CHAPTER 9

Numerical Differentiation and Integration

9.1 NUMERICAL DIFFERENTIATION

Consider a set of values (x_i, y_i) of a function. The process of computing the derivative or derivatives of that function at some values of x from the given set of values is called *Numerical differentiation*. This may be done by first approximating the function by a suitable interpolation formula and then differentiating it as many times as desired.

If the values of x are equispaced and the derivative is required *near the beginning* of the table, we employ *Gregory–Newton forward interpolation formula*. If it is required *near the end* of the table, we use *Gregory–Newton backward interpolation formula*. For the values *near the middle* of the table, the derivative is calculated by means of *central difference interpolation formulae*.

If the values of x are not equispaced, we use, *Newton's divided difference interpolation formula* or *Lagrange's interpolation formula* to get the derivative value.

9.2 DERIVATIVES USING NEWTON'S FORWARD DIFFERENCE FORMULA

Newton's forward interpolation formula is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots \quad (9.1)$$

where $p = \dfrac{x - x_0}{h}$

Differentiating both sides of Eqn (9.1) with respect to p, we have

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \cdots \tag{9.2}$$

Now $\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{dy}{dp}\cdot\frac{1}{h}$ $\quad\left(\because \frac{dp}{dx} = \frac{1}{h}\right)$

$$\therefore \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0\right.$$

$$\left. + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 + \cdots\right] \tag{9.3}$$

At $x = x_0$, $p = 0$. Hence, putting $p = 0$ in Eqn (9.3), we get

$$\left.\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right] \tag{9.4}$$

Differentiating Eqn (9.3) again w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx} = \frac{1}{h}\times\frac{d}{dp}\left(\frac{dy}{dx}\right)$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{6p^2-18p+11}{12}\Delta^4 y_0 + \cdots\right] \tag{9.5}$$

Putting $p = 0$ in Eqn (9.5), we get

$$\left.\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right] \tag{9.6}$$

Similarly,

$$\left.\frac{d^3y}{dx^3}\right]_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right] \tag{9.7}$$

and so on.

Aliter: We know that $1 + \Delta = E = e^{hD}$, where

$$hD = \log(1+\Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \cdots$$

$$\therefore D = \frac{1}{h}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 \cdots\right]$$

$$D^2 = \frac{1}{h^2}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \cdots\right]^2$$

$$= \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \cdots\right]$$

and
$$D^3 = \frac{1}{h^3}\left[\Delta^3 - \frac{3}{2}\Delta^4 + \frac{7}{4}\Delta^5 - \cdots\right]$$

Now applying these identities to y_0,

$$Dy_0 = \left.\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$D^2 y_0 = \left.\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{2}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

and

$$D^3 y_0 = \left.\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \frac{7}{4}\Delta^5 y_0 - \cdots\right]$$

which are same as Eqns (9.4)–(9.7), respectively.

9.3 DERIVATIVES USING NEWTON'S BACKWARD DIFFERENCE FORMULA

Newton's backward interpolation formula is

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots \tag{9.8}$$

$$\text{where } p = \frac{x - x_n}{h} \tag{9.9}$$

$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{dy}{dp}\cdot\frac{1}{n} \qquad \left[\because \frac{dp}{dx} = \frac{1}{n}\right]$$

$$= \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n \cdots\right] \tag{9.10}$$

At $x = x_n$, $p = 0$. Hence, putting $p = 0$ in Eqn (9.10),

$$\left.\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right] \tag{9.11}$$

Again, differentiating Eqn (9.10) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\cdot\frac{dp}{dx} = \frac{1}{h}\cdot\frac{d}{dp}\left(\frac{dy}{dx}\right)$$

$$= \frac{1}{h^2}[\nabla^2 y_n + \frac{6p+6}{3!}\nabla^3 y_n + \frac{6p^2+18p+11}{12}\nabla^4 y_n + \cdots] \quad (9.12)$$

Putting $p = 0$, we get

$$\left.\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots] \quad (9.13)$$

Similarly,

$$\left.\frac{d^3y}{dx^3}\right]_{x=x_n} = \frac{1}{h^3}[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots] \quad (9.14)$$

and so on.

Aliter: We know that $1 - \nabla = E^{-1} = e^{-hD}$

$$-hD = \log(1-\nabla) = -[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \cdots]$$

$$\therefore D = \frac{1}{h}[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 \cdots]$$

$$D^2 = \frac{1}{h^2}[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \cdots]^2$$

$$= \frac{1}{h^2}[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \frac{5}{6}\nabla^5 + \cdots]$$

Similarly,

$$D^3 = \frac{1}{h^3}[\nabla^3 + \frac{3}{2}\nabla^4 + \frac{7}{4}\nabla^5 + \cdots]$$

Applying these identities to y_n, we get

$$Dy_n = \left.\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots]$$

$$D^2 y_n = \left.\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots]$$

and

$$D^3y_n = \left.\frac{d^3y}{dx^3}\right]_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \frac{7}{4}\nabla^5 y_n + \cdots\right]$$

Which are same as Eqns (9.11)–(9.14), respectively.

9.4 DERIVATIVES USING STIRLING'S FORMULA

Stirling's formula is

$$y = y_0 + \frac{p}{1!}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1^2)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$

$$+ \frac{p^2(p^2-1^2)}{4!}\Delta^4 y_{-2} + \cdots\Bigg] \qquad (9.15)$$

$$\text{where } p = \frac{x - x_0}{h} \qquad (9.16)$$

Now $\quad \dfrac{dy}{dx} = \dfrac{dy}{dp}\cdot\dfrac{dp}{dx} = \dfrac{dy}{dp}\cdot\dfrac{1}{h} \qquad \left[\because \dfrac{dp}{dx} = 1\right]$

$$\therefore \frac{dy}{dx} = \frac{1}{h}\left[\left\{\frac{\Delta y_0 + \Delta y_{-1}}{2}\right\} + p\Delta^2 y_{-1} + \frac{3(p^2-1)}{6}\left\{\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right\}\right.$$

$$\left. + \frac{(2p^3 - p)}{12}\Delta^4 y_{-2} + \cdots\right] \qquad (9.17)$$

At $x = x_0$, $p = 0$. Hence, putting $p = 0$, we get

$$\left.\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\left\{\frac{\Delta y_0 + \Delta y_{-1}}{2}\right\} - \frac{1}{6}\left\{\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right\}\right.$$

$$\left. + \frac{1}{30}\left\{\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right\} + \cdots\right] \qquad (9.18)$$

Differentiating Eqn (9.17) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_{-1} + p\left\{\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right\} + \frac{6p^2 - 1}{12}\Delta^4 y_{-2} + \cdots\right]$$

$$\therefore \left.\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} - \cdots] \tag{9.19}$$

Similarly,

$$\left.\frac{d^3y}{dx^3}\right]_{x=x_0} = \frac{1}{h^3}[\frac{1}{2}\{\Delta^3 y_{-1} + \Delta^3 y_{-2}\} + \cdots] \tag{9.20}$$

and so on.

In the same manner, we can use any other interpolation formula for computing the derivatives.

Note: Numerical differentiation should be performed only if it is clear from the tabulated values that differences of some order are constant. Otherwise, the method will involve errors of considerable magnitude and they go on increasing significantly as the derivatives of higher orders are computed. This is due to the fact that the original function $f(x)$ and the approximating function $\phi(x)$ may not differ much at the data points but $f'(x) - \phi'(x)$ may be large.

9.5 MAXIMA AND MINIMA OF TABULATED FUNCTION

Differentiating Newton's forward interpolation formula (Eqn (9.1)) with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{h}[\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \cdots] \tag{9.21}$$

We know that the maximum and minimum values of a function $y = f(x)$ can be found by equating dy/dx to zero and solving for x.

$\therefore$ From Eqn (9.21) $dy/dx = 0$

$$\Rightarrow \quad \Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \cdots = 0$$

Hence, by keeping only upto the third difference, we have

$$\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 = 0$$

Solving this for p, by substituting Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$ (which we get from the difference table), we get x as $x_0 + ph$, at which y is a maximum or minimum.

Example 9.1 Find the first, second and third derivatives of $f(x)$ at $x = 1.5$ if

x	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$	3.375	7.000	13.625	24.000	38.875	59.000

Solution We have to find the derivative at the point $x = 1.5$ which is at the beginning of the given data. Therefore, we use here the derivatives of Newton's forward interpolation formula. The forward difference table is as follows:

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.5	3.375				
		3.625			
2.0	7.000		3.000		
		6.625		0.750	
2.5	13.625		3.750		0
		10.375		0.750	
3.0	24.000		4.500		0
		14.875		0.750	
3.5	38.875		5.250		
		20.125			
4.0	59.000				

Here, $x_0 = 1.5, y_0 = 3.375, \Delta y_0 = 3.625, \Delta^2 y_0 = 3, \Delta^3 y_0 = 0.75$ and $h = 0.5$

Now, from Eqn (9.4), we have

$$\left.\frac{dy}{dx}\right]_{x=x_0} = f'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 - \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$\therefore f'(1.5) = \frac{1}{0.5}\left[3.625 - \frac{1}{2}(3) + \frac{1}{3}(0.75)\right] = 4.75$$

From Eqn (9.6), we have

$$\left.\frac{d^2 y}{dx^2}\right]_{x=x_0} = f''(x_0) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right]$$

$$\therefore \quad f''(1.5) = \frac{1}{(0.5)^2}[3 - 0.75] = 9$$

Again from Eqn (9.7), we have

$$\left.\frac{d^3y}{dx^3}\right]_{x=x_0} = f'''(x_0) = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right]$$

$$\therefore \quad f'''(1.5) = \frac{1}{(0.5)^3}(0.75) = 6$$

Example 9.2 The population of a certain town (as obtained from census data) is shown in the following table.

Year	1951	1961	1971	1981	1991
Population (in thousands)	19.96	36.65	58.81	77.21	94.61

Find the rate of growth of the population in the year 1981.

Solution . Here, we have to find the derivative at 1981 which is near the end of the table. Hence, we use the derivative of Newton's backward difference formula. The table of differences is as follows:

x (year)	y (population)	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1951	19.96				
		16.69			
1961	36.65		5.47		
		22.16		−9.23	
1971	58.81		−3.76		11.99
		18.40		2.76	
1981	77.21		−1		
		17.40			
1991	94.61				

Hence, $h = 10$, $x_n = 1991$, $\nabla y_n = 17.4$, $\nabla^2 y_n = -1$, $\nabla^3 y_n = 2.76$ and $\nabla^4 y_n = 11.99$

We know from Eqn (9.10) that

$$\left.\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2}\nabla^2 y_n + \frac{3p^2+6p+2}{6}\nabla^3 y_n + \frac{2p^3+9p^2+11p+3}{12}\nabla^4 y_0 + \cdots\right] \qquad (1)$$

Now, we have to find out the rate of growth of the population in the year 1981,

i.e. $\left.\frac{dy}{dx}\right]_{x=1981}$ i.e. $x_n + ph = 1981 \therefore p = \frac{1981-1991}{10} = -1$

$\therefore$ Putting $p = -1$, $h = 10$ and the values of ∇y_n, $\nabla^2 y_n$, $\nabla^3 y_n$ and $\nabla^4 y_n$ in Eqn (1), we get

$$y'(1981) = \frac{1}{10}\left[17.4 + \frac{2(-1)+1}{2}(-1) + \frac{3(-1)^2+6(-1)+2}{6}(2.76) + \frac{2(-1)^3+9(-1)^2+11(-1)+3}{12}(11.99)\right]$$

$$= 1/10\,[17.4 + 0.5 - 0.46 - 0.9991666]$$
$$= 1.6440833$$

$\therefore$ The rate of growth of the population in the year 1981 is 1.6440833.

Example 9.3 Obtain the value of $f'(90)$ using Stirling's formula to the following data:

x	60	75	90	105	120
$f(x)$	28.2	38.2	43.2	40.9	37.7

Also find the maximum value of the function from the data.

Solution Since $x = 90$ is in the middle of the table, we use central difference formula and in particular, Stirling's formula.

The central difference table is as given below.

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
60	28.2				
		10			
75	38.2		−5		
		5		−2.3	
90	43.2		−7.3		8.7
		−2.3		6.4	
105	40.9		−0.9		
		−3.2			
120	37.7				

Here, $x_0 = 90$, $y_0 = 43.2$, $\Delta y_0 = -2.3$, $\Delta y_{-1} = 5$, $\Delta^3 y_{-1} = -2.3$, $\Delta^3 y_{-2} = 6.4$ and $h = 15$.

Now, from Eqn (9.18).

$$\left.\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\left\{\frac{\Delta y_0 + \Delta y_{-1}}{2}\right\} - \frac{1}{6}\left\{\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right\} + \cdots\right]$$

$$\therefore f'(90) = \frac{1}{15}\left[\left\{\frac{-2.3+5}{2}\right\} - \frac{1}{6}\left\{\frac{-2.3+6.4}{2}\right\}\right]$$

$$= 1/15\,[\,1.35 - 0.3416666] = 0.0672222.$$

To find the maximum value of the tabular function:

By Stirling's formula,

$$y = y(x_0 + ph) = y_0 + \frac{p}{2}(\Delta y_0 + \Delta y_{-1}) + \frac{p^2}{2!}\Delta^2 y_{-1}$$

$$+ \frac{p(p^2-1^2)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{p(p^2-1^2)}{4!}\Delta^4 y_{-2} + \cdots$$

Substituting the values from the table, we get, after simplification,

$y = 43.2 + 1.35\,p - 3.65\,p^2 + 0.3417\,(p^3 - p)$

or $y = 0.3417\,p^3 - 3.65\,p^2 + 1.0083\,p + 43.2$

If y is maximum, $dy/dp = 0$

i.e. $1.0251\,p^2 - 7.3\,p + 1.0083 = 0$

$$\therefore p = \frac{7.3 \pm \sqrt{(1.3)^2 - 4(1.0251)(1.0083)}}{2(1.0251)} = 6.9803 \text{ or } 0.1409$$

$p = 6.9803$ is out of range. $\therefore p = 0.1409$

Hence, $x = x_0 + ph = 90 + 15\,(0.1409) = 92.1135$

and maximum of y

$= 0.3417\,(0.1409)^3 - 3.65\,(0.1409)^2 + 1.0083\,(0.1409) + 43.2$

$= 43.27$

Example 9.4 Using Bessel's formula, find the derivative of $f(x)$ at $x = 3.5$ from the following table.

x	3.47	3.48	3.49	3.50	3.51	3.52	3.53
$f(x)$	0.193	0.195	0.198	0.201	0.203	0.206	0.208

Solution The central difference table is as follows:

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
3.47	0.193						
		0.002					
3.48	0.195		0.001				
		0.003		−0.001			
3.49	0.198		0.000		0.000		
		0.003		−0.001		0.003	
3.50	0.201		−0.001		0.003		−0.010
		0.002		0.002		−0.007	
3.51	0.203		0.001		−0.004		
		0.003		−0.002			
3.52	0.206		−0.001				
		0.002					
3.53	0.208						

Bessel's formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right] + \frac{\left(p-\frac{1}{2}\right)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\left[\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right]$$

$$+ \frac{\left(p-\frac{1}{2}\right)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-2}$$

$$+ \frac{(p+2)(p+1)p(p-1)(p-2)(p-3)}{6!}\left[\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right] + \cdots \quad \text{(i)}$$

where $p = \frac{x - x_0}{h}$. Differentiating Eqn (i) with respect to x, we get

$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\cdot\frac{dy}{dp} \qquad \left[\because \frac{dp}{dx} = \frac{1}{h}\right]$$

Now, $\left.\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\cdot\left[\frac{dy}{dp}\right]_{p=0}$

$$= \frac{1}{h}\left[\Delta y_0 - \frac{1}{4}(\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{1}{12}\Delta^3 y_{-1}\right.$$

$$\left. + \frac{1}{24}(\Delta^4 y_{-2} + \Delta^4 y_{-1}) - \frac{1}{120}\Delta^5 y_{-2} - \frac{1}{240}(\Delta^6 y_{-3} + \Delta^6 y_{-2})\right]$$

Substituting values from the table in above, we get

$$\left.\frac{dy}{dx}\right]_{x=3.5} = f'(3.5) = \frac{1}{0.01}\left[0.02 - \frac{1}{4}(-0.001 + 0.001) + \frac{1}{12}(0.002)\right.$$

$$+ \frac{1}{24}(-0.004 + 0.003) + \frac{1}{120}(-0.00)$$

$$\left. - \frac{1}{240}(-0.010 + 0)\right]$$

$$= [0.02 - 0 + 0.01666 - 0.04166 + 0.00583 + 0.04166] = 0.22249$$

Example 9.5 Given the following data, find the maximum value of y

x	-1	1	2	3
y	-21	15	12	3

Solution Since the arguments are not equispaced, we will form the divided difference table as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-1	-21			
		18		
1	15		-7	
		-3		1
2	12		-3	
		-9		
3	3			

Using Newton's divided difference formula, we get

$$y = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0$$

$$= -21 + (x + 1)(18) + (x + 1)(x - 1)(-7) + (x + 1)(x - 1)(x - 2)(1)$$

$$= x^3 - 9x^2 + 17x + 6$$

Now for maximum $\dfrac{dy}{dx} = 0 \Rightarrow 3x^2 - 18x + 17 = 0$

$$x = \frac{18 \pm \sqrt{(-18)^2 - 4(3)(17)}}{2(3)} = 4.8257 \text{ or } 1.1743$$

$x = 4.8257$ is out of range $\therefore x = 1.1743$ is the value giving maximum of y.

$\therefore$ Max of y (at $x = 1.1743$) $= (1.1743)^3 - 9(1.1743)^2 + 17(1.1743) + 6$
$= 15.171612$

EXERCISE 9.1

1. Find the first and second derivatives of the function tabulated below at the point $x = 19$

x	1.0	1.2	1.4	1.6	1.8	2.0
$f(x)$	0	0.128	0.544	1.296	2.432	4.00

(*Madras 1991*)

2. The following data gives corresponding values of pressure and specific volume of super-heated steam.

V	2	4	6	8	10
P	105	42.07	25.3	16.7	13

(i) Find the rate of change of pressure with respect to volume when $V = 2$.

(ii) Find the rate of change of volume with respect to pressure when $P = 105$.

3. Find $y'(0)$ and $y''(0)$ from the following table:

x	0	1	2	3	4	5
y	4	8	15	7	6	2

4. From the values in the table given below, find the value of sec 31° using numerical differentiation.

θ °	31	32	33	34
$\tan\theta$	0.6008	0.6249	0.6494	0.6745

5. The table given below reveals velocity V of a body during time 't' specified. Find its acceleration at $t = 1.1$

t	1.0	1.1	1.2	1.3	1.4
v	43.1	47.7	52.1	56.4	60.8

6. A rod is rotating in a plane. The following table gives the angle θ (radians) through which the rod has turned for various values of time t in seconds.

t	0	0.2	0.4	0.6	0.8	1.0	1.2
θ	0	0.122	0.493	0.123	2.022	3.200	4.61

Find the angular velocity and angular acceleration at $t = 0.6$.

7. From the following table of values of x and y, find y' (1.25) and y'' (1.25).

x	1.00	1.05	1.10	1.15	1.20	1.25	1.30
y	1.00000	1.02470	1.04881	1.07238	1.09544	1.11803	1.14017

8. Obtain the value of f' (0.04) using Bessel's formula given the table below:

x	0.01	0.02	0.03	0.04	0.05	0.06
$f(x)$	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

9. Use Stirling's formula to compute f' (0.5) from the following data:

x	0.35	0.40	0.45	0.50	0.55	0.60	0.65
$f(x)$	1.521	1.506	1.488	1.467	1.444	1.418	1.389

10. A slider in a machine moves along a fixed straight rod. Its distance x (cm) along the rod is given below for various values of time t seconds. Find the veocity of the slider and its acceleration when $t = 0.3$.

t	0	0.1	0.2	0.3	0.4	0.5	0.6
x	30.13	31.62	32.87	33.64	33.95	33.81	33.24