2.3 NUMERICAL SOLUTION OF LINEAR SYSTEM OF EQUATIONS: INDIRECT OR ITERATIVE METHODS

The preceding methods of solving simultaneous linear equations are known as direct methods, as these methods yield the solution after a certain amount of fixed computation. On the other hand, an iterative method is that in which we start from an approximation to the true solution and obtain better and better approximations from a computation cycle repeated as often as may be necessary for achieving a desired accuracy. Thus, in an iterative method, the amount of compution depends on the degree of accuracy required.

For large systems, iterative methods may be faster than the direct methods. Even the round-off errors in iterative methods are smaller. In fact, iteration is a set correcting process and any error made at any stage of computation gets automatically corrected in the subsequent steps.

Simple iterative methods can be devised for systems in which the coefficients of the leading diagonal are large as compared to others. We shall describe two particular methods of iteration.

2.3.1 Gauss–Jacobi Method

Consider the system of equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \qquad (2.11)$$

Suppose, in the above, in each equation, the coefficients of the diagonal terms are large compared to other coefficients. Solving for x, y, z respectively, we have

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \\ y &= \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \\ z &= \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \end{aligned}\right\} \quad (2.12)$$

Suppose, $x^{(0)}$, $y^{(0)}$, $z^{(0)}$ are initial estimates for the values of the unknowns x, y, z. Substituting these values in the right sides of (2.12), we have a system of first approximations, given by

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1 y^{(0)} - c_1 z^{(0)})$$

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2\, x^{(0)} - c_2 z^{(0)})$$

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3\, x^{(0)} - b_3 z^{(0)})$$

Substituting the values $x^{(1)}$, $y^{(1)}$, $z^{(1)}$ in (2.12), we have the second approximations given by

$$x^{(2)} = \frac{1}{a_1}(d_1 - b_1 y^{(1)} - c_1 z^{(1)})$$

$$y^{(2)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)} - c_2 z^{(1)})$$

$$z^{(2)} = \frac{1}{c_3}(d_3 - a_3\, x^{(1)} - c_3 y^{(1)})$$

If $x^{(r)}$, $y^{(r)}$, $z^{(r)}$ are the rth iterates, then

$$x^{(r+1)} = \frac{1}{a_1}(d_1 - a_1 y^{(r)} - c_1 z^{(r)})$$

$$y^{(r+1)} = \frac{1}{b_2}(d_2 - a_2 x^{(r)} - c_2 z^{(r)})$$

$$z^{(r+1)} = \frac{1}{c_3}(d_3 - a_3 x^{(r)} - b_3 y^{(r)})$$

The process is continued till convergency is secured.

2.3.2 Gauss–Seidel Method

This is a modification of Gauss–Jacobi method. As before, solving for x, y, z respectively, the system of equations (2.11), we have

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \\ y &= \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \\ z &= \frac{1}{c_3}(d_3 - a_3 x - b_3\ y) \end{aligned}\right\} \qquad (2.13)$$

As before, we start with the initial values $x^{(0)}$, $y^{(0)}$, $z^{(0)}$ for x, y, z respectively. Substituting $y^{(0)}$ and $z^{(0)}$ in the first equation in (2.13), we get

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1 y^{(0)} - c_1 z^{(0)})$$

Then, we substitute $x^{(1)}$ for x and $z^{(0)}$ for z in the second equation and get

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)} - c_2 z^{(0)})$$

We then substitute $x^{(1)}$ for x and $y^{(1)}$ for y in the third equation, we get

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3\ x^{(1)} - b_3 y^{(1)})$$

Thus, as soon as a new value for a variable is found, it is used immediately in the following equations.

If $x^{(r)}$, $y^{(r)}$, $z^{(r)}$ are the rth iterates, then

$$x^{(r+1)} = \frac{1}{a_1}(d_1 - b_1 y^{(r)} - c_1 z^{(r)})$$

$$y^{(r+1)} = \frac{1}{b_2}(d_2 - a_2 x^{(r+1)} - c_2 z^{(r)})$$

$$z^{(r+1)} = \frac{1}{c_3}(d_3 - a_3 x^{(r+1)} - b_3 y^{(r+1)})$$

The process is continued till convergency is secured.

Note 1: Since the current values of the unknowns at each stage of iteration are used in proceeding to the next stage of iteration, the convergence in Gauss–Seidel method will be more rapid than in Gauss–Jacobi method.

Note 2: The rate of convergence of Gauss–Seidel method is roughly twice that of the Gauss–Jacobi. The condition for convergence of Gauss–Jacobi and Gauss–Seidel methods is given by following rule.

The process of iteration will converge if in each equation of the system, the absolute value of the largest coefficient is greater than the sum of the absolute values of all the remaining coefficients.

EXAMPLE 2.9 Obtain the solution of the following system, using the Gauss–Jacobi iteration method.

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18$$

$$2x - 3y + 20z = 25$$

Solution In the given equations we find that the largest coefficient is attached to a different unknown. Also in each equation, the absolute value of the largest coefficient is greater than the sum of the remaining coefficients. So, iteration method can be applied. From the equations, we have

$$x = \frac{1}{20}(17 - y + 2z) \tag{i}$$

$$y = \frac{1}{20}(-18 - 3x + z) \tag{ii}$$

$$z = \frac{1}{20}(25 - 2x + 3y) \tag{iii}$$

We start from an approximation $x_0 = y_0 = z_0 = 0$.
Substituting these values on the right sides of Eqs. (i)–(iii), we get

$$x^{(1)} = \frac{17}{20}$$

$$= 0.85$$

$$y^{(1)} = \frac{-18}{20}$$

$$= -0.9$$

$$z^{(1)} = \frac{25}{20}$$

$$= 1.25$$

Putting these values on the right sides of the Eqs. (i)–(iii), we obtain

$$x^{(2)} = \frac{1}{20}(17 - y^{(1)} + 2z^{(1)})$$

$$= 1.02$$

$$y^{(2)} = \frac{1}{20}(-18 - 3x^{(1)} + z^{(1)})$$

$$= -0.965$$

$$z^{(2)} = \frac{1}{20}(25 - 2x^{(1)} + 3y^{(1)})$$
$$= 1.03$$

Substituting these values on the right sides of Eqs. (i)–(iii), we have

$$x^{(3)} = \frac{1}{20}(17 - y^{(2)} + 2z^{(2)})$$
$$= 1.00125$$

$$y^{(3)} = \frac{1}{20}(-18 - 3x^{(2)} + z^{(2)})$$
$$= -1.0015$$

$$z^{(3)} = \frac{1}{20}(25 - 2x^{(2)} + 3y^{(2)})$$
$$= 1.00325$$

Substituting these values, we get

$$x^{(4)} = \frac{1}{20}(17 - y^{(3)} + 2z^{(3)})$$
$$= 1.0004$$

$$y^{(4)} = \frac{1}{20}(-18 - 3x^{(3)} + 3y^{(3)})$$
$$= -1.000025$$

$$z^{(4)} = \frac{1}{20}(25 - 2x^{(3)} + z^{(3)})$$
$$= 0.9965$$

Putting these values in Eqs. (i)–(iii), we have

$$x^{(5)} = \frac{1}{20}(-17 - y^{(4)} + 2z^{(4)})$$
$$= 0.999966$$

$$y^{(5)} = \frac{1}{20}(18 - 3x^{(4)} + z^{(4)})$$
$$= -1.000078$$

$$z^{(5)} = \frac{1}{20}(25 - 2x^{(4)} + 3y^{(4)})$$
$$= 0.999956$$

Again putting these values in Eqs. (i)–(iii), we have

$$x^{(6)} = \frac{1}{20}(-17 - y^{(5)} + 2z^{(5)})$$

$$= 1.0000$$

$$y^{(6)} = \frac{1}{20}(-18 - 3x^{(5)} + z^{(5)})$$

$$= -0.999997$$

$$z^{(6)} = \frac{1}{20}(25 - 2x^{(5)} + 3y^{(5)})$$

$$= 0.999992$$

The values in the 5th and 6th iterations being practically the same, we can terminate here. So, we have

$$x = 1.0$$

$$y = -1.0$$

$$z = 1.0$$

EXAMPLE 2.10 Solve the following system of equations, by the Gauss–Seidel method of iteration.

$$27x + 6y - z = 85$$

$$6x + 15y + 2z = 72$$

$$x + y + 54z = 110$$

Solution From the given equations, we have

$$x = \frac{1}{27}(85 - 6y - z) \tag{i}$$

$$y = \frac{1}{15}(72 - 2z - 6x) \tag{ii}$$

$$z = \frac{1}{54}(110 - x - y) \tag{iii}$$

For the first iteration:

Putting $y = 0 = z$ 7x the right side of Eq. (i), we get

$$x^{(1)} = \frac{85}{27}$$

$$= 3.14815$$

Putting $z = 0$ and the current value of $x = 3.14815$ in Eq. (ii), we get

$$y^{(1)} = \frac{1}{15}(72 - 6 \times 3.14815)$$
$$= 3.54074$$

Putting $x = 3.14815$, $y = 3.54074$ in Eq. (iii), we get

$$z^{(1)} = \frac{1}{54}(110 - 3.14815 - 3.54074)$$
$$= 1.91317$$

For the second iteration:

$$x^{(2)} = \frac{1}{27}(85 - 6 \times 3.54074 + 1.91317)$$
$$= 2.43218$$

$$y^{(2)} = \frac{1}{15}(72 - 2z^{(1)} - 6x^{(2)})$$
$$= \frac{1}{15}(72 - 2 \times 1.91317 - 6 \times 2.43218)$$
$$= 3.5$$

$$z^{(2)} = \frac{1}{54}(110 - x^{(2)} - y^{(2)})$$
$$= \frac{1}{54}(110 - 2.43218 - 3.57204)$$
$$= 1.92585$$

For the third iteration:

$$x^{(3)} = \frac{1}{27}(85 - 6y^{(2)} + z^{(2)})$$
$$= 2.42569$$

$$y^{(3)} = \frac{1}{15}(72 - 2z^{(2)} - 6x^{(3)})$$
$$= 3.57294$$

$$z^{(3)} = \frac{1}{54}(110 - x^{(3)} - y^{(3)})$$
$$= 1.92595$$

For the fourth iteration:

$$x^{(4)} = \frac{1}{27}(85 - 6y^{(3)} + z^{(3)})$$

$$= 2.42549$$

$$y^{(4)} = \frac{1}{15}(72 - 2z^{(3)} - 6x^{(4)})$$

$$= 3.57301$$

$$z^{(4)} = \frac{1}{54}(110 - x^{(4)} - y^{(4)})$$

$$= 1.92595$$

For the fifth iteration:

$$x^{(5)} = \frac{1}{27}(85 - 6y^{(4)} + z^{(4)})$$

$$= 2.42548$$

$$y^{(5)} = \frac{1}{15}(72 - 2z^{(4)} - 6x^{(5)})$$

$$= 3.57301$$

$$z^{(5)} = \frac{1}{54}(110 - x^{(5)} - y^{(5)})$$

$$= 1.92595$$

We find that the values in the 4th and 5th iteration are same, so we can stop the iteration process. The values are:

$$x = 2.4255$$

$$y = 3.5730$$

$$z = 1.9260$$

EXAMPLE 2.11 Solve the following equations, by the Gauss–Seidel iteration method.

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$

$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$

$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$

$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

Solution Rewriting the given equations as

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4 \quad \text{(i)}$$

$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4 \quad \text{(ii)}$$

$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4 \quad \text{(iii)}$$

$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3 \quad \text{(iv)}$$

For the first iteration:
Taking $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ in Eq. (i), we get

$$x_1 = 0.3$$

Now, taking $x_1 = 0.3$, $x_3 = 0$, $x_4 = 0$ in Eq. (ii), we get

$$x_2 = 1.56$$

Now, taking $x_1 = 0.3$, $x_2 = 1.56$, $x_4 = 0$ in Eq. (iii), we get

$$x_3 = 2.886$$

Again, taking $x_1 = 0.3$, $x_2 = 1.56$, $x_3 = 2.886$ in Eq. (iv), we get

$$x_4 = -0.1368$$

For the second iteration:
Putting $x_2 = 1.56$, $x_3 = 2.886$, $x_4 = -0.1368$ in Eq. (i), we obtain

$$x_1 = 0.8869$$

Putting $x_1 = 0.8869$, $x_3 = 2.886$, $x_4 = -0.1368$ in Eq. (ii), we obtain

$$x_2 = 1.9523$$

Putting $x_1 = 0.8569$, $x_2 = 1.9523$, $x_4 = -0.1368$ in Eq. (iii), we obtain

$$x_3 = 2.9566$$

Putting $x_1 = 0.8869$, $x_2 = 1.9523$, $x_3 = 2.9566$ in Eq. (iv), we obtain

$$x_4 = -0.0248$$

For the third iteration:
Substituting $x_1 = 0.8869$, $x_2 = 1.9523$, $x_3 = 2.886$, $x_4 = -0.1368$, we obtain

$$x_1 = 0.9836$$

Substituting $x_1 = 0.9836$, $x_3 = 2.886$, $x_4 = 0.1368$, we obtain

$$x_2 = 1.9899$$

Substituting $x_1 = 0.9836$, $x_2 = 1.9899$, $x_4 = 0.1368$, we obtain

$$x_3 = 2.9924$$

Substituting $x_1 = 0.9836$, $x_2 = 1.9899$, $x_3 = 2.9924$, we obtain

$$x_4 = -0.0042$$

For the fourth iteration:
Proceeding in the same way, we get

$$x_1 = 0.9968$$

$$x_2 = 1.9982$$

$$x_3 = 2.9987$$

$$x_4 = -0.0008$$

For the fifth iteration:

$$x_1 = 0.9994$$

$$x_2 = 1.9997$$

$$x_3 = 2.9997$$

$$x_4 = -0.0001$$

For the sixth iteration:

$$x_1 = 0.9999$$

$$x_2 = 1.9999$$

$$x_3 = 2.9999$$

$$x_4 = -0.0001$$

Hence, the solution is

$$x_1 = 1$$

$$x_2 = 2$$

$$x_3 = 3$$

$$x_4 = 0$$

EXAMPLE 2.12 Solve the following system of equations, by using the Gauss–Jacobi and Gauss–Seidel methods correct to three decimal places.

$$8x - 3y + 2z = 20 \tag{i}$$

$$4x + 11y - z = 33 \tag{ii}$$

$$6x + 3y + 12z = 35 \tag{iii}$$

Solution Since the diagonal elements are dominant in the coefficient matrix, write x, y, z as follows:

$$x = \frac{1}{8}(20 + 3y - 2z)$$

$$y = \frac{1}{11}(33 - 4x + z)$$

$$z = \frac{1}{12}(35 - 6x - 3y)$$

(i) *By the Gauss–Jacobi method*

For the first iteration:

Let the initial values be $x = 0$, $y = 0$, $z = 0$

Using these values in Eqs. (i), (ii) and (iii), we get

$$x^{(1)} = \frac{1}{8}[20 + 3(0) - 2(0)]$$

$$= 2.5$$

$$y^{(1)} = \frac{1}{11}[33 - 4(0) + 0]$$
$$= 3.0$$
$$z^{(1)} = \frac{1}{12}[35 - 6(0) - 3(0)]$$
$$= 2.9166$$

For the second iteration:
Using these values of $x^{(1)}$, $y^{(1)}$ and $z^{(1)}$ in Eqs. (i), (ii) and (iii), we get

$$x^{(2)} = \frac{1}{8}[20 + 3(3.0) - 2(2.916)]$$
$$= 2.8958$$
$$y^{(2)} = \frac{1}{11}[33 - 4(2.5) + 2.916]$$
$$= 2.35606$$
$$z^{(2)} = \frac{1}{12}[35 - 6(2.5) - 3(3.0)]$$
$$= 0.9167$$

For the third iteration:
Substituting $x^{(2)}$, $y^{(2)}$ and $z^{(2)}$ in Eqs. (i), (ii) and (iii), we get

$$x^{(3)} = \frac{1}{8}[20 + 3(2.356) - (20.967)]$$
$$= 3.1546$$
$$y^{(3)} = \frac{1}{11}[33 - 4(2.8958) + 0.9167]$$
$$= 2.0303$$
$$z^{(3)} = \frac{1}{12}[35 - 6(2.8958) - 3(2.3561)]$$
$$= 0.879$$

Thus, the iteration process is continued. The results are tabulated as follows:

Iteration	x	y	z
1.	2.5	3.0	2.9167
2.	2.8958	2.3561	0.91667
3.	3.1544	2.0303	0.8797
4.	3.0414	1.9329	0.8319

(Contd.)

Iteration	x	y	z
5.	3.0169	1.9697	0.9127
6.	3.0104	1.9859	0.9158
7.	3.0158	1.9886	0.91496
8.	3.0169	1.9865	0.9116
9.	3.0170	1.9858	0.91156
10.	3.0168	1.9858	0.91169

In 9th and 10th iterations the values of x, y, z are same, correct to three decimal places. So, we stop at this iteration.

(ii) *By the Gauss–Seidel Method*

For the first iteration:

We take the initial values as $y = 0$ and $z = 0$ in equation (i), we get

$$x^{(1)} = \frac{1}{8}[20 + 3(0) - 2(0)]$$
$$= 2.5$$

Now, we take $x_1 = 2.5$, $z_1 = 0$ in Eq. (ii), we get

$$y^{(1)} = \frac{1}{11}[33 - 4(2.5) + 0]$$
$$= 2.0909$$

Again, take $x_1 = 2.5$, $y_1 = 2.0909$ in Eq. (iii), we get

$$z^{(1)} = \frac{1}{12}[35 - 6(2.5) - 3(2.0909)]$$
$$= 1.1$$

For the second iteration:

Substitute $x_1 = 2.5$, $y_1 = 2.0909$, $z_1 = 1.143$ in Eq. (i), we get

$$x^{(2)} = \frac{1}{8}[20 + 3(2.0909) - 2(1.143939)]$$
$$= 2.998106$$

Substitute $x_1 = 2.998106$ and $z_1 = 1.1439$ in Eq. (ii), we get

$$y^{(2)} = \frac{1}{11}[33 - 4(2.9981) + 1.143939]$$
$$= 2.013774$$

Substituting the values of $x_1 = 2.998106$ and $y_1 = 2.013774$ in Eq. (iii), we get

$$z^{(2)} = \frac{1}{12}[35 - 6(2.998106) - 3(2.013774)]$$
$$= 0.914170$$

For the third iteration:

$$x^{(3)} = \frac{1}{8}[20 + 3(2.013774) - 2(0.914170)]$$
$$= 3.026623$$

$$y^{(3)} = \frac{1}{11}[33 - 4(3.026623) + 0.91417]$$
$$= 1.982516$$

$$z^{(3)} = \frac{1}{12}[35 - 6(3.016512) - 3(1.985607)]$$
$$= 0.912009$$

Thus, the iteration process is contunued. The results are tabulated as follows:

Iteration	x	y	z
1.	2.5	2.0909	1.143939
2.	2.99816	2.013774	0.914170
3.	3.026623	1.982516	0.907726
4.	3.016512	1.985607	0.912009
5.	3.016600	1.985964	0.911876
6.	3.016767	1.985892	0.911810
7.	3.01675	1.985889	0.911816

Since the 6th and 7th iterations give the same values for x, y and z, correct to four decimal places. Therefore, we stop the iteration here.

$$\therefore \quad x = 3.0168$$
$$y = 1.9859$$
$$z = 0.9118$$

Note

This shows that the convergence is rated in Gauss–Seidel method when compared to Gauss–Jacobi method.

Comparison of Direct and Iterative Methods or Comparison of Gauss elimination and Gauss–Seidel Iteration Methods

1. Gauss elimination method has the advantage that elimination is finite for any non-singular set of equations.

2. Gauss–Seidel iteration method converges only for special systems of equations. For some systems, elimination is the only course. available.
3. In general, the round off error is smaller in iteration methods. Iteration is a self-correcting method. Any errors made at any step in the computation are corrected in the subsequent iterations.
   The computational effort is approximately $(2n^3/3)$ arithmetic operations in each elimination but the effort is approximately $2n^2$ arithmetic operations per iteration.
4. If convergence is achieved in less than n iterations it is significantly superior compared to the elimination method.

EXERCISES

2.5 Using the Gauss–Jacobi iteration method, solve the following systems of equations, correct to three decimal places.

(i) $10x + 2y + z = 9$
$2x + 20y - 2z = -44$
$-2x + 3y + 10z = 22$

[*Ans.* $x = 1.0, y = -2.0, z = 3.0$]

(ii) $1.02x_1 - 0.05x_2 - 0.10x_3 = 0.795$
$-0.11x_1 + 1.03x_2 - 0.05x_3 = 0.849$
$-0.11x_1 + 0.12x_2 + 1.04x_3 = 1.398$

[*Ans.* $x_1 = 0.982, x_2 = 1.005, x_3 = 1.564$]

(iii) $13x + 5y - 3z + 4 = 18$
$2x + 12y + z - 44 = 13$
$3x - 4y + 10z + 4 = 29$
$2x + y - 3z + 94 = 31$

[*Ans.* $x = 2.1934, y = 1.7113,$
$z = 2.5643, u = 3.6216$]

2.6 Using the Gauss–Seidel method, solve the following systems of equations.

(i) $10x_1 - 5x_2 - 2x_3 = 3$
$4x_1 - 10x_2 + 3x_3 = -3$
$x_1 + 6x_2 + 10x_3 = -3$

[*Ans.* $x_1 = 0.342, x_2 = 0.285, x_3 = -0.505$]

(ii) $8x - 3y + 2z = 20$
$6x + 3y + 12z = 35$
$4x + 11y - z = 33$

[*Hint:* Interchange second and third equations]
[*Ans.* $x = 3.02, y = 1.99, z = 0.91$]