OPERATIONS RESEARCH

DUALITY IN LINEAR PROGRAMMING

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DUALITY IN LINEAR PROGRAMMING

- It is interesting that every linear programming model has two forms: The Primal and The Dual.

- The original form of a linear programming model is called the Primal.

- The dual is an alternative model form derived completely from the primal. The dual is useful because it provides the decision maker with an alternative way of looking at a problem.

DUALITY (Cont…)

DUAL OF AN LP PROBLEM:

The dual is derived completely from the primal; for easily getting the dual from the primal; <u>divide the linear programming problems into two forms</u>:

1. Normal Linear Programming Problem
2. Non–Normal Linear Programming Problem

1. NORMAL LINEAR PROGRAMMING PROBLEM:

We call ***normal maximum LP problem*** is an LP problem in which all the variables are required to be non–negative and all the constraints are less than or equal to (≤) form.

For Example: a normal maximum LP problem has the following form:

Maximum: $Z = C_1X_1+C_2X_2+ - - - +C_nX_n$

Subject to:

$$a_{11}X_1+a_{12}X_2+ - - - +a_{1n}X_n \leq b_1$$
$$a_{21}X_1+a_{22}X_2+ - - - +a_{2n}X_n \leq b_2$$
$$-$$
$$-$$
$$-$$
$$a_{m1}X_1+a_{m2}X_2+ - - - +a_{mn}X_n \leq b_m$$

Where, $X_j \geq 0$, $j = 0, 1, 2, - - -, n$.

While, a ***normal minimum LP problem*** is an LP problem in which all the variables are required to be non–negative and all the constraints are greater than or equal to (≥) form.

For Example: a normal minimum LP problem has the following form:

Minimum: $Z = C_1X_1+C_2X_2+ - - - +C_nX_n$

Subject to:

$$a_{11}X_1+a_{12}X_2+ - - - +a_{1n}X_n \geq b_1$$
$$a_{21}X_1+a_{22}X_2+ - - - +a_{2n}X_n \geq b_2$$
$$-$$
$$-$$
$$-$$
$$a_{m1}X_1+a_{m2}X_2+ - - - +a_{mn}X_n \geq b_m$$

Where, $X_j \geq 0$, $j = 0, 1, 2, - - -, n$.

STEPS FOR PRIMAL PROBLEM TO DUAL PROBLEM IN CASE OF NORMAL LP PROBLEM:

Step–1: If the primal problem is in a maximization form then the dual problem will be in minimization form. But if the primal problem is in a minimization form then the dual problem will be in maximization form.

Step–2: The number of decision variables in the dual problem is equal to the number of constraints in the primal problem.

Step–3: The quantities (numerical values) which appear on the right hand side (RHS) of the constraints of the primal problem become the coefficients of the decision variables in the objective function of the dual problem.

Step–4: The number of constraints in the dual problem is equal to the number of variables in the primal problem.

Step–5: The coefficients of the variables in the constraints of the primal problem which appear from left to right be placed from top to bottom in the constraints of the dual problem.

Step–6: If the primal problem has less than or equal to ($\leq$) type constraints then the dual problem will have greater than or equal to ($\geq$) type constraints; But if the primal problem has greater than or equal to ($\geq$) type constraints then the dual problem will have less than or equal to ($\leq$) type constraints.

Step–7: The coefficients of the objective function of the primal problem which appears on the right hand side (RHS) of the constraints of the dual problem.

Step–8: Non–negativity restriction will also apply to decision variable of dual problem.

DUALITY (Cont...)

PRIMAL–DUAL CONSTRUCTION RELATIONSHIP WITH AN EXAMPLE:

PRIMAL	DUAL
Maximum: $Z = 3X_1 + 4X_2$	Minimum: $Z = 16Y_1 + 16Y_2$
Subject to:	Subject to:
$2X_1 + 3X_1 \leq 16$	$2Y_1 + 4Y_2 \geq 3$
$4X_2 + 2X_2 \leq 16$	$3Y_1 + 2Y_2 \geq 4$
Where $X_1, X_2 \geq 0$	Where $Y_1, Y_2 \geq 0$

<table>
<tr><th colspan="2">PRIMAL MAXIMIZATION / DUAL MINIMIZATION</th></tr>
<tr><td>Maximum: $Z = C^TX$
Subject to:
$AX \leq b$
$X \geq 0$</td><td>Minimum: $Z = b^TY$
Subject to:
$A^TY \geq C$
$Y \geq 0$</td></tr>
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Subject to:
$A^TY \leq C$
$Y \geq 0$</td></tr>
</table>

DUALITY (Cont…)

1) Find the dual of the following problem:

Maximize: $Z = 3X_1 + 4X_2$

Subject to:

$$2X_1 + 3X_2 \leq 16$$
$$4X_1 + 2X_2 \leq 16$$
$$X_1, X_2 \geq 0$$

Dual problem in case of normal LP problem;

Minimize: $Z = 16Y_1 + 16Y_2$

Subject to:

$$2Y_1 + 4Y_2 \geq 3$$
$$3Y_1 + 2Y_2 \geq 4$$
$$Y_1, Y_2 \geq 0$$

2) Find the dual of the following problem:

Maximize: $Z = 30X_1 + 35X_2 + 50X_3$

Subject to:

$$6X_1 + 3X_2 + 7X_3 \leq 53$$
$$5X_1 + 3X_2 + 4X_3 \leq 19$$
$$2X_1 + 4X_2 + 5X_3 \leq 11$$
$$X_1, X_2, X_3 \geq 0$$

Dual problem in case of normal LP problem;

Minimize: $Z = 53Y_1 + 19Y_2 + 11Y_3$

Subject to:

$$6Y_1 + 5Y_2 + 2Y_3 \geq 30$$
$$3Y_1 + 3Y_2 + 4Y_3 \geq 35$$
$$7Y_1 + 4Y_2 + 5Y_3 \geq 50$$
$$Y_1, Y_2, Y_3 \geq 0$$

Find the dual of the following problem:

Minimize: $Z = 40X_1 + 200X_2$

Subject to:

$$4X_1 + 40X_2 \geq 160$$
$$3X_1 + 10X_2 \geq 60$$
$$8X_1 + 10X_2 \geq 80$$
$$X_1, X_2 \geq 0$$

Dual problem in case of normal LP problem;

Maximize: $Z = 160Y_1 + 60Y_2 + 80Y_3$

Subject to:

$$4Y_1 + 3Y_2 + 8Y_3 \leq 40$$
$$40Y_1 + 10Y_2 + 10Y_3 \leq 200$$
$$Y_1, Y_2, Y_3 \geq 0$$

STEPS FOR PRIMAL PROBLEM TO DUAL PROBLEM IN CASE OF NON–NORMAL LP PROBLEM:

Step–1: If the given LP problem is in non–normal form then we first convert it into normal form, for converting the non–normal LP problem into normal LP problem adopt the following procedure:

IN CASE OF MAXIMIZATION PROBLEM

Constraints / Variable Type	Procedure
1. If Less than or equal to (≤) type	No Change is required
2. If greater than or equal to (≥) type	Convert the '≥' type inequality into '≤' type by multiplying it by '–1'.
3. If equal to (=) type	a. Convert the equality into two inequalities in which one having '≥' sign while other having '≤' sign. b. Convert the '≥' type inequality into '≤' type by multiplying it by '–1'.
4. Unrestricted–in–sign Decision Variable (X_i)	Any unrestricted in sign decision variable can be rewritten 'X_i' as the difference $X_i = X_i^{/} - X_i^{//}$ of two non–negative decision variables $X_i^{/}$, $X_i^{//}$.

IN CASE OF MINIMIZATION PROBLEM

Constraints / Variable Type	Procedure
1. If Less than or equal to (≤) type	Convert the '≤' type inequality into '≥' type by multiplying it by '–1'.
2. If greater than or equal to (≥) type	No Change is required
3. If equal to (=) type	a. Convert the equality into two inequalities in which one having '≥' sign while other having '≤' sign. b. Convert the '≤' type inequality into '≥' type by multiplying it by '–1'.
4. Unrestricted–in–sign Decision Variable (X_i)	Any unrestricted in sign decision variable can be rewritten 'X_i' as the difference $X_i = X_i^{/} - X_i^{//}$ of two non–negative decision variables $X_i^{/}$, $X_i^{//}$.

STEPS FOR PRIMAL PROBLEM TO DUAL PROBLEM IN CASE OF NON–NORMAL LP PROBLEM (Cont…):

Step–2: Restate the primal problem after taking step–1.

Step–3: If the primal problem is in a maximization form then the dual problem will be in minimization form. But if the primal problem is in a minimization form then the dual problem will be in maximization form.

Step–4: The number of decision variables in the dual problem is equal to the number of constraints in the primal problem.

Step–5: The quantities (numerical values) which appear on the right hand side (RHS) of the constraints of the primal problem become the coefficients of the decision variables in the objective function of the dual problem.

Step–6: The number of constraints in the dual problem is equal to the number of variables in the primal problem.

Step–7: The coefficients of the variables in the constraints of the primal problem which appear from left to right be placed from top to bottom in the constraints of the dual problem.

Step–8: If the primal problem has less than or equal to (≤) type constraints then the dual problem will have greater than or equal to (≥) type constraints; But if the primal problem has greater than or equal to (≥) type constraints then the dual problem will have less than or equal to (≤) type constraints.

Step–9: The coefficients of the objective function of the primal problem which appears on the right hand side (RHS) of the constraints of the dual problem.

Step–10: Non–negativity restriction will also apply to decision variable of dual problem.

Find the dual of the following problem:
Maximize: $Z = 6X_1 + 4X_2 + 6X_3 + X_4$
Subject to:

$$4X_1 + 5X_2 + 4X_3 + 8X_4 = 21$$
$$3X_1 + 7X_2 + 8X_3 + 2X_4 \leq 48$$
$$X_1, X_2, X_3, X_4 \geq 0$$

As the given LP problem is in the form of **maximum non–normal form** so *we need to convert it into* **maximum normal LP problem** in which we need all the constraints in the form of less than or equal to '≤'. So,

Constraint–1: $4X_1 + 5X_2 + 4X_3 + 8X_4 = 21$; Now we convert it into two inequalities that are:

$$4X_1 + 5X_2 + 4X_3 + 8X_4 \leq 21 \qquad 4X_1 + 5X_2 + 4X_3 + 8X_4 \geq 21$$

Inequality having '≥' sign; convert it into '≤' form by multiplying with '–1'; we get: $-4X_1 - 5X_2 - 4X_3 - 8X_4 \leq -21$

Now restating the primal as below:

Maximize: $Z = 6X_1 + 4X_2 + 6X_3 + X_4$

Subject to:

$$4X_1 + 5X_2 + 4X_3 + 8X_4 \leq 21$$
$$-4X_1 - 5X_2 - 4X_3 - 8X_4 \leq -21$$
$$3X_1 + 7X_2 + 8X_3 + 2X_4 \leq 48$$
$$X_1, X_2, X_3, X_4 \geq 0$$

Note: Regarding the step–4 of the algorithm, the number of decision variables in the dual problem is equal to the number of constraints in the primal problem. But we can see that in the given original primal there are two '2' constraints and four '4' variables while in dual there are four '4' constraints and three '3' variables which shows the inconsistency. So in order to remove such an inconsistency: let $(Y_1 - Y_2) = Y^{/}$.

Now follow the remaining steps for converting the primal into dual; we will get the dual of the given problem which is:

Minimize: $Z = 21Y^{/} + 48Y_3$

Subject to:

$$4Y^{/} + 3Y_3 \geq 6$$
$$5Y^{/} + 7Y_3 \geq 4$$
$$4Y^{/} + 8Y_3 \geq 6$$
$$8Y^{/} + 2Y_3 \geq 1$$

Where $Y_3 \geq 0$; $Y^{/}$ **(unrestricted in sign)**

DUALITY (Cont…)

Find the dual of the following problem:

Maximize: $Z = 6X_1 + 8X_2$

Subject to:

$$2X_1 + 3X_2 \le 16$$
$$4X_1 + 2X_2 \ge 16$$
$$2X_1 + X_2 = 16$$
$$X_1, X_2 \ge 0$$

As the given LP problem is in the form of **maximum non–normal form** so *we need to convert it into* **maximum normal LP problem** in which we need all the constraints in the form of less than or equal to '≤'. So,

Now restating the primal as below:

Maximize: $Z = 6X_1 + 8X_2$

Subject to:

$$2X_1 + 3X_2 \le 16$$
$$-4X_1 - 2X_2 \le -16$$
$$2X_1 + X_2 \le 16$$
$$-2X_1 - X_2 \le -16$$
$$X_1, X_2 \ge 0$$

DUAL:

Minimize: $Z = 16Y_1 - 16Y_2 + 16Y_3 - 16Y_4$

Subject to:

$$2Y_1 - 4Y_2 + 2Y_3 - 2Y_4 \ge 6$$
$$3Y_1 - 2Y_2 + Y_3 - Y_4 \ge 8$$
$$Y_1, Y_2, Y_3, Y_4 \ge 0$$

Minimize: $Z = 16Y_1 - 16Y_2 + 16Y^{/}$

Subject to:

$$2Y_1 - 4Y_2 + 2Y^{/} \ge 6$$
$$3Y_1 - 2Y_2 + Y^{/} \ge 8$$

Where, $Y_1, Y_2 \ge 0$; $Y^{/}$ unrestricted in sign

DUALITY (Cont…)

Find the dual of the following problem:

Maximize: $Z = 12X_1 + 15X_2 + 9X_3$

Subject to:

$$8X_1 + 16X_2 + 12X_3 \leq 25$$
$$4X_1 + 8X_2 + 10X_3 \geq 80$$
$$7X_1 + 9X_2 + 8X_3 = 105$$
$$X_1, X_2, X_3 \geq 0$$

Minimize: $Z = 25Y_1 - 80Y_2 + 105Y^/$

Subject to:

$$8Y_1 - 4Y_2 + 7Y^/ \geq 12$$
$$16Y_1 - 8Y_2 + 9Y^/ \geq 15$$
$$12Y_1 - 10Y_2 + 8Y^/ \geq 9$$

Where, $Y_1, Y_2 \geq 0$; $Y^/$ unrestricted in sign

Find the dual of the following problem:

Maximize: $Z = 3X_1 + X_2 + X_3 - X_4$

Subject to:

$$X_1 + X_2 + 2X_3 + 3X_4 \leq 5$$
$$X_3 - X_4 \geq -1$$
$$X_1 - X_2 = -1$$
$$X_1, X_2, X_3, X_4 \geq 0$$

Minimize: $Z = 5Y_1 + Y_2 - Y^/$

Subject to:

$$Y_1 + Y^/ \geq 3$$
$$Y_1 - Y^/ \geq 1$$
$$2Y_1 - Y_2 \geq 1$$
$$3Y_1 + Y_2 \geq -1$$

Where, $Y_1, Y_2 \geq 0$; $Y^/$ (unrestricted in sign)

INTERPRETATION OF THE PRIMAL–DUAL SOLUTION RELATIONSHIP:

PRIMAL PROBLEM: Solve the given LP problem.

Minimize: $Z = 40X_1 + 200X_2$

Subject to:

$4X_1 + 40X_2 \geq 160$

$3X_1 + 10X_2 \geq 60$

$8X_1 + 10X_2 \geq 80$

$X_1, X_2 \geq 0$

Maximize: $Z = -40X_1 - 200X_2 + 0S_1 + 0S_2 + 0S_3 - MA_1 - MA_2 - MA_3$

Subject to:

$4X_1 + 40X_2 - S_1 + A_1 = 160$

$3X_1 + 10X_2 - S_2 + A_2 = 60$

$8X_1 + 10X_2 - S_3 + A_3 = 80$

Where: $X_1, X_2, X_3, S_1, S_2, S_3, A_1, A_2, A_3 \geq 0$

Contribution Per Unit		C_j	–40	–200	0	0	0	–M	–M	–M	
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	**X_1**	**X_2**	**S_1**	**S_2**	**S_3**	**A_1**	**A_2**	**A_3**	**Ratio**
–M	A_1	160	4	40*	–1	0	0	1	0	0	4 ←
–M	A_2	60	3	10	0	–1	0	0	1	0	6
–M	A_3	80	8	10	0	0	–1	0	0	1	8
Total Profit (Z_j)		–300M	–15M	–60M	M	M	M	–M	–M	–M	
Net Contribution ($C_j - Z_j$)			–40+15M	–200+60M ↑	–M	–M	–M	0	0	0	

INTERPRETATION OF THE PRIMAL–DUAL SOLUTION RELATIONSHIP:

Contribution Per Unit		C_j	–40	–200	0	0	0	–M	–M	–M	
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	**X_1**	**X_2**	**S_1**	**S_2**	**S_3**	**A_1**	**A_2**	**A_3**	**Ratio**
–200	X_2	4	1/10	1	–1/40	0	0	1/40	0	0	40
–M	A_2	20	2	0	1/4	–1	0	–1/4	1	0	10
–M	A_3	40	7*	0	1/4	0	–1	–1/4	0	1	40/7 ←
Total Profit (Z_j)		–800–60M	–20–9M	–200	5 – M/2	M	M	–5+M/2	–M	–M	
Net Contribution ($C_j – Z_j$)			–20+9M ↑	0	–5+M/2	–M	–M	5–3M/2	0	0	

Contribution Per Unit		C_j	–40	–200	0	0	0	–M	–M	–M	
C_{Bi}	**B.V. (B)**	**Quantity (Qty)**	**X_1**	**X_2**	**S_1**	**S_2**	**S_3**	**A_1**	**A_2**	**A_3**	**Ratio**
–200	X_2	24/7	0	1	–1/35	0	1/70	1/35	0	–1/70	240
–M	A_2	60/7	0	0	5/28	–1	(2/7)*	–5/28	1	–2/7	30 ←
–40	X_1	40/7	1	0	1/28	0	–1/7	–1/28	0	1/7	---
Total Profit: (Z_j)		(–6400–60M)/7	–40	–200	(120–5M)/28	M	(20–2M)/7	(–120+5M)/28	–M	(–20+2M)/7	
Net Contribution ($C_j – Z_j$)			0	0	(–120+5M)/28	–M	[(–20+2M)/7] ↑	(120–5M)/28	0	(20–2M)/7	

INTERPRETATION OF THE PRIMAL–DUAL SOLUTION RELATIONSHIP:

Contribution Per Unit C_j			–40	–200	0	0	0	–M	–M	–M
C_{Bi}	**B.V. (B)**	**Quantity (Qty)**	**X_1**	**X_2**	**S_1**	**S_2**	**S_3**	**A_1**	**A_2**	**A_3**
–200	X_2	3	0	1	–21/560	1/20	0	21/560	–1/20	0
0	S_3	30	0	0	5/8	–7/2	1	–5/8	7/2	–1
–40	X_1	10	1	0	1/8	–1/2	0	–1/8	1/2	0
Total Profit: (Z_j)		–1000	–40	–200	5/2	10	0	–M+1/8	–M+1/2	–M
Net Contribution ($C_j – Z_j$)			0	0	–5/2	–10	0	–1/8	–1/2	0

Value of 'Z' = (–40)(10) + (–200)(3) = –1000

Optimal solution is –1000 for Max. 'Z'; at $X_1 = 10$, $X_2 = 3$.

Because Min (Z) = –Max (–Z) = 1000; at $X_1 = 10$, $X_2 = 3$.

DUALITY (Cont…)

INTERPRETATION OF THE PRIMAL–DUAL SOLUTION RELATIONSHIP:

PRIMAL PROBLEM: Solve the given LP problem.

Minimize: $Z = 40X_1 + 200X_2$

Subject to:

$$4X_1 + 40X_2 \geq 160$$
$$3X_1 + 10X_2 \geq 60$$
$$8X_1 + 10X_2 \geq 80$$
$$X_1 , X_2 \geq 0$$

DUAL PROBLEM:

The Dual of the above primal is written as follows:

Maximize: $Z = 160Y_1 + 60Y_2 + 80Y_3$

Subject to:

$$4Y_1 + 3Y_2 + 8Y_3 \leq 40$$
$$40Y_1 + 10Y_2 + 10Y_3 \leq 200$$
$$Y_1 , Y_2, Y_3 \geq 0$$

DUALITY (Cont…)

INTERPRETATION OF THE PRIMAL–DUAL SOLUTION RELATIONSHIP:

Maximize: $Z = 160Y_1 + 60Y_2 + 80Y_3 + 0S_1 + 0S_2$

Subject to:

$4Y_1 + 3Y_2 + 8Y_3 + S_1 = 40$

$40Y_1 + 10Y_2 + 10Y_3 + S_2 = 200$

$Y_1, Y_2, Y_3, S_1, S_2 \geq 0$

Contribution Per Unit C_j			160	60	80	0	0	Ratio
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	**Y_1**	**Y_2**	**Y_3**	**S_1**	**S_2**	
0	S_1	40	4	3	8	1	0	10
0	S_2	200	40*	10	10	0	1	5 ←
Total Profit (Z_j)		0	0	0	0	0	0	
Net Contribution ($C_j - Z_j$)			160 ↑	60	80	0	0	

Contribution Per Unit C_j			160	60	80	0	0	Ratio
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	**Y_1**	**Y_2**	**Y_3**	**S_1**	**S_2**	
0	S_1	20	0	2	7*	1	–1/10	20/7 ←
160	Y_1	5	1	1/4	1/4	0	1/40	20
Total Profit (Z_j)		800	160	40	40	0	4	
Net Contribution ($C_j - Z_j$)			0	20	40 ↑	0	–4	

DUALITY (Cont…)

INTERPRETATION OF THE PRIMAL–DUAL SOLUTION RELATIONSHIP:

Contribution Per Unit C_j			160	60	80	0	0	Ratio
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	**Y_1**	**Y_2**	**Y_3**	**S_1**	**S_2**	
80	Y_3	20/7	0	(2/7)*	1	1/7	–1/70	10 ←
160	Y_1	30/7	1	5/28	0	–1/28	1/35	24
Total Profit (Z_j)		6400/7	160	360/7	80	40/7	24/7	
Net Contribution ($C_j – Z_j$)			0	60/7 ↑	0	–40/7	–24/7	

Contribution Per Unit C_j			160	60	80	0	0
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	**Y_1**	**Y_2**	**Y_3**	**S_1**	**S_2**
60	Y_2	10	0	1	7/2	1/2	–1/20
160	Y_1	5/2	1	0	–5/8	–1/8	3/80
Total Profit (Z_j)		1000	160	60	110	10	3
Net Contribution ($C_j – Z_j$)			0	0	–30	–10	–3

Value of 'Z' = (60)(10) + (160)(5/2) = 1000; at Y_1 = 5/2, Y_2 = 10.

DUALITY (Cont...)

INTERPRETATION OF THE PRIMAL–DUAL SOLUTION RELATIONSHIP:

COMPARISON OF THE PRIMAL AND DUAL OPTIMAL TABLE

PRIMAL OPTIMAL SOLUTION (After deleting artificial variable columns)

Contribution Per Unit		C_j	–40	–200	0	0	0
C_{Bi}	**B.V. (B)**	**Quantity (Qty)**	**X_1**	**X_2**	**S_1**	**S_2**	**S_3**
–200	X_2	3	0	1	–21/560	1/20	0
0	S_3	30	0	0	5/8	–7/2	1
–40	X_1	10	1	0	1/8	–1/2	0
Total Profit: (Z_j)		–1000	–40	–200	5/2	10	0
Net Contribution ($C_j – Z_j$)			0	0	–5/2	–10	0

DUAL OPTIMAL SOLUTION

Contribution Per Unit		C_j	160	60	80	0	0
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	**Y_1**	**Y_2**	**Y_3**	**S_1**	**S_2**
60	Y_2	10	0	1	7/2	1/2	–1/20
160	Y_1	5/2	1	0	–5/8	–1/8	3/80
Total Profit (Z_j)		1000	160	60	110	10	3
Net Contribution ($C_j – Z_j$)			0	0	–30	–10	–3

DUALITY (Cont…)

INTERPRETATION OF THE PRIMAL–DUAL SOLUTION RELATIONSHIP:

COMPARISON OF THE PRIMAL AND DUAL OPTIMAL TABLE

Note: The respective coefficients of S_1, S_2 & Y_3 in the $(C_j – Z_j)$ row (ignoring the sign) from the Dual optimal solution table are 10, 3 & 30.

Primal Optimal Solution		Dual Optimal Solution	
$(C_j - Z_j)$	Basic Variable Solution	Basic Variable Solution	$(C_j - Z_j)$
0	$X_1 = 10$	$Y_1 = 5/2$	0
0	$X_2 = 3$	$Y_2 = 10$	0
		$Y_3 = 0$	30
5/2	$S_1 = 0$	$S_1 = 10$	10
10	$S_2 = 0$	$S_2 = 3$	3
0	$S_3 = 30$		
	$\underline{Z^P = 1000}$	$\underline{Z^D = 1000}$	

- Now, it is clear that the primal and dual lead to the same solution, even though they are formulated differently.
- It is also clear that in the final simplex table of primal problem, the absolute value of the numbers in $(C_j – Z_j)$ row under the slack variable represents the solutions to the dual problem. In another words it also happens that the absolute value of the $(C_j – Z_j)$ values of the slack variable in the optimal dual solution represent the optimal values of the primal 'X_1' and 'X_2' variables.
- The minimum opportunity cost derived in the dual must always be equal the maximum profit derived in the primal.

PRIMAL–DUAL RELATIONSHIPS:

WEAK DUALITY THEOREM:

- If '$X^{/}$' is a feasible solution for primal problem and '$Y^{/}$' is a feasible solution for dual problem, then:
  - (Primal Objective function value) $C^T X^{/} \leq b^T Y^{/}$ (Dual Objective function value) if the primal is in a maximization case &
  - (Primal Objective function value) $C^T X^{/} \geq b^T Y^{/}$ (Dual Objective function value) if the primal is in a minimization case
- This is called the weak duality.
- The non–negative quantity $\beta = \left| b^T Y^{/} - C^T X^{/} \right|$ is called the duality gap. So, when $\beta = 0$, we can say that there is no duality gap.

STRONG DUALITY:

- If 'X^*' is an optimal solution for primal problem and 'Y^*' is an optimal solution for dual problem, then the optimal objective value of the primal is the same as the optimal objective value of the dual: $C^T X^* = b^T Y^*$; in this case, the duality gap is zero so this is called the strong duality.

- **DUAL OF THE DUAL IS PRIMAL**

DUAL SIMPLEX METHOD

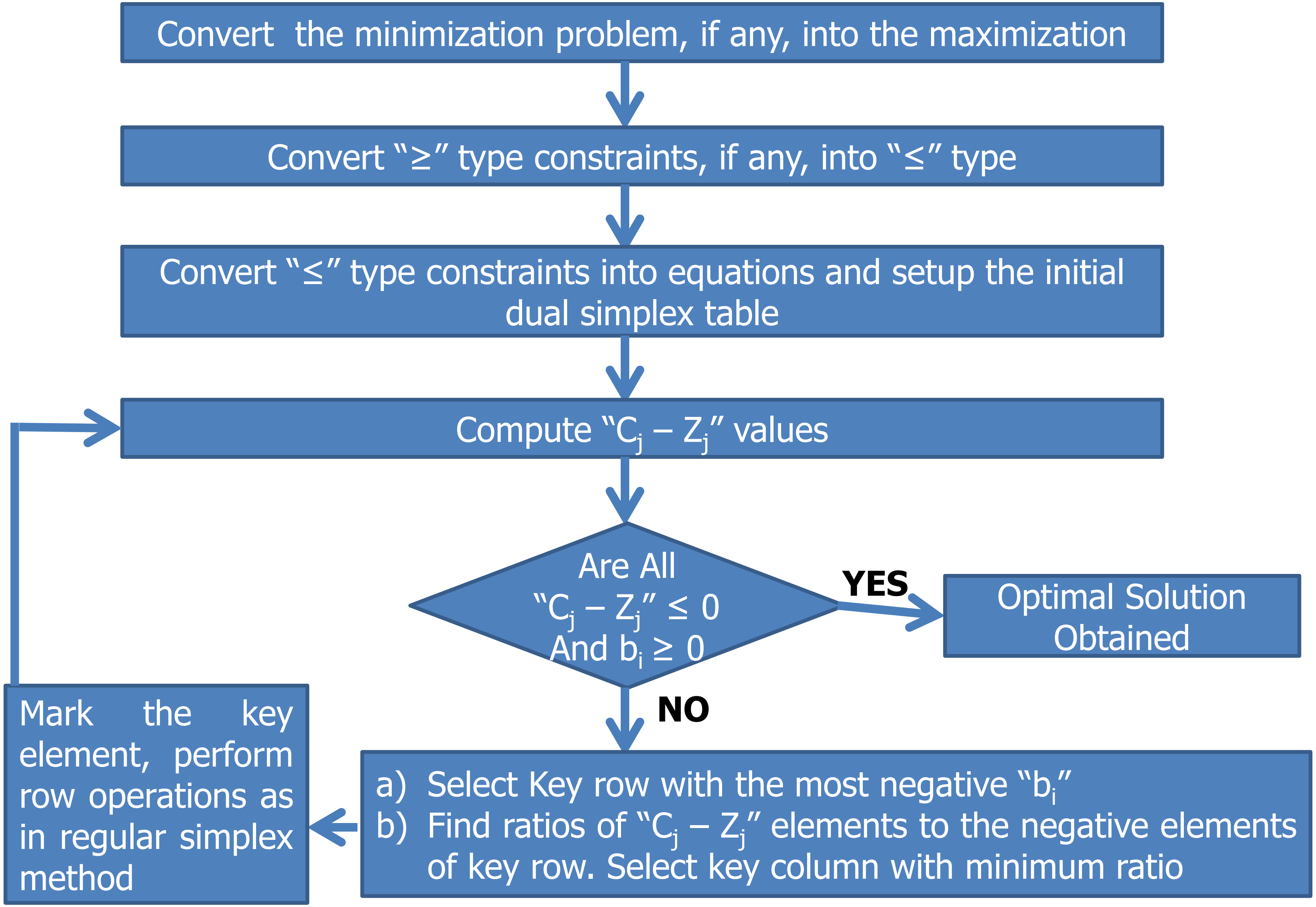
- Dual simplex method, developed by C.E. Lemte, is very similar to the regular simplex method.

- The only differences lies in the criterion used for selecting a variable to enter the basis (Basic Variables) and the leave the basis (Basic Variables). In dual simplex method, we first select the variable to leave the basis (Basic Variables) and then the variable to enter the basis (Basic Variables).

- In this method the <u>solution starts from optimum but infeasible</u> and remains infeasible until the true optimum is reached at which the solution becomes feasible.

- The advantage of this method is avoiding the artificial and surplus variables introducing in the constraints, as the constraint is in the form of greater than or equal to '≥' converted into less than or equal to '≤'.

DUAL SIMPLEX ALGORITHM



DUAL SIMPLEX METHOD

EXAMPLE : Use Dual Simplex method to solve the LP problem.

Maximize: $Z = -3X_1 - X_2$

Subject to:

$$X_1 + X_2 \geq 1$$
$$2X_1 + 3X_2 \geq 2$$
$$X_1, X_2 \geq 0$$

Step–1: Given problem is already in the form of maximization. So go to the next step.

Step–2: Convert the Given constraints into (≤) form by multiplying with '–1' both sides.

Maximize: $Z = -3X_1 - X_2$

Subject to:

$$-X_1 - X_2 \leq -1$$
$$-2X_1 - 3X_2 \leq -2$$
$$X_1, X_2 \geq 0$$

Step–3: After introducing slack variables (S_1, S_2) standard form of the given problem is:

Maximize: $Z = -3X_1 - X_2 + 0S_1 + 0S_2$

Subject to:

$$-X_1 - X_2 + S_1 = -1$$
$$-2X_1 - 3X_2 + S_2 = -2$$
$$X_1, X_2, S_1, S_2 \geq 0$$

DUAL SIMPLEX METHOD (Cont…)

Now, we display the initial simplex table:

COntribuTion Per Unit		C_j	–3	–1	0	0
C_{Bi}	**Bas)c VarIables (B)**	**Quantity (Qty)**	**X_1**	**X_2**	**S_1**	**S_2**
0	S_1	–1	–1	–1	1	0
0	S_2	–2	–2	–3	0	1
Total Profit (Z_j)		0	0	0	0	0
Net Contribution ($C_j - Z_j$)			–3	–1	0	0

Step–4: Since all (C_j – Z_j) ≤ 0 and all the values in the quantity (Qty) column less than (<) zero. So, the current solution is not optimum basic feasible solution.

Step–5:

Contribution Per Unit		C_j	–3	–1	0	0
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	**X_1**	**X_2**	**S_1**	**S_2**
0	S_1	–1	–1	–1	1	0
0	S_2	← –2	–2	–3*	0	1
Total Profit (Z_j)		0	0	0	0	0
Net Contribution ($C_j - Z_j$)			–3	–1	0	0
Replacement Ratio			3/2	(1/3) ↑	---	---

Step–6: Pivot row indicates that outgoing (leaving) variable is 'S_2' While Pivot column indicates that incoming (entering) variable is 'X_2'. So, we replace the outgoing (leaving) variable 'S_2' by the incoming (entering) variable 'X_2' together with its contribution per unit.

DUAL SIMPLEX METHOD (Cont…)

Now, preparing the next duaL simplex table

Contribution Per Unit		C_j	−3	−1	0	0
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	**X_1**	**X_2**	**S_1**	**S_2**
0	S_1	← −1/3	−1/3	0	1	(−1/3)*
−1	X_2	2/3	2/3	1	0	−1/3
Total Profit (Z_j)		−2/3	−2/3	−1	0	1/3
Net Contribution ($C_j - Z_j$)			−7/3	0	0	−1/3
Replacement Ratio			7	---	---	1 ↑

Contribution Per Unit		C_j	−3	−1	0	0
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	**X_1**	**X_2**	**S_1**	**S_2**
0	S_2	1	1	0	−3	1
−1	X_2	1	1	1	−1	0
Total Profit (Z_j)		−1	−1	−1	1	0
Net Contribution ($C_j - Z_j$)			−2	0	−1	0

Since all $(C_j - Z_j) \leq 0$ and all the values in the quantity (Qty) column greater than or equal to zero (≥ 0); so, the current solution is optimum feasible solution.

Thus, the optimal feasible solution to the given LP problem is:

Maximum $Z = -1$; at $X_1 = 0$, $X_2 = 1$.

DUAL SIMPLEX METHOD (Cont…)

PRACTICE QUESTION:

Use Dual Simplex method to solve the LP problem.

Minimize: $Z = X_1 + 2X_2 + 3X_3$

Subject to:

$$X_1 - X_2 + X_3 \geq 4$$

$$X_1 + X_2 + 2X_3 \leq 8$$

$$X_2 - X_3 \geq 2$$

$$X_1, X_2, X_3 \geq 0$$

QUESTIONS