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Numerical Solution of Algebraic and Transcendental Equations

1.1 INTRODUCTION

A problem of great importance in applied mathematics and engineering is that of determining the roots of an equation of the form

$$f(x) = 0 \tag{1.1}$$

where x and $f(x)$ may be real, complex or vector values. Such type of problems arise in the analysis of flexible chains, determination of minimum axial load on a column, natural frequencies of an elastic string, flexural vibrations, etc.

Equation (1.1) may belong to one of the following types of equations:

(i) Polynomial equations
(ii) Algebraic equations
(iii) Transcendental equations

1.1.1 Polynomial Equations

An expression of the form

$$f(x) = a_0x^n + a_1x^{n-1} + a_2 x^{n-2} + \dots + a_n \tag{1.2}$$

where n is a positive integer and $a_0, a_1, a_2, \dots a_n$ are all constants, called a polynomial in x of the nth degree, if $a_0 \neq 0$.

If the value p of x which satisfies $f(x) = 0$, then p is called a root of $f(x) = 0$.

If p is a root of $f(x) = 0$ of multiplicity m, then we can write Eq. (1.1) as

$$f(x) = (x - p)^m\, g(x) = 0$$

where $g(x)$ is bounded and $g(p) \neq 0$.

1.1.2 Algebraic Equations

An equation of the type, $y = f(x)$, is said to be algebraic if it can be expressed in the form

$$f_0 + f_1 y_1 + f_2 y_2 + \ldots + f_n y_n = 0$$

where f_k is a kth degree polynomial in x.

EXAMPLE

(i) $5x + 3y - 88 = 0$ (linear)

(ii) $101y - 10x + 55xy = 0$ (nonlinear)

These equations have an infinite number of pairs of x and y which satisfy them.

1.1.3 Transcendental Equations

Any nonalgebraic equations is called a transcendental equation, i.e. it contains some other functions, such as trigonometric, logarithmic, exponential etc.

EXAMPLE

(i) $ae^x + bx \tan x + c = 0$

(ii) $3x^2 - \log x^2 + \sin x + e^x - 1 = 0$

Solution of an equation $f(x) = 0$ means we have to find its roots or zeros. This is a problem of great importance in scientific, engineering and applied mathematics.

If $f(x)$ is a quadratic expression, then we have a simple formula to find the roots of $f(x) = 0$, i.e. if

$$f(x) = ax^2 + bx + c = 0 \tag{1.3}$$

Then
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

are the roots of the Eq. (1.3)

Suppose $x = \alpha$ is a particular value of x, then the cubic polynomial can be factorized and can be written in the form

$$(x - \alpha)\,(a_1 x^2 + b_1 x + c_1) = 0$$

from which we can find the other two roots, by solving the quadratic equation

$$a_1 x^2 + b_1 x + c_1 = 0$$

We note that if the degree of the polynomial increases, then it is very difficult to evaluate the root of the given equation using direct method. And also we cannot use direct methods for transcendental equations.

In view of all these considerations the direct methods are of no use for higher degree algebraic equations or transcendental equations and hence there is a need for other numerical methods for solving such type of equations.

In this chapter, we shall discuss some numerical methods for the solution of algebraic and transcendental equations. Before we develop various numerical methods, we shall list below some of the basic properties of an algebraic equation.

(i) Every algebraic equation of nth degree, where n is a positive integer, has n and only n roots.

(ii) Complex roots occur in pairs, i.e. if $(a + ib)$ is a root of $f(x) = 0$, then $a - ib$ is also a root of this equation.

(iii) If $x = \alpha$ is a root of $f(x) = 0$, a polynomial of degree n, then $(x - \alpha)$ is a factor of $f(x)$. On dividing $f(x)$ by $(x - \alpha)$, we obtain a polynomial of degree $(n - 1)$.

(iv) *Descarte's rule of signs:* The number of positive roots of an equation $f(x) = 0$ with real coefficients cannot exceed the number of changes in sign of the coefficients in the polynomial $f(x) = 0$. Similarly, the number of negative roots of $f(x) = 0$ cannot exceed the number of changes in the sign of the coefficients of $f(-x) = 0$, for example consider the equation

$$f(x) = 2x^3 - 8x^2 + 5x - 6 = 0$$

Signs of $f(x)$ are: $+ \quad - \quad + \quad -$

There are three changes in sign, also the degree of the equation is three, and hence the given equation may have three positive roots.

(v) In any numerical computation, we have the following types of errors.

(a) *Inherent errors:* Errors which are already present in the statement of a problem before its solution are called inherent errors.

(b) *Rounding errors:* Rounding errors arise from the process of rounding off the numbers during the computation.

(c) *Truncation errors:* Truncation errors are caused by using approximate results or on replacing an infinite process by a finite one. If we are using a decimal computer having a fixed word length of 4 digits, rounding off of 13.658 gives 13.66 whereas truncation gives 13.65. For example, if

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^3}{3!} + \ldots + \infty = x \text{ (say) is replaced by}$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = x' \text{ (say)}$$

then the truncation error is $X - X'$. Truncation error is a type of algorithm error.

(d) *Absolute, relative and percentage errors:* If X is the true value of a quantity and X' is its approximate value, then $|X - X'|$ is called the absolute error E_a.

The relative error is defined by $E_r = \left|\frac{X - X'}{X}\right|$

and the percentage error by

$$E_p = 100E_r = 100\left|\frac{X - X'}{X}\right|$$

If $\overline{X}$ is an upper limit on the magnitude of absolute error and measures the absolute accuracy.

1.2 BISECTION METHOD

This method is based on the repeated application of the following intermediate value theorem.

If $f(x)$ is continuous in the interval (a, b) and if $f(a)$ and $f(b)$ are of opposite signs then the equation $f(x) = 0$ will have atleast one real root between a and b.

Bisection method is one of the simplest and most reliable of all iterative methods for the solution of nonlinear equations. This method is also known as *half-interval method.*

Let the function $f(x)$ be continuous in the range (a, b) with $a < b$, and let $f(a)$ be +ve and $f(b)$ be –ve. Then by intermediate value theorem, there exist at least one real root between a and b. Let its approximate value be given by

$$x_0 = \frac{a + b}{2}$$

If $f(x_0) = 0$, then x_0 is a root of $f(x) = 0$. Otherwise, if $f(x_0)$ is –ve, the root lies between a and x_0.

If $f(x_0)$ is +ve, the root lies between x_0 and b. Suppose we assume that $f(x_0)$ is +ve as shown in Figure 1.1, then the root lies between x_0 and b, and take the root as

$$x_1 = \frac{x_0 + b}{2}$$

Now, $f(x_1)$ is –ve (as in Figure 1.1), hence the root lies between x_0 and x_1.

Let the root be

$$x_2 = \frac{x_0 + x_1}{2}$$

Now, $f(x_2)$ is –ve as shown in Figure 1.1, then the root lies between x_0 and x_2 and let

$$x_3 = \frac{x_0 + x_2}{2}$$

and so on. Continue in this way, we form a sequence of approximate roots $x_0, x_1, x_2, \ldots$ whose limit of convergence is the exact root. This process can be repeated until the interval containing the root is as small as we desire. The bisection method is a simple but slowly convergent method.

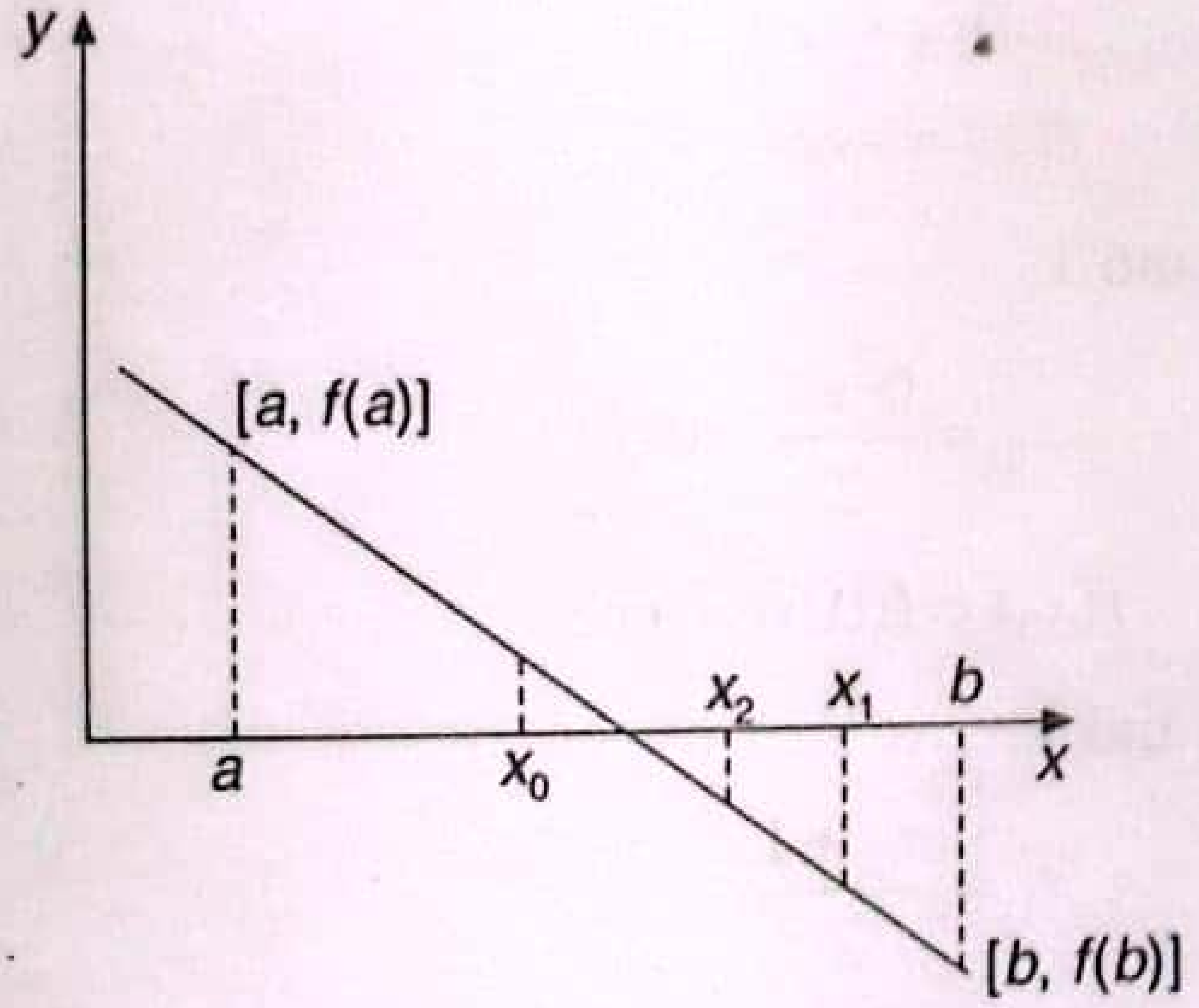

Figure 1.1 Illustrating the bisection method.

Notes

(i) After n bisections, the length of the subinterval, which contains x_n, is $\dfrac{b-a}{2^n}$.

(ii) If the permissible error is ε, then the approximate number of iterations required, may be determined from the relation

$$\frac{b-a}{2^n} \le \varepsilon$$

i.e.

$$2^n \ge \frac{b-a}{\varepsilon}$$

$$\log 2^n \ge \log\left(\frac{b-a}{\varepsilon}\right)$$

$$n \log 2 \ge \log\left(\frac{b-a}{\varepsilon}\right)$$

$$n \ge \frac{\log\left(\dfrac{b-a}{\varepsilon}\right)}{\log 2}$$

$$n \ge \frac{\log(b-a) - \log \varepsilon}{\log 2}$$

EXAMPLE 1.1 Find the positive root of the equation $\cos x = xe^x$, by the bisection method, correct to three decimal places.

Solution

$$f(x) = \cos x - xe^x$$

$$f(0) = +\text{ve}$$

$$f(1) = -\text{ve}$$

$\therefore$ The root lies between 0 and 1.

Take $$x_0 = \frac{0+1}{2} = 0.5$$

$$f(x_0) = f(0.5) = +\text{ve}$$

$\therefore$ The root lies between x_0 and 1.

Take $$x_1 = \frac{x_0 + 1}{2}$$

$$= \frac{0.5+1}{2}$$

$$= 0.75$$

$$f(x_1) = f(0.75) = -\text{ve}$$

$\therefore$ The root lies between x_0 and x_1.

Take $$x_2 = \frac{x_0 + x_1}{2} = \frac{0.5 + 0.75}{2}$$

$$= 0.625$$

$$f(0.625) = -\text{ve}$$

$\therefore$ The root lies between x_0 and x_2

$\therefore$ $$x_3 = \frac{x_0 + x_2}{2} = \frac{0.5 + 0.625}{2}$$

$$= 0.5625$$

$$f(0.5625) = +\text{ve}$$

$\therefore$ The root lies between x_3 and x_2.

$$x_4 = \frac{x_3 + x_2}{2}$$

$$= \frac{0.5625 + 0.625}{2}$$

$$x_4 = 0.5938$$

$$f(x_4) = -\text{ve}$$

$\therefore$ The root lies between x_3 and x_4.

$$x_5 = \frac{x_3 + x_4}{2}$$

$$= \frac{0.5625 + 0.5938}{2}$$

$$= 0.5782$$

$$f(x_5) = -\text{ve}$$

$\therefore$ The root lies between x_3 and x_5.

$$x_6 = \frac{x_3 + x_5}{2}$$

$$= \frac{0.5625 + 0.5782}{2}$$

$$= 0.5704$$

$$f(x_6) = -\text{ve}$$

$\therefore$ The root lies between x_3 and x_6.

$$x_7 = \frac{x_3 + x_6}{2}$$

$$= \frac{0.5625 + 0.5704}{2}$$

$$= 0.5665$$

$$f(x_7) = -\text{ve}$$

$\therefore$ The root lies between x_3 and x_7.

$$x_8 = \frac{x_3 + x_7}{2}$$

$$= \frac{0.5625 + 0.5665}{2}$$

$$= 0.5645$$

$$f(x_8) = -\text{ve}$$

$\therefore$ The root lies between x_3 and x_8.

$$x_9 = \frac{x_3 + x_8}{2}$$

$$= \frac{0.5625 + 0.5645}{2}$$

$$= 0.5635$$

$$f(x_9) = -\text{ve}$$

∴ The root lies between x_3 and x_9.

$$x_{10} = \frac{x_3 + x_9}{2}$$

$$= \frac{0.5625 + 0.5635}{2}$$

$$= 0.563$$

$$f(x_{10}) = -\text{ve}$$

∴ The root lies between x_3 and x_{10}.

$$x_{11} = \frac{x_3 + x_{10}}{2}$$

$$= \frac{0.5625 + 0.563}{2}$$

$$= 0.5628$$

∴ $x_{10} = 0.563$

and $x_{11} = 0.5628$

∴ The approximate root is 0.563.

EXAMPLE 1.2 Find the negative root of $x^3 - 21x + 3500 = 0$, by the bisection method, correct to three decimal places.

Solution

$$f(x) = x^3 - 21x + 3500 = 0$$

$$f(-x) = -x^3 + 21x + 3500$$

The negative root of $f(x) = 0$ is a positive root of $f(-x) = 0$.
∴ We will find the positive root of $f(-x) = 0$.

Take $f(-x) = \phi(x) = -x^3 + 21x + 3500 = 0$

$$\phi(x) = x^3 - 21x - 3500 = 0$$

$$\phi(0) = -\text{ve}$$

$$\phi(1) = -\text{ve}$$

$$\phi(2) = -\text{ve}$$

$$\vdots$$

$$\phi(14) = -\text{ve}$$

$$\phi(15) = -\text{ve}$$

$$\phi(16) = +\text{ve}$$

$\therefore$ A root lies between 15 and 16.

Take
$$x_0 = \frac{15+16}{2}$$
$$= 15.5$$
$$f(15.5) = -\text{ve}$$

$\therefore$ The root lies between 15.5 and 16.

$$x_1 = \frac{15.5+16}{2}$$
$$= 15.75$$
$$f(x_1) = +\text{ve}$$

$\therefore$ The root lies between 15.5 and 15.75.

$$x_2 = \frac{15.5+15.75}{2}$$
$$= 15.625$$
$$f(x_2) = -\text{ve}$$

$\therefore$ The root lies between 15.625 and 15.75.

$$x_3 = \frac{15.625+15.75}{2}$$
$$= 15.6875$$
$$f(x_3) = +\text{ve}$$

$\therefore$ The root lies between 15.625 and 15.6875

$$x_4 = \frac{15.625+15.6875}{2}$$
$$= 15.6563$$
$$f(x_4) = +\text{ve}$$

$\therefore$ The root lies between 15.625 and 15.6563.

$$x_5 = \frac{15.625 + 15.6563}{2}$$
$$= 15.6407$$
$$f(x_5) = -\text{ve}$$

$\therefore$ The root lies between 15.6407 and 15.6563.

$$x_6 = \frac{15.6407 + 15.6563}{2}$$
$$= 15.6485$$
$$f(x_6) = +\text{ve}$$

$\therefore$ The root lies between 15.6407 and 15.6485.

$$x_7 = \frac{15.6407 + 15.6485}{2}$$
$$= 15.6446$$
$$f(x_7) = +\text{ve}$$

$\therefore$ The root lies between 15.6407 and 16.6446.

$$x_8 = \frac{15.6407 + 15.6446}{2}$$
$$= 15.6427$$
$$f(x_8) = -\text{ve}$$

$\therefore$ The root lies between 15.6427 and 15.646.

$$x_9 = \frac{15.6427 + 15.646}{2}$$
$$= 15.644$$
$$f(x_9) = +\text{ve}$$

$\therefore$ The root lies between 15.6427 and 15.6444.

$$x_{10} = \frac{15.6427 + 15.6444}{2}$$
$$= 15.64355$$
$$= 15.644$$
$$x_9 = 15.6444$$

and $$x_{10} = 15.64355$$

$\therefore$ The approximate positive root is 15.644.

∴ The positive root of $\phi(x)$ is 15.644, hence the negative root of the given equation is –15.644.

The above iterations can be tabulated as follows:

Iteration	x_1	x_2	$f(x_1)$	$f(x_2)$	x	$f(x)$
1	15	16	–ve	+ve	15.5	+ve
2	15.5	16	–ve	+ve	15.75	+ve
3	15.5	15.75	–ve	+ve	15.625	–ve
4	15.625	15.75	–ve	+ve	15.6875	+ve
5	15.625	15.6875	–ve	+ve	15.6563	+ve
6	15.625	15.6563	–ve	+ve	15.6407	–ve
7	15.6407	15.6563	–ve	+ve	15.6485	+ve
8	15.6407	15.6485	–ve	+ve	15.6446	+ve
9	15.6407	15.6446	–ve	+ve	15.6427	–ve
10	15.6427	15.646	–ve	+ve	15.6444	+ve
11	15.6427	15.6444	–ve	+ve	15.64355	

EXAMPLE 1.3 Compute the real root of $x \log_{10} x = 1.2$, which lies between 2 and 3, correct to two decimal places, using the bisection method.

Solution

$$f(x) = x \log_{10} x - 1.2$$

$$f(2) = 2 \log_{10} 2 - 1.2 = -0.5980 = -\text{ve}$$

$$f(3) = 3 \log_{10} 3 - 1.2 = 0.231364 = +\text{ve}$$

∴ The root lies between 2 and 3.

$$\therefore \quad x_0 = \frac{2+3}{2}$$

$$= 2.5$$

$$f(2.5) = 2.5 \log_{10} 2.5 - 1.2$$

$$= -0.20515$$

$$f(2.5) = -\text{ve}$$

Hence, the root lies between 2.5 and 3.

Next approximation is given by

$$x_1 = \frac{2.5+3}{2}$$

$$= 2.75$$

$$f(x_1) = +\text{ve}$$

∴ A root lies between 2.5 and 2.75.

Similarly, we get:

Iteration	x_1	x_2	$f(x_1)$	$f(x_2)$	x	$f(x)$
1	2	3	–ve	+ve	2.5	–ve
2	2.5	3	–ve	+ve	2.75	+ve
3	2.5	2.75	–ve	+ve	2.625	–ve
4	2.625	2.75	–ve	+ve	2.6875	–ve
5	2.6875	2.75	–ve	+ve	2.71875	–ve
6	2.71875	2.75	–ve	+ve	2.7344	+ve
7	2.71875	2.7344	–ve	+ve	2.7266	

∴ The root is 2.73, correct to two decimal places.

EXAMPLE 1.4 Using the bisection method, calculate a root of the equation $x^3 - 4x - 9 = 0$.

Solution

$$f(x) = x^3 - 4x - 9$$

$$f(0) = -9 = -\text{ve}$$

$$f(1) = 1 - 4 - 9 = -12 = -\text{ve}$$

$$f(2) = 8 - 8 - 9 = -9 = -\text{ve}$$

$$f(3) = 27 - 12 - 9 = 6 = +\text{ve}$$

∴ A root lies between 2 and 3.

Take $$x_0 = \frac{2+3}{2} = 2.5$$

$$f(x_0) = f(2.5) = -3.375 = -\text{ve}$$

Hence, a root lies between x_0 and 3.

Take $$x_1 = \frac{x_0 + 3}{2}$$

$$= \frac{2.5 + 2.75}{2}$$

$$= 2.75$$

$$f(x_1) = (2.75)^3 - (4)(2.75)$$

$$= 0.796875$$

$$f(x_1) = +\text{ve}$$

$\therefore$ A root lies between x_0, and x_1.

Take
$$x_2 = \frac{x_0 + x_1}{2}$$
$$= \frac{2.5 + 2.75}{2}$$
$$= 2.625$$
$$f(x_2) = -1.412109 = -\text{ve}$$

$\therefore$ A root lies between x_1 and x_2.

Take
$$x_3 = \frac{x_1 + x_2}{2}$$
$$= \frac{2.75 + 2.625}{2}$$
$$= 2.6875$$
$$f(x_3) = -0.339111 = -\text{ve}$$

$\therefore$ A root lies between x_1 and x_3.

$$x_4 = \frac{x_1 + x_3}{2}$$
$$= \frac{2.75 + 2.6875}{2}$$
$$= 2.71875$$
$$f(x_4) = (2.71875)^3 - (4 \times 2.71875 - 9)$$
$$= 220917$$
$$f(x_4) = +\text{ve}$$

$\therefore$ A root lies between x_3 and x_4.

Take
$$x_5 = \frac{x_3 + x_4}{2}$$
$$= \frac{2.6875 + 2.71875}{2}$$
$$= 2.703125$$
$$f(x_5) = -0.061077 = -\text{ve}$$

$\therefore$ A root lies between x_4 and x_5.

Take
$$x_6 = \frac{x_4 + x_5}{2}$$

$$= \frac{2.71875 + 2.703125}{2}$$

$$= 2.7109375$$

$$f(x_6) = 0.079423 = +\text{ve}$$

$\therefore$ A root lies between x_5 and x_6.

Take
$$x_7 = \frac{x_5 + x_6}{2}$$

$$= \frac{2.703125 + 2.7109375}{2}$$

$$= 2.70703125$$

$$f(x_7) = 0.009048 = +\text{ve}$$

$\therefore$ A root lies between x_5 and x_7.

Take
$$x_8 = \frac{x_5 + x_7}{2}$$

$$= \frac{2.703125 + 2.70703125}{2}$$

$$= 2.705078$$

$$f(x_8) = -0.026047 = -\text{ve}$$

$\therefore$ A root lies between x_7 and x_8.

Take
$$x_9 = \frac{x_7 + x_8}{2}$$

$$= \frac{2.70703125 + 2.705078}{2}$$

$$= 2.706054$$

$$f(x_9) = -0.008518 = -\text{ve}$$

$\therefore$ A root lies between x_7 and x_9.

Take
$$x_{10} = \frac{x_7 + x_9}{2}$$

$$= \frac{2.70703125 + 2.706054}{2}$$

$$= 2.70654$$

$$f(x_{10}) = 0.00216 = +\text{ve}$$

$\therefore$ A root lies between x_9 and x_{10}.

$$x_{11} = \frac{x_9 + x_{10}}{2}$$

$$= \frac{2.706054 + 2.70654}{2}$$

$$= 2.706297$$

$$f(x_{11}) = -0.004157 = -\text{ve}$$

$\therefore$ A root lies between x_{10} and x_{11}.

Take $$x_{12} = \frac{x_{10} + x_{11}}{2}$$

$$= \frac{2.70654 + 2.706297}{2}$$

$$= 2.7064185$$

$$f(x_{12}) = -0.001967 = -\text{ve}$$

$\therefore$ A root lies between x_{10} and x_{12}.

Take $$x_{13} = \frac{x_{10} + x_{12}}{2}$$

$$= \frac{2.70654 + 2.70642}{2}$$

$$= 2.70648$$

$$f(x_{13}) = -0.000862 = -\text{ve}$$

$\therefore$ A root lies between x_{10} and x_{13}.

Take $$x_{14} = \frac{x_{10} + x_{13}}{2}$$

$$= \frac{2.70654 + 2.70651}{2}$$

$$= 2.70651$$

$$f(x_{14}) = -0.000323 = -\text{ve}$$

$\therefore$ A root lies between x_{10} and x_{14}.

Take $$x_{15} = \frac{x_{10} + x_{14}}{2}$$

$$= \frac{2.70654 + 2.70651}{2}$$

$$= 2.706525$$

$$x_{14} = 2.70651$$

and $$x_{15} = 2.706525$$

$\therefore$ The approximate root is 2.7065.

EXAMPLE 1.5 Find the root of the equation $x - \cos x = 0$, by the bisection method.

Solution

Let $$f(x) = x - \cos x$$

$$f(0) = -\text{ve}$$

$$f(0.5) = -0.37758 = -\text{ve}$$

$$f(1) = 0.45970 = +\text{ve}$$

$\therefore$ The root lies between 0.5 and 1.

Take $$x_0 = \frac{0.5 + 1}{2}$$

$$= 0.75$$

$$f(x_0) = f(0.75) = 0.018311 = +\text{ve}$$

$\therefore$ A root lies between 0.5 and x_0.

Take $$x_1 = \frac{0.5 + 0.75}{2}$$

$$= 0.625$$

$$f(0.625) = -0.18596 = -\text{ve}$$

$\therefore$ A root lies between x_1 and x_0.

$$x_2 = \frac{0.625 + 0.75}{2}$$

$$= 0.6875$$

$$f(x_2) = f(0.6875) = -0.085335 = -\text{ve}$$

$\therefore$ The root lies between x_2 and x_0.

Take $$x_3 = \frac{1}{2}(0.6875 + 0.75)$$

$$= 0.71875$$

$$f(0.71875) = -0.033879 = -\text{ve}$$

$\therefore$ The root lies between x_3 and x_0.

Take $$x_4 = \frac{0.71875 + 0.75}{2}$$

$$= 0.73438$$

$$f(x_4) = f(0.73438) = -0.0078664 = -\text{ve}$$

$\therefore$ The root lies between x_4 and x_0.

$$x_5 = \frac{0.73438 + 0.75}{2}$$

$$= 0.742190$$

$$f(x_5) = f(0.742190) = 0.0051999 = +\text{ve}$$

$\therefore$ The root lies between x_4 and x_5.

Take $$x_6 = \frac{1}{2}(0.73438 + 0.742190)$$

$$= 0.73829$$

$$f(x_6) = f(0.73829) = -0.0013305 = -\text{ve}$$

$\therefore$ The root lies between x_6 and x_5.

Take $$x_7 = \frac{0.73829 + 0.74219}{2}$$

$$= 0.7402$$

$$f(x_7) = f(0.7402) = 0.0018663 = +\text{ve}$$

$\therefore$ The root lies between x_6 and x_7.

Take $$x_8 = \frac{0.73829 + 0.7402}{2}$$

$$= 0.73925$$

$$f(x_8) = f(0.73925) = 0.00027593 = +\text{ve}$$

$\therefore$ The root lies between x_6 and x_8.

Take $$x_9 = \frac{0.73829 + 0.73925}{2}$$

$$= 0.73877$$

$\therefore$ The approximate root is 0.7388, i.e. 0.739.

EXAMPLE 1.6 Assuming that a root of $x^3 - 9x + 1 = 0$ lies in the interval (2, 4), find that root, by the bisection method.

Solution

Let $$f(x) = x^3 - 9x + 1$$

$$f(2) = -\text{ve}, f(4) = +\text{ve}$$

$\therefore$ A root lies between 2 and 4.

Take
$$x_0 = \frac{a+b}{2}$$
$$x_0 = \frac{2+4}{2}$$
$$= 3$$
$$f(x) = x^3 - 9x + 1$$
$$f(x_0) = f(3) = 27 - 27 + 1 = 1 = +\text{ve}$$

$\therefore$ The root lies between 2 and 3.
$$x_1 = \frac{2+3}{2}$$
$$= 2.5$$
$$f(x_1) = f(2.5) = -\text{ve}$$

$\therefore$ The root lies between 2.5 and 3.
$$x_2 = \frac{2.5+3}{2}$$
$$= 2.75$$
$$f(x_2) = f(2.75) = -\text{ ve}$$

$\therefore$ The root lies between 2.75 and 3.
$$x_3 = \frac{1}{2}(2.75+3)$$
$$= 2.875$$
$$f(x_3) = f(2.875) = -\text{ve}$$

$\therefore$ The root lies between 2.875 and 3.
$$x_4 = \frac{2.875+3}{2}$$
$$= 2.9375$$
$$f(x_4) = f(2.9375) = -\text{ve}$$

$\therefore$ The root lies between 2.9375 and 3.
$$x_5 = \frac{2.9375+3}{2}$$
$$= 2.9688$$
$$f(x_5) = f(2.9688) = +\text{ve}$$

∴ The root lies between 2.9688 and 2.9375.

$$x_6 = \frac{2.9375 + 2.9688}{2}$$

$$= 2.9532$$

$$f(x_6) = f(2.9532) = -\text{ve}$$

∴ The root lies between 2.9375 and 2.9532.

$$x_7 = \frac{1}{2}(2.9375 + 2.9532)$$

$$= 2.9454$$

$$f(x_7) = f(2.9454) = +\text{ve}$$

∴ The root lies between 2.9375 and 2.9454.

$$x_8 = \frac{2.9375 + 2.9454}{2}$$

$$= 2.9415$$

$$f(2.9415) = -\text{ve}$$

∴ The root lies between 2.9415 and 2.9454.

$$x_9 = \frac{1}{2}(2.9415 + 2.9454)$$

$$= 2.9435$$

$$f(2.9435) = +\text{ve}$$

∴ The root lies between 2.9415 and 2.9435.

$$x_{10} = 2.9425$$

$$f(2.9425) = -\text{ve}$$

∴ The root lies between 2.9425 and 2.9435.

$$x_{11} = 2.9430$$

$$f(2.9430) = +\text{ve}$$

∴ The root lies between 2.9425 and 2.9430.

$$x_{12} = 2.94275$$

$$f(2.94275) = -\text{ve}$$

∴ The root lies between 2.94275 and 2.9430.

$$x_{13} = 2.942875$$

∴ The approximate root is 2.9429.

1.3 REGULA FALSI OR FALSE POSITION METHOD

This is the oldest method for finding the root of an equation and closely resembles the bisection method.

Consider the equation $f(x) = 0$. Let $f(a)$ and $f(b)$ be of opposite signs. Also let $a < b$. The curve $y = f(x)$ will meet the x-axis at some point between $P[a, f(a)]$ and $Q[b, f(b)]$. Equation of the chord PQ is

$$\frac{y - f(a)}{x - a} = \frac{f(a) - f(b)}{a - b}$$

The x-coordinate of the point of intersection of this chord with x-axis gives an approximate value for the root of $f(x) = 0$. Setting $y = 0$ in the chord equation, we have

$$\frac{-f(a)}{x - a} = \frac{f(a) - f(b)}{a - b}$$

$$(x - a)\,[f(a) - f(b)] = (a - b)\,[-f(a)]$$

$$x[f(a) - f(b)] - af(a) + af(b) = -af(a) + bf(a)$$

$$x[f(a) - f(b)] = b\,f(a) - a\,f(b)$$

$$\therefore \qquad x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Hence, the first approximation to the root is given by

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

This value of x_1 gives an approximate value of the root of

$$f(x) = 0 \; (a < x_1 < b)$$

Now, $f(x_1)$ and $f(a)$ are of opposite signs or $f(x_1)$ and $f(b)$ are of opposite signs.

If $f(x_1)\,f(a) < 0$, then x_2 lies between x_1 and a.

Hence $$x_2 = \frac{af(x_1) - x_1 f(a)}{f(x_1) - f(a)}$$

Continuing in this process, we get $x_3, x_4, x_5, \ldots$.

This sequence $x_1, x_2, x_3, \ldots$ will converge to the required root.

1.3.1 Geometrical Interpretation of Regula Falsi Method

Consider the curve $y = f(x)$ between $P[a, f(a)]$ and $Q[b, f(b)]$ such that $f(a)$ and $f(b)$ are opposite in sign (Figure 1.2).

The chord AB cuts x-axis at $x = x_1$. This x_1 is the first approximate root of $f(x) = 0$. Now, $R[x_1, f(x_1)]$ is on the curve.

If $f(a).f(x_1) < 0$, join the chord AC which cuts x-axis at $x = x_2$. Then x_2 is the second approximate root of $f(x) = 0$. The procedure is continued till the root is obtained to the desired degree of accuracy. The points of intersection of the successive chords with x-axis namely B_1, R_2, ... tend to coincide with M, the point where the curve $y = f(x)$ cuts the x-axis and so we get successive approximate values of the root. Its rate of convergence is faster than that of the bisection method.

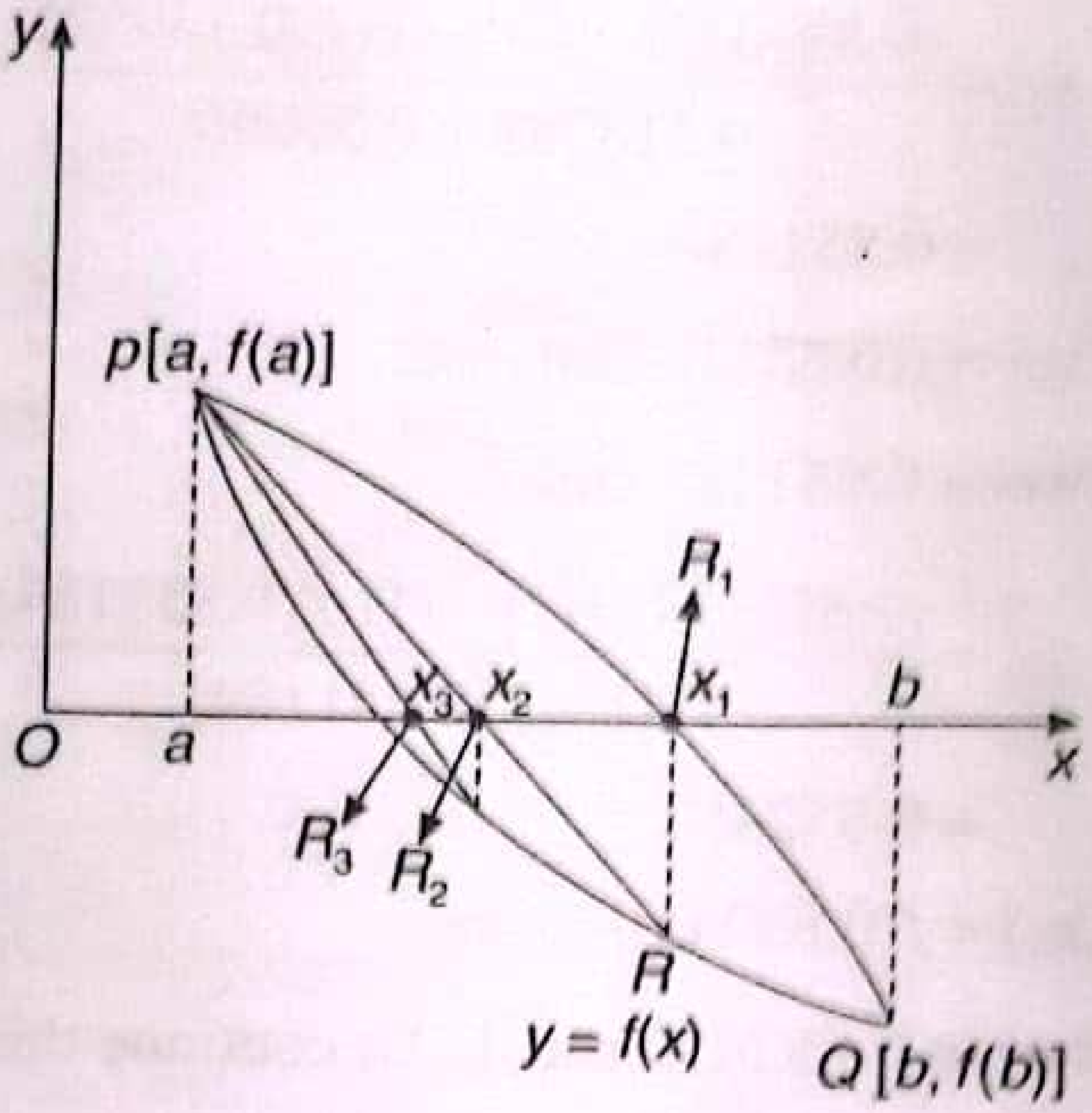

Figure 1.2 Illustrating the regula falsi method.

1.1

EXAMPLE 1.7 Find a root of $xe^x - 2 = 0$, by the regula falsi method.

Solution

Let
$$f(x) = xe^x - 2$$
$$f(0) = -2 = +\cancel{v}e$$ -ive
$$f(1) = 0.718282 = +\text{ve}$$

$\therefore$ A root lies between 0 and 1.

Hence, $a = 0$, $b = 1$

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{0 - f(0)}{f(1) - f(0)}$$
$$= \frac{2}{2.718282}$$
$$= 0.735759$$
$$f(x_1) = f(0.735759) = -\text{ve}$$

$\therefore$ The root lies between 0.735759 and 1.

$$x_2 = \frac{(0.735759)f(1) - (1)f(0.735759)}{f(1) - f(0.735759)}$$

$$= 0.839521$$

$$f(x_2) = f(0.839521) = -0.056293$$

$\therefore$ The root lies between 0.839521 and 1.

$$x_3 = \frac{(0.839521)\ f(1) - (1)f(0.839521)}{0.718282 + 0.56293}$$

$$= 0.851184$$

$$f(x_3) = f(0.851184) = -\text{ve}$$

$\therefore$ The root lies between 0.851184 and 1.

$$x_4 = \frac{(0.851184)f(1) - (1)\ f(0.851184)}{f(1) - f(0.851184)}$$

$$= 0.852452$$

$$f(x_4) = f(0.85245) = -\text{ve}$$

$\therefore$ The root lies between 0.852452 and 1. To continue this process, we get

$$x_5 = 0.85261$$

$$x_6 = 0.85261$$

$\therefore$ The root of the equation $xe^x - 2$ is 0.85261.

EXAMPLE 1.8 Solve the equation $x \log_{10} x - 1.2 = 0$, by the regula falsi method.

Solution

Let $f(x) = x \log_{10} x - 1.2$

$$f(1) = -1.2 = -\text{ve}$$

$$f(2) = 2 \log_{10} 2 - 1.2 = -0.59794 = -\text{ve}$$

$$f(3) = 3 \log_{10} 3 - 1.2 = 0.23136 = +\text{ve}$$

$\therefore$ A root lies between 2 and 3.

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{2f(3) - 3f(2)}{f(3) - f(2)}$$

$$= \frac{(2)(0.23136) - (3)(-0.59794)}{0.23136 + 0.59794}$$

$$= 2.721014$$

$$f(x_1) = f(2.7210) = -0.017104$$

$\therefore$ The root lies between x_1 and 3.

$$x_2 = \frac{(x_1)f(3) - (3)f(x_1)}{f(3) - f(x_1)}$$

$$= \frac{(2.721014)(0.231364) - (3)(-0.017104)}{0.23136 + 0.017104}$$

$$= 2.740211$$

$$f(x_2) = f(2.740211) = -0.00038905$$

$\therefore$ The root lies between 2.740211 and 3.

$$x_3 = \frac{(2.7402)f(3) - (3)f(2.7402)}{f(3) - f(2.7402)}$$

$$= \frac{0.63514}{0.23175}$$

$$= 2.740627$$

$$f(x_3) = f(2.740627) = 0.00011998$$

$\therefore$ The root lies between 2.740211 and 2.740627.

$$x_4 = \frac{(2.7402)\,f(2.7406) - (2.7406)\,f(2.7402)}{f(2.7406) - f(2.7402)}$$

$$= \frac{0.0013950}{0.00050903}$$

$$= 2.7405$$

$\therefore$ The root is 2.7405.

1.4 ITERATION METHOD OR FIXED POINT ITERATIVE METHOD OR METHOD OF SUCCESSIVE APPROXIMATIONS

This method of iteration is particularly useful for finding the real roots of an equation given in the form of an infinite series.

EXAMPLE The real root of the equation

$$x - \frac{x^3}{3} + \frac{x^5}{70} - \frac{x^7}{42} + \frac{x^9}{216} - \frac{x^{11}}{1320} = 0.4431135$$

is 0.4769, correct to four decimal places.

The method of iteration can be applied to find a real root of the equation $f(x) = 0$, by rewriting the same in the form $x = F(x)$.

EXAMPLE

$$f(x) = \cos x - 4x + 5 = 0$$

It can be written as

$$x = \frac{1}{4}(\cos x + 3) = F(x)$$

1.4.1 Sufficient Condition for Convergence of Iterations

To find the roots of the equation

$$f(x) = 0 \tag{1.4}$$

by the successive approximations method.

Equation (1.4) can be written in the form

$$x = F(x) \tag{1.5}$$

Let $x = x_0$ be an initial approximation to the root a, then the first approximation x_1 is given by

$$x_1 = F(x_0) \tag{1.6}$$

Now, treating x_1 as the initial value, the second approximation is

$$x_2 = F(x_1)$$

In general,

$$x_{k+1} = F(x_k) \tag{1.7}$$

where $k = 0, 1, 2, \ldots$, will give further approximations. This formula (1.7) is known as the *iteration formula*. This is valid if $x_1, x_2, \ldots, x_k, x_{k+1}, \ldots$ converges to a.

Subtracting (1.6) from (1.5), we get

$$x_1 - x = F(x_0) - F(x)$$

$$x - x_1 = F(x) - F(x_0) \tag{1.8}$$

By mean value theorem

$$\frac{F(x) - F(x_0)}{x - x_0} = F'(\varepsilon_0)$$

where $x < \varepsilon_0 < x_1$.

Equation (1.8) becomes

$$x - x_1 = (x - x_0)\, F'(\varepsilon_0)$$

Similarly,

$$x - x_2 = (x - x_1)\, F'(\varepsilon_1)$$

$$x - x_3 = (x - x_2)\, F'(\varepsilon_2)$$

....

....

....

$$x - x_n = (x - x_{n-1})\, F'(\varepsilon_{n-1})$$

Multiplying all these equations, we get

$$(x - x_1)(x - x_2)(x - x_3) \ldots (x - x_{n-1})$$
$$(x - x_n) = (x - x_0)(x - x_1)(x - x_2) \ldots (x - x_{n-1})\, F'(\varepsilon_0)\, F'(\varepsilon_1)\, F'(\varepsilon_2) \ldots F'(\varepsilon_{n-1})$$

$\therefore$
$$(x - x_n) = (x - x_0)\, F'(\varepsilon_0)\, F'(\varepsilon_1)\, F'(\varepsilon_2) \ldots F'(\varepsilon_{n-1}) \tag{1.9}$$

If the maximum value of $|F'(x)| < 1$, then each of the quantities

$$|F'(\varepsilon_0)|, |F'(\varepsilon_1)|, \ldots |F'(\varepsilon_{n-1})| < 1$$

i.e. less than a proper fraction λ, then

$$|F'(\varepsilon_0)|\,|F'(\varepsilon_1)| \ldots |F'(\varepsilon_{n-1})| < \lambda^n$$

Equation (1.9) implies

$$|x - x_n| = |x - x_0|\,|F'(\varepsilon_0)|\,|F'(\varepsilon_1)|\,|F'(\varepsilon_2)| \ldots |F'(\varepsilon_{n-1})|$$
$$|x - x_n| < |x - x_0|\lambda^n$$

Since λ is a proper fraction

$$\lim_{n \to \infty} |x - x_0|\,\lambda^n = 0$$

$\therefore$ The sequence of approximations $x_0, x_1, x_2, \ldots x_k, x_{k+1}, \ldots$ converges to the exact root a, if $|F'(x)| < 1$ for all values of x in the interval in which the root lies. The sequence will converge rapidly if $|F'(x)|$ is very small.

Note

If $|F'(x)| > 1$, $|x - x_n|$ will become indefinitely large and the sequence will not converge.

1.4.2 Order of Convergence of an Iterative Method

Let $x_0, x_1, x_2, \ldots x_n, \ldots$ be the successive approximations of the root α of $f(x) = 0$. Let e_i be the error in the root x_i, $i = 1, 2, 3, \ldots$.

If α is the exact root, then

$$e_i = x_i - \alpha$$

and

$$e_{i+1} = x_{i+1} - \alpha$$

If $m \geq 1$ can be found out such that

$$|e_{i+1}| \leq |e_i|^m k$$

where k is a positive constant for every i, then m is called the *order of convergence*.

If $m = 1$, the convergence is linear, and if $m = 2$, it is quadratic.

EXAMPLE 1.9 Find a real root of the equation $x^3 - x - 1 = 0$, correct to four decimal places, by using the iteration method.

Solution

Let
$$f(x) = x^3 - x - 1 = 0$$
$$f(0) = -1 = -\text{ve}$$
$$f(1) = -1 = -\text{ve}$$
$$f(2) = 5 = +\text{ve}$$

$\therefore$ The root lies between 1 and 2.

The given equation can be expressed as
$$x^3 - x = 1$$
$$-x = 1 - x^3$$
$$x = x^3 - 1$$

Hence,
$$F(x) = x^3 - 1$$
$$F'(x) = 3x^2$$

Now,
$$|F'(1)| \nless 1 \text{ and } |F'(2)| \nless 1$$

Hence, this way of writing $f(x)$, will not give any valid iteration process.

The given equation can be written as
$$x^3 = 1 + x$$
$$x \cdot x^2 = 1 + x$$
$$x = \frac{1+x}{x^2}$$

i.e.
$$F(x) = \frac{1+x}{x^2}$$
$$F'(x) = -\frac{2}{x^3} - \frac{1}{x^2}$$

Now,
$$|F'(1)| \nless 1 \quad \text{and} \quad |F'(2)| \nless 1$$

$\therefore$ This way of writing $f(x)$, iteration process is not valid.

Another way of writing the equation is
$$x^3 = 1 + x$$
$$x = (1 + x)^{1/3}$$

Hence,

$$F(x) = (1 + x)^{1/3}$$

$$F'(x) = \frac{1}{3}(1 + x)^{-2/3}$$

Now,

$$|F'(1)| < 1 \quad \text{and} \quad |F'(2)| < 1$$

i.e. $|F'(x)| < 1$ for all x in $(1, 2)$

Hence, the iterative method can be applied.

Take $x_0 = 1.3$ as starting value

$$x_1 = F(x_0)$$
$$= F(1.3)$$
$$= (1 + 1.3)^{1/3}$$
$$= 1.3200$$
$$x_2 = F(x_1)$$
$$= (1 + 1.3200)^{1/3}$$
$$= 1.3238$$
$$x_3 = F(x_2)$$
$$= (1 + 1.3238)^{1/3}$$
$$= 1.3245$$
$$x_4 = F(x_3)$$
$$= (1 + 1.3245)^{1/3}$$
$$= 1.3247$$
$$x_5 = F(x_4)$$
$$= (1 + 1.3247)^{1/3}$$
$$= 1.3247$$

Hence, the root is 1.3247, correct to four decimal places.

EXAMPLE 1.10 Using the iteration method, calculate the root of the equation $4x = e^x$.

Solution

Let

$$f(x) = e^x - 4x = 0$$

$$f(0) = 1 - 0 = 1 = +\text{ve}$$

$$f(1) = e - 4 = -1.281718172$$

$\therefore$ $$f(1) = -\text{ve}$$

$\therefore$ A root lies between 0 and 1.

$$e^x - 4x = 0$$

$$-4x = -e^x$$

$$x = \frac{1}{4}e^x$$

$\therefore$ $$F(x) = \frac{1}{4}e^x$$

$$F'(x) = \frac{1}{4}e^x$$

$$|F'(0)| = \frac{1}{4} < 1 \quad \text{and} \quad |F'(1)| = 0.6796 < 1$$

$\therefore$ $|F'(x)| < 1$ for all values of $x \in (0, 1)$

Take $x_0 = 0.3$ as starting value

$$x_1 = F(x_0) = \frac{1}{4}e^{0.3}$$

$$= 0.3375$$

$$x_2 = F(x_1)$$

$$= \frac{1}{4}e^{0.3375}$$

$$= 0.3504$$

$$x_3 = F(x_2) = \frac{1}{4}e^{0.3504}$$

$$= 0.3549$$

$$x_4 = F(x_3) = \frac{1}{4}e^{0.3549}$$

$$= 0.3565$$

$$x_5 = F(x_4) = \frac{1}{4}e^{0.3565}$$

$$= 0.3571$$

$$x_6 = F(x_5) = \frac{1}{4}e^{0.3571}$$

$$= 0.3573$$

$$x_7 = F(x_6) = \frac{1}{4} e^{0.3573}$$

$$= 0.3574$$

$$x_8 = F(x_7) = \frac{1}{4} e^{0.3573}$$

$$= 0.3574$$

We get $x_7 = x_8 = 0.3574.$

Hence, the root is 0.3574, correct to four decimal places.

EXAMPLE 1.11 Find the root of the equation $3x - \sqrt{1+\sin x} = 0$, by the iteration method.

Solution

Let
$$f(x) = 3x - \sqrt{1+\sin x} = 0$$

$$f(0) = -1 = -\text{ve}$$

$$f(1) = 3 - \sqrt{1+\sin 1}$$

$$= 3 - 1.841470985$$

$$= 1.158529015$$

$\therefore$
$$f(1) = +\text{ve}$$

$$3x - \sqrt{1+\sin x} = 0$$

$$3x = \sqrt{1+\sin x}$$

$$x = \frac{1}{3}\sqrt{1+\sin x}$$

$\therefore$
$$F(x) = \frac{1}{3}\sqrt{1+\sin x}$$

$$F'(x) = \frac{\cos x}{6\sqrt{1+\sin x}}$$

$$F'(0) = \frac{1}{6} < 1$$

$\therefore$
$$|F'(0)| < 1$$

$$F'(1) = \frac{\cos 1}{6\sqrt{1+f\sin 1}}$$

$$= \frac{0.540302305}{6(1.841470985)}$$

$$= \frac{0.540302305}{11.04882591}$$

$$F'(1) = 0.048901332$$

$\therefore$ $$|F'(1)| < 1$$

We have

$$|F'(0)| < 1 \quad \text{and} \quad |F'(1)| < 1$$

$\therefore$ $|F'(x)| < 1$ for all the values of $x \in (0, 1)$

Take $x_0 = 0.4$

$$x_1 = F(x_0)$$
$$= \frac{1}{3}\sqrt{1 + \sin(0.4)}$$
$$= 0.39291$$
$$x_2 = F(x_1)$$
$$= \frac{1}{3}\sqrt{1 + \sin(0.39291)}$$
$$= 0.39199$$
$$x_3 = F(x_2)$$
$$= \frac{1}{3}\sqrt{1 + \sin(0.39199)}$$
$$= 0.39187$$
$$x_4 = F(x_3)$$
$$= \frac{1}{3}\sqrt{1 + \sin(0.39187)}$$
$$= 0.39185$$
$$x_5 = F(x_4)$$
$$= \frac{1}{3}\sqrt{1 + \sin(0.39185)}$$
$$= 0.39185$$

$\therefore$ The root is 0.39185.

EXAMPLE 1.12 Find the root of the equation

$$1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \ldots = 0$$

correct to two decimal places.

Solution The given equation can be written as

$$x = 1 + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \ldots = \phi(x)$$

Neglecting x^2 and higher powers of x, we get

$x = 1$, (approximately)

Taking $x_0 = 1$, we have

$$x_1 = \phi(x_0)$$

$$= 1 + \frac{1}{(2!)^2} - \frac{1}{(3!)^2} + \frac{1}{(4!)^2} - \frac{1}{(5!)^2} + \ldots$$

$$= 1.2239$$

$$x_2 = \phi(x_1)$$

$$= 1 + \frac{(1.2239)^2}{(2!)^2} - \frac{(1.2239)^3}{(3!)^2} + \frac{(1.2239)^4}{(4!)^2} - \frac{(1.2239)^5}{(5!)^2}$$

$$= 1.3263$$

$$x_3 = \phi(x_2)$$

$$= 1 + \frac{(1.3263)^2}{(2!)^2} - \frac{(1.3263)^3}{(3!)^2} + \frac{(1.3263)^4}{(4!)^2} - \frac{(1.3263)^5}{(5!)^2}$$

$$= 1.38$$

$$x_4 = \phi(x_3)$$

$$= 1 + \frac{(1.38)^2}{(2!)^2} - \frac{(1.38)^3}{(3!)^2} + \frac{(1.38)^4}{(4!)^2} - \frac{(1.38)^5}{(5!)^2}$$

$$= 1.409$$

$$x_5 = \phi(x_4)$$

$$= 1 + \frac{(1.409)^2}{(2!)^2} - \frac{(1.409)^3}{(3!)^2} + \frac{(1.409)^4}{(4!)^2} - \frac{(1.409)^5}{(5!)^2}$$

$$= 1.425$$

$$x_6 = \phi(x_5)$$

$$= 1 + \frac{(1.425)^2}{(2!)^2} - \frac{(1.425)^3}{(3!)^2} + \frac{(1.425)^4}{(4!)^2} - \frac{(1.425)^5}{(5!)^2}$$

$$= 1.434$$

$$x_7 = 1.439$$

Similarly

$$x_8 = 1.442$$

$\therefore$ The root is 1.44, correct to two decimal places.

1.5 NEWTON RAPHSON METHOD OR NEWTON'S METHOD

When the derivative of $f(x)$ is a simple expression and easily found, the roots of $f(x) = 0$ can be computed rapidly by a process called the *Newton–Raphson Method*. Of all the methods available for the numerical solution of algebraic and transcendental equations, Newton–Raphson method seems the one most generally satisfactory and is very easy for computer program also. This method is very powerful method for finding the real root of an equation in the form $f(x) = 0$. It is also applicable to the solution of both algebraic and transcendental equations. It can also be used when the roots are complete. This method is also called *method of tangents*.

Suppose $x = x_0$ is an approximate root of the equation $f(x) = 0$.

If $x_1 = x_0 + h$ be the exact root of $f(x) = 0$ then $f(x_1) = 0$, where h is small, positive or negative.

By Taylor's expansion,

$$f(x_1) = f(x_0 + h)$$

$$= f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \ldots = 0$$

Since h is very small, neglecting, higher powers $h^2, h^3, h^4, \ldots$ etc., we get

$$f(x_0) + hf'(x_0) = 0$$

$$h = 1 - \frac{f(x_0)}{f'(x_0)}, \quad \text{if } f'(x_0) \neq 0$$

$$\therefore \quad x = x_0 + h$$

$$= x_0 - \frac{f(x_0)}{f'(x_0)}, \quad \text{where } h = \frac{-f(x_0)}{f'(x_0)} \text{ approximately}$$

Let this value be x_1.

$$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

This value x_1 is a closer approximation to the root of $f(x) = 0$ than x_0.

Similarly, starting with x_1 we can get a better approximation x_2 to the root is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Continuing this process until $|x_{k+1} - x_k|$ is less than the quantity desired.

$\therefore$ In general,

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \qquad (k = 0, 1, 2, 3, \ldots)$$

which is known as the *Newton-Raphson formula* or *Newton's iteration formula.*

1.5.1 Geometrical Interpretation of Newton–Raphson Method

Consider the graph (Figure 1.3) of the function $y = f(x)$ and let $x = \alpha$ be an exact root of $f(x) = 0$.

The curve cuts the x-axis at the point R whose x-coordinate is α. Let x_0 be a point closer to the root α of the equation $f(x) = 0$. Then the equation of the tangent at R_0 is

$$y - f(x_0) = f'(x_0)\,(x - x_0) \tag{1.10}$$

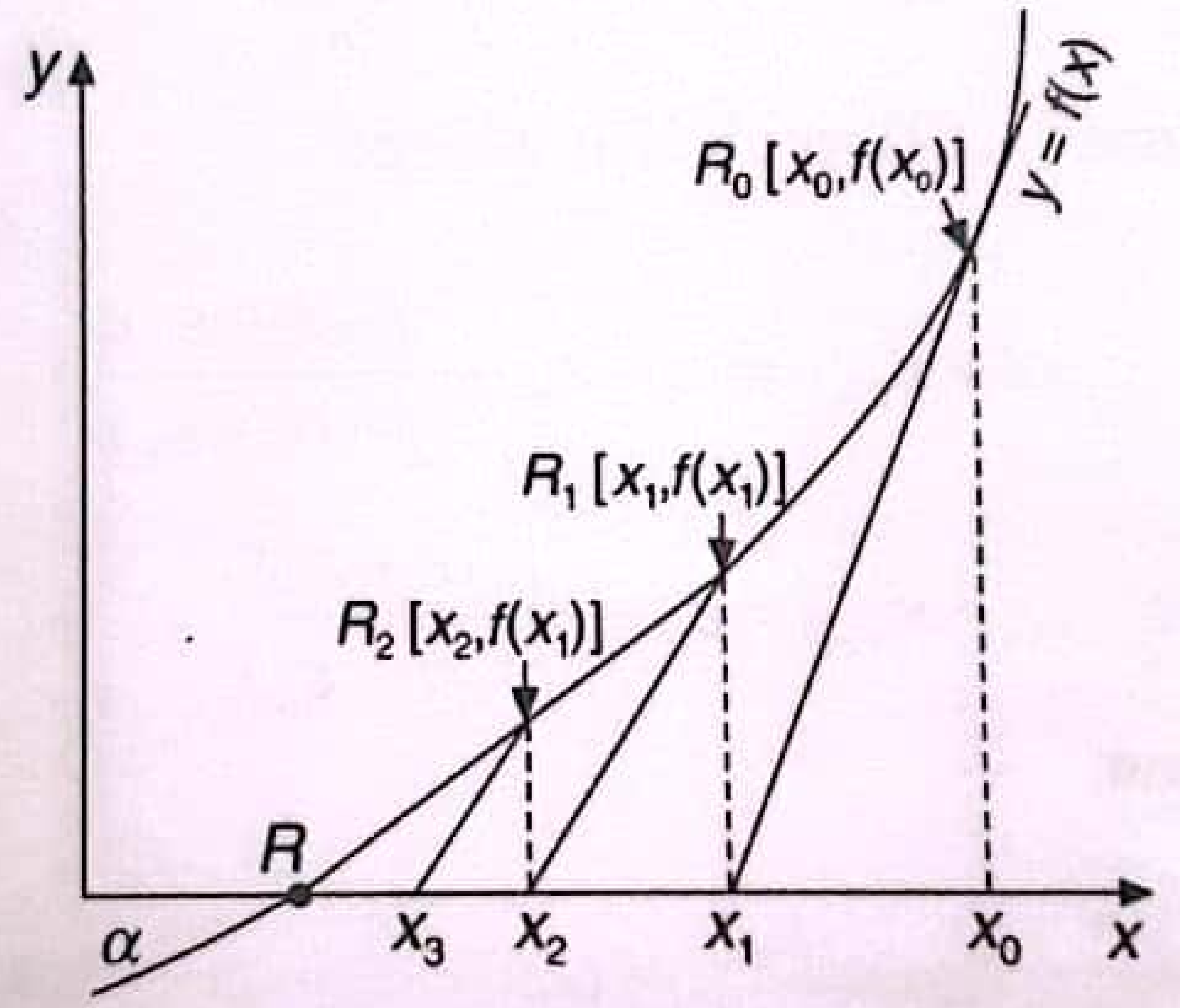

Figure 1.3 Illustrating the Newton–Raphson method.

This cuts the x-axis at $x = x_1$

$\therefore$ Put $y = 0$ in Eq. (1.10)

$$-f(x_0) = f'(x_0)\,(x - x_0)$$

$$\therefore \qquad x - x_0 = \frac{-f(x_0)}{f'(x_0)}$$

$$\therefore \qquad x = x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\therefore \qquad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

which is the first approximation to the root α.

If R_1 is the point on the curve corresponding to $x = x_1$, then the tangent at R_1 will cut the x-axis at x_2 which is nearer to α than x_1. So x_2 is a second approximation to the root.

Continuing this process, we get a sequence $x_0, x_1, x_2, x_3, \ldots$ and the limit of this sequence is α.

1.5.2 Rate of Convergence of Newton–Raphson Method

Let α be the exact value of the root of the equation $f(x) = 0$. Suppose x_n differs from the root α by a small quantity ε_n so that

$$x_0 = \alpha + \varepsilon_n \tag{1.11}$$

and

$$x_{n+1} = \alpha + \varepsilon_{n+1} \tag{1.12}$$

By Newton's formula, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1.13}$$

Substituting (1.11) and (1.12) in (1.13), we get

$$\alpha + \varepsilon_{n+1} = \alpha + \varepsilon_n - \frac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)}$$

$$\therefore \quad \varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)}$$

By Taylor's expansion

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha) + \varepsilon_n f'(\alpha) + \dfrac{\varepsilon_n^2}{2!} f''(\alpha) + \ldots}{f'(\alpha) + \varepsilon_n f''(\alpha) + \dfrac{\varepsilon_n^2}{2!} f''(\alpha) + \ldots}$$

$$= \varepsilon_n - \frac{f(\alpha) + \varepsilon_n f'(\alpha) + \dfrac{\varepsilon_n^2 f''(\alpha)}{2}}{f'(\alpha) + \varepsilon_n f''(\alpha)} \quad \text{(neglecting higher powers of } \varepsilon_n)$$

$$= \frac{\varepsilon_n f'(\alpha) + \varepsilon_n^2 f''(\alpha) - f(\alpha) - \varepsilon_n f'(\alpha) - \dfrac{\varepsilon_n^2 f''(\alpha)}{2}}{f'(\alpha) + \varepsilon_n f''(\alpha)}$$

$$= \frac{\varepsilon_n^2 f''(\alpha) - \dfrac{\varepsilon_n^2 f''(\alpha)}{2}}{f'(\alpha) + \varepsilon_n f''(\alpha)} \quad [\because \alpha \text{ is a root of } f(x) = 0 \Rightarrow f(a) = 0]$$

$$= \frac{\frac{1}{2}\varepsilon_n^2 f''(\alpha)}{f'(\alpha) + \varepsilon_n f''(\alpha)}$$

$$= \frac{\varepsilon_n^2 f''(\alpha)}{2[f'(\alpha) + \varepsilon_n f''(\alpha)]}$$

$$= \frac{\varepsilon_n^2 f''(\alpha)}{2f'(\alpha)\left[1 + \frac{\varepsilon_n f''(\alpha)}{f'(\alpha)}\right]}$$

$$= \frac{\varepsilon_n^2 f''(\alpha)}{2f'(\alpha)}$$

$\left(\because \frac{\varepsilon_n f''(\alpha)}{f'(\alpha)}\right.$ is a very small quantity and, therefore, we neglect it$\left.\right)$

$$\varepsilon_{n+1} = k\,\varepsilon_n^2$$

where $$k = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$$

This shows that the subsequent error at each step is proportional to the square of the previous error and such the convergence is quadratic. Therefore, Newton–Raphson method has a quadratic convergence or its order of convergence is two.

1.5.3 Condition for Convergence of Newton–Raphson Method

Newton's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore \qquad x_{n+1} = \phi(x_n) \tag{1.14}$$

which is similar to the iteration method. The general form of Eq. (1.14) is

$$x = \phi(x) \tag{1.15}$$

We know that the iteration method given by equation (1.15) converges, if $|\phi'(x)| < 1$.

$$\phi(x) = x - \frac{f(x)}{f'(x)}$$

$$\phi'(x) = \frac{d}{dx}\left(x - \frac{f(x)}{f'(x)}\right)$$

$$\therefore \quad \phi'(x) = 1 - \frac{[f'(x)]^2 - f(x)\, f''(x)}{[f'(x)]^2}$$

$$= \frac{[f'(x)]^2 - [f'(x)]^2 + f(x)\, f''(x)}{[f'(x)]^2}$$

$$\phi'(x) = \frac{f(x)\, f''(x)}{[f'(x)]^2}$$

$$|\phi'(x)| = \left| \frac{f(x)\, f''(x)}{[f'(x)]^2} \right|$$

Hence, the condition for convergence of Newton–Raphson method is $|\phi'(x)| < 1$

$$\Rightarrow \quad \left| \frac{f(x)\, f''(x)}{[f'(x)]^2} \right| < 1$$

$$\Rightarrow \quad |f(x)\, f''(x)| < [f'(x)]^2$$

This is the condition for convergence of Newton–Raphson method.

Notes

(i) Newton's formula converges, provided the initial approximation x_0, is chosen sufficiently close to the root.

(ii) Newton's method cannot be used if $f'(x) = 0$ in the neighbourhood of the root. In such cases the regula falsi method should be used.

(iii) If the initial approximation to the root is not given, then we can find any two values of x say p and q such that $f(p)$ and $f(q)$ are of opposite signs.
If $|f(p)| < |f(q)|$, then p is taken as the first approximation to the root.

(iv) Newton's method is also applicable in the case of large values of $f'(x)$, i.e. when the graph of $f(x) = 0$ is nearly vertical where it crosses the x-axis.

EXAMPLE 1.13 Find the real root of $3x - \cos x - 1 = 0$, by the Newton–Raphson method.

Solution

Let

$$f(x) = 3x - \cos x - 1$$

$$f(0) = -2 = -\text{ve}$$

$$f(1) = 1.459697694 = +\text{ve}$$

$\therefore$ A root lies between 0 and 1.

$$f'(x) = 3 + \sin x$$

The first approximation is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Take $$x_0 = 0.5$$

$$x_1 = 0.5 - \frac{f(0.5)}{f'(0.5)}$$

$$= 0.5 - \frac{3(0.5) - \cos(0.5) - 1}{3 + \sin(0.5)}$$

$$= 0.608518649$$

$$= 0.60852$$

The second approximation to the root is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.60852 - \frac{f(0.60852)}{f'(0.60852)}$$

$$= 0.607101879$$

$$= 0.607102$$

The third approximation to the root is given by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.607102 - \frac{f(0.607102)}{f'(0.607102)}$$

$$= 0.607101648$$

$$= 0.607102$$

∴ The root is 0.607102.

EXAMPLE 1.14 Find the root of the equation $x(1 - \log_e x) = 0.5$ lies between 0.1 and 0.2, by the Newton–Raphson method.

Solution

Let

$$f(x) = x(1 - \log_e x) - 0.5$$

$$f(0.1) = -0.1697 = -\text{ve}$$

$$f(0.2) = 0.02188 = +\text{ve}$$

∴ A root lies between 0.1 and 0.2.

$$f'(x) = 1 - (1 - \log_e x) - x(1/x)$$
$$= 1 - \log_e x - 1 = -\log_e x$$

Take
$$x_0 = 0.2$$

The first approximation is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 0.2 - \frac{0.2(1 - \log_e 0.2) - 0.5}{-\log_e 0.2}$$
$$= 0.1864$$

The second approximation is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 0.1864 - \frac{0.1864(1 - \log_e 0.1864) - 0.5}{-\log_e 0.1864}$$
$$= 0.1866$$

The third approximation is given by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 0.1866 - \frac{0.1866\,(1 - \log_e 0.1866) - 0.5}{-\log_e 0.1866}$$
$$= 0.1866$$

∴ The root is 0.1866.

EXAMPLE 1.15 Evaluate $\sqrt{12}$, using the Newton–Raphson method.

Solution

Let
$$x = \sqrt{12}$$
$$x^2 = 12 \Rightarrow x^2 - 12 = 0$$

Let
$$f(x) = x^2 - 12$$
$$f(0) = -\text{ve}$$
$$f(1) = -\text{ve}$$
$$f(2) = -\text{ve}$$
$$f(3) = -\text{ve}$$
$$f(4) = +\text{ve}$$

$\therefore$ The root lies between 3 and 4.

Take $$x_0 = 3, f'(x) = 2x$$

The first approximation is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 3 - \frac{f(3)}{f'(3)}$$

$$= 3 - \frac{(3^2 - 12)}{(2)(3)}$$

$$= 3.5$$

The second approximation is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 3.5 - \frac{f(3.5)}{f'(3.5)}$$

$$= 3.5 - \frac{(3.5)^2 - 12}{(2)(3.5)}$$

$$= 3.464285714$$

$$= 3.46429$$

The third approximation is given by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 3.46429 - \frac{(3.46429)^2 - 12}{2(3.46429)}$$

$$= 3.46410162$$

The fourth approximation is given by

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 3.46410162 - \frac{(3.46410162)^2 - 12}{2(3.46410162)}$$

$$= 3.464101615$$

$\therefore$ The value of $\sqrt{12}$ is 3.4641016.

EXAMPLE 1.16 Calculate the general formula to find $\sqrt{N}$ and hence find $\sqrt{5}$.

Solution

Take
$$x = \sqrt{N}$$
$$x^2 = N$$
$$x^2 - N = 0$$

Let
$$f(x) = x^2 - N$$
$$f'(x) = 2x$$

By Newton's method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$
$$= x_k - \frac{(x_k^2 - N)}{2x_k}$$
$$= \frac{2x_k^2 - x_k^2 + N}{2x_k}$$
$$= \frac{x_k^2 + N}{2x_k}$$
$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{N}{x_k}\right), \quad \text{where } k = 0, 1, 2, \ldots$$

This is the general formula for $\sqrt{N}$. Next to find $\sqrt{5}$

$$\text{Put } N = 5$$
$$\text{Take } x_0 = 2$$

The first approximation is given by

$$x_1 = \frac{1}{2}\left(x_0 + \frac{5}{x_0}\right)$$
$$= \frac{1}{2}\left(2 + \frac{5}{2}\right)$$
$$= 2.25$$

The second approximation is given by

$$x_2 = \frac{1}{2}\left(x_1 + \frac{5}{x_1}\right)$$

$$= \frac{1}{2}\left(2.25 + \frac{5}{2.25}\right)$$

$$= 2.236111111$$

The third approximation is given by

$$x_3 = \frac{1}{2}\left(x_2 + \frac{5}{x_2}\right)$$

$$= \frac{1}{2}\left(2.2361111 + \frac{5}{2.236111}\right)$$

$$= 2.236067978$$

$$= 2.2361$$

$\therefore$ The value of $\sqrt{5}$ is 2.2361.

Generalized Newton–Raphson Method

If μ is a root of $f(x) = 0$ with multiplicity p then the general formula corresponding to

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

is taken as

$$x_{n+1} = x_n - \frac{p\,f(x_n)}{f'(x_n)}$$

This means that $\frac{1}{p}f'(x_n)$ is the slope of the line through (x_n, y_n) and intersecting the axis of x at $(x_{n+1}, 0)$.

If μ is a root of $f(x) = 0$ with multiplicity p, then μ is also a root of $f'(x) = 0$ with multiplicity $(p-1)$ and it is a root of $f''(x) = 0$ with multiplicity $(p-2)$ and so on:

Therefore,

$$x_0 - p\frac{f(x_0)}{f'(x_0)},$$

$$x_0 - (p-1)\frac{f'(x_0)}{f''(x_0)},$$

$$x_0 - (p-2)\frac{f''(x_0)}{f'''(x_0)}$$

will have the same value.

EXAMPLE 1.17 Find the double root of $x^3 - 5.4x^2 + 9.24x - 5.096 = 0$, correct to first decimal place, given that it is near 1.5.

Solution

Let
$$f(x) = x^3 - 5.4x^2 + 9.24x - 5.096$$

$$f'(x) = 3x^2 - 10.8x + 9.24$$

$$x - \frac{pf(x)}{f'(x)} = x - \frac{f(x)}{f'(x)}$$

$$= x - \frac{2(x^3 - 5.4x^2 + 9.24x - 5.096)}{3x^2 - 10.8x + 9.24}$$

$$= \frac{x^3 - 9.24x + 10.192}{3x^2 - 10.8x + 9.24}$$

Hence,
$$x_{k+1} = \frac{x_k^3 - 9.24x_k + 10.192}{3x_{10}^2 - 10.8x_k + 9.24}$$

Take
$$x_0 = 1.5$$

$$x_1 = \frac{(1.5)^3 - (9.24)(1.5) + 10.192}{3(1.5)^2 - (10.8)(1.5) + 9.24}$$
$$= 1.3952$$

$$x_2 = \frac{(1.395)^3 - (9.24)(1.3952) + 10.192}{3(1.3952)^2 - (10.8)(1.3952) + 9.24}$$
$$= 1.3966$$

$$x_3 = \frac{(1.3966)^3 - 9.24(1.3966) + 10.192}{3(1.3966)^2 - 10.8(1.3966) + 9.24}$$
$$= 1.4024$$

$$x_4 = \frac{(1.4024)^3 - (9.24)(1.4024) + 10.192}{3(1.4024)^2 - 10.8(1.4024) + 9.24}$$
$$= 1.4211$$

$$x_5 = \frac{(1.4211)^3 - (9.24)(1.4211) + (10.192)}{3(1.4211)^2 - 10.8(1.4211) + (9.24)}$$
$$= 1.4016$$

From x_1, x_2, x_3, x_4, x_5 it is clear that the root is 1.4, correct to one decimal place.

1.6 SECANT METHOD

This method is an improvement over the method of regula falsi as it does not require the condition $f(x_0) f(x) < 0$. The convergence of the secant method is slow.

Consider the graph (Figure 1.4) of the function $y = f(x)$ and taking x_0, x_1 as the initial limits of the interval. Slope of the secant line passing through x_0 and x_1 is given by

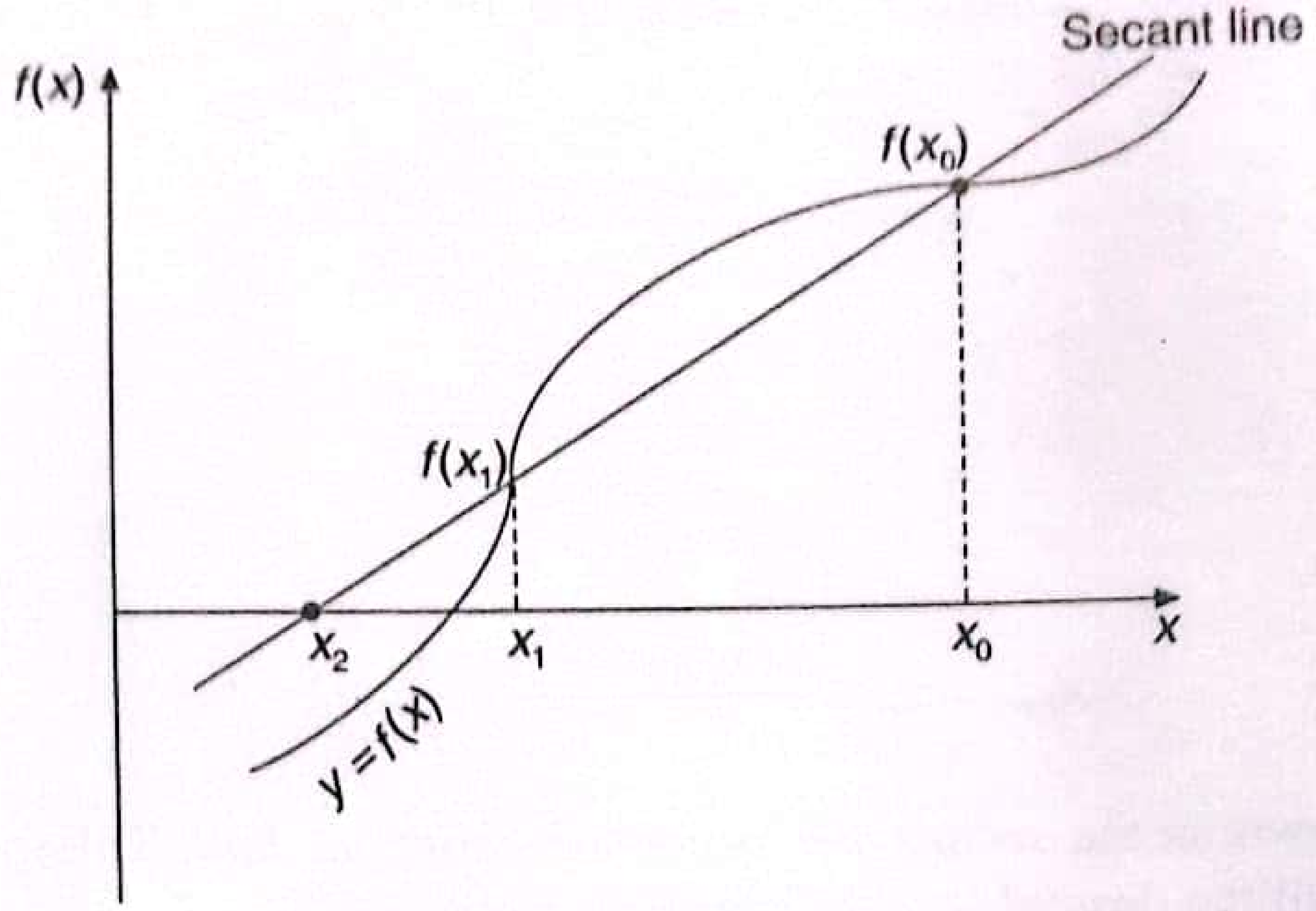

Figure 1.4 Illustrating the secant method.

$$\frac{f(x_0)}{x_0 - x_2} = \frac{f(x_1)}{x_1 - x_2}$$

where x_2 is the point of intersection between the secant line and x-axis which represents the approximate root of $f(x)$.

$$\therefore \quad (x_1 - x_2) f(x_0) = f(x_1)(x_0 - x_2)$$

$$x_1 f(x_0) - x_2 f(x_0) = x_0 f(x_1) - x_2 f(x_1)$$

$$x_2 f(x_1) - x_2 f(x_0) = x_0 f(x_1) - x_1 f(x_0)$$

$$x_2 [f(x_1) - f(x_0)] = x_0 f(x_1) - x_1 f(x_0)$$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \tag{1.16}$$

Adding and subtracting $f(x_1)\, x_1$ to the numerator in Eq. (1.16)

$$\therefore \quad x_2 = \frac{x_0 f(x_1) - x_1 f(x_0) + x_1 f(x_1) - x_1 f(x_1)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{x_0 f(x_1) + x_1[f(x_1) - f(x_0)] - x_1 f(x_1)}{f(x_1) - f(x_0)}$$

$$= \frac{x_1[f(x_1) - f(x_0)]}{f(x_1) - f(x_0)} + \frac{x_0 f(x_1) - x_1 f(x_1)}{f(x_1) - f(x_0)}$$

$$= x_1 + \frac{x_0 f(x_1) - x_1 f(x_1)}{f(x_1) - f(x_0)} \tag{1.17}$$

$$= x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

Similarly, replace x_0 and x_1 by and x_2, respectively in Eq. (1.17), we have the next approximate root of $y = f(x)$

$$x_3 = x_2 + \frac{x_1 f(x_2) - x_2 f(x_2)}{f(x_2) - f(x_1)}$$

$$= x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

In general,

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}, \quad k \geq 1$$

This is known as the *secant method* or the *chord method*. This procedure is continued till the desired level of accuracy is obtained.

EXAMPLE 1.18 Using the secant method, find the root of the equation $xe^x = \cos x$.

Solution

Let

$$f(x) = xe^x - \cos x = 0$$

Taking the initial approximations

$$x_0 = 0$$

$$x_1 = 1$$

So that

$$f(x_0) = f(0)$$

$$= -\cos 0 = -1$$

$$f(x_1) = f(1)$$

$$= e - \cos 1$$

$$= 2.177979523$$

By secant method we have

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 1 - \frac{f(1)(1-0)}{f(1) - f(0)}$$

$$= 1 - \frac{2.17798}{3.17798}$$

$$= 1 - 0.685334709$$

$= 0.314665291$

$= 0.31467$

$f(x_2) = f(0.31467)$

$= -0.519861335$

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

$$= 0.31467 - \frac{(-0.519861335)(-0.68533)}{(-0.519861335) - (2.17798)}$$

$= 0.31467 + 0.132059867$

$= 0.446729867$

$= 0.4467$

$f(x_3) = f(0.44673)$

$= -0.20353978$

$= -0.20354$

$$x_4 = x_3 - \frac{(x_3 - x_2) f(x_3)}{f(x_3) - f(x_2)}$$

$$= 0.44673 - \frac{(0.13206)(-0.20354)}{0.316321335}$$

$= 0.44673 + 0.084975274$

$= 0.531705274$

$= 0.53171$

$f(x_4) = f(0.53171)$

$= 0.042943982$

$= 0.04294$

$$x_7 = x_6 - \frac{f(x_6)(x_6 - x_5)}{f(x_6) - f(x_5)}$$

$$= 0.53171 - \frac{(0.04294)(0.08501)}{0.24648}$$

$= 0.53171 - 0.01480984$

$= 0.51690016$

$= 0.51690$

$f(x_5) = f(0.51690)$

$= -0.002606334582$

$f(x_5) = -0.00261$

$$x_6 = x_5 - \frac{f(x_5)(x_5 - x_4)}{f(x_5) - f(x_4)}$$

$$= 0.51690 - \left[\frac{(-0.00261)(-0.01481)}{-0.04555}\right]$$

$$= 0.51690 + 0.0008486081229$$

$$= 0.517748608$$

$$= 0.51775$$

$$f(x_6) = f(0.51775)$$

$$= -0.0000224011$$

$$= -0.00002$$

$$x_7 = x_6 - \frac{f(x_6)(x_6 - x_5)}{f(x_6) - f(x_5)}$$

$$= 0.51775 - \left[\frac{(-0.00002)(0.00085)}{0.00259}\right]$$

$$= 0.51775 + 0.000006563706564$$

$$= 0.517756563$$

$$= 0.51775$$

$\therefore$ The root is 0.51775, correct to four decimal places.

Notes

(i) The rate of convergence of secant method is 1.618.

(ii) By secant method, the successive approximations is given by

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}, \quad k \geq 1$$

If at any iteration $f(x_k) = f(x_{k-1})$, this method fails and shows that it does not converge necessarily. This is a drawback of secant method over the regula falsi method which always converges.

1.7 MULLER'S METHOD

Muller's method is an extension of the secant method. This method has some advantages compared to Newton's method and method of regula falsi. This method does not require the evaluation of derivatives as in Newton–Raphson method. Using this method we can calculate the complex roots

EXERCISES

1.1 Find a positive root of each of the following equations, by the bisection method.

(i) $x^2 - 5x + 2 = 0$ [*Ans.* 0.4384]

(ii) $x^3 - x - 1 = 0$ [*Ans.* 1.3437]

(iii) $x^3 + x^2 - 1 = 0$ [*Ans.* 0.754869]

(iv) $2x = 3 + \cos x$ [*Ans.* 1.524]

(v) $x^4 - x^3 - 2x^2 - 6x - 4 = 0$ [*Ans.* 2.7315]

(vi) $x^3 - 18 = 0$ [*Ans.* 2.621]

1.2 Find a real root of each of the following equations, by the method of false position.

(i) $x^6 - x^4 - x^3 - 1 = 0$ [*Ans.* 1.4036]

(ii) $x^2 - \log_e x - 12 = 0$ [*Ans.* 3.646]

(iii) $xe^x - 2 = 0$ [*Ans.* 0.8526]

(iv) $x^3 - 4x - 9 = 0$ [*Ans.* 2.7065]

(v) $xe^x = \cos x$ [*Ans.* 0.51776]

(vi) $x \tan x = -1$ [*Hint: Root lies between* (2.5 and 3.0)]
[*Ans.* 2.798]

(vii) $2x - 3 \sin x = 5$ [*Ans.* 2.8832]

1.3 Find a root of each of the following equations, by the method of iteration.

(i) $2x - \log_{10} x = 3$ [*Ans.* 1.6024]

(ii) $5x^5 = 5x + 1$ [*Ans.* 1.0448]

(iii) $3x = 1 - \cos x$ [*Ans.* 0.6071]

(iv) $xe^x = \cos x$ [*Ans.* 0.5177]

(v) $x^3 = 2x + 5$ [*Ans.* 2.0945]

(vi) $x - \dfrac{x^3}{3} + \dfrac{x^5}{10} - \dfrac{x^7}{42} + \dfrac{x^9}{216} - \dfrac{x^{11}}{1320} + \ldots = 0.443$ [*Ans.* 0.47680]

1.4 Find a root of each of the following equations by the secant method, correct to three decimal places.

(i) $x^3 + x^2 + x + 7 = 0$ [*Ans.* – 2.0625]

(ii) $x - e^x = 0$ [*Ans.* 0.567]

1.5 Find a root of each of the following equations by the Newton–Raphson method.

(i) $2x - 3 \sin x = 5$ [*Ans.* 2.283]

(ii) $2x^3 + 3x - 10 = 0$ [*Ans.* 1.4209]

(iii) $\sin x = x/2$ — the root lies between $(\pi/2, \pi)$ [*Ans.* 1.895]

(iv) $x^3 - 5x - 6 = 0$ [*Ans.* 0.689]

(v) $e^x = x^3 + \cos 25x$ — which is near 4.5 [*Ans.* 4.545]

(vi) Find a positive root of $\sqrt{24}$ and $\sqrt{17}$. [*Ans.* 2.2360, 2.571]

(vii) Using Newton's method, show that the iteration formula for finding

(a) the pth root of a is $x_{n+1} = \dfrac{(p-1)\, x_n^{\,p} + a}{p x_n^{\,p-1}}$

(b) the reciprocal of the pth root of a is $x_{n+1} = \dfrac{x_n (p + 1 - a x_n^{\,p})}{p}$

Hence find the values of $(10)^{1/3}$ and $\dfrac{1}{\sqrt{\sin (0.5)}}$.

(viii) Find the least positive root of the equation $\tan x - x = 0$ to an accuracy of 0.0001, by the Newton–Raphson method. [*Ans.* 4.4934]

1.6 Find a root of each of the following methods, using Muller's method.

(i) $x^3 - x - 1 = 0$ [*Ans.* 1.3247]

(ii) $x^3 - x^2 - x - 1 = 0$ [*Ans.* 1.8393]

1.7 Find a root of each of the following equations, using Horner's method.

(i) $x^3 - 3x^2 + 2.5 = 0$ [*Ans.* 2.641 or 1.168]

(ii) $10x^3 - 15x + 3 = 0$ that lies between 1 and 2 [*Ans.* 1.11]

(iii) $h^3 + 3h^2 - 12h - 11 = 0$ [*Ans.* 2.7689]

1.8 Solve the following equations by using Graffe's method.

(i) $x^3 - 4.5x^2 + 6.56x - 3.12 = 0$ [*Ans.* 2.012, 1.364, 1.137]

(ii) $x^4 - x^3 + 3x^2 + x - 4 = 0$

[*Ans.* 1.0912, – 0.9172, 0.413 ± 1.956*i*]

(iii) $x^3 + 3x^2 - 4 = 0$ [*Ans.* – 2, – 2, 1]

(iv) $x^3 - 5x^2 - 17x + 20 = 0$ [*Ans.* 7.017, – 2.974, 0.958]