Engineering Plasticity(ID5040)- Project 2

Umair Hussain (ME19D400)

Solution of a Boundary Value Problem using a non-associative Drucker-Prager plasticity model

1 Introduction

The behavior of geomaterials can be better described as an elasto-plastic behavior. But unlike most metals, this behavior is sensitive to the pressure (Hydrostatic stress). Such behavior can be very well modeled using the Drucker-Prager plasticity model (1952)(1). The deformation in granular materials can be understood as two grains sliding over each other. This sliding depends on the friction between them which depends on the pressure. Such deformation also accompanies volume changes with it, which will be evident from the plastic strain formula. The criterion is based on the assumption that the octahedral shear stress at failure depends linearly on the octahedral normal stress through material constants. This model is generalized for all kind of pressure dependent materials. Different modifications have been done to fit for specific materials. Albrate studied the model for polymers (2). By manipulating values of the constants, Gibson etal. used the model for foams (3).

2 Formulation

2.1 Governing Equations

- Elastic Constitutive Law:

$$\dot{\sigma}_{\mathrm{ij}} = C_{\mathrm{ijkl}}(\dot{\epsilon}_{\mathrm{kl}} - \dot{\epsilon^{\mathrm{p}}}_{\mathrm{kl}}) \tag{1}$$

  where, σ_{ij} is the stress tensor; ϵ_{kl} is the total strain tensor; $\epsilon^{\mathrm{p}}{}_{\mathrm{kl}}$ is the plastic strain tensor; and C_{ijkl} is the fourth-order elastic stifness tensor. The dot above represents the rate form of the constitutive equation.

- Yield Criteria:

$$f = \sqrt{3J_2} + \alpha I_1 - \kappa \leq 0 \tag{2}$$

  where, J_2 is the second invariant of the deviatoric component of the stress tensor; I_1 is the first invariant of the stress tensor; α and κ are material constants.

- Flow Rule:

$$\dot{\epsilon^{\mathrm{p}}}_{\mathrm{kl}} = \dot{\lambda}\frac{\partial g}{\partial \sigma_{\mathrm{kl}}} \tag{3}$$

  where, $\dot{\lambda}$ is the plastic multiplier; g is the plastic potential function, given as

$$g = \sqrt{3J_2} + \beta I_1 - \kappa \tag{4}$$

  So,

$$\dot{\epsilon^{\mathrm{p}}}_{\mathrm{kl}} = \dot{\lambda}\left(\frac{3}{2\sqrt{3J_2}}\tau_{\mathrm{kl}} + \frac{\beta}{3}\delta_{\mathrm{kl}}\right) \tag{5}$$

- Kuhn-Tucker condition:

$$f \leq 0; \dot{\lambda} \geq 0 \tag{6}$$

  So,

$$f\dot{\lambda} = 0 \tag{7}$$

- Consistency condition:

$$\dot{f} = 0 \tag{8}$$

2.2 Derivation for Plastic multiplier

Using the consistency condition:

$$\begin{aligned}
\dot{f} &= 0 \\
\frac{\partial f}{\partial \tau_{\mathrm{ij}}} \dot{\tau_{\mathrm{ij}}} &= 0 \\
\frac{3}{2\sqrt{3J_2}} \tau_{\mathrm{ij}} \dot{\tau_{\mathrm{ij}}} + \frac{\alpha}{3} \delta_{\mathrm{ij}} \dot{\sigma_{\mathrm{ij}}} &= 0 \\
\frac{3}{2\sqrt{3J_2}} \tau_{\mathrm{ij}} \dot{\sigma_{\mathrm{ij}}} + \frac{\alpha}{3} \delta_{\mathrm{ij}} \dot{\sigma_{\mathrm{ij}}} &= 0
\end{aligned}$$

Substituting value of $\dot{\sigma_{\mathrm{ij}}}$ from (1):

$$(\frac{3}{2\sqrt{3J_2}} \tau_{\mathrm{ij}} + \frac{\alpha}{3} \delta_{\mathrm{ij}})(C_{\mathrm{ijkl}}(\dot{\epsilon_{\mathrm{kl}}} - \dot{\epsilon^{\mathrm{p}}_{\mathrm{kl}}})) = 0$$

Substituting value of plastic strain from (5):

$$\begin{aligned}
(\frac{3}{2\sqrt{3J_2}} \tau_{\mathrm{ij}} + \frac{\alpha}{3} \delta_{\mathrm{ij}}) C_{\mathrm{ijkl}}(\dot{\epsilon_{\mathrm{kl}}} - \dot{\lambda}(\frac{3}{2\sqrt{3J_2}} \tau_{\mathrm{kl}} + \frac{\beta}{3} \delta_{\mathrm{kl}})) &= 0 \\
\dot{\lambda} = \frac{C_{\mathrm{ijkl}} \dot{\epsilon_{\mathrm{kl}}} (\frac{3}{2\sqrt{3J_2}} \tau_{\mathrm{ij}} + \frac{\alpha}{3} \delta_{\mathrm{ij}})}{C_{\mathrm{ijkl}} (\frac{3}{2\sqrt{3J_2}} \tau_{\mathrm{ij}} + \frac{\alpha}{3} \delta_{\mathrm{ij}})(\frac{3}{2\sqrt{3J_2}} \tau_{\mathrm{kl}} + \frac{\beta}{3} \delta_{\mathrm{kl}})}
\end{aligned} \tag{9}$$

2.3 Derivation for Tangent Modulus

From (1):

$$\dot{\sigma_{\mathrm{ij}}} = C_{\mathrm{ijkl}}(\dot{\epsilon_{\mathrm{kl}}} - \dot{\lambda}(\frac{3}{2\sqrt{3J_2}} \tau_{\mathrm{kl}} + \frac{\beta}{3} \delta_{\mathrm{kl}}))$$

Substituting value of $\dot{\lambda}$ from (9):

$$\dot{\sigma_{\text{ij}}} = C_{\text{ijkl}}(\dot{\epsilon_{\text{kl}}} - (\frac{3}{2\sqrt{3J_2}}\tau_{\text{kl}} + \frac{\beta}{3}\delta_{\text{kl}})(\frac{C_{\text{ijkl}}\dot{\epsilon_{\text{kl}}}(\frac{3}{2\sqrt{3J_2}}\tau_{\text{ij}} + \frac{\alpha}{3}\delta_{\text{ij}})}{C_{\text{ijkl}}(\frac{3}{2\sqrt{3J_2}}\tau_{\text{ij}} + \frac{\alpha}{3}\delta_{\text{ij}})(\frac{3}{2\sqrt{3J_2}}\tau_{\text{kl}} + \frac{\beta}{3}\delta_{\text{kl}})}))$$

$$\dot{\sigma_{\text{ij}}} = (C_{\text{ijkl}} - C_{\text{ijkl}}(\frac{3}{2\sqrt{3J_2}}\tau_{\text{kl}} + \frac{\beta}{3}\delta_{\text{kl}})(\frac{C_{\text{ijkl}}(\frac{3}{2\sqrt{3J_2}}\tau_{\text{ij}} + \frac{\alpha}{3}\delta_{\text{ij}})}{C_{\text{ijkl}}(\frac{3}{2\sqrt{3J_2}}\tau_{\text{ij}} + \frac{\alpha}{3}\delta_{\text{ij}})(\frac{3}{2\sqrt{3J_2}}\tau_{\text{kl}} + \frac{\beta}{3}\delta_{\text{kl}})}))\dot{\epsilon_{\text{kl}}}$$

$$\frac{\partial\dot{\sigma_{\text{ij}}}}{\partial\dot{\epsilon_{\text{kl}}}} = C^{\text{t}}{}_{\text{ijkl}}$$

Hence, consistent tangent modulus is:

$$C^{\text{t}}{}_{\text{ijkl}} = (C_{\text{ijkl}} - C_{\text{ijkl}}(\frac{3}{2\sqrt{3J_2}}\tau_{\text{kl}} + \frac{\beta}{3}\delta_{\text{kl}})(\frac{C_{\text{ijkl}}(\frac{3}{2\sqrt{3J_2}}\tau_{\text{ij}} + \frac{\alpha}{3}\delta_{\text{ij}})}{C_{\text{ijkl}}(\frac{3}{2\sqrt{3J_2}}\tau_{\text{ij}} + \frac{\alpha}{3}\delta_{\text{ij}})(\frac{3}{2\sqrt{3J_2}}\tau_{\text{kl}} + \frac{\beta}{3}\delta_{\text{kl}})})) \tag{10}$$

3 Results

3.1 Stress-strain behavior of the model under simple loading conditions

Using cutting plane algorithm, to model a simple loading condition for a Drucker-Prager model, following results are generated. The value of coefficients have been optimised using the values suggested by Walid A. Al-Kutti(1) and also based on the convergence of the program. The expressions fro α and β used are:

$$\alpha = \frac{2sin\phi}{\sqrt{3}(3 - sin\phi)}$$

$$\beta = \frac{2sin\theta}{\sqrt{3}(3 - sin\theta)}$$

where, ϕ= 35° and θ= 28° are suitably chosen values for easy convergence of code.

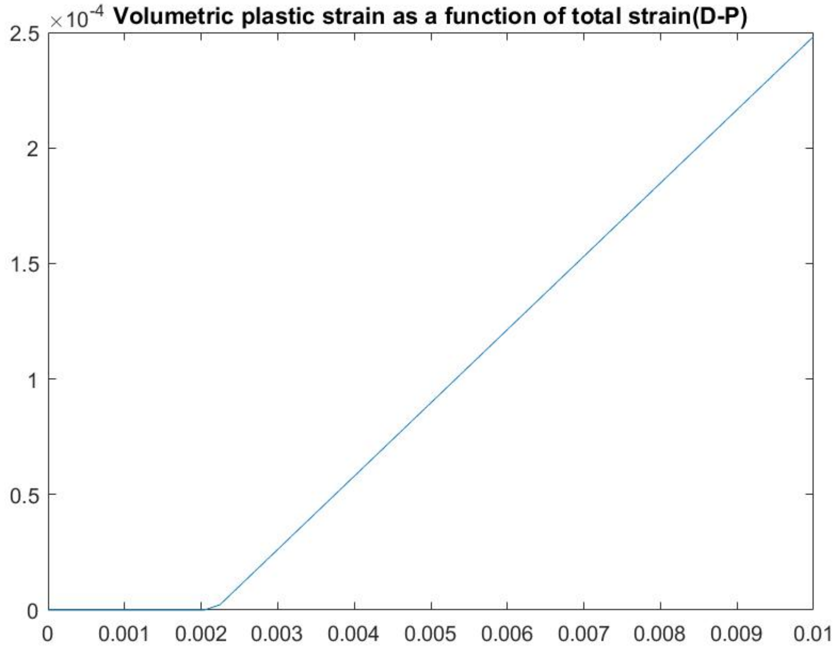

Figure 1: Volumetric Plastic strain with increase in total strain.

A prominent thing about Drucker-Prager model is that the volumetric or mean plastic strain is non-zero opposite to other models for metals, since porous materials exhibit a considerable volume change due to sliding of grains. This fact is evident from the volumetric plastic strain curve above. While, for J_2 model, the curve suggests negligible plastic volumetric strain.

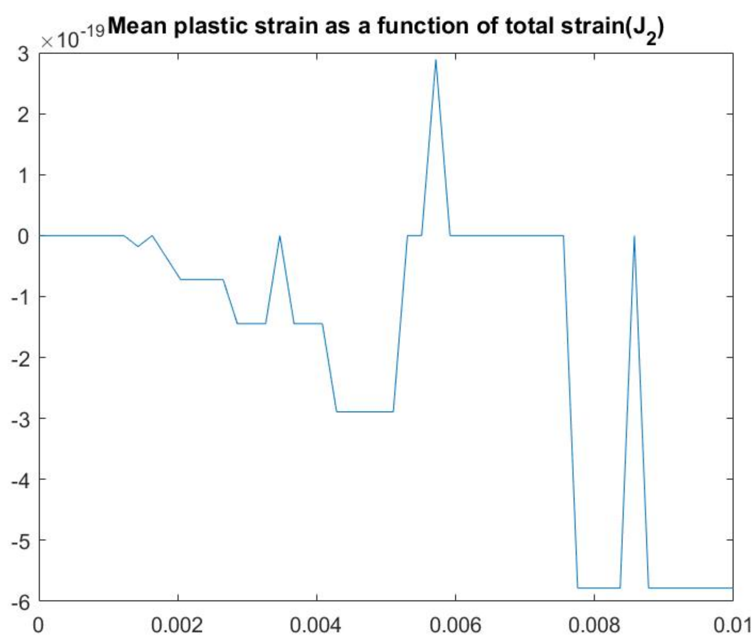

Figure 2: Volumetric Plastic strain with increase in total strain for J2 model.

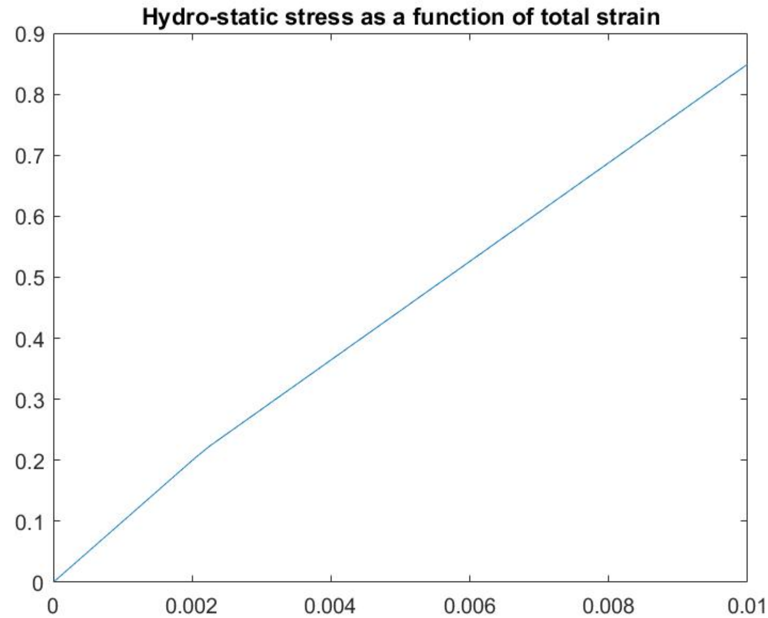

Figure 3: Hydro-static stress with increase in total strain.

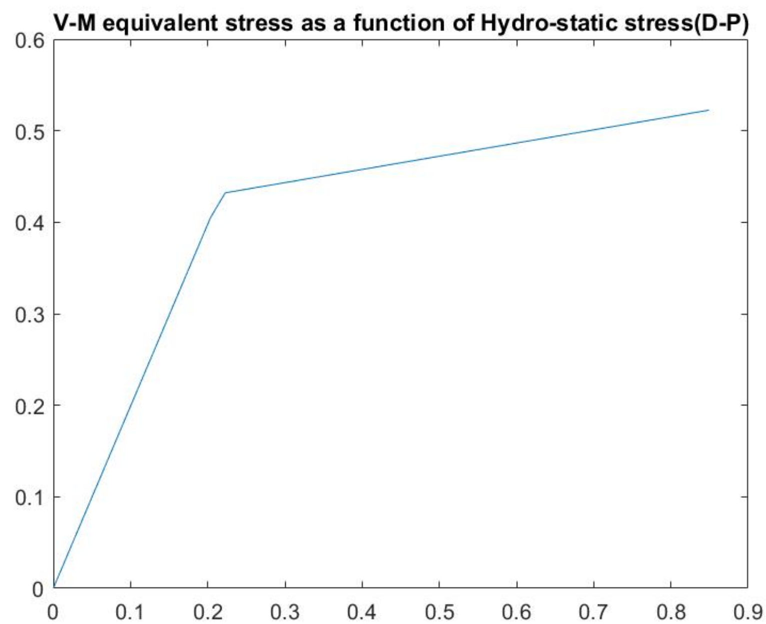

Figure 4: Von-Mises equivalent stress as a function of Hydro-static stress.

The plot of Von-mises equivalent and mean stress represents the Meridian plane for Drucker-Prager yield surface. As the yield point is achieved, the curve suggests a conical yield surface. If we increase the total strain further, we can reach the apex of cone at mean stress axis. On. the other hand it remains constant after yielding for the case J_2 model, suggesting the cylindrical yield surface for it.

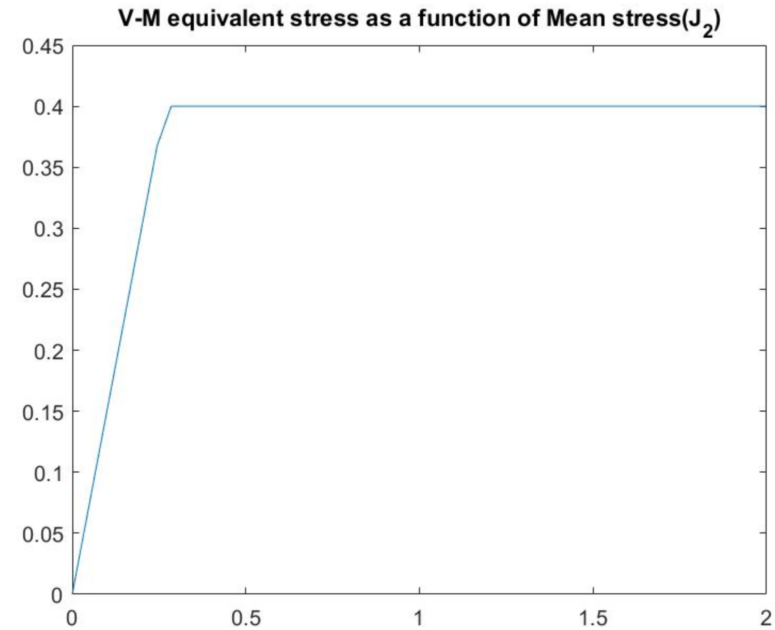

Figure 5: Von-Mises equivalent stress as a function of Hydro-static stress for J2 model.

3.2 Solution of plate with hole under tension using Drucker-Prager model

Following are the plots for Equivalent and Hydro-static stress and plastic strain distribution in the plate using Drucker-Prager model and their counter-parts using J_2 model:

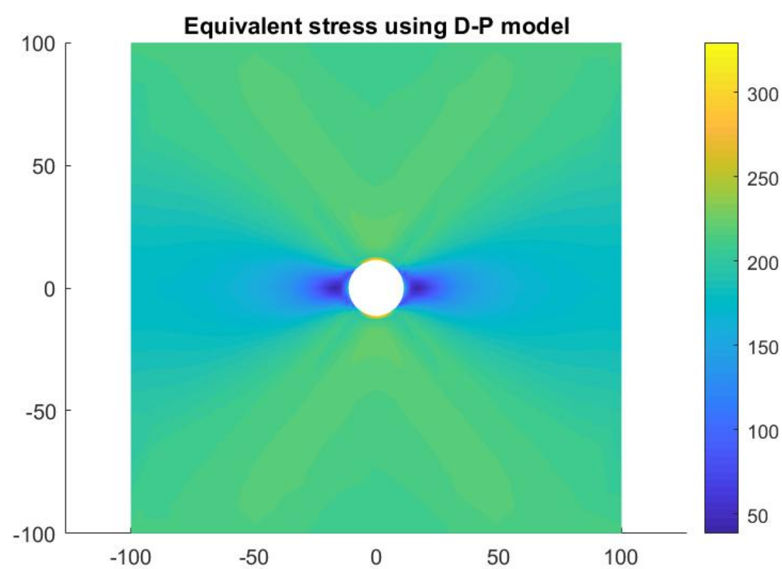

Figure 6: Equivalent stress distribution in domain using Drucker-Prager model.

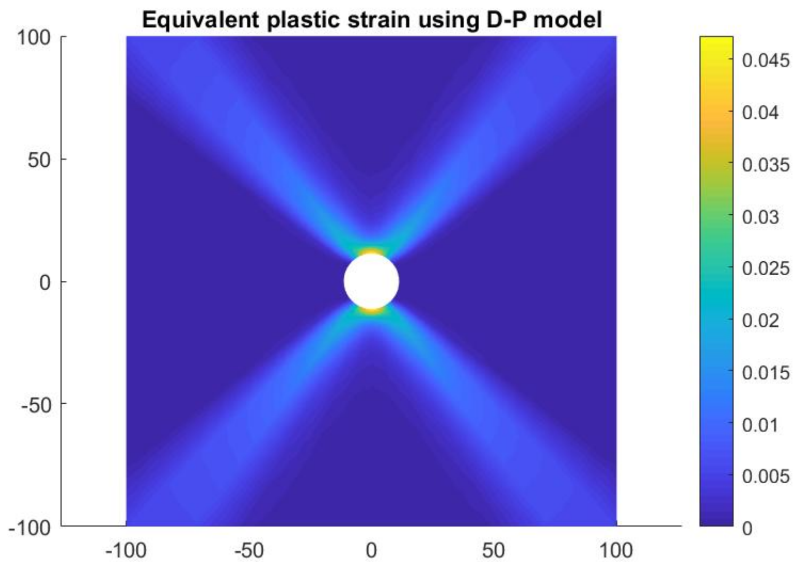

Figure 7: Equivalent Plastic strain distribution in domain using Drucker-Prager model.

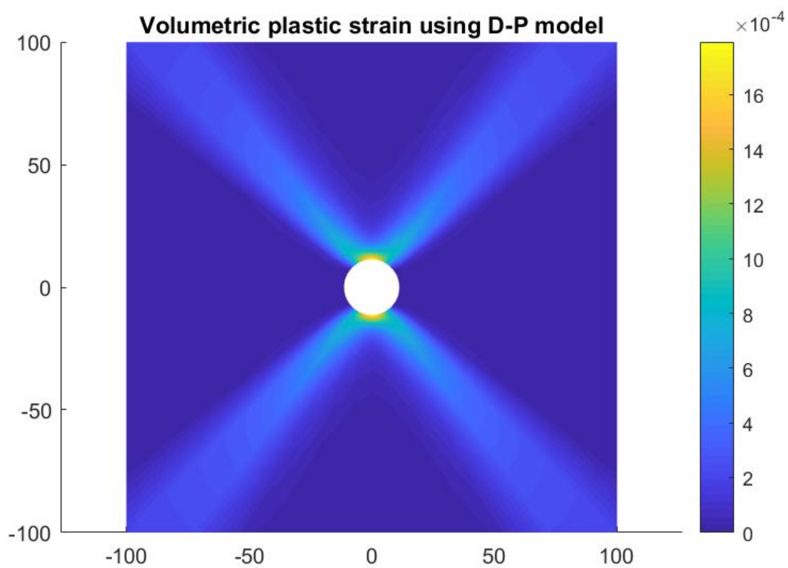

Figure 8: Volumetric Plastic strain distribution in domain using Drucker-Prager model.

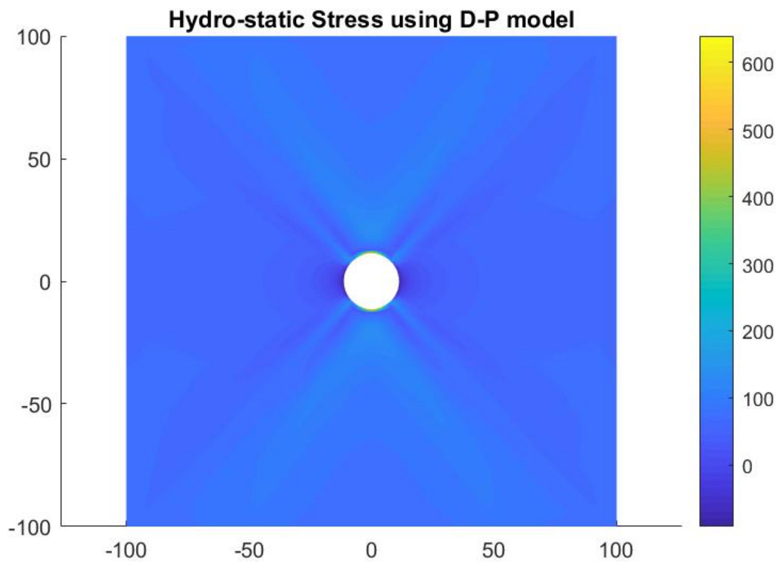

Figure 9: Hydro-static stress distribution in domain using Drucker-Prager model.

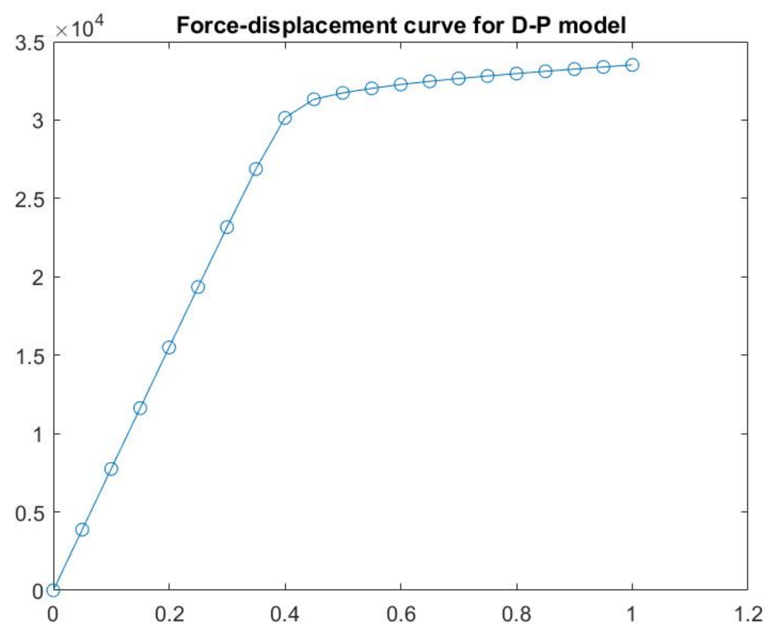

Figure 10: Load versus Displacement.

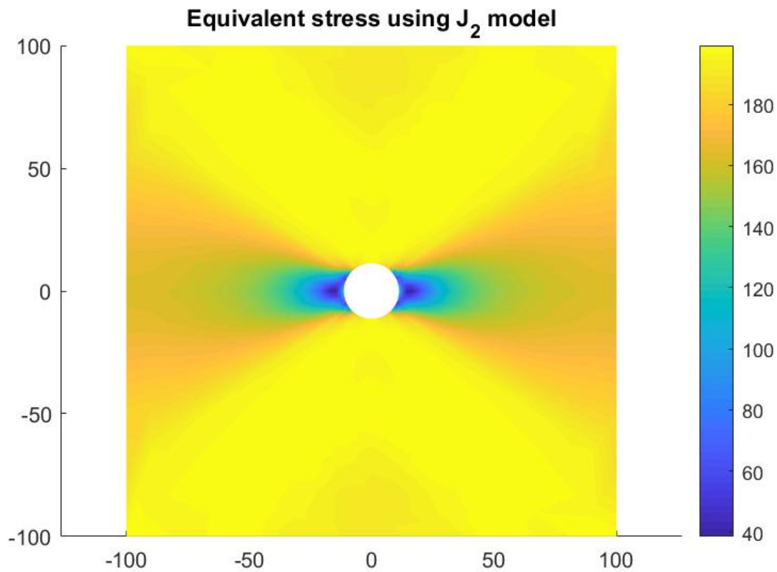

Figure 11: Equivalent stress distribution in domain using J2 model.

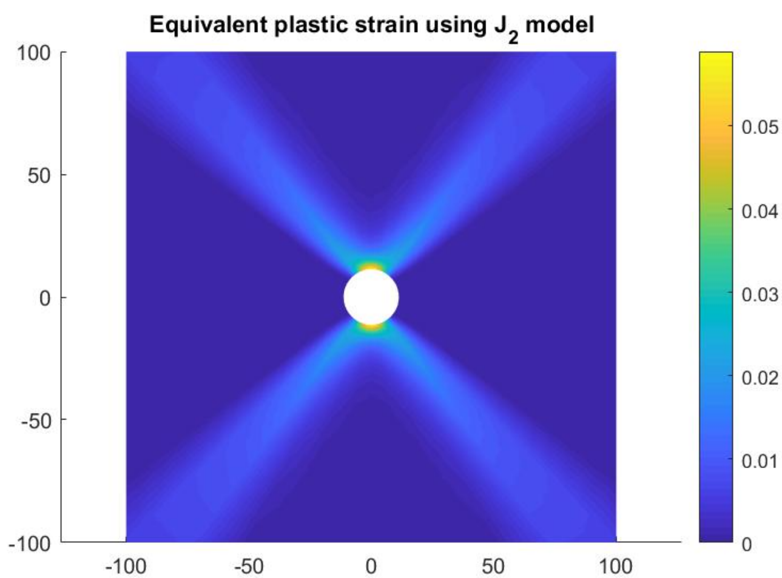

Figure 12: Equivalent Plastic strain distribution in domain using J2 model.

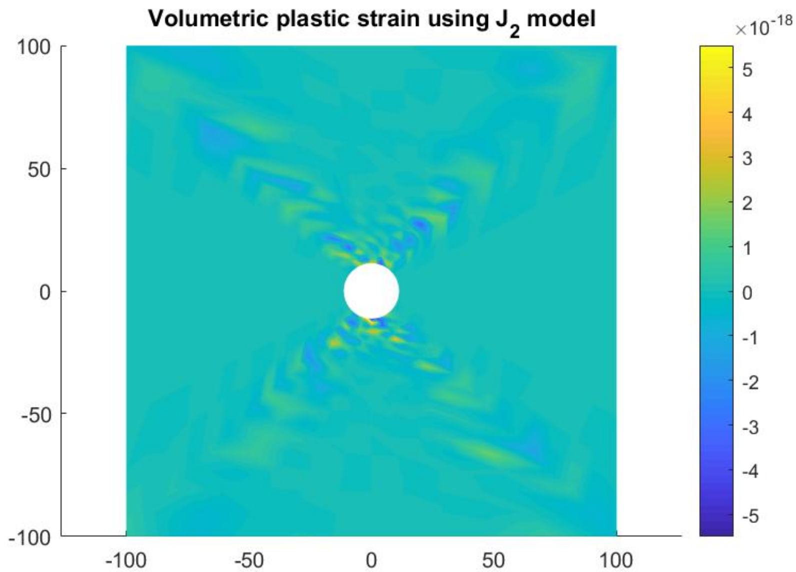

Figure 13: Volumetric Plastic strain distribution in domain using J2 model.

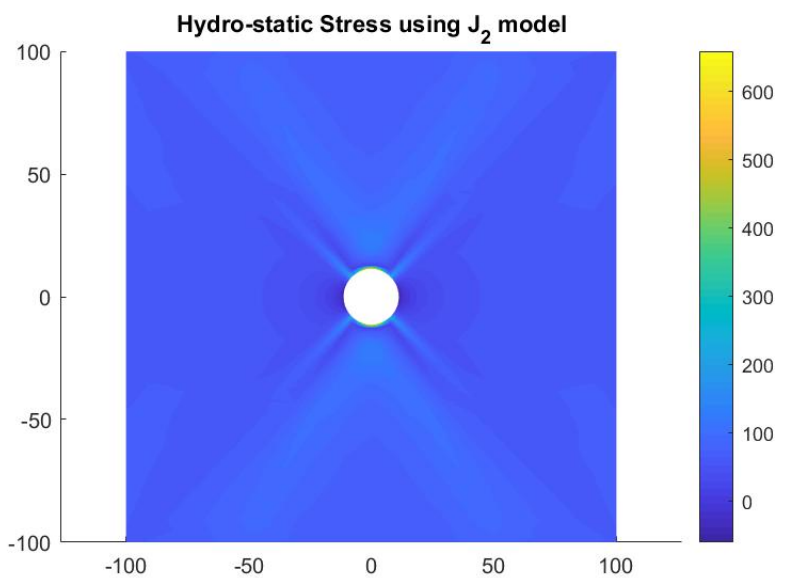

Figure 14: Hydro-static stress distribution in domain using J2 model.

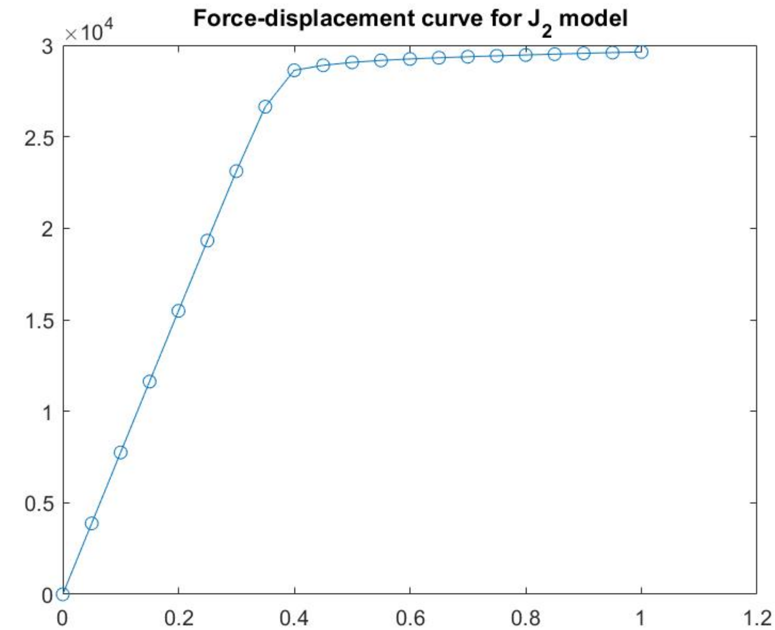

Figure 15: Load versus Displacement.

4 Conclusion

From the results it is evident that there will be some volume expansion under the Drucker-Prager model, while almost negligible expansion from the J_2 model. Also, the equivalent stress tends to increase at the end of the loading in Drucker-Prager model as suggested by the conical yield surface, as also evident from the simple loading behaviour. Although, the equivalent plastic strain and hydro-static stress does not show much difference because of isotropic nature of both the models.

References

[1] D. C. Drucker and W. Prager, Soil Mechanics And Plastic Analysis Or Limit Design, Quarterly of Applied Mathematics, Vol. 10, No. 2 (July, 1952), pp. 157-165.

[2] Abrate, S., Criteria for yielding or failure of cellular materials, Journal of Sandwich Structures and Materials, (2008) vol. 10. pp. 5–51.

[3] Gibson, L.J., Ashby, M.F., Zhang, J. and Triantafilliou, T.C., Failure surfaces for cellular materials under multiaxial loads. I. Modeling, International Journal of Mechanical Sciences, (1989) vol. 31, no. 9, pp. 635–665.

[4] Walid A. Al-Kutti, Parameters estimation of Drucker-Prager plasticity criteria for steel confined circular concrete columns in compression, MATEC Web of Conferences 149(2):01048 · January 2018

[5] M. Hjiaj, J. Fortin, G. de Saxc, A complete stress update algorithm for the non-associated Drucker–Prager model including treatment of the apex, International Journal of Engineering Science 41 (2003) 1109–1143.

[6] Leandro R. Alejano, Antonio Bobet, Drucker–Prager Criterion, Rock Mech Rock Eng (2012) 45:995–999