

→ Week 1: Lecture 1

Error

round off $5.43512 \approx 5.44$

truncate $5.43512 \approx 5.43$ (simply cut off)

1. Round off error

term > 5 , add 1 to previous term

term < 5 , previous term remains same

2. Truncation error

approximation of an infinite mathematical procedure

3. Absolute error

$$\text{Absolute} = |\text{exact} - \text{approximate}|$$

E_a

4. Relative error

$$E_r = \left| \frac{E_a}{\text{exact}} \right|$$

5. Percentage error

$$E_r \times 100$$

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⇒ Week 1: Lecture 2

→ Interpolation

Interpolation

equally spaced data unequally spaced data

• Newton forward difference

• Newton backward difference

→ Newton forward difference formula

Let x_0, x_1, \dots, x_n

$$y_0, y_1, \dots, y_m \text{ such that } y = f(x)$$

$$\begin{array}{cccc} y_0 & y_1 & y_2 & \dots \\ x_0 & x_1 = x_0 + h & x_2 = x_0 + 2h & x = x_0 + ph \end{array}$$

$$x_p = x_0 + ph$$

$$p = \frac{x - x_0}{h}$$

$$\text{here } x = x_p$$

$$E^p f(x) = f(x + ph)$$

$$E^p f(x_0) = f(x_0 + ph)$$

$$\therefore x_p = x_0 + ph$$

$$E^p y_0 = f(x)$$

$$f(x) = E^p y_0$$

$$y = (1 + \Delta)^p y_0$$

$$(1+x)^m = 1 + mx + \frac{n(n-1)}{2!} x^2 \dots$$

$$E^p f(x) = f(x + ph)$$

$$F = 1 + \Delta$$

$$y = \left[1 + p\Delta + \frac{p(p-1)}{2!} \Delta^2 + \frac{p(p-1)(p-2)}{3!} \Delta^3 + \dots \right] y_0$$

$$y = \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots \right]$$

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x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0			
x_1	y_1	$\Delta y_0 = y_1 - y_0$	$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	
x_2	y_2	$\Delta y_1 = y_2 - y_1$	$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
x_3	y_3	$\Delta y_2 = y_3 - y_2$		

- how much formula evaluated depends on the number of datapoints.

- 2 datapoints \rightarrow linear

- 3 datapoints \rightarrow quadratic

polynomial is 1 less order of # of datapoints

• Question

The following are data from the steam table

Temperature	140	150	160	170	180
Pressure	3.685	4.854	6.302	8.076	10.225

- Using Newton's formula find pressure of the steam for a temperature of $x^\circ C$

Step 1: Calculate h

$$h = 10$$

Step 2: Calculate p

$$p = \frac{x - x_0}{h}$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
140	3.685				
150	4.854	1.169	0.279	0.047	0.002
160	6.302	1.448	0.326	0.049	
170	8.076	1.774	0.375		
180	10.225	2.149			

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$$y = 3.685 + \frac{P(1.169)}{2} + \frac{P(P-1)0.279}{6} + \frac{P(P-1)(P-2)0.047}{24} + \frac{P(P-1)(P-2)(P-3)0.002}{120}$$

$$= 3.685 + 1.169P + 0.1395 P(P-1) + 0.00733 P(P-1)(P-2) + 0.000083 P(P-1)(P-2)(P-3)$$

$$\Delta y = y_1 - y_0$$

$$\Delta^2 y = \Delta y_1 - \Delta y_0$$

$$\Delta^3 y = \Delta^2 y_1 - \Delta^2 y_0$$

$$\Delta^4 y = \Delta^3 y_1 - \Delta^3 y_0$$

→ Week 2: Lecture 1

→ Newton Backward difference formula

x	y	∇y
x_0	y_0	$\nabla y_0 = y_1 - y_0$
x_1	y_1	$\nabla y_1 = y_2 - y_1$
x_2	y_2	$\nabla y_2 = y_3 - y_2$
x_3	y_3	

In backward, $\Delta y_i = y_i - y_{i-1}$ where $i = n+1$ in start (start from end/back)

$$P = \frac{x - x_m}{h}$$

$$\text{Formula: } y = y_n + p \nabla y + \frac{P(P+1)}{2!} \nabla^2 y_n + \frac{P(P+1)(P+2)}{3!} \nabla^3 y_n + \dots$$

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Question: Apply ~~Newton's~~ forward formula to find value of $f(x)$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
2.5	24.145					
3.0	22.043	$\Delta y_0 = -2.102$	0.284	-0.47		
3.5	20.225	$\Delta y_1 = -1.81$	0.237		0.009	
4.0	18.644	$\Delta y_2 = -1.58$	0.199	-0.038	0.006	-0.003
4.5	17.262	$\Delta y_3 = -1.38$	0.167	-0.032		
5.0	16.047	$\Delta y_4 = -1.215$				

1. Equally spaced data so Newton's forward/backward used

$$2. p = \frac{x - x_0}{h}$$

$$= \frac{x - 2.5}{0.5}$$

$$= 2x - 5$$

3. Now by using NDDF (finding polynomial)

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \Delta^4 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)(p-4)}{5!} \Delta^5 y_0$$

$$= 24.145 + (2x-5)(-2.102) + \frac{(2x-5)(2x-5-1)(0.284)}{2!} +$$

$$+ \frac{(2x-5)(2x-5-1)(2x-5-2)(-0.47)}{3!} + \frac{(2x-5)(2x-5-1)(2x-5-2)(2x-5-3)0.009}{4!}$$

$$+ \frac{(2x-5)(2x-5-1)(2x-5-2)(2x-5-3)(2x-5-4)(-0.003)}{5!}$$

=

If $x = 3.2$ given in the start

$$p = \frac{x - x_0}{h} \quad (\text{find value of } p)$$

$$= \frac{3.2 - 2.5}{0.5}$$

$$= 1.4$$

Now by using NODF formula, substitute value of p and find specific y .

if we have to find functional value in the starting region of the values we use Newton's forward, and if in ending region, we use Newton's backward. (answer same in both cases)

→ Gauss formulas

Gauss forward and backward formula is used for finding values in the central region.

In central region, the formula we chose depends on the value of p

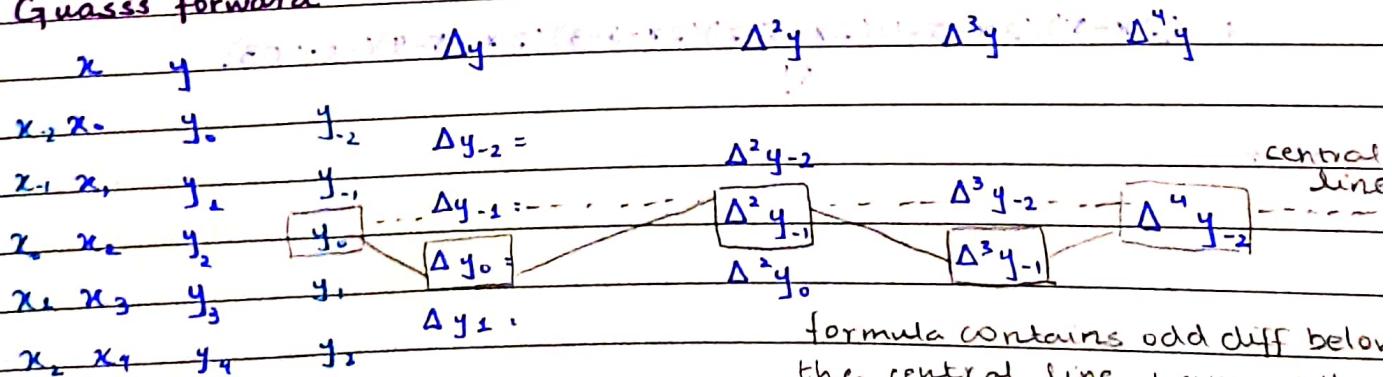
→ Gauss forward $\therefore p: 0-1$

→ Gauss backward $p: -1-0$

$p = \frac{x - x_0}{h}$ for both forward and backward

→ Week 2: Lecture 2

→ Gauss's forward



formula contains odd diff below the central line + even on the central line

Formula

$$y_p = y_0 + p \Delta y_0 + \left[\frac{p(p-1)}{1 \cdot 2} \right] \Delta^2 y_{-1} + \left[\frac{(p+1)p(p-1)}{1 \cdot 2 \cdot 3} \right] \Delta^3 y_{-2} + \dots$$

$$\left[\frac{(p+1)p(p-1)(p-2)}{1 \cdot 2 \cdot 3 \cdot 4} \right] \Delta^4 y_{-3} + \dots$$

here $0 < p < 1$

→ Gauss's backward formula

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0	y_1	y_2	y_3	y_4
x_1	y_1	y_2	y_3	y_4	
x_2	y_2	y_3	y_4		
x_3	y_3	y_4			
x_4	y_4				

--- central line

This formula contains odd differences above the central line and even differences on the line

Formula:

$$y_p = y_0 + (p) \Delta y_{-1} + \left[\frac{(p+1)p}{1 \cdot 2} \right] \Delta^2 y_{-1} + \left[\frac{(p+1)p(p-1)}{1 \cdot 2 \cdot 3} \right] \Delta^3 y_{-2} + \left[\frac{(p+2)(p+1)p(p-1)}{1 \cdot 2 \cdot 3 \cdot 4} \right] \Delta^4 y_{-3} + \dots$$

In Gauss's forward formula we chose the upward value as x_0

In Gauss's backward formula we chose the ^{down}backward value as x_0

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→ Stirling's formula

Average of two Gauss's formula gives the Stirling's formula:

Formula:

$$y_p = y_0 + p \left[\frac{\Delta y_0 + \Delta y_{-1}}{2} \right] + \left[\frac{p^2}{2!} \right] (\Delta^2 y_{-1}) + \left[\frac{p(p^2 - 1^2)}{3!} \right] \left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right] \\ + \left[\frac{p^2(p^2 - 1^2)}{4!} \right] (\Delta^4 y_{-2}) + \dots$$

It is used to interpolate the values of the function from the value

$$P \left(-\frac{1}{2} \leq p \leq \frac{1}{2} \right)$$

Good estimates if $\frac{-1}{4} \leq p \leq \frac{1}{4}$

Note:

1. Formula involves means of odd differences just above and below the central line and even differences on this line

y_0	Δy_{-1}	$\Delta^2 y_{-1}$	$\Delta^3 y_{-2}$	$\Delta^4 y_{-2}$	$\Delta^5 y_{-3}$	$\Delta^6 y_{-3}$	central line:
	↓ Mean	↓ Mean	↓ Mean	↓ Mean	↓ Mean	↓ Mean	

2. p must lie between $-\frac{1}{2}$ to $\frac{1}{2}$ while using this formula

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⇒ Week 3: Lecture 1

Gauss's Forward: $0 < p < 1$

Gauss's Backward: $-1 < p \leq 0$

Stirling's formula: $-0.5 < p < 0.5$

best estimation: $-0.25 \leq p < 0.25$

Bessel's formula: $0.25 \leq p < 0.75$

→ Bessel's Formula

Formula:

$$y_p = \frac{y_0 + y_1}{2} + \left(p - \frac{1}{2}\right)(\Delta y_0) + \left[\frac{p(p-1)}{2!}\right] \left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right]$$

$$+ \left[\frac{(p-\frac{1}{2})p(p-1)}{3!}\right] \Delta^3 y_{-1} +$$

$$\left[\frac{(p+1)p(p-1)(p-2)}{4!}\right] \left[\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right] + \dots$$

$$\bullet \quad \frac{1}{4} \leq p \leq \frac{3}{4}$$

- formula involves odd differences below the central line and means of the even differences on and below the line

$$\begin{array}{c} y_0 \\ y_1 \\ y_2 \end{array} \left. \begin{array}{c} \Delta y_0 \\ \Delta^2 y_0 \end{array} \right\} \left. \begin{array}{c} \Delta^2 y_{-1} \\ \Delta^3 y_{-1} \end{array} \right\} \left. \begin{array}{c} \Delta^3 y_{-2} \\ \Delta^4 y_{-2} \\ \Delta^4 y_{-1} \end{array} \right\}$$

↑ ↑ ↑

Mean Mean Mean

- coefficients of all odd differences are 0 when $p = \frac{1}{2}$. Thus, special case is known as the formula for interpolating to highs.

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→ Newton Divided Difference formula for unequal intervals

divided differences ($\Delta f(x_0), (x_0, x_1), f(x_0, x_1)$) representation

1st order differences

$$\Delta y_0 = \frac{y_1 - y_0}{x_1 - x_0} \quad (x_0, x_1)$$

2nd DD

$$\Delta^2 y_0 = \frac{\Delta y_1 - \Delta y_0}{x_2 - x_0} \quad (x_0, x_1, x_2)$$

3rd DD

$$\Delta^3 y_0 = \frac{\Delta^2 y_1 - \Delta^2 y_0}{x_3 - x_0} \quad (x_0, x_1, x_2, x_3)$$

$$\Delta y_1 = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Delta^2 y_1 = \frac{\Delta y_2 - \Delta y_1}{x_3 - x_1}$$

$$\Delta^3 y_1 = \frac{\Delta^2 y_2 - \Delta^2 y_1}{x_4 - x_1}$$

$$\Delta y_2 = \frac{y_3 - y_2}{x_3 - x_2}$$

$$\Delta^2 y_2 = \frac{\Delta y_3 - \Delta y_2}{x_4 - x_2}$$

$$\Delta y_3 = \frac{y_4 - y_3}{x_4 - x_3}$$

4th DD

$$\Delta^4 y_0 = \frac{\Delta^3 y_1 - \Delta^3 y_0}{x_4 - x_0} \quad (x_0, x_1, x_2, x_3, x_4)$$

→ Question:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 5$	150	$\Delta y_0 = 121$			
$x_1 = 7$	392		$\Delta^2 y_0 = 24$		
$x_2 = 11$	1452	$\Delta y_1 = 265$	$\Delta^2 y_1 = 32$	$\Delta^3 y_0 = 1$	
$x_3 = 13$	2366	$\Delta y_2 = 457$	$\Delta^2 y_2 = 42$	$\Delta^3 y_1 = 1$	$\Delta^4 y_0 = 0$
$x_4 = 17$	5202	$\Delta y_3 = 709$			

Formula:

$$y = 150 + (x-5)(121) + (x-5)(x-7)(24) + (x-5)(x-7)(x-1)(1)$$

$$= 150 + 121x - 605 + (x^2 - 12x + 35)(24) + (x^2 - 12x + 35)(x-11)$$

$$= 150 + 121x - 605 + 24x^2 - 288x + 840 + x^3 - 11x^2 - 12x^2 + 132x + 35x - 35$$

$$= x^3 + x^2 + 350$$

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→ Week 3: Lecture 2

→ Lagrange's Formula

Formula: $y = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} y_1 + \dots + \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} y_3 + \dots$

- use when datapoints are less (5-6)

→ Lagrange's Inverse Formula

Formula: $x = \frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)(y_0-y_4)} x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)(y-y_4)}{(x_0-x_1)(x-x_2)(x-x_3)(x-x_4)} + \dots$

→ Numerical Differentiation

Formula: $y_p = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots$

where $p = \frac{x-x_0}{h}$

for more accuracy we take derivative to find values rather than derivative of polynomial (in equal intervals)
for unequal intervals,

$$y' = \frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}$$

Guany & Bessel derivatives

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$$\frac{dy}{dp} = 0 + \Delta y_0(1) + \frac{\Delta^2 y_0}{2!}(2p-1) + \frac{\Delta^3 y_0}{3!}(3p^2 - 6p + 2) + \dots$$

$$= \frac{\Delta y_0}{2!} + \frac{(2P-1)\Delta^2 y_0}{3!} + \dots$$

$$\frac{dy}{dx} = \frac{1}{b}$$

$$\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dz}$$

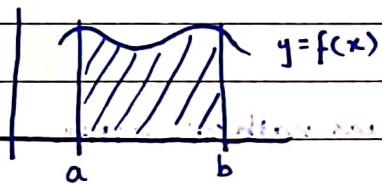
$$= \frac{1}{h} \left[\Delta y_0 + \frac{(2P-1)}{2!} \Delta^2 y_0 + \frac{(3P^2 - 6P + 2)}{3!} \Delta^3 y_0 + \dots \right]$$

⇒ Week 4: Lecture 1

Numerical Integration

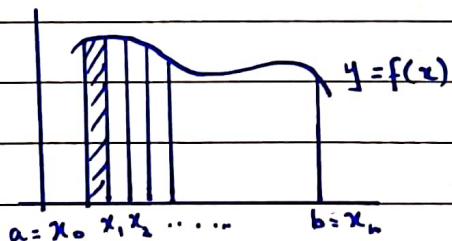
$$\int \frac{x^2 e^{x^2/2}}{x+1} dx \quad \text{can't be integrated by parts}$$

$$f(x) = \frac{x^2 e^{x/2}}{x+1}$$



Trapezoidal Rule

1. Divide the interval $[a, b]$ into n equal sub-intervals.



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→ Limits x_0, x_1

$$\int_{x_0}^{x_1} f(x) dx \approx \int_{x_0}^{x_1} (y_0 + p\Delta y_0) dx \quad \text{Newton forward formula}$$
$$p = \frac{x - x_0}{h}$$

substitution

After substitution

$$= h \int (y_0 + p\Delta y_0) dp$$

$$= h \left[y_0 p + \frac{p^2}{2} \Delta y_0 \right]_0$$

$$= h \left[y_0 + \frac{1}{2} (y_1 - y_0) \right]$$

$$= h \left[y_0 + \frac{y_1 - y_0}{2} \right]$$

$$= \frac{h}{2} (y_0 + y_1)$$

$$x = x_0 + ph$$

$$\frac{dx}{dp} = h$$

$$dx = h dp$$

$$\text{when } x \rightarrow x_0, p \rightarrow 0$$

$$x \rightarrow x_1, p \rightarrow 1$$

$$y_1 = y_0 + ph$$
$$x_0 + ph = x_1 + ph$$

$$p = 1$$

→ Limits x_1, x_2

$$\int_{x_1}^{x_2} f(x) dx \approx \int_{x_1}^{x_2} (y_1 + p\Delta y_1) dx$$

limits

$$p = \frac{x - x_1}{h}$$

After substitution

$$= h \int (y_1 + p\Delta y_1) dp$$

⋮

$$= \frac{h}{2} (y_1 + y_2)$$

$$x = x_1 + ph$$

$$dx = h dp$$

$$x \rightarrow x_1, p \rightarrow 0$$

$$x \rightarrow x_2, p \rightarrow 1$$

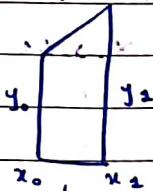
$$\frac{h}{2} (y_1 + y_2)$$

$$\frac{h}{2} (y_{n-1} + y_n)$$

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Formula:

$$\int_a^b f(x) dx = \int_{x_0}^{x_1} f(x) dx + \int_{x_1}^{x_2} f(x) dx + \dots + \int_{x_{n-1}}^{x_n} f(x) dx$$
$$= \frac{h}{2} (y_0 + y_1) + \frac{h}{2} (y_1 + y_2) + \frac{h}{2} (y_2 + y_3) + \dots + \frac{h}{2} (y_{n-1} + y_n)$$
$$= \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})]$$



$$\text{Area } \square = hy$$

$$\text{Area } \Delta = \frac{1}{2} h (y_2 - y_0)$$

$$\text{Area} = hy_0 + \frac{h}{2} (y_2 - y_0)$$

$$= \frac{h}{2} (y_2 - y_0) + hy_0$$

Question: Calculate $\int_0^1 \frac{1}{1+x^2} dx$, using trapezoidal rule

consider $n=4$

$$a=0, b=1$$

$$h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$$

$$x_0 = a = 0$$

$$x_1 = x_0 + h = 0 + \frac{1}{4} = \frac{1}{4}$$

$$x_2 = x_1 + h = \frac{1}{4} + \frac{1}{4} = \frac{2}{4}$$

$$x_3 = x_2 + h = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$$

$$x_4 = x_3 + h = \frac{3}{4} + \frac{1}{4} = 1$$

$$\int_a^b f(x) dx$$

$$y = \frac{1}{1+x^2}$$

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$$y_0 = \frac{1}{1+x_0^2} = \frac{1}{1+0} = 1 \quad y_3 = \frac{1}{1+x_3^2} = \frac{1}{1+(\frac{3}{4})^2} = \frac{16}{25}$$

$$y_2 = \frac{1}{1+x_1^2} = \frac{1}{1+(\frac{1}{4})^2} = \frac{16}{17} \quad y_4 = \frac{1}{1+x_4^2} = \frac{1}{1+1^2} = \frac{1}{2}$$

$$y_1 = \frac{1}{1+x_2^2} = \frac{1}{1+(\frac{1}{2})^2} = \frac{4}{5}$$

In formula,

$$= \frac{h}{2} \left[(y_0 + y_4) + 2(y_2 + y_3) \right]$$

$$= \frac{1}{8} \left[(1 + \frac{1}{2}) + 2(\frac{16}{17} + \frac{4}{5} + \frac{16}{25}) \right]$$

→ Week 5: Lecture 1

Numerical Integration

Simpson's Method

$$\int_{x_0}^{x_2} f(x) dx = \int_{x_0}^{x_2} \left(y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 \right) dx \rightarrow 1.$$

$$p = \frac{x - x_0}{h}$$

$$x = x_0 + ph$$

$$dx = h$$

$$dp$$

$$dx = h dp$$

$$\text{when } x \rightarrow x_0, p \rightarrow 0$$

$$x \rightarrow x_2, p \rightarrow 2$$

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$$1 \Rightarrow \int_{x_0}^{x_2} f(x) dx = h^2 \int_{y_0}^{y_2} (y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0) dp$$

$$= h \left[y_0 p + \frac{p^2}{2!} \Delta y_0 + \left(\frac{p^3}{3} - \frac{p^2}{2} \right) \frac{\Delta^2 y_0}{2} \right]_0^2$$

$$= h \left[2y_0 + 2\Delta y_0 + \left(\frac{8}{3} - 2 \right) \frac{\Delta^2 y_0}{2} \right]$$

$$= h [2y_0 + 2(y_1 - y_0) + \frac{2}{3} (y_2 - 2y_1 + y_0)]$$

$$= h \left[\frac{y_0}{3} + \frac{4y_1}{3} + \frac{y_2}{3} \right]$$

$$= \frac{h}{3} (y_0 + 4y_1 + y_2)$$

$$\Delta^2 y_0 = \Delta(\Delta y_0)$$

$$= \Delta(y_1 - y_0)$$

$$\Delta y_1 - \Delta y_0$$

$$= y_2 - y_1 - y_1 + y_0$$

$$= y_2 - 2y_1 + y_0$$

$$\frac{h}{3} (y_0 + 4y_1 + y_2)$$

$$\frac{h}{3} (y_2 + 4y_3 + y_4)$$

⋮

$$\text{Formula: } \frac{h}{3} [(y_0 + y_1) + 4(y_1 + y_3 + y_5 + \dots + y_{2N-1}) + 2(y_2 + y_4 + \dots + y_{2N-2})]$$

$$y = f(x) = e^{-x}$$

$$\text{Question: } \int_0^1 e^{-x} dx \quad h = 0.5, \quad h = 0.25, \quad h = 0.125$$

$$2N = \frac{b-a}{h}$$

$$x_0 = 0, 0.5, 1$$

$$2N = \frac{1-0}{0.5}$$

$$2N = 2$$

~~approximate~~

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→ Errors

Trapezoidal

$$E = -\frac{(b-a)}{12} f''(\eta) h^2 \quad a < \eta < b$$

$$\eta = \text{cata}$$

h power tells accuracy

n & c & b

Simpson

$$E = -\frac{(b-a)}{180} f''(\eta) h^4 \quad a < \eta < b$$

$$C\left(\frac{h}{2}\right)^2 = \frac{ch^2}{4}$$

→ for previous question

$$\text{Trapezoidal: } E = -\frac{(1-0)}{12} f''(\eta)$$

$$f(x) = e^{-x}$$

$$|E| = \frac{1}{12} e^{-\eta} \quad 0 < \eta < 1$$

$$f'(x) = -e^{-x}$$

$$-\frac{e^{-\eta}}{12} < |E| < \frac{e^{-\eta}}{12}$$

$$f''(x) = e^{-x}$$

$$-\frac{e^{-1}}{12} < |E| < \frac{1}{12}$$

lower bound

upper bound

→ Week 5: Lecture 2

Romberg Integration

Step size h

$$h = 1$$

$O(h^2)$

$$I_h^{(0)} = 0.683939721$$

$O(h^4) m=1$

$$I_h^{(1)} = 0.63233680$$

$O(h^6) m=2$

$$\frac{h}{2} = 0.5$$

$$I_{h/2}^{(0)} = 0.645235190$$

$$I_{h/2}^{(1)} = 0.63134175$$

$$I_{h/2}^{(2)} = 0.632120875$$

$$\frac{h}{4} = 0.25$$

$$I_{h/4}^{(0)} = 0.635409429$$

$$I_{h/4}^{(1)} = 0.63212414$$

$$\frac{h}{8} = 0.125$$

$$I_{h/8}^{(0)} = 0.632943418$$

$$I_{h/8}^{(2)} =$$

Stop here as required
accuracy achieved
(no need to even complete
column)

Romberg Formula based on Trapezoidal Rule

$$I_r^m(h) = 4^m I_r^{m-1}\left(\frac{h}{2}\right) - I_r^{m-1}(h)$$

$$4^{m-1}$$

$$m = 1, 2, \dots$$

where

$$I_r^0(h) = I_h$$

$$I_r(h) = I_h$$

$$\int e^{-x} dx = 0.6321205588$$

$$h = 1, 0.5, 0.25, 0.125$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$I_h \quad I_{h/2} \quad I_{h/4} \quad I_{h/8}$$

continue solving the table until
required accuracy is achieved (6 dp)

Calculation

$$m = 1$$

$$I_r^1(h) = 4^1 I_r^0\left(\frac{h}{2}\right) - I_r^0(h) =$$

$$I_r^1\left(\frac{h}{2}\right) = 4 I_r^0\left(\frac{h}{4}\right) - I_r^0\left(\frac{h}{2}\right) =$$

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$$m = 2$$

$$I_b^2 = \frac{4^2 I_{h_2}^1 - I_h^1}{15} =$$

In the above table (last page)

Extrapolations: 4

Functional evaluations: 8

Romberg formula based on Simpson's Rule

$$I_s^m(h) = \frac{4^{m+1} I_s^{m-1}(h_2) - I_s^{m-1}(h)}{4^{m+1} - 1}$$

$$\text{where } I_s^0(h_2) = I_{h_2}$$

$$I_s^0(h) = I_h$$

⇒ Week 6: Lecture 2

Finding Root of the equation

1. Bisection Method

• Root of the equation: $f(x) = 0$

• Intermediate Value Theorem

Statement:

If f is a function which is continuous at every point of the interval $[a, b]$ and $f(a) < 0, f(b) > 0$

then there exists a pt 'c' such that

$$f(c) = 0$$

Question

$$f(x) = x^3 - 2x + 1$$

Put values of x in $f(x)$ and find one x where $f(x) > 0$ and one x where $f(x) < 0$

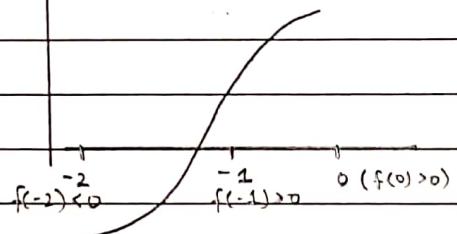
$$\rightarrow x = -2 \rightarrow f(x) = -3$$

$$\rightarrow x = 0 \rightarrow f(x) = 1$$

- $[-2, 0]$

$$\begin{matrix} \downarrow \\ a \end{matrix} \quad \begin{matrix} \downarrow \\ b \end{matrix}$$

$$c = \frac{a+b}{2} = \frac{-2+0}{2} = -1$$



$f(c) = f(-1) = 2 > 0$ hence we will consider interval $[-2, -1]$

- $[-2, -1]$

$$c = \frac{-2-1}{2} = -1.5$$

* smaller the initial interval less the # of iterations required.

$$f(-1.5) = 0.625 > 0$$

∴ new interval $[-2, -1.5]$

- $[-2, -1.5]$

$$c = \frac{-2-1.5}{2} = -1.75$$

$$f(-1.75) \leq 0$$

∴ new interval $[-1.75, -1.5]$

- $[-1.75, -1.5]$

$$c = \frac{-1.75-1.5}{2} = -1.625$$

$$f(-1.625) = -0.04 < 0$$

∴ new interval $[-1.625, -1.5]$

- $[-1.625, -1.5]$

$$c = \frac{-1.625-1.5}{2} = -1.5625$$

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$$f(-1.5625) = 0.3103 > 0$$

∴ new interval $[-1.625, -1.5625]$

$$\cdot c = \frac{-1.625 - 1.5625}{2} = -1.59375$$

$$f(-1.59375) = 0.1393 > 0$$

∴ new interval $[-1.625, -1.59375]$

$$\cdot c = \frac{-1.625 - 1.59375}{2} = -1.609375$$

$$f(-1.609375) = 0.0503 > 0$$

∴ new interval $[-1.625, -1.609375]$

$$\cdot c = \frac{-1.625 - 1.609375}{2} = -1.6171875$$

$$f(-1.6171875) < 0$$

∴ new interval $[-1.6171875, -1.609375]$

$$\cdot c = \frac{-1.6171875 - 1.609375}{2} = -1.61328125$$

$$f(-1.61328125) > 0$$

∴ new interval $[-1.6171875, -1.61328125]$

$$\cdot c = \frac{-1.6171875 - 1.61328125}{2} = -1.615234375$$

$$f(c) > 0$$

∴ $[-1.6171875, -1.615234375]$

$$\cdot c = \frac{-1.6171875 - 1.615234375}{2} = -1.61621125$$

$$f(c) > 0$$

∴ $[-1.6171875, -1.61621125]$

$$\cdot c = \frac{-1.6171875 - 1.61621125}{2} = -1.6166995$$

$$f(c) > 0$$

∴ $[-1.617, -1.617]$ accurate upto 3 dp

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To find no. of intervals

$$\frac{b-a}{2^n} < \epsilon$$

n : # of iterations

$$\log(b-a) - \log(2^n) < \log \epsilon$$

$$\log(b-a) - n \log 2 < \log \epsilon$$

$$\log(b-a) - \log \epsilon < n \log 2$$

$$\log(b-a) - \log \epsilon < n \log 2$$

for correct to 3 dp, put $\epsilon = 10^{-3}$

→ Week 7: Lecture 1

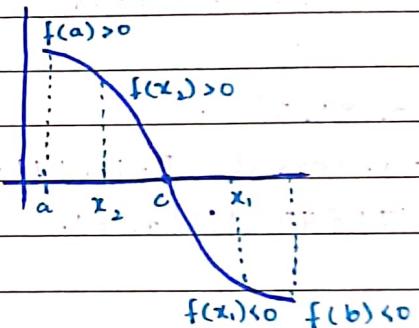
1. Bisection Method

consider $f(x) = 0$

Step 1: $f(a) \cdot f(b) < 0$

Step 2:

$$x_1 = \frac{a+b}{2}$$



$$f(x_1) < 0$$

$$\therefore [a, x_1]$$

$$f(a) + f(x) - f(b)$$

$$x_2 = \frac{a+x_1}{2}$$

$$f(x_2) < 0$$

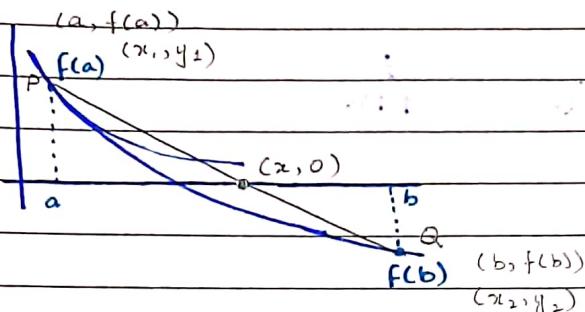
$$\therefore [a, x_2]$$

$$f(a) + f(x_2) - f(x_1)$$

2. Regular Falsi Method

Consider the equation

$$f(x) = 0$$



$$\text{Step 1: } f(b) \cdot f(a) < 0$$

Step 2: Find the first approximation to the root x , by joining the chord AC for which cuts axis

$$\frac{b-a}{2h} < \epsilon$$

Equation of the chord PQ is

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

$$x_1 = a, x_2 = b$$

$$y_1 = f(a), y_2 = f(b)$$

$$x = x, y = 0$$

$$\frac{-f(a)}{x-a} = \frac{f(b)-f(a)}{b-a}$$

$$-f(a)(b-a) = (f(b)-f(a))(x-a)$$

$$-bf(a) + af(a) = xf(b) - xf(a) - af(b) + af(a)$$

$$-bf(a) = x(f(b)-f(a)) - af(b)$$

$$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

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Question: Find the root of $xe^x = 2$, by the regular falsi method.

→ Step 1:

$$f(x) = 0$$

$$f(x) = xe^x - 2 = 0$$

$$f(0) < 0$$

$$f(1) > 0 \quad [0, 1]$$



→ Step 2:

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= 0 - \frac{1 \cdot f(0)}{f(1) - f(0)}$$

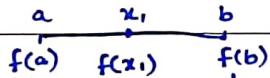
$$= \frac{2}{f(1) - f(0)}$$

$$= \frac{2}{0.718282 - (-2)}$$

$$x_1 = 0.735759$$

$$f(x_1) < 0$$

$$\therefore [x_1, b]$$



$$[0.735759, 1]$$

$$x_2 = \frac{x_1 f(b) - b f(x_1)}{f(b) - f(x_1)}$$

$$= 0.735759(0.718282) - 1(-0.4644228)$$

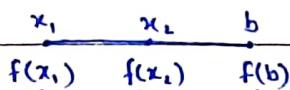
$$0.718282 - (-0.4644228)$$

$$= 0.839521$$

$$f(x_2) < 0$$

$$\therefore [x_2, b]$$

$$[0.839521, 1]$$



Date: _____

$$\begin{aligned}x_3 &= \frac{x_2 f(b) - b f(x_2)}{f(b) - f(x_2)} \\&= \frac{0.839521(0.718282) - 1(0.943707)}{0.718282 - (-0.056293)} = 0.851184\end{aligned}$$

$$\begin{aligned}f(x_3) &= -0.00617087 < 0 \\&\therefore [0.851184, 1]\end{aligned}$$

$$\begin{aligned}x_4 &= \frac{x_3 f(b) - b f(x_3)}{f(b) - f(x_3)} \\&= \frac{0.851184(0.718282) - 1(-0.00617087)}{0.718282 - (-0.00617087)} = 0.852452\end{aligned}$$

$$\begin{aligned}f(x_4) &= -0.000668827 < 0 \\&\therefore [0.852452, 1]\end{aligned}$$

$$\begin{aligned}x_5 &= \frac{x_4 f(b) - b f(x_4)}{f(b) - f(x_4)} \\&= \frac{0.852452(0.718282) - 1(-0.000668827)}{0.718282 - (-0.000668827)} = 0.852589\end{aligned}$$

$$\begin{aligned}f(x_5) &= -0.000070577594 < 0 \\&\therefore [0.852589, 1]\end{aligned}$$

$$\begin{aligned}x_6 &= \frac{x_5 f(b) - b f(x_5)}{f(b) - f(x_5)} \\&= \frac{0.852589(0.718282) - 1(-0.000070578)}{0.718282 - (-0.000070578)} = 0.852603\end{aligned}$$

$$\begin{aligned}f(x_6) &= -8.776 \times 10^{-6} < 0 \\&\therefore [0.852603, 1]\end{aligned}$$

$$\begin{aligned}x_7 &= \frac{x_6 f(b) - b f(x_6)}{f(b) - f(x_6)} \\&= \frac{0.852603(0.718282) - 1(-8.776 \times 10^{-6})}{0.718282 - (-8.776 \times 10^{-6})} = 0.852605 \approx 0.85261\end{aligned}$$

$$\begin{aligned}f(x_6) &> 0 \\&\therefore [0.85260, 0.85261]\end{aligned}$$

→ Week 7: Lecture 2

Iteration Method or Fixed point iterative method

Step 0: $f(x) = 0$

Step 1: $f(a)f(b) < 0$

Step 2: Rewrite the given function in the form of
 $x = F(x)$

$$\text{eg } f(x) = \cos x - 4x + 5 = 0$$

$$x = \frac{1}{4}(\cos x + 5)$$

\downarrow

$F(x)$

$$x_0 = F(x_0)$$

$$x_1 = F(x_1)$$

$$1. 4x = \cos x + 5$$

$$x = \frac{1}{4} \cos x + 5$$

OR

$$2. \cos x = 4x - 5$$

$$x = \cos^{-1}(4x - 5)$$

Condition for convergence

$$|F'(x)| < 1 \quad [a, b]$$

$$x_k = F(x_{k+1})$$

where $k = 1, 2, \dots$

Question: $f(x) = x^3 - x - 1 = 0$

$[a, b]$

$$f(1) = -ve$$

$$f(1)f(2) < 0$$

$$f(2) = +ve$$

∴ interval $[1, 2]$

Step 3: $x = F(x)$

$$① x = x^3 - 1 \rightarrow F(x)$$

$$F'(x) = 3x^2$$

Date: _____

$$|F'(x)| = 3x^2 \quad [1, 2]$$

$3(1) = 3 > 1$ hence reject this $F(x)$

② $x^3 - x - 1 = 0$

$$x = (x+1)^{1/3}$$

↓
 $F(x)$

$$|F'(x)| = \frac{1}{3} (x+1)^{-2/3}$$
$$= \frac{1}{3(x+1)^{2/3}}$$

$$F'(1) < 1$$

$$F'(2) < 1 \quad \text{hence accept this } F(x)$$

$$F(x) = (x+1)^{1/3}$$

Take an initial guess, it can be any middle value, to take extreme point values a few conditions must be specified.

$$x_0 = 1.5$$

$$x_1 = F(x_0)$$

$$= F(1.5)$$

$$F(1.5) = (1.5 + 1)^{1/3} = 1.3572$$

$$x_2 = F(x_1) = 1.3309$$

$$x_3 = 1.3259$$

$$x_4 = 1.3249$$

$$x_5 = 1.3248\dots$$

$$x_6 = 1.3247$$

$$x_7 = 1.3247$$

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Proof

$$x = F(x) \rightarrow A$$

initial guess = x_0

$$x_1 = F(x_0) \rightarrow B$$

$$x_2 = F(x_1)$$

⋮

$$x_{k+1} = F(x_k) \rightarrow 1.$$

where $k = 0, 1, 2, \dots$ will give further approximations

Formula (1) is known as the iteration formula

Subtracting B from A

$$x_1 - x_0 = F(x_0) - F(x)$$

$$x - x_1 = F(x) - F(x_0)$$

Mean value theorem

$$\frac{F(x) - F(x_0)}{x - x_0} = F'(E_0) \quad x < E_0 < x_0$$

$$F(x) - F(x_0) = F'(E_0)(x - x_0)$$

$$2 \Rightarrow x - x_1 = F'(E_0)(x - x_0)$$

$$x - x_2 = F'(E_1)(x - x_1)$$

$$x - x_3 = F'(E_2)(x - x_2)$$

⋮

$$x - x_n = F'(E_{n-1})(x - x_{n-1})$$

Now multiply all the LHS and RHS values,

$$(x - x_1)(x - x_2)(x - x_3) \dots (x - x_n) = F'(E_0)(x - x_0) F'(E_1)(x - x_1) \dots F'(E_{n-1})(x - x_{n-1})$$

$$(x - x_n) = (x - x_0) F'(E_0) F'(E_1) \dots F'(E_{n-1})$$

↓ all cancel out on
RHS + LHS except 1 on
each side ..

If the maximum value of $|F'(x)| < 1$, then

$$|F'(E_0)|, |F'(E_1)|, \dots, |F'(E_{n-1})| < 1$$

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⇒ Weeks 8: Lecture 1

Sufficient Condition for convergence of fixed point method

Statement: If $x = \alpha$ is exact root of $f(x) = 0$ and I is the interval which contains α . If $F(x)$ and $F'(x)$ are continuous in interval where $x = F(z)$ is equivalent form of $f(x) = 0$.

Now if $|F'(z)| < 1$, $\forall z \in [a, b]$, then the sequence of approximations x_1, x_2, \dots, x_{n+1} converges to root exact of α .

Proof: To find the root of the equation $f(x) = 0$ by fixed point method $f(x) = 0$ can be written as

$$x = F(x) \rightarrow (iii)$$

Let $x = x_0$ be an initial approximation to a root α , then the 1st approximation x_1 is given

$$x_1 = F(x_0) \rightarrow (iii)$$

and

$$x_2 = F(x_1) \rightarrow (iv)$$

In general

$$x_{k+1} = F(x_k) \rightarrow (v)$$

where $k = 0, 1, 2, \dots$

Now, subtracting (iii) from (ii) we get

$$x_1 - x = F(x_0) - F(x)$$

$$x - x_1 = F(x) - F(x_0) \rightarrow (vi)$$

by mean value theorem

$$\frac{F(x) - F(x_0)}{x - x_0} = F'(E_0) \quad x < E_0 < x_0$$

$$\Rightarrow F(x) - F(x_0) = (x - x_0) F'(E_0)$$

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$$(Vii) \rightarrow x - x_1 = (x - x_0) F'(E_0)$$

$$x - x_2 = (x - x_1) F'(E_1)$$

$$x - x_3 = (x - x_2) F'(E_2)$$

:

$$x - x_n = (x - x_{n-1}) F'(E_{n-1})$$

Multiplying all these equations we get

$$x - x_n = (x - x_0) F'(E_0) F'(E_1) F'(E_2) \dots F'(E_{n-1}) \quad (Vii)$$

If the maximum value of $|F'(x)| < 1$, then each of the quantities

$$|F'(E_0)|, |F'(E_1)|, |F'(E_2)| \dots < 1$$

$$vii \rightarrow x - x_n < (x - x_0) \lambda \cdot \lambda \cdot \lambda \dots \lambda$$

$$|x - x_n| < |(x - x_0)| \lambda^n$$

$$\lim_{n \rightarrow \infty} (x - x_0) \lambda^n = 0$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

$$x - x_n \approx 0$$

$$x \approx x_n$$

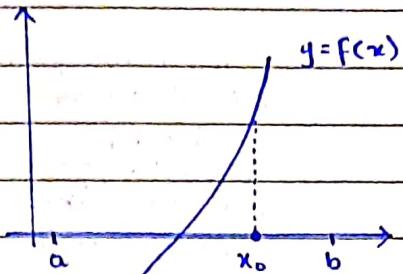
: The system of approximations $x_0, x_1, x_2, \dots, x_k, x_{k+1}, \dots$ converges to exact root α

if $|F'(x)| < 1, \forall [a, b]$

Date: _____

4. \Rightarrow Newton Raphson Method

Let $f(x) = 0$



1. $[a, b]$

$$f(a) \cdot f(b) < 0$$

2. x_0, x_1

\downarrow
chosen
randomly

$$x_0, f(x_0)$$

$$x_1, f(x_1)$$

The equation of the tangent at (x_0, y_0)

The tangent line cuts the x-axis at $x = x_1$, where $y = 0$

$$y - y_0 = f'(x_0)(x - x_0)$$

$$-y_0 = f'(x_0)(x - x_0)$$

$$-\frac{y_0}{f'(x_0)} = x - x_0$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

⋮

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$k = 0, 1, 2, \dots$$

Date: _____

Section: Find the real root of $3x = \cos x + 1$ by using the Newton Raphson Method.

$$f(x) = 0$$

$$3x - \cos x - 1 = 0$$

$$f'(x) = 3 + \sin x$$

$$[0, 1]$$

$$f(a) \cdot f(b) < 0$$

-ve	+ve
0	1

$$x_0 = 0.5 \text{ (guess)}$$

Extreme points as initial guess

$$[a, b]$$

If $f(a) < f(b)$ then a is taken as 1st approximation else b

$$f(x_0) = -0.377583 = -ve$$

Calculation in Radians:

∴ new interval $[0.5, 1]$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.5 - \frac{-0.377583}{3.47943} = 0.60852$$

$$f(x_1) = 0.005060 = +ve$$

∴ new interval $[0.5, 0.60852]$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.60852 - \frac{0.005061}{3.571654} = 0.607103$$

$$f(x_2) = 4.86029 \times 10^{-6} = +ve$$

∴ new interval $[0.5, 0.607103]$

$$x_3 = 0.607103 - \frac{4.86029 \times 10^{-6}}{3.570491} = 0.6071016$$

$$f(x_3) = -3.33544 \times 10^{-8}$$

∴ new interval $[0.6071016, 0.607103]$

$$x_4 = 0.6071016 + \frac{-3.33544 \times 10^{-8}}{3.570489} = 0.607101609$$

∴ interval $[0.607102, 0.607103]$

Date: _____

⇒ Week 8 : Lecture 2

Rate of Convergence of Newton-Raphson Method

Quadratic

Let α be the exact value of the root of the equation $f(x) = 0$

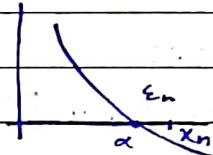
$$x_n = \alpha + \epsilon_n \rightarrow 1.$$

$$x_{n+1} = \alpha + \epsilon_{n+1} \rightarrow 2$$

By Newton's formula

$$1. x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\alpha + \epsilon_{n+1} = \alpha + \epsilon_n - \frac{f(\alpha + \epsilon_n)}{f'(\alpha + \epsilon_n)}$$



$$\epsilon_{n+1} = \epsilon_n - \frac{f(\alpha + \epsilon_n)}{f'(\alpha + \epsilon_n)}$$

$$= \epsilon_n - \frac{f(\alpha) + \epsilon_n f'(\alpha) + \frac{\epsilon_n^2}{2!} f''(\alpha) + \dots}{f'(\alpha) + \epsilon_n f''(\alpha) + \frac{\epsilon_n^2}{2!} f'''(\alpha) + \dots}$$

$$= \epsilon_n - \frac{f(\alpha) + \epsilon_n f'(\alpha) + \frac{\epsilon_n^2 f''(\alpha)}{2}}{f'(\alpha) + \epsilon_n f''(\alpha)}$$

$$= \frac{f(\alpha) + \epsilon_n f'(\alpha) + \frac{\epsilon_n^2 f''(\alpha)}{2}}{f'(\alpha) + \epsilon_n f''(\alpha)}$$

$$f(\alpha) = 0$$

$$= \frac{\epsilon_n f'(\alpha) + \frac{\epsilon_n^2 f''(\alpha)}{2} - f(\alpha) - \epsilon_n f'(\alpha) - \frac{\epsilon_n^2 f''(\alpha)}{2}}{f'(\alpha) + \epsilon_n f''(\alpha)}$$

$$= \frac{\epsilon_n^2 f''(\alpha)}{f'(\alpha) + \epsilon_n f''(\alpha)}$$

$$\epsilon_{n+1} = \frac{\epsilon_n^2 f''(\alpha)}{2 [f'(\alpha) + \epsilon_n f''(\alpha)]}$$

$$= \frac{\epsilon_n^2 f''(\alpha)}{2 f'(\alpha) \left[1 + \frac{\epsilon_n f''(\alpha)}{f'(\alpha)} \right]}$$

(Rate at which
exact and approxi-
mate converge → order
of convergence)

→ neglect this value.

$$= K \epsilon_n^2$$

$$\text{where } K = \frac{f''(\alpha)}{2 f'(\alpha)}$$

Date: _____

Condition of convergence of Newton-Raphson

Newton's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = \phi(x_n)$$

where

$$x = \phi(x)$$

$$\phi(x) = x - \frac{f(x)}{f'(x)}$$

$$|f'(x)| < 1$$

$$|\phi'(x)| < 1$$

$$\phi'(x) = 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2}$$

→ (quotient rule for derivative)

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

$$|\phi'(x)| = \left| \frac{[f'(x)]^2 - [f'(x)]^2 + f(x)f''(x)}{[f'(x)]^2} \right|$$

$$|\phi'(x)| = \frac{|f(x)f''(x)|}{|f'(x)|^2} < 1$$

Hence the condition for convergence of Newton-Raphson method is

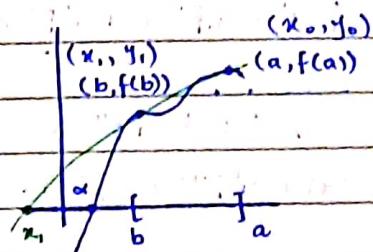
$$|\phi'(x)| < 1$$

5. Secant Method

$$f(x) = 0$$

$$f(a) \cdot f(b) \leq 0 \quad x$$

$$[a, b] \quad \checkmark$$



$$3. x_0, x_1, f(x_0), f(x_1)$$

$$4. x_2 = ?$$

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}$$

$$x - x_0 = x_1 - x_0$$

$$(x_0, y_0), (x_1, y_1), (x_2, 0)$$

$$0 - y_0 = \frac{y_1 - y_0}{x_1 - x_0}$$

$$x_2 - x_0 = \frac{x_1 - x_0}{y_1 - y_0}$$

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$$x_2 = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0}$$

$$= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

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→ Numerical Solution of linear system of equations

1. Direct Method

2. Indirect or Iterative Method → Gauss Jacobi

↳ Gauss Seidel

Indirect or Iterative Method

1: Gauss Jacobi Method

⇒ Step 1: Given

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

⇒ Step 2 : Check diagonally dominant condition

$$|a_1| > |b_1| + |c_1|$$

$$|b_2| > |a_2| + |c_2|$$

(if not alter/rearrange equations)

$$|c_3| > |a_3| + |b_3|$$

if still not satisfied, this method can't be used

Step 3: eg

$$8x - 3y + 2z = 20$$

$$|8| > |-3| + |2| \quad \checkmark$$

$$4x + 11y - z = 33$$

$$|4| > |11| + |-1| \quad \checkmark$$

$$6x + 3y + 12z = 35$$

$$|6| > |3| + |12| \quad \checkmark$$

Step 3:

$$x^{(k+1)} = \frac{1}{a_1} (d_1 - b_1 y^{(k)} - c_1 z^{(k)})$$

$$y^{(k+1)} = \frac{1}{b_2} (d_2 - a_2 x^{(k)} - c_2 z^{(k)})$$

$$z^{(k+1)} = \frac{1}{c_3} (d_3 - a_3 x^{(k)} - b_3 y^{(k)}) \quad k = 0, 1, 2, \dots$$

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→ Step 4:

$$k = 0$$

$$x^{(0)} = \frac{1}{a_1} (d_1 - b_1 y^{(0)} - c_1 z^{(0)})$$

$$y^{(0)} = \frac{1}{b_2} (d_2 - a_2 x^{(0)} - c_2 z^{(0)})$$

$$z^{(0)} = \frac{1}{c_3} (d_3 - a_3 x^{(0)} - b_3 y^{(0)})$$

eq

$$x^{(0)} = \frac{1}{8} (20 + 3y^{(0)} - 2z^{(0)}) = 2.5 \quad (x_0, y_0, z_0) = (0, 0, 0)$$

$$y^{(0)} = \frac{1}{11} (33 - 4x^{(0)} + z^{(0)}) = 3.0$$

$$z^{(0)} = \frac{1}{12} (35 - 6x^{(0)} - 3y^{(0)}) = 2.9166$$

Up to 10 iterations required

$$\rightarrow x^{(2)} = \frac{1}{8} (20 + 3(3) - 2(2.9166)) = 2.89585$$

$$y^{(2)} = \frac{1}{11} (33 - 4(2.5) + (2.9166)) = 2.356054$$

$$z^{(2)} = \frac{1}{12} (35 - 6(2.5) - 3(3)) = 0.916667$$

$$\rightarrow x^{(3)} = \frac{1}{8} (20 + 3(2.356054) - 2(0.916667)) = 3.1543535$$

$$y^{(3)} = \frac{1}{11} (33 - 4(2.89585) + (0.916667)) = 2.030297$$

$$z^{(3)} = \frac{1}{12} (35 - 6(2.89585) - 3(2.356054)) = 0.8797281$$

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$$\rightarrow x^{(4)} = \frac{1}{8} (20 + 3(2.030297) - 2(0.8797281)) = 3.0414$$

$$y^{(4)} = \frac{1}{11} (33 - 4(3.1543535) + 0.8797281) = 1.9329$$

$$z^{(4)} = \frac{1}{12} (35 - 6(3.1543535) - 3(2.030297)) = 0.8319$$

$$\rightarrow x^{(5)} = \frac{1}{8} (20 + 3(1.9329) - 2(0.8319)) = 3.0169$$

$$y^{(5)} = \frac{1}{11} (33 - 4(3.0414) + 0.8319) = 1.9697$$

$$z^{(5)} = \frac{1}{12} (35 - 6(3.0414) - 3(1.9329)) = 0.9127$$

$$\rightarrow x^{(6)} = \frac{1}{8} (20 + 3(1.9697) - 2(0.9127)) = 3.0104$$

$$y^{(6)} = \frac{1}{11} (33 - 4(3.0169) + 0.9127) = 1.9859$$

$$z^{(6)} = \frac{1}{12} (35 - 6(3.0169) - 3(1.9697)) = 0.9158$$

$$\rightarrow x^{(7)} = \frac{1}{8} (20 + 3(1.9859) - 2(0.9158)) = 3.0158$$

$$y^{(7)} = \frac{1}{11} (33 - 4(3.0104) + 0.9158) = 1.9886$$

$$z^{(7)} = \frac{1}{12} (35 - 6(3.0104) - 3(1.9859)) = 0.91496$$

$$\rightarrow x^{(8)} = \frac{1}{8} (20 + 3(1.9886) - 2(0.91496)) = 3.0169$$

$$y^{(8)} = \frac{1}{11} (33 - 4(3.0158) + 0.91496) = 1.9865$$

$$z^{(8)} = \frac{1}{12} (35 - 6(3.0158) - 3(1.9886)) = 0.9116$$

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$$\rightarrow x^{(9)} = 3.0170$$

$$y^{(9)} = 1.9858$$

$$z^{(9)} = 0.91156$$

$$\rightarrow x^{(10)} = 3.0168$$

$$y^{(10)} = 1.9858$$

$$z^{(10)} = 0.91169$$

Formula

If $x^{(r)}, y^{(r)}, z^{(r)}$ are the r^{th} iterates, then

$$x^{(r+1)} = \frac{1}{a_1} (d_1 - a_1 y^{(r)} - c_1 z^{(r)})$$

$$y^{(r+1)} = \frac{1}{b_2} (d_2 - a_2 x^{(r)} - c_2 z^{(r)})$$

$$z^{(r+1)} = \frac{1}{c_3} (d_3 - a_3 x^{(r)} - b_3 y^{(r)})$$

The process is continued till convergency is secured.

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2. Gauss Seidel Method

(Step 1 and Step 2 same as Gauss Jacobi

Step 3

$$x^{(k+1)} = \frac{1}{a_1} (d_1 - b_{11}y^{(k)} - c_{12}z^{(k)})$$

$$y^{(k+1)} = \frac{1}{b_2} (d_2 - a_{21}x^{(k+1)} - c_{23}z^{(k)})$$

$$z^{(k+1)} = \frac{1}{c_3} (d_3 - a_{31}x^{(k+1)} - b_{32}y^{(k+1)})$$

this will reduce the number of iterations by half

z 's value will get accurate first (2 recent values), then y 's, then x 's

→ Week 9: Lecture 2

Direct Method (exact solution)

1. Gauss Elimination Method

Given,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = d_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = d_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = d_3$$

Make $a_{21} = 0$

$$R_2 - \frac{a_{21}}{a_{11}} R_1 \rightarrow R_2$$

$$a_{21} - \frac{a_{21}}{a_{11}} \cdot a_{11} = 0$$

$$a_{22} - \frac{a_{21}}{a_{11}} \cdot a_{12} = a_{22}^{(2)}$$

$$a_{23} - \frac{a_{21}}{a_{11}} \cdot a_{13} = a_{23}^{(2)}$$

$$d_2 - \frac{a_{21}}{a_{11}} \cdot d_1 = d_2^{(2)}$$

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Make $a_{31} = 0$

$$R_3 - \frac{a_{31}}{a_{11}} R_1 \rightarrow R_3$$

$$\rightarrow a_{31} - \frac{a_{31}}{a_{11}} \cdot a_{11} = 0$$

$$\rightarrow a_{32} - \frac{a_{31}}{a_{11}} \cdot a_{12} = a_{32}^{(2)}$$

$$\rightarrow a_{33} - \frac{a_{31}}{a_{11}} \cdot a_{13} = a_{33}^{(2)}$$

After the above 2 operations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = d_1$$

$$0 + a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 = d_2^{(2)}$$

$$0 + a_{32}^{(2)}x_2 + a_{33}^{(2)}x_3 = d_3^{(2)}$$

Now make $a_{32} = 0$

$$R_3 - \frac{a_{32}}{a_{22}} R_2 \rightarrow R_3$$

$$0 - \frac{a_{32}(0)}{a_{22}} = 0$$

$$a_{32} - \frac{a_{32}}{a_{22}} a_{22}^{(2)} = 0$$

$$a_{33}^{(2)} - \frac{a_{32}}{a_{22}} a_{23}^{(2)} = a_{33}^{(3)}$$

$$d_3^{(2)} - \frac{a_{32}}{a_{22}} d_2^{(2)} = d_3^{(3)}$$

After this

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = d_1$$

$$0 + a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 = d_2^{(2)}$$

$$0 + 0 + a_{33}^{(3)}x_3 = d_3^{(3)}$$

Date: _____

Example

$$0x_1 + 8x_2 + 2x_3 = -7$$

$$3x_1 + 5x_2 + 2x_3 = 8$$

$$6x_1 + 2x_2 + 8x_3 = 26$$

Step 1 : $R_2 \leftrightarrow R_1$

$$3x_1 + 5x_2 + 2x_3 = 8$$

$$0 + 8x_2 + 2x_3 = -7$$

$$6x_1 + 2x_2 + 8x_3 = 26$$

Step 2 :

$$R_3 - \frac{6}{3} R_1 \rightarrow R_3$$

$$6 - 2(3) = 0$$

$$2 - 2(5) = -8$$

$$8 - 2(2) = 4$$

$$26 - 2(8) = 10$$

$$\left[\begin{array}{ccc|c} 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \\ 0 & -8 & 4 & 10 \end{array} \right]$$

Step 3 :

$$R_3 - \frac{8}{-8} R_2 \rightarrow R_3$$

$$\left[\begin{array}{ccc|c} 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \\ 0 & 0 & 6 & 3 \end{array} \right]$$

$$R_3 + R_2 \rightarrow R_3$$

$$\text{Eqn. } 3x_1 + 5x_2 + 2x_3 = 8$$

$$8x_2 + 2x_3 = -7$$

$$6x_3 = 3$$

$$\text{Solve: } x_3 = \frac{1}{2}$$

$$x_2 = -1$$

$$x_1 = 4$$

Date: _____

→ Week 10: Lecture 1

Crass Elimination Method

Partial pivoting

$$\begin{cases} 3x_1 - 4x_2 + 5x_3 = -1 \\ -3x_1 + 2x_2 + x_3 = 1 \\ \boxed{6x_1} + 8x_2 - x_3 = 35 \end{cases}$$

$R_1 \geq R_3$ Biggest value in col 1: check ms
 $|1|, |1-3|, |16|$

$$6x_1 + 8x_2 - x_3 = 35$$

$$-3x_1 + 2x_2 + x_3 = 1$$

$$3x_1 - 4x_2 + 5x_3 = -1$$

$$\begin{array}{l} R \sim \left[\begin{array}{ccc|c} 6 & 8 & -1 & : 35 \\ 0 & 6 & 0.5 & : 37/2 \\ 0 & \boxed{-8} & 5.5 & : -37/2 \end{array} \right] \quad R_2 - \frac{-3}{6} R_1 \\ \quad R_2 + \frac{1}{2} R_1 \\ \quad R_3 - \frac{1}{2} R_1 \end{array}$$

$$\begin{array}{l} R \sim \left[\begin{array}{ccc|c} 6 & 8 & -1 & : 35 \\ 0 & -8 & 5.5 & : -37/2 \\ 0 & 6 & 0.5 & : 37/2 \end{array} \right] \quad R_3 + \frac{6}{8} R_2 \\ \quad R_3 + \frac{3}{4} R_2 \end{array}$$

$$\begin{array}{l} R \sim \left[\begin{array}{ccc|c} 6 & 8 & -1 & : 35 \\ 0 & -8 & 5.5 & : -37/2 \\ 0 & 0 & 37/8 & : 37/8 \end{array} \right] \quad R_3 : \frac{8}{37} \\ \quad R_3 \end{array}$$

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Total Pivoting

$$3x_1 - 4x_2 + 5x_3 = -1$$

$$-3x_1 + 2x_2 + x_3 = 1$$

$$6x_1 + 8x_2 - x_3 = 35$$

$$C = \max_{k \leq i \leq n} |a_{ik}^{(k)}|$$

$$|a_{kk}^{(k)}| = C$$

$$C_1 = 5 \quad 3$$

$$C_2 = 3 \quad 3$$

$$C_3 = 8 \quad 6$$

max value = pivot element

(if no such option, this method fails.
shift to partial pivoting
(if this scenario occurs)

$$-3x_1 + 2x_2 + x_3 = 1$$

$$3x_1 - 4x_2 + 5x_3 = -1$$

$$6x_1 + 8x_2 - x_3 = 35$$

$$\left[\begin{array}{ccc|c} -3 & 2 & 1 & : 1 \\ 3 & -4 & 5 & : -1 \\ 6 & 8 & -1 & : 35 \end{array} \right] \quad R_2 + R_1, \quad R_3 + 2R_1$$

$$\sim \left[\begin{array}{ccc|c} -3 & 2 & 1 & : 1 \\ 0 & -2 & 6 & : 0 \\ 0 & 12 & -1 & : 37 \end{array} \right] \quad C_2 = 6 \quad 2 \\ C_3 = 12 \quad 12$$

$$\sim \left[\begin{array}{ccc|c} -3 & 2 & 1 & : 1 \\ 0 & 12 & -1 & : 37 \\ 0 & -2 & 6 & : 0 \end{array} \right] \quad R_3 + \frac{R_2}{6}$$

$$\sim \left[\begin{array}{ccc|c} -3 & 2 & 1 & : 1 \\ 0 & 12 & -1 & : 37 \\ 0 & 0 & \frac{35}{6} & : \frac{37}{6} \end{array} \right]$$

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→ Ill Conditioning

$$x_1 + 3x_2 = 4$$

$$3x_1 + 9.0000x_2 = 12.00001$$

Exact Solution

$$x_1 = x_2 = 1$$

$$x_1 = 1.0, x_2 = -2$$

If we change the no. of dp and answer changes a lot, then the system is highly sensitive.

→ Week 10 : Lecture 2

Direct Factorization

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$Ax = B$$

$$\left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right] = \left[\begin{array}{c} b_1 \\ b_2 \\ b_3 \end{array} \right]$$

$$\text{Step 2 } A = LU$$

$$\left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right] = \left[\begin{array}{ccc} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{array} \right] \left[\begin{array}{ccc} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{array} \right]$$

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Take either $U_{11}, U_{22}, U_{33} = 1 \rightarrow$ DoLittle Method

or $U_{11}, U_{22}, U_{33} \neq 1 \rightarrow$ Crout's Method

$$A = \begin{bmatrix} U_{11}L_{11} & L_{12}U_{12} & L_{13}U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + L_{22}U_{22} & L_{21}U_{13} + L_{22}U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + L_{33}U_{33} \end{bmatrix}$$

Computing values here

We know that,

$$AX = B \text{ and } A = LU$$

$$\therefore LUX = B$$

$$\text{Let } UX = Z \rightarrow (i)$$

then $LZ = B$ (find Z from here and put it in (i) to find X)

→ Week 11: Lecture 1

Q. Solve the following system using Direct (Do Little Method) and using Indirect method.

$$4x_1 + 10x_2 + 8x_3 = 44$$

$$10x_1 + 26x_2 + 26x_3 = 128$$

$$8x_1 + 26x_2 + 61x_3 = 214$$

Direct Factorization

$$\text{Step 1: } AX = B \quad \left[\begin{array}{ccc|c} 4 & 10 & 8 & 44 \\ 10 & 26 & 26 & 128 \\ 8 & 26 & 61 & 214 \end{array} \right]$$

Step 2:

$$A = LU \quad \left[\begin{array}{ccc|c} 4 & 10 & 8 & 1 & 0 & 0 \\ 10 & 26 & 26 & L_{21} & 1 & 0 \\ 8 & 26 & 61 & L_{31} & L_{32} & 1 \end{array} \right] \left[\begin{array}{ccc} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{array} \right]$$

Date:

$$\begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + (l_{32}u_{23} + u_{33}) \end{bmatrix}$$

$$u_{11} = 4 \quad u_{12} = 10 \quad ; \quad u_{13} = 8$$

$$l_{21} = \frac{10}{4} = 2.5 \quad ; \quad l_{21}u_{12} + u_{22} = 26 - 2.5 \cdot 10 = 1 \quad ; \quad u_{23} = 26 - 2.5(8) = 6$$

$$l_{31} = \frac{8}{4} = 2 \quad l_{32} = \frac{26 - (2)(10)}{9} = 6 \quad u_{33} = 61 - 2(8) = 6(6)$$

$$U = \begin{bmatrix} 4 & 10 & 8 \\ 0 & 1 & 6 \\ 0 & 0 & 9 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2.5 & 1 & 0 \\ 2 & 6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.5 & 1 & 0 \\ 2 & 6 & 1 \end{bmatrix} \begin{bmatrix} 4 & 10 & 8 \\ 0 & 1 & 6 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix}$$

$$\text{Let } UX = Z$$

$$\begin{bmatrix} 4 & 10 & 8 \\ 0 & 1 & 6 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix} \rightarrow (i)$$

$$LZ = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.5 & 1 & 0 \\ 2 & 6 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix} \quad ; \quad z_1 = 44 \rightarrow \text{put in (i)}$$

$$z_2 = 18 \quad ; \quad z_3 = 18 \quad ;$$

$$UX = Z \quad \begin{bmatrix} 4 & 10 & 8 \\ 0 & 1 & 6 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 44 \\ 18 \\ 18 \end{bmatrix} \quad ; \quad x_1 = 11.8 \quad ; \quad x_2 = 6 \quad ; \quad x_3 = 2$$

Gauss Seidel Method

$$\left[\begin{array}{ccc|c} 4 & 10 & 8 & : 44 \\ 10 & 26 & 26 & : 128 \\ 8 & 26 & 61 & : 214 \end{array} \right]$$

Checking condition

$$|a_{11}| > |b_{11}| + |c_{11}|$$

This condition doesn't come true for any of the three equations hence we cannot solve this by indirect method.

→ Week 11: Lecture 2

Symmetric Factorization

(Cholesky Method / Square root Method.)

$$A = LU$$

$$Eq. 7 Pg. # 172$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$AX = B$$

$$A = LL^T$$

$$A = UU^T$$

$$AX = B$$

$$LL^T X = B$$

$$\text{Let } L^T X = Z$$

$$LZ = B$$

Example 9:

$$\left[\begin{array}{cccc|c} 4 & -1 & 0 & 0 & x_1 \\ -1 & 4 & -1 & 0 & x_2 \\ 0 & -1 & 4 & -1 & x_3 \\ 0 & 0 & -1 & 4 & x_4 \end{array} \right] = \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right]$$

→ Differential Equations

Given $\frac{dy}{dx} = f(x, y)$

$$y(x_0) = y_0$$

$$y(x) = ?$$

Single Step Method

1. Taylor series method for 1st ODE, 2nd ODE system of eq;
2. Picard Method
3. Euler Method, Euler Modified, Euler improved Method
4. RK Method of 1, 2, 3, 4

~~Order of accuracy and error analysis~~

Multistep Method

Predictor and Corrector Method

1. Adam Bashforth
2. Milne's Method

→ Taylor Series Method

1st ODE

$$\frac{dy}{dx} = f(x, y) \quad y(x_0) = y_0$$

$$y(x) = ?$$

$$y(x) = y_0 + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \dots$$

$$y(x_1) = y(x_0 + h) = y(x_0) + hy'(x_0) + \frac{h^2}{2!} y''(x_0) + \dots$$

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Week 12: Lecture 1

→ Taylor Series Method

Example

$$\frac{dy}{dx} = -xy, \quad y(0) = 1, \quad \text{find } y(0.1) = ?$$

by using Taylor Series Method

$$y' = -xy, \quad x_0 = 0, \quad y_0 = 1, \quad x_1 = 0.1, \quad h = 0.1 - 0 = 0.1$$

$$h = x_1 - x_0$$

$$y(x) = y_0 + hy'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \dots \quad (\text{wrt } x)$$

$$y_1 = y(x_1) = 1 + 0.1y'_0 + \frac{(0.1)^2}{2!} y''_0 + \frac{(0.1)^3}{3!} y'''_0$$

$$y' = -xy$$

$$y'' = -[xy' + y]$$

$$y''' = -[xy'' + 2y']$$

$$y' = -xy \rightarrow y'_0 = -x_0 y_0 = 0$$

$$y'' = -[xy' + y] \rightarrow y''_0 = -[x_0 y'_0 + y_0] = -1$$

$$y''' = -[xy'' + 2y'] \rightarrow y'''_0 = x_0 y''_0 + 2y'_0 \\ = 2(0) = 0$$

(3 non zero derivatives required)

Week 13: Lecture 1

Taylor Series Method (system of eq)
(First order)

$$y, z \left\{ \begin{array}{l} \frac{dy}{dx} = f(x, y, z) \quad y(x_0) = y_0 \\ \frac{dz}{dx} = g(x, y, z) \quad z(x_0) = z_0 \end{array} \right.$$

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$$y(x) = y_0 + hy'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \dots$$

$$z(x) = z_0 + hz'_0 + \frac{h^2}{2!} z''_0 + \frac{h^3}{3!} z'''_0 + \dots$$

Example

$$y(x) \quad \frac{dy}{dx} = x^2 z + y \quad ; \quad y(1) = 1 \quad y(1.1) = ?$$

$$z(x) \quad \frac{dz}{dx} = -xy \quad ; \quad z(1) = 2 \quad z(1.1) = ?$$

$$y(x) = y_0 + hy'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \dots$$

$$\begin{array}{ll} z_0 = 2 & z_1 = ? \\ y_0 = 1 & y_1 = ? \\ x_0 = 1 & x_1 = 1.1 \end{array}$$

$$z(x) = z_0 + hz'_0 + \frac{h^2}{2!} z''_0 + \frac{h^3}{3!} z'''_0 + \dots$$

$$h = x_1 - x_0$$

$$= 1.1 - 1$$

$$= 0.1$$

for y

$$y' = x^2 z + y \quad = 2+1=3 \quad (\text{at } x_0, y_0 = z_0)$$

$$y'' = 2xz + x_2 z' + y' \quad = 4+1^2(-1)+3 = 6$$

$$y''' = 2xz' + 2z + 2x_2 z' + x^2 z'' + y'' \quad = 2(1)(-1) + 4 + 2(1)(-1) + (-4) + 6 \\ \dots = 2$$

for z

$$z' = -yz \quad = -1$$

$$z'' = -xy' + -y \quad = -1(3) - 1 = -4$$

$$z''' = -xy'' - y' - y'$$

$$z''' = -xy''' - 2y' \quad = (-1)(6) - 2(3) = -12$$

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Taylor Series 2nd ODE

$$y(x) \frac{d^2y}{dx^2} = f(x, y, y') \quad \text{convert 2nd ODE to 1st ODE}$$

$$y(x_0) = y_0$$

$$y'(x_0) = y'_0$$

$$\text{Let } \frac{dy}{dx} = z ; \quad y(x_0) = y_0$$

$$\frac{d^2y}{dx^2} = z' ; \quad z(x_0) = z_0$$

$$\frac{dz}{dx} = f(x, y, z) \quad z(x) \quad \left. \begin{array}{l} \text{solve this as 1st ODE} \\ \dots \end{array} \right\}$$

$$\frac{dy}{dx} = z = g(x, y, z) \quad y(x) \quad \left. \begin{array}{l} \text{use derivatives of } z, \text{ no need to} \\ \text{find } z(x) \\ \text{only } y(x) \text{ required} \end{array} \right\}$$

Example

$$\frac{d^2y}{dx^2} = xy' + 3y \rightarrow 1. \quad y(0) = 1 \quad y(0.2) = ?$$

$$y'(0) = 2$$

$$\text{Let } \frac{dy}{dx} = z \quad \left. \begin{array}{l} \dots \\ \dots \end{array} \right\} \quad \text{put in eq (1)}$$

$$\frac{d^2y}{dx^2} = z' = \frac{dz}{dx}$$

$$1 \rightarrow \frac{dz}{dx} = xz + 3y$$

$$\Rightarrow \frac{dy}{dx} = z \quad \left. \begin{array}{l} x_0 \uparrow \\ y(0) = 1 \end{array} \right\} \quad \text{system of equations}$$

$$\frac{dz}{dx} = xz + 3y \quad z(0) = 2$$

$$\downarrow \quad \downarrow$$

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$$h = 0.2 - 0$$

$$= 0.2$$

$$y(x) = y_0 + hy'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \dots$$

$$y' = z = 2$$

$$z' = xz + 3y = 3$$

$$y'' = z' = 3$$

$$z'' = xz' + z + 3y' = 8$$

$$y''' = z'' = 8$$

$$z''' = z' + z' + xz'' + 3y''$$

$$y(x) = y_0 + hy'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 +$$

$$= 1 + 0.2(2) + \frac{0.2^2}{2!}(3) + \frac{0.2^3}{3!}(8)$$

$$= 1.47066$$

⇒ Week 13: Lecture 2

Picard Method

$$\text{Let } \frac{dy}{dx} = f(x, y) ; y(x_0) = y_0$$

$$\int_{x_0}^x \frac{dy}{dx} dx = \int_{x_0}^x f(x, y) dx$$

$$y(x) \Big|_{x_0}^x = \int_{x_0}^x f(x, y) dx$$

$$y(x) - y(x_0) = \int_{x_0}^x f(x, y) dx$$

$$y(x) = y(x_0) + \int_{x_0}^x f(x, y) dx$$

General form

$$y(x_n) = y(x_0) + \int_{x_{n-1}}^x f(x, y) dx$$

$$n = 1, 2, 3 \dots$$

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Question: Given the equation $\frac{dy}{dx} = 3x^2 + y$ with $y(1) = 2$
 Estimate $y(2)$ by Picard Method

$$\begin{aligned}
 y(x_1) &= y_0 + \int_{x_0}^{x_1} f(t, y_0) dt \\
 &= 2 + \int_{1}^{x_1} f(t, 2) dt \\
 &= 2 + \int_{1}^{x_1} (3t^2 + 2) dt \\
 &= 2 + \left[\frac{3t^3}{3} + 2t \right]_1^{x_1} \\
 &= 2 + [t^3 + 2t]_1^{x_1} \\
 &= 2 + [(x_1^3 + 2x_1) - (1^3 + 2(1))] \\
 &= 2 + [x_1^3 + 2x_1 - 3] \\
 &= x_1^3 + 2x_1 - 1
 \end{aligned}$$

$n = 2$

$$\begin{aligned}
 y(x_2) &= 2 + \int_{1}^{x_2} f(t, y_1) dt \\
 &= 2 + \int_{1}^{x_2} (3t^2 + t^3 + 2t - 1) dt \\
 &\quad \rightarrow f(t, y_2) = 3t^2 + y_2 \\
 &= 2 + \left[\frac{3t^3}{3} + \frac{t^4}{4} + \frac{2t^2}{2} - t \right]_1^{x_2} \\
 &= 2 + \left[t^3 + \frac{t^4}{4} + t^2 - t \right]_1^{x_2} \\
 &= 2 + \left[(x_2^3 + \frac{x_2^4}{4} + x_2^2 - x_2) - (1 + \frac{1}{4} + 1 - 1) \right] \\
 &= 2 + \left[\frac{x_2^4}{4} + x_2^3 + x_2^2 - x_2 - \frac{5}{4} \right] \\
 &= \frac{x_2^4}{4} + x_2^3 + x_2^2 - x_2 + \frac{3}{4}
 \end{aligned}$$

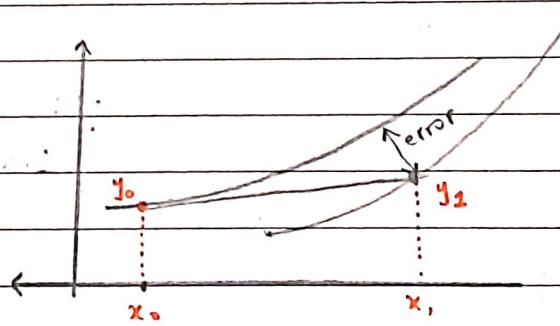
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$$y(2) = \frac{2^4}{4} + 2^3 + 2^2 - 2 + 3 \\ = 14.75$$

Euler Method

Let $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$

$$m = \frac{dy}{dx} = f(x, y)$$



$$(x_0, y_0) \quad (x_1, y_1)$$

$$y_1 - y_0 = m(x_1 - x_0)$$

$$\text{here } x_1 - x_0 = h$$

$$y_1 - y_0 = hf(x_0, y_0)$$

$$f(x_0, y_0) = m$$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_2 = y_1 + hf(x_1, y_1)$$

⋮

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

Date: _____

Question: Given the equation $\frac{dy}{dx} = 3x^2 + y$, with $y(1) = 2$.
Estimate $y(2)$ by Euler Method.

Iteration process

$$y_1 = y_0 + h f(x_0, y_0)$$

$$\frac{dy}{dx} = f(x, y) = 3x^2 + y$$

$$f(x_0, y_0) = 3x_0^2 + y_0$$

$$= 3(1)^2 + 2$$

$$= 5$$

$$y_1 = 2 + 1(5)$$

$$= 7$$

$$\begin{matrix} x_0 & y_0 \\ \uparrow & \uparrow \\ y(1) & = 2 \end{matrix}$$

$$y(2) = ?$$

$$\downarrow$$

$$h = x_1 - x_0 = 1$$

i) $h = 0.5$

$$\begin{array}{ccc} y_0 = 2 & y_1 = ? & y_2 = ? \\ \hline x_0 = 1 & x_1 = 1.5 & x_2 = 2 \\ h = 0.5 & & x_n = 2 \end{array}$$

$$y_1 = 2 + 0.5(5)$$

$$= 4.5$$

$$y(1) = 2$$

$$y(1.5) = 4.5$$

$$f(x_1, y_1) = 3x_1^2 + y_1$$

$$= 3(1.5)^2 + 4.5$$

$$= 11.25$$

$$y_2 = y_1 + h f(x_1, y_1)$$

$$= 4.5 + 0.5(11.25)$$

$$= 7.125$$

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ii) 0.25

$$y_0 = 2 \quad y_1 = 3.25$$
$$x_0 = 1 \quad x_1 = 1.25 \quad x_2 = 1.5 \quad x_3 = 1.75 \quad x_4 = 2$$

$$\rightarrow f(x_0, y_0) = 3x_0^2 + y_0 \\ = 3(1)^2 + 2 = 5$$

$$y_1 = y_0 + hf(x_0, y_0) \\ = 2 + 0.25(5) \\ = 3.25$$

$$\rightarrow f(x_1, y_1) = 3x_1^2 + y_1 \\ = 3(1.25)^2 + 3.25 \\ = 7.9375$$

$$y_2 = y_1 + hf(x_1, y_1) \\ = 3.25 + 0.25(7.9375) \\ = 5.234375$$

$$\rightarrow f(x_2, y_2) = 3x_2^2 + y_2 \\ = 3(1.5)^2 + 5.234375 = \\ = 11.984315$$

$$y_3 = y_2 + hf(x_2, y_2) \\ = 5.234375 + 0.25(11.984315) \\ = 8.230469$$

$$\rightarrow f(x_3, y_3) = 3x_3^2 + y_3 \\ = 3(1.75)^2 + 8.230469 \\ = 17.41797$$

$$y_4 = y_3 + hf(x_3, y_3) \\ = 8.230469 + 0.25(17.41797) \\ = 12.58496$$

$$\Rightarrow f(x_{11}, y_{11}) = 3x_{11}^2 + y_{11} \\ = 3(2)^2 + 12.58496$$

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→ Week 14 : Lecture 1

Euler Modified Method

$$y_{n+1} = y_n + h \left\{ f \left[x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n) \right] \right\}$$

Euler Improved Method

$$y_{n+1} = y_n + \frac{1}{2} h \left[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n)) \right]$$

Question: Given $\frac{dy}{dt} = \frac{y-t}{y+t}$ with the initial condition $y=1$ at $t=0$.

Find y approximately at $t=0.1$ in 5 steps using Euler Improved and Euler Modified.

• Euler Method Improved

$$\frac{dy}{dt} = f(t, y) = \frac{y-t}{y+t}$$

$$y_0 = 1, t_0 = 0$$

$$t_1 = t_0 + h$$

$$= 0 + 0.02$$

$$= 0.02$$

$$h = \frac{b-a}{n} = \frac{0.1-0}{5}$$

$$\begin{array}{cccccc} y_0 & = & 1 & , & y_1 & , \\ t_0 & = & 0 & , & t_1 & = 0.02 \\ t_2 & & & & t_2 & = 0.04 \\ t_3 & & & & t_3 & = 0.06 \\ t_4 & & & & t_4 & = 0.08 \\ t_5 & & & & t_5 & = 0.1 \end{array}$$

$$> 0.02$$

$$y_1 = y_0 + \frac{h}{2} \left[f(t_0, y_0) + f(t_0 + h, y_0 + hf(t_0, y_0)) \right]$$

$$= 1 + \frac{0.02}{2} \left[f(0, 1) + f(0 + 0.02, 1 + 0.02f(0, 1)) \right]$$

$$= 1 + 0.01 [1 + f(0.02, 1.02)]$$

$$= 1 + 0.01 (1.019615)$$

$$\approx 1.01961$$

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$$\bullet y_2 = y_1 + \frac{h}{2} [f(t_1, y_1) + f\{t_1+h, y_1 + hf(t_1, y_1)\}]$$

$$= 1.01961 + 0.01 [f(0.02, 1.01961) + f\{0.02+h, 1.01961 + 0.02f(0.02, 1.01961)\}]$$

$$= 1.02 + 0.01 \left[\frac{1.02 - 0.02}{0.02+0.02} + f(0.04,$$

$$= 1.02 + \frac{0.01}{0.04} (0.96 + 1.0392)$$

$$= 1.04$$

$$\bullet y_3 = y_2 + 0.01 [f(t_2, y_2) + f\{t_2+h, y_2 + hf(t_2, y_2)\}]$$

$$= 1.04 + 0.01 [f(0.04, 1.04) + f\{0.06, 1.04 + 0.02f(0.04, 1.04)\}]$$

$$= 1.04 + 0.01 [0.93 + f(0.06, 1.059)]$$

$$= 1.058 \approx 1.06$$

$$\bullet y_4 = y_3 + 0.01 [f(t_3, y_3) + f\{t_3+h, y_3 + hf(t_3, y_3)\}]$$

$$= 1.06 + 0.01 [f(0.06, 1.06) + f(0.08, 1.06 + 0.02f(0.06, 1.06))]$$

$$= 1.06 + 0.01 [0.893 + f(0.08, 1.08)]$$

$$= 1.0775 \approx 1.08$$

$$\bullet y_5 = y_4 + 0.01 [f(t_4, y_4) + f(t_4+h, y_4 + hf(t_4, y_4))]$$

$$= 1.08 + 0.01 [f(0.08, 1.08) + f(1, 1.08 + 0.02f(0.08, 1.08))]$$

$$= 1.08 + 0.01 [0.862 + f(1, 1.0972)]$$

$$\approx 1.089$$

Euler Modified Method

(same procedure only change the formula)

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→ RK - Method of Order 1

$$y_{n+1} = y_n + \underbrace{hf(x_n, y_n)}_{k_1}$$

[Same as Euler Method]

→ RK - Method of Order 2

$$y_{n+1} = y_n + \Delta y$$

and $\Delta y = k_2$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_1 = hf(x_n, y_n)$$

$$h = 0$$

~~$$y_1 = y_0 + \Delta y$$~~

~~$$\Delta y = k_2$$~~

→ RK - Method of Order 3

$$y_{n+1} = y_n + \Delta y$$

$$\Delta y = \frac{1}{6} (k_1 + 4k_2 + k_3)$$

where

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + h, y_n + 2k_2 - k_1\right)$$

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→ RK - Method of Order 4

$$y_{n+1} = y_n + \Delta y$$

$$\Delta y = \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

⇒ Week 14: Lecture 2

→ Question: Using RK Method of Order 4, Find the solution of the given DE at $x = 1.2$, $f(x, y) = y' = \frac{2xy + e^x}{x^2 + xe^x}$; $y(1) = 0$

$$y_0 = 0, x_0 = 1, x_1 = 1.2, y_1 = ?$$

(If h not given then simply x_0 and x_1)

$$y_1 = y_0 + h$$

$$\Delta y = \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = hf(x_0, y_0)$$

$$= 0.2 \left[\frac{2(1)(0) + e^1}{1^2 + 1(e)^1} \right]$$

$$= 0.2(0.73106)$$

$$= 0.14621$$

$$h = 1.2 - 1 = 0.2$$

$$\therefore k_1 = 0.2 \cdot 0.73106 = 0.14621$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.2 f(1.1, 0.0731)$$

$$= 0.1402$$

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$$\begin{aligned} k_3 &= hf \left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2} \right) \\ &= 0.2 f(1.1, 0.1402) \\ &= 0.1399 \end{aligned}$$

$$\begin{aligned} k_4 &= hf(x_0 + h, y_0 + k_3) \\ &= 0.2 f(1.2, 0.1399) \\ &= 0.1348 \end{aligned}$$

$$\begin{aligned} \Delta y &= \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \\ &= \frac{1}{6} (0.14621 + 2(0.1402) + 2(0.1399) + 0.1348) \\ &= 0.1402 \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + \Delta y \\ &= 0 + 0.1402 = 0.1402 \quad y(1-2) = 0.1402 \end{aligned}$$

→ RK System of Equations

$$y(x) \leftarrow \frac{dy}{dx} = f(x, y, z), \quad y(x_0) = y_0.$$

(can be solved using
any order of RK)

$$z(x) \leftarrow \frac{dz}{dx} = g(x, y, z), \quad z(x_0) = z_0.$$

RK-4

$$y_1(x) = y_0 + \Delta y$$

$$\Delta y = \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = hf(x_0, y_0, z_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

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$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3, z_0 + l_3\right)$$

$$z_1 = z_0 + \Delta z$$

$$\Delta z = \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4)$$

$$l_1 = hg(x_0, y_0, z_0)$$

$$l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) \quad \text{(Some how k's and l's simultaneously)}$$

$$l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$l_4 = hg\left(x_0 + h, y_0 + k_3, z_0 + l_3\right)$$

→ Second Order Differential Equations

$$\frac{d^2y}{dx^2} = f(x, y, y')$$

$$y(x_0) = y_0$$

$$y'(x_0) = y'_0$$

$$\frac{dy}{dx} = z = g(x, y, z)$$

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

$$\frac{dz}{dx} = g(x, y, z)$$

$$\left\{ \begin{array}{l} \frac{dy}{dx} = z = g(x, y, z) \\ \frac{dz}{dx} = g(x, y, z) \end{array} \right.$$

$$\left\{ \begin{array}{l} \frac{dy}{dx} = z = g(x, y, z) \\ \frac{dz}{dx} = g(x, y, z) \end{array} \right.$$

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Question: Using the RK Method of Order 1, 2, 3, 4, find the solution of the system at $x = 0.2$

$$f(x, y, z) \quad \frac{dy}{dx} = x + z \quad ; \quad y(0) = 0 \quad f(x, y, z)$$

$$\frac{dz}{dx} = x - y \quad ; \quad z(0) = 1 \quad g(x, y, z)$$

Using RK-4

$$y(0.2) \quad z(0.2)$$

$$h = x_1 - x_0$$

$$= 0.2 - 0 = 0.2$$

- $k_1 = hf(x_0, y_0, z_0)$
 $= 0.2f(0, 0, 1)$
 $= 0.2(1)$
 $= 0.2$

- $l_1 = hg(x_0, y_0, z_0)$
 $= 0.2g(0, 0, 1)$
 $= 0.2(0 - 0)$
 $= 0$

- $K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$
 $= 0.2f(0.1, 0.1, 1)$
 $= 0.22$

- $L_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$
 $= 0.2g(0.1, 0.1, 1)$
 $= 0$

- $k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$
 $= 0.2f(0.1, 0.11, 1)$
 $= 0.22$

- $l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$
 $= 0.2g(0.1, 0.11, 1)$
 $= -0.002$

- $k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$
 $= 0.2f(0.2, 0.22, 0.998)$
 $= 0.2396$

- $l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$
 $= 0.2g(0.2, 0.22, 0.998)$
 $= -0.004$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.2 + 2(0.22) + 2(0.22) + 0.2396) = 0.2199$$

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$$\begin{aligned}y_1 &= y_0 + \Delta y \\&= 0 + 0.2199 \\&= 0.2199\end{aligned}$$

$$\Delta z = \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4)$$

$$\begin{aligned}&= \frac{1}{6} (0 + 0 + 2(-0.002) - 0.004) \\&= -0.00133\end{aligned}$$

$$\begin{aligned}z_1 &= z_0 + \Delta z \\&= 0.9986\end{aligned}$$

$$y(0.2) = 0.2199 \quad z(0.2) = 0.9986$$

→ Week 15: Lecture 1

Predictor and Corrector Method

Multistep Method

$y(x_0) = y_0, y'_0, y'_1, y'_2, y'_3$
first use previous methods to find these

1. Milne's Method P-C

$$n = 3$$

$$y_{4,p} = y_0 + \frac{4h}{3} (2y'_1 - y'_2 + 2y'_3)$$

$$P: y_{n+1,p} = y_{n-3} + \frac{4h}{3} (2y'_{n-2} - y'_{n-1} + 2y'_n)$$

$$C: y_{n+1,c} = y_{n-1} + \frac{h}{3} (y'_{n-1} + 4y'_n + y'_{n+1})$$

$$y_{4,c} = y_2 + \frac{h}{3} (y'_2 + 4y'_3 + y'_4)$$

Predictor

Corrector pair:

Assignment Box

Question : Given $y' = \frac{1}{x+y}$,

$$y(0) = 2, y(0.2) = 2.0933, y(0.4) = 2.1755, y(0.6) = 2.2496$$

(i) Find $y(0.8)$ by Milne's Predictor-corrector method.

→ Milne's Predictor

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3} (2y'_{n-2} - y'_{n-1} + 2y'_n)$$

$$n=3 \quad y_{4,p} = y_0 + \frac{4h}{3} (2y'_1 - y'_2 + 2y'_3) \rightarrow (i)$$

$$\bullet \quad y'_1 = \frac{1}{x_1 + y_1} = \frac{1}{0.2 + 2.0933} \\ = 0.43605$$

$$\bullet \quad y'_2 = \frac{1}{x_2 + y_2} = \frac{1}{0.4 + 2.1755} \\ = 0.38827$$

$$\bullet \quad y'_3 = \frac{1}{x_3 + y_3} = \frac{1}{0.6 + 2.2496} \\ = 0.35096$$

Put in (i)

$$y_{4,p} = 2 + \frac{4(0.2)}{3} (2(0.43605) - 0.38827 + 2(0.35096))$$

$$= 2.3161 \rightarrow \cancel{y(0.8)} \quad y(0.8) = 2.3161 \rightarrow y_4$$

→ Milne's Corrector

$$y_{4,c} = y_2 + \frac{h}{3} (y'_1 + 4y'_3 + y'_4) \rightarrow (ii)$$

$$y'_4 = \frac{1}{x_4 + y_4} = \frac{1}{0.8 + 2.3161} \\ = 0.320914$$

if y' is not given
calculate through
previous methods,
i.e Taylor
expansion
RK

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put in iii)

$$y_{4,c} = 2.1755 + \frac{0.2}{3} (0.38827 + 4(0.35096) + 0.320914) \\ = 2.3162$$

$$y(0.8) = 2.3162$$

→ Week 15 Lecture 2

→ Adam Bashforth Predictor Corrector Method

$$P: y_{n+1,p} = y_n + \frac{h}{24} [55y'_n - 59y'_{n-1}]$$

$$C: y_{n+1,c} = y_n + \frac{h}{24} [9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}]$$

- In these methods, the first four values of y are given, if not they can be calculated using Taylor Series or RK.

→ Finite Difference Method

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!} y''(x) + \dots \quad | \text{ optional}$$

$$y(x+h) - y(x) = h(y'(x) + O(h))$$

$$y'(x) = \frac{y(x+h) - y(x)}{h}$$

$$y(x-h) = y(x) - hy'(x) + \frac{h^2}{2!} y''(x) - \frac{h^3}{3!} y'''(x) + \dots$$

$$y'(x) = \frac{y(x) - y(x-h)}{h}$$

$$\frac{y(x+h) - y(x-h)}{2h} = y'(x)$$

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$$y'(x) = \frac{y_{i+1} - y_{i-1}}{2h} \rightarrow (i)$$

$$y''(x) = \frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} \rightarrow (ii)$$

Suppose a boundary value problem

$$y'' + a(x)y' + b(x)y(x) = c(x)$$

together with the boundary conditions

$$y(x_0) = \alpha, y(x_n) = \beta \quad x \in (x_0, x_n)$$

Put values of $y'(x)$ [i] and $y''(x)$ [ii] in boundary value problem

$$\frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} + a(x) \frac{y_{i+1} - y_{i-1}}{2h} + b(x)y(x_i) = c(x_i)$$

Simplifying

$$y_{i+1} \left[1 + \frac{h}{2} a_i \right] + y_i (h^2 b_i - 2) + y_{i-1} \left[1 - \frac{h}{2} a_i \right] = a_i h^2 + c_i$$

where $i = 1, 2, \dots, n-1$

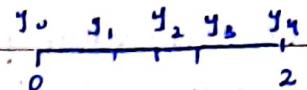
$$y_0 = \alpha, y_n = \beta, a_i = a(x_i), b_i = b(x_i), c_i = c(x_i)$$

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Question: Using the finite difference method, solve $\frac{d^2y}{dx^2} = y$ in $(0, 2)$ given $y(0) = 0$, $y(2) = 3.63$ subdividing the range of x in four equal parts.

$$h = \frac{b-a}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\frac{d^2y}{dx^2} = y_i$$



from (ii) we know

$$y''(x) = \frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} = y_i$$

~~Put~~ Simplify

$$y''(x) = y_{i+1} + y_{i-1} - 2y_i = h^2 y_i$$

$$y_{i+1} + y_{i-1} = y_i (h^2 + 2)$$

$$y_{i+1} - y_i (h^2 + 2) + y_{i-1} = 0$$

$$\text{Put } h = \frac{1}{2}$$

$$y_{i+1} - \frac{9}{4} y_i + y_{i-1} = 0 \quad (i=1, 2, 3) \rightarrow (i)$$

Hence the equations to be solved

$$y_2 - \frac{9}{4} y_1 + y_0 = 0 \rightarrow (ii)$$

$$y_3 - \frac{9}{4} y_2 + y_1 = 0 \rightarrow (iii)$$

$$y_4 - \frac{9}{4} y_3 + y_2 = 0 \rightarrow (iv)$$

using $y_0 = 0$ ($y(0) = 0$) $y_4 = 3.63$ ($y(2) = 3.63$)

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$$y_2 - \frac{9}{4} y_1 = 0 \rightarrow (v)$$

$$y_3 - \frac{9}{4} y_2 + y_1 = 0 \rightarrow (vi)$$

$$3.63 - \frac{9}{4} y_3 + y_2 = 0 \rightarrow (vii)$$

Eliminating y_1 from (v) and (vi)

$$(v) \rightarrow y_1 = \frac{4}{9} y_2$$

$$\text{in (vi)} \rightarrow y_3 - \frac{9}{4} y_2 + \frac{4}{9} y_2 = 0$$

$$y_3 - \frac{65}{36} y_2 = 0 \rightarrow (viii)$$

from (vii) and (viii)

$$(viii) \rightarrow y_3 = \frac{65}{36} y_2$$

$$\text{in (vii)} \rightarrow 3.63 - \frac{9}{4} \left(\frac{65}{36} y_2 \right) + y_2 = 0$$

$$y_2 = 1.1853$$

put y_2 in (viii)

$$y_3 = \frac{65}{36} (1.1853)$$

$$= 2.1401$$

$$(v) \rightarrow y_1 = \frac{4}{9} y_2 = \frac{4}{9} (2.1401)$$

$$= 0.5268$$

x	0	0.5	1	1.5	2	
y	0	0.5268	1.1853	2.1401	3.63	