OPERATIONS RESEARCH

INTEGER LINEAR PROGRAMMING

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INTEGER PROGRAMMING: AN INTRODUCTION

- An integer programming model is one where one or more of the decision variables has to take on an integer value in the final solution
- Solving an integer programming problem is much more difficult than solving an LP problem
- Even the fastest computers can take an excessively long time to solve big integer programming problems
- If requiring integer values is the only way in which a problem deviates from a linear programming formulation, then it is an *integer programming (IP) problem. (The more complete name is integer linear programming, but the adjective linear normally is dropped except when this* problem is contrasted with the more esoteric integer nonlinear programming problem
- So, The mathematical model for integer programming is the linear programming model with the one additional restriction that the variables must have integer values.

TYPES OF INTEGER PROGRAMMING PROBLEMS

- **PURE-INTEGER PROBLEMS**
  - require that *all* decision variables have integer solutions.

- **MIXED-INTEGER PROBLEMS**
  - Require some, but not all, of the decision variables to have integer values in the final solution, whereas others need not have integer values.

- **0–1 INTEGER PROBLEMS**
  - Require integer variables to have value of 0 or 1, such as situations in which decision variables are of the yes-no type.

INTEGER PROGRAMMING: FORMULATION

- **Pure ILP Problem:**

A jewelery shop in the city specializes in ornaments and the manger has planned to limit the use of diamonds to the artistic configuration of diamond rings, diamond earnings and diamond necklaces. The three items require the following specifications:

ORNAMENT	DIAMOND	
	½ Carat	¼ Carat
Ring	4	6
Earring (Pair)	3	5
Necklace	10	9
Availability	**150**	**160**

The jeweler does no want to configure the diamond into more than 50 items. The per unit profit for the rings is Rs. 1500, for earnings is Rs. 2400 and for necklace is Rs. 3600. Formulate the problem as an ILP model for maximizing the profit.

INTEGER PROGRAMMING: FORMULATION

- **Decision Variables:** Let X_1 = Number of diamond rings, X_2 = Number of pair of earrings, X_1 = Number of necklaces

- **Objective Function:** Max. $Z = 1500X_1 + 2400X_2 + 3600X_3$

Subject to:

$4X_1 + 3X_2 + 10X_3 \leq 150$ (1/2 Carat Diamond)

$6X_1 + 5X_2 + 9X_3 \leq 160$ (1/4 Carat Diamond)

$X_1 + X_2 + X_3 \leq 50$ (Total Number of items)

With $X_1, X_2, X_3 \geq 0$; X_1, X_2, X_3 are integers

INTEGER PROGRAMMING: FORMULATION

- **Mixed ILP Problem:**

A textile company can use any or all of three different processes for weaving in standard white polyester fabric. Each of these production processes has a weaving machine setup cost and per square-meter processing cost. These costs and the capacities of each of the three production processes are shown below:

Process Number	Weaving machine Set –Up cost (Rs.)	Processing Cost (Rs.)	Maximum daily capacity (Sq. meter)
1	150	15	2000
2	240	10	3000
3	300	8	3500

The daily demand forecasts for its white polyester fabric is 4000 Sq. meter. The company's production manager wants to determine the optimal combination of the production processes and their actual daily production levels such that the total production cost is minimized.

INTEGER PROGRAMMING: FORMULATION

- **Decision Variables:** Let X_j be the production level for process j (j = 1, 2, 3) also let
  - $Y_j = 1$ if process j is used, and
  - $Y_j = 0$ if process j is not used

- **Objective Function:**
  - Minimize $Z = (15X_1 + 10X_2 + 8X_3) + (150Y_1 + 240Y_2 + 300Y_3)$

Subject to:

$X_1 + X_2 + X_3 = 4000$ (Daily Diamond)

$X_1 - 2000Y_1 \leq 0$ (Daily Capacity of Process-1)

$X_2 - 3000Y_2 \leq 0$ (Daily Capacity of Process-2)

$X_3 - 3500Y_3 \leq 0$ (Daily Capacity of Process-3)

With $X_1, X_2, X_3 \geq 0$; $Y_j = 0$ or 1, $j = 1, 2, 3$

INTEGER PROGRAMMING: FORMULATION

- **Zero–One ILP Problem:**

A real estate development firm, Peterson and Johnson, is considering five possible development projects. The following table shows the estimated long-run profit (net present value) that each project would generate, as well as the amount of investment required to undertake the project, in units of millions of dollars.

	Development Project				
	1	2	3	4	5
Estimated profit	1	1.8	1.6	0.8	1.4
Capital required	6	12	10	4	8

The owners of the firm, Dave Peterson and Ron Johnson, have raised $20 million of investment capital for these projects. Dave and Ron now want to select the combination of projects that will maximize their total estimated long-run profit (net present value) without investing more that $20 million. Formulate a Binary Integer Programming (0–1) model for this problem.

INTEGER PROGRAMMING: FORMULATION

Let $x_1 = 1$ if invest in project 1; 0 if not

$x_2 = 1$ if invest in project 2; 0 if not

$x_3 = 1$ if invest in project 3; 0 if not

$x_4 = 1$ if invest in project 4; 0 if not

$x_5 = 1$ if invest in project 5; 0 if not

Maximize $NPV = x_1 + 1.8x_2 + 1.6x_3 + 0.8x_4 + 1.4x_5$

subject to $6x_1 + 12x_2 + 10x_3 + 4x_4 + 8x_5 \leq 20$

and $x_1,\ x_2,\ x_3,\ x_4,\ x_5$, are binary variables.

METHODS FOR SOLVING ILP PROBLEMS

1. Rounding–Off A non–integer solution

2. Cutting–Plane Method
   - (developed by: Ralph E. Gomory)

3. Branch–and–Bound Method
   - (Developed By: A.H. Land and A. G. Doing)

HARRISON ELECTRIC COMPANY EXAMPLE OF INTEGER PROGRAMMING

The Company produces two products popular with home renovators, old-fashioned chandeliers and ceiling fans Both the chandeliers and fans require a two-step production process involving wiring and assembly It takes about 2 hours to wire each chandelier and 3 hours to wire a ceiling fan Final assembly of the chandeliers and fans requires 6 and 5 hours respectively The production capability is such that only 12 hours of wiring time and 30 hours of assembly time are available Each chandelier produced nets the firm \$7 and each fan \$6 Harrison's production mix decision can be formulated using LP as follows:

Maximize profit = $\$7X_1 + \$6X_2$

subject to

$2X_1 + 3X_2 \leq 12$ **(wiring hours)**

$6X_1 + 5X_2 \leq 30$ **(assembly hours)**

$X_1, X_2 \geq 0$ **(nonnegative)**

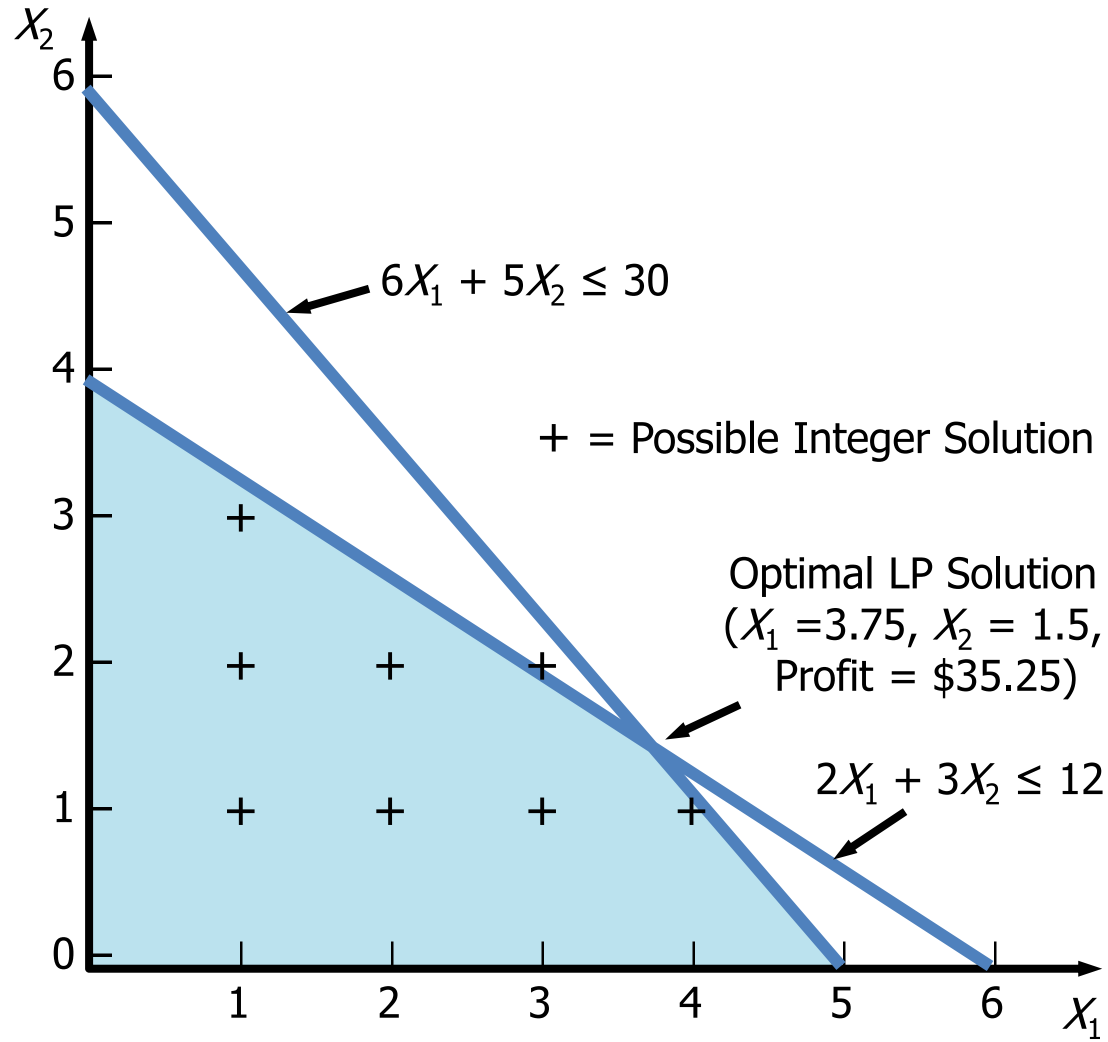
where

X_1 **= number of chandeliers produced**

X_2 **= number of ceiling fans produced**

HARRISON ELECTRIC COMPANY EXAMPLE OF INTEGER PROGRAMMING

- The Harrison Electric Problem



HARRISON ELECTRIC COMPANY EXAMPLE OF INTEGER PROGRAMMING

- The production planner Wes recognizes this is an integer problem
- His first attempt at solving it is to round the values to $X_1 = 4$ and $X_2 = 2$
- However, this is not feasible
- Rounding X_2 down to 1 gives a feasible solution, but it may not be *optimal*
- This could be solved using the *enumeration* method
- Enumeration is generally not possible for large problems

HARRISON ELECTRIC COMPANY EXAMPLE OF INTEGER PROGRAMMING

- **INTEGER SOLUTIONS**

CHANDELIERS (X_1)	CEILING FANS (X_2)	PROFIT ($\$7X_1 + \$6X_2$)
0	0	$0
1	0	7
2	0	14
3	0	21
4	0	28
5	0	35 ← Optimal solution to integer programming problem
0	1	6
1	1	13
2	1	20
3	1	27
4	1	34 ← Solution if rounding is used
0	2	12
1	2	19
2	2	26
3	2	33
0	3	18
1	3	25
0	4	24

HARRISON ELECTRIC COMPANY EXAMPLE OF INTEGER PROGRAMMING

- The rounding solution of $X_1 = 4$, $X_2 = 1$ gives a profit of \$34
- The optimal solution of $X_1 = 5$, $X_2 = 0$ gives a profit of \$35
- The optimal integer solution is less than the optimal LP solution
- An integer solution can *never* be better than the LP solution and is *usually* a lesser solution

METHODS FOR SOLVING ILP PROBLEMS

1. Rounding–Off A non–integer solution
2. Cutting–Plane Method
   - (developed by: Ralph E. Gomory)
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THE CUTTING–PLANE ALGORITHM

An Algorithm for solving Pure integer and mixed integer programming problems has been developed by Ralph E. Gomory

1. Relax the integer requirements.
2. Solve the resulting LP problem using Simplex Method.
3. If all the basic variables have integer values, Optimality of the Integer programming problem is reached. So go step 7; otherwise go to step 4.
4. Examine the constraints corresponding to the current optimal solution. For each Basic Variable with non-integer solution in the current optimal table, find the fractional part ,f_i , Therefore, $b_i = [b_i] + f_i$, where $[b_i]$ is the integer part of b_i, and f_i is the fractional part of b_i.
5. Choose the largest fraction among various f_i ; i.e. Max (f_i). Treat the constraint corresponding to the maximum fraction as the source row (equation). Based on the source equation, develop an additional constraint (Gomory's constraint / fractional cut) as shown:
   - $-f_i = S_i$ – Summation ((f_i)(Non–Basic Variable))
6. Add the fractional cut as the last row in the latest optimal table and proceed further using dual simplex method, and find the new optimum solution. If the new optimum solution is integer then go to step 7; otherwise go to step 4.
7. Print the integer solution [X's and Z – Values]

THE CUTTING–PLANE ALGORITHM

EXAMPLE:

Max. $Z = 5X_1 + 8X_2$

Subject to:

$X_1 + 2X_2 \leq 8$

$4X_1 + X_2 \leq 10$

$X_1, X_2 \geq 0$ and integers

Standard Form:

Max. $Z = 5X_1 + 8X_2 + 0S_1 + 0S_2$

Subject to:

$X_1 + 2X_2 + S_1 = 8$

$4X_1 + X_2 + S_2 = 10$

$X_1, X_2, S_1,$ and $S_2 \geq 0$ and integers

THE CUTTING–PLANE ALGORITHM (Cont...)

Initial Table:

Contribution Per Unit C_j		5	8	0	0		
C_{Bi}	Basic Variables (B)	X_1	X_2	S_1	S_2	SOLUTION	Ratio
0	S_1	1	2	1	0	8	8/2 = 4*
0	S_2	4	1	0	1	10	10/1 = 10
Total Profit (Z_j)		0	0	0	0	0	
Net Contribution ($C_j – Z_j$)		5	8*	0	0		

Iteration # 1:

Contribution Per Unit C_j		5	8	0	0		
C_{Bi}	Basic Variables (B)	X_1	X_2	S_1	S_2	SOLUTION	Ratio
8	X_2	1/2	1	1/2	0	4	8
0	S_2	7/2	0	-1/2	1	6	12/7*
Total Profit (Z_j)		4	8	4	0	32	
Net Contribution ($C_j – Z_j$)		1*	0	-4	0		

THE CUTTING–PLANE ALGORITHM (Cont…)

Iteration # 2:

CB_i	C_j	5	8	0	0	Solution
	Basic variable	X_1	X_2	S_1	S_2	
8	X_2	0	1	4/7	−1/7	22/7
5	X_1	1	0	−1/7	2/7	12/7
	Z_j	5	8	27/7	2/7	236/7
	$C_j - Z_j$	0	0	−27/7	−2/7	

All the Values of $(C_j - Z_j) \leq 0$; So, the current solution is optimal for linear programming.

$$\mathbf{X_1 = 12/7, X_2 = 22/7 \text{ and } Z = 236/7}$$

Since the values of the decision variables X_1 & X_2 are not integers, so, the solution is not optimum for Integer Programming.

STEP #4: **Summary of Integer & Fractional Parts**

Basic Variable in the above Optimal table	b_i	$[b_i] + f_i$
X_1	12/7	1 + (5/7)
X_2	22/7	3 + (1/7)

THE CUTTING–PLANE ALGORITHM (Cont...)

STEP # 5: Because, the fractional part,f_1, is the maximum. So, Select the Row "X_1" as the Source row for developing first cut.

$$12/7 = X_1 - 1/7S_1 + 2/7S_2 \rightarrow (1+ 5/7) = X_1 + (-1+6/7)S_1 + (0+2/7)S_2$$

The Corresponding fractional cut is:

$$-f_i = S_i - \text{Summation } ((f_i)(\text{Non–Basic Variable}))$$

$$\mathbf{-5/7 = S_3 - 6/7S_1 - 2/7S_2}$$

STEP # 6: This cut is added to the table which we get in Iteration # 2 (Optimal Table Solution for Linear Programming); and further solved using dual simplex method.

CB_i	C_j	5	8	0	0	0	
	Basic variable	X_1	X_2	S_1	S_2	S_3	Solution
8	X_2	0	1	4/7	−1/7	0	22/7
5	X_1	1	0	−1/7	2/7	0	12/7
0	S_3	0	0	−6/7	$\boxed{-2/7}$	1	−5/7*
	Z_j	5	8	27/7	2/7	0	236/7
	$C_j - Z_j$	0	0	−27/7	−2/7*	0	

THE CUTTING–PLANE ALGORITHM (Cont...)

Only the third row (Containing S_3) has a negative solution value. Therefore, S_3 (LEAVING Variable) leaves the basis.

CB_i	C_j / Basic variable	5 X_1	8 X_2	0 S_1	0 S_2	0 S_3	Solution
8	X_2	0	1	4/7	−1/7	0	22/7
5	X_1	1	0	−1/7	2/7	0	12/7
0	S_3	0	0	−6/7	$\boxed{-2/7}$	1	−5/7*
	Z_j	5	8	27/7	2/7	0	236/7
	$C_j - Z_j$	0	0	−27/7	−2/7*	0	

For ENTERING Variable;

Ratio = $(C_j - Z_j)$ / (Pivot Row <0)

--	**--**	**9/2**	**1**	**--**

The smallest ratio is "1" and the corresponding variable is "S_2". So, the variable "S_2" enters the basis.

THE CUTTING–PLANE ALGORITHM (Cont...)

Contribution Per Unit	C_j	5	8	0	0	0	
C_{Bi}	Basic Variables (B)	X_1	X_2	S_1	S_2	S_3	SOLUTION
8	X_2	0	1	1	0	- 1/2	7/2
5	X_1	1	0	-1	0	1	1
0	S_2	0	0	3	1	-7/2	5/2
Total Profit (Z_j)		5	8	3	0	1	33
Net Contribution ($C_j – Z_j$)		0	0	-3	0	-1	

The Solution is still non-integer. So, develop a fractional cut. The Basic variables X_2 and S_2 are not integers.

STEP #4:

Summary of Integer & Fractional Parts

Basic Variable in the above Optimal table	b_i	$[b_i] + f_i$
X_2	7/2	3 + 1/2
S_2	5/2	2 + 1/2

STEP # 5: Here, the fractional parts are the same for X_2 & S_2. But, we preferred the fractional part of the X_2. So, Select the Row "X_2" as the Source row for developing Cut.

THE CUTTING–PLANE ALGORITHM (Cont…)

$7/2 = X_2 + S_1 - 1/2S_3 \rightarrow (3+1/2) = (1+0)X_1 + (1+0)S_1 + (-1+1/2)S_3$

The Corresponding fractional cut is:

$-f_i = S_i -$ Summation $((f_i)(\text{Non–Basic Variable}))$

$\mathbf{-1/2 = S_4 - 1/2S_3}$

STEP # 6: This cut is added to the above table; and further solved using dual simplex method.

CB_i	C_j	5	8	0	0	0	0	
	Basic variable	X_1	X_2	S_1	S_2	S_3	S_4	Solution
8	X_2	0	1	1	0	−1/2	0	7/2
5	X_1	1	0	−1	0	1	0	1
0	S_2	0	0	3	1	−7/2	0	5/2
0	S_4	0	0	0	0	$\boxed{-1/2}$	1	−1/2*
	Z_j	5	8	3	0	1	0	33
	$C_j - Z_j$	0	0	−3	0	−1*	0	

For ENTERING Variable;

Ratio = $(C_j - Z_j)$ / (Pivot Row <0)

--	--	--	--	2	--

The smallest positive ratio is "2" and the corresponding variable is "S_3". So, the variable "S_3" enters the basis.

THE CUTTING–PLANE ALGORITHM (Cont…)

Contribution Per Unit C_j		5	8	0	0	0	0	
C_{Bi}	Basic Variables (B)	X_1	X_2	S_1	S_2	S_3	S_4	SOLUTION
8	X_2	0	1	1	0	0	-1	4
5	X_1	1	0	-1	0	0	2	0
0	S_2	0	0	3	1	0	-7	6
0	S_3	0	0	0	0	1	-2	1
Total Profit (Z_j)		5	8	3	0	0	2	32
Net Contribution ($C_j - Z_j$)		0	0	-3	0	0	-2	

So, The values of all the basic variables are integers. So, the optimality is reached and the corresponding results are summarized as follows:

$X_1 = 0$, $X_2 = 4$ and Z (Optimum) = 32

PRACTICE QUESTION

The Company produces two products popular with home renovators, old-fashioned chandeliers and ceiling fans Both the chandeliers and fans require a two-step production process involving wiring and assembly It takes about 2 hours to wire each chandelier and 3 hours to wire a ceiling fan Final assembly of the chandeliers and fans requires 6 and 5 hours respectively The production capability is such that only 12 hours of wiring time and 30 hours of assembly time are available Each chandelier produced nets the firm $7 and each fan $6 Harrison's production mix decision can be formulated using LP as follows:

Maximize profit = $\$7X_1 + \$6X_2$

subject to

$2X_1 + 3X_2 \leq 12$ **(wiring hours)**

$6X_1 + 5X_2 \leq 30$ **(assembly hours)**

$X_1, X_2 \geq 0$ **(nonnegative)**

where

X_1 **= number of chandeliers produced**

X_2 **= number of ceiling fans produced**

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BRANCH–AND–BOUND METHOD

- Creates and solves a sequence of sub-problems to the original problem that are increasingly more restrictive until an optimal solution is found

- **BRANCHING:**
  - Selection of an integer value of a decision variable to examine for a possible integer solution to a problem
  - "If the solution to the linear programming problem contains non-integer values for some or all decision variables, then the solution space is reduced by introducing constraints with respect to any one of those decision variables. If the value of the decision variable "X_1" is 2.5, then two more problems will be created by using each of the following constraints. $X_1 \leq 2$ and $X_1 \geq 3$.

- **BOUND:**
  - An upper or lower limit on the value of the objective function at a given stage of the analysis of an integer programming problem.
  - **LOWER BOUND:** The lower bound at a node is the value of the objective function corresponding to the <u>truncated values (integer parts)</u> of the decision variables of the problem in that node.
  - **UPPER BOUND:** The upper bound at a node is the value of the objective function corresponding to the linear programming solution in that node.

BRANCH–AND–BOUND METHOD (Cont…)

- **FATHOMED SUBPROBLEM / NODE:** A problem is said to be fathomed if any one of the following three conditions is true:
  1. The values of the decision variables of the problem are integer.
  2. The upper bound of the problem which has non–integer values for its decision variables is not greater than the current best lower bound.
  3. The problem has infeasible solution.

  This means that further branching from this type of fathomed nodes is not necessary.

- **CURRENT BEST LOWER BOUND:** This is the best lower bound (highest in the case of maximization problem among the lower bounds of all the fathomed nodes. Initially, it is assumed as infinity for the root node.

BRANCH–AND–BOUND METHOD (Cont…)

BRANCH & BOUND ALGORITHM APPLIED TO MAXIMIZATION PROBLEM:

1. Solve the given linear programming problem graphically or using iterative method. Set, the current best lower bound Z_B as ∞.
2. Check, Whether the problem has integer solution. If yes, print the current solution as the optimal solution and stop; Otherwise go to Step–3.
3. Identify the variable X_k which has the maximum fractional part as the branching variable. (In case of tie, select the variable which has the highest objective function coefficient.)
4. Create two more problems by including each of the following constraints to the current problem and solve them.
    a. $X_k \leq$ Integer part of X_k
    b. $X_k \geq$ Next Integer of X_k
5. If any one of the new sub-problems has infeasible solution or fully integer values for the decision variables, the corresponding node is fathomed. If a new node has integer values for the decision variables, update the current best lower bound as the lower bound of that node if its lower bound is greater than the previous current best lower bound.
6. Are all terminal nodes fathomed? If answer is yes, go to step–7; otherwise, identify the node with the highest lower bound and go to step–3.
7. Select the solution of the problem with respect to the fathomed node whose lower bound is equal to the current best lower bound as the optimal solution.

BRANCH–AND–BOUND METHOD (Cont…)

Max. $Z = 10X1 + 20X2$

Subject to:

$$6X1 + 8X2 \le 48$$
$$X1 + 3X2 \le 12$$

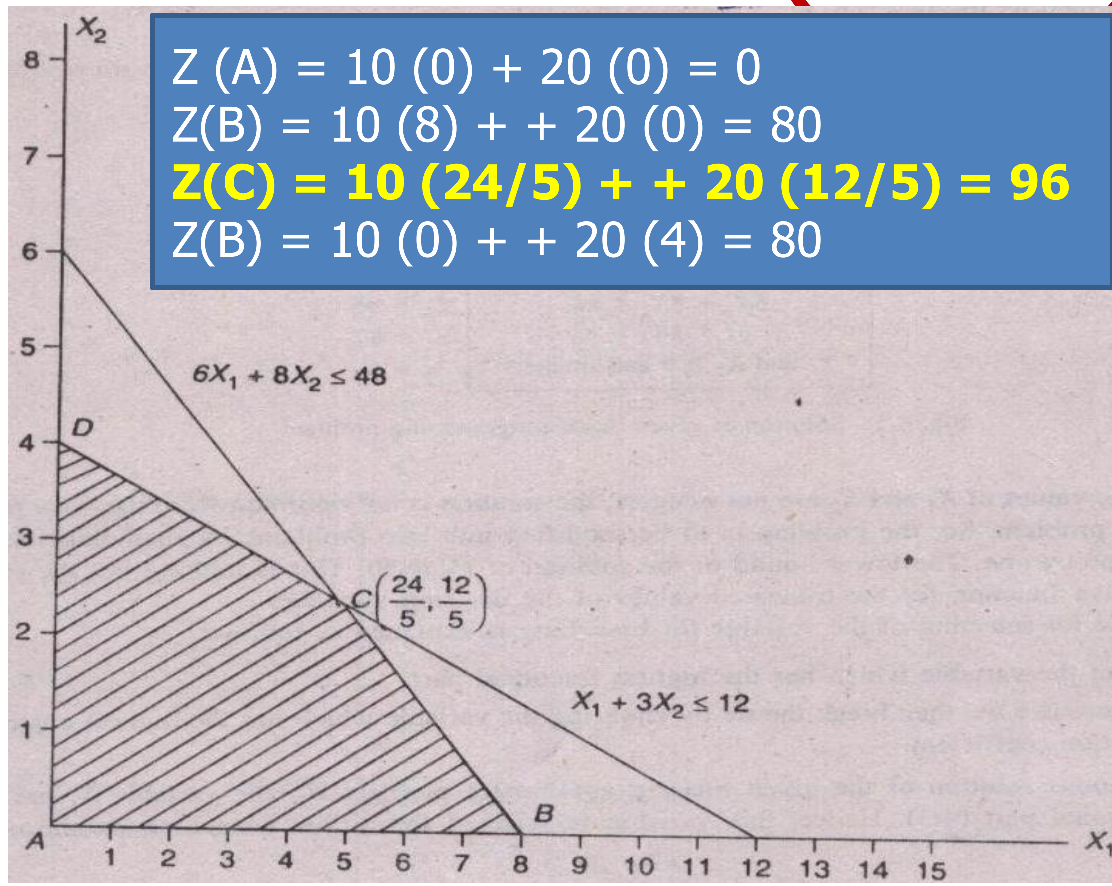
$X1, X2 \ge 0$ and integers

Z (A) = 10 (0) + 20 (0) = 0

Z(B) = 10 (8) + + 20 (0) = 80

Z(C) = 10 (24/5) + + 20 (12/5) = 96

Z(B) = 10 (0) + + 20 (4) = 80



- Z_U = Upper bound = Z (Optimum) of LP Problem.
- Z_L = Lower bound w. r. t. the truncated values of the decision variables
- Z_B = Current Best Lower Bound

P_1

Maximize $Z = 10X_1 + 20X_2$ subject to	$X_1 = 24/5$
$6X_1 + 8X_2 \le 48$	$X_2 = 12/5$
$X_1 + 3X_2 \le 12$	$Z_U = 96$
	$Z_L = 80$
X_1 and $X_2 \ge 0$ and integers	$Z_B = \infty$

BRANCH–AND–BOUND METHOD (Cont…)

P_1

Maximize $Z = 10X_1 + 20X_2$
subject to

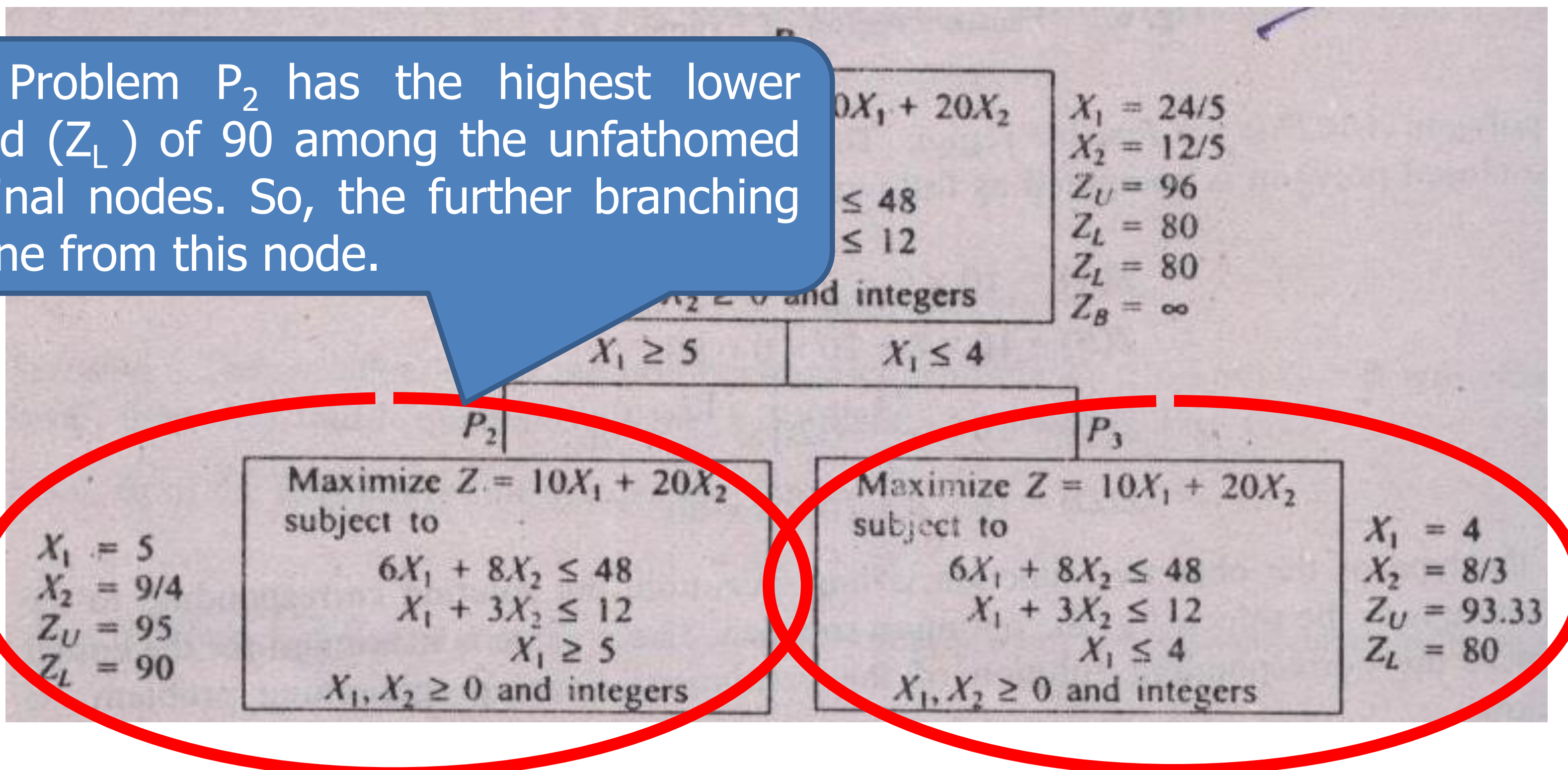
$$6X_1 + 8X_2 \le 48$$
$$X_1 + 3X_2 \le 12$$

X_1 and $X_2 \ge 0$ and integers

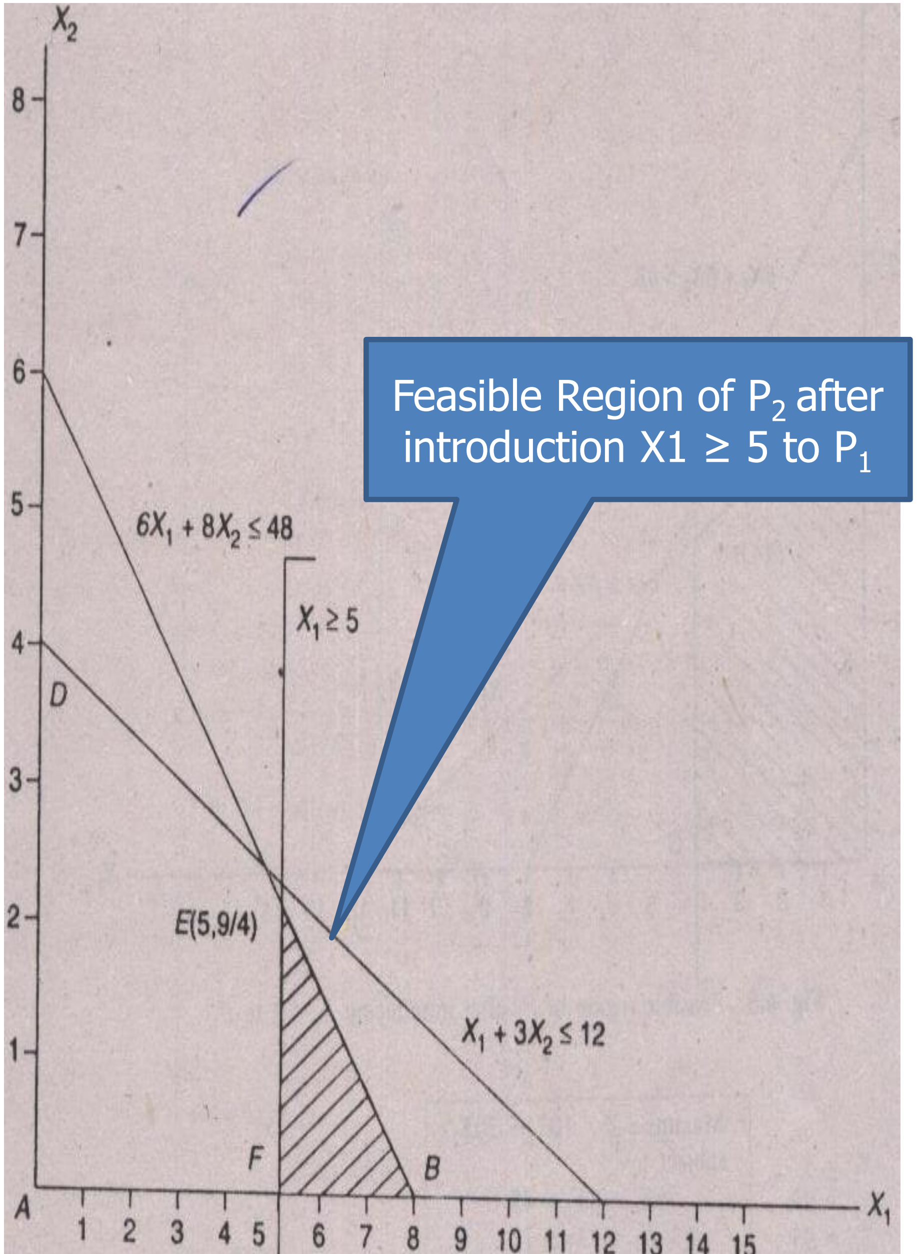
$X_1 = 24/5$
$X_2 = 12/5$
$Z_U = 96$
$Z_L = 80$
$Z_B = \infty$

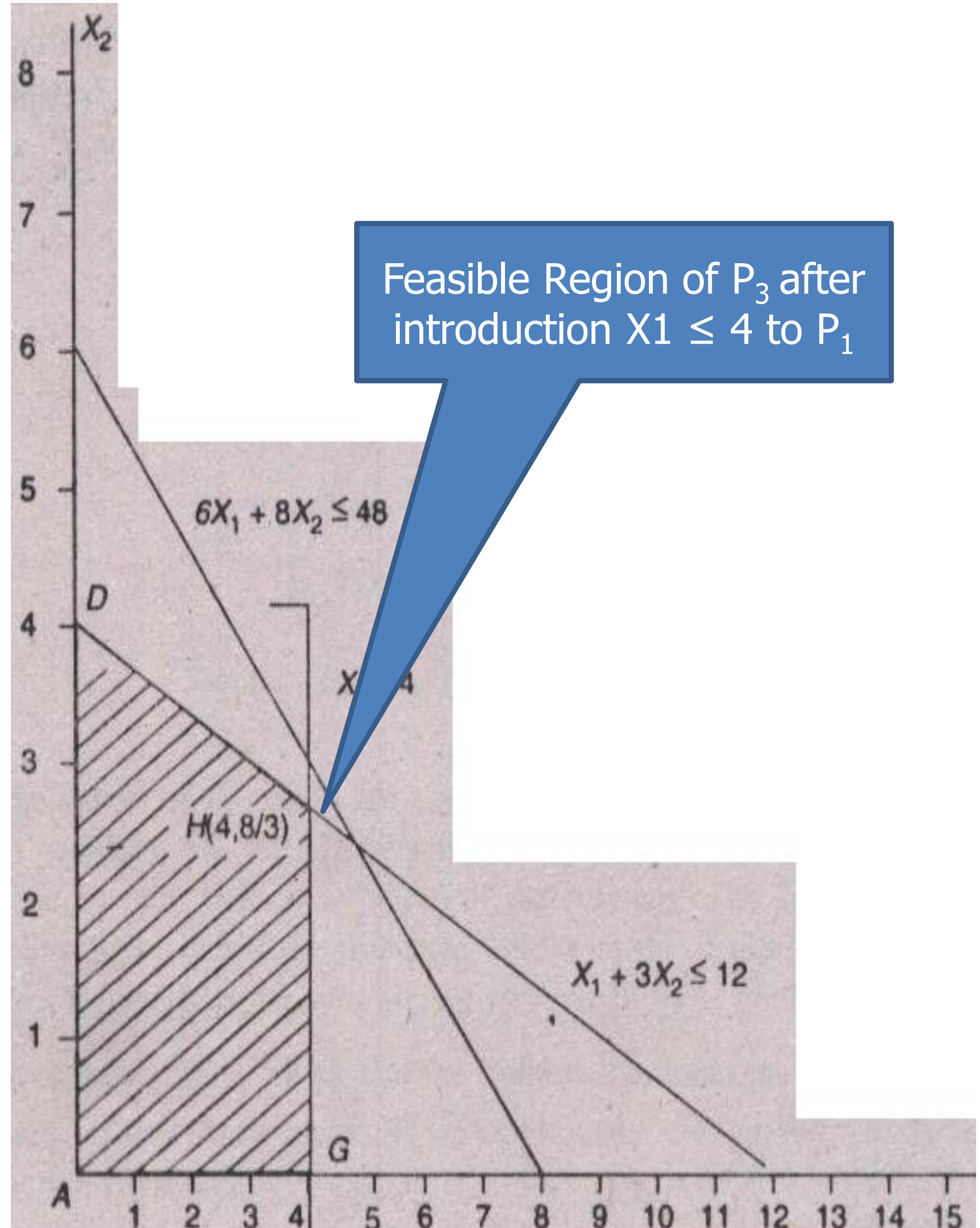
In Problem (P_1), X_1 has the highest fractional part 4/5. Hence; "X_1" is selected for further branching.

The Problem P_2 has the highest lower bound (Z_L) of 90 among the unfathomed terminal nodes. So, the further branching is done from this node.



BRANCH–AND–BOUND METHOD (Cont…)





BRANCH–AND–BOUND METHOD (Cont…)

P_2

$X_1 = 5$
$X_2 = 9/4$
$Z_U = 95$
$Z_L = 90$

Maximize $Z = 10X_1 + 20X_2$
subject to
$$6X_1 + 8X_2 \le 48$$
$$X_1 + 3X_2 \le 12$$
$$X_1 \ge 5$$
$$X_1, X_2 \ge 0 \text{ and integers}$$

$X_2 \ge 3$ $\quad$ $X_2 \le 2$

P_4

Maximize $Z = 10X_1 + 20X_2$
subject to
$$6X_1 + 8X_2 \le 48$$
$$X_1 + 3X_2 \le 12$$
$$X_1 \ge 5$$
$$X_2 \ge 3$$
$$X_1, X_2 \ge 0 \text{ and integers}$$

(Infeasible solution)
(fathomed)

P_5

Maximize $Z = 10X_1 + 20X_2$
subject to
$$6X_1 + 8X_2 \le 48$$
$$X_1 + 3X_2 \le 12$$
$$X_1 \ge 5$$
$$X_2 \le 2$$
$$X_1, X_2 \ge 0 \text{ and integers}$$

$X_1 = 16/3$
$X_2 = 2$
$Z_U = 93.33$
$Z_L = 90$

Fig. 6.6 Branching from P_2.

BRANCH–AND–BOUND METHOD (Cont…)

P_2

$X_1 = 5$
$X_2 = 9/4$
$Z_U = 95$
$Z_L = 90$

Maximize $Z = 10X_1 + 20X_2$
subject to
$6X_1 + 8X_2 \leq 48$
$X_1 + 3X_2 \leq 12$
$X_1 \geq 5$
$X_1, X_2 \geq 0$ and integers

$X_2 \geq 3$ | $X_2 \leq 2$

P_4

Maximize $Z = 10X_1 + 20X_2$
subject to
$6X_1 + 8X_2 \leq 48$
$X_1 + 3X_2 \leq 12$
$X_1 \geq 5$
$X_2 \geq 3$
$X_1, X_2 \geq 0$ and integers

P_5

Maximize $Z = 10X_1 + 20X_2$
subject to
$6X_1 + 8X_2 \leq 48$
$X_1 + 3X_2 \leq 12$
$X_1 \geq 5$
$X_2 \leq 2$
$X_1, X_2 \geq 0$ and integers

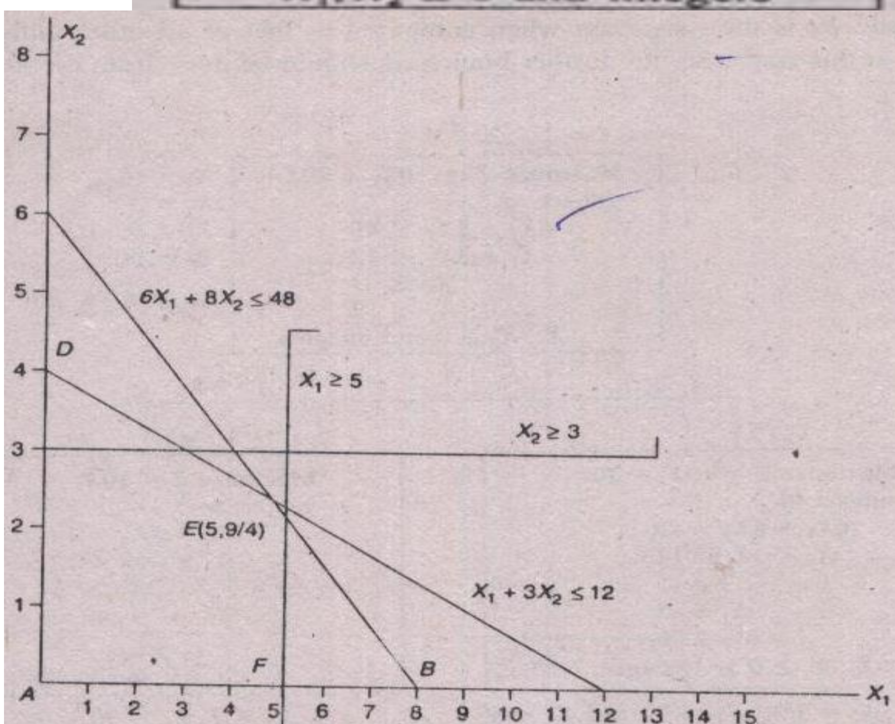


Fig. 6.7 Infeasible region of P_4 after introducing $X_2 \geq 3$ to P_2.

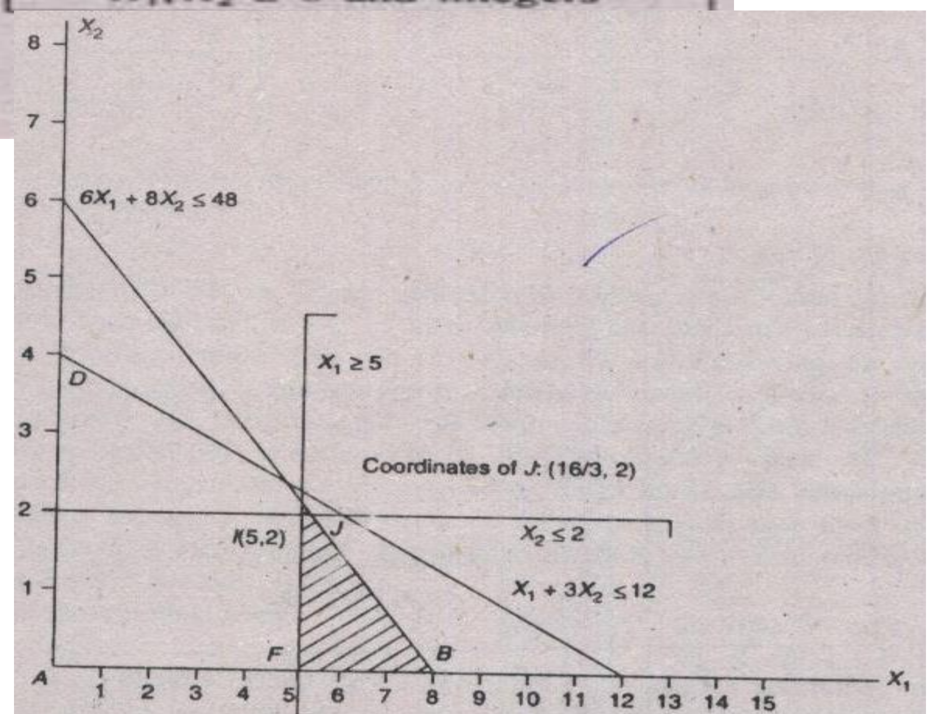


Fig. 6.8 Feasible region of P_5 after introducing $X_1 \leq 2$ to P_2.

BRANCH–AND–BOUND METHOD (Cont…)

P_5

Maximize $Z = 10X_1 + 20X_2$
subject to
$$6X_1 + 8X_2 \le 48$$
$$X_1 + 3X_2 \le 12$$
$$X_1 \ge 5$$
$$X_2 \le 2$$
$X_1, X_2 \ge 0$ and integers

$X_1 = 16/3$
$X_2 = 2$
$Z_U = 93\ 1/3$
$Z_L = 90$

$X_1 \ge 6$ $\quad$ $X_1 \le 5$

P_6

Maximize $Z = 10X_1 + 20X_2$
subject to
$$6X_1 + 8X_2 \le 48$$
$$X_1 + 3X_2 \le 12$$
$$X_1 \ge 5$$
$$X_2 \le 2$$
$$X_1 \ge 6$$
$X_1, X_2 \ge 0$ and integers

$X_1 = 6,\ X_2 = 3/2,\ Z_U = 90$ and $Z_L = 80$
(Fathomed)

P_7

Maximize $Z = 10X_1 + 20X_2$
subject to
$$6X_1 + 8X_2 \le 48$$
$$X_1 + 3X_2 \le 12$$
$$X_1 \ge 5$$
$$X_2 \le 2$$
$$X_1 \le 5$$
$X_1, X_2 \ge 0$ and integers

$X_1 = 5,\ X_2 = 2,\ Z_U = 90 = Z_L,\ Z_B = 90$
(Fathomed)

Fig. 6.9 Branching from P_5.

BRANCH–AND–BOUND METHOD (Cont...)

P_5

Maximize $Z = 10X_1 + 20X_2$
subject to
$6X_1 + 8X_2 \le 48$
$X_1 + 3X_2 \le 12$
$X_1 \ge 5$
$X_2 \le 2$
$X_1, X_2 \ge 0$ and integers

$X_1 = 16/3$
$X_2 = 2$
$Z_U = 93\ 1/3$
$Z_L = 90$

$X_1 \ge 6$ | $X_1 \le 5$

P_6

Maximize $Z = 10X_1 + 20X_2$
subject to
$6X_1 + 8X_2 \le 48$
$X_1 + 3X_2 \le 12$
$X_1 \ge 5$
$X_2 \le 2$
$X_1 \ge 6$
$X_1, X_2 \ge 0$ and integers

$X_1 = 6,\ X_2 = 3/2,\ Z_U = 90$ and $Z_L = 80$
(Fathomed)

P_7

Maximize $Z = 10X_1 + 20X_2$
subject to
$6X_1 + 8X_2 \le 48$
$X_1 + 3X_2 \le 12$
$X_1 \ge 5$
$X_2 \le 2$
$X_1 \le 5$
$X_1, X_2 \ge 0$ and integers

$X_1 = 5,\ X_2 = 2,\ Z_U = 90 = Z_L,\ Z_B = 90$
(Fathomed)

Fig. 6.9 Branching from P_5.

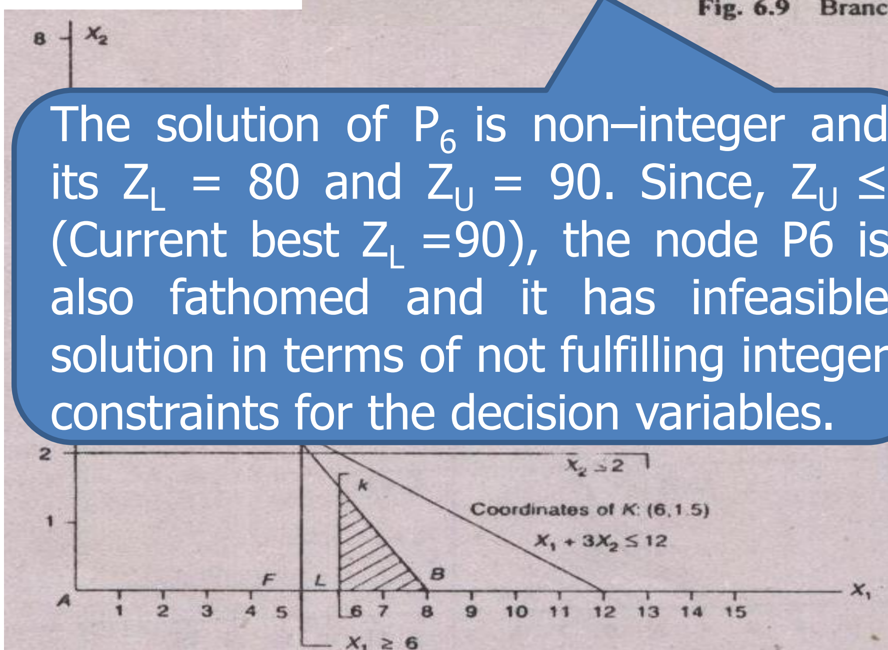
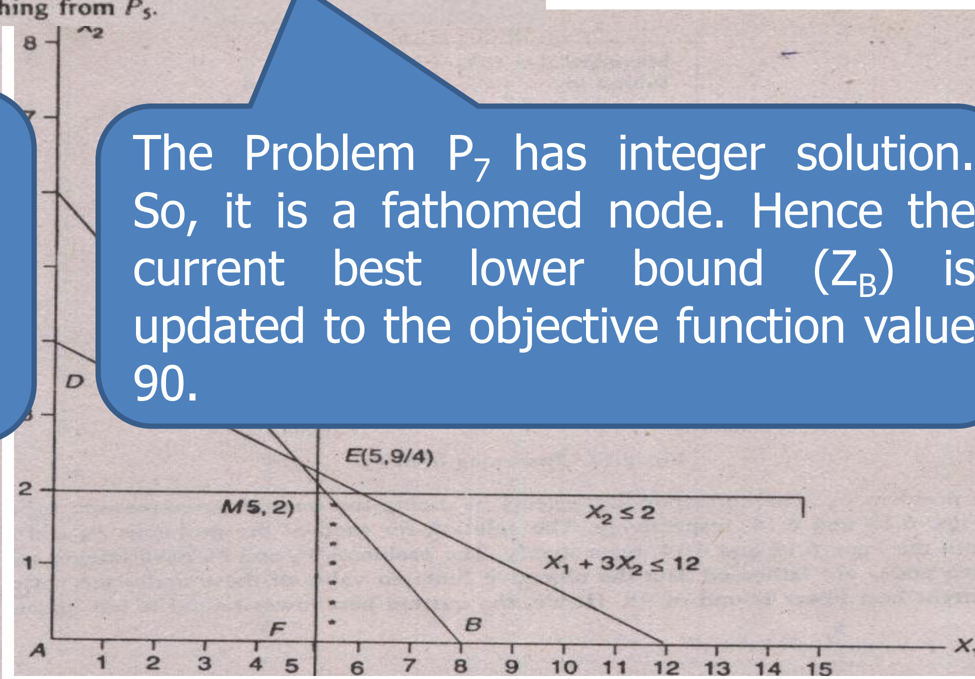


Fig. 6.10 Feasible region of P_6 after introducing $X_1 \ge 6$ to P_5.

The solution of P_6 is non–integer and its Z_L = 80 and Z_U = 90. Since, $Z_U \le$ (Current best Z_L =90), the node P6 is also fathomed and it has infeasible solution in terms of not fulfilling integer constraints for the decision variables.



The Problem P_7 has integer solution. So, it is a fathomed node. Hence the current best lower bound (Z_B) is updated to the objective function value 90.

BRANCH–AND–BOUND METHOD (Cont…)

Now, the Only unfathomed terminal node is P_3. The further branching from this node is shown below:

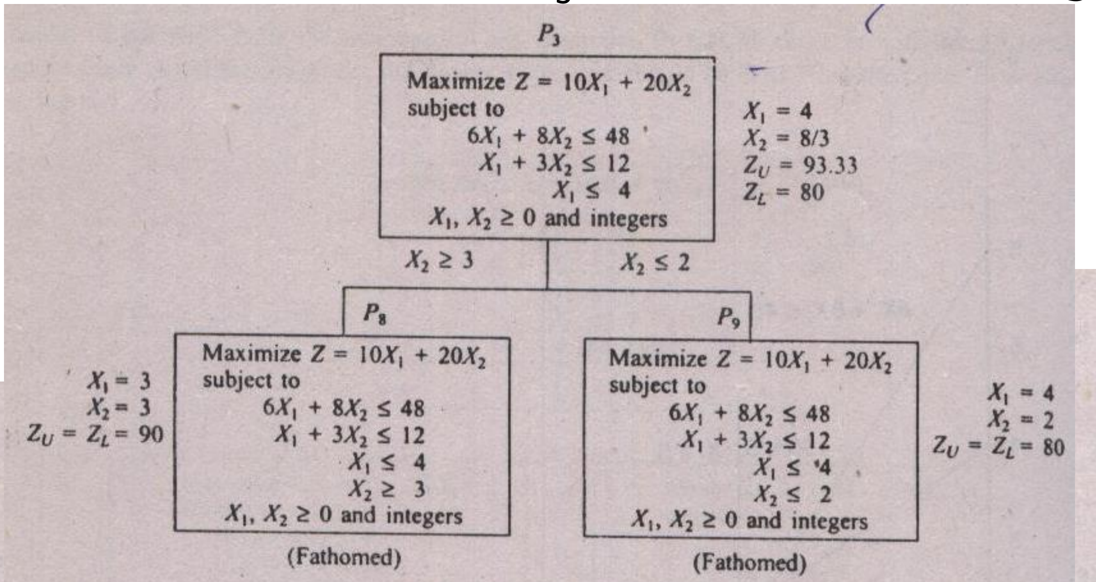


Fig. 6.12 Branching from P_3.

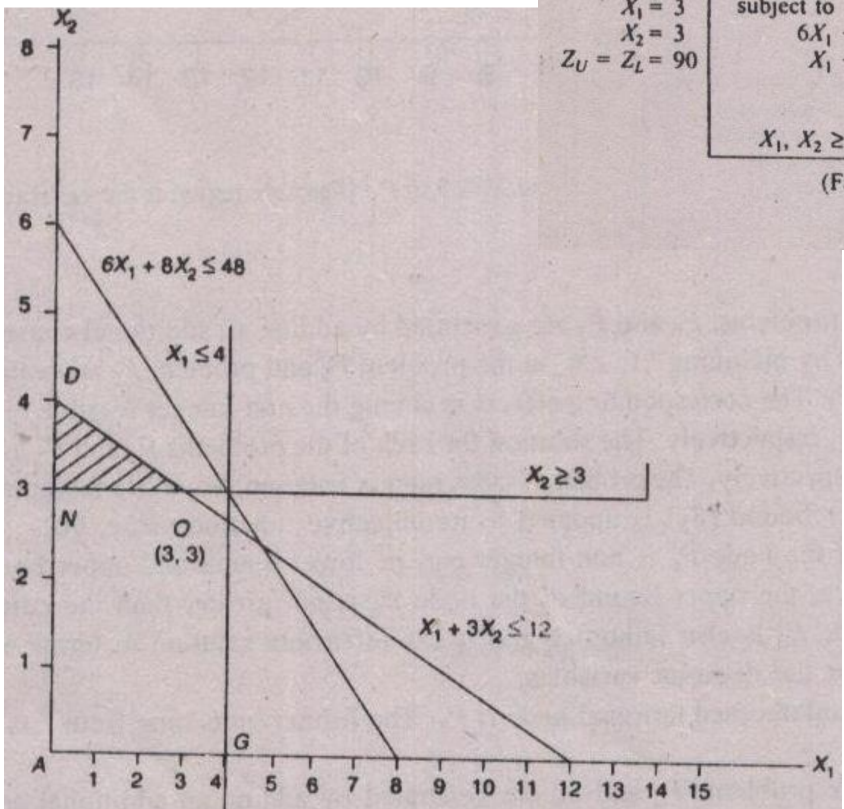


Fig. 6.13 Feasible region of P_8 after introducing $X_2 \ge 3$ to P_3.

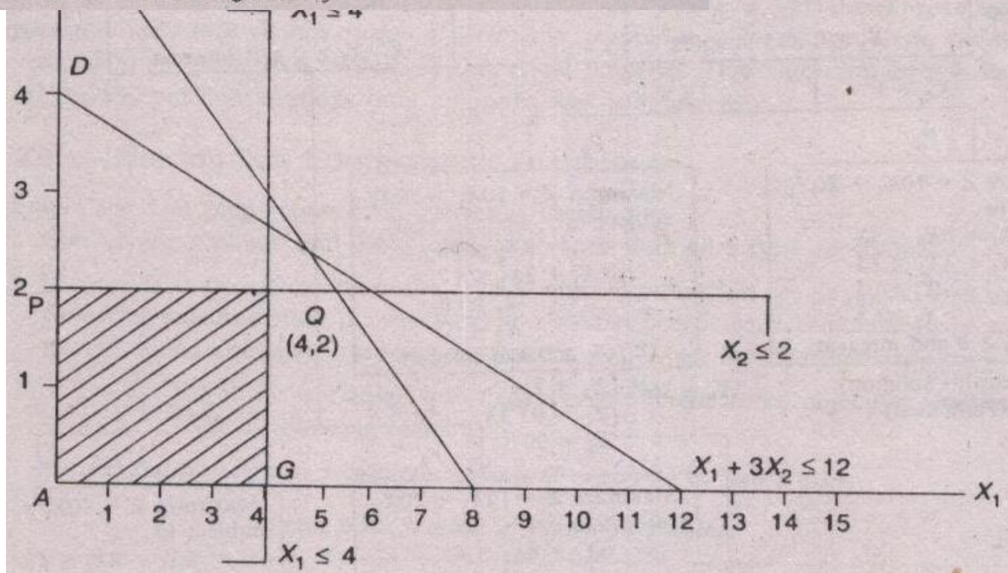


Fig. 6.14 Feasible region of P_9 after introducing $X_2 \le 2$ to P_3.

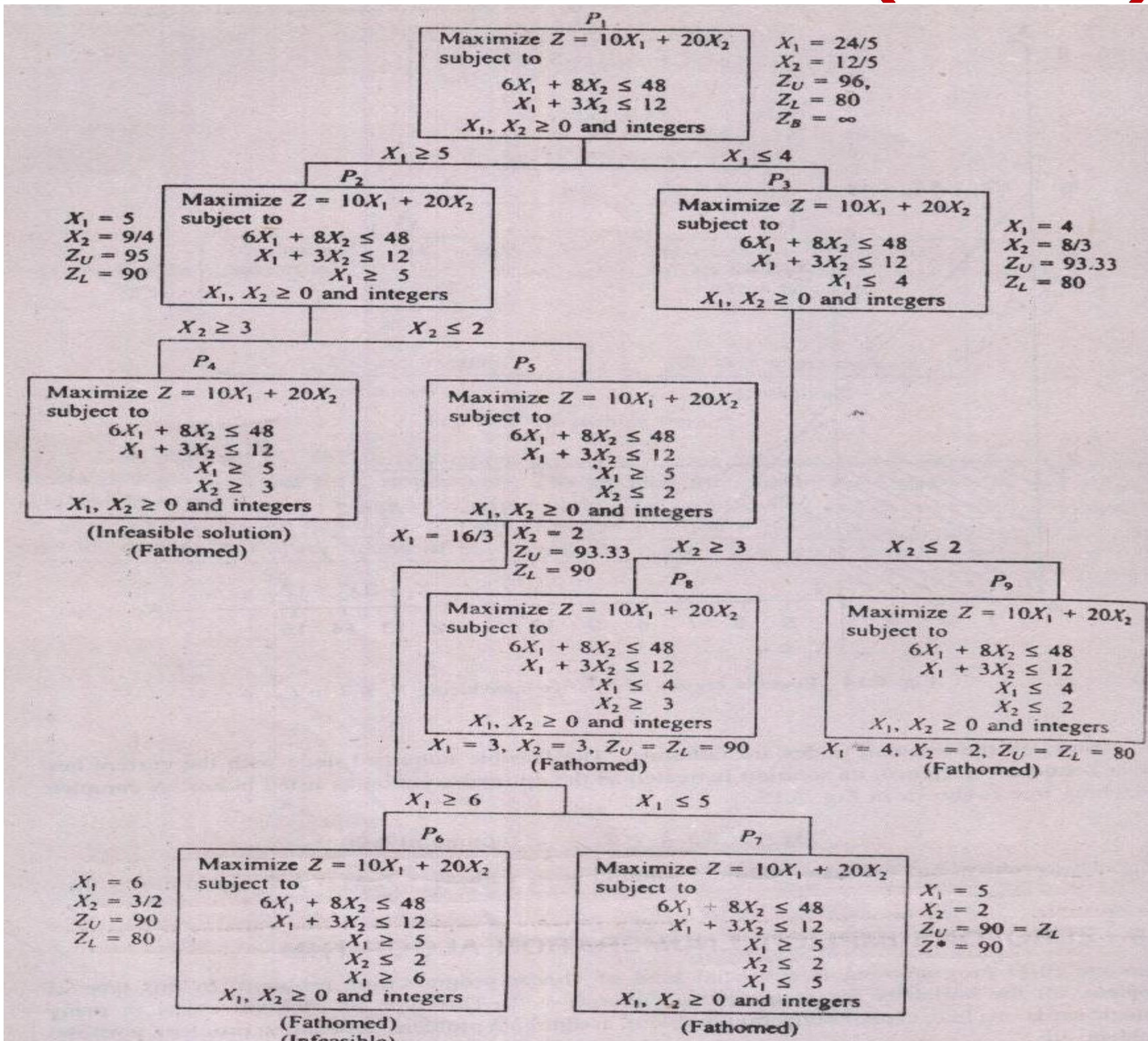
BRANCH–AND–BOUND METHOD (Cont…)

- The problems P_8 and P_9 have integer solution. So, these two nodes are fathomed.
- But the objective function value of these nodes are not greater than the current best lower bound of 90. Hence, the current best lower bound is not updated.
- Now, all the terminal nodes are fathomed. The feasible fathomed node with the current best lower bound is P_7.
- Hence, its solution is treated as the optimal solution: $X_1=5$, $X_2=2$, Z(Optimum) = 90

- NOTE: This Problem has alternative optimum solution at P_8 with $X_1=3$, $X_2=3$, Z(Optimum)=90

BRANCH–AND–BOUND METHOD (Cont…)

- Complete Tree:



BRANCH–AND–BOUND METHOD (Cont…)

❑PRACTICE QUESTION:

Max.: $Z = 6X1 + 8X2$

Subject to:

$$4X1 + 5X2 \leq 22$$

$$5X1 + 8X2 \leq 30$$

$X1, X2 \geq 0$ and integers

INTEGER PROGRAMMING PROBLEMS AND SENSITIVITY ANALYSIS

- Integer programming problems do not readily lend themselves to sensitivity analysis as only a relatively few of the infinite solution possibilities in a feasible solution space will meet integer requirements.

- Trial-and-error examination of a range of reasonable alternatives involving completely solving each revised problem is required

QUESTIONS