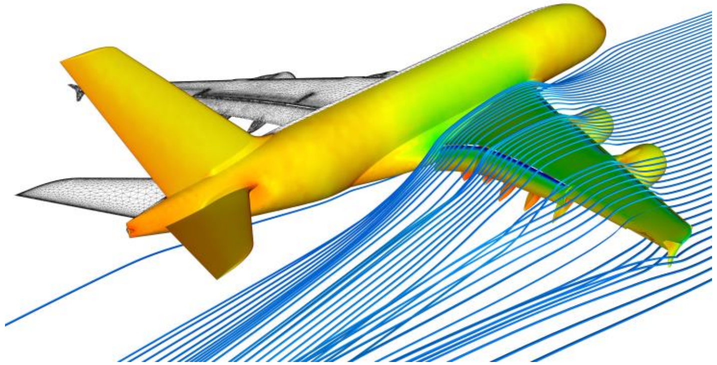


+ many more !

So How to CFD?

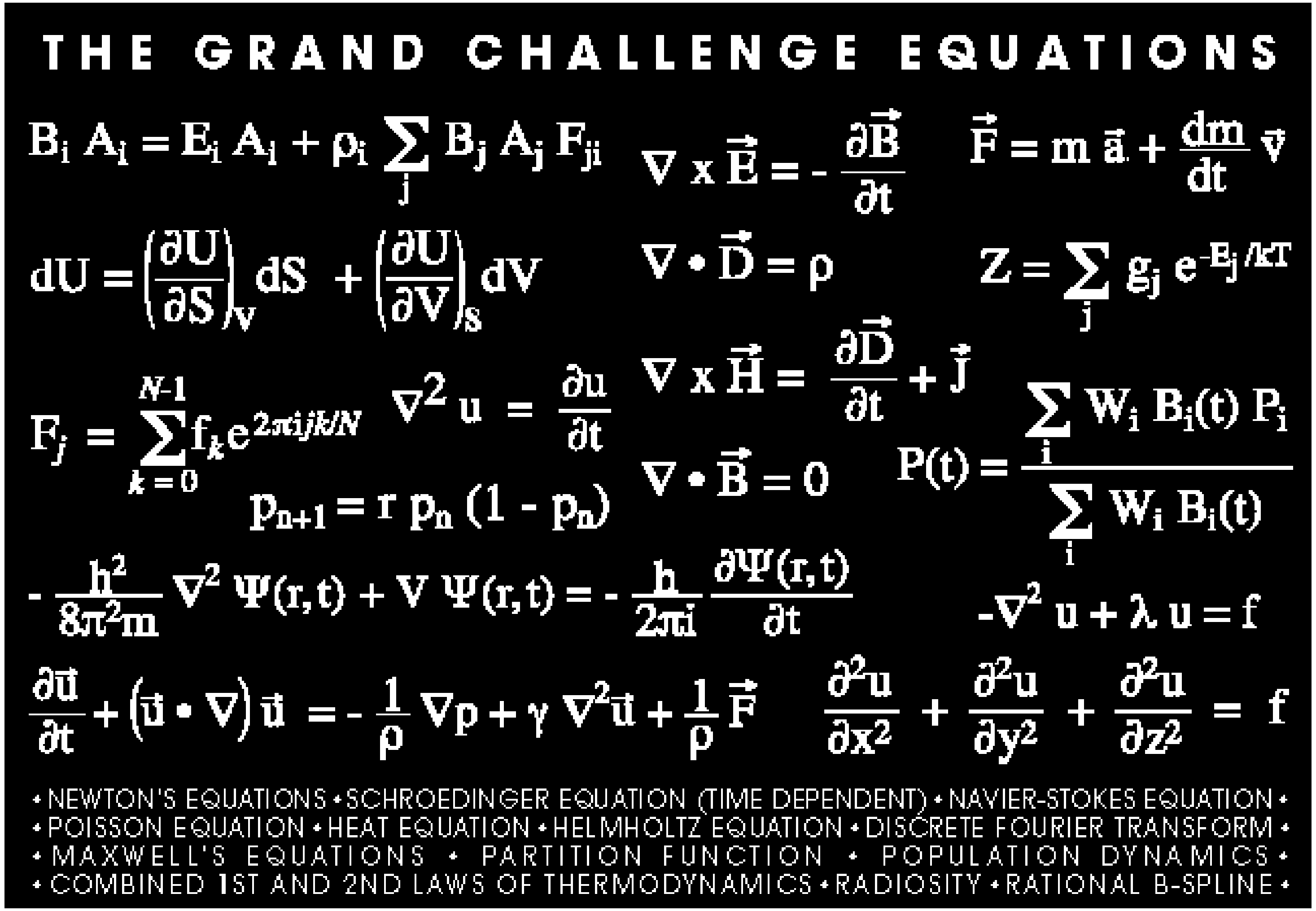
CFD involves:

- Identifying the physical phenomenon, that includes Governing equations, Initial & Boundary conditions.

- Breaking down the continuous problem to a discrete representation.

- Solving the discrete set of equations using adequate numerical methods.

- Post processing the results.



THE GRAND CHALLENGE EQUATIONS

$$B_i A_i = E_i A_i + \rho_i \sum_j B_j A_j F_{ji}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{F} = m\vec{a} + \frac{dm}{dt}\vec{v}$$

$$dU = \left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S dV$$

$$\nabla \cdot \vec{D} = \rho$$

$$Z = \sum_j g_j e^{-E_j/kT}$$

$$F_j = \sum_{k=0}^{N-1} f_k e^{2\pi i jk/N}$$

$$\nabla^2 u = \frac{\partial u}{\partial t}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$

$$\nabla \cdot \vec{B} = 0$$

$$P(t) = \frac{\sum_i W_i B_i(t) P_i}{\sum_i W_i B_i(t)}$$

$$p_{n+1} = r\, p_n (1 - p_n)$$

$$-\frac{h^2}{8\pi^2 m}\nabla^2 \Psi(r,t) + V\,\Psi(r,t) = -\frac{h}{2\pi i}\frac{\partial \Psi(r,t)}{\partial t}$$

$$-\nabla^2 u + \lambda u = f$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{1}{\rho}\nabla p + \gamma \nabla^2 \vec{u} + \frac{1}{\rho}\vec{F}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f$$

• NEWTON'S EQUATIONS • SCHROEDINGER EQUATION (TIME DEPENDENT) • NAVIER-STOKES EQUATION •
• POISSON EQUATION • HEAT EQUATION • HELMHOLTZ EQUATION • DISCRETE FOURIER TRANSFORM •
• MAXWELL'S EQUATIONS • PARTITION FUNCTION • POPULATION DYNAMICS •
• COMBINED 1ST AND 2ND LAWS OF THERMODYNAMICS • RADIOSITY • RATIONAL B-SPLINE •

What are we solving?

Navier Stokes equations

Continuity equation: $\partial_t \rho + \nabla.(\rho \boldsymbol{u}) = 0$

Momentum equation: $\partial_t \rho \boldsymbol{u} + \nabla.(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p - \nabla.\boldsymbol{\tau} = \boldsymbol{f}$

Energy equation: $\partial_t E + \nabla.[(E + p)\boldsymbol{u}] - \nabla.(\boldsymbol{\tau u}) - \nabla.(\kappa \nabla T) = S$

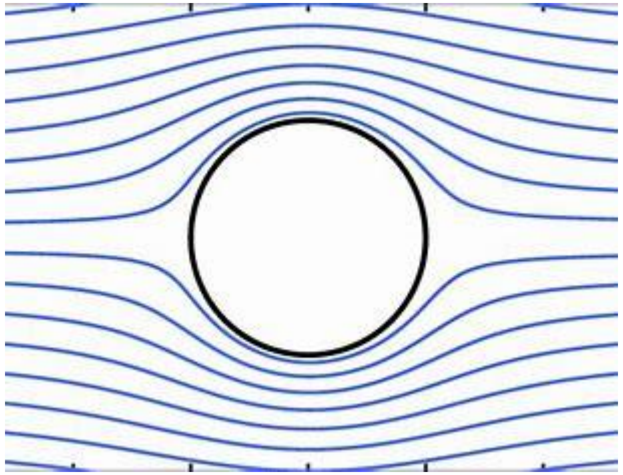
Depending on the nature of physics governing the fluid motion one or more terms might be negligible.

Presence of each term and their combinations determines the appropriate solution algorithm and the numerical procedure.

Classification of PDEs

Elliptic

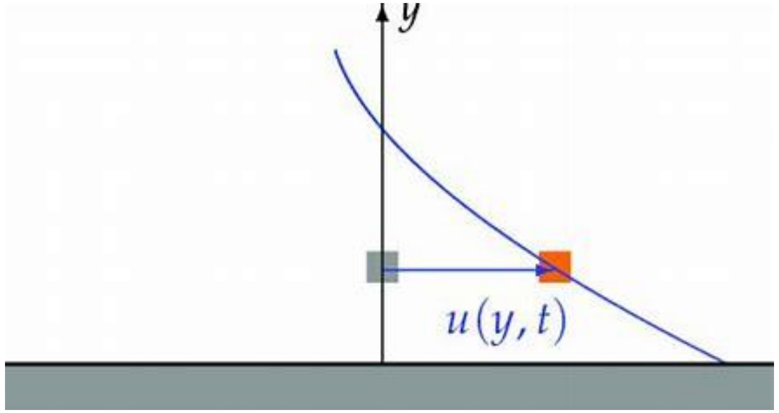
$$\nabla^2 u \; = 0$$



Potential Flow

Parabolic

$$\frac{\partial u}{\partial t} = \nu \nabla^2 u$$



Flow over an oscillating plate
(Stokes 2nd Problem)

Hyperbolic

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

Wave motion

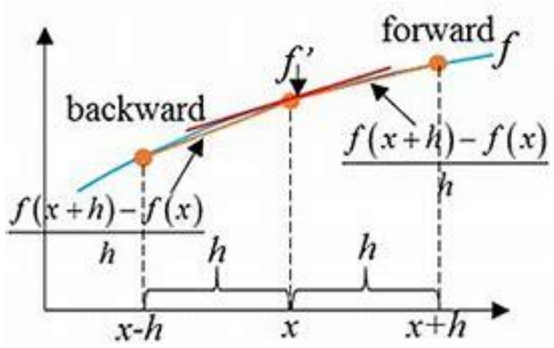
Equations belonging to each of these classifications behave in different ways both *physically* and *numerically*.

Techniques for Numerical Discretization

Governing Equations $\xrightarrow{\text{Discretization}}$ Numerical Analogue

Commonly used discretization methods

Finite Difference Methods



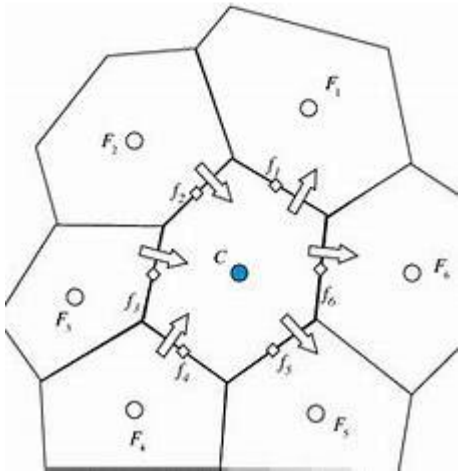
Spectral Methods

$$G(f)=\Im\{\cos(2\pi At)\}=\int_{-\infty}^{\infty}\frac{e^{i2\pi At}+e^{-i2\pi At}}{2}e^{-i2\pi ft}\,dt$$

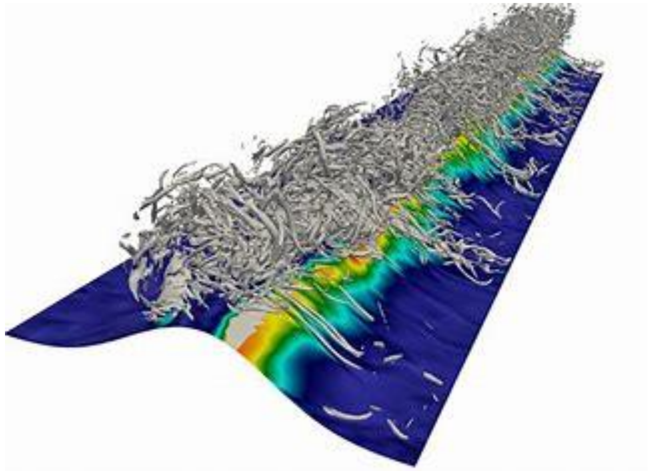
$$=\frac{1}{2}\left[\int_{-\infty}^{\infty}e^{i2\pi At}e^{-i2\pi ft}\,dt+\int_{-\infty}^{\infty}e^{-i2\pi At}e^{-i2\pi ft}\,dt\right]$$

$$=\frac{1}{2}\left[\delta(f-A)+\delta(f+A)\right]$$

Finite Volume Methods



Finite/Spectral Element Methods



Techniques for Numerical Discretization

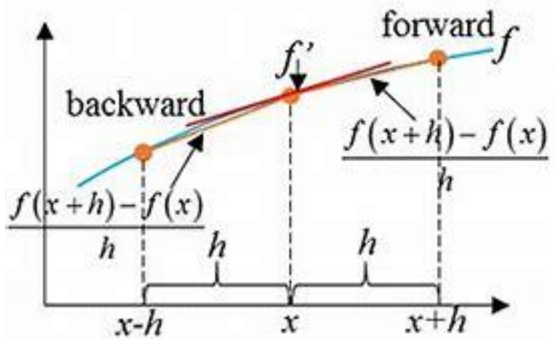
$$\text{Governing Equations} \xrightarrow{\text{Discretization}} \text{Numerical Analogue}$$

Commonly used discretization methods

TODAY

TOMORROW

Finite Difference Methods



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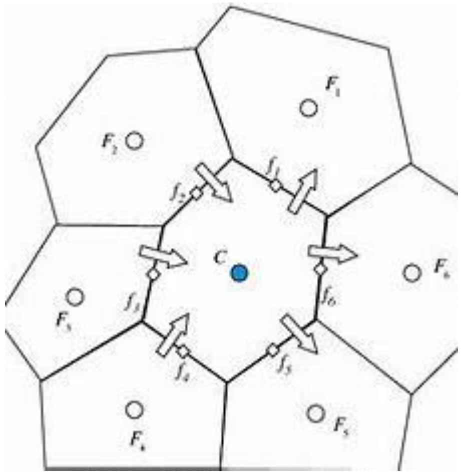
Finite Volume Methods



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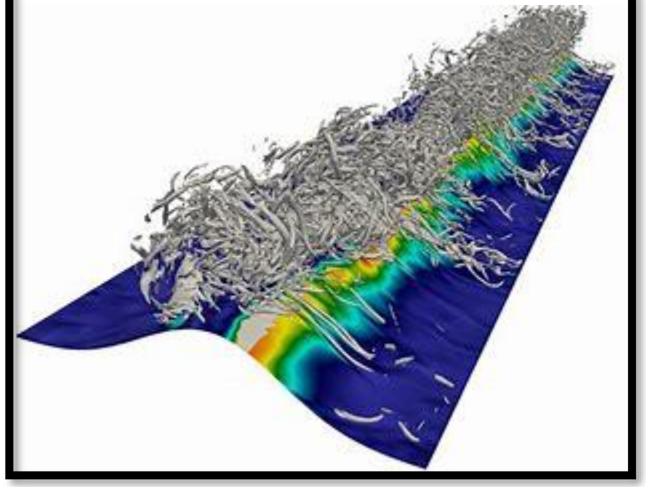

Finite Difference Method

Definition of a derivative

Exact

$$\frac{df}{dx} = \lim_{dx \to 0} \frac{f(x+dx) - f(x)}{dx}$$

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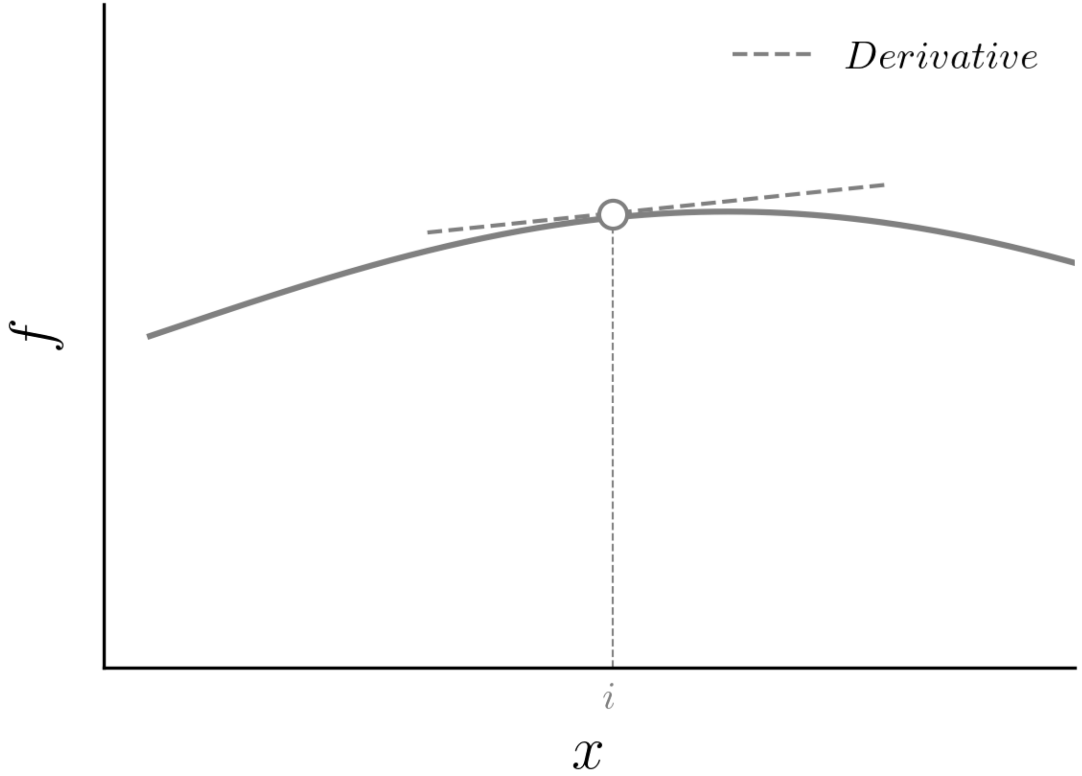

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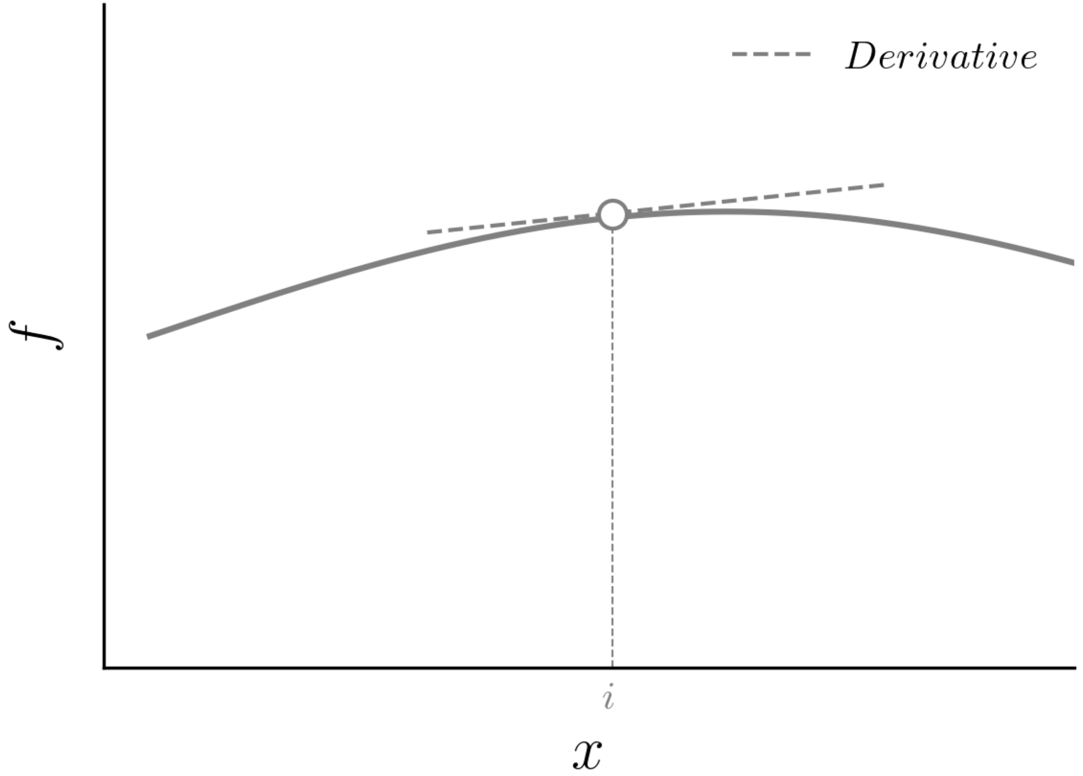

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$$\frac{df}{dx} \approx \frac{f(x+dx)-f(x)}{dx}$$

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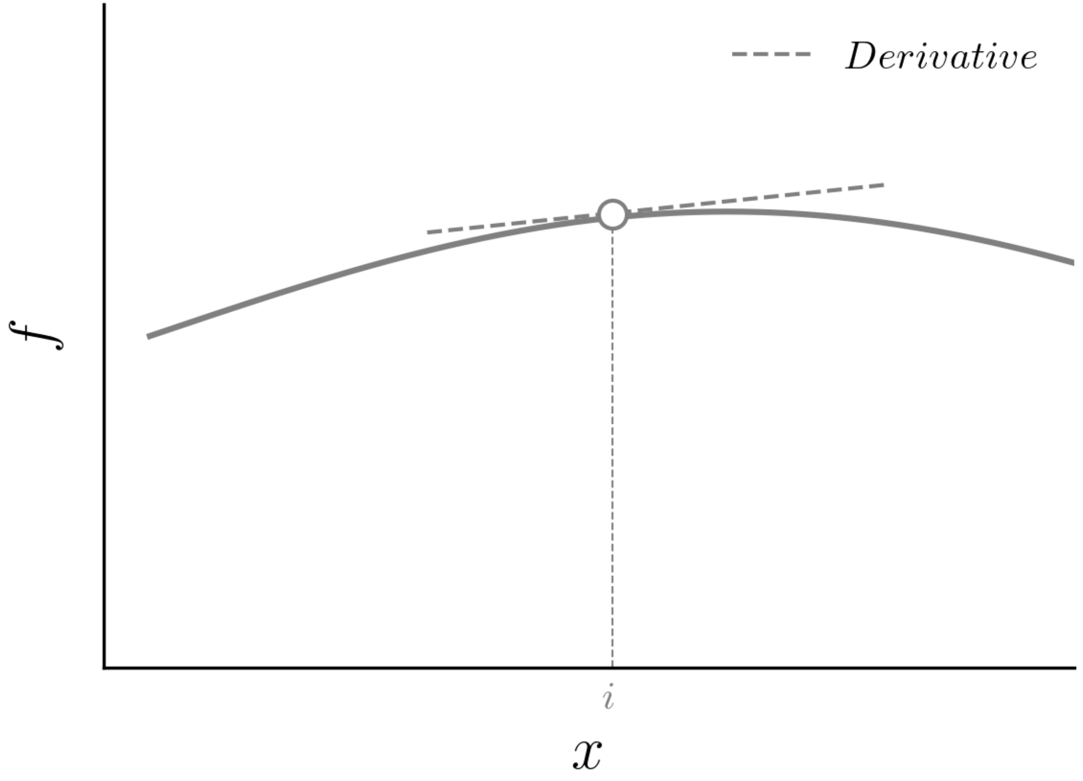

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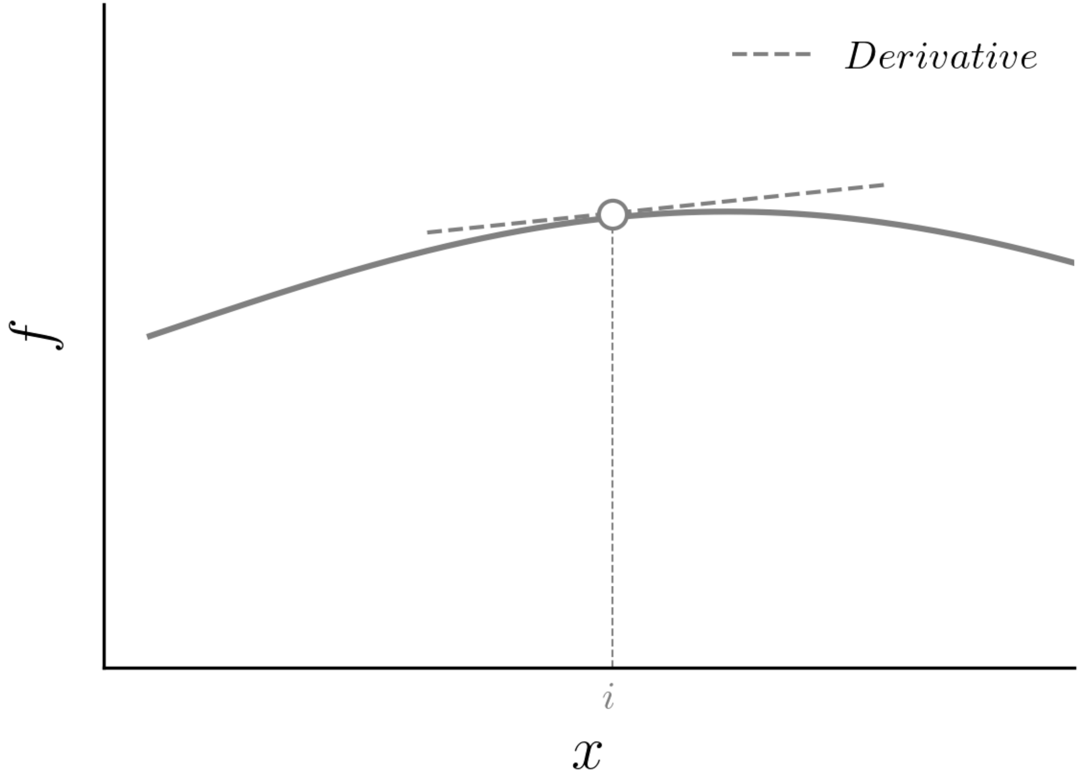

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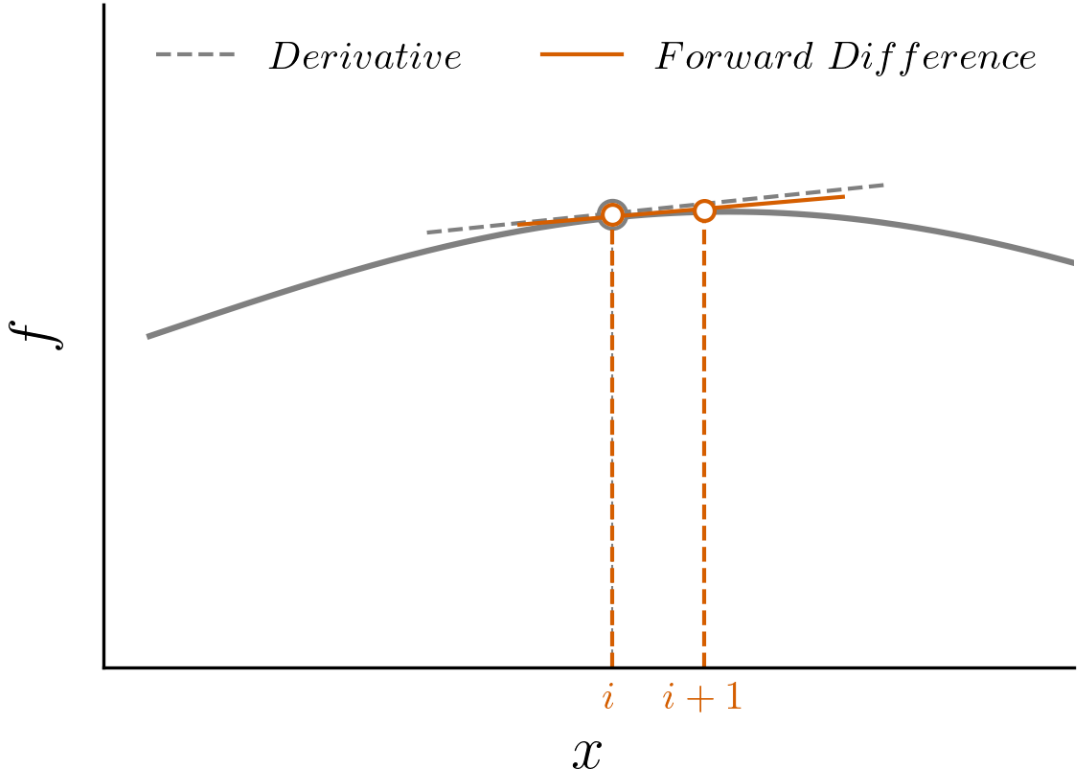

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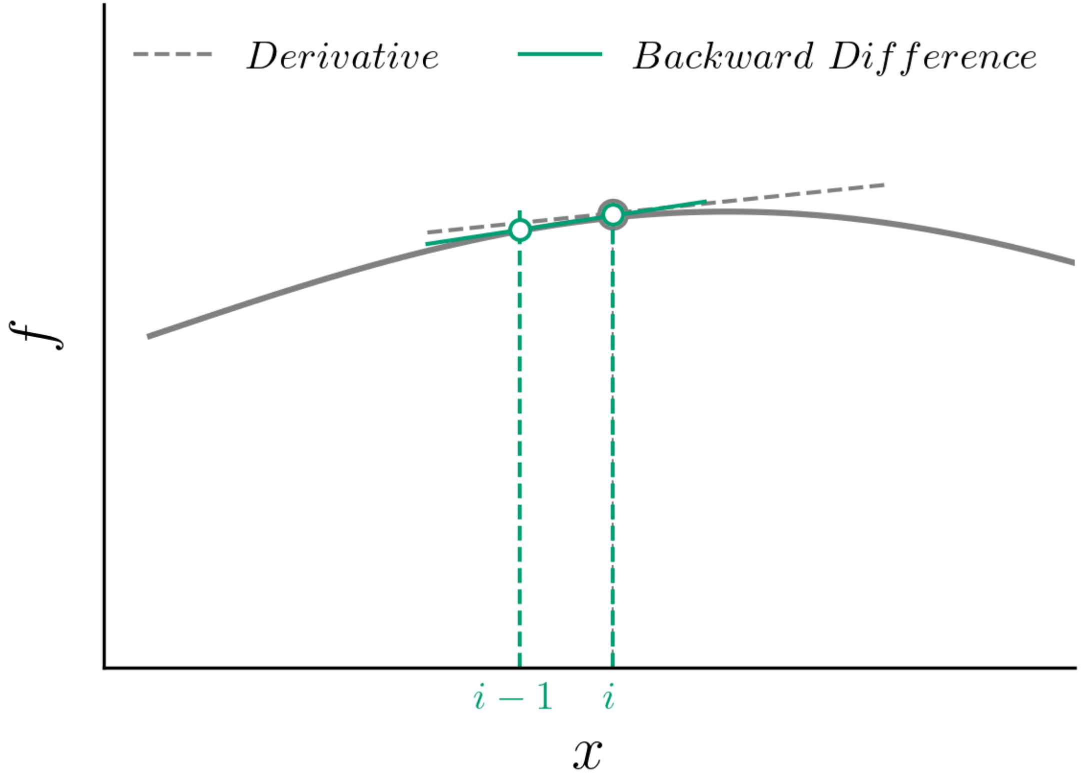

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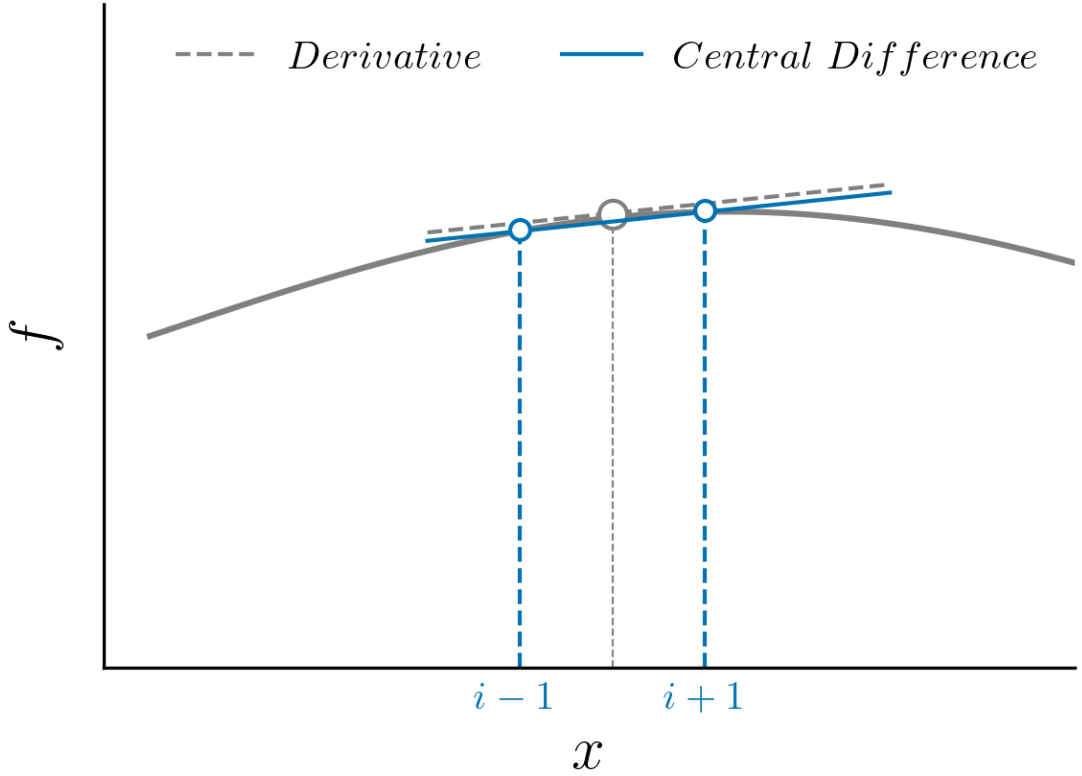

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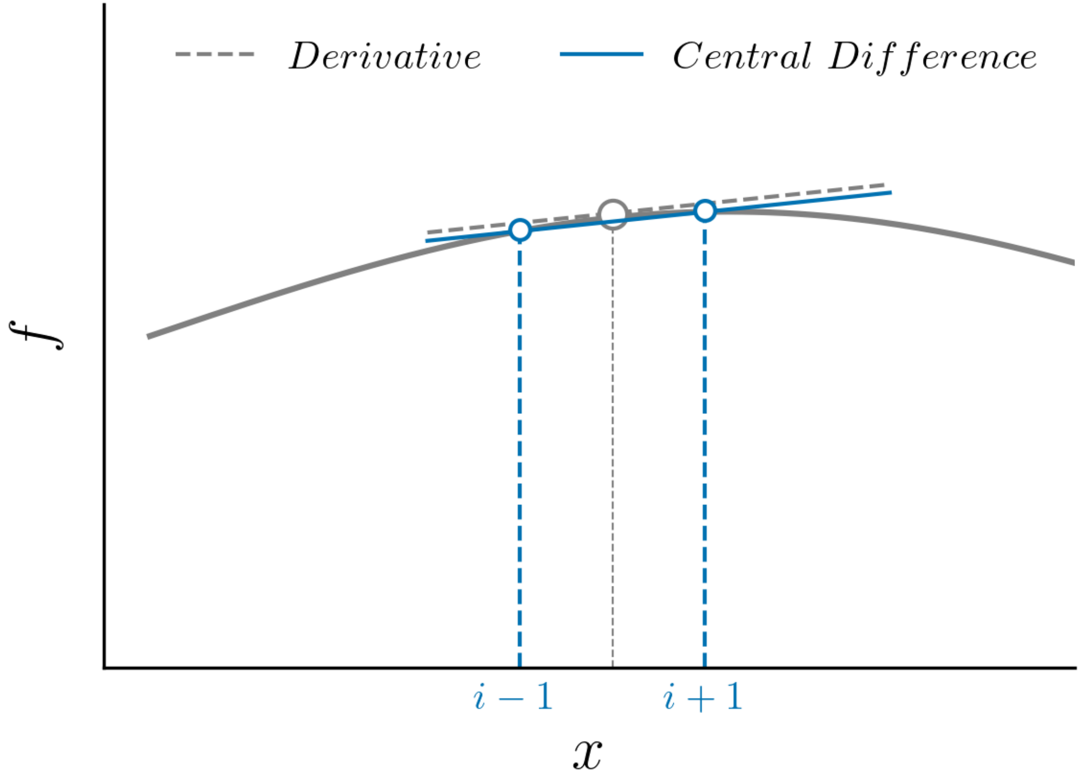

How good is the approximation?

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Taylor Series

$$f(x_0 + dx) = f(x_0) + \frac{df}{dx}\bigg|_{x_0} dx + \frac{1}{2!}\frac{d^2f}{dx^2}\bigg|_{x_0} dx^2 + \frac{1}{3!}\frac{d^3f}{dx^3}\bigg|_{x_0} dx^3 + \cdots$$

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$$\frac{f(x_0+dx)-f(x_0)}{dx}=\frac{df}{dx}\bigg|_{x_0}+O(dx)$$

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How good is the approximation?

Taylor Series

$$f(x_0+dx) = f(x_0) + \frac{df}{dx}\bigg|_{x_0} dx + \frac{1}{2!}\frac{d^2f}{dx^2}\bigg|_{x_0} dx^2 + \frac{1}{3!}\frac{d^3f}{dx^3}\bigg|_{x_0} dx^3 + \cdots$$

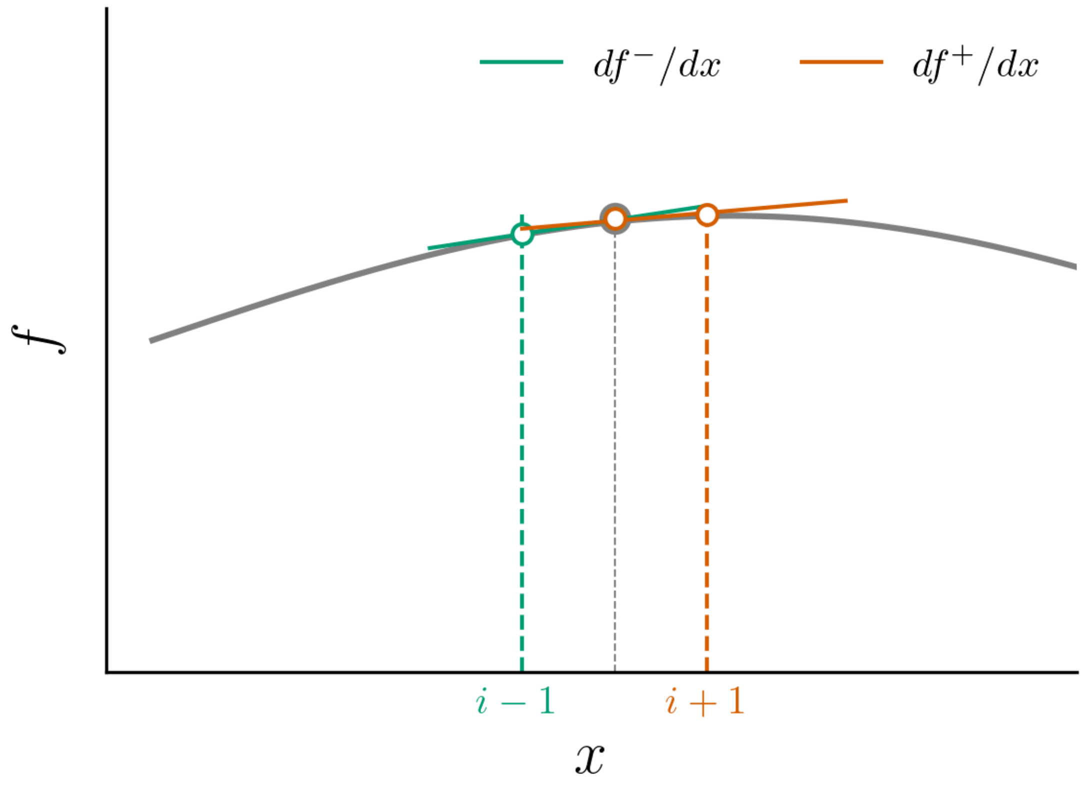
$$\frac{f(x_0+dx) - f(x_0)}{dx} = \frac{df}{dx}\bigg|_{x_0} + \frac{1}{2!}\frac{d^2f}{dx^2}\bigg|_{x_0} dx + \frac{1}{3!}\frac{d^3f}{dx^3}\bigg|_{x_0} dx^2 + \cdots$$

$$\frac{f(x_0+dx) - f(x_0)}{dx} = \frac{df}{dx}\bigg|_{x_0} + O(dx)$$

1st order accurate

Second derivative

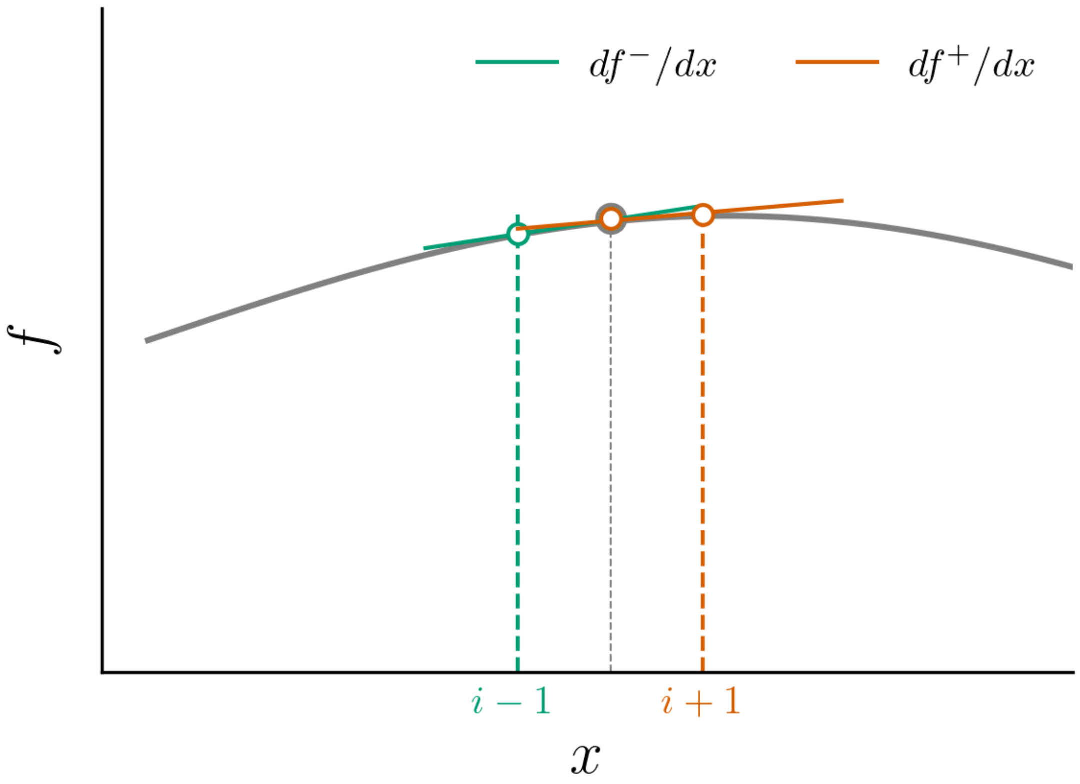
$$\frac{\left.\frac{df}{dx}\right|^{+} - \left.\frac{df}{dx}\right|^{-}}{dx} \approx \frac{d^2 f}{dx^2}$$



Second derivative

$$\frac{\left.\frac{df}{dx}\right|^{+} - \left.\frac{df}{dx}\right|^{-}}{dx} \approx \frac{d^2 f}{dx^2}$$

$$\frac{\frac{f(x+dx)-f(x)}{dx} - \frac{f(x+dx)-f(x)}{dx}}{dx} \approx \frac{d^2 f}{dx^2}$$



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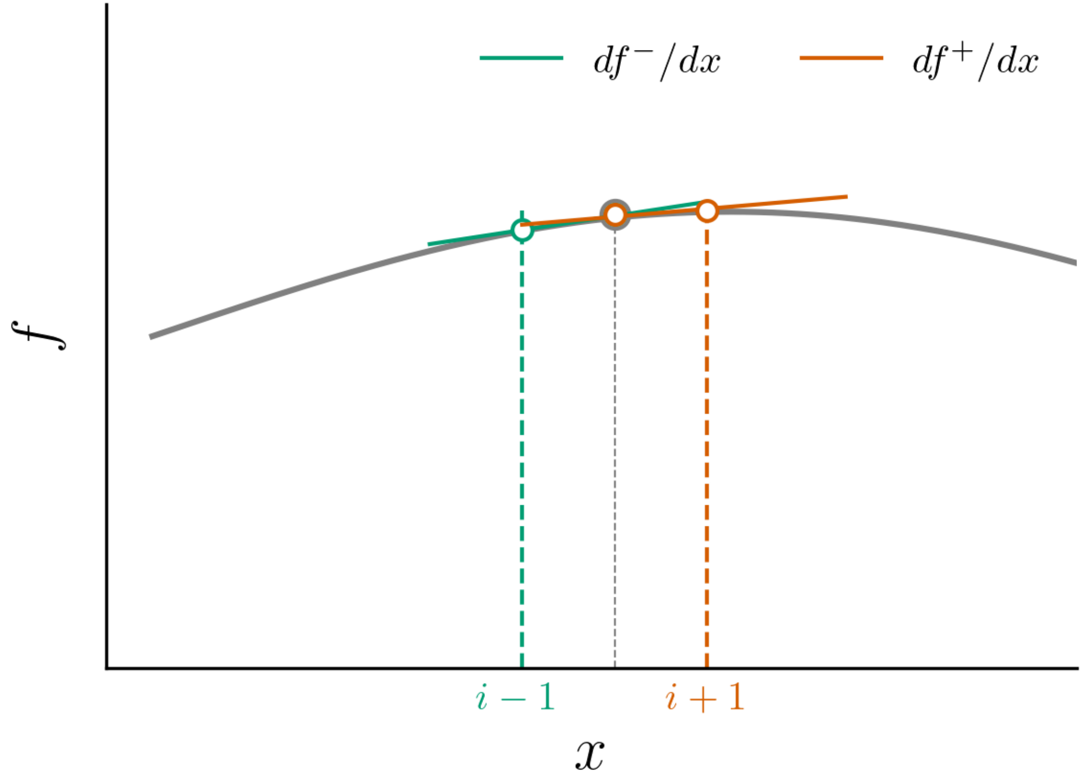
$$\frac{f(x+dx)-2f(x)+f(x-dx)}{dx^2} \approx \frac{d^2 f}{dx^2}$$



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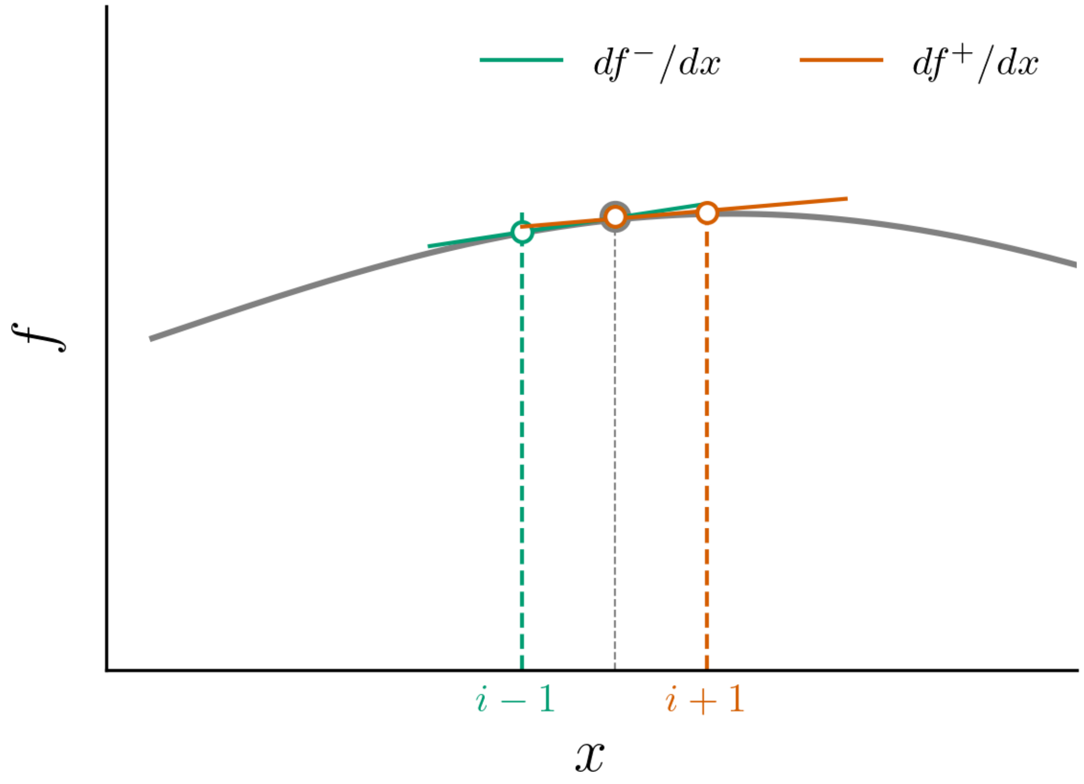

Central Differencing formula for 2nd derivative

Derivative Operators

Taylor Series

$$[f(x+dx)] = \left[f(x) + f'(x)dx + \frac{1}{2!}f''(x)dx^2 + \cdots\right]$$

$$[f(x)] = [f(x)]$$

$$[f(x-dx)] = \left[f(x) - f'(x)dx + \frac{1}{2!}f''(x)dx^2 + \cdots\right]$$

Derivative Operators

Taylor Series

$$a[f(x+dx)] = a\left[f(x) + f'(x)dx + \frac{1}{2!}f''(x)dx^2 + \cdots\right]$$

$$b[f(x)] = b[f(x)]$$

$$c[f(x-dx)] = c\left[f(x) - f'(x)dx + \frac{1}{2!}f''(x)dx^2 + \cdots\right]$$

Derivative Operators

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$$af(x+dx) + bf(x) + cf(x-dx) \approx (a+b+c)f(x) + (a-c)f'(x)dx + \frac{(a+c)}{2}f''(x)dx^2$$

Derivative Operators

$$af(x+dx)+bf(x)+cf(x-dx)\approx(a+b+c)f(x)+(a-c)f'(x)dx+\frac{(a+c)}{2}f''(x)dx^2$$

Derivative Operators

$$af(x+dx)+bf(x)+cf(x-dx)\approx(a+b+c)f(x)+(a-c)f'(x)dx+\frac{(a+c)}{2}f''(x)dx^2$$

1st Derivative

$$(a+b+c)=0$$

$$(a\qquad -c)=1/dx$$

$$(a\qquad +c)=0$$

Derivative Operators

$$af(x+dx)+bf(x)+cf(x-dx)\approx(a+b+c)f(x)+(a-c)f'(x)dx+\frac{(a+c)}{2}f''(x)dx^2$$

1st Derivative

$$\begin{aligned}(a+b+c)&=0\\(a\quad\;\;-c)&=1/dx\\(a\quad\;\;+c)&=0\end{aligned}$$

$$\begin{array}{cccc} A & w & & s \\ \begin{bmatrix}1 & 1 & 1\\ 1 & 0 & -1\\ 1 & 0 & 1\end{bmatrix} & \begin{pmatrix}a\\ b\\ c\end{pmatrix} & = & \begin{pmatrix}0\\ 1/dx\\ 0\end{pmatrix}\end{array}$$

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$$a=\frac{1}{2dx}$$
$$b=0$$
$$c=\frac{-1}{2dx}$$

Derivative Operators

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2nd Derivative

$$(a+b+c)=0$$
$$(a\qquad -c)=0$$
$$(a\qquad +c)=2!/dx^2$$

$$\overset{A}{\begin{bmatrix}1&1&1\\1&0&-1\\1&0&1\end{bmatrix}}\overset{w}{\begin{pmatrix}a\\b\\c\end{pmatrix}}=\overset{s}{\begin{pmatrix}0\\0\\2/dx^2\end{pmatrix}}\xrightarrow{\text{Inversion}}\quad a=\frac{1}{dx^2}\quad b=\frac{-2}{dx^2}\quad c=\frac{1}{dx^2}$$

Finite Difference in action

1D Linear Convection

$$\frac{du}{dt} + c\frac{du}{dx} = 0$$

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$$u(x,0) = u_0(x)$$

Exact Solution:

$$u(x,t) = u_0(x - ct)$$

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With an Initial Condition (given as a wave), the equation represents propagation of a wave with a speed *c*, *without change in its shape*!

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Backward Difference

$$\frac{du}{dx} \approx \frac{u(x) - u(x - \Delta x)}{\Delta x}$$

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$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

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Let's write out 1st CFD code !

Barba et al., (2018). CFD Python: the 12 steps to Navier-Stokes equations. Journal of Open-Source Education, 1(9), 21. https://doi.org/10. 21105/jose.00021

1st Code: 1D Linear Convection

```python
import numpy as np
import matplotlib.pyplot as plt
import time, sys
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nt = 25
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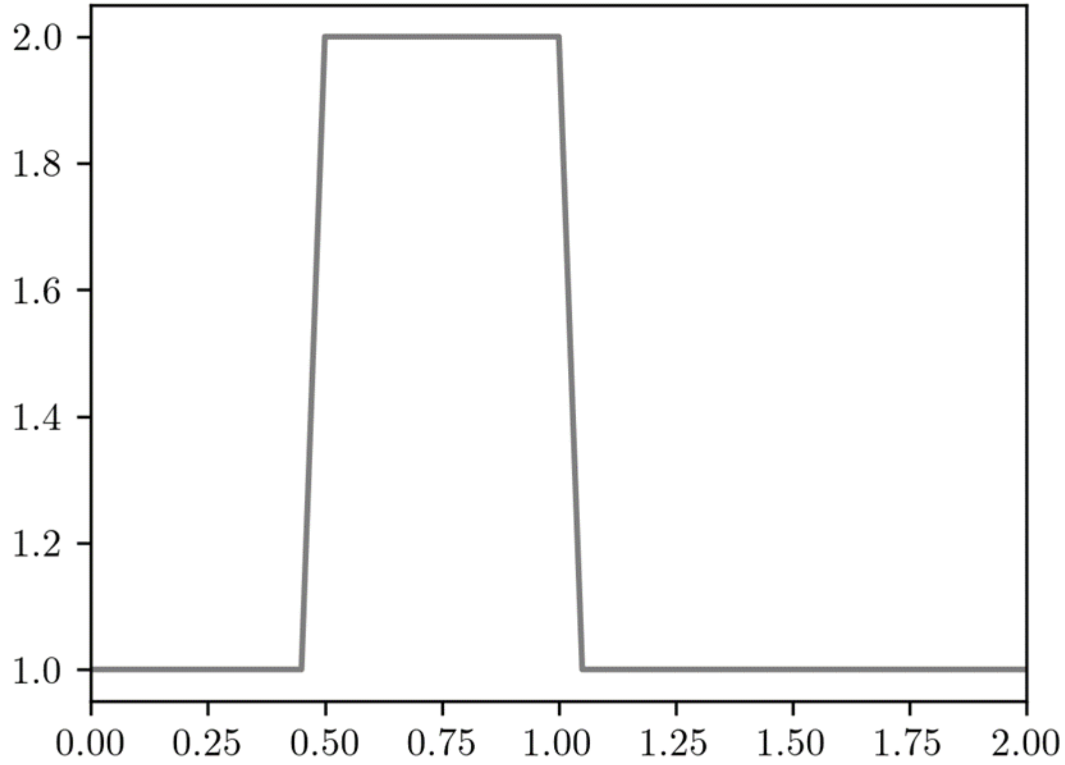
The initial velocity u_0 is given as 2 in the interval $0.5 \leq x \leq 1$ and 1 elsewhere in $(0,2)$ $\Longrightarrow$ Hat function



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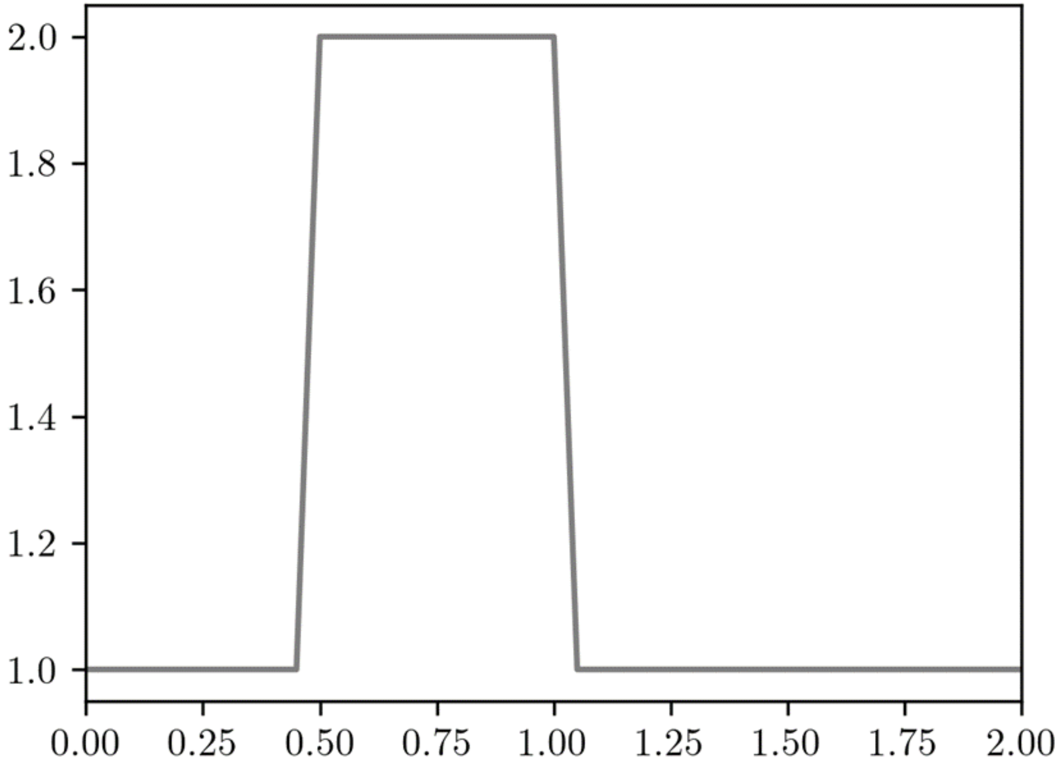

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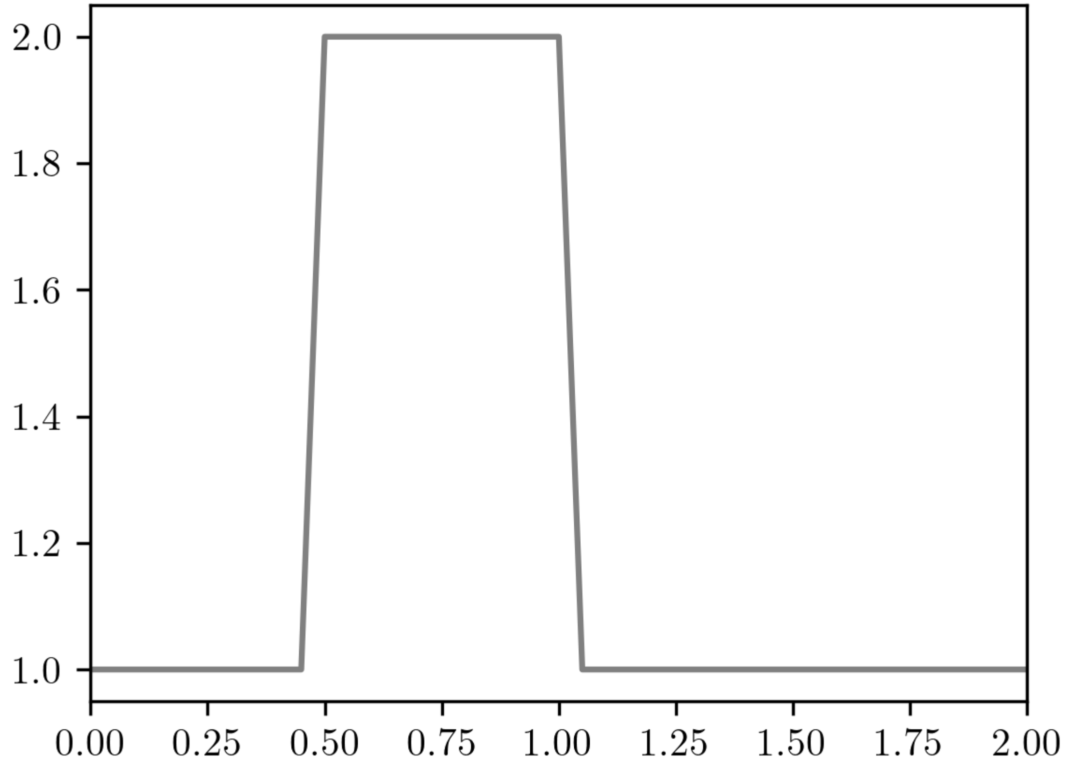

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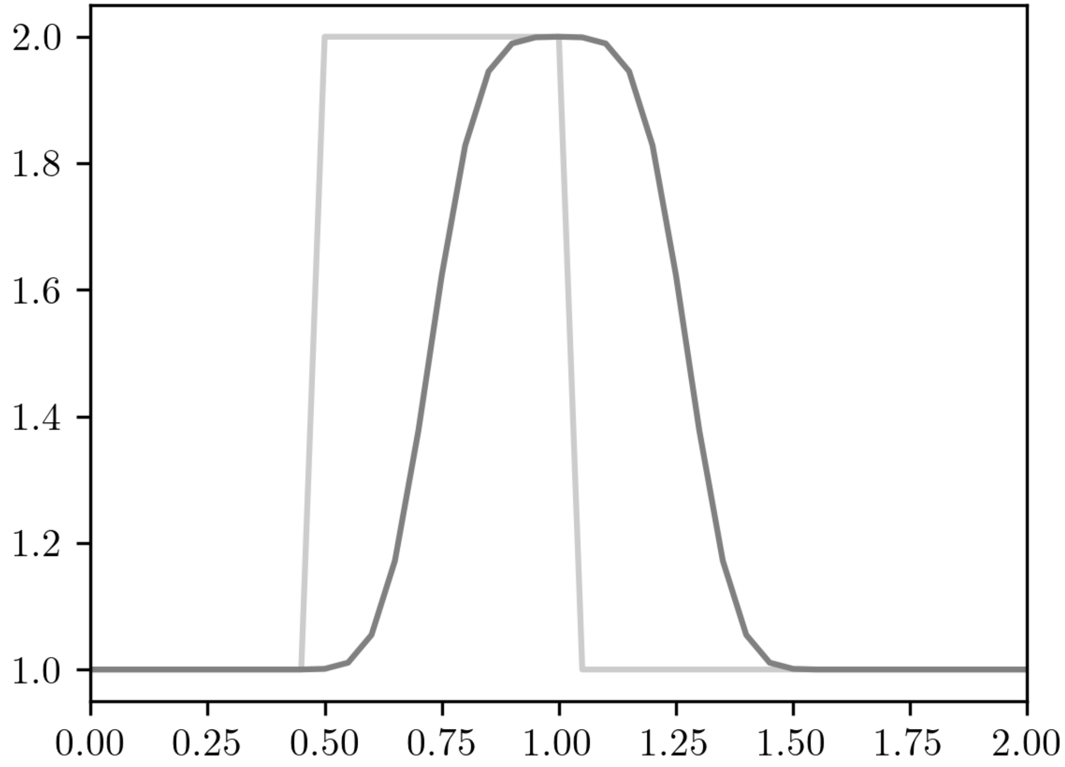

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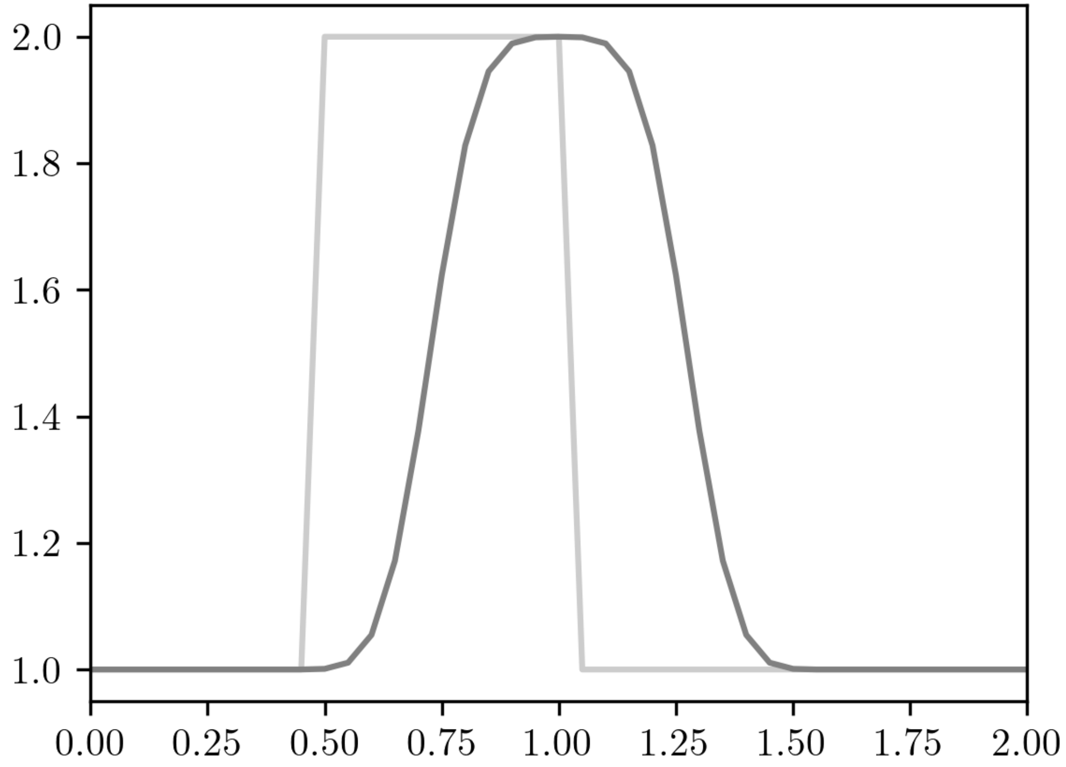

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Let's try with central differencing !

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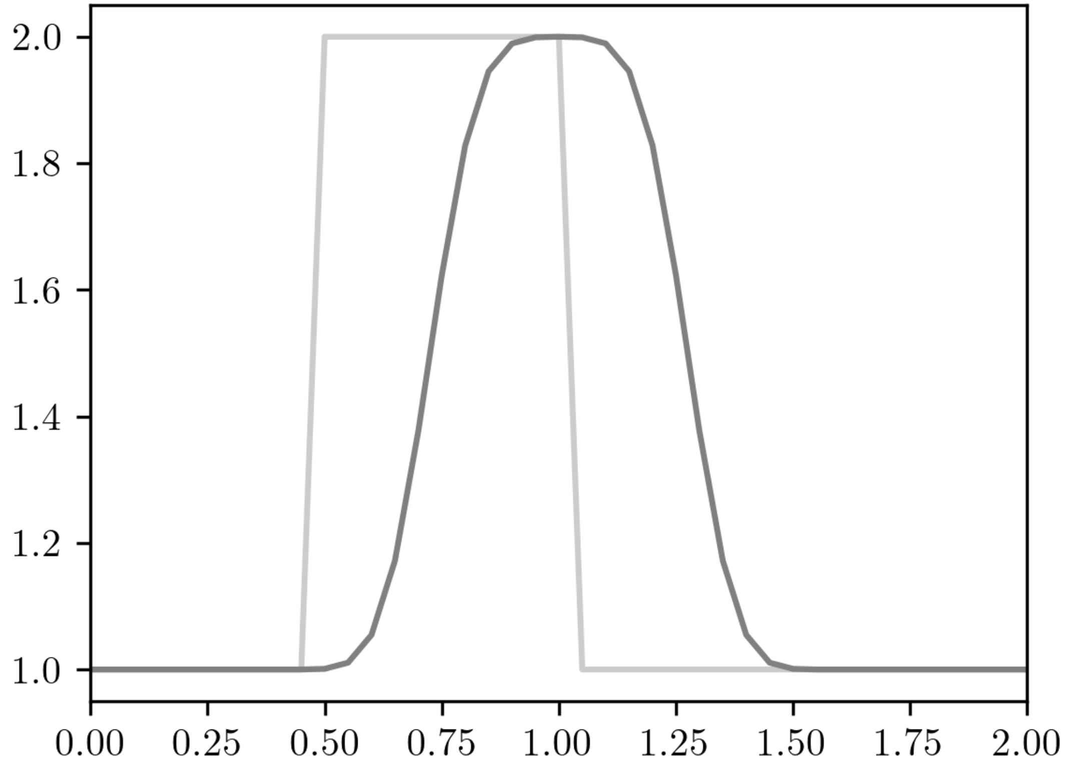

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Change these !

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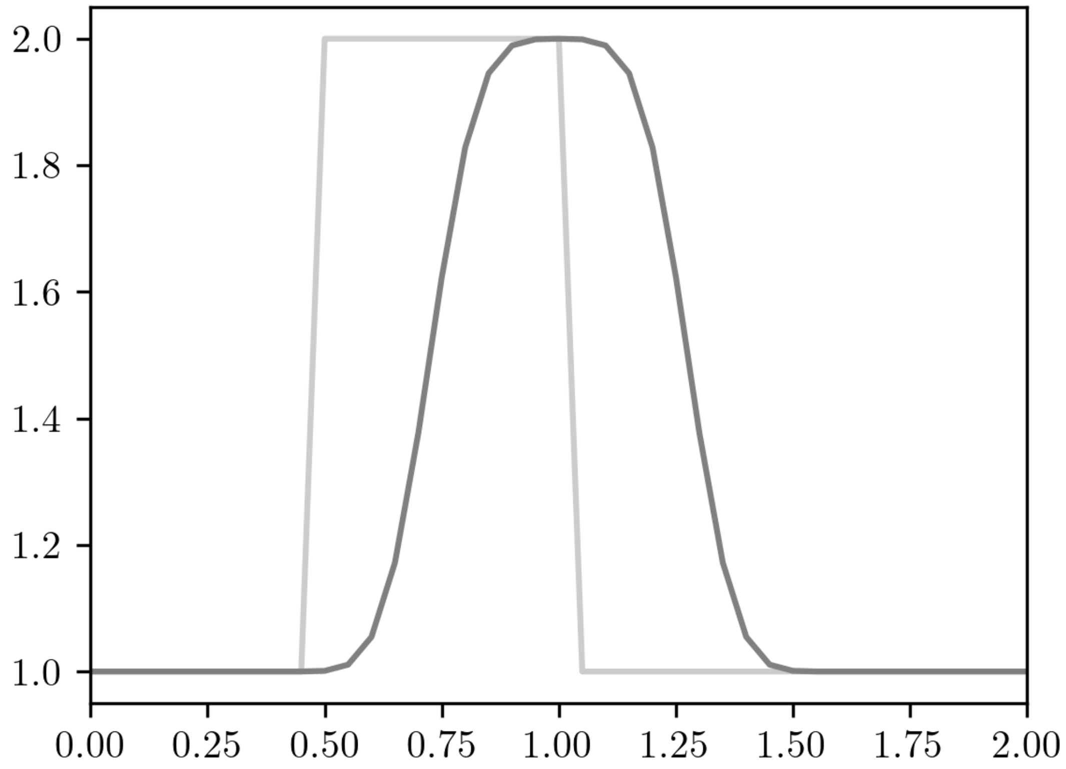

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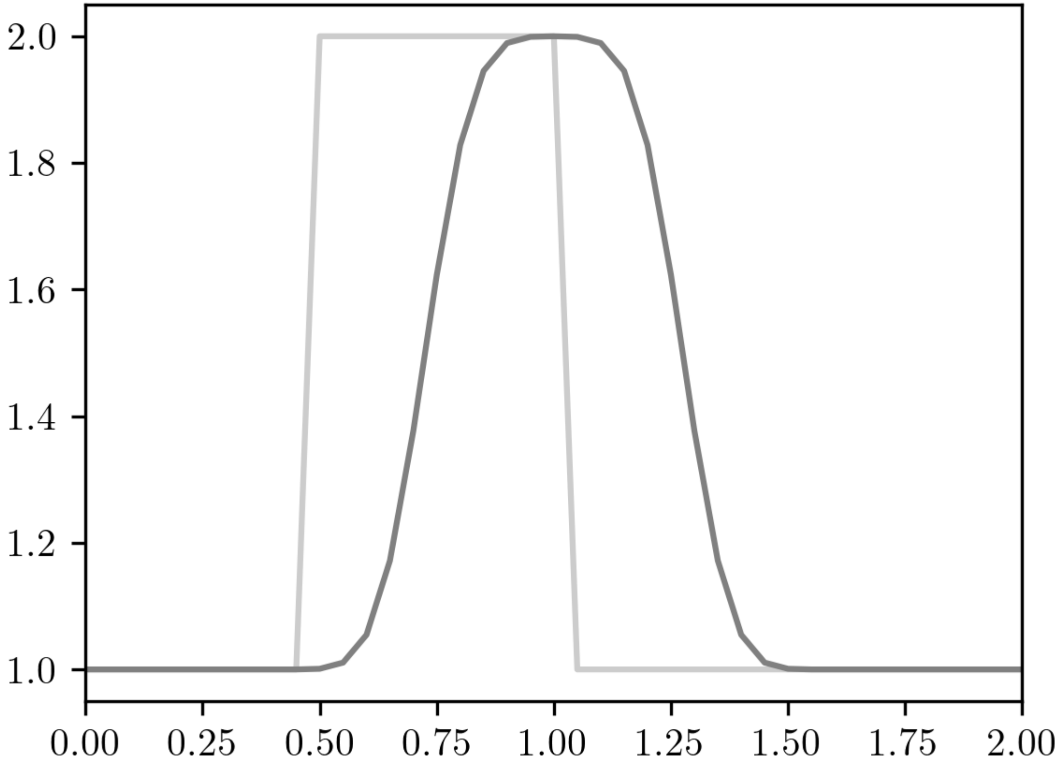

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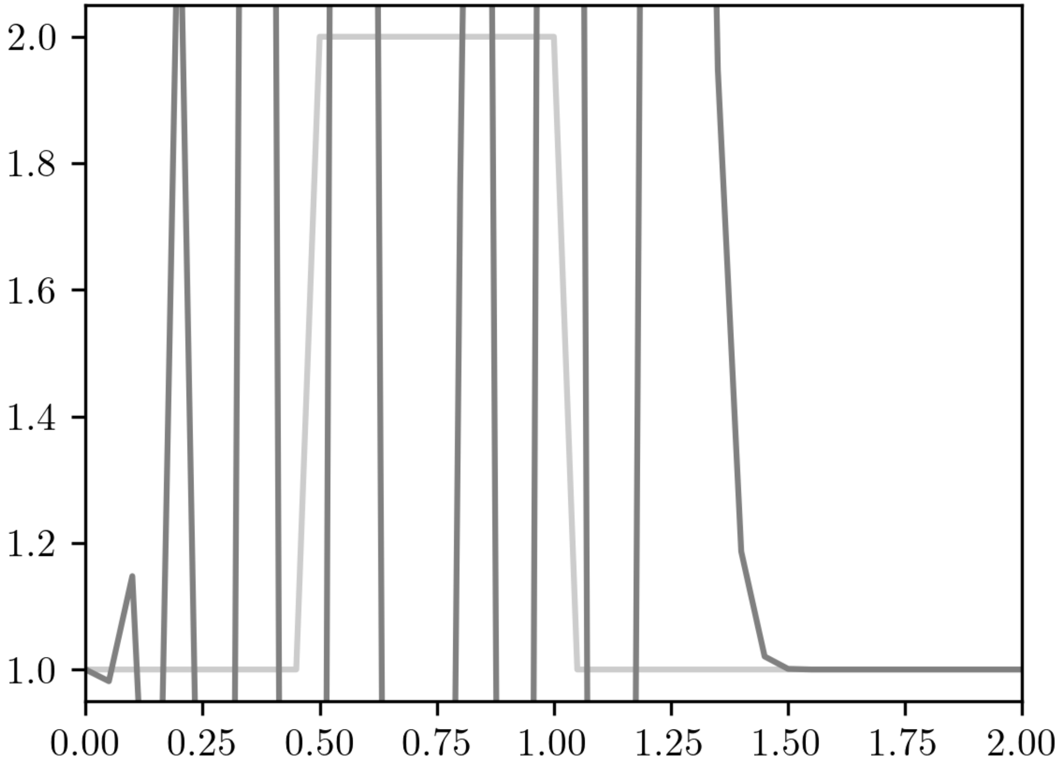

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What's that !

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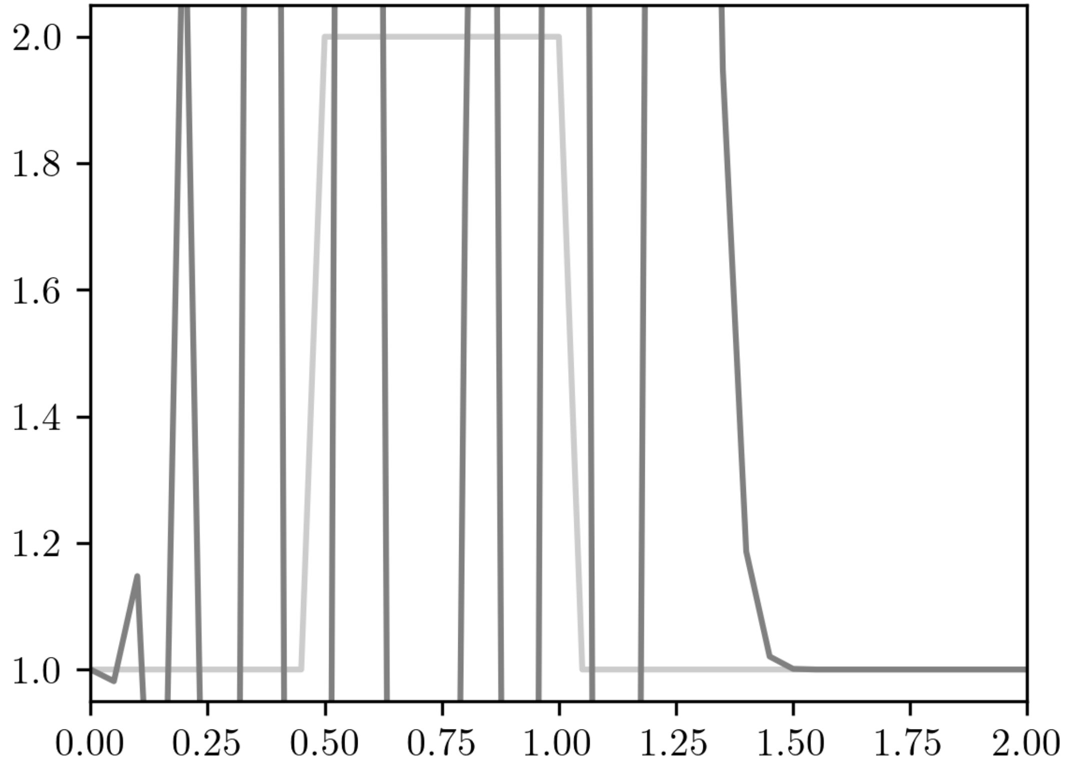

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1D Linear Convection with Central Differencing in Space

$$\frac{du}{dt} + c\frac{du}{dx} = 0 \longrightarrow \frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{2\Delta \mathrm{x}}(u_{i+1}^n - u_{i-1}^n) = 0$$

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1D Linear Convection with Central Differencing in Space

$$\frac{du}{dt} + c\frac{du}{dx} = 0 \longrightarrow \frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{2\Delta \mathrm{x}}(u_{i+1}^n - u_{i-1}^n) = 0$$

$$u_i^{n+1} = u_i^n + \Delta t \frac{\partial u}{\partial t}\bigg|_i^n + \frac{\Delta t^2}{2!}\frac{\partial^2 u}{\partial t^2}\bigg|_i^n + \frac{\Delta t^3}{3!}\frac{\partial^3 u}{\partial t^3}\bigg|_i^n + \cdots$$

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Truncation Error (ϵ_t)

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Truncation Error (ϵ_t)

As Δt & $\Delta x \longrightarrow 0$: $\epsilon_T \longrightarrow 0 \Rightarrow$ Numerical Scheme is Consistent

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NOT a convection equation, It is a Convection-Diffusion equation !!

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Explosion : Unstable Scheme !!

↑

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Convection-Diffusion equation !!

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Diffusion Coefficient must be +ve !!

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Courant-Friedrich-Lewy (CFL) Number!!

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Poor Accuracy !!

2nd Code: 1D non-Linear Convection

```
import numpy as np
import matplotlib.pyplot as plt
import time, sys

nx = 41
dx = 2/(nx-1)
nt = 25
dt = 0.025
c = 1

u = np.ones(nx)
u[int(0.5/dx):int(1/dx+1)] = 2
print(u)
plt.plot(np.linspace(0,2,nx), u)

un = np.ones(nx)
for n in range(nt):
    un=u.copy()
    for i in range(1,nx):
        u[i] = un[i] – c * dt/dx * (un[i]-un[i-1])
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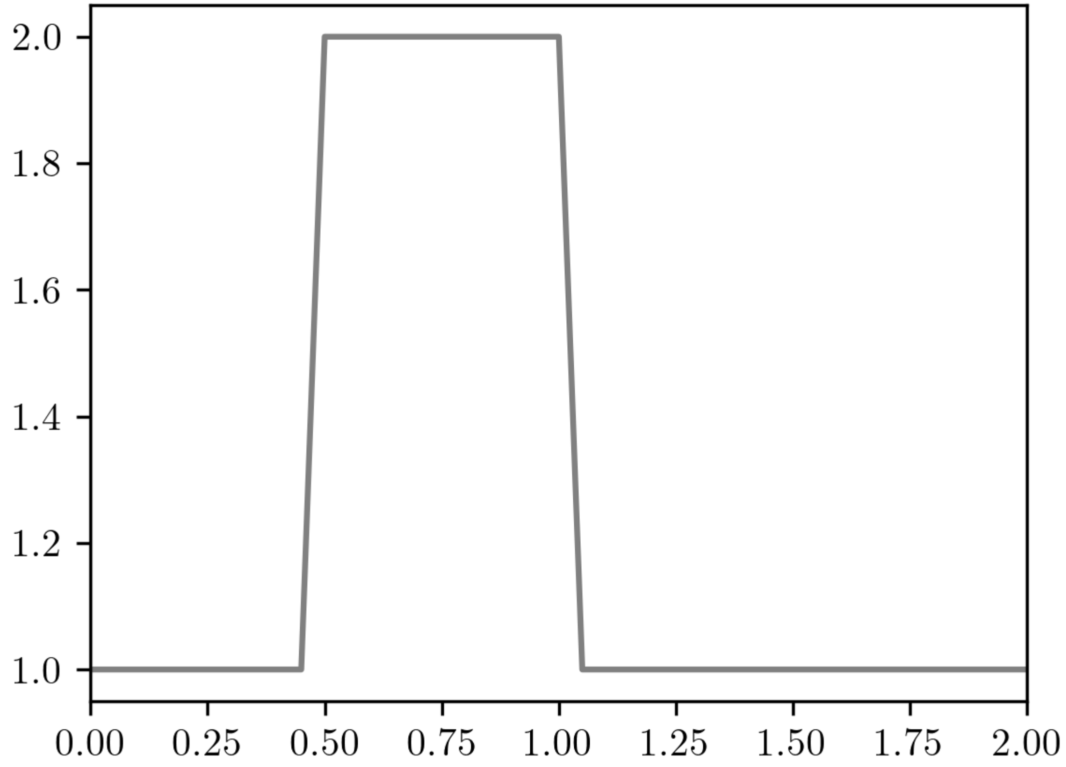
```

The initial velocity u_0 is given as 2 in the interval $0.5 \leq x \leq 1$ and 1 elsewhere in $(0,2)$ $\Longrightarrow$ Hat function



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Change this only !

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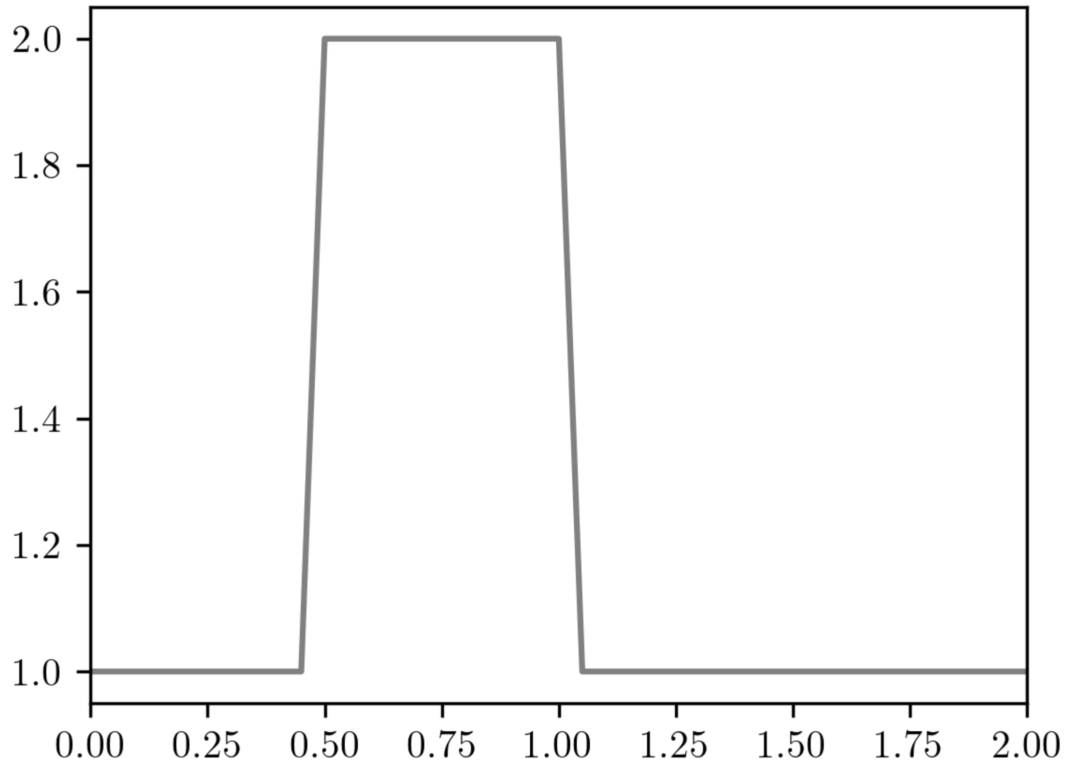

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nt = 25
dt = 0.025
c = 1

u = np.ones(nx)
u[int(0.5/dx):int(1/dx+1)] = 2
print(u)
plt.plot(np.linspace(0,2,nx), u)

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for n in range(nt):
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```
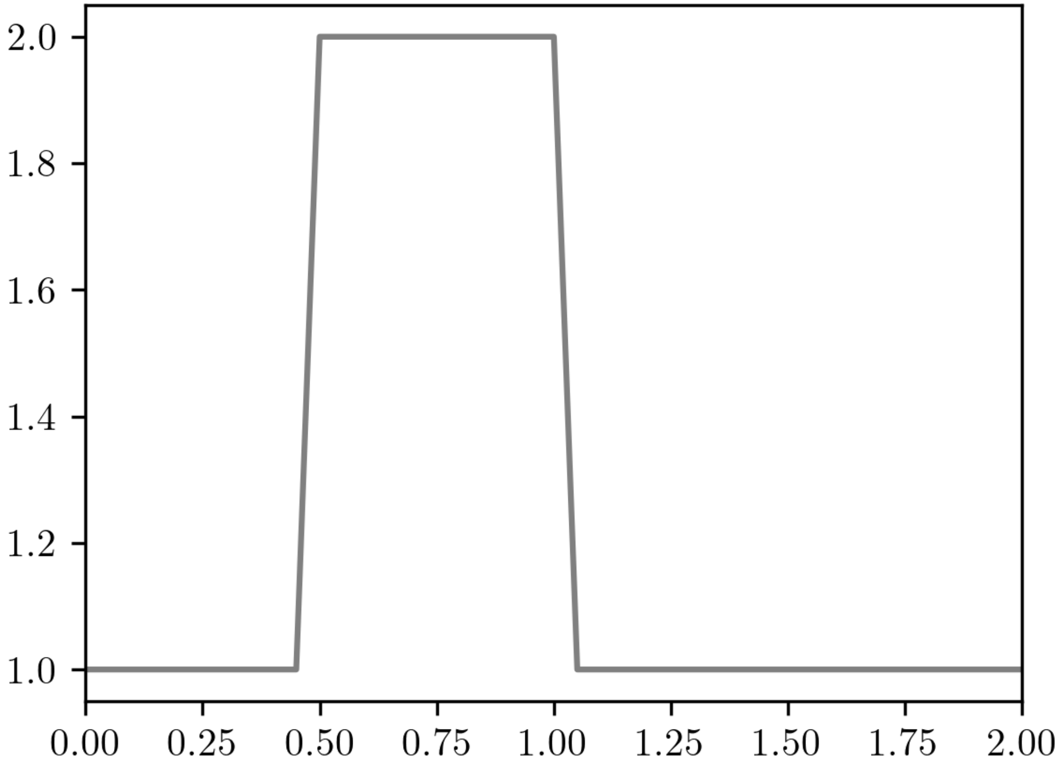

The initial velocity u_0 is given as 2 in the interval $0.5 \leq x \leq 1$ and 1 elsewhere in $(0,2)$ $\Longrightarrow$ Hat function



2nd Code: 1D non-Linear Convection

```
import numpy as np
import matplotlib.pyplot as plt
import time, sys

nx = 41
dx = 2/(nx-1)
nt = 25
dt = 0.025
c = 1

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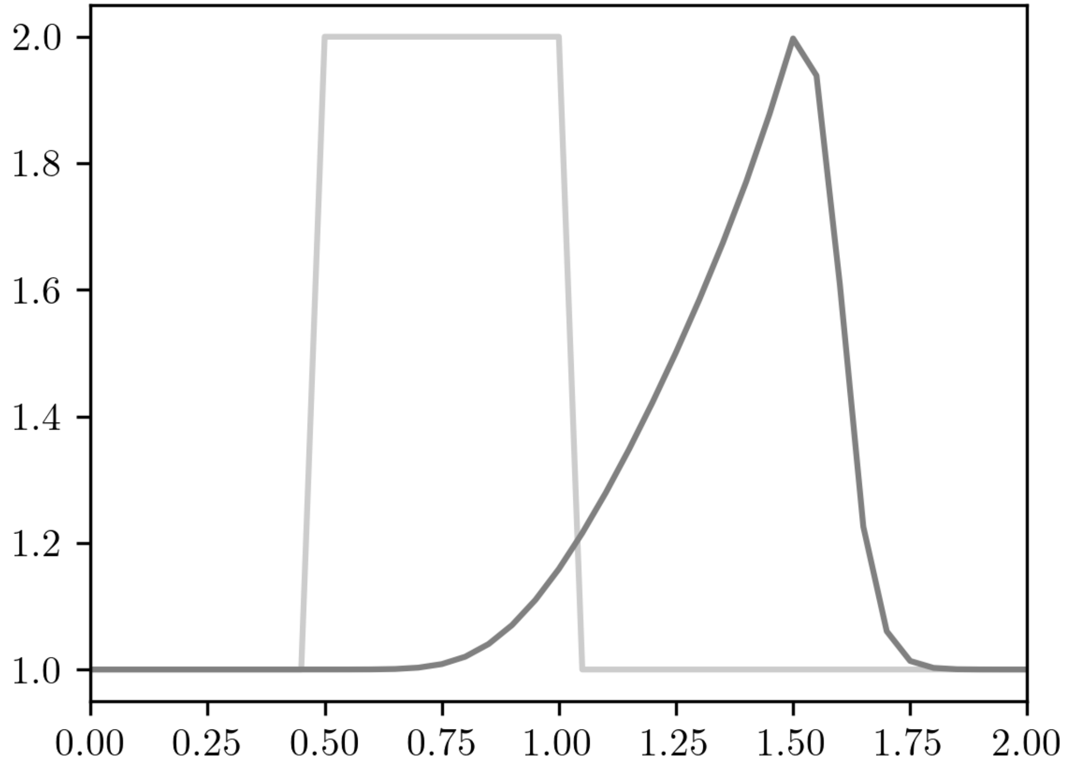

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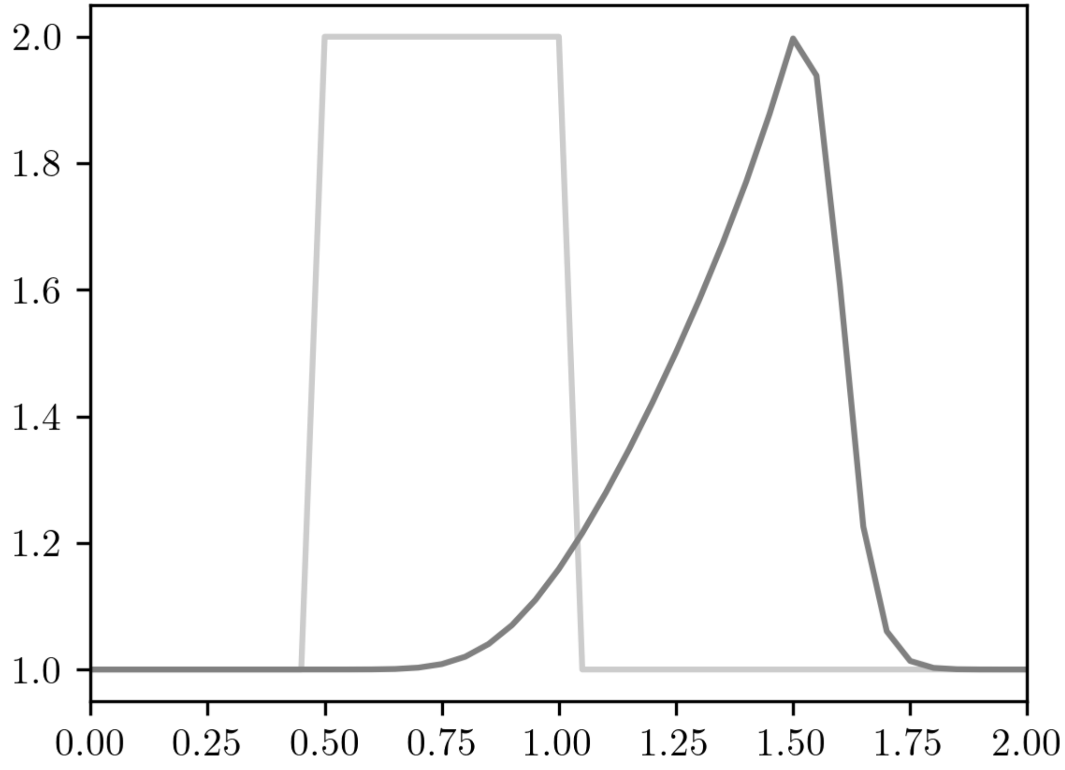
Let's introduce CFL

The initial velocity u_0 is given as 2 in the interval $0.5 \leq x \leq 1$ and 1 elsewhere in $(0,2) \Longrightarrow$ Hat function



2nd Code: 1D non-Linear Convection

```
import numpy as np
import matplotlib.pyplot as plt
import time, sys

nx = 41
dx = 2/(nx-1)
nt = 25
CFL = 0.9

u = np.ones(nx)
u[int(0.5/dx):int(1/dx+1)] = 2
print(u)
plt.plot(np.linspace(0,2,nx), u)

dt = CFL*dx/max(abs(u))

un = np.ones(nx)
for n in range(nt):
    un=u.copy()
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        u[i] = un[i] – un[i] * dt/dx * (un[i]-un[i-1])
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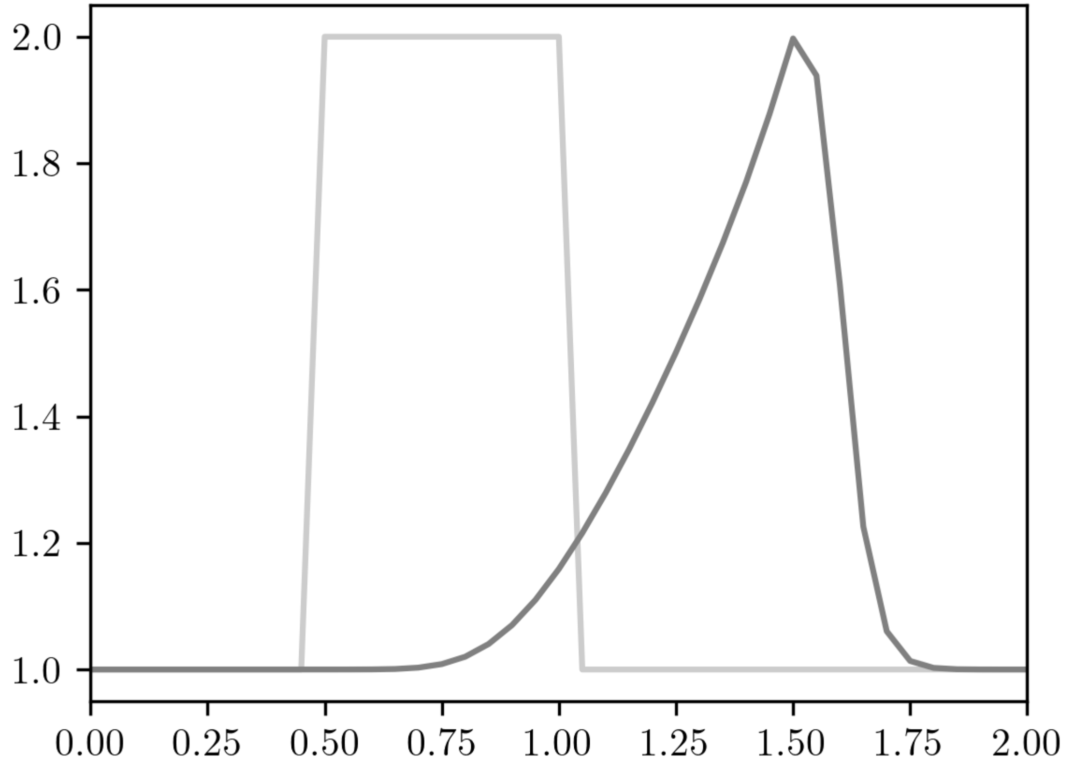

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```

Let's introduce CFL

Play a bit & see what happens!

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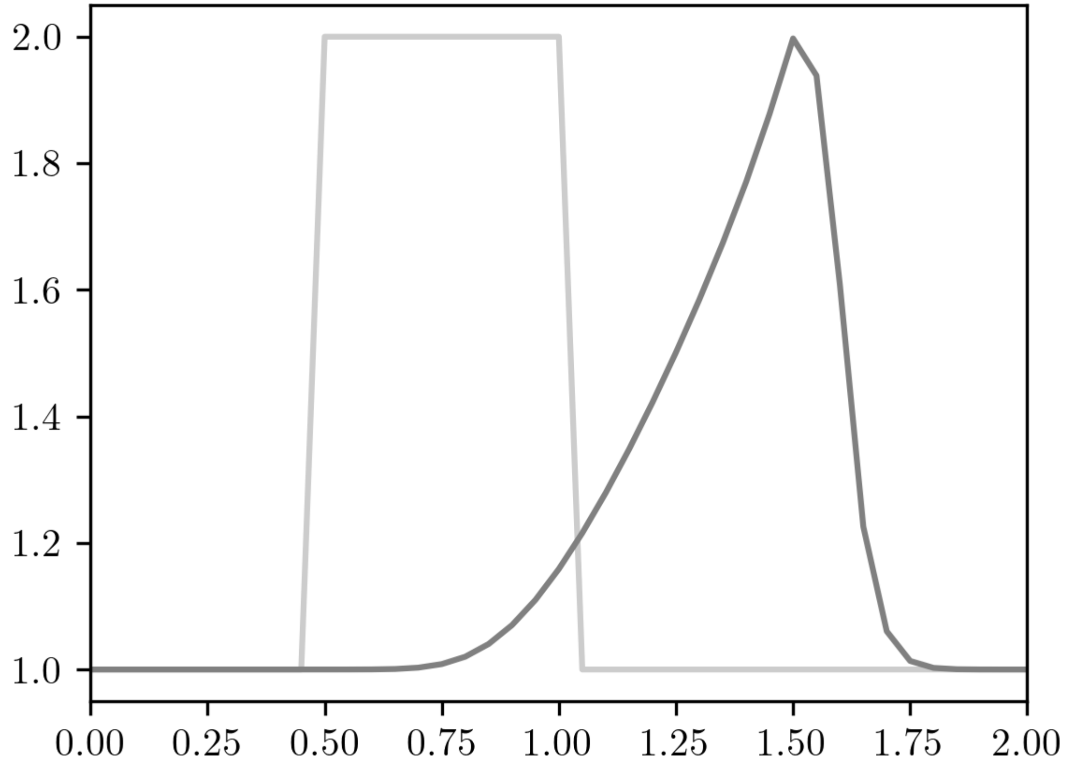

3rd Code: 1D Diffusion

$$\frac{du}{dt} = \nu \frac{d^2u}{dx^2}$$

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Leading to the solution: $u = \hat{u}e^{i\kappa x}e^{-\nu\kappa^2 t}$

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Exponential Damping for $\nu > 0$

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Consider solution of type: $u = \hat{u}e^{i(\kappa x - \omega t)}$

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Diffusion is isotropic in nature

Leading to the solution: $u = \hat{u}e^{i\kappa x}\boxed{e^{-\nu\kappa^2 t}}$

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No Directional Bias

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No Directional Bias

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Central-Differencing

3rd Code: 1D Diffusion

$$\frac{du}{dt} = \nu \frac{d^2u}{dx^2}$$

Central Difference $$\frac{d^2u}{dx^2} \approx \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{\Delta x^2}$$

Forward Euler $$\frac{du}{dt} \approx \frac{u^{n+1}(x) - u^n(x)}{\Delta t}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

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$$\boxed{u_i^{n+1} = u_i^n + \frac{\nu\Delta t}{\Delta \mathrm{x}^2}(u_{i-1}^n - 2u_i^n + u_{i-1}^n)}$$

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Let's Code It !

3^{rd} Code: 1D Diffusion

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sigma = 0.2
dt = sigma * dx**2 / nu

u = np.ones(nx)
u[int(0.5/dx):int(1/dx+1)] = 2
print(u)
plt.plot(np.linspace(0,2,nx), u)
```

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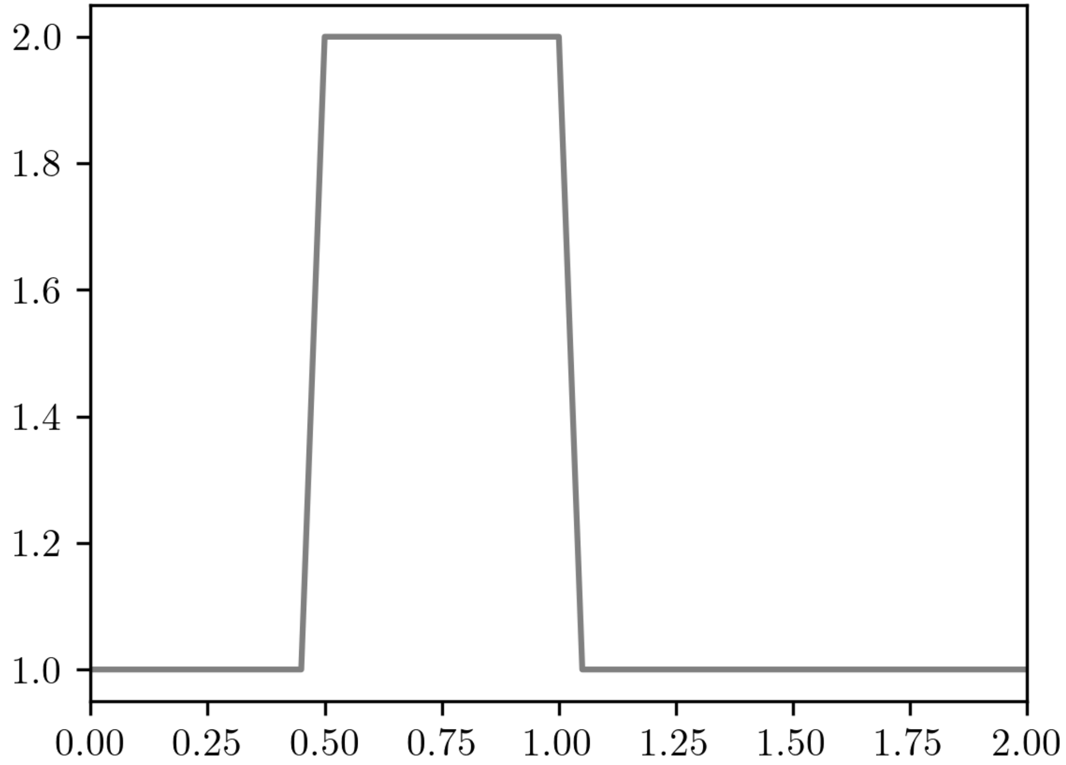

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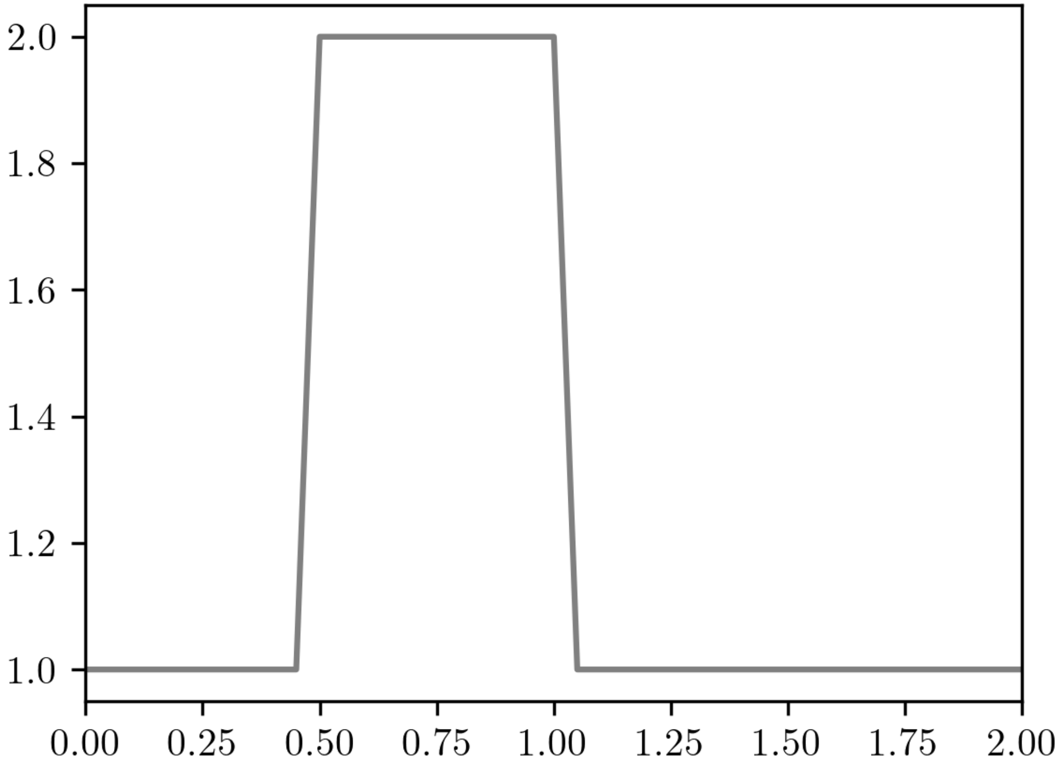

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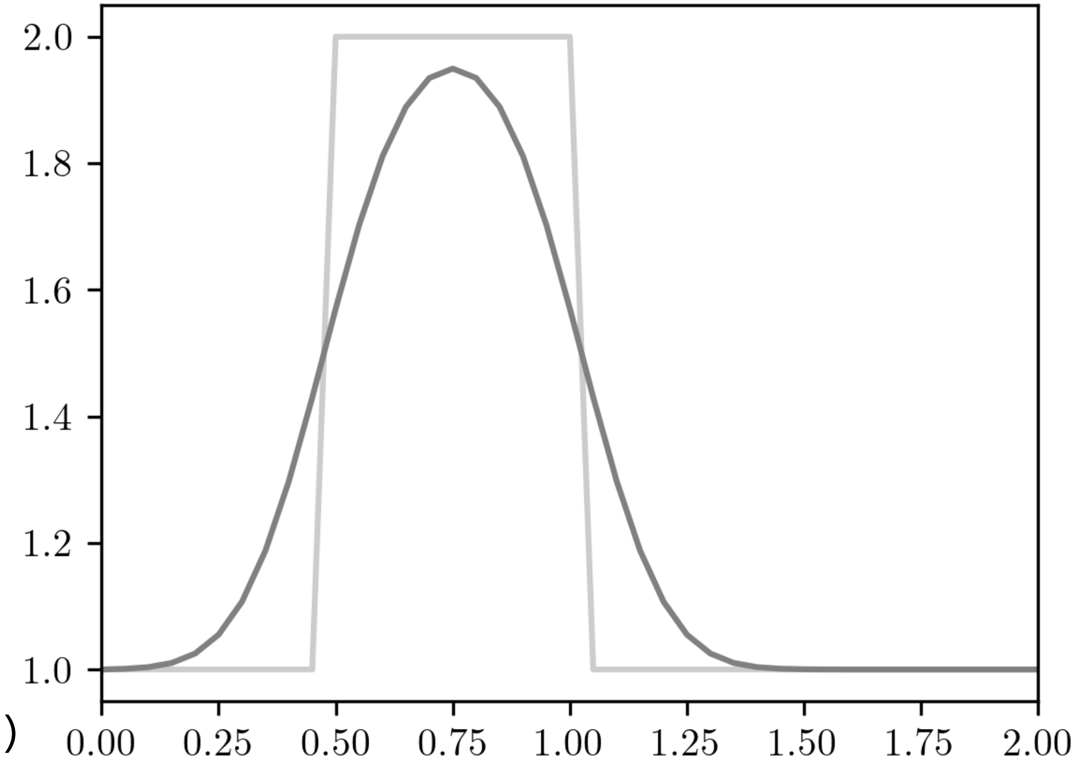

1D Diffusion (Caution!)

$$\frac{du}{dt} = \nu \frac{d^2u}{dx^2}$$

1D Diffusion (Caution!)

Parabolic PDE

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Characteristic lines are lines of constant 't'

↓

All information at a given time should affect the solution

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Our Numerical Formulation:

$$u_i^{n+1} = u_i^n + \frac{\nu\Delta t}{\Delta \mathrm{x}^2}(u_{i-1}^n - 2u_i^n + u_{i-1}^n)$$

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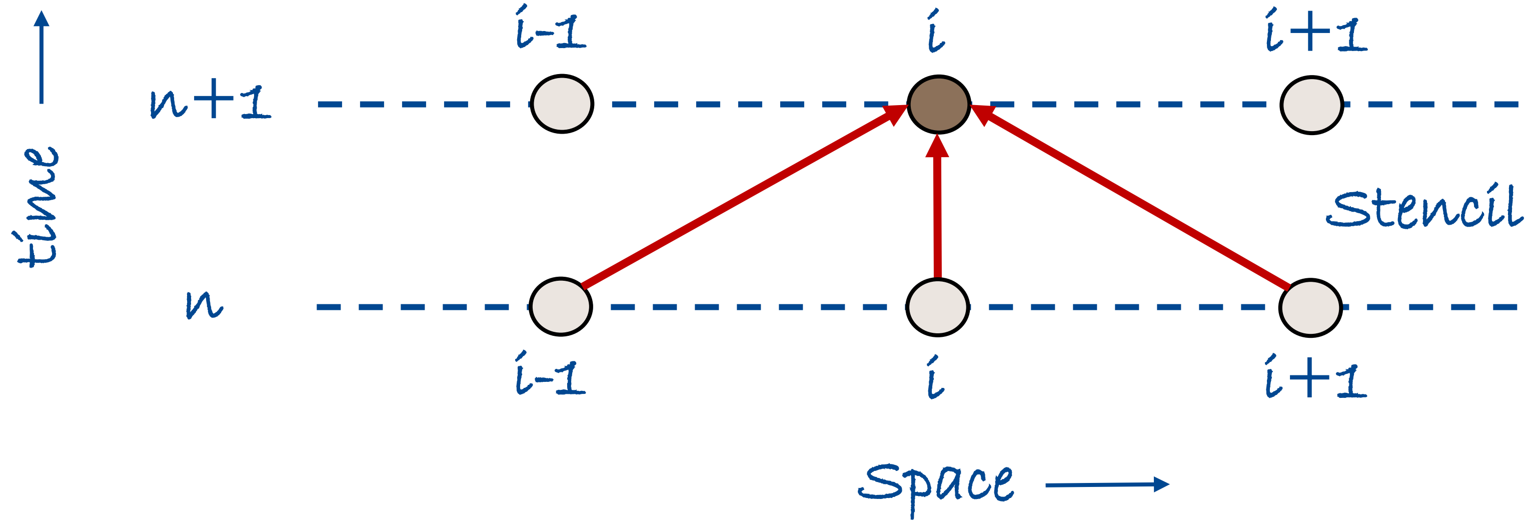

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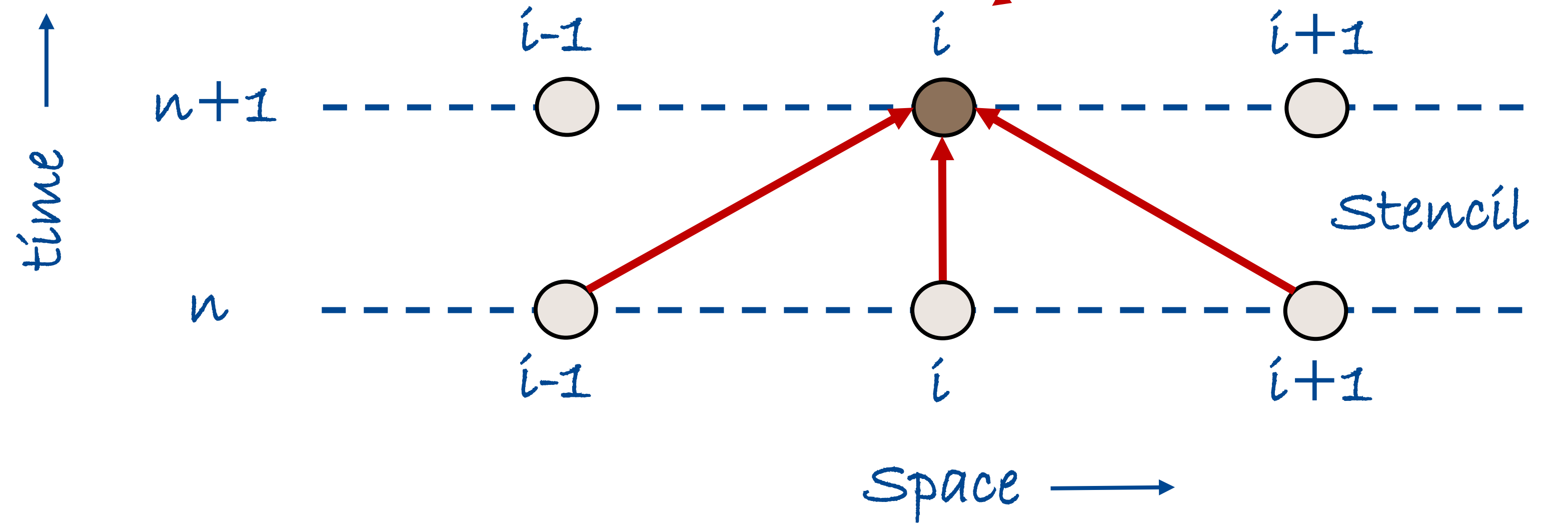

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"Explicit Formulation"

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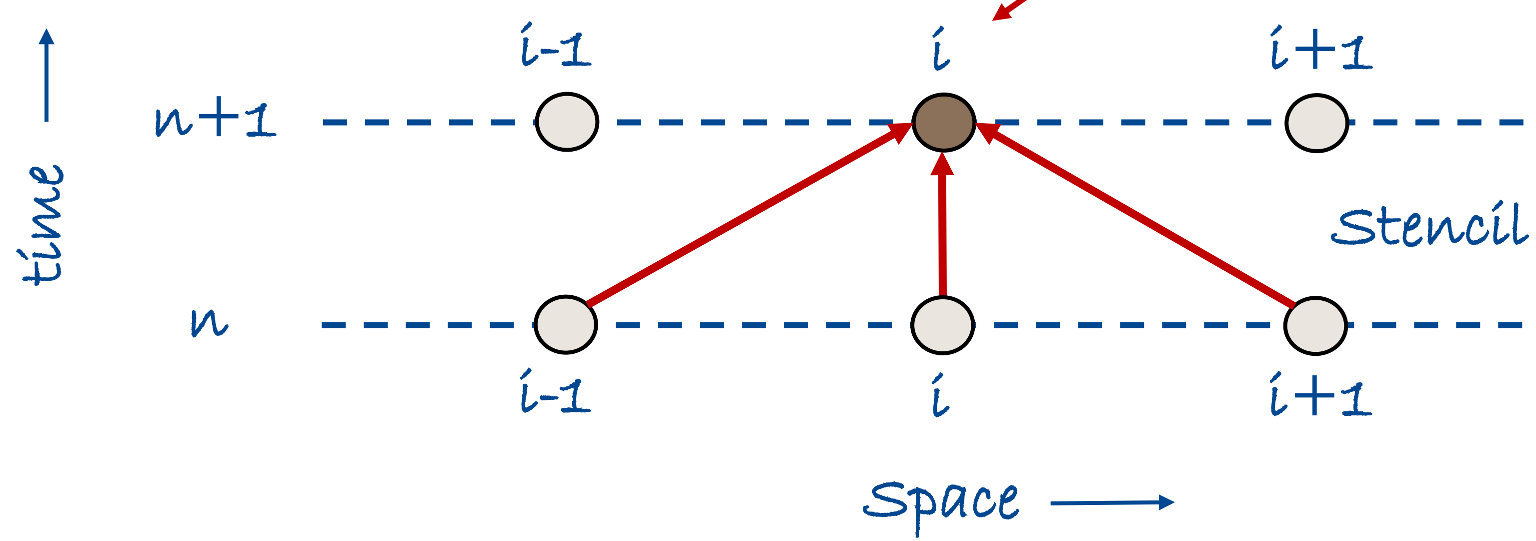

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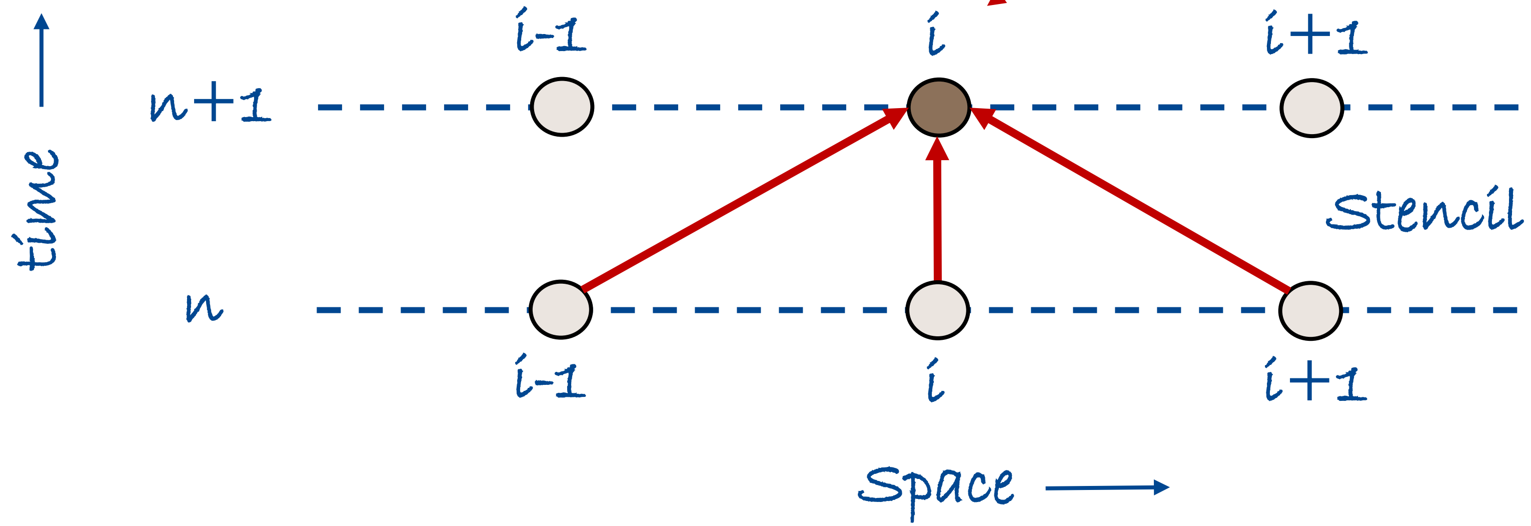

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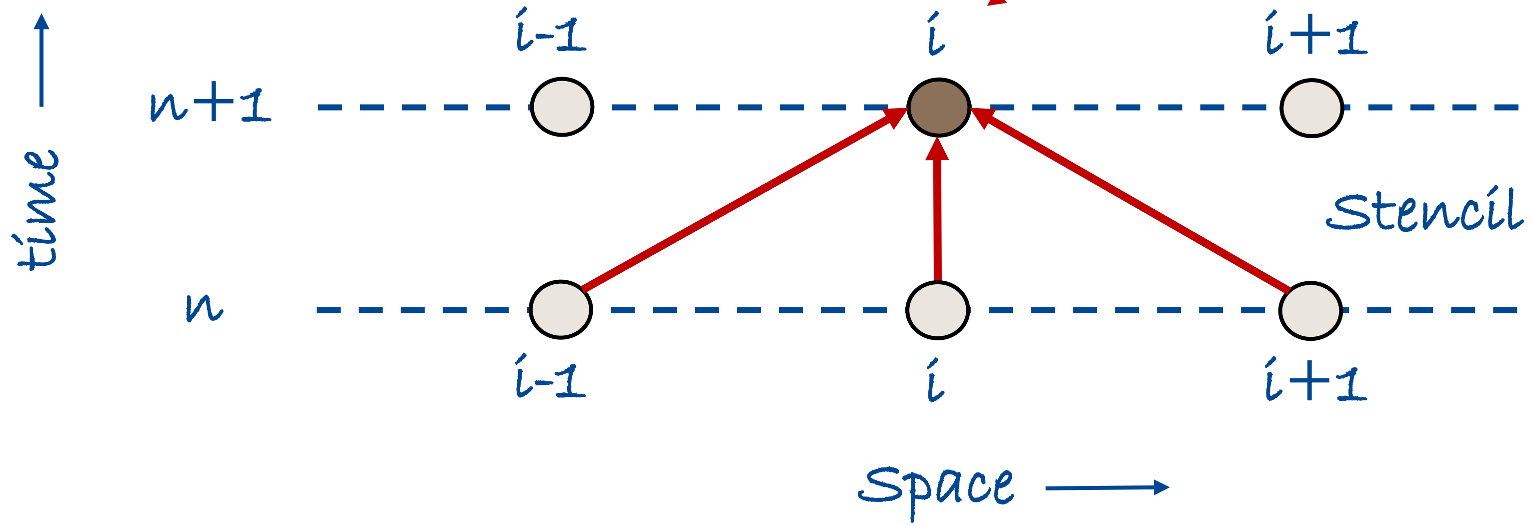
Problem if Boundary Conditions are time-dependent!



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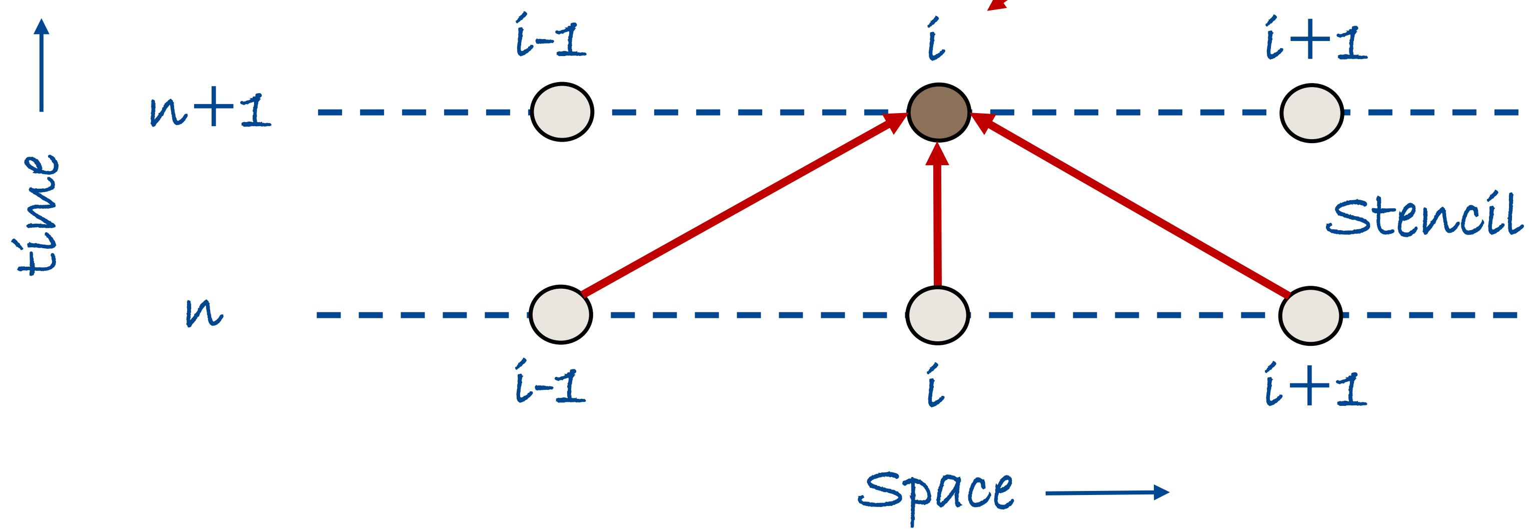
Problem if Boundary Conditions are time-dependent!

How can we include BCs at current time?



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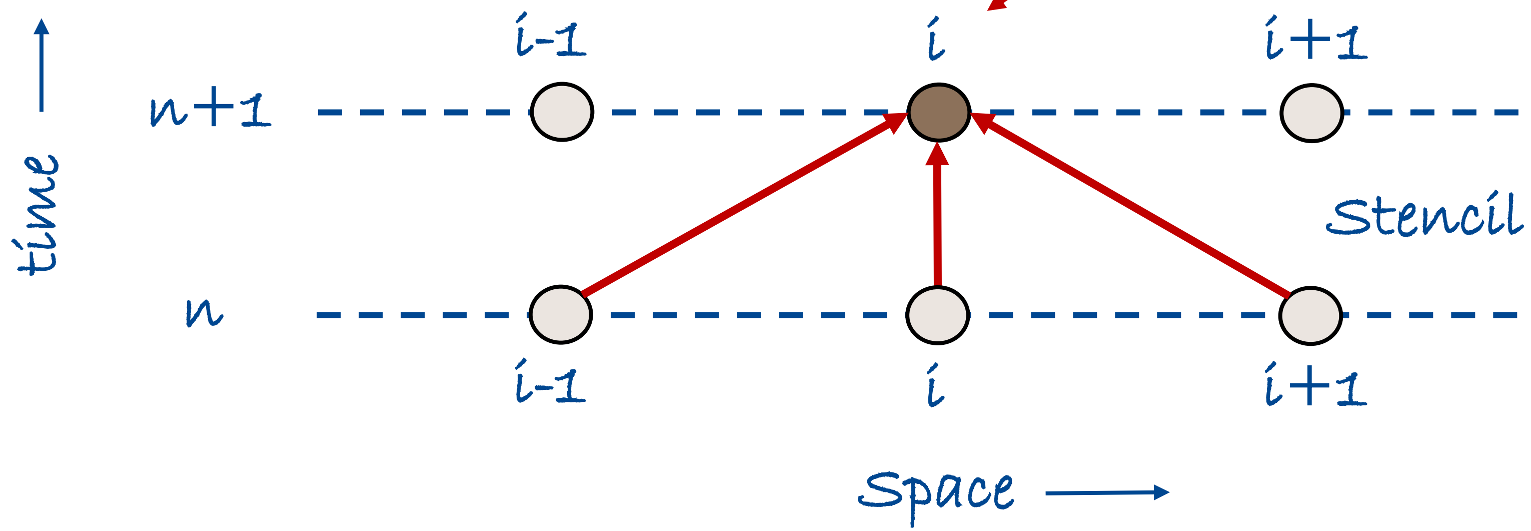

1D Diffusion (Implicit Formulation)

Parabolic PDE

$$\frac{du}{dt} = \nu \frac{d^2u}{dx^2}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\nu}{\Delta \mathrm{x}^2}\left(u_{i-1}^n - 2u_i^n + u_{i-1}^n\right)$$

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1D Diffusion (Implicit Formulation)

Parabolic PDE

$$\frac{du}{dt} = \nu \frac{d^2u}{dx^2} \qquad \frac{u_i^{n+1} \quad - u_i^n}{\Delta t} = \frac{\nu}{\Delta \mathrm{x}^2}\left(u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right)$$

Rearrange:

$$\frac{\nu\Delta t}{\Delta \mathrm{x}^2} u_{i-1}^{n+1} - \left(1 + \frac{2\nu\Delta t}{\Delta \mathrm{x}^2}\right) u_i^{n+1} - \frac{\nu\Delta t}{\Delta \mathrm{x}^2} u_{i+1}^{n+1} = -u_i^n$$

1D Diffusion (Implicit Formulation)

Parabolic PDE

$$\frac{du}{dt} = \nu \frac{d^2u}{dx^2} \qquad \frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\nu}{\Delta x^2}\left(u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right)$$

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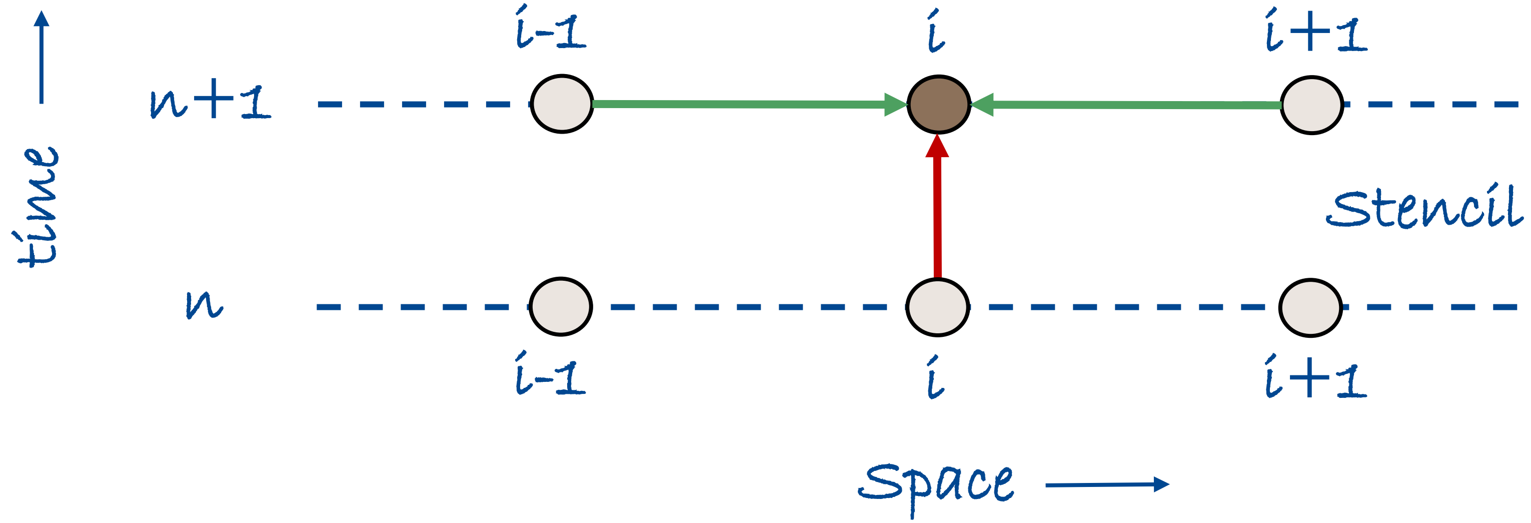

1D Diffusion (Implicit Formulation)

Parabolic PDE

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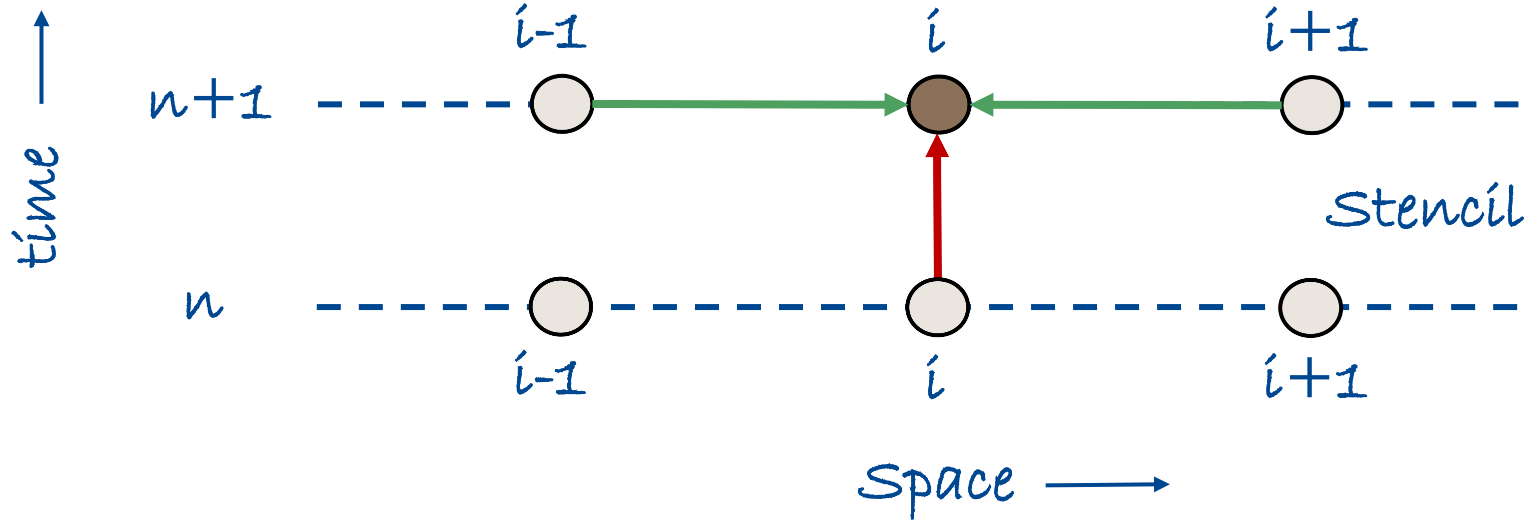

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Parabolic PDE

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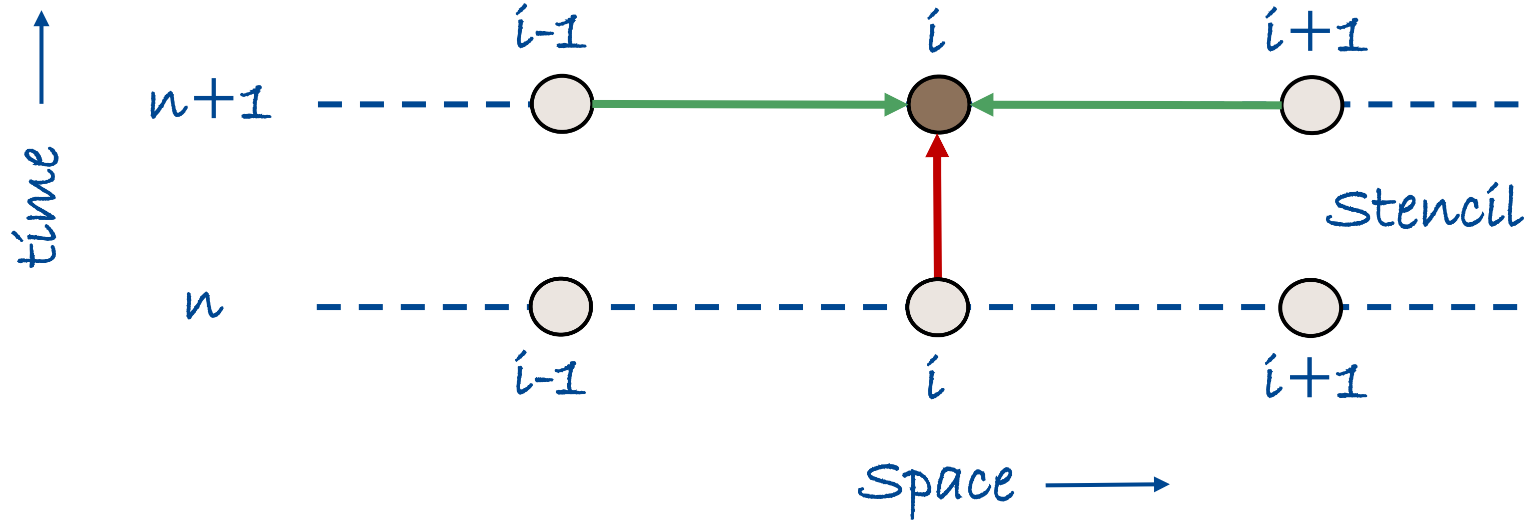
Need Algorithms to solve such a sparse matrix system!



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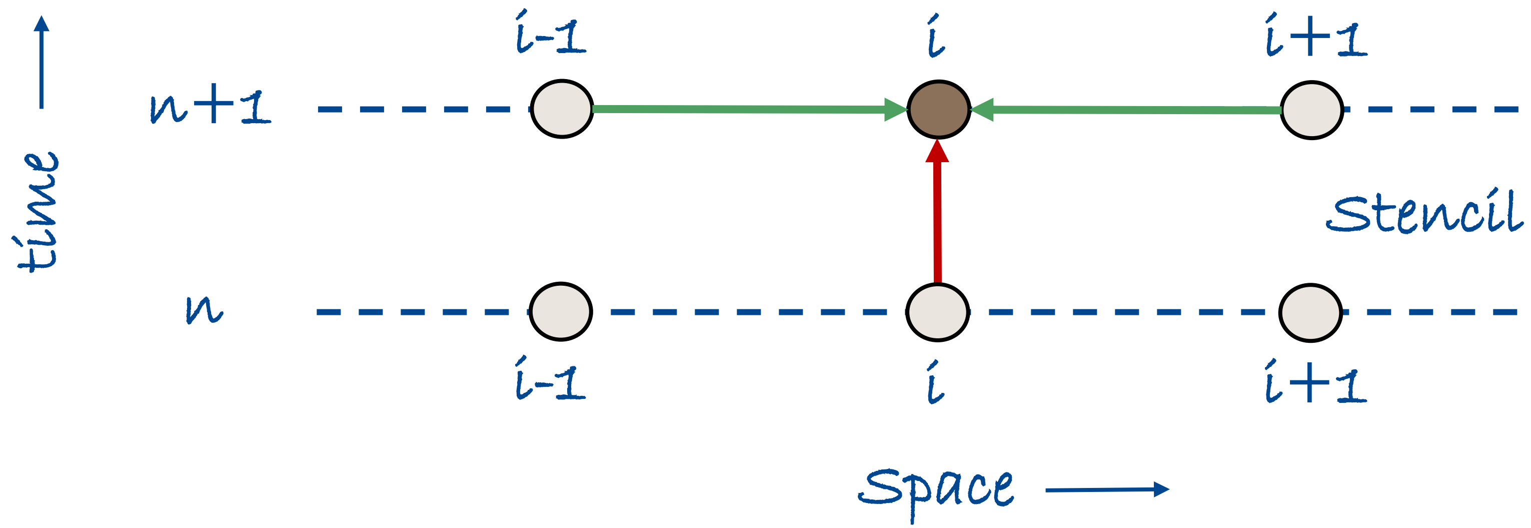
Rearrange: $$\frac{\nu\Delta t}{\Delta x^2} u_{i-1}^{n+1} - \left(1 + \frac{2\nu\Delta t}{\Delta x^2}\right) u_i^{n+1} - \frac{\nu\Delta t}{\Delta x^2} u_{i+1}^{n+1} = -u_i^n$$ "Implicit Formulation"



Need Algorithms to solve such a sparse matrix system!

Direct or Iterative Methods

1D Diffusion (Implicit Formulation)

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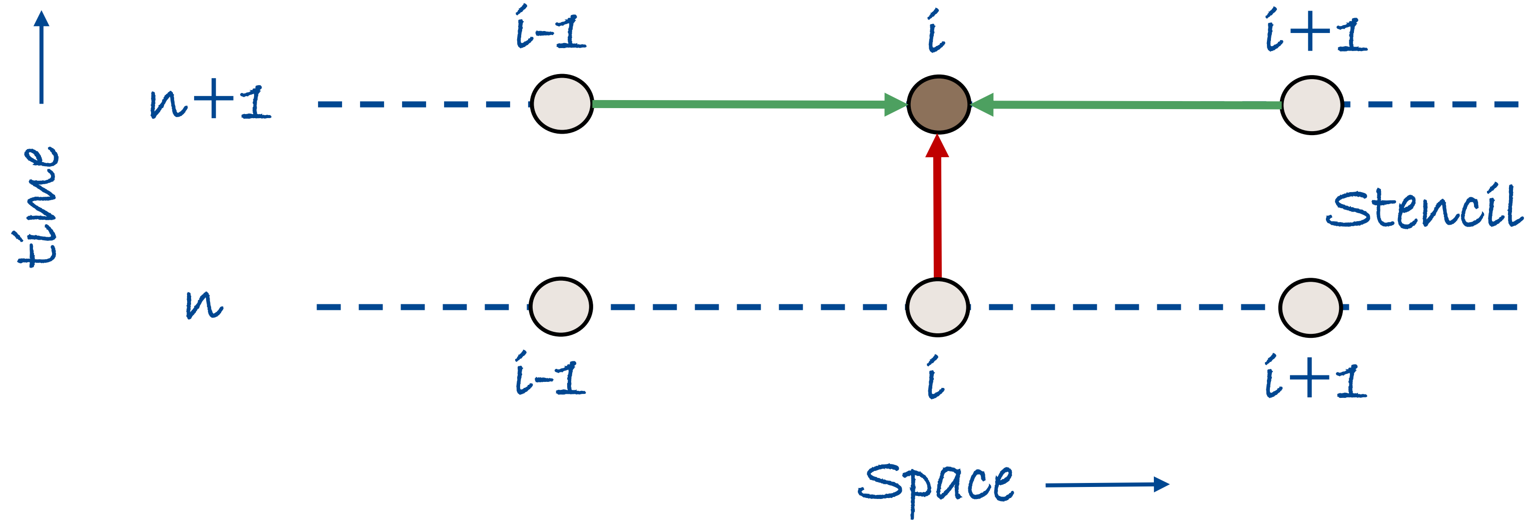

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Direct or Iterative Methods

Can be Expensive!

Crank-Nicholson Method

Well-known for Parabolic PDEs

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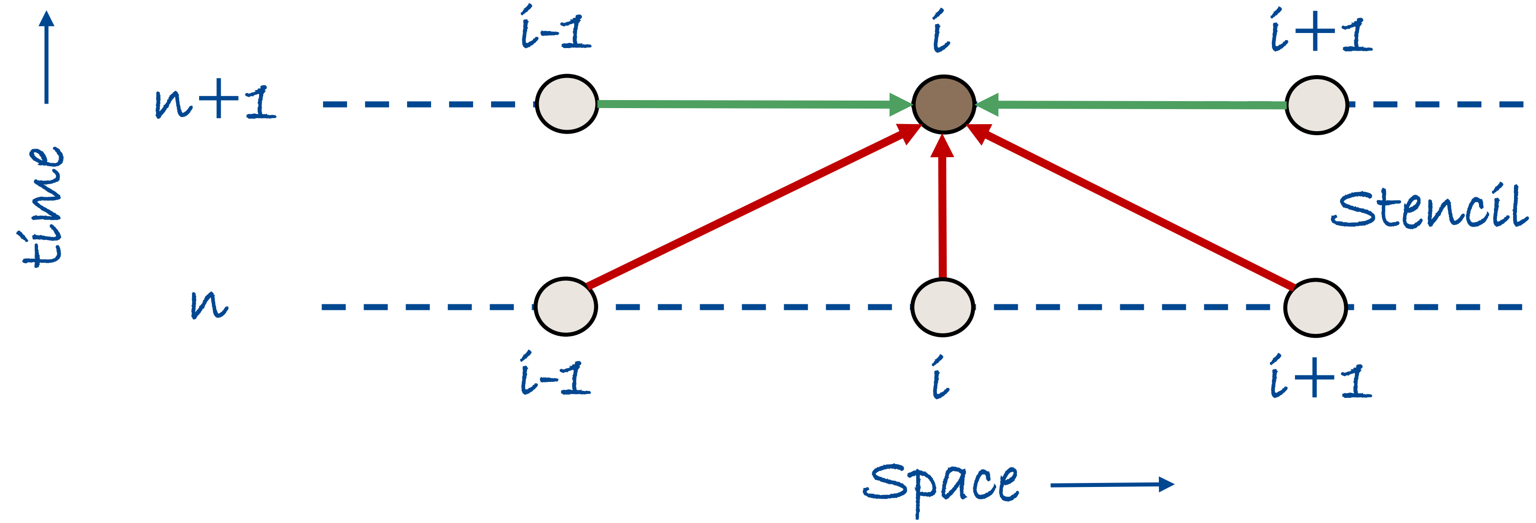

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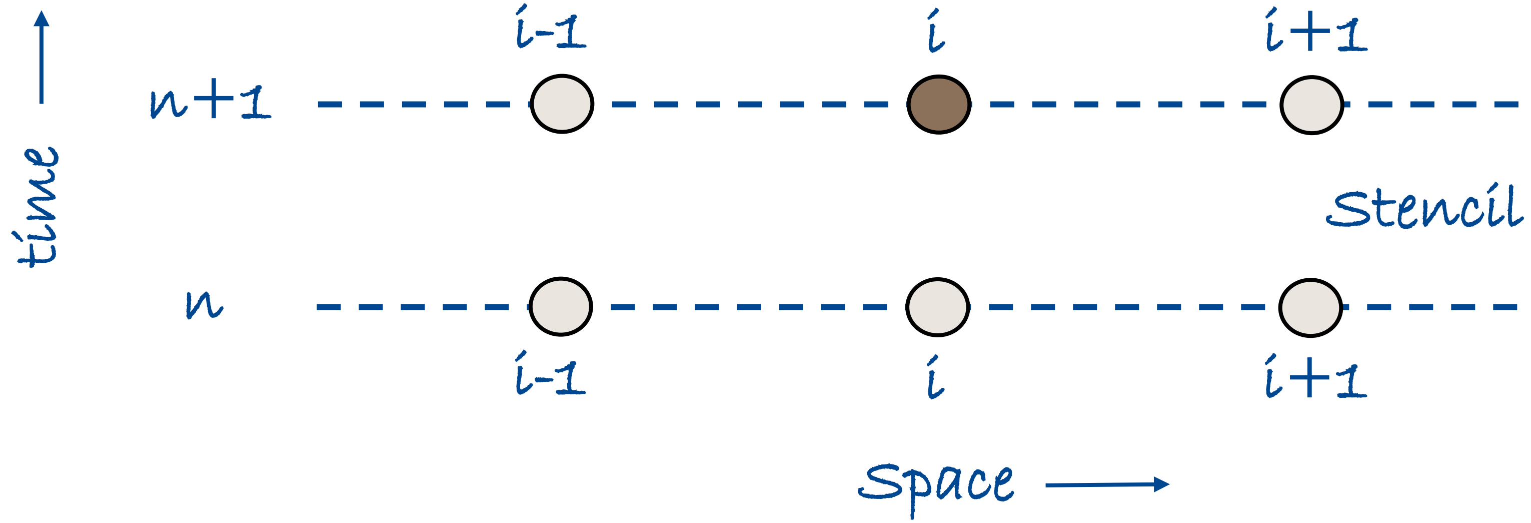

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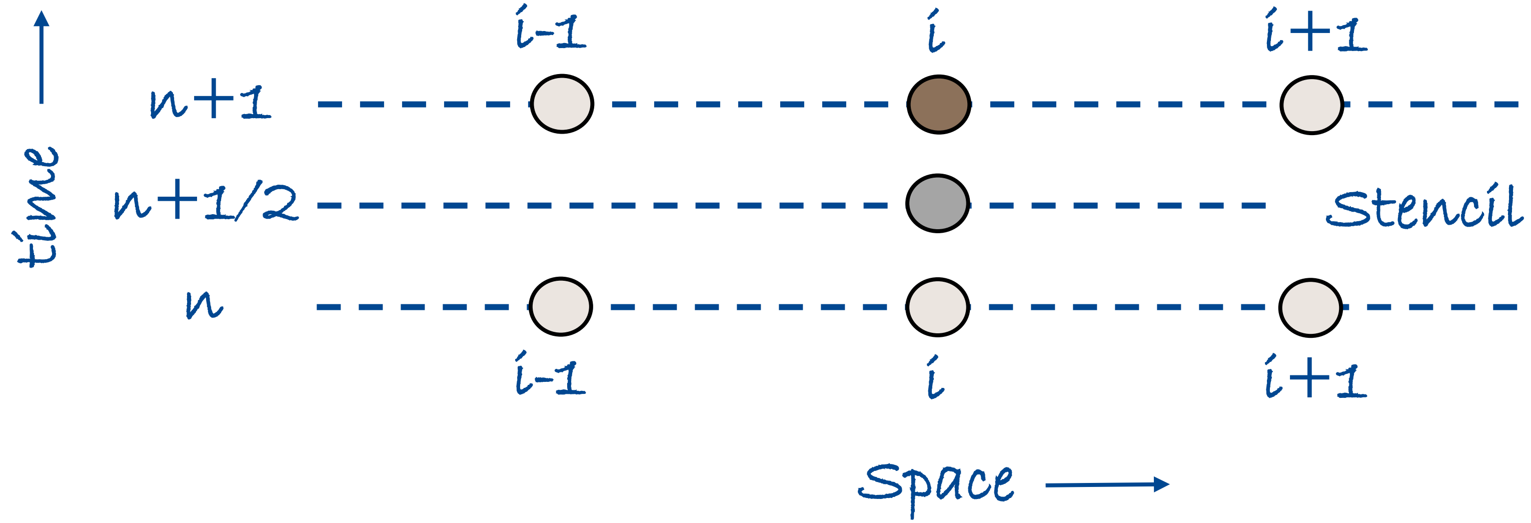

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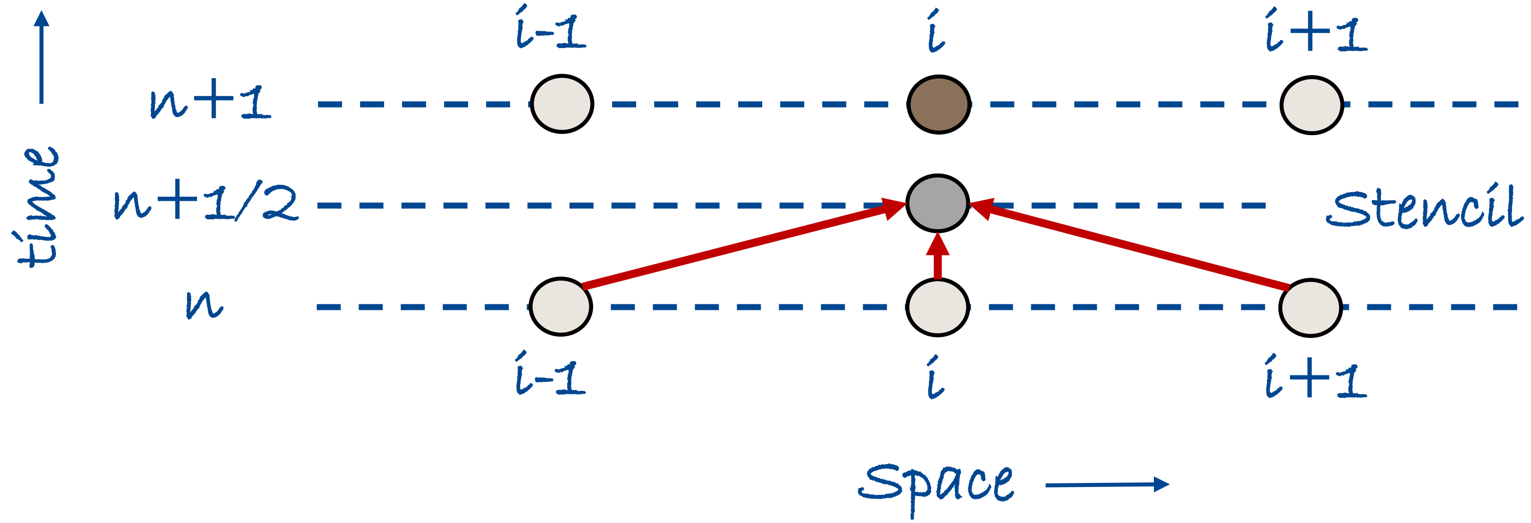

Explicit:

$$\frac{u_i^{n+1/2} - u_i^n}{\Delta t/2} = \frac{\nu}{\Delta x^2}\left(u_{i-1}^{n} - 2u_i^{n} + u_{i-1}^{n}\right)$$

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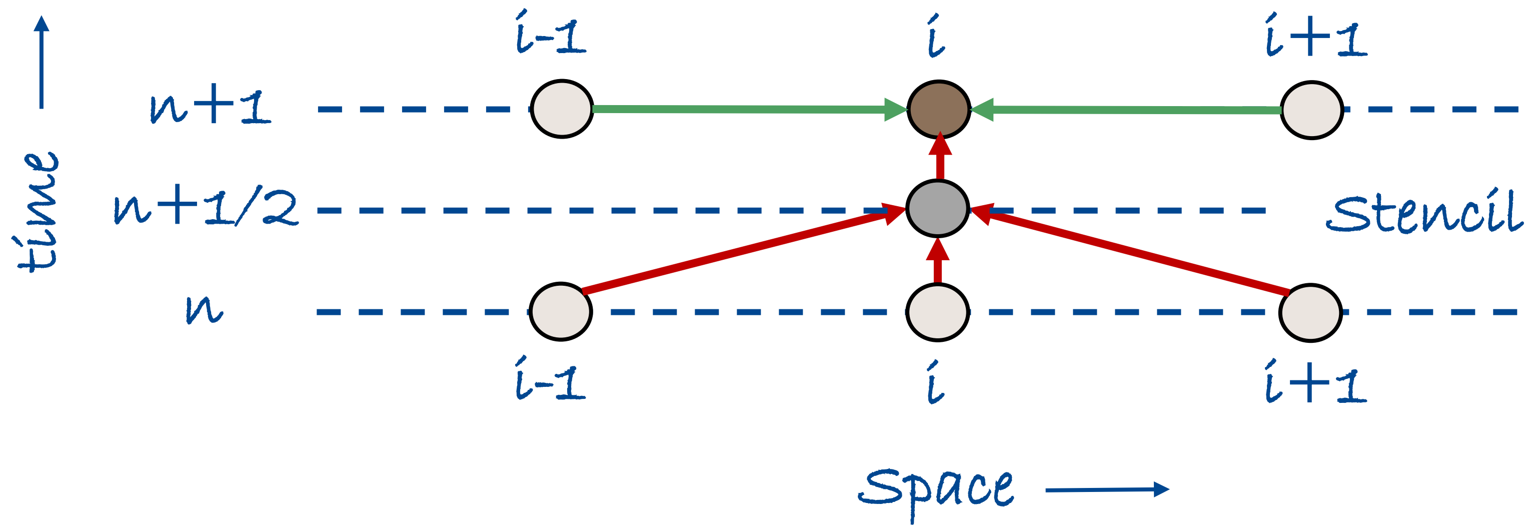

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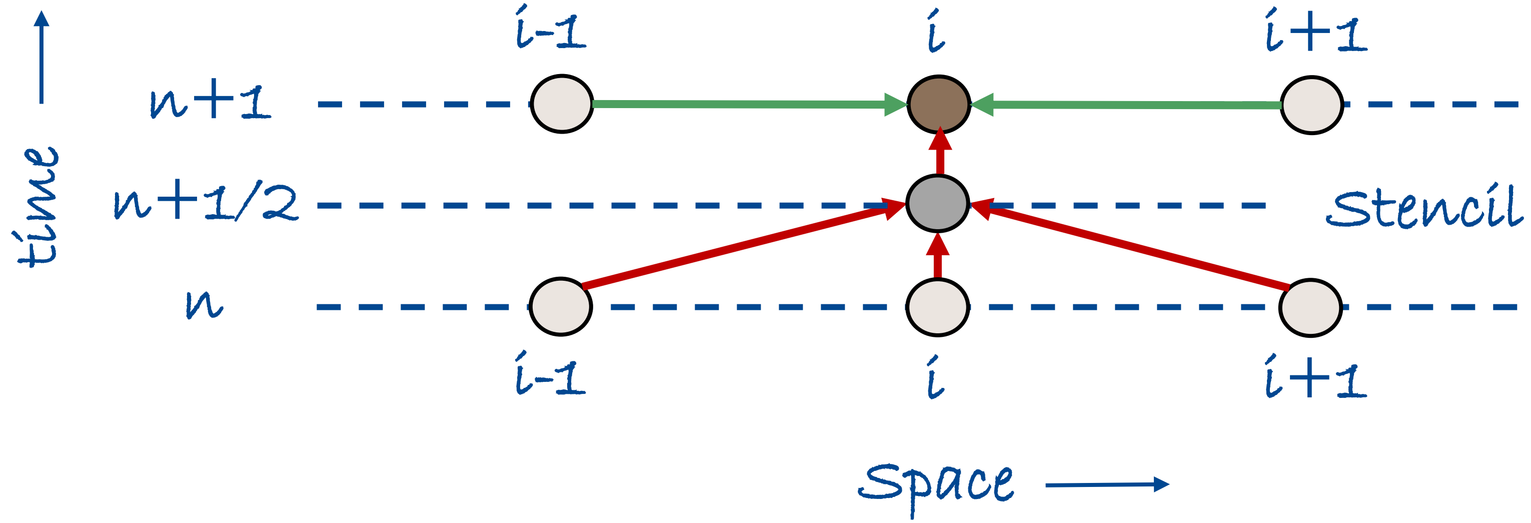

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✓ 2nd order in time & space !

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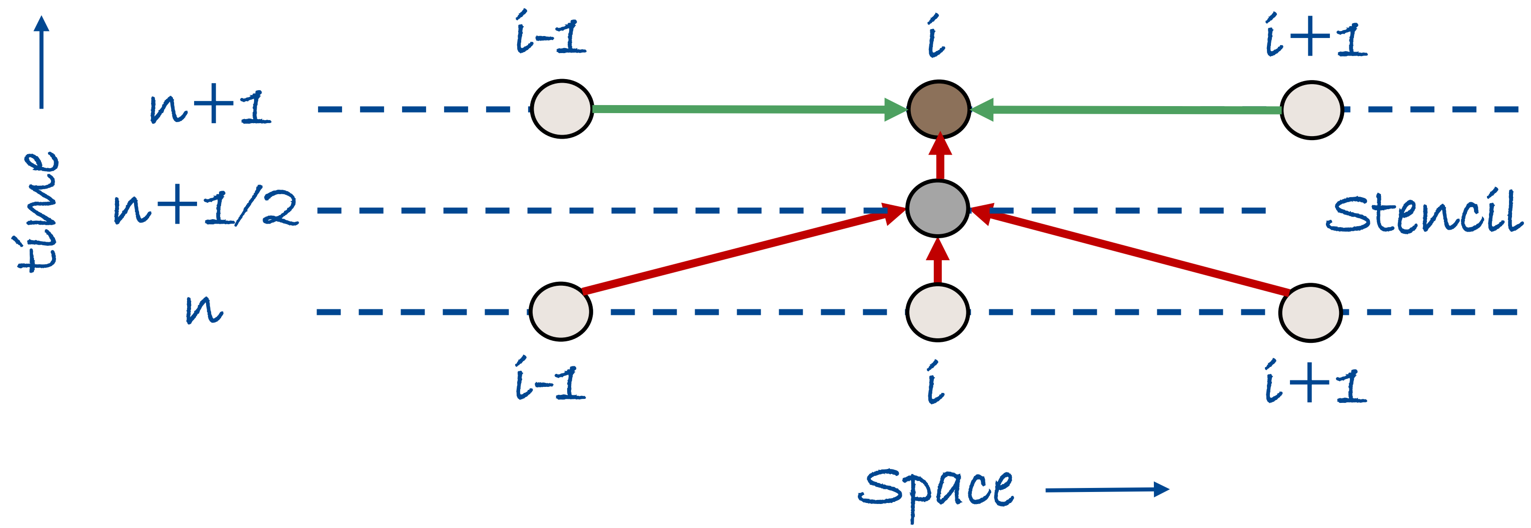

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✓ 2nd order in time & space !

⚠ Tridiagonal system to solve!

4th Code: 1D Burger's Equation

$$\frac{du}{dt} + u\frac{du}{dx} = \nu\frac{d^2u}{dx^2}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n\frac{u_i^n - u_{i-1}^n}{\Delta x} = \nu\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

$$\boxed{u_i^{n+1} = u_i^n - \frac{c\Delta t}{\Delta \mathrm{x}}(u_i^n - u_{i-1}^n) + \frac{c\Delta t}{\Delta \mathrm{x}^2}(u_{i-1}^n - 2u_i^n + u_{i-1}^n)}$$

Initial Condition: $u(x,0) = -\frac{2\nu}{\phi}\frac{\partial\phi}{\partial x} + 4$ $\quad\phi = \exp\left(-\frac{x^2}{4\nu}\right) + \exp\left(-\frac{(\mathrm{x} - 2\pi)^2}{4\nu}\right)$

Boundary Condition: $u(0) = u(2\pi)$

Periodic BC!

Exact Solution: $u = -\frac{2\nu}{\phi}\frac{\partial\phi}{\partial x} + 4$ $\quad\phi = \exp\left(-\frac{(\mathrm{x} - 4\mathrm{t})^2}{4\nu(t+1)}\right) + \exp\left(-\frac{(\mathrm{x} - 4\mathrm{t} - 2\pi)^2}{4\nu(t+1)}\right)$

Let's Code It!

4th Code: 1D Burger's Equation

```
import numpy as np
import sympy as sp
import pylab as pl
pl.ion()
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x, nu, t = sp.symbols('x nu t')
phi = sp.exp(-(x-4*t)**2/(4*nu*(t+1))) + sp.exp(-(x-4*t-2*sp.pi)**2/(4*nu*(t+1)))

phiprime = phi.diff(x)
u = -2*nu*(phiprime/phi)+4

from sympy.utilities.lambdify import lambdify
ufunc = lambdify ((t, x, nu), u)
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nx = 101
dx = 2*np.pi/(nx-1)
nt = 100
nu = 0.07
dt = dx*nu
T = nt*dt

grid = np.linspace(0, 2*np.pi, nx)
u = np.empty(nx)
t = 0
u = np.asarray([ufunc(t, x, nu) for x in grid])


pl.figure(figsize=(11,7), dpi=100)
pl.plot(grid,u, marker='o', lw=2)
pl.xlim([0,2*np.pi])
pl.ylim([0,10])
pl.xlabel('X')
pl.ylabel('Velocity')
pl.title('1D Burgers Equation - Initial condition')
```

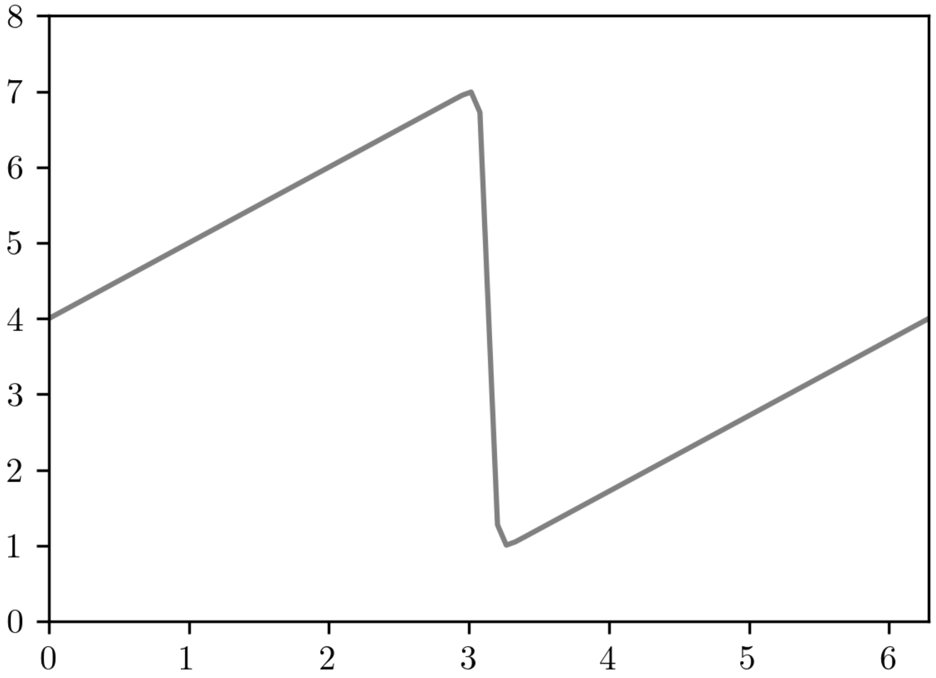

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```
for n in range(nt):
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            nu * dt/(dx**2) * (un[i+1] - 2*un[i] + un[i-1])
    # infer the periodicity
    u[-1] = un[-1] - un[-1] * dt/dx * (un[-1]-un[-2]) + \
            nu * dt/(dx**2) * (un[0] - 2*un[-1] + un[-2])

u_analytical = np.asarray([ufunc(T, xi, nu) for xi in grid])

pl.figure(figsize=(11,7), dpi=100)
pl.plot(grid, u, marker='o', lw=2, label='Computational')
pl.plot(grid, u_analytical, label='Analytical')
pl.xlim([0, 2*np.pi])
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pl.legend()
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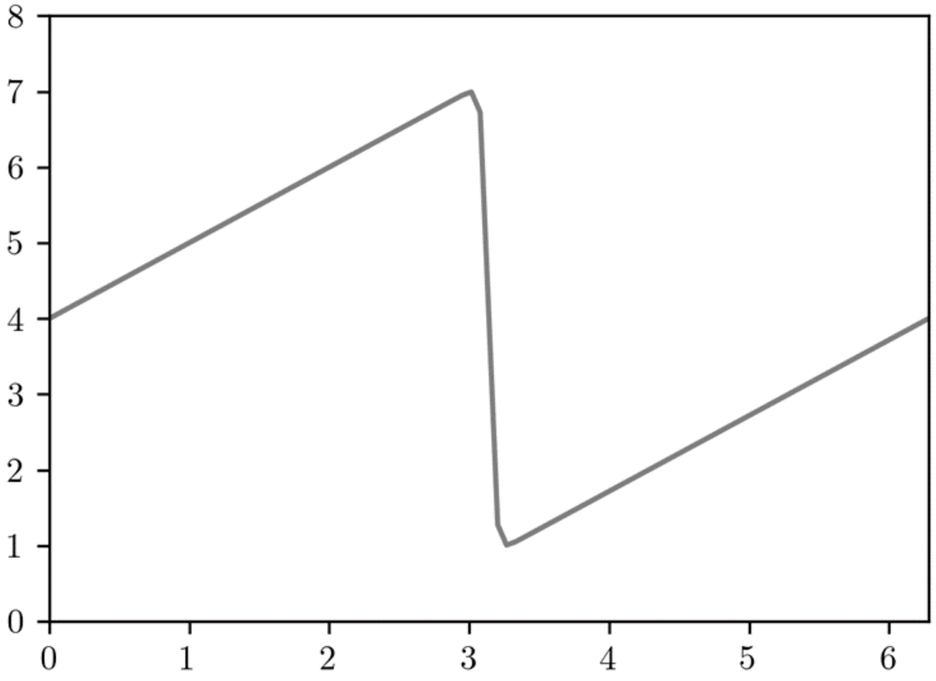

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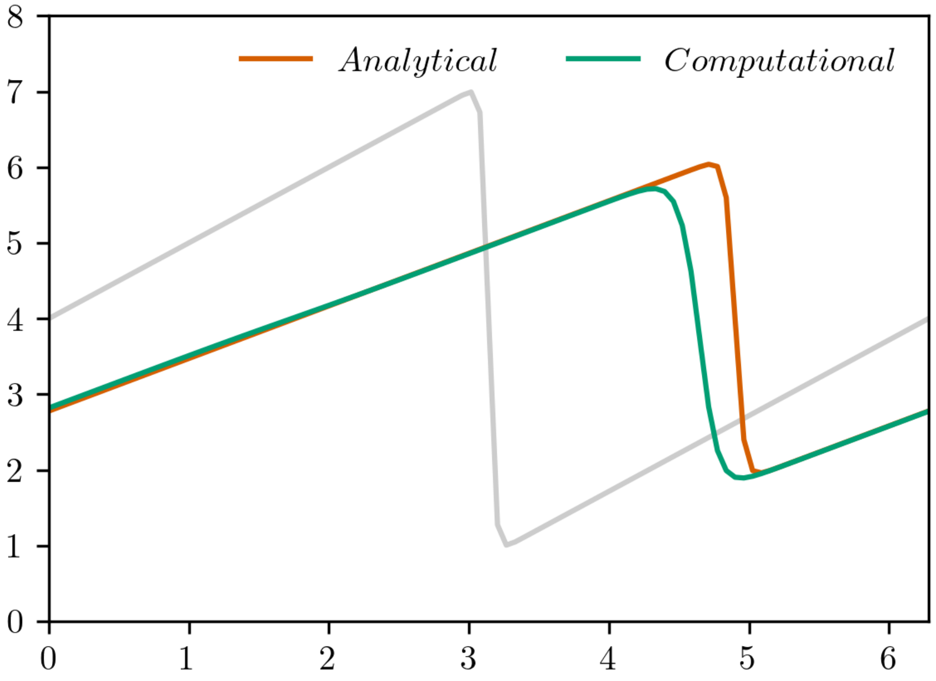

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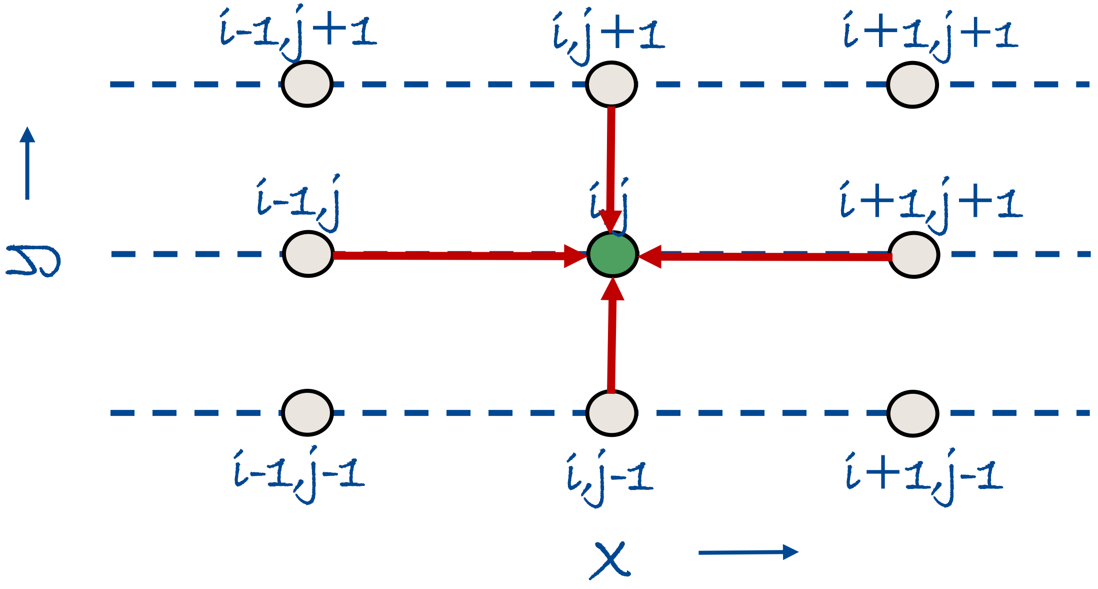
5 points stencil



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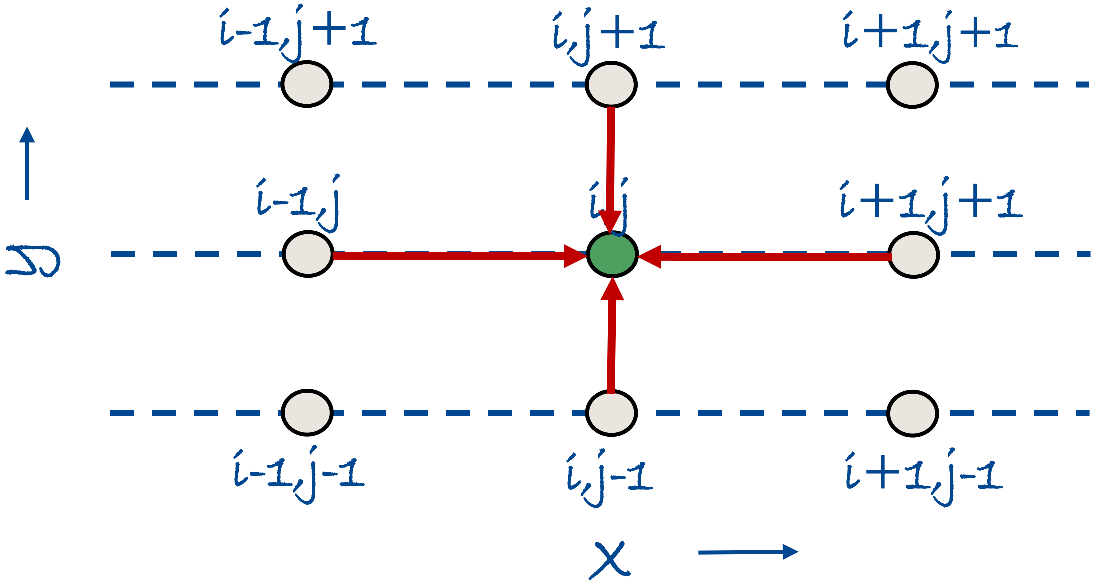

5th Code: 2D Laplace Equation

No time dependence !

Calculates an equilibrium state given the specified BCs

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0$$

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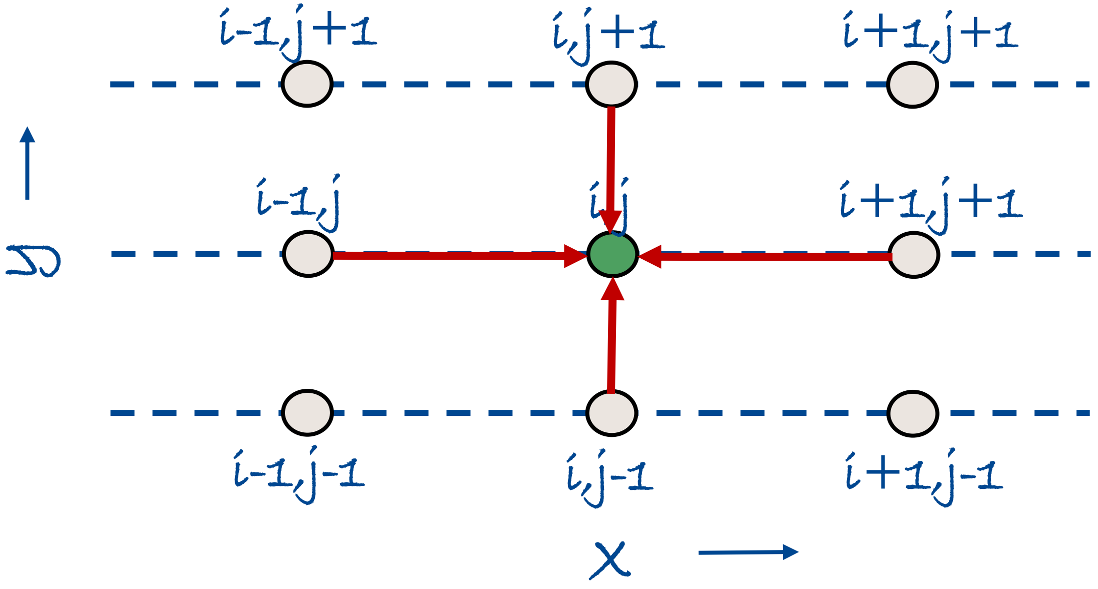

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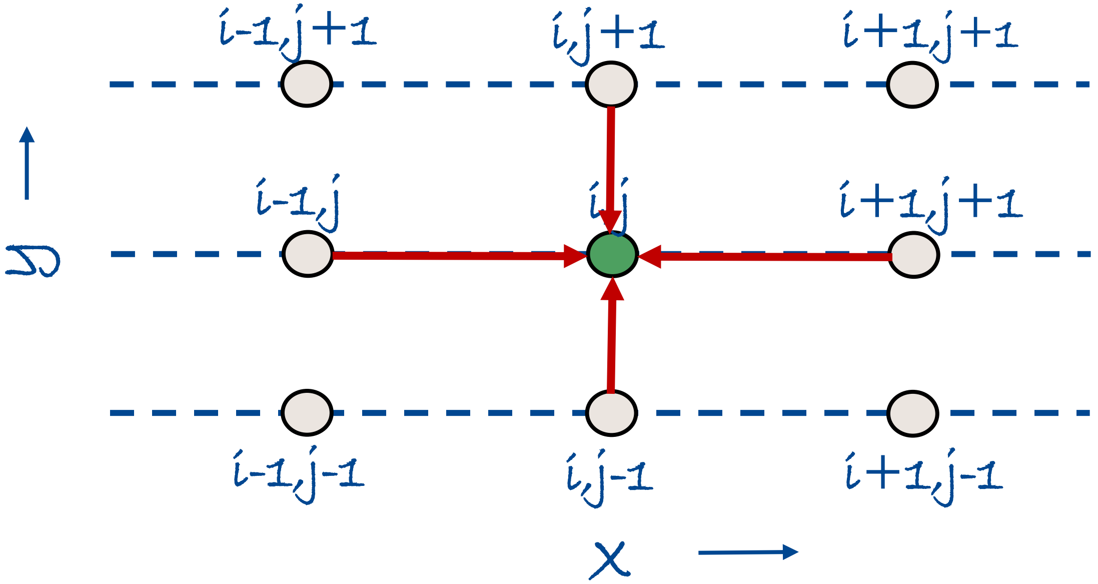
Need to solve iteratively

Will reach equilibrium as iterations → ∞

Need to specify a threshold!

5 points stencil



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BCs: $p = 0 \; at \; x = 0$

$p = y \; at \; x = 2$

$\partial p / \partial y = 0 \; at \; y = 0,1$

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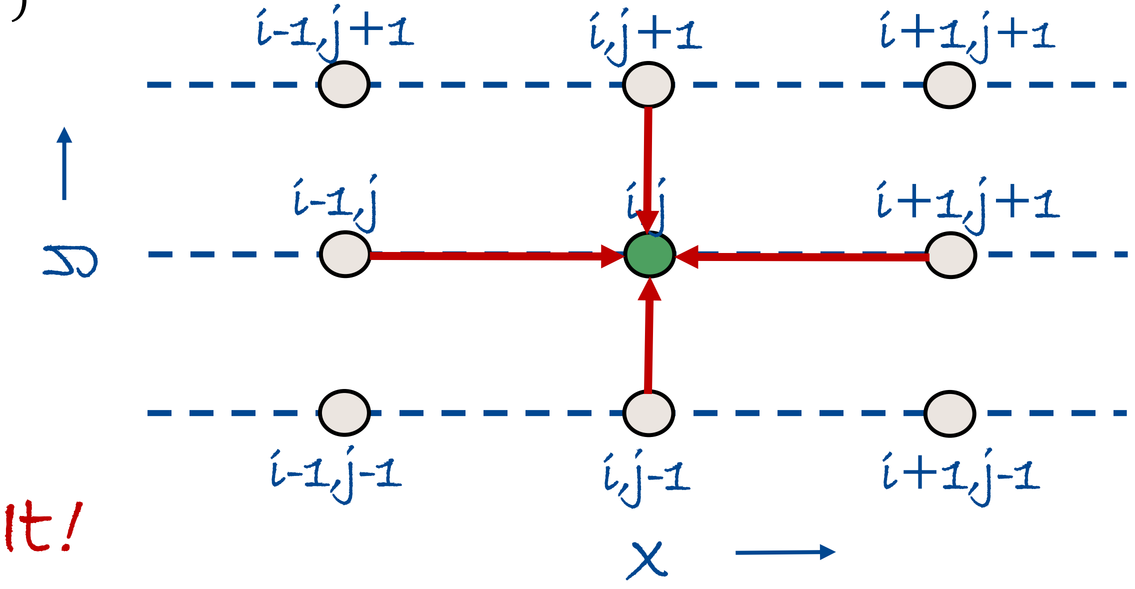


Let's Code It!

5th Code: 2D Laplace Equation

No time dependence !

Calculates an equilibrium state given the specified BCs

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0$$

BCs: $p = 0 \ at\ x = 0$

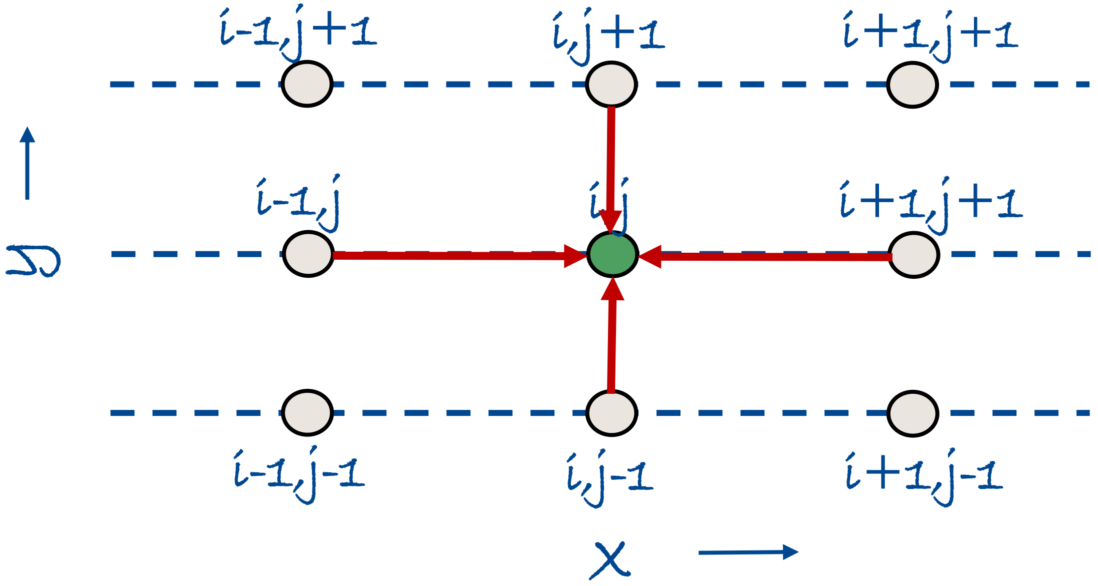
$p = y \ at\ x = 2$

$\partial p/\partial y = 0 \ at\ y = 0{,}1$

$$\frac{p^n_{i+1,j} - 2p^n_{i,j} + p^n_{i-1,j}}{\Delta x^2} + \frac{p^n_{i,j+1} - 2p^n_{i,j} + p^n_{i,j-1}}{\Delta y^2} = 0$$

$$p^n_{i,j} = \frac{\Delta y^2 (p^n_{i+1,j} + p^n_{i-1,j}) + \Delta x^2 (p^n_{i,j+1} + p^n_{i,j-1})}{2(\Delta x^2 + \Delta y^2)}$$

5 points stencil



Need to solve iteratively

Will reach equilibrium as iterations → ∞

Need to specify a threshold!

5th Code: 2D Laplace Equation

```
import numpy as np
from matplotlib import pyplot, cm
from mpl_toolkits.mplot3d import Axes3D
```

5th Code: 2D Laplace Equation

```
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from matplotlib import pyplot, cm
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def plot2D(x,y,p):
    fig = pyplot.figure( figsize=(11,7), dpi=100 )
    ax  = fig.gca(projection=‘3d’)
    X, Y = np.meshgrid(x,y)
    surf = ax.plot_surface( X, Y, p[:], rstride=1, cstride=1, cmap=cm.viridis,
                            linewidth=0, antialiased=False )
    ax.set_xlabel(“$x$”);  ax.set_xlim(0,2)
    ax.set_ylabel(“$y$”);  ax.set_ylim(0,1)
    ax.view_init(30,225)
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def laplace2D(p,y,dx,dy,target_norm):
    norm=1
    pn = np.empty_like(p)
    while norm > target_norm:
        pn = p.copy()
        p[1:-1,1:-1] = (
                    (dy**2 * (pn[1:-1,2:] – pn[1:-1,:-2]))   +
                    (dx**2 * (pn[2:,1:-1] – pn[:-2,1:-1]))
                   ) / (2* (dx**2 + dy**2))
        p[:,0] = 0              # p=0 at x=0
        p[:,-1]= y              # p=y at x=2
        p[0,:] = p[1,:]         # dp/dy=0 at y=0
        p[-1,:] = p[-2,:]       # dp/dy=0 at y=1
        norm = (np.sum(np.abs(p)) – np.sum(np.abs(pn)))/(np.sum(np.abs(pn)))
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nx = 31; ny = 31; dx = 2/(nx-1); dy =1/(ny-1)
x = np.linspace(0, 2, nx); y = np.linspace(0, 1, ny)
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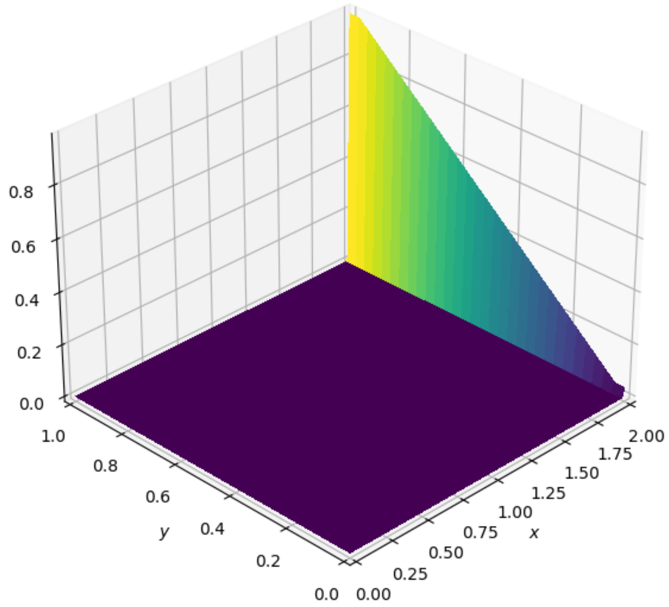
plot2D(x,y,p)
```



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plot2D(x,y,p)
p = laplace2D(p,y,dx,dy,1e-4)
plot2D(x,y,p)
```

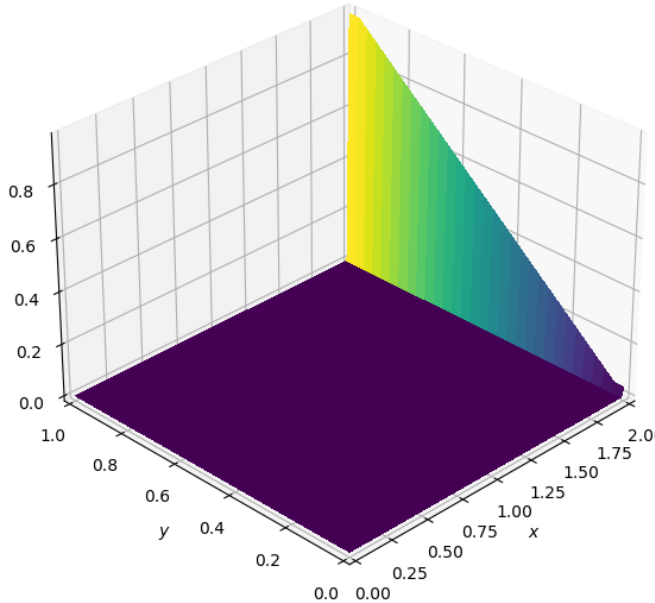

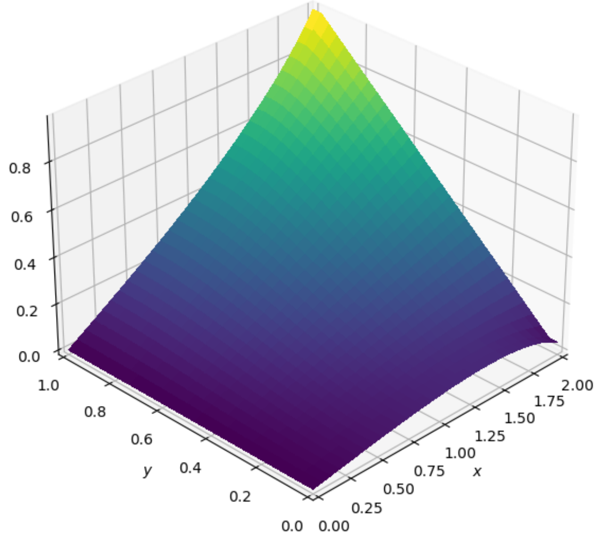

Navier-Stokes equations for incompressible flows

Continuity equation: $\nabla.\vec{v} = 0$

Momentum equation: $$\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v}$$

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No equation for pressure !

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No equation for pressure !

There is no obvious way of coupling these equations !

Navier-Stokes equations for incompressible flows

Continuity equation: $\nabla . \vec{v} = 0$ ← Provides a kinematic constraint that requires pressure field to evolve such that the velocity field remains solenoidal !

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Remedy: “Construct” a pressure field that guarantees continuity constraint!

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$$\cancelto{0}{\frac{\partial \nabla . \vec{v}}{\partial t}} + \nabla . (\vec{v}.\nabla)\vec{v} = -\nabla^2 p + \nu \nabla^2 \cancelto{0}{\nabla . \vec{v}}$$

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Pressure Poisson Equation

$$\boxed{\nabla^2 p = -\nabla.(\vec{v}.\nabla)\vec{v}}$$

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Pressure Poisson Equation

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Let's look at the 6th code to learn how to solve a Poisson's equation.

6th Code: 2D Poisson's Equation

Obtained by adding a source term on the right-hand side of the Laplace's equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = b$$

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Poisson's equation act to "relax" the initial sources in the field

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Poisson's equation act to "relax" the initial sources in the field

Discretized form:

$$\frac{p_{i+1,j}^{n} - 2p_{i,j}^{n} + p_{i-1,j}^{n}}{\Delta x^2} + \frac{p_{i,j+1}^{n} - 2p_{i,j}^{n} + p_{i,j-1}^{n}}{\Delta y^2} = b_{i,j}^{n}$$

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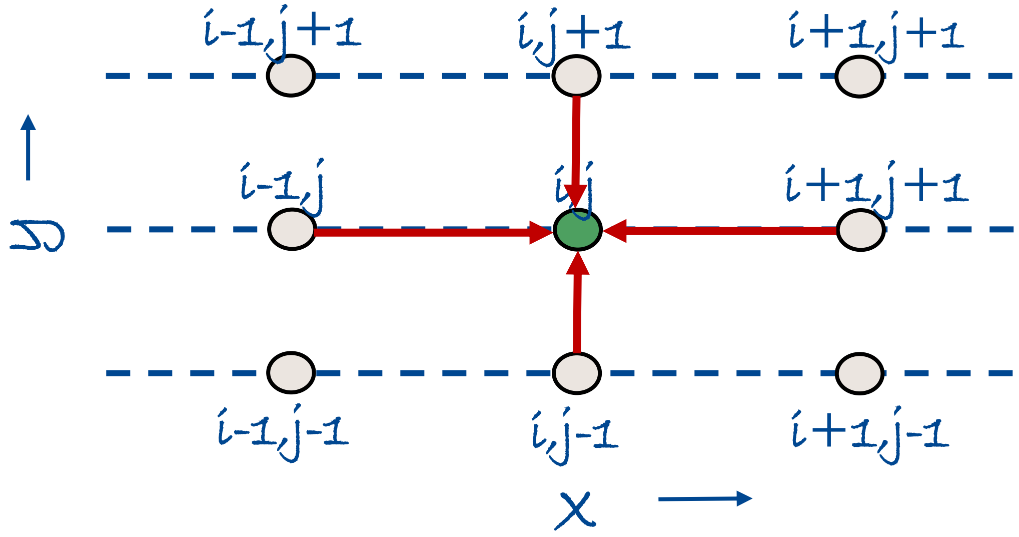
$$p^n_{i,j} = \frac{\Delta y^2 (p^n_{i+1,j} + p^n_{i-1,j}) + \Delta x^2 (p^n_{i,j+1} + p^n_{i,j-1}) - b^n_{i,j}\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}$$

5 points stencil



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Source term:

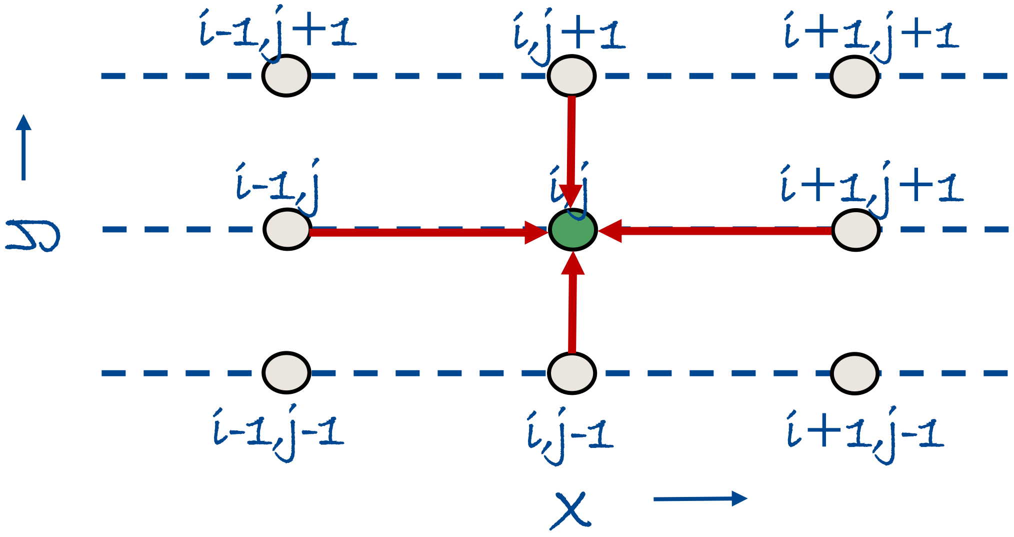
2 Spikes

$$b_{i,j} = -100 \; at \; i = \frac{nx}{4}, j = \frac{3ny}{4}$$

$$b_{i,j} = 100 \; at \; i = \frac{3nx}{4}, j = \frac{ny}{4}$$

$$b_{i,j} = 0 \;\; elsewhere$$

5 points stencil



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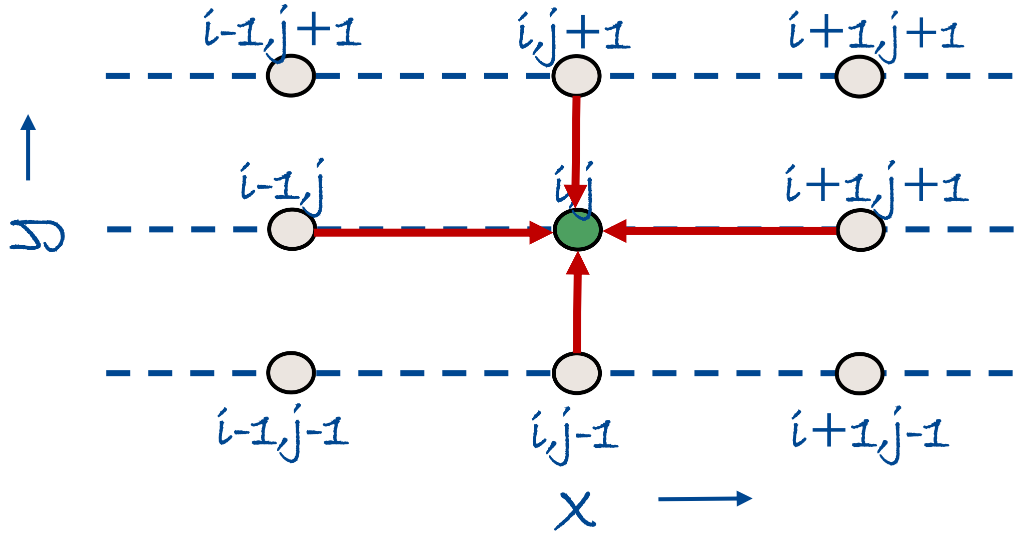
Source term: $b_{i,j}=-100\ at\ i=\frac{nx}{4}, j=\frac{3ny}{4}$

2 Spikes $b_{i,j}=100\ at\ i=\frac{3nx}{4}, j=\frac{ny}{4}$

$b_{i,j}=\ 0\ \ elsewhere$

The iteration will relax the initial spikes!

5 points stencil



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BCs:

$$p = 0 \; at \; x = 0{,}2$$

$$p = 0 \; at \; y = 0{,}2$$

$$p^n_{i,j} = \frac{\Delta y^2 (p^n_{i+1,j} + p^n_{i-1,j}) + \Delta x^2 (p^n_{i,j+1} + p^n_{i,j-1}) - b^n_{i,j} \Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}$$

Source term:

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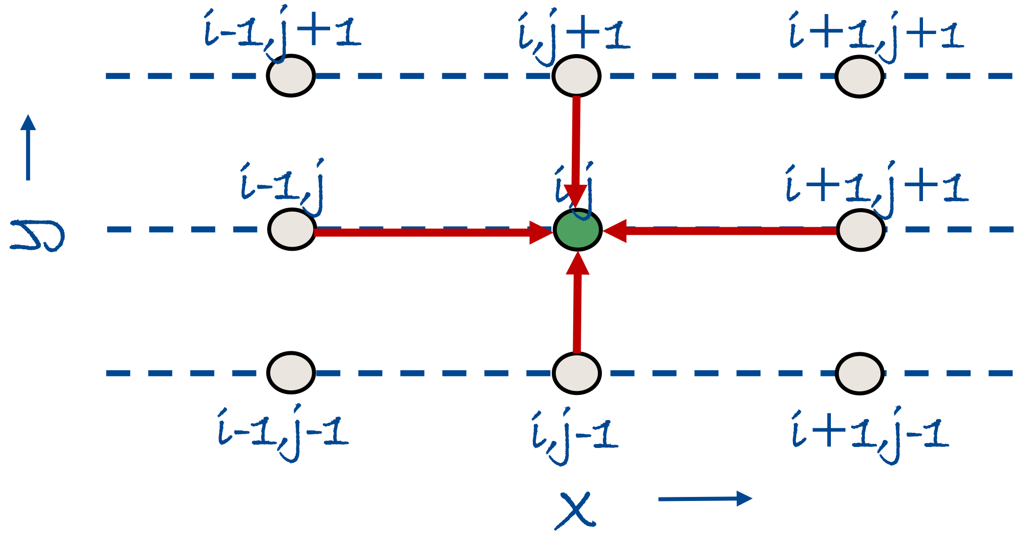
$$b_{i,j} = 0 \;\; elsewhere$$

The iteration will relax the initial spikes!

5 points stencil



6th Code: 2D Poisson's Equation

Obtained by adding a source term on the right-hand side of the Laplace's equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = b$$

Poisson's equation act to "relax" the initial sources in the field

Discretized form:

$$\frac{p_{i+1,j}^n - 2p_{i,j}^n + p_{i-1,j}^n}{\Delta x^2} + \frac{p_{i,j+1}^n - 2p_{i,j}^n + p_{i,j-1}^n}{\Delta y^2} = b_{i,j}^n$$

BCs:

$$p = 0 \; at \; x = 0{,}2$$

$$p = 0 \; at \; y = 0{,}2$$

$$p_{i,j}^n = \frac{\Delta y^2 (p_{i+1,j}^n + p_{i-1,j}^n) + \Delta x^2 (p_{i,j+1}^n + p_{i,j-1}^n) - b_{i,j}^n \Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}$$

Source term:

2 Spikes

$$b_{i,j} = -100 \; at \; i = \frac{nx}{4}, j = \frac{3ny}{4}$$

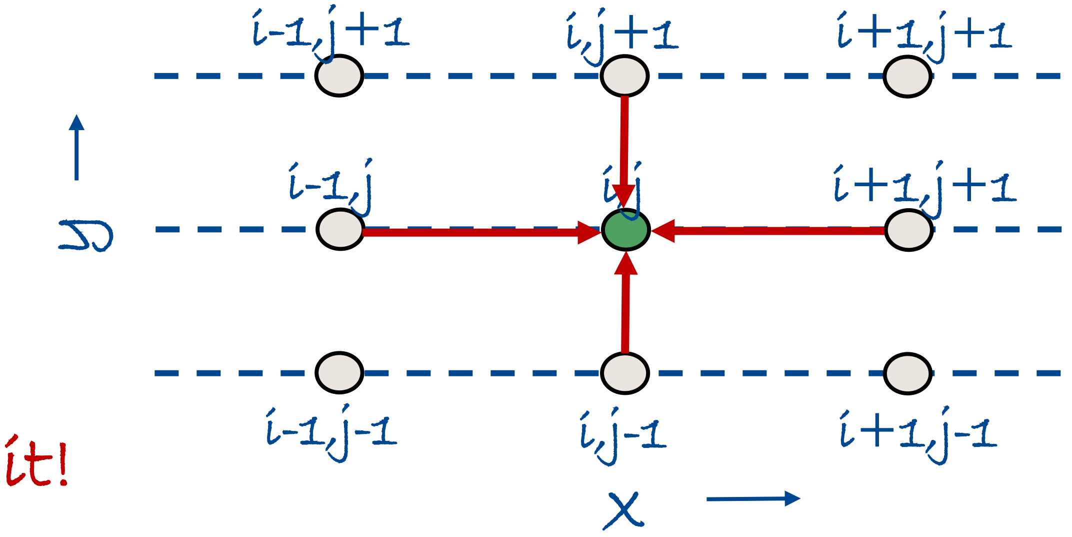
$$b_{i,j} = 100 \; at \; i = \frac{3nx}{4}, j = \frac{ny}{4}$$

$$b_{i,j} = 0 \;\; elsewhere$$

The iteration will relax the initial spikes!

Let's Code it!

5 points stencil



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    ax  = fig.gca(projection=‘3d’)
    X, Y = np.meshgrid(x,y)
    surf = ax.plot_surface( X, Y, p[:], rstride=1, cstride=1, cmap=cm.viridis,
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    ax.set_xlabel(“$x$”);  ax.set_xlim(0,2)
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plot2D(x,y,p)
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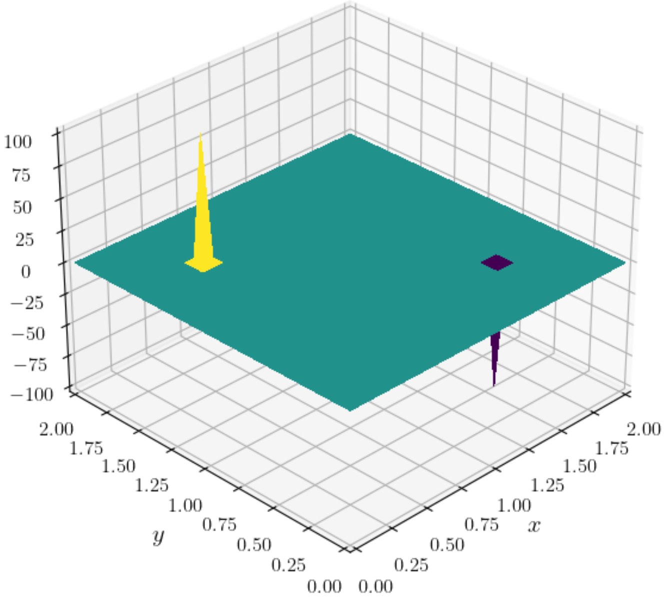

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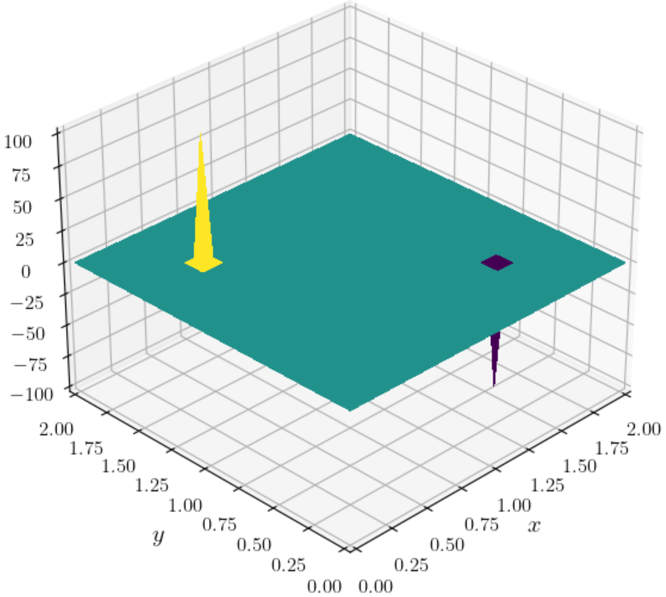

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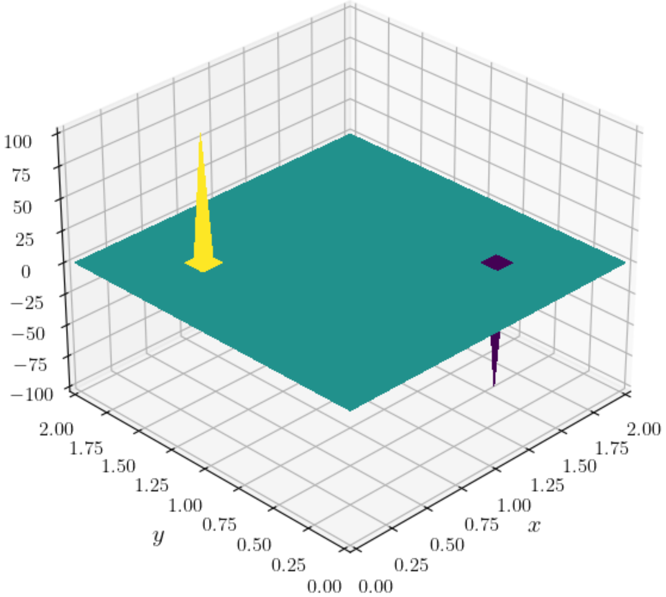
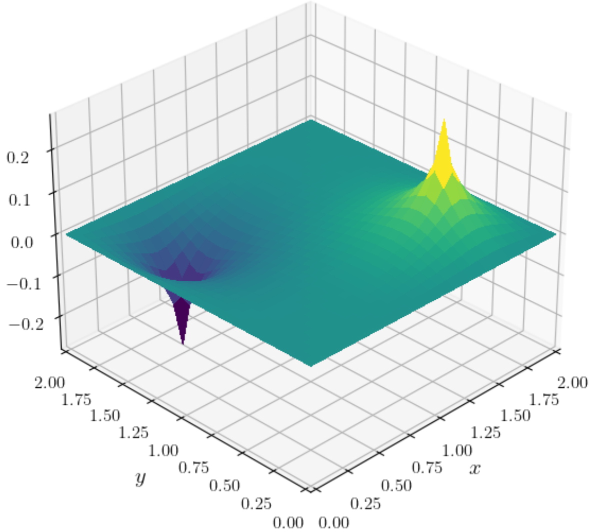
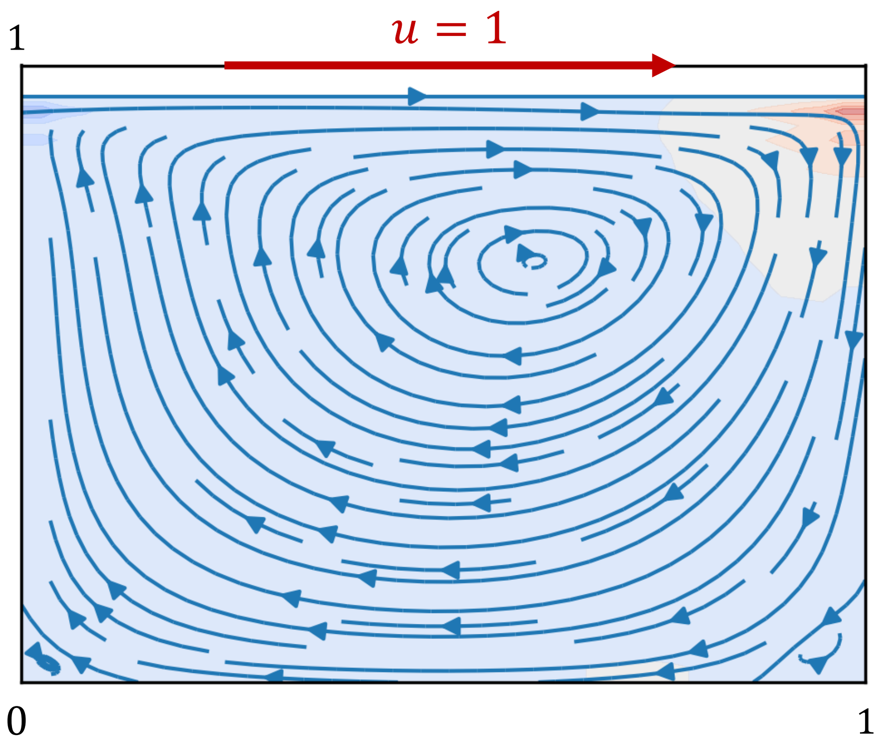




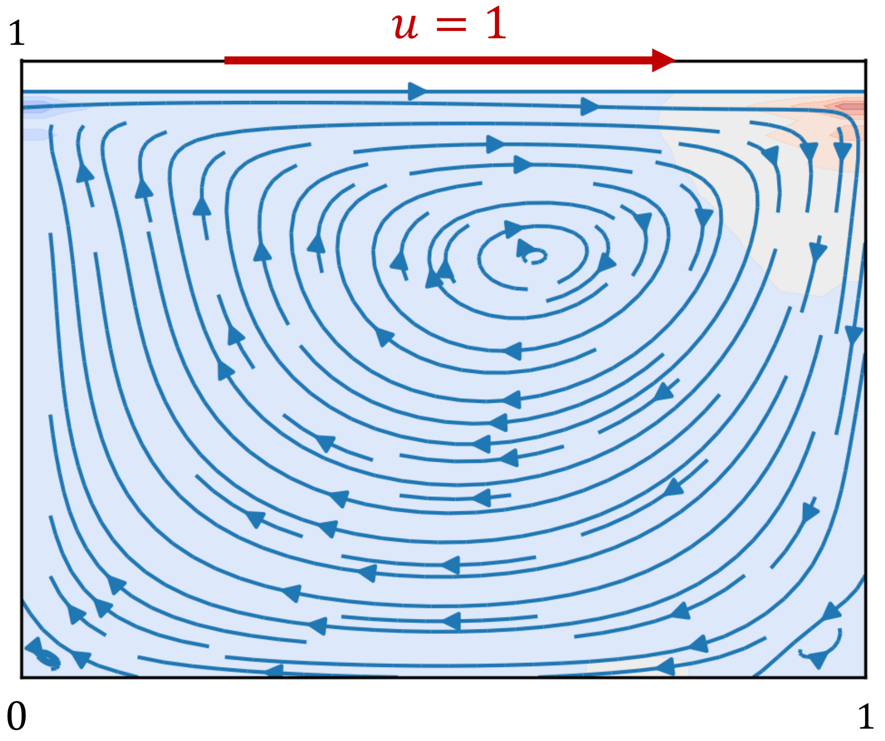
Practical Session: 2D Lid Driven Cavity Flow



Practical Session: 2D Lid Driven Cavity Flow

Continuity: $\nabla . \vec{v} = 0$

Momentum: $\dfrac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v} = -\dfrac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v}$



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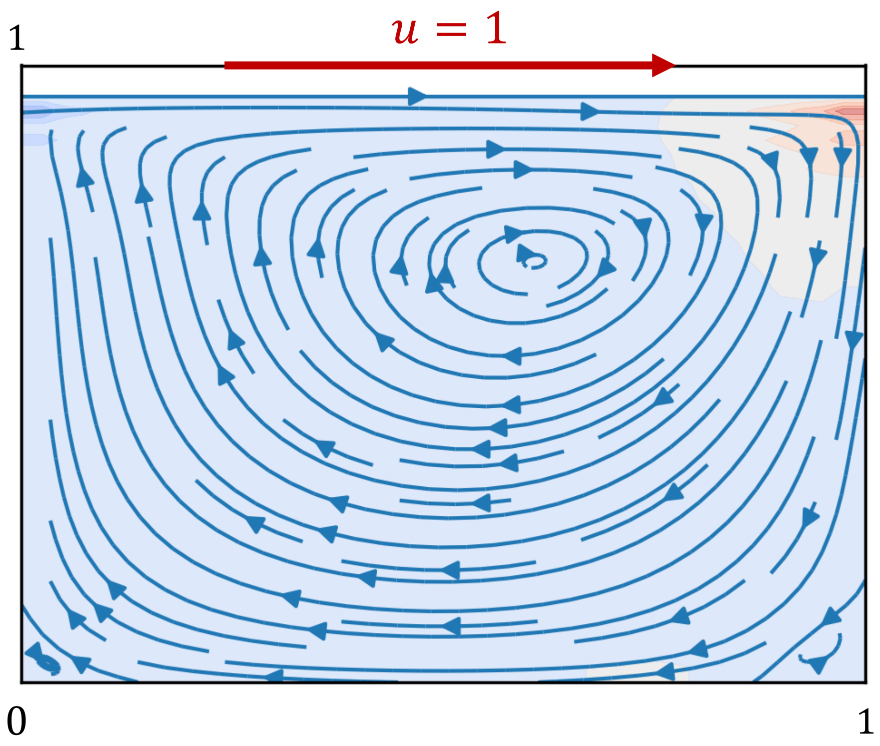

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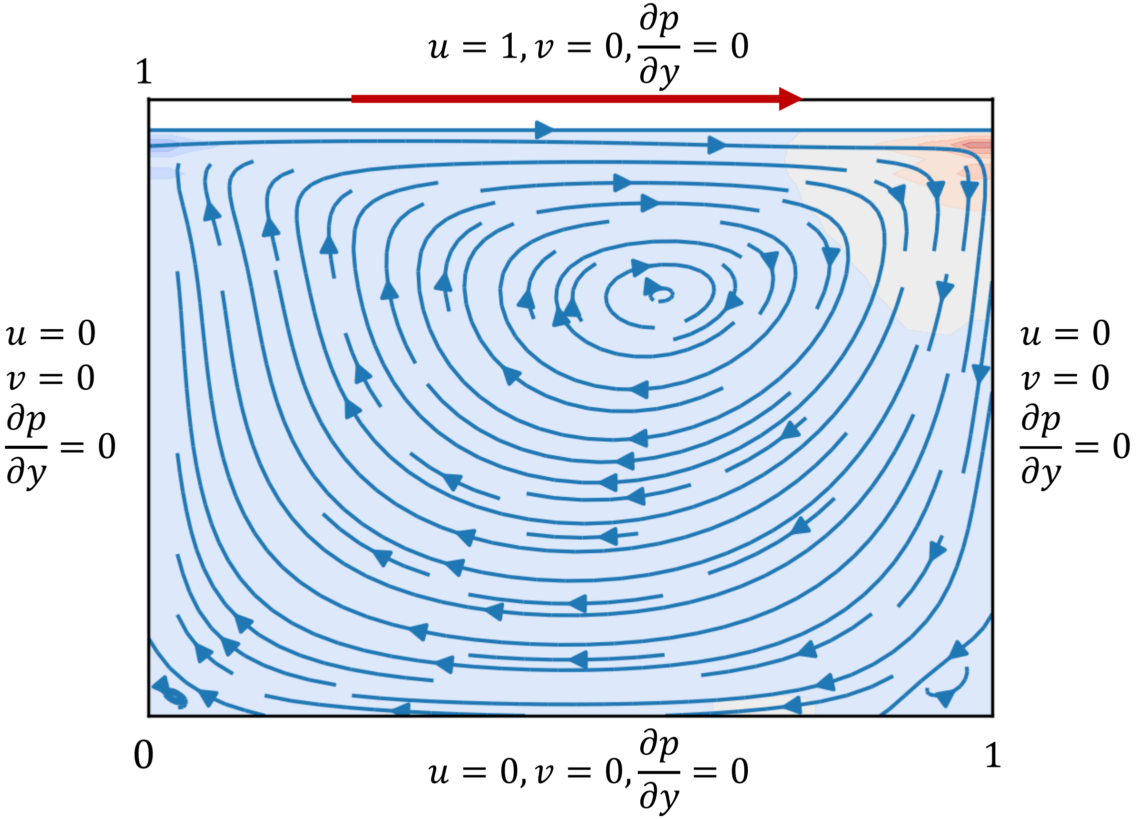

$u = 1, v = 0, \dfrac{\partial p}{\partial y} = 0$
1
$u = 0$
$v = 0$
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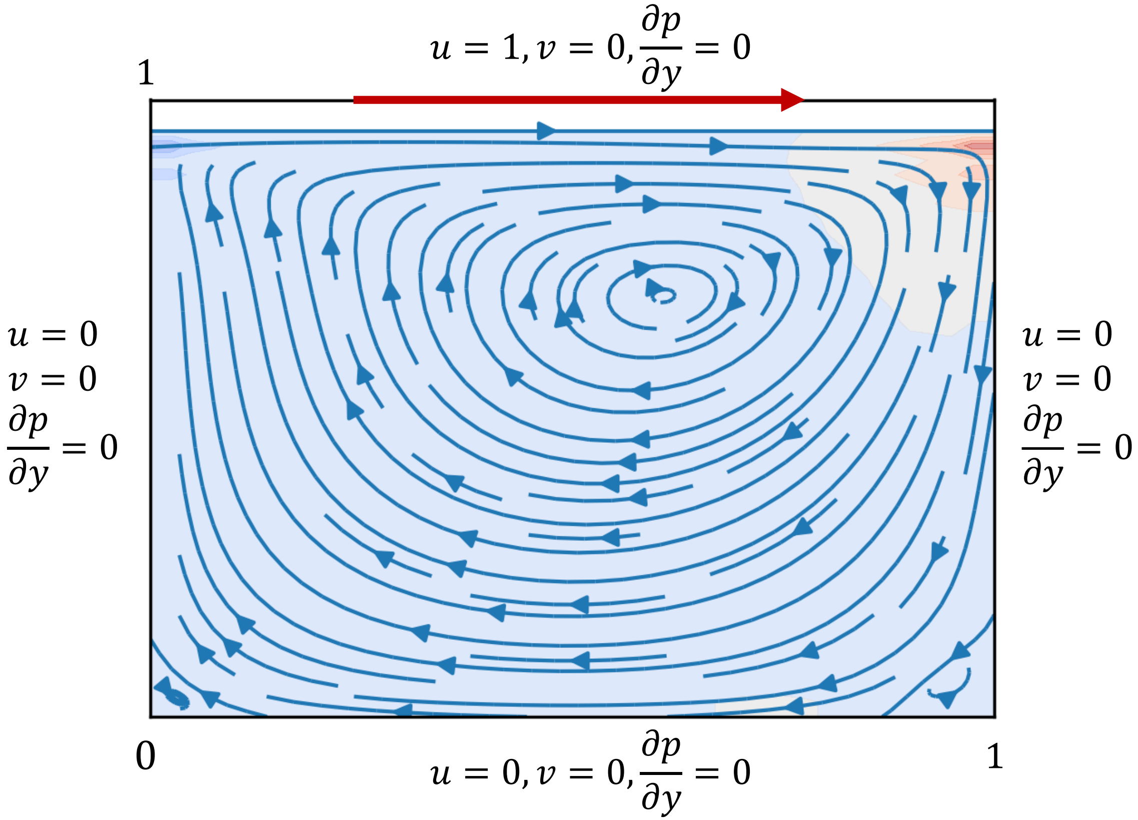
What about the pressure?



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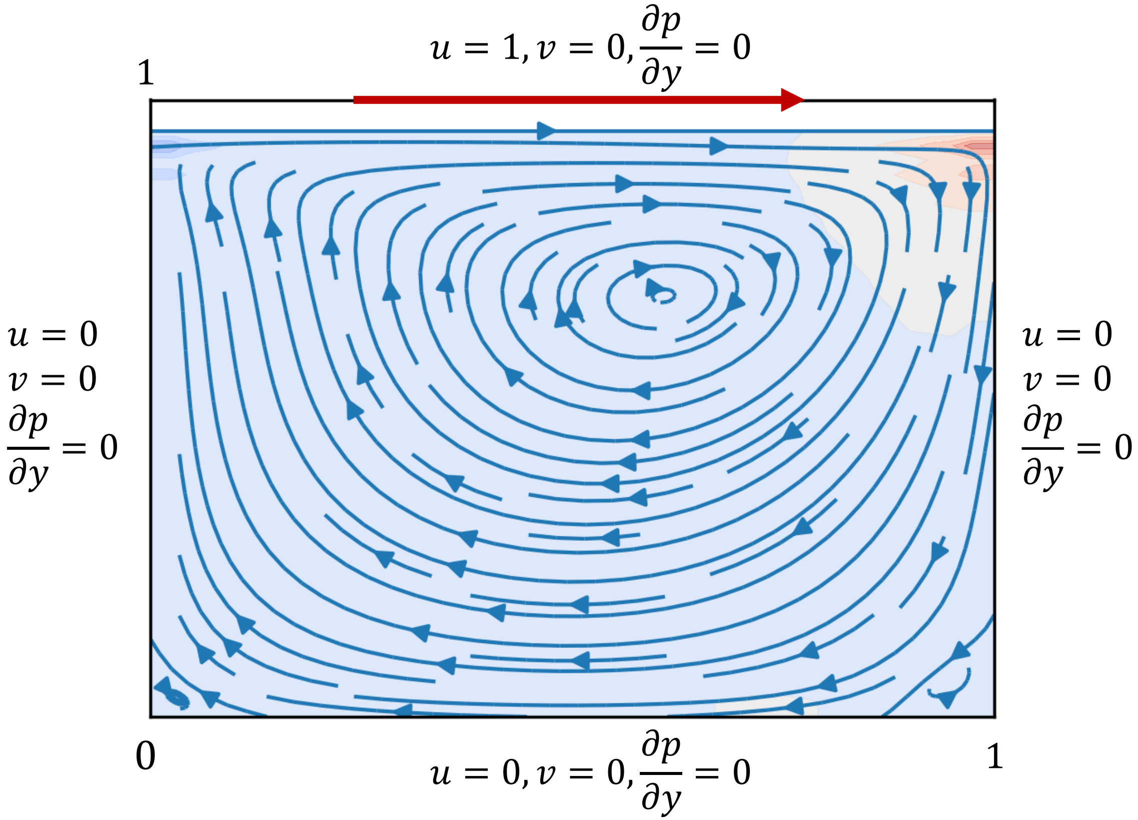

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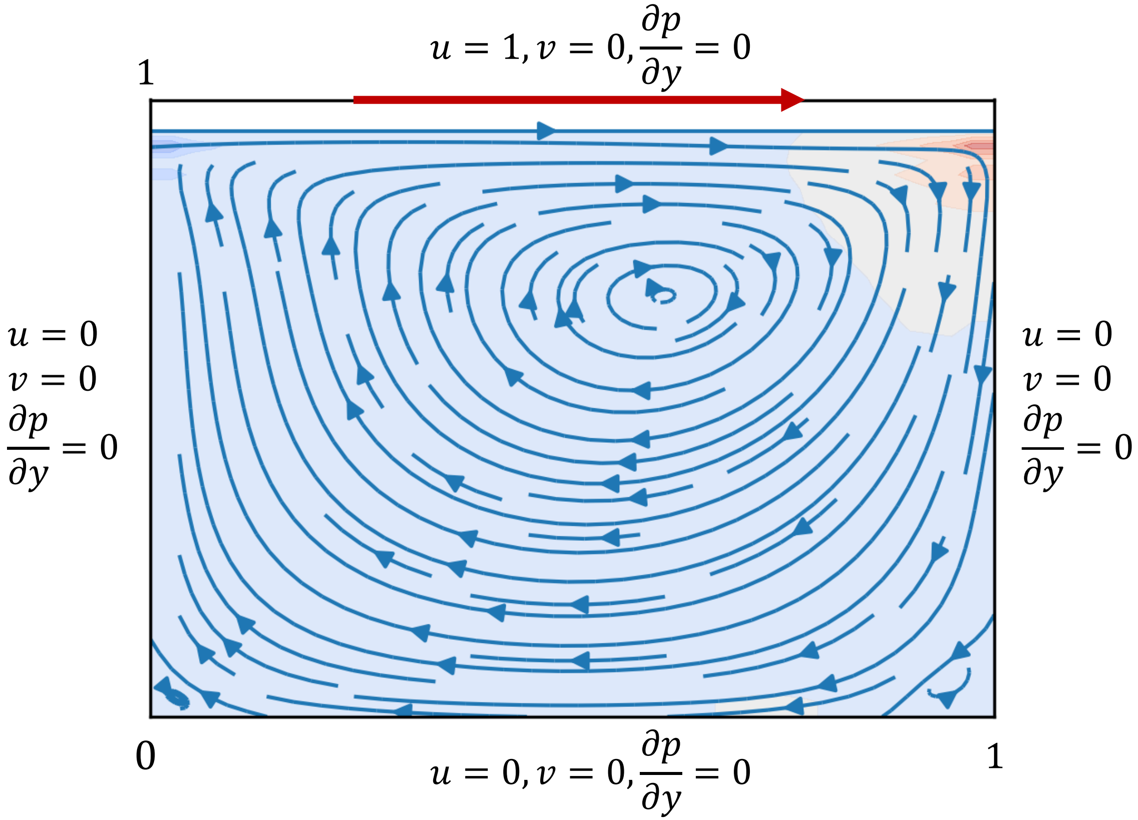

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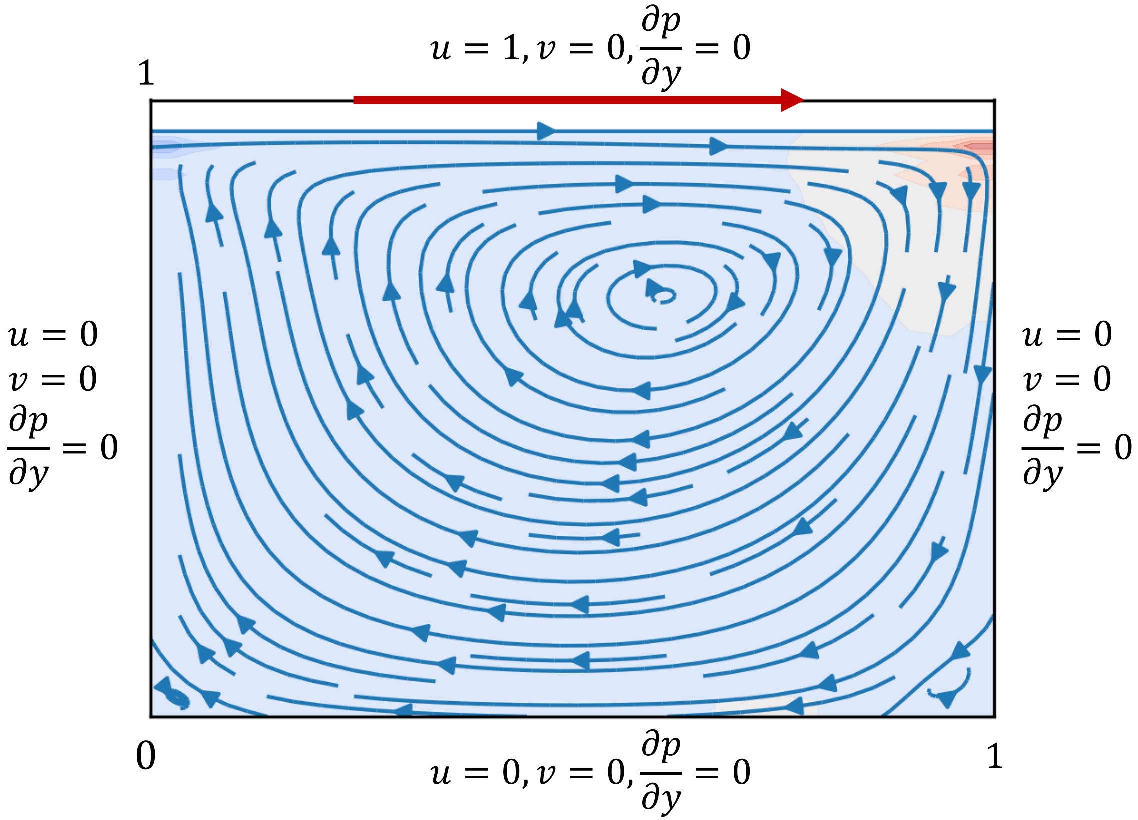

Now it's your turn to discretize the equations, code it & visualize the results!

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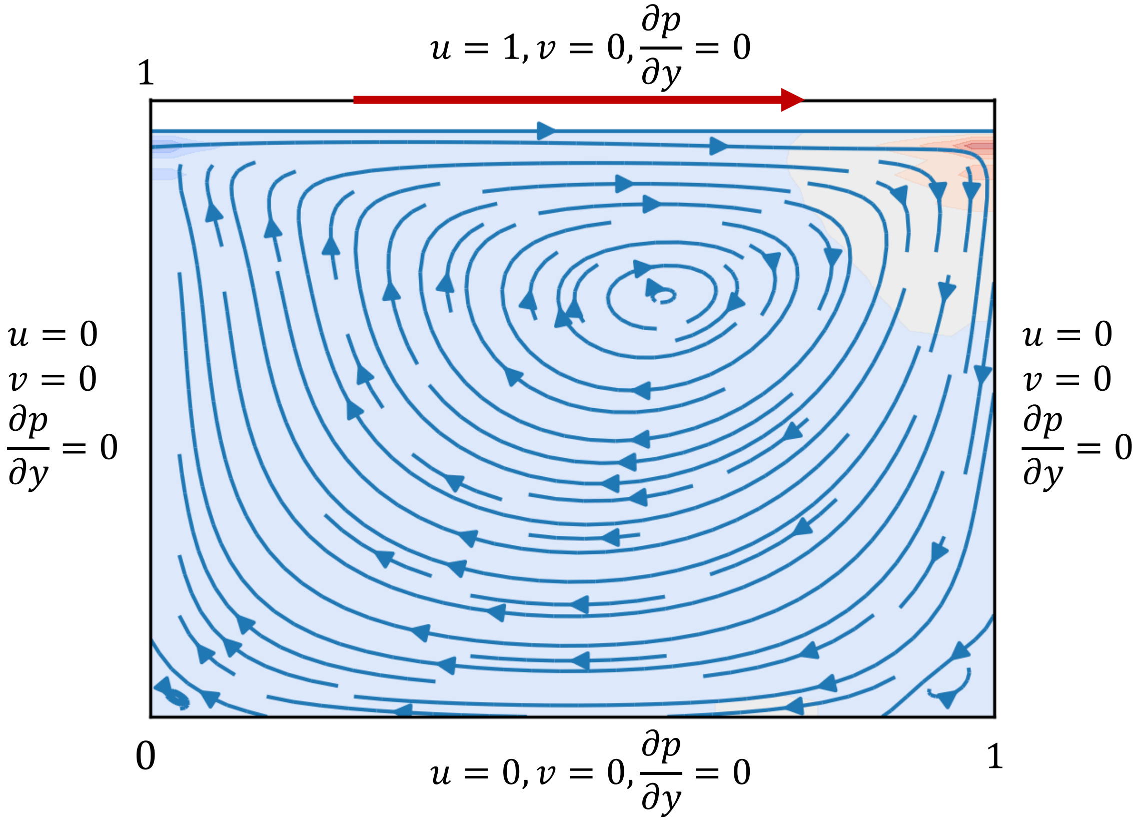
Recall: The pressure Poisson's Equation

$$\nabla^2 p = -\nabla.(\vec{v}.\nabla)\vec{v}$$

$$\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) = -\rho\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\right)$$



Now it's your turn to discretize the equations, code it & visualize the results!

Good Luck!