NUMERICAL ANALYSIS

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PEARS

Numerical Analysis

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Dedicated
to
Our beloved Parents

Brief Contents

Contents

CHAPTER 7 NUMERICAL INTEGRATION 407

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Preface

Applications of mathematics invariably boil down to numerical results. The aim of this book, *Numerical Analysis*, is to present the important principles, methods and processes used for obtaining these results in a systematic and simple way with accuracy and clarity. The book can cater to the needs of undergraduate and postgraduate students of science and engineering courses offered in Indian universities. Keeping the treatment simple, the procedures are illustrated with typical examples and problems taken from various university question papers.

The topics of the book are presented in 11 chapters.

Chapter 1 gives a brief discussion of errors in approximations.

Chapter 2 deals with the direct and iterative methods of solving algebraic and transcendental equations. These methods include Ramanujan's method, Muller's method, Chebyshev's method, Graffe's root square method and Lin–Bairstow's method.

Chapter 3 spells out the direct and iterative methods of solving simultaneous linear equations. It also expounds on eigen value problems and factorization methods.

Chapter 4 elucidates the steps involved in polynomial interpolation when the arguments are spaced equally and unequally. The chapter also encompasses interpolation of cubic splines.

Chapter 5 examines the different methods of inverse interpolation.

Chapter 6 is devoted to numerical differentiation and maxima and minima.

Chapter 7 elaborates on the concepts of numerical integration and double integration.

Chapter 8 delves into the different methods of curve fitting, including the method of sum of exponentials.

Chapter 9 focuses on the numerical solution of ordinary differential equations.

Chapter 10 seeks to determine numerical solutions of partial differential equations of the types, namely elliptic, parabolic and hyperbolic.

Chapter 11 the last chapter, sheds light on the solution of difference equations.

Mathematics is a subject that can be mastered only through hard work and practice. The key word in the learning process of mathematics is practice. Remember the maxim: **"Mathematics without practice is blind and practice without understanding is futile".**

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We are grateful to the members of our family for lending us their support for the successful completion of this book. Special thanks are due to our daughter S. Sathiyapriya, our niece V. S. Anushanthi and our student J. Jayalakshmi for their help in proofreading the chapters.

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About the Authors

P. Sivaramakrishna Das started his career as Assistant Professor of Mathematics at Ramakrishna Mission Vivekananda College, Chennai, his alma mater, and retired as Professor and Head of the Department of Mathematics from the same college after an illustrious career spanning 36 years. Currently, he is Professor of Mathematics and Head of the Department of Science and Humanities, K.C.G. College of Technology, Chennai (a unit of Hindustan group of colleges).

Professor Das has done pioneering research work in the field of "Fuzzy Algebra". His paper on fuzzy groups and level sub-groups was a fundamental paper on the subject. It was the second paper published on this topic at the international level and the first paper from India in this subject.

With a teaching experience spanning over 45 years at various arts and science and engineering colleges, Prof. Das is an accomplished teacher of Mathematics at undergraduate and postgraduate levels and has guided several students in their research leading to the M.Phil. degree. He has also coached students in their preparation for IIT–JEE examinations and was a visiting professor at a few leading IIT–JEE training centres in Andhra Pradesh. Along with his wife, C. Vijayakumari, retired Professor of Mathematics, he has written 10 books covering various aspects of Engineering Mathematics catering to the syllabus of Anna University, Chennai.

C. Vijayakumari began her career as Assistant Professor of Mathematics at Government Arts College for Women, Thanjavur, and has taught at various government arts and science colleges across Tamil Nadu before retiring as Professor of Mathematics, Queen Mary's College, Chennai, with 40 years of teaching experience behind her.

As Visiting Professor of Mathematics, Prof. Vijayakumari has taught the students at two engineering colleges in Chennai. An accomplished teacher of Mathematics and Statistics at both undergraduate and postgraduate levels, she has also guided several students in their M.Phil. degree courses.

Along with her husband, Prof. P. Sivaramakrishna Das, she has written several books on Engineering Mathematics.

Errors in Numerical Computations

INTRODUCTION

In practice, the applications of Mathematics ultimately results in numerical form. These results may be from the evolution of formulae, solutions of equations or inferences drawn from tabulated data.

Numerical analysis may be described as that branch of Mathematics which provides with convenient methods for obtaining numerical solutions to such problems.

Suppose for a certain experiment the heights of a set of university students are measured. The numbers representing the heights are only approximations, true to two or three decimal places. In general the data represent approximations. Sometimes, the process used to deal with the data is approximate. So, the error in a computed result may be due to errors in the data or errors in the methods or both.

Numerical analysis deals with methods of which errors in computation is reduced to a minimum. With the advent of computers the demand for numerical methods increased rapidly in the applications to engineering and scientific fields. Today numerical methods have become an indispensable tool.

ACCURACY OF NUMBERS

The numbers $1, 2, 3, \ldots, \frac{1}{3}, \frac{3}{4}, \frac{8}{7}, \ldots$ and $\sqrt{3}, \sqrt{\frac{5}{3}}, \sqrt{\pi}$, e etc written in this form are exact numbers. If $\sqrt{3}$ is written as 1.732, then it is an approximate number. We can also write $\sqrt{3}$ as 1.73205, which is a better approximation. However, we cannot write the exact value of $\sqrt{3}$ by finite number of digits, because $\sqrt{3}$ has infinite number of decimals. Thus we deal with two types of numbers exact and approximate.

Significant Figures

The digits used to express a number meaningfully are called significant figures or significant digits. For instance the digits 1,2,3,...,9 are all significant digits in any number. But 0 may or may not be a significant figure. If 0 is used to fix the decimal point or to fill places of unknown or discarded digits, then 0 is not a significant figure.

In the number of 0.000567, the significant figures are 5, 6, 7. The zeros are not significant, because they are used to fix the decimal point. But in the number 4.5037 all the five digits are significant. Here 0 is also a significant digit.

In 0.5000, only 5 is the significant figure.

The numbers 0.0028, 0.000035 contain only two significant figures 2, 8, and 3, 5 respectively.

Rounding Off of Numbers

We know $\sqrt{3} = 1.732...$ is a never ending decimal. To use such a number in a practical computation we must cut down to a usable form such as 1.7 or 1.73 or 1.732 etc. This process of cutting off the extra digits is called rounding off of numbers.

Usually rounding off a number is done according to the following rules.

"To round off a number to n significant figures, discard all the digits to the right of the n^{th} digit, and if the $(n + 1)^{th}$ digit is

(i) less than 5, then leave the n^{th} digit unaltered
(ii) greater than 5, then increase the n^{th} digit by 1
(iii) equal to 5, then the n^{th} digit is unaltered if it is even and increase the n^{th} digit by 1 if it is odd"

A number rounded off according to this rule is said to be correct to n significant places.

For example, the following numbers are rounded off to five significant figures.

23.876345	becomes	23.876
0.876345	becomes	0.87634
4.782250	becomes	4.7822
1.823156	becomes	1.8232
76.69954	becomes	76.700

A Safe Rule

In numerical analysis, often, we have to perform a sequence of arithmetical operations on numbers. During the computation retain one more figure (or decimal place) than that given in the data and round off after the last operation has been performed. When this practice is followed, no attention is paid to rounding off rule.

ERRORS AND THEIR ANALYSIS

In numerical computations we always look for the accuracy of the result obtained.
The size of the error in the computed value is usually expressed in two ways:

(1) absolute error and (2) relative error

Absolute error = |true value – approximate value|

$= |x - x_1|$, where x is the true value and x_1 is the approximate value and it is denoted by E_a.

$\therefore$ $$E_a = |x - x_1|$$

$$\text{Relative error} = \frac{\text{absolute error}}{|\text{True value}|}$$ and it is denoted by E_r.

$\therefore$ $$E_r = \frac{E_a}{|x|} = \left|\frac{x - x_1}{x}\right|$$

Sometimes, we express as percentages which will enable us for comparison.

$\therefore$ Percentage error $= E_r \times 100$ and it is denoted by E_p.

$\Rightarrow$ $$E_p = E_r \times 100$$

Note:

(1) Absolute error is in terms of the unit used, where as relative error and percentage error are pure numbers independent of the unit of measurement.

(2) It is obvious that the absolute error is related to the number of decimal places, whereas relative error is related to the number of significant figures.

(3) If a number is correct to n decimal places, then absolute error is $\leq \frac{1}{2}10^{-n}$.

If a number is correct to n significant figures, then the relative error is $\leq \frac{1}{2}10^{1-n}$.

In other words, a number x_1 is an approximation to x to n significant figures if $\frac{|x - x_1|}{|x|} \leq \frac{1}{2}10^{1-n}$.

Classification of Errors

In numerical analysis errors are classified into two major categories.

(i) Inherent error
(ii) Truncation error

Inherent error: It is the error that is present in the statement of the problem before its analysis and solution. The inherent errors arise due to the simplication assumptions that are used in the mathematical formulation of the problem or due to the errors in measurements of the parameters of the problem.

Truncation error: It is due to those errors caused by the method. For example:

If $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$ is approximated by the cubic $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$ and computed then the error in the result is due to truncating the series.

The remainder R_4 after the fourth term in the Maclaurin's series of e^x is the **truncation error.**

We shall state some theorems without proof.

Theorem 1.1

If a number is correct to n significant figures and if a_1 is the first significant figure, then the relative error is less than $\frac{1}{a_1 \times (10)^{n-1}}$ i.e. $E_r < \frac{1}{a_1 (10)^{n-1}}$.

Theorem 1.2

If x is a number having n decimal digits and if x_1 is got by truncating to k digits, then the absolute error is $= |x - x_1| < 10^{n-k}$.

Absolute error due to rounding off to k digits $= |x - x_1| < \frac{1}{2} 10^{n-k}$.

Theorem 1.3

If the relative error of any number is $\leq \frac{1}{2 \times 10^n} = \frac{1}{2} 10^{-n}$, then the number is correct to n significant figures.

WORKED EXAMPLES

Example 1

If the number 23.876 is correct to 5 significant figures, then find the relative error.

Solution

Given the number is 23.876.

$\therefore$ the first significant figure is 2 and the number of significant digits is 5, $n = 5$, $a_1 = 2$

$\therefore$ the relative error $E_r \leq \frac{1}{2} 10^{1-n} = \frac{1}{2} 10^{1-5} = \frac{1}{2} 10^{-4}$ ■

Example 2

Consider the number 52.43, which is correct to four significant figures. Find E_a and E_r. Also find the percentage error.

Solution

Given the number is 52.43. The first significant number is 5 and the number of significant figures is 4

$$\therefore \qquad a_1 = 5, \quad n = 4$$

$\therefore$ by theorem 1 the relative error $\quad E_r < \dfrac{1}{a_1 \times 10^{n-1}}$

$$\Rightarrow \quad E_r < \frac{1}{5 \times 10^{4-1}}$$

$$= \frac{1}{5 \times 10^3}$$

$$= 0.0002$$

The n^{th} place is $\quad \dfrac{1}{100} = 0.01$

$\therefore$ the absolute error $\quad E_a < 0.01 \times \dfrac{1}{2} = 0.005 \; = 0.005$

Percentage error $E_p = E_r \times 100 \; = 0.0002 \times 100 = 0.02$ ■

Example 3

Find the absolute error and relative error in taking $\pi = 3.141593$ as $\dfrac{22}{7}$.

Solution

Given $\quad x = 3.141593$

$$x_1 = \frac{22}{7} = 3.142857$$

$$\therefore \quad E_a = |x - x_1| = |3.141593 - 3.1428571|$$

$$= |-1.258 \times 10^{-3}| = 1.258 \times 10^{-3}$$

$$E_r = \left|\frac{x - x_1}{x}\right| = \frac{1.258 \times 10^{-3}}{3.141593} = 0.0004$$ ■

Example 4

Round off the numbers 784652 and 78.4625 to four significant digits and compute E_a, E_r, E_p.

Solution

(i) Given $x = 784652$
Rounding off to 4 significant figures, we get $x_1 = 784600$

$$\therefore \quad E_a = |x - x_1| = |784652 - 784600| = 52$$

$$E_r = \left|\frac{x - x_1}{x}\right| = \frac{52}{784652} = 6.627 \times 10^{-5}$$

and $\quad E_p = E_r \times 100 = 6.627 \times 10^{-3}$

(ii) Given $x = 78.4625$

Rounding off to 4 significant figures, we get $x_1 = 78.4600$

$\therefore$
$$E_a = |x - x_1| = |78.4625 - 78.4600| = 2.5$$
$$E_r = \left|\frac{x - x_1}{x}\right| = \frac{2.5}{78.4625} = 3.186 \times 10^{-5}$$
and
$$E_p = E_r \times 100 = 3.186 \times 10^{-3}$$

Example 5

A person measured a length as 3450 cm, where as its actual length is 3445 cm and another length as 145 cm where as its actual length is 140 cm Compare their absolute and relative errors.

Solution

(i) Given $x = 3445$, $x_1 = 3450$

$\therefore$
$$E_a = |x - x_1| = |3445 - 3450| = 5$$
$$E_r = \frac{|x - x_1|}{|x|} = \frac{5}{3445} = 0.00145$$

(ii) Given $x = 140$, $x_1 = 145$

$\therefore$
$$E_a = |x - x_1| = |140 - 145| = 5$$
$$E_r = \frac{|x - x_1|}{|x|} = \frac{5}{140} = 0.0357$$

Though the absolute errors are same for both measurements, their relative errors differ more than 24 times.

A GENERAL FORMULA FOR ERROR

Let $y = f(x_1, x_2, x_3, \ldots, x_n)$ be a function of several variables $x_1, x_2, \ldots, x_n$ and be differentiable.

Let $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ be the errors in the variables so that the error in y is Δy.

We have to find the error relation.

We know that the total differential
$$dy = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2 + \ldots + \frac{\partial y}{\partial x_n} dx_n$$

$\therefore$ the error relation is
$$\Delta y = \frac{\partial y}{\partial x_1} \Delta x_1 + \frac{\partial y}{\partial x_2} \Delta x_2 + \ldots + \frac{\partial y}{\partial x_n} \Delta x_n$$

The relative error of y is $E_r = \dfrac{\Delta y}{y} = \dfrac{\partial y}{\partial x_1}\dfrac{\Delta x_1}{y} + \dfrac{\partial y}{\partial x_2}\dfrac{\Delta x_2}{y} + \ldots + \dfrac{\partial y}{\partial x_n}\dfrac{\Delta x_n}{y}$

$\therefore$ The maximum value of $E_r = \left|\dfrac{\Delta y}{y}\right| \le \left|\dfrac{\partial y}{\partial x_1}\dfrac{\Delta x_1}{y}\right| + \left|\dfrac{\partial y}{\partial x_2}\dfrac{\Delta x}{y}\right| + \ldots + \left|\dfrac{\partial y}{\partial x_n}\dfrac{\Delta x_n}{y}\right|$

Note: Suppose $y = k.\dfrac{x^m z^n}{u^p v^q}$, then $\log y = \log k + m\log x + n\log z - p\log u - q\log v$

The differential relation is

$$\frac{1}{y}dy = m\cdot\frac{1}{x}dx + n\cdot\frac{1}{z}dz - p\frac{1}{u}du - q\cdot\frac{1}{v}dv$$

$\therefore$ the error relation is

$$\frac{\Delta y}{y} = m\frac{\Delta x}{x} + n\frac{\Delta z}{z} - p\frac{\Delta u}{u} - q\frac{\Delta u}{v}$$

The relative error in y is

$$E_r = \frac{\Delta y}{y}$$

$$\Rightarrow \qquad E_r = m\frac{\Delta x}{x} + n\frac{\Delta z}{z} - p\frac{\Delta u}{u} - q\frac{\Delta v}{v}$$

Maximum relative error is $= \left|m\dfrac{\Delta x}{x}\right| + \left|n\dfrac{\Delta z}{z}\right| + \left|P\dfrac{\Delta u}{u}\right| + \left|q\dfrac{\Delta u}{v}\right|$

WORKED EXAMPLES

Example 6

Let $y = \dfrac{3x_1^2 x_2}{x_3^4}$. Find the maximum error and relative error in y if $x_1 = x_2 = x_3 = 1$ and $\Delta x_1 = \Delta x_2 = \Delta x_3 = 0.01$.

Solution

Given $$y = \frac{3x_1^2 x_2}{x_3^4}$$

$$\therefore \qquad dy = \frac{\partial y}{\partial x_1}dx_1 + \frac{\partial y}{\partial x_2}dx_2 + \frac{\partial y}{\partial x^3}dx_3$$

$\therefore$ the error relation is $\Delta y = \dfrac{\partial y}{\partial x_1}\Delta x_1 + \dfrac{\partial y}{\partial x_2}\Delta x_2 + \dfrac{\partial y}{\partial x^3}\Delta x_3$

$\therefore$ the maximum error relation Δy is $= \left|\dfrac{\partial y}{dx_1}\Delta x_1\right| + \left|\dfrac{\partial y}{\partial x_2}\Delta x_2\right| + \left|\dfrac{\partial y}{\partial x_2}\Delta x_3\right|$

But $$\frac{\partial y}{\partial x_1} = \frac{6x_1x_2}{x_3}$$

$$\frac{\partial y}{\partial x_2} = \frac{3x_1^2}{x_3}, \quad \frac{\partial y}{\partial x_3} = 3x_1^2x_2\left(-\frac{4}{x_3^5}\right)$$

At the point $x_1 = 1, x_2 = 1, x_3 = 1$

$$\frac{\partial y}{\partial x_1} = 6, \quad \frac{\partial y}{\partial x_2} = 3, \quad \frac{\partial y}{\partial x_3} = -12$$

and given that $\Delta x_1 = \Delta x_2 = \Delta x_3 = 0.01.$

When $x_1 = 1, \quad x_2 = 1, \quad x_3 = 1, \quad y = 3$

$\therefore$ maximum error is $= |6(0.01)| + |3(0.01)| + |-12(0.01)|$

$$= 0.06 + 0.3 + 0.12 = 0.21$$

Maximum relative error $= \dfrac{\text{Max}\,\Delta y}{y} = \dfrac{0.21}{3} = 0.07$

The general error formula can be used to find error in fundamental operations of arithmetic.
If $u = u_1 + u_2 + \ldots + u_n$, then $\Delta u = \Delta u_1 + \Delta u_2 + \ldots + \Delta u_n$

$\therefore E_a = \Delta u$ is the algebraic sum of absolute errors. ■

Example 7

Find the sum of the numbers 681.32, 521.7, 94.853, 5.9271, 0.0034, each being correct to its last digit. Also find the absolute error.

Solution

Here 52.7 has one decimal place

The absolute error $= \dfrac{1}{2}10^{-1} = 0.05$

This number is the one with greatest absolute

So, we round off all the numbers to two decimals.

$\therefore$ the round off numbers are 681.32, 94.85, 5.93, 0.00

So, their sum $S = 681.32 + 521.7 + 94.85 + 5.93 + 0.00 = 1303.8$

E_a = sum of absolute errors in each number

$$= \frac{1}{2}10^{-2} + \frac{1}{2}10^{-1} + \frac{1}{2}10^{-2} + \frac{1}{2}10^{-2}$$

$$= 0.005 + 0.05 + 0.005 + 0.005 = 0.065 = 0.07 \text{ to 2 places of decimal}$$

Rounding off error is 0.01.

$\therefore$ total error in S is $0.07 + 0.01 = 0.08$

$$\therefore \quad S = 1303.8 \pm 0.08$$ ■

Example 8

Evaluate the number $\sqrt{3}+\sqrt{5}+\sqrt{6}$ correct to 4 significant digits and find its absolute and relative error.

Solution

Evaluate the numbers $\sqrt{3},\sqrt{5},\sqrt{6}$ to 4 significant digits using calculator $\sqrt{3}=1.732$, $\sqrt{5}=2.236$, $\sqrt{6}=2.449$

$\therefore$ their sum $S=\sqrt{3}+\sqrt{5}+\sqrt{6}=6.417$

We know that the absolute error in a number correct to 3 decimal places is $\frac{1}{2}10^{-3}=0.0005$

$\therefore$ the absolute error in the sum of the 3 numbers is

$$E_a = 0.0005+0.0005+0.0005=0.0015=1.5\times10^{-3}$$

This shows that the sum $S=\sqrt{3}+\sqrt{5}+\sqrt{6}$ is correct to 3 significant figures only,

$$\therefore \quad S = 6.41$$

Then
$$E_r = \frac{E_a}{S} = \frac{1.5\times10^{-3}}{6.41} = 0.000234$$ ■

ERROR IN SERIES APPROXIMATION

Power series expansion of a function is a very useful technique in theory and applications. The general method for expanding functions into power series is by means of Taylor's formula.

(i) Taylor's formula for $f(x)$ about $x=a$, under valid conditions is

$$f(x)=f(a)+\frac{(x-a)}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\ldots+\frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)$$
$$+\frac{(x-a)^n}{n!}f^n[a+\theta(x-a)],\quad 0<\theta<1$$

(ii) If $a=0$, then the expansion about the origin is

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^{n-1}}{(n-1)!}f^{(n)}(0)+\frac{x^n}{n!}+f^{(n)}(\theta x),\ 0<\theta<1$$

This is called Maclaurin's formula.

The remainder after n terms is denoted by R_n and $R_n(x) = \frac{x^n}{n!} f^{(n)}(\theta_x)$

If $R_n \to 0$ as $n \to \infty$, then the series converges.

If we approximate $f(x)$ by the first n terms, then the maximum error committed in the truncation is R_n.

Conversely, if the accuracy required is given in advance, the we can find the number of terms n to be taken.

Error in Some Important Series

1. Logarithmic series

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots,$$

where $|x| < 1$ or $x = 1$; then the error committed in truncating is less than the first term neglected

$$\therefore \qquad R_n < \left|\frac{x^{n+1}}{n+1}\right|.$$

2. Binomial series

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots + \frac{m(m-1)\ldots(m-n+1)}{(n-1)!} + R_n$$

where $R_n = \frac{m(m-1)(m-2)\ldots(m-n+1)}{n!}x^n(1+\theta x)^{m-n}, \quad 0 < \theta < 1$

If $x > 0$, then $\qquad R_n < \left|\frac{m(m-1)(m-2)\ldots(m-n+1)}{n!}x^n\right|$

If $x < 0$, $n > m$, then $\quad R_n < \left|\frac{m(m-1)(m-2)\ldots(m-n+1)}{n!}\right| \frac{x^n}{(1+x)^{n-m}}.$

3. The exponential series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}e^{\theta x}, \quad 0 < \theta < 1$$

where $\qquad R_n = \frac{x^n}{n!}e^{\theta x}$

Error R_n is maximum when $\theta = 1$.

If $x > 0$, maximum. relative error $= \frac{R_n}{e^x} = \frac{x^n}{n!}$

WORKED EXAMPLES

Example 9

Compute $\log_e 1.02$ truncating after the third term. Find the error.

Solution

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + R_3$$

where
$$R_3 < \left|\frac{x^4}{4}\right|$$

Put
$$x = 0.02.$$

$\therefore$
$$\log_e(1.02) = 0.02 - \frac{(0.02)^2}{2} + \frac{(0.02)^3}{3} + R_3$$

where
$$R_3 < \left|\frac{(0.02)^4}{4}\right|$$
$$= 0.00000004$$

and
$$\log_e(1.02) = 0.02 - 0.002 + 0.0000026666$$
$$= 0.01980266$$

which is correct to 7 decimal places, because in R_3, the first significant value is 4 occurring in the 8th place.

Note: Using calculator $\log_e 1.02 = 0.019802627$ ■

Example 10

Find the number of terms n to be taken in the expansion of e^x correct to 7 significant figures, when $x = 1$.

Solution

The Maclaurin's series for

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n+1}}{(n-1)!} + \frac{x^n}{n!}e^{\theta x}, \quad 0 < \theta < 1$$

Maximum relative error $= \dfrac{x^n}{n!}$

Maximum relative error, when $x = 1$ is $\dfrac{1}{n!}$.

For 7 significant figure accuracy, we must have, by theorem 2

$$\frac{1}{n!} < \frac{1}{2 \times 10^7}$$

$\Rightarrow$
$$n! > 2 \times 10^7 = 20000000$$

$$\text{But } 11! = 39916800 > 2 \times 10^7$$
$$10! = 3628800 < 2 \times 10^7$$
$$\therefore \quad n = 11$$

This means we have to take 11 terms of the series for 7 significant figure accuracy. ■

Exercises 1.1

(1) Round of the following numbers correctly to four significant figure. 23.7642, 53266, 0.070037, 0.0052725.

(2) Find the sum of the number which are correct to the number of significant figure given 142.6, 26.23, 0.23425, 220.44, 3.42.

Answers 1.1

(1) 23.76, $53270 = 5327 \times 10$, 0.07004, 0.005272 (2) 392.92

SHORT ANSWER QUESTIONS

1. The value of π is 3.1416 correct to four decimal places, find the error.
2. Round off the numbers 34789 and 3.7256 to three significant figures.
3. Find the relative error if the number 852.43 is correct to five significant figure.
4. If the number 0.0700 is correct to 3 significant figures, find the relative error.
5. Find the absolute error if the number 0.0033543 is truncated to three decimal places.
6. If the number 25.34217 is rounded off to four significant figures then find E_a, E_r.
7. Find the absolute error in $\sqrt{6} + \sqrt{7} + \sqrt{8}$ correct to 4 significant digits.

Solution of Algebraic and Transcendental Equations

2.0 INTRODUCTION

Finding solutions of an equation of the form $f(x) = 0$ is frequently encountered in science and engineering. If $f(x)$ is a quadratic $ax^2 + bx + c$, then $f(x) = 0$ has solutions $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ in terms of the coefficients.

If $f(x)$ is cubic or biquadratic in x, then we have standard formulae to find solutions in terms of the coefficients of $f(x)$. If the degree of $f(x)$ is greater than or equal to 5, then there is no formula to solve $f(x) = 0$, only approximate solutions can be found.

Equations involving trigonometric, logarithmic, exponential, inverse trigonometric functions etc. are called transcendental equations.

For example, $e^x + 2x = 0$, $\log x + x^2 = 10$, $x - \cos x = 0$ are transcendental equations.

There is no general method to solve the transcendental equations. When the coefficients of the algebraic and the transcendental equations are pure numbers, approximate solutions can be found to any desired degree of accuracy by using numerical methods.

These methods are based on successive approximations or iterations starting with an initial approximation. For this initial approximation we locate an interval in which the root lies by using the intermediate value theorem started below.

Theorem 2.1

Let $f(x)$ be a continuous function on $[a, b]$. If $f(a)$ and $f(b)$ have opposite signs, then $f(x) = 0$ has at least one root α between a and b.

Definition 2.1: Rate of convergence of iteration method

Let α be the actual root of an equation $f(x) = 0$. An iterative method is said to be of **order p** or has **the rate of convergence p**, if p is the largest positive real number for which there exists a positive number $k \neq 0$ such that $|\epsilon_{n+1}| = k|\epsilon_n|^p$, where $\epsilon_n = x_n - \alpha$, $\epsilon_{n+1} = x_{n+1} - \alpha$ and x_n, x_{n+1} are the n^{th} and $(n+1)^{th}$ iterative roots of $f(x) = 0$.

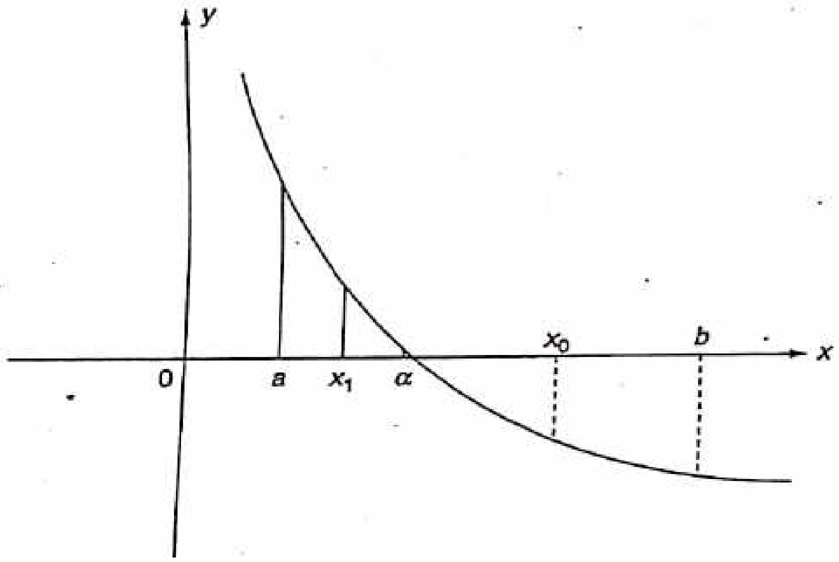
The constant k is called the **asymptotic error constant** and usually depends on derivatives of $f(x)$ at $x = \alpha$.

If $p = 1$, the convergence is said to be linear.
If $p = 2$, the convergence is said to be quadratic.

BISECTION METHOD OR BOLZANO METHOD

Let $f(x) = 0$ be the equation for which a root is required. Suppose $f(a)$ and $f(b)$ have opposite signs or $f(a)f(b) < 0$ and $f(x)$ is continuous in $[a,b]$, then there is a root α between a and b.

Bisect the interval $[a,b]$ and take $x_0 = \dfrac{a+b}{2}$ as the initial approximate value of the root α. If $f(x_0) = 0$, then x_0 is the root α. If $f(x_0) \neq 0$, then the root lies in (a, x_0) or (x_0, b).



From the figure, the root lies in (a, x_0), since $f(a) > 0$ and $f(x_0) < 0$. Bisect this interval and take the first approximation as $x_1 = \dfrac{a+x_0}{2}$. Now the root is in (x_1, x_0) and bisect this interval and take the second approximation $x_2 = \dfrac{x_1+x_0}{2}$.

Proceed like this till two successive approximations coincide upto the desired degree of accuracy.

Note:

(1) This method is the simplest of all successive approximation methods. At each stage only one functional value is computed. The convergence is linear with order or rate $\dfrac{1}{2}$.

(2) In this method the sequence of approximations $\{x_0, x_1, x_2, \ldots, x_n\}$ always converge to the root α. But the convergence is very slow, which is the draw back of this method.

(3) After n bisections, the root will lie in an interval of length $\frac{b-a}{2^n}$, which is small for large n. So, the error after n iterations is less than $\left|\frac{b-a}{2^n}\right|$.

(4) In all the successive approximations or the iteration methods, the starting point x_0 should be chosen close to the root for quick convergence.

WORKED EXAMPLES

Example 1

Find a root of the equation $x^3 - 4x - 9 = 0$ correct to three decimal places.

Solution

Let $f(x) = x^3 - 4x - 9$

We have $f(0) = -9 < 0, f(1) = 1 - 4 - 9 = -12 < 0$

$$f(2) = 8 - 8 - 9 = -9 < 0$$
$$f(3) = 27 - 12 - 9 = 6 > 0$$

So a root lies between 2 and 3 we chose a smaller interval.
Now

$$f(2.5) = (2.5)^3 - 4(2.5) - 9$$
$$= 15.625 - 10 - 9 = -3.375 < 0$$
$$f(2.7) = (2.7)^3 - 4(2.7) - 9 = 19.683 - 10.8 - 9 = -0.117 < 0$$

$\therefore$ the root lies between 2.7 and 3.
$\therefore$ the initial approximation is

$$\mathbf{x_0 = \frac{2.7 + 3}{2} = 2.85}$$

Now $$f(2.85) = (2.85)^3 - 4(2.85) - 9$$
$$= 23.149 - 11.4 - 9 = 2.749 > 0$$

and $$f(2.7) < 0$$

$\therefore$ the root lies between 2.7 and 2.85

I approximation is $$x_1 = \frac{2.7 + 2.85}{2} = 2.775$$

Now $$f(2.775) = (2.775)^3 - 4(2.775) - 9$$
$$= 21.3692 - 11.1 - 9 = 1.2692 > 0$$

and $$f(2.7) < 0$$

So, the root lies between 2.7 and 2.775

II approximation is $x_2 = \dfrac{2.7+2.775}{2} = \dfrac{5.475}{2} = 2.7375$

Now $f(2.7375) = (2.7375)^3 - 4(2.7375) - 9$

$= 20.51456 - 10.95 - 9 = 0.5646 > 0$

and $f(2.7) < 0$

$\therefore$ the root lies between 2.7 and 2.7375

III approximation is $x_3 = \dfrac{2.7+2.7375}{2} = \dfrac{5.4375}{2} = 2.7187$

Now $f(2.7187) = (2.7187)^3 - 4(2.7187) - 9$

$= 20.0948 - 10.8748 - 9 = 0.22 > 0$

and $f(2.7) < 0$

$\therefore$ the root lies between 2.7 and 2.7187

IV approximation is $x_4 = \dfrac{2.7+2.7187}{2} = \dfrac{5.4187}{2} = 2.7094$

Now $f(2.7094) = (2.7094)^3 - 4(2.7094) - 9$
$= 19.8893 - 10.8376 - 9 = +0.0517 > 0$

and $f(2.7) < 0$

$\therefore$ the root lies between 2.7 and 2.7094

V approximation is $x_5 = \dfrac{2.7+2.7094}{2} = \dfrac{5.4094}{2} = 2.7047$

Now $f(2.7047) = (2.7047)^3 - 4(2.7047) - 9$

$= 19.78596 - 10.8188 - 9 = -0.0328 < 0$

and $f(2.7094) > 0$

$\therefore$ the root lies between 2.7047 and 2.7094

VI approximation is $x_6 = \dfrac{2.7047+2.7094}{2} = \dfrac{5.4141}{2} = 2.7071$

Now $f(2.7071) = (2.7071)^3 - 4(2.7071) - 9$

$= 19.8387 - 10.8284 - 9 = 0.0103 > 0$

and $f(2.7047) < 0$

$\therefore$ the root lies between 2.7047 and 2.7071

VII approximation is $x_7 = \dfrac{2.7047 + 2.7071}{2} = \dfrac{5.4118}{2} = 2.7059$

Now $f(2.7059) = (2.7059)^3 - 4(2.7059) - 9$

$= 19.8123 - 10.8236 - 9 = -0.0113 < 0$

and $f(2.7071) > 0$

$\therefore$ the root lies between 2.7059 and 2.7071

VIII approximation is $x_8 = \dfrac{2.7059 + 2.7071}{2} = \dfrac{5.4130}{2} = 2.7065$

Now $f(2.7065) = (2.7065)^3 - 4(2.7065) - 9$

$= 19.8255 - 10.826 - 9 = -0.0005 < 0$

and $f(2.7071) > 0$

$\therefore$ the root lies between 2.7065 and 2.7071

IX approximation is $x_9 = \dfrac{2.7065 + 2.7071}{2} = \dfrac{5.4136}{2} = 2.7068$

Since x_8 and x_9 coincide upto three decimal places, the root is $\boldsymbol{x = 2.706}$ ∎

Example 2

Find a root of the transcendental equation $3x + \sin x - e^x = 0$.

Solution

Let $f(x) = 3x + \sin x - e^x$.

We have $f(0) = 0 + 0 - 1 = -1 < 0$

$f(1) = 3 + \sin 1 - e = 3 + 0.8417 - 2.71828 = 1.12319 > 0$

Now $f(0.5) = 1.5 + \sin(0.5) - e^{0.5}$

$= 1.5 + 0.4794 - 1.6487 = 0.3307 > 0$

So, the root lies between 0 and 0.5. Choose a still smaller interval.

Now $f(0.3) = 3(0.3) + \sin(0.3) - e^{0.3}$

$= 0.9 + 0.2955 - 1.3498 = -0.1543 < 0$

$\therefore f(0.3) < 0$ and $f(0.5) > 0$

So, the root lies between 0.3 and 0.5

Take $\boldsymbol{x_0 = \dfrac{0.3 + 0.5}{2} = \dfrac{0.8}{2} = 0.4}$

Now $f(0.4) = 3(0.4) + \sin(0.4) - e^{+0.4}$

$= 1.2 + 0.3894 - 1.4918 = 0.0976 > 0$

and $f(0.3) < 0$

So, the root lies between 0.3 and 0.4

I approximation is $x_1 = \dfrac{0.3+0.4}{2} = \dfrac{0.7}{2} = 0.35$

Now
$$f(0.35) = 3(0.35) + \sin(0.35) - e^{+0.35}$$
$$= 1.05 + 0.3429 - 1.4191 = -0.0262 < 0$$

and $f(0.4) > 0$

$\therefore$ the root lies between 0.35 and 0.4

II approximation is $x_2 = \dfrac{0.35+0.4}{2} = \dfrac{0.75}{2} = 0.375$

Now
$$f(0.375) = 3(0.375) + \sin(0.375) - e^{0.375}$$
$$= 1.125 + 0.3663 - 1.4550 = +0.0363 > 0$$

and $f(0.35) < 0$

$\therefore$ the root lies between 0.35 and 0.375

III approximation is $x_3 = \dfrac{0.35+0.375}{2} = \dfrac{0.725}{2} = 0.3625$

Now
$$f(0.3625) = 3(0.3625) + \sin(0.3625) - e^{0.3625}$$
$$= 1.0895 + 0.3546 - 1.4369 = 0.0052 > 0$$

and $f(0.35) < 0$

$\therefore$ the root lies between 0.35 and 0.3625

IV approximation is $x_4 = \dfrac{0.35+0.3625}{2} = \dfrac{0.7125}{2} = 0.35625 = 0.3563$

Now
$$f(0.3563) = 3(0.3563) + \sin(0.3563) - e^{0.3563}$$
$$= 1.0689 + 0.3488 - 1.4280 = -0.0103 < 0$$

and $f(0.3625) > 0$

$\therefore$ the root lies between 0.3563 and 0.3625

V approximation is $x_5 = \dfrac{0.3563+0.3625}{2} = \dfrac{0.7188}{2} = 0.3594$

Now
$$f(0.3594) = 3(0.3594) + \sin(0.3594) - e^{0.3594}$$
$$= 1.0782 + 0.3517 - 1.4325 = -0.0026 < 0$$

and $f(0.3625) > 0$

$\therefore$ the root lies between 0.3594 and 0.3625

VI approximation is $x_6 = \dfrac{0.3594+0.3625}{2} = \dfrac{0.7219}{2} = 0.361$

Now
$$f(0.361) = 3(0.361) + \sin(0.361) - e^{0.361}$$
$$= 1.083 + 0.3532 - 1.4348 = 0.0014 > 0$$

and $f(0.3594) < 0$

$\therefore$ the root lies between 0.3594 and 0.361

VII approximation is $x_7 = \dfrac{0.3594 + 0.361}{2} = \dfrac{0.7204}{2} = 0.3602$

Now $\quad f(0.3602) = 3(0.3602) + \sin(0.3602) - e^{0.3602}$

$= 1.0806 + 0.3525 - 1.4336 = -0.0005 < 0$

and $\quad f(0.361) > 0$

$\therefore$ the root lies between 0.3602 and 0.361.

VIII approximation is $\quad x_8 = \dfrac{0.3602 + 0.361}{2} = \dfrac{0.7212}{2} = 0.3606$

and $\quad f(0.3606) = 3(0.3606) + \sin(0.3606) - e^{0.360}$

$= 1.0818 + 0.3528 - 1.4342 = 0.0004 > 0$

and $\quad f(0.3602) < 0$

$\therefore$ the root lies between 0.3602 and 0.3606.

IX approximation is $\quad x_9 = \dfrac{0.3602 + 0.3606}{2} = \dfrac{0.7208}{2} = 0.3604$

Now $\quad f(0.3604) = 3(0.3604) + \sin(0.3604) - e^{0.3604}$

$= 1.812 + 0.3520 - 1.4339 = -0.0001 < 0$

and $\quad f(0.3606) > 0$

$\therefore$ the root lies between 0.3604 and 0.3606

X approximation is $\quad x_{10} = \dfrac{0.3604 + 0.3606}{2} = \dfrac{0.721}{2} = 0.3605$

Now $\quad f(0.3605) = 3(0.3605) + \sin(0.3605) - e^{0.3605}$

$= 1.0815 + 0.3527 - 1.4340 = 0.0002 > 0$

and $\quad f(0.3604) < 0$

$\therefore$ the root lies between 0.3604 and 0.3605.

XI approximation is $\quad x_{11} = \dfrac{0.3604 + 0.3605}{2} = \dfrac{0.7209}{2} = 0.3605$

So, x_{10} and x_{11} are equal upto four places of decimals.

$\therefore$ the root correct to four decimal places is $\boldsymbol{x = 0.3605}$

Note:

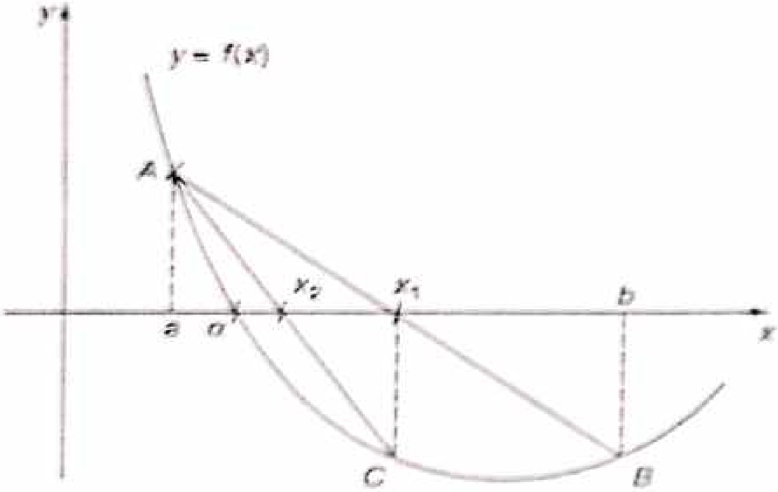
(1) We have $f(0.3605) = 0.0002$

So, the error is 2×10^{-4}.

(2) Error after n iterations is $< \left|\dfrac{b-a}{2^n}\right| < \dfrac{0.4 - 0.3}{\ldots} = 0.0004$

METHOD OF FALSE POSITION OR REGULA-FALSI (IN LATIN)

This method is the oldest for finding a real root of $f(x)=0$. First decide an interval (a, b) in which a root lies. If α is the root lying in (a, b), then α is the intersection of the curve $y=f(x)$ and the x-axis.



Let A be $(a, f(a))$ and B be $(b, f(b))$

Then the equation of the chord AB is $\dfrac{y-f(a)}{f(a)-f(b)}=\dfrac{x-a}{a-b}$ (1)

To find the intersection with the x-axis put $y=0$ in (1)

$$\therefore \qquad -\frac{f(a)}{f(a)-f(b)}=\frac{x-a}{a-b}$$

$$\Rightarrow \qquad (x-a)(f(a)-f(b))=-(a-b)f(a)$$

$$\Rightarrow \qquad x[f(a)-f(b)]=af(a)-af(b)-af(a)+bf(a)=bf(a)-af(b)$$

$$\therefore \qquad x=\frac{af(b)-bf(a)}{f(b)-f(a)}$$

$\therefore$ the first approximation root is $\quad x_1=\dfrac{af(b)-bf(a)}{f(b)-f(a)}$

Now find the sign of $f(x_1)$ and decide the root is in (a, x_1) or in (x_1, b).

Repeat the process to find the second approximation x_2, third approximation x_3 and so on. The sequence of approximations $x_1, x_2, x_3, \ldots$ approach the actual root α.

Note:

(1) The simplicity of this method lies in the fact that we replace the curve between A and B by the chord AB, the curve between A and C by the chord AC and so on. The points of intersection of these chords with the x-axis are the successive approximations of the root.

(2) The Regula-Falsi method require the evaluation of one function value per iteration.

(3) The order of convergence of Regula-Falsi method is 1. That is, it has linear rate of convergence.

WORKED EXAMPLES

Example 1

Find the positive root of $x^3 - 2x - 5 = 0$ by false position method.

Solution

Given equation is $x^3 - 2x - 5 = 0$

Let $\quad f(x) = x^2 - 2x - 5$

We have $\quad f(2) = 8 - 4 - 5 = -1 < 0$

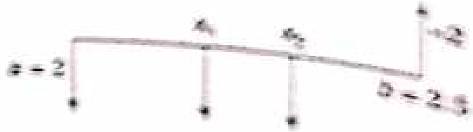
$f(3) = 27 - 6 - 5 = 16 > 0$ and $f(2.5) = 5.625 > 0$

So, the root is between 2 and 2.5 and it is close to 2.

Take $a = 2$, $b = 2.5$

$$\therefore \quad x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{2 \times 5.625 - 2.5 \times (-1)}{5.625 - 1(-1)} = \frac{13.75}{6.625} = 2.07547$$



Now $\quad f(x_1) = f(2.07547) = -0.21070 < 0$ and $f(2.5) > 0$

So, the root x_2 lies between 2.07547 and 2.5

Take $a = 2.07547$, $b = 2.5$

The second approximation is

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2.07547(2.5) - 2.5f(2.07547)}{f(2.5) - f(2.07547)}$$

$$= \frac{2.07547(5.625) - 2.5(-0.21070)}{5.625 - (-0.21070)}$$

$$= \frac{11.67452 + 0.52675}{5.8357} = \frac{12.20127}{5.8357} = 2.0908$$

$$f(x_2) = f(2.0908) = (2.0908)^3 - 2\times 2.0908 - 5 = -0.0418 < 0$$

and $\quad f(2.5) > 0$

So, the root lies between 2.0908 and 2.5

Take $a = 2.0908$ and $b = 2.5$

The third approximation is

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2.0908 f(2.5) - 2.5 f(2.0908)}{f(2.5) - f(2.0908)}$$

$$= \frac{2.0908\times 5.625 - 2.5(-0.0418)}{5.625 - (-0.0418)}$$

$$= \frac{11.86525}{5.6668} = 2.0938$$

Now $\quad f(x_3) = f(2.0938) = (2.0938)^3 - 2\times 2.0938 - 5 = -0.008384 < 0$

and $\quad f(2.5) > 0$

$\therefore$ the root lies between 2.0938 and 2.5

Take $a = 2.0938$ and $b = 2.5$

The fourth approximation is

$$x_4 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2.0938\times f(2.5) - 2.5\times f(2.0938)}{f(2.5) - f(2.0938)}$$

$$= \frac{2.0938\times 5.625 - 2.5\times(-0.008384)}{5.625 - (-0.008384)}$$

$$= \frac{11.7986}{5.6334} = 2.0944$$

Now $\quad f(x_4) = f(2.0944) = (2.0944)^3 - 2\times 2.0944 - 5 = -0.00169 < 0$

and $\quad f(2.5) > 0$

$\therefore$ the root lies between 2.0944 and 2.5

Take $a = 2.0944$, $b = 2.5$

The fifth approximation is

$$x_5 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2.0944\times f(2.5) - 2.5 f(2.0944)}{f(2.5) - f(2.0944)}$$

$$= \frac{2.0944\times 5.625 - 2.5\times(-0.00169)}{5.625 - (-0.00169)}$$

$$= \frac{11.7852}{5.6267} = 2.0945$$

Now $f(x_5) = f(2.0945) = -0.0005746 < 0$ and $f(2.5) > 0$

$\therefore$ the root lies between 2.0945 and 2.5

Take $a = 2.0945$ and $b = 2.5$

The sixth approximation is

$$x_6 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2.0945f(2.5) - 2.5f(2.0945)}{f(2.5) - f(2.0945)}$$

$$= \frac{2.0945 \times 5.625 - 2.5 \times (-0.0005746)}{5.625 - (-0.0005746)}$$

$$= \frac{11.7830}{5.6256} = 2.0945$$

Since $x_5 = x_6$ upto 4 decimal places, the root is $x = \mathbf{2.0945}$ ■

Example 2

Find the real root of $x\log_{10} x - 1.2 = 0$ correct to 4 decimal places using false position method.

Solution

Given $x\log_{10} x - 1.2 = 0$

Let $f(x) = x\log_{10} x - 1.2$

We have $f(1) = -1.2 < 0, f(2) = -0.5979 < 0$

$f(3) = 0.2314 > 0.$

Now $f(2.5) = -0.2052 < 0$

$\therefore$ the root lies between 2.5 and 3.

Take $a = 2.5$, $b = 3$

The first approximation is

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2.5f(3) - 3f(2.5)}{f(3) - f(2.5)}$$

$$= \frac{2.5 \times 0.2314 - 3 \times (-0.2052)}{0.2314 - (-0.2052)}$$

$$= \frac{1.1941}{0.4366} = 2.7350$$

Now $f(x_1) = f(2.7350) = 2.7350\log_{10} 2.7350 - 1.2 = -0.004922 < 0$

and $f(3) > 0$

So, the root lies between 2.7350 and 3.

Take $a = 2.7350$ and $b = 3$.

The second approximation is

$$x_2 = \frac{af(b)-bf(a)}{f(b)-f(a)} = \frac{2.7350f(3)-3f(2.7350)}{f(3)-f(2.7350)}$$

$$= \frac{2.7350\times 0.2314-3\times(-0.004922)}{0.2314-(-0.004922)}$$

$$= \frac{0.6476}{0.2363} = 2.7406$$

Now $\quad f(x_2) = f(2.7406) = 2.7406\log 2.7406 - 1.2 = -0.0000402 < 0$

and $\quad f(3) > 0$

∴ the root lies between 2.7406 and 3.

Take $a = 2.7406$, $b = 3$

The third approximation is

$$x_3 = \frac{af(b)-bf(a)}{f(b)-f(a)} = \frac{2.7406f(3)-3f(2.7406)}{f(3)-f(2.7406)}$$

$$= \frac{2.7406\times 0.2314-3\times(-0.00004)}{0.2314-(-0.00004)}$$

$$= \frac{0.63429}{0.23144} = 2.74062$$

Since $x_2 = x_3$, upto 4 decimal places, the root is $x = \mathbf{2.7406}$ ■

Example 3

Solve for a position root of $x - \cos x = 0$ by Regula-Falsi method.

Solution

Given $\quad x - \cos x = 0$

Let $\quad f(x) = x - \cos x.$

Then $\quad f(0) = 0 - \cos 0 = -1 < 0$

$\quad f(1) = 1 - \cos 1 = 0.4597 > 0$

∴ a root lies between 0 and 1.

Take $a = 0$, $b = 1$

The first approximation is

$$x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)} = \frac{0-1f(0)}{f(1)-f(0)} = \frac{-(-1)}{0.4597-(-1)} = \frac{1}{1.4597}$$

$$= 0.6851$$

Now $$f(x_1) = f(0.6851) = 0.6851 - \cos(0.6851)$$
$$= 0.6851 - 0.7744 = -0.0893 < 0$$

and $$f(1) > 0$$

So, the root lies between 0.6851 and 1.

Take $a = 0.6851$, $b = 1$.

The second approximation is

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0.6851 f(1) - f(0.6851)}{f(1) - f(0.6851)}$$
$$= \frac{0.6851 \times 0.4597 - (0.0893)}{0.4597 - (0.0893)} = \frac{0.40424}{0.549} = 0.7363$$

Now $$f(x_2) = f(0.7363) = 0.7363 - \cos(0.7363) = -0.0047 < 0$$

and $$f(1) > 0.$$

So, the root lies between 0.7363 and 1.

Take $a = 0.7363$, $b = 1$.

The third approximation is

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0.7363 f(1) - f(0.7363)}{f(1) - f(0.7363)}$$
$$= \frac{0.7363 \times 0.4597 - (-0.0047)}{0.4597 - (-0.0047)} = \frac{0.3432}{0.4644} = 0.7390$$

Now $$f(x_3) = f(0.7390) = 0.7390 - \cos(0.7390) = -0.0014 < 0$$

and $$f(1) > 0$$

So, the root lies between 0.7390 and 1.

Take $a = 0.7390$, $b = 1$

The fourth approximation is

$$x_4 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0.7390 f(1) - f(0.7390)}{f(1) - f(0.7390)}$$
$$= \frac{0.7390 \times 0.4597 - (0.00014)}{0.4597 - (-00014)} = 0.7391$$

Now $$f(x_4) = f(0.7391) = 0.00003 > 0$$

Since $f(0.7390) = -0.00014 < 0$ and $f(0.7391) = 0.00003 > 0$, the root is between 0.7390 and 0.7391 and is nearer to 0.7391.
So, correct to 4 decimal places, we take the root as $x = \mathbf{0.7391}$ ■

Example 4

Find a positive root of $xe^x = 2$ by Regula-Falsi method.

Solution

Given equation is $xe^x - 2 = 0$

Let $f(x) = xe^x - 2.$

Now $f(0) = -2 < 0$ and $f(1) = e - 2 = 2.71828 - 2 = 0.71828 = 0.7183 > 0$

Now $f(0.7) = -0.59037 = -0.5904 < 0$

Since $f(0.7)$ and $f(1)$ have opposite signs, the root lies between 0.7 and 1.

Take $a = 0.7$ and $b = 1$

The first approximation is

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0.7f(1) - 1.f(0.7)}{f(1) - f(0.7)}$$

$$= \frac{0.7 \times 0.7183 - (-0.5904)}{0.7183 - (-0.5904)} = \frac{1.0932}{1.3087} = 0.8353$$

Now $\quad f(x_1) = f(0.8353) = 0.8353e^{0.8353} - 2$

$$= 1.9258 - 2 = -0.0742 < 0$$

and $\quad f(1) > 0$

$\therefore$ the root lies between 0.8353 and 1.

Take $a = 0.8353$, $b = 1$.

Then the second approximation is

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0.8353f(1) - 1f(0.8353)}{f(1) - f(0.8353)}$$

$$= \frac{0.8353 \times 0.7183 - (-0.0742)}{0.7183 - (-0.0742)} = \frac{0.6742}{0.7925} = 0.8507$$

Now $\quad f(x_2) = f(0.8507) = 0.8507e^{0.8507} - 2 = -0.0083 < 0$

and $\quad f(1) > 0.$

$\therefore$ the root lies between 0.8507 and 1.

Take $a = 0.8507$, $b = 1$

The third approximation is

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0.8507f(1) - 1f(0.8507)}{f(1) - f(0.8507)}$$

$$= \frac{0.8507 \times 0.7183 - (-0.0083)}{0.7183 - (-0.0083)} = \frac{0.6194}{0.7266} = 0.8525$$

Now $f(x_3) = f(0.8525) = 0.8525e^{0.8525} - 2 = -0.000458 < 0$

and $f(1) > 0.$

$\therefore$ the root lies between 0.8525 and 1.

Take $a = 0.8525$, $b = 1$

The fourth approximation is

$$x_4 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0.8525f(1) - 1f(0.8525)}{f(1) - f(0.8525)}$$

$$= \frac{0.8525 \times 0.7183 - (-0.000458)}{0.7183 - (-0.000458)} = \frac{0.6128}{0.7188} = 0.8525$$

Since $x_3 = x_4$, the root correct to 4 decimal places is $\boldsymbol{x = 0.8525}$ ■

THE SECANT METHOD OR THE CHORD METHOD

The secant method also replaces the curve in the neighbourhood of the root by a straight line. So it is same as Regula-Falsi method except for the omission of the condition $f(a)f(b) < 0$.

So $x_1 = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$ and repeat it.

For iteration purpose, we take, $\boldsymbol{a = x_0, b = x_1}$, then

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{x_1[f(x_1) - f(x_0)] - f(x_1)[x_1 - x_0]}{f(x_1) - f(x_0)}$$

$$= x_1 - \frac{f(x_1)[x_1 - x_0]}{f(x_1) - f(x_0)}$$

$$= x_1 + \frac{(x_1 - x_0)f(x_1)}{f(x_0) - f(x_1)}$$

More generally,

$$x_n = x_{n-1} + \frac{(x_{n-1} - x_{n-2})f(x_{n-1})}{f(x_{n-2}) - f(x_{n-1})}, \quad n = 2, 3, \ldots$$

At any stage of the iteration, if $f(x_r) = f(x_{r-1})$, then this method fails.

So, the secant method does not converge always, where as Regular-falsi method will always converge.

However, when the secant method converges, it converges faster than the Regular-falsi method.

Note: The order of convergence of the secant method is 1.62 approximately

WORKED EXAMPLES

Example 1

Find a real root of $x^3 - 2x - 5 = 0$ between 2 and 3, to three places of decimals.

Solution

Let
$$f(x) = x^3 - 2x - 5$$
$$f(2) = 2^3 - 2\times 2 - 5$$
$$= 8 - 4 - 5 = -1 < 0$$
$$f(3) = 3^3 - 2\times 3 - 5 = 27 - 6 - 5 = 16 > 0$$

Now
$$f(2.5) = (2.5)^3 - 2(2.5) - 5$$
$$= 15.625 - 5 - 5 = 5.625 > 0$$

Let $x_0 = 2$ and $x_1 = 2.5$ be two approximate values.

$\therefore$ the iteration formula is

$$x_n = x_{n-1} + \frac{(x_{n-1} - x_{n-2}) + (x_{n-1})}{f(x_{n-a}) - f(x_{n-1})}, \qquad n = 2, 3, 4, 5, \ldots \qquad (1)$$

I iteration: Put $n = 2$ in (1)

$\therefore$
$$x_2 = x_1 + \frac{(x_1 - x_0) f(x_1)}{f(x_0) - f(x_1)}$$
$$= 2.5 + \frac{(2.5 - 2) f(2.5)}{f(2) - f(2.5)}$$
$$= 2.5 + \frac{0.5 \times 5.625}{-1 - 5.625}$$
$$= 2.5 - \frac{02.8125}{6.625} = 2.5 - 0.4245 = 2.0755$$

II Iteration: Put $n = 3$ in (1)

$\therefore$
$$x_3 = x_2 + \frac{(x_2 - x_1) f(x_2)}{f(x_1) - f(x_2)}$$

But
$$f(x_2) = (2.0755)^3 - 2(2.0755) - 5$$
$$= 8.9406 - 4.151 - 5 = -0.2104$$

$\therefore$ $$x_3 = 2.0755 + \frac{(2.0755-2.5)(-0.2104)}{5.625+0.2104}$$

$$= 2.0755 + \frac{0.08931}{5.8354} = 2.0755 + 0.0153 = 2.0908$$

III Iteration: Put $n = 4$ in (1)

$\therefore$ $$x_4 = x_3 + \frac{(x_3 - x_2)f(x_3)}{f(x_2) - f(x_3)}$$

But $$f(x_3) = f(2.0908)$$

$$= (2.0908)^3 - 2(2.0908) - 5$$

$$= 9.139816 - 4.1816 - 5 = -0.0418$$

$\therefore$ $$x_4 = 2.0908 + \frac{(2.0908-2.0755)(-0.0418)}{-0.2104-(-0.0418)}$$

$$= 2.0908 - \frac{0.0006395}{-0.1686} = 2.0908 + 0.00379 = 2.0946$$

VI Iteration: Put $n = 5$ in (1)

$\therefore$ $$x_5 = x_4 + \frac{(x_4 - x_3)f(x_4)}{f(x_3) - f(x_4)}$$

$$= 2.0946 + \frac{(2.0946-2.0908)f(2.0946)}{f(2.0908) - f(2.0946)}$$

But $$f(2.0946) = (2.0946)^3 - 2(2.0946) - 5$$

$$= 9.18974 - 4.1892 - 5 = 0 = -0.0005415$$

$\therefore$ $$x_5 = 2.0946 + \frac{(2.0946-2.0908)(-0.00054)}{-0.0418-(-0.00054)}$$

$$= 2.0946 + \frac{0.0000021}{0.04126} = 2.0946 + 0.000051 = 2.094651$$

Since $x_4 = x_5$, upto 4 places of decimals,

the root correct to three places of decimals is $x = \mathbf{2.095}$ ■

Example 2

Find the real root of $x^4 - x - 10 = 0$ which lies between 1 and 2.

Solution

Let $f(x) = x^4 - x - 10$

To find the real root between 1 and 2

$$f(1) = 1 - 1 - 10 = -10 < 0$$

$$f(2) = 2^4 - 2 - 10 = 16 - 2 - 10 = 4 > 0$$

We find that the root is nearer to 2

$$f(1.8) = (1.8)^4 - 1.8 - 10$$
$$= 10.4976 - 1.8 - 10 = -1.3024 < 0$$

$\therefore$

So, the root lies between 1.8 and 2.

Take $x_0 = 1.8$ and $x_1 = 2$

The iteration formula is

$$x_n = x_{n-1} + \frac{(x_{n-1} - x_{n-2}) f(x_{n-1})}{f(x_{n-2}) - f(x_{n-1})}, \quad n = 2, 3, \ldots \quad (1)$$

First iteration: Put $n = 2$ in (1)

$$x_2 = x_1 + \frac{(x_1 - x_0) f(x_1)}{f(x_0) - f(x_1)}$$

$\therefore$

$$x_2 = 2 + \frac{(2 - 1.8)(4)}{-1.3024 - 4}$$
$$= 2 - \frac{0.8}{5.3024} = 2 - 0.15088 = 1.8491$$

Second iteration: Put $n = 3$ in (1)

$$x_3 = x_2 + \frac{(x_2 - x_1) f(x_2)}{f(x_1) - f(x_2)}$$

$\therefore$

Now

$$f(x_2) = (1.8491)^4 - 1.8491 - 10$$
$$= 11.6907 - 1.8491 - 10 = -0.1584$$

$\therefore$

$$x_3 = 1.8491 + \frac{(1.8491 - 2)(-0.1584)}{4 - (0.1584)}$$
$$= 1.8491 + \frac{0.02390256}{4.1584}$$
$$= 1.8491 + 0.005748 = 1.8548$$

Now

$$f(x_3) = f(1.8548) = (1.8548)^4 - 1.8548 - 10$$
$$= -0.01925$$

Third iteration: Put $x = 4$ in (1)

$\therefore$

$$x_4 = x_3 + \frac{(x_3 - x_2) f(x_3)}{f(x_2) - f(x_3)}$$
$$= 1.8548 + \frac{(1.8548 - 1.8291)(-0.01925)}{-0.1584 - (-0.01925)}$$
$$= 1.8548 + \frac{(-0.000109725)}{(-0.13915)} = 1.8556$$

$\vdots$

Now $$f(x_4) = f(1.8556) = (1.8556)^4 - 1.8556 - 10$$
$$= 11.85598 - 1.8556 - 10 = 0.00038$$

Fourth iteration: Put $n = 5$ in (1)

$$\therefore \quad x_5 = x_4 + \frac{(x_4 - x_3)f(x_4)}{f(x_3) - f(x_4)}$$
$$= 1.8556 + \frac{(1.8556 - 1.8548)(0.00038)}{-0.01925 - 0.00038}$$
$$= 1.8556 - 0.000015 = 1.8556$$

So $x_4 = x_5$ upto four places of decimals.

$\therefore$ the root correct to four places of decimals is $x = \mathbf{1.8556}$

2.4 THE METHOD OF ITERATION OR FIXED POINT ITERATION: $x = \phi(x)$ METHOD

The fixed-point iteration method is a very useful method of finding a real root of an equation $f(x) = 0$. In this method the equation $f(x) = 0$ is rewritten as $x = \phi(x)$, suitably. First we find an interval in which a root lies. Let x_0 be an approximate value of the root in this interval. A better approximation or the first approximation x_1 is given by $x_1 = \phi(x_0)$

The second approximation x_2 is given by $x_2 = \phi(x_1)$ and so on.

Thus, we get the iteration formula

$$x_{n+1} = \phi(x_n), \qquad n = 0, 1, 2, \ldots$$

The sequence of approximations $x_0, x_1, x_2, \ldots$ approach the root α.

The function $\phi(x)$ is called an **iteration function**. The form $x = \phi(x)$ can be obtained in several ways. The convergence of the iteration process depends on the suitable choice of the function $\phi(x)$ and the choice of x_0.

The condition for convergence of the iteration process is $|\phi'(x)| < 1$ in a neighbourhood of x_0.

Some of the iteration methods (for example, Newton's method) depend on the fixed point method for convergence. So this method is also known as **general iteration method** or **the iteration method**.

The fixed point iteration is specially useful to find roots of an equation given in the form of an infinite series.

The rate of convergence of the fixed point method is 1

ie. it has a linear rate of convergence.

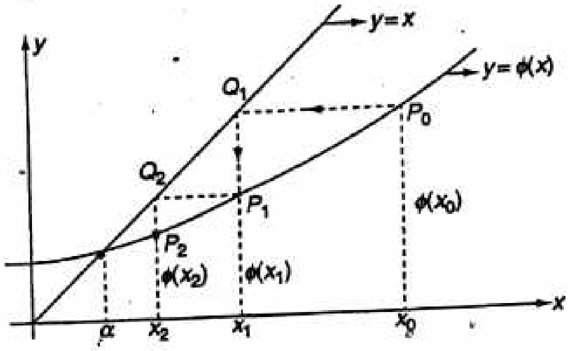
Geometrical meaning of fixed point iteration

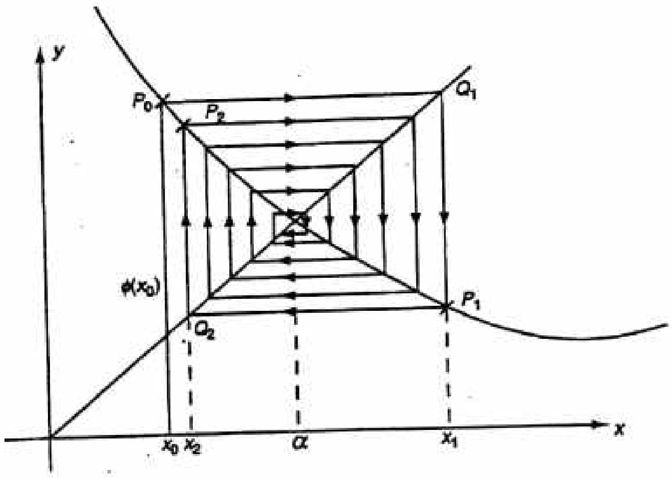
The successive approximations of the root are $x_0, x_1, x_2 \cdots$

where
$$x_1 = \phi(x_0)$$
$$x_2 = \phi(x_1)$$
$$x_3 = \phi(x_2) \text{ and so on}$$

Draw the graph of $y = x$ and $y = \phi(x)$

Since $|\phi'(x)| < 1$ near the root, the inclination of the graph of $\phi(x)$ should be less than 45°





Draw the ordinate $\phi(x_0)$, it meets the curve in P_0 and draw P_0Q_1 parallel to the x-axis. Then Q_1 is $(x_1, \phi(x_0))$, Q_2 is $(x_2, \phi(x_1))$,... we find $x_0, x_1, x_2, \ldots$, approach the root α.

We shall state the theorem for convergence.

Theorem 2.2

Let α be a root of $f(x) = 0$. In a neighbourhood I of α, let the equation be written as $x = \phi(x)$. If $\phi(x)$ is continuous in I and $|\phi^1(x)| < 1 \forall x \in I$, then the sequence of approximations $x_0, x_1, x_3, \ldots$ given by $x_n = \phi(x_n)$ converges to α, where $x_0 \in I$.

WORKED EXAMPLES

Example 1

Compute to four decimals, a root of the equation $x^3 - 10x - 5 = 0$, by the iteration method.

Solution

Given $x^3 - 10x - 5 = 0$

Let $f(x) = x^3 - 10x - 5$

Now $f(3) = 3^3 - 10\times3 - 5 = -8 < 0$

and $f(4) = 4^3 - 40 - 5 = 19 > 0$

But $f(3.5) = (3.5)^3 - 35 - 5 = 2.875 > 0$

So, the root is between 3 and 3.5 and is nearer to 3.5.

Take $x_0 = 3.5$

$f(x) = 0$ can be rewritten as $x^3 = 10x + 5$

$$\Rightarrow \quad x = (10x+5)^{1/3} \tag{1}$$

Let
$$\phi(x) = (10x+5)^{1/3}$$

$$\therefore \quad \phi'(x) = \frac{1}{3}(10x+5)^{-2/3}.10 = \frac{10}{3}(10x+5)^{-2/3}$$

and
$$\phi'(x_0) = \phi'(3.5) = \frac{10}{3}(40)^{-2/3} = \frac{10}{3(40)^{2/3}} = 0.285 < 1$$

$\therefore$ the process will converge in a neighbourhood of 3.5

$$\therefore \quad x_1 = \phi(x_0) = \phi(3.5) = (10\times3.5+5)^{1/3} = (40)^{1/3} = 3.41995$$

$$x_2 = \phi(x_1) = \phi(3.41995) = (10\times3.41995+5)^{1/3} = (39.1995)^{1/3} = 3.39698$$

$$x_3 = \phi(x_2) = \phi(3.39698) = (10\times3.39698+5)^{1/3} = (38.9698)^{1/3} = 3.3903$$

$$x_4 = \phi(x_3) = \phi(3.3903) = (10\times3.3903+5)^{1/3} = (38.903)^{1/3} = 3.3884$$

$$x_5 = \phi(x_4) = \phi(3.3884) = (10\times3.3884+5)^{1/3} = (38.884)^{1/3} = 3.3878$$

$$x_6 = \phi(x_5) = \phi(3.3878) = (10\times3.3878+5)^{1/3} = (38.878)^{1/3} = 3.3877$$

$$x_7 = \phi(x_6) = \phi(3.3877) = (10\times3.3877+5)^{1/3} = (38.877)^{1/3} = 3.3876$$

$$x_8 = \phi(x_7) = \phi(3.3876) = (10\times3.3876+5)^{1/3} = (38.876)^{1/3} = 3.3876$$

Since $x_7 = x_8$, the root correct to 4 decimal places is $\boldsymbol{x = 3.3876}$

Note:

(1) Suppose we rewrite the equation as $x = \frac{1}{10}(x^3 - 5)$.

If $\phi(x) = \frac{1}{10}(x^3 - 5)$, then $\phi'(x) = \frac{3x^2}{10}$

$$\therefore \quad \phi'(3.5) = \frac{3.(3.5)^2}{10} = 3.675 \not< 1$$

So in this form the process will not converge.

(2) If x_0 is chosen close to the root, in few approximations we can get the root. In the above problem we could have chosen x_0 as 3.3. ■

Example 2

Find a negative root of the equation $x^3 - 2x + 5 = 0$ to 4 decimal places.

Solution

Given equation is $$x^3 - 2x + 5 = 0 \qquad (1)$$

The negative root of (1) is the positive root of $(-x)^3 - 2(-x) + 5 = 0$ [Replace by x by $-x$]

$$\Rightarrow \quad -x^3 + 2(x) + 5 = 0$$

$$\Rightarrow \quad x^3 - 2x - 5 = 0 \qquad (2)$$

Let $f(x) = x^3 - 2x - 5$

We find, $f(2) = 8 - 4 - 5 = -1 < 0$ and $f(3) = 27 - 6 - 5 = 16 > 0$

$\therefore$ the root is between 2 and 3 and is nearer to 2.

Take $x_0 = 2$

The equation (2) can be rewritten as $x^3 = 2x + 5$

$$\Rightarrow \quad x = (2x+5)^{1/3}$$

Let $\phi(x) = (2x+5)^{1/3}$ $\therefore \phi'(x) = \frac{1}{3}(2x+5)^{-2/3}.2$

Now $\phi'(2) = \frac{2}{3}(9)^{-2/3} = \frac{2}{3(81)^{1/3}} = 0.1541 < 1$

$\therefore$ the process converges in a neighbourhood of 2

$\therefore$

$$x_1 = \phi(x_0) = \phi(2) = (2 \times 2 + 5)^{1/3} = 9^{1/3} = 2.0801$$
$$x_2 = \phi(x_1) = \phi(2.0801) = (4.1602 + 5)^{1/3} = (9.1602)^{1/3} = 2.0924$$
$$x_3 = \phi(x_2) = \phi(2.0924) = (4.1848 + 5)^{1/3} = (9.1848)^{1/3} = 2.0942$$
$$x_4 = \phi(x_3) = \phi(2.0942) = (4.1884 + 5)^{1/3} = (9.1884)^{1/3} = 2.0945$$
$$x_5 = \phi(x_4) = \phi(2.0945) = (4.1890 + 5)^{1/3} = (9.1890)^{1/3} = 2.0945$$

Since $x_4 = x_5$, the root correct to 4 decimal places is 2.0945

$\therefore$ the negative root of the given equation is $\boldsymbol{x = -2.0945}$ ■

Example 3

Solve $2x - \log_{10} x = 7$ by the method of interation.

Solution

Given equation is $2x - \log_{10} x - 7 = 0$ (1)

Let $f(x) = 2x - \log_{10} x - 7$

We find $f(3) = 6 - \log_{10} 3 - 7 = -1.477 < 0$

and $f(4) = 8 - \log_{10} 4 - 7 = -0.3979 > 0$

Now $f(3.5) = 7 - \log_{10} 3.5 - 7 = -0.544 < 0$

So the root lies between 3.5 and 4 and it is nearer to 4.

We shall take $x_0 = 3.7$

Rewriting the equation (1), we get $x = \frac{1}{2}(\log_{10} x + 7)$

Let $\phi(x) = \frac{1}{2}(\log_{10} x + 7)$

We know $\frac{d}{dx}(\log_{10} x) = \frac{1}{x}.\log_{10} e]$ $[\because \log_{10} x = \log_e x \log_{10} e]$

Then $\phi'(x) = \frac{1}{2}.\frac{1}{x}.\log_{10} e$

$\therefore$ $\phi'(3.7) = \frac{1}{7.4}\log_{10} 2.71828 = 0.05 < 1$

$\therefore$ the process converges in a neighbourhood of 3.7

Now $x_1 = \phi(x_0) = \frac{1}{2}(\log_{10} 3.7 + 7) = 3.7841$

$x_2 = \phi(x_1) = \frac{1}{2}(\log_{10} 3.7841 + 7) = 3.78898$

$x_3 = \phi(x_2) = \frac{1}{2}(\log_{10} 3.78898 + 7) = 3.78926$

$x_4 = \phi(x_3) = \frac{1}{2}(\log_{10} 3.78926 + 7) = 3.78927$

$x_5 = \phi(x_4) = \frac{1}{2}(\log_{10} 3.78927 + 7) = 3.78927$

Since $x_4 = x_5$, the root correct to five decimal places is $\boldsymbol{x = 3.78927}$ ■

Example 4

Solve $e^x - 3x = 0$ by the method of iteration.

Solution

Given equation is $e^x - 3x = 0$ (1)

Let $f(x) = e^x - 3x$

Now $f(0) = 1 > 0$

$f(1) = e - 3 = 2.7183 - 3 = -0.2817 < 0$

and $f(0.5) = e^{0.5} - 1.5 = 0.1487 > 0$

$\therefore$ the root lies between 0.5 and 1 and it is nearer to 0.5

Take $x_0 = 0.6$

The equation (1) can be rewritten as $x = \frac{1}{3}e^x$

Let $\phi(x) = \frac{1}{3}e^x$ $\quad \therefore \phi'(x) = \frac{1}{3}e^x$

and $\phi'(0.6) = \frac{1}{3}e^{0.6} = 0.6074 < 1$

$\therefore$ the process converges in a neighbourhood 0.6

$$x_1 = \phi(x_0) = \phi(0.6) = \frac{1}{3}e^{0.6} = 0.6074$$
$$x_2 = \phi(x_1) = \phi(0.6074) = \frac{1}{3}e^{0.6074} = 0.6119$$
$$x_3 = \phi(x_2) = \phi(0.6119) = \frac{1}{3}e^{0.6119} = 0.6146$$
$$x_4 = \phi(x_3) = \phi(0.6146) = \frac{1}{3}e^{0.6146} = 0.6163$$
$$x_5 = \phi(x_4) = \phi(0.6163) = \frac{1}{3}e^{0.6163} = 0.6174$$
$$x_6 = \phi(x_5) = \phi(0.6174) = \frac{1}{3}e^{0.6174} = 0.6180$$
$$x_7 = \phi(x_6) = \phi(0.6180) = \frac{1}{3}e^{0.6180} = 0.6184$$
$$x_8 = \phi(x_7) = \phi(0.6184) = \frac{1}{3}e^{0.6184} = 0.6187$$
$$x_9 = \phi(x_8) = \phi(0.6187) = \frac{1}{3}e^{0.6187} = 0.6188$$
$$x_{10} = \phi(x_9) = \phi(0.6188) = \frac{1}{3}e^{0.6188} = 0.6190$$
$$x_{11} = \phi(x_{10}) = \phi(0.6190) = \frac{1}{3}e^{0.6190} = 0.6190$$

Since $x_{10} = x_{11}$, the root correct to 4 decimal places is $\boldsymbol{x = 0.6190}$ ■

Example 5

Find an approximate value of a root of the equation $1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+\ldots=0$, which is very close to 2.4.

Solution

Given $$1-\frac{x^2}{2^2}+\frac{1}{(1.2)^2}\left(\frac{x^2}{2^2}\right)^2-\frac{1}{(1.2.3)^2}\left(\frac{x^2}{2^2}\right)^3+\ldots=0$$

Put $$y=\frac{x^2}{2^2} \text{ in (1)}$$

$\therefore$ $$1-y+\frac{y^2}{(2!)^2}-\frac{y^3}{(3!)^2}+\ldots=0$$

$\Rightarrow$ $$y=1+\frac{y^2}{(2!)^2}-\frac{y^3}{3(!)^2}+\ldots$$

Let $$\phi(y)=1+\frac{y^2}{(2!)^2}-\frac{y^3}{(3!)^2}+\ldots$$

Given $x_0=2.4$

$\therefore$ $$y_0=\frac{(2.4)^2}{4}=1.44$$

$\therefore$ $$y_1=\phi(y_0)=\phi(1.44)$$
$$=1+\frac{(1.44)^2}{(2!)^2}-\frac{(1.44)^3}{(3!)^2}+\ldots$$
$$=1+0.5184-0.0829+\ldots=1.4355$$
$$y_2=\phi(y_1)=\phi(1.4355)$$
$$=1+\frac{(1.4355)^2}{(2!)^2}-\frac{(1.4355)^3}{(3!)^2}+\ldots$$
$$=1+0.5152-0.0822+\ldots=1.433$$
$$y_3=\phi(y_2)=\phi(1.433)$$
$$=1+\frac{(1.433)^2}{(2!)^2}-\frac{(1.433)^3}{(3!)^2}+\ldots$$
$$=1+0.5134-0.0817+\ldots=1.4317$$
$$y_4=\phi(y_3)=\phi(1.4317)$$
$$=1+\frac{(1.4317)^2}{(2!)^2}-\frac{(1.4317)^3}{(3!)^2}+\ldots$$
$$=1+0.5124-0.0815+\ldots=1.4309$$

$$y_5 = \phi(y_4) = \phi(1.4309)$$
$$= 1 + \frac{(1.4309)^2}{(2!)^2} - \frac{(1.4309)^3}{(3!)^2} + \ldots$$
$$= 1 + 0.5118 - 0.0814 + \ldots = 1.4305$$

Since y_4 and y_5 are very close, we shall take the value of y as $y = 1.4305$.

$$\therefore \quad \frac{x^2}{2^2} = 1.4305$$
$$\Rightarrow \quad x^2 = 4 \times 1.4305$$
$$\Rightarrow \quad x = \sqrt{4 \times 1.4305} = 2.392$$

$\therefore$ the root is $x = 2.392$ ■

NEWTON-RAPHSON METHOD OR NEWTON'S METHOD OF FINDING A ROOT OF $f(x) = 0$

Let x_0 be a value close to the actual root α of $f(x) = 0$

Let $x_1 = x_0 + h$ be the actual root of $f(x) = 0$

Then, $\qquad f(x_1) = 0 \qquad \Rightarrow f(x_0 + h) = 0.$

Using Taylor's series expansion for $f(x_0 + h)$, we get

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots = 0$$

Since x_1 is close to x_0 and h is small, $h^2, h^3, \ldots$ are very small.
So, neglecting $h^2, h^3, \ldots$, we get

$$f(x_0) + hf'(x_0) = 0, \text{ approximately}$$

$$\Rightarrow \quad h = -\frac{f(x_0)}{f'(x_0)} \qquad \text{if } f'(x_0) \neq 0$$

$\therefore$ the better root is the first approximation root, which is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Now starting with x_1 and proceeding as above we will get the second approximation root.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \text{ and so on.}$$

More generally,
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots \tag{1}$$

This equation (1) is called Newton-Raphson iteration formula or Newton's iteration formula

Note:

(1) In order to reach the actual root α in lesser number of iterations we choose x_0 very close to the actual root. We stop the iteration when two successive iterations coincide upto the desired degree of accuracy.

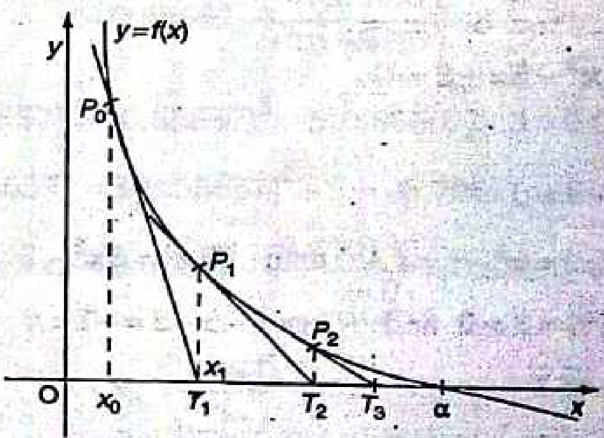
(2) Newton-Raphson method is also known as **method of tangents**.

(3) The **rate of convergence** of Newton's method is 2.
So, it is also referred as **quadratic convergence.**

(4) Condition for convergence is $|f(x)f''(x)| < |f'(x)|^2$ in the neighbourhood of the root.

Geometrical meaning of Newton-Raphson method

Let $f(x) = 0$ be the given equation.
Then the graph of the function is the curve $y = f(x)$.
The intersection of the graph with x-axis is the root α.
Let x_0 be the initial approximation.
Let P_0 be the point on the curve corresponding to $x = x_0$.



Then P_0 is $(x_0, f(x_0))$
Equation of the tangent at P_0 is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

Let this tangent meet the x-axis at T_1.

Put $y = 0;$

$\therefore$ $$-f(x_0) = f'(x_0)(x - x_0)$$

$\Rightarrow$ $$x - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$\Rightarrow$ $$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

That is the first approximation is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$\therefore$ $$OT_1 = x_1$$

Let P_1 be the point on the curve corresponding to $x = x_1$

Then P_1 is $[x_1, f(x_1)]$.

If the tangent at P_1 meets the x-axis at T_2, then $OT_2 = x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ and so on.

$\therefore$ OT_1, OT_2, OT_3 ... are the successive approximations $x_1, x_2, x_3, \ldots$ of the root.

Thus at each stage, we find the graph of the function is replaced by tangent at the point and so this method is also known as **tangent method**.

Note: Newton's-Raphson method should not be used if $f'(x) = 0$ or $f''(x) = 0$ near the root.

WORKED EXAMPLES

Example 1

Find a positive root of $x^3 - 5x + 3 = 0$.

Solution

Given equation is $x^3 - 5x + 3 = 0$

Let $f(x) = x^3 - 5x + 3$, then $f'(x) = 3x^2 - 5$

Now $f(0) = 3 > 0$ and $f(1) = 1 - 5 + 3 = -1 < 0$

So, a root lies between 0 and 1

Further $f(0.5) = 0.125 - 2.5 + 3 = 0.625$, which is closer to zero.

So, the root is actually between 0.5 and 1 and it is nearer to 0.5.

Take $x_0 = 0.5$

Newtone's iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, \ldots$$

$\therefore$

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I approximation is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.5 - \frac{f(0.5)}{f'(0.5)}$$

$$= 0.5 - \frac{0.625}{0.75-5} = 0.5 + \frac{0.625}{4.25} = 0.5 + 0.147 = 0.647$$

II approximation is $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.647 - \frac{f(0.647)}{f'(0.647)}$$

But
$$f(0.647) = (0.647)^3 - 5(0.647) + 3 = 0.03584$$

$$f'(0.647) = 3(0.647)^2 - 5 = -3.7442$$

$\therefore$
$$x_2 = 0.647 - \frac{0.03584}{(-3.7442)} = 0.647 + 0.009572 = 0.65657$$

III approximation is $$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.65657 - \frac{f(0.65657)}{f'(0.65657)}$$

But
$$f(0.65657) = (0.65657)^3 - 5(0.65657) + 3 = 0.00019$$

$$f'(0.65657) = 3(0.65657)^2 - 5 = -3.7067$$

$$x_3 = 0.65657 - \frac{0.00019}{(-3.7067)} = 0.65659 + 0.000051 = 0.6566$$

IV approximation is $$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 0.6560 - \frac{f(0.6566)}{f'(0.6566)}$$

$$= 0.6566 - \frac{0.000076}{-3.70662} = 0.6566 + 0.0000202 = 0.65662$$

Since $x_3 = x_4$ upto four decimal places, the root is $x = \mathbf{0.6566}$ ■

Example 2

Find by Newton-Raphson method a positive root of the equation $3x - \cos x - 1 = 0$.

Solution

Given equation is $3x - \cos x - 1 = 0$

Let $\quad f(x) = 3x - \cos x - 1 \qquad \therefore f'(x) = 3 + \sin x$

$f(0) = -2 < 0 \qquad$ and $\qquad f(1) = 3 - \cos 1 - 1 = 2 - \cos 1 = 1.4597 > 0$

$\therefore$ a root lies between 0 and 1

Now $f(0.5) = 3(0.5) - \cos 0.5 - 1 = -0.3776$, which is closer to 0.

So, the root is indeed between 0.5 and 1 and is nearer to 0.5.

Take $x_0 = 0.6$

Newton's iteration formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, \ldots$

I approximation is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.6 - \frac{f(0.6)}{f'(0.6)}$$

But

$$f(0.6) = 3(0.6) - \cos(0.6) - 1$$

$$= 1.8 - 0.8253356 - 1 = -0.0253356$$

$$f'(0.6) = 3 + \sin(0.6) = 3 + 0.564642 = 3.564642$$

$$x_1 = 0.6 - \frac{(-0.0253356)}{3.564642} = 0.6 + 0.0071 = 0.6071$$

II approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.6071 - \frac{f(0.6071)}{f'(0.6071)}$$

But

$$f(0.6071) = 3(0.6071) - \cos(0.6071) - 1$$

$$= 0.8213 - 0.821306 = -0.000006$$

$$f'(0.6071) = 3 + \sin(0.6071) = 3.570488$$

$\therefore$

$$x_2 = 0.6071 - \frac{(-0.000006)}{3.570488}$$

$$= 0.6071 + 0.00000168 = 0.60710168$$

Since $x_1 = x_2$ upto 4 decimal places, the root is $\boldsymbol{x = 0.6071}$ ■

Example 3

Using Newton's method, find a real root of $x\log_{10}x = 1.2$ correct to 4 decimals.

Solution

Given equation is $x\log_{10}x = 1.2$

$\Rightarrow$ $x\log_{10}x - 1.2 = 0$

Let $f(x) = x\log_{10}x - 1.2$

$\therefore$ $f'(x) = x\frac{1}{x}\log_{10}e + \log_{10}x = \log_{10}e + \log_{10}x$

$= 0.43429 + \log_{10}x$

Now
$$f(1) = -1.2 < 0$$
$$f(2) = 2\log_{10}2 - 1.2 = -0.59 < 0$$
$$f(3) = 3\log_{10}3 - 1.2 = 0.23 > 0$$

So, the root is between 2 and 3, but nearer to 3.

So, **take $x_0 = 2.7$**

Newton's iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0,1,2,\ldots$$

I approximation is $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$$= 2.7 - \frac{f(2.7)}{f'(2.7)}$$

But
$$f(2.7) = 2.7\log_{10}2.7 - 1.2 = -0.03532$$
$$f'(2.7) = 0.43429 + \log_{10}2.7 = 0.86566$$

$\therefore$
$$x_1 = 2.7 - \frac{(-0.03532)}{0.86566}$$
$$= 2.7 + 0.0408 = 2.74$$

II approximation is $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.74 - \frac{f(2.74)}{f'(2.74)}$

But
$$f(2.74) = 2.74\log_{10}2.74 - 1.2 = -0.0005635$$
$$f'(2.74) = 0.43429 + \log_{10}2.74 = 0.872041$$

$\therefore$
$$x_2 = 2.74 - \frac{(-0.0005635)}{0.872041}$$
$$= 2.74 + 0.000646 = 2.74064$$

III approximation is $$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.7406 - \frac{f(2.7406)}{f'(2.7406)}$$

But $$f(2.7406) = 2.7406 \log_{10} 2.7406 - 1.2 = -0.00004$$
$$f'(2.7406) = 0.43429 + \log_{10} 2.7406 = 0.872136$$
$$x_3 = 2.7406 - \frac{(-0.0004)}{0.872136} = 2.7406$$

Since $x_2 = x_3$, upto four decimal places, the root is $x = \mathbf{2.7406}$ ■

Example 4

Find a solution of $3x + \sin x - e^x = 0$ correct to four decimal places by Newton's method.

Solution

Given $3x + \sin x - e^x = 0$

Let $$f(x) = 3x + \sin x - e^x, \; f'(x) = 3 + \cos x - e^x$$
$$f(0) = -e^0 = -1 < 0$$
$$f(1) = 3 + \sin 1 - e$$
$$= 3 + 0.841 - 2.718 = 1.123 > 0$$

So, the root lies between 0 and 1.

Now $f(0.5) = 3(0.5) + \sin(.5) - e^{0.5} = 1.5 + 0.479 - 1.649 = 0.33$ which is nearer to 0.

$\therefore$ the root lies between 0 and 0.5 and it is nearer to 0.5

Take $x_0 = 0.5$

Newton's iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0,1,2,\ldots$$

I approximation is $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{f(0.5)}{f'(0.5)}$$

$$f'(0.5) = 3 + \cos 0.5 - e^{0.5} = 3.87758 - 1.64872 = 2.2288$$

$\therefore$ $$x_1 = 0.5 - \frac{0.33}{2.23} = 0.5 - 0.148 = 0.352$$

II approximation is $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.352 - \frac{f(0.352)}{f'(0.352)}$$

But $$f(0.352) = 3(0.352) + \sin(0.352) - e^{0.352} = -0.02113$$
$$f'(0.352) = 3 + 0.93869 - 1.42191 = 2.51678$$
$$x_2 = 0.352 - \frac{(-0.02113)}{2.51678}$$
$$= 0.352 + 0.00839 = 0.3604$$

III approximation is $$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.3604 - \frac{f(0.3604)}{f'(0.3604)}$$

But $$f(0.3604) = 3(0.3604) + \sin(0.3604) - e^{0.3604}$$
$$= 1.4338 - 1.4339 = -0.0001$$

But $$f'(0.3604) = 3 + 0.93576 - 1.4339 = 2.50186$$

$\therefore$ $$x_3 = 0.3604 - \frac{(-0.0001)}{2.50186}$$
$$= 0.3604 + 0.0000399 = 0.360439$$

Since $x_2 = x_3$ upto four decimal places, the root is $\boldsymbol{x = 0.3604}$ ■

Example 5

Find the negative root of $x^3 - 2x + 5 = 0$ by Newton-Raphson method correct to 3 places of decimals.

Solution

Given equation is $x^3 - 2x + 5 = 0$

The negative root of $x^3 - 2x + 5 = 0$ is the positive root of

$$(-x)^3 - 2(-x) + 5 = 0$$

$\Rightarrow$ $$-x^3 + 2x + 5 = 0$$

$\Rightarrow$ $$x^3 - 2x - 5 = 0$$

Let $$f(x) = x^3 - 2x - 5$$

$\therefore$ $$f'(x) = 3x^2 - 2$$

Now $f(0) = -5, < 0,$ $\quad f(1) = 1 - 2 - 5 = -6 < 0$

and $f(2) = 8 - 4 - 5 = -1 < 0,$ $\quad f(3) = 27 - 6 - 5 = 16 > 0$

So, the root lies between 2 and 3 and it is nearer to 2.

Take $x_0 = 2$

Newton's iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, \ldots$$

I approximation is $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 2 - \frac{f(2)}{f'(2)} = 2 - \frac{(-1)}{10} = 2.1$$

II approximation is $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.1 - \frac{f(2.1)}{f'(2.1)}$$

But $$f(2.1) = (2.1)^3 - 2(2.1) - 5 = 9.261 - 9.2 = 0.061$$
$$f'(2.1) = 3(2.1)^2 - 2 = 11.23$$
$$x_2 = 2.1 - \frac{0.061}{11.23} = 2.1 - 0.0054 = 2.0946$$

III approximation is $$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.0946 - \frac{f(2.0946)}{f'(2.0946)}$$

But $$f(2.0946) = (2.0946)^3 - 2(2.0946) - 5$$
$$= 9.18974 - 4.1892 - 5 = 0.00054$$
$$f'(2.0946) = 3(2.0946)^2 - 2 = 11.16205$$

$\therefore$ $$x_3 = 2.0946 - \frac{0.00054}{11.16205}$$
$$= 2.0946 - 0.000048 = 2.09455$$

$\therefore$ the root correct to 3 decimal places is 2.095

$\therefore$ the negative root of the given equation is $\boldsymbol{x = -2.095}$ ■

Example 6

Find an iterative formula to find $\sqrt{N}$, where N is a positive integer, using Newton's method and hence find $\sqrt{11}$.

Solution

Let $x = \sqrt{N}$. Then $x^2 = N \Rightarrow x^2 - N = 0$

Let $f(x) = x^2 - N$ $\therefore$ $f'(x) = 2x$

Newton's iteration formula is $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, 3, \ldots$$

$\therefore$ $$x_{n+1} = x_n - \frac{x_n^2 - N}{2x_n}$$
$$= \frac{2x_n^2 - x_n^2 + N}{2x_n}$$
$$x_{n+1} = \frac{x_n^2 + N}{2x_n} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

To find value of $\sqrt{11}$: Here $N = 11$

We know $x = \sqrt{11}$ lies between 3 and 4 $\quad \left[\because \sqrt{9} = 3, \sqrt{16} = 4 \text{ and } 3 < \sqrt{11} < \sqrt{16}\right]$

Now $f(3.5) = (3.5)^2 - 11 = 1.25 > 0$.

So, the root lies between 3 and 3.5.

Take $x_0 = 3.3$

$\therefore$

$$x_1 = \frac{1}{2}\left(3.3 + \frac{11}{3.3}\right) = 3.31666$$

$$x_2 = \frac{1}{2}\left(3.31666 + \frac{11}{3.31666}\right) = 3.31662$$

$$x_3 = \frac{1}{2}\left(3.31662 + \frac{11}{3.31662}\right) = 3.31662$$

$\therefore \sqrt{11} = \mathbf{3.3166}$ correct to 4 decimal places. ■

Example 7

Find an iterative formula for $\sqrt[3]{N}$, where N is a positive integer, using Newton's method and hence find $\sqrt[3]{24}$, $\sqrt[3]{41}$ to 6 places of decimals.

Solution

Let $x = \sqrt[3]{N}$, then $x^3 = N \Rightarrow x^3 - N = 0$

Let $f(x) = x^3 - N$, $\therefore f'(x) = 3x^2$

Newton's iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, \ldots.$$

$$= x_n - \frac{(x_n^3 - N)}{3x_n^2}$$

$$= \frac{3x_n^3 - x_n^3 + N}{3x_n^2}$$

$$x_{n+1} = \frac{2x_n^3 + N}{3x_n^2} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right), \qquad n = 0, 1, 2, \ldots$$

which is the iteration formula

(i) **To find $\sqrt[3]{24}$** : Here $N = 24$.

We know $x = \sqrt[3]{24}$ lies between 2 and 3. $\left[\because \sqrt[3]{8} = 2, \sqrt[3]{27} = 3 \text{ and } 2 < \sqrt[3]{24} < 3\right]$

$$f(2) = 8 - 24 = -16 < 0 \quad \text{and} \quad f(3) = 27 - 24 = 3 > 0$$

Now $f(2.5) = (2.5)^3 - 24 = -8.375 < 0$

So, the root is between 2.5 and 3 and it is closer to 3.

Take $x_0 = 2.8$

$\therefore$
$$x_1 = \frac{1}{3}\left(2(2.8) + \frac{24}{(2.8)^2}\right)$$
$$= \frac{1}{3}(5.6 + 3.06122) = 2.88707$$
$$x_2 = \frac{1}{3}\left[2(2.88707) + \frac{24}{(2.88707)^2}\right]$$
$$= \frac{1}{3}[5.77414 + 2.879364]. = 2.88450$$
$$x_3 = \frac{1}{3}\left[2x_2 + \frac{24}{x_2^2}\right]$$
$$= \frac{1}{3}\left[2(2.88450) + \frac{24}{(2.88450)^2}\right]$$
$$= \frac{1}{3}[5.76900 + 2.88450] = 2.884499$$

Similarly, $x_4 = 2.884499$

$$\sqrt[3]{24} = 2.884499$$

which is correct upto 6 places.

(ii) $\sqrt[3]{41}$; Here $N = 41$ Let $x = \sqrt[3]{41}$ $\Rightarrow x^3 = 41$ $\Rightarrow x^3 - 41 = 0$

Let $f(x) = x^3 - 41$

we know $\sqrt[3]{41}$ lies between 3 and 4. $\left[\because 3 = \sqrt[3]{27} \text{ and } 4 = \sqrt[3]{64}\right]$

Now $f(3) = 27 - 41 = -14 < 0$ and $f(4) = 64 - 51 = 23 > 0$

But $f(3.5) = 1.875 > 0$ so, the root is between 3 and 3.5 and closer to 3.5

Take $x_0 = 3.4$

$\therefore$
$$x_1 = \frac{1}{3}\left[2x_0 + \frac{41}{x_0^2}\right]$$
$$= \frac{1}{3}\left[2(3.4) + \frac{41}{(3.4)^2}\right]$$
$$= \frac{1}{3}[6.8 + 3.5467] = 3.4489$$

$$x_2 = \frac{1}{3}\left[2(3.4489) + \frac{41}{(3.4489)^2}\right]$$

$$= \frac{1}{3}[6.8978 + 3.44685] = 3.4482$$

$$x_3 = \frac{1}{3}\left[2(3.4482) + \frac{41}{(3.4482)^2}\right]$$

$$= \frac{1}{3}[6.8964 + 3.4482517] = 3.44821724$$

Similarly $\quad x_4 = 3.448217$

$\therefore \quad \sqrt[3]{41} = 3.448217$ upto 6 decimal places. ■

Example 8

Find the iterative formula by Newton's formula for (i) $\frac{1}{N}$ (ii) $\frac{1}{\sqrt{N}}$, where N is a positive integer. Hence find $\frac{1}{31}$ and $\frac{1}{\sqrt{15}}$.

Solution

(i) Let $x = \frac{1}{N} \quad \Rightarrow \frac{1}{x} - N = 0$

Let $f(x) = \frac{1}{x} - N$, then $f'(x) = -\frac{1}{x^2}$

Newton's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0,1,2,\ldots$$

$$\therefore \quad x_{n+1} = x_n - \frac{\left(\frac{1}{x_n} - N\right)}{-\frac{1}{x_n^2}}$$

$$= x_n + x_n^2\left(\frac{1}{x_n} - N\right)$$

$$= x_n + x_n - Nx_n^2$$

$$x_{n+1} = 2x_n - Nx_n^2 = x_n(2 - Nx_n), \qquad n = 0,1,2,\ldots \tag{1}$$

To find $\frac{1}{31}$: Here $N = 31$, Since $\frac{1}{31} \simeq 0.03$, **take $x_0 = 0.03$,**

$\therefore$

$$x_1 = x_0(2-31x_0) \qquad \text{[From(1)]}$$
$$= 0.03(2-31\times 0.03) = 0.0321$$
$$x_2 = x_1(2-31x_1)$$
$$= 0.0321(2-31\times 0.0321) = 0.03226$$
$$x_3 = 0.03226(2-31\times 0.03226 = 0.032258)$$

$\therefore$ the value of $\frac{1}{31}$ is **0.03226** correct to 4 significant places.

(ii) **Let $x = \frac{1}{\sqrt{N}}$** $\Rightarrow x^2 = \frac{1}{N} \Rightarrow x^2 - \frac{1}{N} = 0$

Let $f(x) = x^2 - \frac{1}{N}$, then $f'(x) = 2x$

Newton's formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0,1,2,\ldots$

$$= x_n - \frac{\left(x_n^2 - \frac{1}{N}\right)}{2x_n}$$

$$x_{n+1} = \frac{2x_n^2 - x_n^2 + \frac{1}{N}}{2x_n} = \frac{x_n^2 + \frac{1}{N}}{2x_n} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right), \qquad n = 0,1,2,\ldots$$

(iii) **To find $\frac{1}{\sqrt{15}}$:** Here $N = 15$.

Since $\frac{1}{\sqrt{15}}$ close to $\frac{1}{\sqrt{16}} = \frac{1}{4} = 0.25$, we shall **take $x_0 = 0.25$**

Then

$$x_1 = \frac{1}{2}\left(0.25 + \frac{1}{15\times 0.25}\right) = \frac{1}{2}(0.25 + 0.26666) = 0.25833$$
$$x_2 = \frac{1}{2}\left(0.25833 + \frac{1}{15\times 0.25833}\right) = 0.25819$$
$$x_3 = \frac{1}{2}\left(0.25819 + \frac{1}{15\times 0.25819}\right) = 0.25819889$$

$\therefore$ the value of $\frac{1}{\sqrt{15}}$ is **0.25819** correct to 5 places of decimals.

Example 9

The equation $2e^{-x} = \frac{1}{x+2} + \frac{1}{x+1}$ has two roots greater than -1. Calculate these two roots correct to five decimal places.

Solution

The given equation is

$$2e^{-x} - \frac{1}{x+2} - \frac{1}{x+1} = 0$$

Let $$f(x) = 2e^{-x} - \frac{1}{x+2} - \frac{1}{x+1}$$

Now $$f(0) = 2 - \frac{1}{2} - 1 = \frac{1}{2} > 0$$

$$f(1) = 2e^{-1} - \frac{1}{3} - \frac{1}{2} = -0.09757 < 0$$

So, there is a root between 0 and 1 and the root is nearer to 1.

Now $$f(-0.8) = 2e^{0.8} - \frac{1}{-0.8+2} - \frac{1}{-0.8+1}$$

$$= 2e^{0.8} - \frac{1}{1.2} - \frac{1}{0.2} = -1.38225 < 0$$

$\therefore$ another root lies between -0.8 and 0 and the root is close to zero, since $f(0) = 0.5$. and $f(0.8) = -1.38225$.

We shall find these roots by Newton-Raphson method.

(i) **First we shall find the root between -0.8 and 0**

Take $x_0 = -0.6$

Newton's iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0,1,2,3...$$

First approximation: Put $n = 0$

$$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

But $$f(x) = 2e^{-x} - \frac{1}{x+2} - \frac{1}{x+1}$$

$$\therefore \quad f'(x) = -2e^{-x} - \frac{(-1)}{(x+2)^2} - \frac{(-1)}{(x+1)^2} \qquad \left[\because \frac{dy}{dx} = \frac{-1}{(x+a)^2} \text{ if } y = \frac{1}{x+a}\right]$$

$$= -2e^{-x} + \frac{1}{(x+2)^2} + \frac{1}{(x+1)^2}$$

Now $$f(-0.6) = 2e^{0.6} - \frac{1}{-0.6+2} - \frac{1}{-0.6+1}$$
$$= 2e^{0.6} - \frac{1}{1.4} - \frac{1}{0.4}$$
$$= 2(1.822119) - 0.714286 - 2.5$$
$$= 3.644238 - 0.714286 - 2.5 = 0.429952$$

$$f'(-0.6) = -2e^{+0.6} + \frac{1}{(-0.6+2)^2} + \frac{1}{(-0.6+1)^2}$$
$$= -2(1.822119) + \frac{1}{(1.4)^2} + \frac{1}{(0.4)^2}$$
$$= -3.644238 + 0.510204 + 6.25$$
$$= 3.115966$$

$\therefore$ $$x_1 = -0.6 - \frac{0.429952}{3.11597}$$
$$= -0.6 - 0.13798 = -0.73798$$

Second approximation is: Put $n = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -0.73798 - \frac{f(-0.73798)}{f'(-0.73798)}$$

Now $$f(-0.73798) = 2e^{0.73798} - \frac{1}{-0.73798+2} - \frac{1}{-0.73798+1}$$
$$= 2.e^{0.73798} - \frac{1}{1.26202} - \frac{1}{0.26202}$$
$$= 4.183412 - 0.79238 - 3.81650 = -0.42547$$

$$f'(-0.73798) = -2.e^{0.73798} + \frac{1}{(1.26202)^2} + \frac{1}{(0.26202)^2}$$
$$= -4.183412 + 0.62787 + 14.56569 = 11.010150$$

$\therefore$ $$x_2 = -0.73798 - \frac{(-0.42547)}{11.010150} = -0.69934$$

Third approximation: Put $n = 2$

$\therefore$ $$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -0.69934 - \frac{f(-0.69934)}{f(-0.69934)}$$

Now $$f(-0.69934) = 2e^{0.69934} - \frac{1}{1.30066} - \frac{1}{0.30066}$$
$$= 4.02485 - 0.76884 - 3.32602 = -0.07001$$

$$f'(-0.69934) = -2e^{0.69934} + \frac{1}{(1.30066)^2} + \frac{1}{(0.30066)^2}$$
$$= -4.02485 + 0.59112 + 11.06238 = 7.62865$$

$\therefore$
$$x_3 = -0.69934 - \left[\frac{-0.07001}{7.62865}\right]$$
$$= -0.69934 + 0.009177 = -0.69016$$

Fourth approximation: Put $n = 3$

$\therefore$
$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = -0.69016 - \frac{f(-0.69016)}{f'(-0.69016)}$$

Now
$$f(x_3) = 2e^{0.69016} - \frac{1}{1.30984} - \frac{1}{0.30984}$$
$$= 3.98807 - 0.76345 - 3.22747$$
$$= -0.00285$$

$$f'(x_3) = -2e^{0.69016} + \frac{1}{(1.30984)^2} + \frac{1}{(0.30984)^2}$$
$$= -3.98807 + 0.58286 + 10.41658 = 7.01137$$

$\therefore$
$$x_4 = -0.69016 - \frac{(-0.00285)}{7.01137} = -0.68975$$

Fifth approximation: Put $n = 4$

$\therefore$
$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = -0.68975 - \frac{f(0.68975)}{f'(-0.68975)}$$

Now
$$f(-0.68975) = 2.e^{0.68975} - \frac{1}{(1.31025)} - \frac{1}{0.31025}$$
$$= 3.98643 - 0.76321 - 3.22321 = -0.00001$$

$$f'(-0.68975) = -2e^{0.68975} + \frac{1}{(1.31025)^2} + \frac{1}{(0.31025)^2}$$
$$= -3.98643 + 0.58249 + 10.38906 = 6.98512$$

$\therefore$
$$x_5 = -0.68975 - \frac{(0.00001)}{6.98512}$$
$$= -0.68975 + 0.0000014316 = -0.68975$$

Since $x_4 = x_5$, the root between -1 and 0 correct to four decimal places is $\mathbf{x = -0.68975}$

(ii) The other root lies between 0 and 1 and it is nearer to 1.

Take $x_0 = \mathbf{0.8}$

I approximation is $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 0.8 - \dfrac{f(0.8)}{f'(0.8)}$

$$f(x_0) = f(0.8) = 2e^{-0.8} - \frac{1}{2.8} - \frac{1}{1.8}$$
$$= 0.89866 - 0.35714 - 0.55556 = -0.01404$$

$$f'(x_0) = f'(0.8) = -2e^{-0.8} + \frac{1}{(2.8)^2} + \frac{1}{(1.8)^2}$$
$$= -0.89866 + 0.12755 + 0.30864 = -0.46247$$

$\therefore$ $$x_1 = 0.89866 - \frac{(-0.01404)}{(-0.46247)} = 0.89866 - 0.03036 = 0.8683$$

II approximation is $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 0.8683 - \dfrac{f(0.8683)}{f'(0.8683)}$

Now $$f(0.8683) = 2e^{-0.8683} - \frac{1}{0.8683+2} - \frac{1}{0.8683+1}$$
$$= 0.83933 - \frac{1}{2.8683} - \frac{1}{1.8683+1}$$
$$= 0.83933 - 0.34864 - 0.53525 = -0.04456$$

$$f'(0.8683) = -2e^{-0.8683} + \frac{1}{(2.8683)^2} + \frac{1}{(1.8683)^2}$$
$$= -0.83933 + 0.12155 + 0.28649 = -0.43129$$

$\therefore$ $$x_2 = 0.8683 - \frac{(-0.04456)}{(-0.43129)} = 0.8683 - 0.10332 = 0.76498$$

III approximation is $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = 0.76498 - \dfrac{f(0.76498)}{f'(0.76498)}$

Now $$f(0.76498) = 2e^{-0.76498} - \frac{1}{0.76498+2} - \frac{1}{0.76498+1}$$
$$= 0.93069 - 0.36167 - 0.56658 = 0.00244$$

$$f'(0.76498) = -2e^{-0.76498} + \frac{1}{(2.76498)^2} + \frac{1}{(1.76498)^2}$$
$$= -0.93069 + 0.1308 + 0 + 0.32101 = -0.47888$$

$\therefore$ $$x_3 = x_2 - \frac{f(x)}{f'(x_2)} = 0.76498 - \frac{0.00244}{(-0.47888)} = 0.76999$$

IV approximation is

$$x_4 = x_3 - \frac{f(x_3)}{f'(x)} = 0.76999 - \frac{f(0.76999)}{f'(0.76999)}$$

Now

$$f(0.76999) = 2e^{-0.76999} - \frac{1}{0.76999+2} - \frac{1}{0.76999+1}$$

$$= 2(0.4630) - 0.3610 - 0.56497 = 0.00003$$

$$f'(0.76999) = -2e^{-0.76999} + \frac{1}{(0.76999+2)^2} + \frac{1}{(0.76999+1)^2}$$

$$= -2(0.4630) + 0.1303 + 0.3192 = -0.4765$$

$\therefore$

$$x_4 = 0.76999 - \frac{0.00003}{-0.4765} = 0.76999 + 0.000063 = 0.770053$$

$\therefore$ the second root between 0 and 1, correct to 5 decimal places is $x = \mathbf{0.77005}$ ■

Exercises 2.1

(I) Find a root by bi-section method.

(1) $x^4 - x^3 - 2x^2 - 6x - 1 = 0$, between 2 and 3 to four decimal place.

(2) $x^4 - 2x - 10 = 0$, which lies between 1 and 2.

(II) Find a root by Regular-Falsi method.

(3) $x^3 - 5x + 1 = 0$

(4) the smallest positive root of $x^2 - \log_e x - 12 = 0$

(5) $xe^x - 3 = 0$

(6) $xe^x = \cos x$

(7) $x^3 - 3x - 5 = 0$

(8) $e^{-x} - \sin x = 0$

(9) $x^3 + x - 1 = 0$

(10) $xe^x - \sin x = 0$ to 3 decimal places.

(III) Find a root of the following by the fixed point iteration method.

(11) $x^3 + x^2 - 1 = 0$

(12) $x^3 + x^2 - 100 = 0$

(13) $2x - 10\log_{10} x - 7 = 0$

(14) $x^x + 5x = 1000$

[Hint: $x = (1000 - 5x)^{1/x}$, $\phi(x) = (1000 - 5x)^{1/x}$]

(15) Given that 0.44 is approximate root of the equation

$$p-\frac{p^3}{3}+\frac{p^5}{10}-\frac{p^7}{42}+\frac{p^9}{216}-\frac{p^{11}}{1320}=0.4431135$$

Improve it by two iterations.

(16) Find the smallest root of

$$1+x+\frac{x^3}{(2!)^2}-\frac{x^3}{(3!)^3}+\frac{x^4}{(4!)^4},\ldots=0$$

(IV) By Newton's method find a root of the following equations.

(17) $3x^3-9x^2+8=0$

(18) $x^3-6x+4=0$ between 0 and 1 to 5 places of decimals.

(19) $x^3-3x+1=0$ between 1 and 2.

(20) $xe^x-2=0$

(21) $2x^3-3x-6=0$ between 1 and 2 to five places of decimals.

(22) $x\log_{10}x=12.34$

(23) $e^x=4x$ between 2 and 3.

(24) $x^2+4\sin x=0 (x\neq 0)$

(25) $x^3-2x-5=0$

(26) $x-2\sin x=0$ near 1.9 to 3 places of decimals.

(27) $e^{-x}-\sin x=0$

(28) $\log_{10}x=\cos x$

(29) $x^4-x-10=0$

(30) Negative root of $x^3-5x+11=0$ correct to 2 places of decimals.

(31) $2x-\log_{10}x=7$

(32) Find $\sqrt[3]{10}$ by iteration.

(V) By the secant method find a root of the following equations.

(33) $x^3+x^2-3x-3=0$ between 1 and 2 correct to four decimal places.

(34) $3x+\sin x-e^x=0$, correct to four decimal places.

Answers 2.1

(1) 2.7321	(2) 1.9295	(3) 0.2016	(4) 3.646	(5) 1.050
(6) 0.517	(7) 2.279	(8) 0.5885	(9) 0.686	(10) –0.134
(11) 0.7549	(12) 4.3311	(13) 4.7892	(14) 4.5465	(15) 0.476
(16) 1.4453	(17) 1.226	(18) 0.73205	(19) 1.5321	(20) 0.8526
(21) 1.78377	(22) 11.5949	(23) 2.1533	(24) –1.9338	(25) 2.0946
(26) 1.896	(27) 0.5885	(28) 1.3029	(29) 1.856	(30) –2.95
(31) 3.7892	(32) 2.15466	(33) 1.7351	(34) 0.3604	

GENERALISED NEWTON-RAPHSON METHOD OR MODIFIED NEWTON'S METHOD

Suppose $x = \alpha$ is a root of $f(x) = 0$ of multiplicity P, then $f(x) = (x-\alpha)^p \phi(x)$, where $\phi(\alpha) \neq 0$.

Then the iteration formula is $x_{n+1} = x_n - p\dfrac{f(x_n)}{f'(x_n)}$, $\quad n = 0,1,2,\ldots$

Note:

(1) When $p = 1$, we get Newton-Raphson formula.

(2) Since α is a root of $f(x)=0$ of multiplicity p, α is a root of $f'(x) = 0$ of multiplicity $p-1$, α is a root of $f''(x) = 0$ of multiplicity $p-2$ and so on.

If the initial approximation x_0 is chosen closer to the root, then the expressions

$x_0 - p\dfrac{f(x_0)}{f'(x_0)}$, $\quad x_0 - (p-1)\dfrac{f'(x_0)}{f''(x_0)}$, $\quad x_0 - (p-2)\dfrac{f''(x_0)}{f'''(x_0)}$,... must have the same value.

WORKED EXAMPLES

Example 1

Find the double root of $x^3 - 7x^2 + 16x - 12 = 0$ by general Newton's formula with $x_0 = 1.5$.

Solution

Let $\quad f(x) = x^3 - 7x^2 + 16x - 12 \quad$ Given $x_0 = 1.5$

$\therefore \quad f'(x) = 3x^2 - 14x + 16$

Since there is a double root near $x_0 = 1.5$, the generalized Newton-Raphson iteration formula is

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)}, \quad n = 0,1,2,\ldots$$

Putting $n = 0$, the first approximation is

$$x_1 = x_0 - \frac{2f(x_0)}{f'(x_0)} = 1.5 - \frac{2f(1.5)}{f'(1.5)}$$

Now
$$f(x_0) = f(1.5) = (1.5)^3 - 7(1.5)^2 + 16(1.5) - 12$$
$$= 3.375 - 15.75 + 24 - 12 = -0.375$$

and
$$f'(x_0) = f'(1.5) = 3(1.5)^2 - 14(1.5) + 16$$
$$= 6.75 - 21 + 16 = 1.75$$

$$\therefore \quad x_1 = 1.5 - \frac{2(-0.375)}{1.75} = 1.5 + \frac{0.75}{1.75} = 1.5 + 0.42857 = 1.9286$$

Second approximation is

$$x_2 = x_1 - \frac{2f(x_1)}{f'(x_1)} = 1.9286 - \frac{2f(1.9286)}{f'(1.9286)}$$

Now $$f(x_1) = f(1.9286) = (1.9286)^3 - 7(1.9286)^2 + 16(1.9286) - 12$$
$$= 7.173423 - 26.036485 + 30.8576 - 12 = -0.005462$$

$$f'(1.9286) = 3(1.9286)(1.9286)^2 - 14(1.9286) + 16$$
$$= 11.158494 - 27.0004 + 16 = 0.158094 = 0.1581$$

$\therefore$ $$x_2 = 1.9286 - \frac{2(-0.005462)}{0.15814}$$
$$= 1.9286 + 0.069096 = 1.9977$$

Third approximation is

$$x_3 = x_2 - 2\frac{f(x_0)}{f'(x_0)} = 1.9977 - \frac{2f(1.9977)}{f'(1.9977)}$$

Now $$f(1.9977) = (1.9977)^3 - 7(1.9977)^2 + 16(1.9977) - 12$$
$$= 7.9724317 - 27.93563703 + 31.9632 - 12 = -0.0000053$$

$$f'(1.9977) = 3(1.9977)^2 - 14(1.9977) + 16$$
$$= 11.97241587 - 27.9678 + 16 = 0.004616$$

$\therefore$ $$x_3 = 1.9977 - \frac{2(-0.0000053)}{0.004616} = 1.9977 + 0.002296 = 1.999996$$

$\therefore$ the multiple root is $\boldsymbol{x = 2}$ approximately ■

Example 2

Find the root of multiplicity 2 near 0.5 for the equation $x^3 - x^2 - x + 1 = 0$.

Solution

Let $$f(x) = x^3 - x^2 - x + 1$$

$\therefore$ $$f'(x) = 3x^2 - 2x - 1$$

Since there is a double root near $x_0 = 0.5$, the generalized Newton's-Raphson iteration formula is

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, \ldots$$

Putting $n = 0$, the first approximation is

$$x_1 = x_0 - \frac{2f(x_0)}{f'(x_0)} = 0.5 - \frac{2f(0.5)}{f'(0.5)}$$

$$f(0.5) = (0.5)^3 - (0.5)^2 - 0.5 + 1$$
$$= 0.125 - 0.25 - 0.5 + 1 = 0.375$$

$$f'(0.5) = 3(0.5)^2 - 2(0.5) - 1 = 0.75 - 1 - 1 = -1.25$$

$\therefore$
$$x_1 = 0.5 - 2\frac{(0.375)}{-1.25} = 0.5 + 0.6 = 1.1$$

Putting $n = 1$, the second approximation is

$$x_2 = x_1 - 2\frac{f(x_1)}{f'(x_1)} = 1.1 - 2\frac{f(1.1)}{f'(1.1)}$$

Now
$$f(1.1) = (1.1)^3 - (1.1)^2 - 1.1 + 1$$
$$= 1.331 - 1.21 - 1.1 + 1 = 0.021$$

$$f'(1.1) = 3(1.1)^2 - 2(1.1) - 1 = 3.63 - 2.2 - 1 = 0.43$$

$\therefore$
$$x_2 = 1.1 - \frac{2(0.021)}{0.43} = 1.1 - 0.09767 = 1.0023$$

Putting $n = 2$, the third approximation is

$$x_3 = x_2 - \frac{2f(x_2)}{f'(x_2)}$$
$$= 1.0023 - \frac{2 \times f(1.0023)}{f'(1.0023)}$$

$$f(1.0023) = (1.0023)^3 - (1.0023)^2 - 1.0023 + 1$$
$$= 1.006915882 - 1.00460529 - 1.0023 + 1$$
$$= 0.00001059 = 0.00001$$

$$f'(1.0023) = 3(1.0023)^2 - 2(1.0023) - 1$$
$$= 3.013815 - 2.0046 - 1 = 0.00922$$

$\therefore$
$$x_3 = 1.0023 - \frac{2(0.00001)}{0.00922}$$
$$= 1.0023 - 0.00217 = 1.00013$$

$\therefore$ the double root is $x = \mathbf{1.0013}$

Note that 1 is the double root of the given equation and 1.00013 is very close to the actual root ■

RAMANUJAN'S METHOD

The year 2012 was declared as National Mathematical year by our Prime Minister Dr. Manmohan Singh in honour of our Mathematical prodigy, "Srinivasa Ramanujan" on his 125th birth day. So, we thought it is appropriate to include Ramanujan's method in this text book.

He has given an iterative method to find the smallest root of $f(x) = 0$, where $f(x)$ can be written in the form

$$f(x) = 1 - \left(a_1 x + a_2 x^2 + a_3 x^3 + \ldots\right) \tag{1}$$

If x is small then $\left[1-\left(a_1x+a_2x^2+a_3x^3+\ldots\right)\right]^{-1}$ can be expanded using binomial expansion as a series of the form $b_1+b_2x+b_3x^2+b_4x^4+\ldots$

$$\therefore \quad 1+\left(a_1x+a_2x^2+a_3x^3+\ldots\right)+\left(a_1x+a_2x^2+a_3x^3+\ldots\right)^2+\left(a_1x+a_2x^2+a_3x^3+\ldots\right)^3+\ldots$$
$$=b_1+b_2x+b_3x^2+b_4x^3+b_5x^4+\ldots$$

$$\Rightarrow \quad 1+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots$$
$$+a_1^2x^2+2a_1a_2x^3+a_2^2x^4+2a_1a_3x^4+a_1^3x^3+3a_1^2a_2x^4+\ldots$$
$$=b_1+b_2x+b_3x^2+b_4x^3+b_5x^4+\ldots$$

$$\Rightarrow \quad 1+a_1x+\left(a_2+a_1^2\right)x^2+\left(a_3+2a_1a_2+a_1^3\right)x^3+\left(a_4+a_2^2+2a_1a_3+3a_1^2a_2\right)x^4+\ldots$$
$$=b_1+b_2x+b_3x^2+b_4x^3+b_5x^4+\ldots$$

Equating coefficients of like powers on both sides, we get

$$b_1=1$$
$$b_2=a_1$$
$$b_3=a_2+a_1^2$$
$$b_4=a_3+2a_1a_2+a_1^3$$
$$b_5=a_4+a_2^2+2a_1a_3+3a_1^2a_2+a_1^4 \text{ and so on.}$$

Now we shall rewrite the R.H.S interms of a's and b's in a special way.

$$b_1=1$$
$$b_2=a_1.1=a_1b_1$$
$$b_3=a_1a_1+a_2.1=a_1b_2+a_2b_1 \qquad [\because a_1=b_2]$$
$$b_4=a_1\left(a_1^2+a_2\right)+a_2a_1+a_3.1=a_1b_3+a_2b_2+a_3b_1$$
$$b_5=a_1\left(a_1^3+2a_1a_2+a_3\right)+a_1^2a_2+a_1a_3+a_2^2+a_4$$
$$=a_1\left(a_1^3+2a_1a_2+a_3\right)+a_2\left(a_1^2+a_2\right)+a_1a_3+a_4.1$$
$$=a_1b_4+a_2b_3+a_3b_2+a_4b_1 \text{ and so on.}$$

From the above relations, we observe that the R.H.S is in terms of a's and b's.

The sum of the suffixes in each term on the R.H.S is the same suffix of b on the L.H.S. From this pattern, we can write

$$b_n=a_1b_{n-1}+a_2b_{n-2}+a_3b_{n-3}+\ldots+a_{n-1}b_1, \qquad n=2,3,\ldots$$

Srinivasa Ramanujan was gifted with powerful intuition and computational skill.

He has simply stated this method (without rigorous proof) the successive convergents, namely

$$\frac{b_n}{b_{n+1}}, \quad n=1,2,3,4,\ldots$$

ie. $\frac{b_1}{b_2}, \frac{b_2}{b_3}, \frac{b_3}{b_4}, \frac{b_4}{b_5}, \ldots$ approach a root of $f(x)=0$, where $f(x)$ is given by (1)

WORKED EXAMPLES

Example 1

Find the smallest root of $\sin x = 1-x$ by Ramanujan's method to 4 decimal places.

Solution

Given $\quad \sin x = 1-x \quad (1)$

$$\Rightarrow \quad 1-x-\sin x=0$$

$$\Rightarrow \quad 1-x-\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\right]=0$$

$$\Rightarrow \quad 1-\left[2x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\right]=0$$

Here
$$f(x)=1-\left(2x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\right)$$

By Ramanujan's method

$$\left[1-\left(2x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}+\ldots\right)\right]^{-1}$$
$$=b_1+b_2x+b_3x^2+b_4x^3+b_5x^4+b_6x^5+b_7x^6+\ldots$$

Here

$$a_1=2, \quad a_2=0, \quad a_3=-\frac{1}{6}, \quad a_4=0, \quad a_5=\frac{1}{120}, \quad a_6=0, \quad a_7=-\frac{1}{5040}$$

$\therefore \quad b_1=1$

$$b_2=a_1b_1=2\times1=2$$

$$b_3=a_1b_2+a_2b_1=2\times2+0\times1=4$$

$$b_4=a_1b_3+a_2b_2+a_3b_1=2\times4+0+(-\frac{1}{6})\times1=8-\frac{1}{6}=\frac{47}{6}$$

$$b_5=a_1b_4+a_2b_3+a_3b_2+a_4b_1$$
$$=2\times\frac{47}{6}+0+(-\frac{1}{6})\times2+0=\frac{47}{3}-\frac{1}{3}=\frac{46}{3}$$

$$b_6=a_1b_5+a_2b_4+a_3b_3+a_4b_2+a_5b_1$$
$$=2\times\frac{46}{3}+0+(-\frac{1}{6})\times4+0+\frac{1}{120}\times1=\frac{92}{3}-\frac{2}{3}+\frac{1}{120}=30+\frac{1}{120}=\frac{3601}{120}$$

$$b_7=a_1b_6+a_2b_5+a_3b_4+a_4b_3+a_5b_2+a_6b_1$$
$$=2\times\frac{3601}{120}+0+(-\frac{1}{6})\times\frac{47}{6}+0+\frac{1}{120}\times2=\frac{3602}{60}-\frac{47}{36}+\frac{1}{60}=\frac{3602}{60}-\frac{47}{36}=58.72778$$

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Now the iterative formula is $\dfrac{b_n}{b_{n+1}}$, $\qquad n = 1,2,3,4,\ldots$

That is $\dfrac{b_1}{b_2}, \dfrac{b_2}{b_3}, \dfrac{b_3}{b_4}, \dfrac{b_4}{b_5}, \dfrac{b_5}{b_6}, \dfrac{b_6}{b_7}\ldots$ approach a real root of $\sin x = 1 - x$.

$$\frac{b_1}{b_2} = \frac{1}{2} = 0.5 \qquad\qquad \frac{b_2}{b_3} = \frac{2}{4} = 0.5$$

$$\frac{b_3}{b_4} = \frac{4}{\frac{47}{6}} = \frac{24}{47} = 0.51064 \qquad \frac{b_4}{b_5} = \frac{\frac{47}{6}}{\frac{46}{3}} = \frac{47}{92} = 0.51087$$

$$\frac{b_5}{b_6} = \frac{\frac{46}{3}}{\frac{3601}{120}} = \frac{46 \times 40}{3601} = 0.51097$$

$$\frac{b_6}{b_7} = \frac{\frac{3601}{120}}{58.72778} = \frac{3601}{120 \times 58.72778} = 0.51097$$

$\therefore$ the root correct to five decimal places is $\boldsymbol{x = 0.51097}$ ■

Example 2

Find a real root of $x e^x = 2$.

Solution

Given
$$x e^x = 2$$

$$\Rightarrow \qquad x\left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots\right] = 2$$

$$\Rightarrow \qquad 2 - \left[x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + \frac{x^6}{120} + \ldots\right] = 0$$

$$\Rightarrow \qquad 1 - \left[\frac{x}{2} + \frac{x^2}{2} + \frac{x^3}{4} + \frac{x^4}{12} + \frac{x^5}{48} + \frac{x^6}{240} + \ldots\right] = 0$$

Here
$$f(x) = 1 - \left(\frac{x}{2} + \frac{x^2}{2} + \frac{x^3}{4} + \frac{x^4}{12} + \frac{x^5}{48} + \frac{x^6}{240} + \ldots\right)$$

By Ramanujan's method

$$\left[1 - \left(\frac{x}{2} + \frac{x^2}{2} + \frac{x^3}{4} + \frac{x^4}{12} + \frac{x^5}{48} + \frac{x^6}{240} + \ldots\right)\right]^{-1}$$
$$= b_1 + b_2 x + b_3 x^2 + b_4 x^3 + b_5 x^4 + \ldots$$

Here

$$a_1 = \frac{1}{2},\ a_2 = \frac{1}{2},\ a_3 = \frac{1}{4},\ a_4 = \frac{1}{12},\quad a_5 = \frac{1}{48},\ a_6 = \frac{1}{240}$$

$\therefore$

$$b_1 = 1$$

$$b_2 = a_1 b_1 = \frac{1}{2}.1 = \frac{1}{2}$$

$$b_3 = a_1 b_2 + a_2 b_1 = \frac{1}{2}.\frac{1}{2} + \frac{1}{2}.1 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

$$b_4 = a_1 b_3 + a_2 b_2 + a_3 b_1 = \frac{1}{2}.\frac{3}{4} + \frac{1}{2}.\frac{1}{2} + \frac{1}{4}.1$$

$$= \frac{3}{8} + \frac{1}{4} + \frac{1}{4} = \frac{3}{8} + \frac{1}{2} = \frac{7}{8}$$

$$b_5 = a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1$$

$$= \frac{1}{2}.\frac{7}{8} + \frac{1}{2}.\frac{3}{4} + \frac{1}{4}.\frac{1}{2} + \frac{1}{12}.1$$

$$= \frac{7}{16} + \frac{3}{8} + \frac{1}{8} + \frac{1}{12} = \frac{21+18+6+4}{48} = \frac{49}{48}$$

Now, the interative formula is $\dfrac{b_n}{b_{n+1}}$, $\qquad n = 1, 2, 3, \ldots$

That is $\dfrac{b_1}{b_2}, \dfrac{b_2}{b_3}, \dfrac{b_3}{b_4}, \dfrac{b_4}{b_5}, \ldots$ approach to a real root of $xe^x = 2$.

$$\frac{b_1}{b_2} = \frac{1}{y_2} = 2 \qquad \frac{b_2}{b_3} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3} = 0.66667$$

$$\frac{b_3}{b_4} = \frac{\frac{3}{4}}{\frac{7}{8}} = \frac{6}{7} = 0.857143 \qquad \frac{b_4}{b_5} = \frac{\frac{7}{8}}{\frac{49}{48}} = \frac{42}{49} = 0.857143$$

$\therefore$ the root correct to six decimal places is $x = \mathbf{0.857143}$ ■

Example 3

Find the smallest root of $\cos x - xe^x = 0$ correct to 3 decimal places by Ramanujan's method.

Solution

Given equation is $\qquad \cos x - xe^x = 0$

$$\Rightarrow \qquad 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-x\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\frac{x^7}{7!}+\ldots\right)=0$$

$$\Rightarrow \qquad 1-x-\frac{3}{2}x^2-\frac{x^3}{2}+\left(\frac{1}{24}-\frac{1}{6}\right)x^4-\frac{x^5}{24}-\left(\frac{1}{720}+\frac{1}{120}\right)x^6-\frac{x^7}{720}+\left(\frac{1}{8!}-\frac{1}{7!}\right)x^8+\ldots=0$$

$$\Rightarrow \qquad 1-\left(x+\frac{3}{2}x^2+\frac{x^3}{2}+\frac{1}{8}x^4+\frac{x^5}{24}+\frac{7}{720}x^6+\frac{1}{720}x^7+\frac{7x^8}{8!}+\ldots\right)=0$$

Here

$$f(x)=1-\left(x+\frac{3}{2}x^2+\frac{x^3}{2}+\frac{1}{8}x^4+\frac{x^5}{24}+\frac{7}{720}x^6+\frac{x^7}{720}+\frac{7x^8}{8!}+\ldots\right)$$

By Ramanujan's method,

$$\left[1-\left(x+\frac{3}{2}x^2+\frac{x^3}{2}+\frac{x^4}{8}+\frac{x^5}{24}+\frac{7}{720}x^6+\frac{x^7}{720}+\frac{7x^8}{8!}+\ldots\right)\right]^{-1}$$

$$=b_1+b_2x+b_3x^2+b_4x^3+b_5x^4+b_6x^5+b_7x^6+b_8x^7+b_9x^8+\ldots$$

Here $\qquad a_1=1,\ a_2=\frac{3}{2},\ a_3=\frac{1}{2},\ a_4=\frac{1}{8},\ a_5=\frac{1}{24},\ a_6=\frac{7}{720},\ a_7=\frac{1}{720},\ a_8=\frac{7}{8!},\ldots$

$\therefore \qquad b_1=1$

$$b_2=a_1.b_1=1.1=1$$

$$b_3=a_1b_2+a_2b_1=1.1+\frac{3}{2}.1=1+\frac{3}{2}=\frac{5}{2}$$

$$b_4=a_1b_3+a_2b_2+a_3b_1$$

$$=1.\frac{5}{2}+\frac{3}{2}.1+\frac{1}{2}.1=\frac{5}{2}+\frac{3}{2}+\frac{1}{2}=\frac{9}{2}$$

$$b_5=a_1b_4+a_2.b_3+a_3b_2+a_4b_1$$

$$=1\times\frac{9}{2}+\frac{3}{2}\times\frac{5}{2}+\frac{1}{2}\times1+\frac{1}{8}\times1$$

$$=\frac{9}{2}+\frac{15}{4}+\frac{1}{2}+\frac{1}{8}=\frac{36+30+4+1}{8}=\frac{71}{8}$$

$$b_6=a_1b_5+a_2b_4+a_3b_3+a_4b_2+a_5b_1$$

$$=1\times\frac{71}{8}+\frac{3}{2}\times\frac{9}{2}+\frac{1}{2}\times\frac{5}{2}+\frac{1}{8}\times1+\frac{1}{24}\times1$$

$$=\frac{71}{8}+\frac{27}{4}+\frac{5}{4}+\frac{1}{8}+\frac{1}{24}=\frac{72}{8}+\frac{32}{4}+\frac{1}{24}=17+\frac{1}{24}=\frac{409}{24}$$

$$b_7=a_1b_6+a_2b_5+a_3b_4+a_4b_3+a_5b_2+a_6b_1$$

$$=1\times\frac{409}{24}+\frac{3}{2}\times\frac{71}{8}+\frac{1}{2}\times\frac{9}{2}+\frac{1}{8}\times\frac{5}{2}+\frac{1}{24}\times1+\frac{7}{720}\times1$$

$$= \frac{410}{24} + \frac{213}{16} + \frac{9}{4} + \frac{5}{16} + \frac{7}{720}$$
$$= 17.0833 + 13.3125 + 2.25 + 0.3125 + 0.009722$$
$$= 32.968056$$

$$b_8 = a_1 b_7 + a_2 b_6 + a_3 b_5 + a_4 b_4 + a_5 b_3 + a_6 b_2 + a_7 b_1$$
$$= 1 \times 32.968056 + \frac{3}{2} \cdot \frac{409}{24} + \frac{1}{2} \cdot \frac{71}{8} + \frac{1}{8} \cdot \frac{9}{2} + \frac{1}{24} \cdot \frac{5}{2} + \frac{7.1}{720} + \frac{1}{720} .1$$
$$= 32.968056 + 25.5625 + 4.4375 + 0.5625 + 0.1041667 + 0.009722 + 0.0013889$$
$$= 63.645833$$

$$b_9 = a_1 b_8 + a_2 b_7 + a_3 b_6 + a_4 b_5 + a_5 b_4 + a_6 b_3 + a_7 b_2 + a_8 b_1$$
$$= 1 \times 63.645833 + \frac{3}{2} \times 32.968056 + \frac{1}{2} \times \frac{409}{24} + \frac{1}{8} \times \frac{71}{8} + \frac{1}{24} \times \frac{9}{2} + \frac{7}{720} \times \frac{5}{2}$$
$$+ \frac{1}{720} \times 1 + \frac{7}{8!} \times 1$$
$$= 63.645833 + 49.452084 + 8.520833 + 1.109375 + 0.1875 + 0.0243056$$
$$+ 0.0013889 + 0.00017361 = 122.941490$$

Now the iterative formula is $\frac{b_n}{b_{n+1}}$, $n = 1, 2, 3, 4, \ldots$

That is $\frac{b_1}{b_2}, \frac{b_2}{b_3}, \frac{b_3}{b_4}, \frac{b_4}{b_5}, \ldots$ approach a real root of (1)

$$\frac{b_1}{b_2} = 1 \qquad \frac{b_2}{b_3} = \frac{1}{\frac{5}{2}} = \frac{2}{5} = 0.4$$

$$\frac{b_3}{b_4} = \frac{\frac{5}{2}}{\frac{9}{2}} = \frac{5}{9} = 0.55555 \qquad \frac{b_4}{b_5} = \frac{\frac{9}{2}}{\frac{71}{8}} = \frac{36}{71} = 0.507042$$

$$\frac{b_5}{b_6} = \frac{\frac{71}{8}}{\frac{409}{24}} = \frac{3 \times 71}{409} = 0.52078 \qquad \frac{b_6}{b_7} = \frac{\frac{409}{24}}{32.968056} = 0.5169$$

$$\frac{b_7}{b_8} = \frac{32.968056}{63.645833} = 0.51799 \qquad \frac{b_8}{b_9} = \frac{63.645833}{122.941490} = 0.51769$$

$\therefore$ the root correct to three decimal places is $x = \mathbf{0.517}$ ■

MULLER'S METHOD

In the iteration methods we have considered so far, the function $f(x)$ is approximated, by a straight line in the neighbourhood of the root. But in Muller's method, the function $f(x)$ is approximated by a parabola or a quadratic polynomial in the neighbourhood of the root. So, the roots of this quadratic are taken as the approximate roots of $f(x)=0$.

Let $f(x)=ax^2+bx+c, a\neq 0$ in the neighbourhood of the root α of $f(x)=0$. Let x_0, x_1, x_2 be three approximate values of the root and let y_0, y_1, y_2 be the values of $y=f(x)$.

$\therefore y=f(x)$ passes through the points $(x_0, y_0), (x_1, y_1), (x_2, y_2)$

Then by Lagrange's formula, the second degree polynomial is given by

$$y=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2 \qquad \ldots(1)$$

For simplicity, put

$$\lambda=\frac{x-x_2}{x_2-x_1}, \qquad \lambda_2=\frac{x_2-x_1}{x_1-x_0}, \qquad \delta_2=\frac{x_2-x_0}{x_1-x_0}$$

$$\therefore \qquad 1+\lambda_2=1+\frac{x_2-x_1}{x_1-x_0}=\frac{x_1-x_0+x_2-x_1}{x_1-x_0}, \quad =\frac{x_2-x_0}{x_1-x_0}=\delta_2$$

the equation (1) becomes

$$y=\frac{y_0\lambda_2^2-y_1\lambda_2\delta_2+y_2\lambda_2}{\delta_2}\lambda^2+\frac{y_0\lambda_2^2-y_1\delta_2^2+y_2(\lambda_2+\delta_2)}{\delta_2}\lambda+y_2 \qquad (2)$$

Since $$\lambda=\frac{x-x_2}{x_2-x_1} \quad \Rightarrow \lambda(x_2-x_1)=x-x_2 \quad \Rightarrow x=x_2+\lambda(x_2-x_1) \qquad (3)$$

Since λ is unknown, it should be determined so as to get the best value of the root.

To find λ, put $y=0$ in (2) $\qquad [\because f(x)=0]$

$\therefore$ we get $$\left(y_0\lambda_2^2-y_1\lambda_2\delta_2+y_2\lambda_2\right)\lambda^2+\left(y_0\lambda_2^2-y_1\delta_2^2+y_2\left(\lambda_2+\delta_2\right)\right)\lambda+y_2\delta_2=0$$

Put $$c_2=y_0\lambda_2^2-y_1\lambda_2\delta_2+y_2\lambda_2=\lambda_2(y_0\lambda_2-y_1\delta_2+y_2$$

and $$g_2=y_0\lambda_2^2-y_1\delta_2^2+y_2(\lambda_2+\delta_2)$$

$$\therefore \quad c_2\lambda^2+g_2\lambda+y_2\delta_2=0 \qquad (4)$$

$$\therefore \quad \lambda=-\frac{g_2\pm\sqrt{g_2^2-4c_2\delta_2y_2}}{2c_2}$$

$$=-\frac{\left[g_2\mp\sqrt{g_2^2-4c_2\delta_2y_2}\right]\left[g_2\pm\sqrt{g_2^2-4c_2\delta_2y_2}\right]}{2c_2\left[g_2\pm\sqrt{g_2^2-4c_2\delta_2y_2}\right]}$$

$$= -\frac{\left[g_2^2 - \left(g_2^2 - 4c_2\delta_2 y_2\right)\right]}{2c_2\left[g_2 \pm \sqrt{g_2^2 - 4c_2\delta_2 y_2}\right]}$$

$$= -\frac{4c_2\delta_2 y_2}{2c_2\left[g_2 \pm \sqrt{g_2^2 - 4c_2\delta_2 y_2}\right]}$$

$$\lambda = -\frac{2\delta_2 y_2}{g_2 \pm \sqrt{g_2^2 - 4c_2\delta_2 y_2}} = \lambda_3, \text{say} \tag{5}$$

The sign of the denominator in (5) is chosen such that λ_3 has the smallest absolute value.

Thus (3) $\Rightarrow x = x_2 + (x_2 - x_1)\lambda_3$ is better approximation.

We take this as x_3.

$$\therefore \qquad x_3 = x_2 + (x_2 - x_1)\lambda_3$$

The successive approximations are obtained this way.
Therefore the iteration formula is

$$x_{n+1} = x_n + (x_n - x_{n-1})\lambda_{n+1},$$

which is called Muller's formula,

where $$\lambda_{n+1} = -\frac{2\delta_n y_n}{\left[g_n \pm \sqrt{g_n^2 - 4c_n\delta_n y_n}\right]}$$

where $$c_n = \lambda_n(\lambda_n y_{n-2} - \delta_n y_{n-1} + y_n)$$

and $$g_n = y_{n-2}\lambda_n^2 - y_{n-1}\delta_n^2 + (\lambda_n + \delta_n)y_n$$

$$\lambda_n = \frac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}}$$

and $$\delta_n = \frac{x_n - x_{n-2}}{x_{n-1} - x_{n-2}} = 1 + \lambda_n, \quad n = 2, 3, 4, \ldots$$

Working Procedure: To solve $f(x) = 0$

1. Find a small interval $[a, b]$ in which a root lies, by intermediate value theorem.

   Take $a = x_0$, $b = x_2$, $x_1 = \dfrac{x_0 + x_2}{2}$

2. Let $y = f(x)$ and $y_0 = f(x_0)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$

3. Iteration formula is

$$x_{n+1} = x_n + (x_n - x_{n-1})\lambda_{n+1}$$

where $$\lambda_{n+1} = -\frac{2\delta_n y_n}{g_n \pm \sqrt{g_n^2 - 4c_n\delta_n y_n}}$$

If $g_n > 0$, then + sign is taken in the denominator and if $g_n < 0$, then − sign is taken in the denominator.

$$c_n = \left(y_{n-2}\lambda_n^2 - y_{n-1}\delta_n + y_n\right)\lambda_n$$

$$\lambda_n = \frac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}}.$$

$$g_n = y_{n-2}\lambda_n^2 - y_{n-1}\delta_n^2 + (\lambda_n + \delta_n)y_n, \quad \delta_n = 1 + \lambda\delta_n, \quad n = 2,3,4,\ldots$$

Note:

(1) The Muller's method starts with three initial approximations x_0, x_1, x_2. The method converges for all initial values.

(2) If no better approximations are known, then we can choose $x_0 = -1$, $x_1 = 0$, $x_2 = 1$.

(3) This method also requires only one functional value in each iteration like earlier methods.

(4) The rate of convergence of Muller's method is 1.84.

WORKED EXAMPLES

Example 1

Find a real root of Wallis's equation $x^3 - 2x - 5 = 0$ by Muller's method.

Solution

Let $f(x) = x^3 - 2x - 5$

we have $f(2) = 2^3 - 2\times2 - 5 = 8 - 4 - 5 = -1 < 0$

and $f(3) = 3^3 - 2\times3 - 5 = 27 - 6 - 5 = 16 > 0$

Now $f(2.5) = (2.5)^3 - 2\times2.5 - 5 = 15.625 - 5 - 5 = 5.625 > 0$

So, the root lies between 2 and 2.5

Take $x_0 = 2$, $x_2 = 2.5$

$$\therefore \quad x_1 = \frac{x_0 + x_2}{2} = \frac{2 + 2.5}{2} = 2.25$$

$$y_0 = f(x_0) = f(2) = -1$$

$$y_1 = f(x_1) = f(2.25)$$

$$= (2.25)^3 - 2(2.25) - 5 = 11.3906 - 4.5 - 5 = 1.8906$$

$$y_2 = f(x_2) = f(2.5) = 5.625$$

Mullers iteration formula is

$$\therefore \quad x_{n+1} = x_n + (x_n - x_{n-1})\lambda_{n+1}$$

where $$\lambda_{n+1} = -\frac{2\delta_n y_n}{g_n \pm \sqrt{g_n^2 - 4c_n\delta_n y_n}}$$

$$c_n = (y_{n-2}\lambda_n - y_{n-1}\delta_n + y_n)\lambda_n$$

$$\lambda_n = \frac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}}, \quad \delta_n = 1 + \lambda_n$$

$$g_n = y_{n-2}\lambda_n^2 - y_{n-1}\delta_n^2 + (\lambda_n + \delta_n)y_n, \quad n = 2, 3, 4, \ldots$$

I iteration: Put $n = 2$

$\therefore$ $$x_3 = x_2 + (x_2 - x_1)\lambda_3$$

where $$\lambda_3 = -\frac{2\delta_2 y_2}{g_2 \pm \sqrt{g_2^2 - 4c_2\delta_2 y_2}}$$

$$c_2 = (y_0\lambda_2 - y_1\delta_2 + y_2)\lambda_2$$

$$\lambda_2 = \frac{x_2 - x_1}{x_1 - x_0}, \quad \delta_2 = 1 + \lambda_2$$

$$g_2 = y_0\lambda_2^2 - y_1\delta_2^2 + (\lambda_2 + \delta_2)y_2$$

Now $$\lambda_2 = \frac{2.5 - 2.25}{2.25 - 2} = \frac{0.25}{0.25} = 1$$

$$\delta_2 = 1 + \lambda_2 = 1 + 1 = 2$$

$$c_2 = (-1 \times 1 - 1.8906 \times 2 + 5.625)1$$

$$= -1 - 3.7812 + 5.625 = 0.8438$$

$$g_2 = (-1) \times 1^2 - 1.8906 \times 2^2 + (1 + 2)(5.625)$$

$$= -1 - 7.5624 + 16.875 = 8.3126$$

$\therefore$ $$\lambda_3 = \frac{-2 \times 2 \times 5.625}{8.3126 \pm \sqrt{(8.3126)^2 - 4(0.8438)(2)(5.625)}}$$

$$= -\frac{22.5}{8.3126 \pm \sqrt{69.0993 - 37.971}}$$

$$= -\frac{22.5}{8.3126 \pm 5.5793}$$

Since $g_2 > 0$, we take positive sign in the denominator.

$\therefore$ $$\lambda_3 = -\frac{22.5}{8.3126 + 5.5793} = -1.6196$$

$$x_3 = 2.5 + (2.5 - 2.25)(-1.6196) = 2.5 - 0.4049 = 2.0951$$

$\therefore$ $$y_3 = (2.0951)^3 - 2 \times 2.0951 - 5 = 9.196324 - 4.1902 - 5 = 0.006124$$

II iteration: Put $n = 3$,

$$x_4 = x_3 + (x_3 - x_2)\lambda_4$$

where $$\lambda_4 = -\frac{2\delta_3 y_3}{g_3 \pm \sqrt{g_3^2 - 4c_3\delta_3 y_3}}$$

$$c_3 = (y_1\lambda_3 - y_2\delta_3 + y_3)\lambda_3$$

$$\lambda_3 = \frac{x_3 - x_2}{x_2 - x_1}, \quad \delta_3 = 1 + \lambda_3$$

$$g_3 = y_1\lambda_3^2 - y_2\delta_3^2 + (\lambda_3 + \delta_3)y_3$$

Now $$\lambda_3 = \frac{2.0951 - 2.5}{2.5 - 2.25} = -\frac{0.4049}{0.25} = -1.6196$$

$$\delta_3 = 1 + \lambda_3 = 1 - 1.6196 = -0.6196$$

$$c_3 = [(1.8906)(-1.6196) - 5.625(-0.6196) + 0.006124](-1.6196)$$
$$= [-3.0620 + 3.48525 + 0.006124][-1.6198]$$
$$= 0.429374(-1.6196) = -0.6954$$

$$g_3 = 1.8906(-1.6196)^2 - 5.625(-0.6196)^2 + (-1.6196 - 0.6196)(0.006124)$$
$$= 4.95924 - 2.15946 - 0.01371286 = 2.7861$$

$\therefore$ $$\lambda_4 = \frac{-2(-0.6196)(0.006124)}{(2.7861) \pm \sqrt{(2.7861)^2 - 4(-0.6954)(-0.6196)(0.006124)}}$$
$$= \frac{0.007589}{2.7861 \pm \sqrt{7.76235 - 0.010554}}$$
$$= \frac{0.007589}{2.7861 \pm 2.7842}$$

Since $g_3 > 0$, we take the positive sign in the deneminator

$$\lambda_4 = \frac{0.007589}{2.7861 + 2.7842} = \frac{0.007589}{5.5703} = 0.001362$$

$\therefore$ $$x_4 = 2.0951 + (2.0951 - 2.5)(0.001362)$$
$$= 2.0951 - 0.00055147 = 2.0945$$

$\therefore$ $$y_4 = (2.0945)^3 - 2 \times 2.0945 - 5 = 9.188425 - 4.1890 - 5 = 0.0005746$$

III iteration: Put $n = 4$,

$$x_5 = x_4 + (x_4 - x_3)\lambda_5$$

$$\lambda_5 = -\frac{2\delta_4 y_4}{g_4 \pm \sqrt{g_4^2 - 4c_4 + \delta_4 y_4}}$$

where $$c_4 = (y_2\lambda_4 - y_3\delta_4 + y_4)\lambda_4$$

$$\lambda_4 = \frac{x_4 - x_3}{x_3 - x_2}, \quad \delta_4 = 1 + \lambda_4$$

$$g_4 = y_2\lambda_4^2 - y_3\delta_4^2 + (\lambda_4 + \delta_4)y_4$$

and $\lambda_4 = \dfrac{2.0945 - 2.0951}{2.0951 - 2.5} = 0.001481$

$$c_4 = [5.625 \times 0.001481 - 0.006124 \times 1.001481 - 0.0005746](0.001481)$$
$$= 0.0000024$$

$$g_4 = 5.625(0.001481)^2 - 0.006124(1.001481)^2 + (0.001481 + 1.001481)(-0.0005746)$$
$$= -0.006706$$

Since $g_4 < 0$, we take negative sign in the denominator.

$$\therefore \quad \lambda_5 = \frac{-2(1.001481)(-0.0005746)}{-0.006706 - \sqrt{(0.006706)^2 - 4(0.0000024)(1.001481)(-0.0005746)}}$$
$$= \frac{0.001151}{-0.006706 - 0.006706}$$
$$= -\frac{0.001151}{0.013412} = -0.0858$$

$$\therefore \quad x_5 = 2.0945 + (2.0945 - 2.0951)(-0.0858)$$
$$= 2.09455$$

Since $x_4 = x_5$ upto four decimal places, the root is $x = \mathbf{2.0945}$ ■

Example 2

Find the root of $\cos x - xe^x = 0$, correct to four decimal places, using Muller's iteration method.

Solution

Let $f(x) = \cos x - xe^x$

We shall use the initial approximations $x_0 = -1, x_1 = 0, x_2 = 1$

$$\therefore \quad y_0 = f(x_0) = \cos(-1) - (-1)e^{-1}$$
$$= \cos 1 + e^{-1} = 0.5403 + 0.3679 = 0.9082$$
$$y_1 = f(x_1) = \cos 0 - 0 = 1$$
$$y_2 = f(x_2) = \cos 1 - 1e^1 = 0.5403 - 2.7183 = -2.178$$

Muller's iteration formula is

$$x_{n+1} = x_n + (x_n - x_{n-1})\lambda_{n+1},$$

where
$$\lambda_{n+1} = -\frac{2\delta_n y_n}{g_n \pm \sqrt{g_n^2 - 4c_n\delta_n y_n}}$$
$$c_n = (y_{n-2}\lambda_n - y_{n-1}\delta_n + y_n)\lambda_n$$
$$\lambda_n = \frac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}}, \qquad \delta_n = 1 + \lambda_n$$

and
$$g_n = y_{n-2}\lambda_n^2 - y_{n-1}\delta_n^2 + (\lambda_n + \delta_n)y_n, \qquad n = 2, 3, \ldots$$

I iteration: Put $n = 2$

$$\therefore \quad x_3 = x_2 + (x_2 - x_1)\lambda_3$$

$$\lambda_3 = \frac{-2\delta_2 y_2}{g_2 \pm \sqrt{g_2^2 - 4c_2 d_2 y_2}}$$

$$\lambda_2 = \frac{x_2 - x_1}{x_1 - x_0} = \frac{1-0}{0-(-1)} = 1, \qquad \delta_2 = 1 + 1 = 2$$

$$c_2 = (y_0\lambda_2 - y_1\delta_2 + y_2)\lambda_2$$

$$c_2 = [0.9082 - 1\times 2 + (-2.178)].1 = 0.9082 - 2 - 2.178 = -3.2698$$

and

$$g_2 = y_0\lambda_2^2 - y_1\delta_2^2 + (\lambda_2 + \delta_2)y_2$$

$$g_2 = 0.9082(1)^2 - 1\times 2^2 + (1+2)(-2.178) = 0.9082 - 4 - 6.534 = -9.6258$$

$$\therefore \quad \lambda_3 = \frac{-2\times 2(-2.178)}{-9.6258 \pm \sqrt{(-9.6258)^2 - 4(-3.2698)2(-2.178)}}$$

$$= \frac{8.712}{-9.6258 \pm \sqrt{92.6560 - 56.97299}}$$

$$= \frac{8.712}{-9.6258 \pm \sqrt{35.68301}} = \frac{8.712}{-9.6258 \pm 5.9735}$$

Since g_2 is negative, we take negative sign in denominator.

$$\therefore \quad \lambda_3 = \frac{8.712}{-9.6258 - 5.9735} = \frac{8.712}{-15.5993} = -0.5585$$

$$\therefore \quad x_3 = 1 + (1-0)(-0.5585) = 1 - 0.5585 = 0.4415$$

and

$$y_3 = f(x_3) = \cos(0.4415) - (0.4415)e^{0.4415}$$

$$= 0.9041 - 0.6865 = 0.2176$$

II Iteration: Put $n = 3$

$$\therefore \quad x_4 = x_3 + (x_3 - x_2)\lambda_4$$

where

$$\lambda_4 = \frac{-2\delta_3 y_3}{g_3 \pm \sqrt{g_3^2 - 4c_3\delta_3 y_3}}$$

$$\lambda_3 = \frac{x_3 - x_2}{x_2 - x_1}, \qquad \delta_3 = 1 + \lambda_3$$

$$c_3 = (y_1\lambda_3 - y_2\delta_3 + y_3)\lambda_3, \qquad g_3 = y_1\lambda_3^2 - y_2\delta_3^2 + (\lambda_3 + \delta_3)y_3$$

Now

$$\lambda_3 = \frac{0.4415 - 1}{1 - 0} = -0.5585$$

$$\delta_3 = 1 - 0.5585 = 0.4415$$

$$c_3 = [1(-0.5585) - (-2.178)(0.4415) + (0.2176)](-0.5585)$$

$$= (-0.5585 + 0.961587 + 0.2176)(-0.5585)$$

$$= (0.620957)(-0.5585) = -0.3467$$

$$g_3 = 1(-0.5585)^2 - (-2.178)(0.4415)^2 + (-0.5585 + 0.4415)(0.2176)$$
$$= (0.311922 + 0.42454) - (0.117)(0.2176)$$
$$= 0.73422 - 0.0254592 = 0.711$$

Since $g_3 > 0$, we take positive sign in the denominator of λ_4.

$$\lambda_4 = \frac{-2(0.4415)(0.2176)}{0.711 + \sqrt{(0.711)^2 - 4(-0.3467)(0.4415)(0.2176)}}$$
$$= -\frac{0.192141}{0.711 + \sqrt{0.505521 + 0.13323043}}$$
$$= -\frac{0.192141}{0.711 + \sqrt{0.63875143}} = -\frac{0.192141}{1.5102} = -0.1272$$

$\therefore$ $$x_4 = 0.4415 + (0.4415 - 1)(-0.1272)$$

$\Rightarrow$ $$x_4 = 0.4415 + (-0.5585)(-0.1272) = 0.4415 + 0.0710412 = 0.51254$$

$\therefore$ $$y_4 = \cos(0.51254) - (0.51254)e^{0.51254}$$
$$= 0.87150 - 1.66953 \times 0.51254 = 0.015799$$

III Iteration: Put $n = 4$

$\therefore$ $$x_5 = x_4 + (x_4 - x_3)\lambda_5,$$

where $$\lambda_5 = -\frac{2\delta_4 y_4}{g_4 \pm \sqrt{g_4^2 - 4c_4\delta_4 y_4}}$$

$$\lambda_4 = \frac{x_4 - x_3}{x_3 - x_2}, \qquad \delta_4 = 1 + \lambda_4$$

$$c_4 = (y_2\lambda_4 - y_3\delta_4 + y_4)\lambda_4, \qquad g_4 = y_2\lambda_4^2 - y_3\delta_4^2 + (\lambda_4 + \delta_4)y_4$$

Now $$\lambda_4 = \frac{0.51254 - 0.4415}{0.4415 - 1} = \frac{0.07104}{-0.5585} = -0.1272$$

$$\delta_4 = 1 + (-0.1272) = 0.8728$$

$$c_4 = [(-2.178)(-0.1272) - (0.2176)(0.8728) + 0.015799][-0.1272]$$
$$= [0.2770416 - 0.18992128 + 0.015799][-0.1272]$$
$$= [0.10291032][-0.1272] = -0.01309$$

$$g_4 = (-2.178)(-0.1272)^2 - (0.2176)(0.8728)^2 + (0.7456)(0.015799)$$
$$= -0.035239691 - 0.165763214 + 0.0117791 = -0.18922$$

Since $g_4 < 0$, we take negative sign in the denominator of λ_5.

$$\therefore \quad \lambda_5 = \frac{-2(0.8728)(0.015799)}{-0.18922 - \sqrt{(-0.18922)^2 - 4(-0.01309)(0.8728)(0.0157)}}$$

$$= \frac{-0.027579}{-0.18922 - \sqrt{0.035804 + 0.000722}}$$

$$= \frac{-0.027579}{-0.18922 - 0.19112} = \frac{0.027579}{0.38034} = 0.07251$$

$$\therefore \quad x_5 = 0.51254 + (0.51254 - 0.4415)(0.00251)$$

$$= 0.51254 + 0.005151 = 0.51769$$

$$\therefore \quad y_5 = f(x_5) = \cos(0.5769) - 0.51769e^{0.51769}$$

$$= 0.86896 - 0.86876 = 0.0002$$

IV Iteration: Put $n = 5$

$$\therefore \quad x_6 = x_5 + (x_5 - x_4)\lambda_6$$

where

$$\lambda_6 = \frac{-2\delta_5 y_5}{g_5 \pm \sqrt{g_5^2 - 4c_5\delta_5 y_5}}$$

$$\lambda_5 = \frac{x_5 - x_4}{x_4 - x_3}, \qquad \delta_5 = 1 + \delta_5$$

$$c_5 = (y_3\lambda_5 - y_4\delta_5 + y_5)\lambda_5, \qquad g_5 = y_3\lambda_5^2 - y_4\delta_5^2 + (\lambda_5 + \delta_5)y_5$$

Now

$$\therefore \quad \lambda_5 = \frac{0.51769 - 0.51254}{0.51254 - 0.4415} = \frac{0.00515}{0.07104} = 0.07249$$

$$\delta_5 = 1 + \lambda_5 = 1 + 0.07249 = 1.07249$$

$$c_5 = [0.2176(0.07249) - 0.015799(1.07249) + 0.0002](0.07249)$$

$$= [0.0157738 - 0.016944269 + 0.0002](0.07249)$$

$$= (-0.000970469)(0.07249) = -0.00007035$$

$$g_5 = 0.2176(0.07249)^2 - 0.015799(1.07249)^2 + (0.7249 + 1.07249)(0.0002)$$

$$= 0.00143445 - 0.018172559 + 0.000228996 = -0.0168$$

Since $g_5 < 0$, we take negative sign in the denominator of λ_6.

$$\therefore \qquad \lambda_6 = \frac{-2(1.07249)(0.0002)}{-0.0168-\sqrt{(-0.0168)^2-4(-0.0007035)(1.0249)(0.0002)}}$$

$$= \frac{-0.000429}{-0.0168-\sqrt{0.00028224+0.000000603597}}$$

$$= \frac{-0.000429}{-0.03360} = 0.012768$$

$$\therefore \qquad x_6 = 0.51769+(0.51769-0.51254)(0.012768)$$

$$= 0.51769+0.0000\dot{6}575 = 0.517756$$

Since x_5 and x_6 are equal upto four places of decimal, the root is $\boldsymbol{x = 0.5177}$

Note that $\qquad y_6 = f(x_6)$

$$= \cos(0.5177) - 0.5177e^{0.5177}$$

$$= 0.868959 - 0.868785 = 0.000174$$

which is very close to zero.
So, the root 0.5177 is a very good approximation.

Asad Ali

CHEBYSHEV'S METHOD

In Chebyshev's method of finding a root of $f(x) = 0$ the function $f(x)$ is approximated by a quadratic polynominal in the neighbourhood of the root as in Muller's method.

This is same as approximating $f(x)$ by Taylor's theorem in the neighbourhood of the root upto second degree terms.

If x_0 is the initial approximation of a root of $f(x) = 0$, which is chosen close to the root then $x = x_0 + h$ is the actual root, where h is small.

$$\therefore \qquad f(x_0 + h) = 0$$

Then, by Taylor's theorem, we get

$$f(x_0) + \frac{h}{1!}f'(x_0) + \frac{h^2}{2!}f''(x_0) + \ldots = 0 \tag{1}$$

Omitting h^2 and higher powers of h, we get

$$f(x_0) + hf'(x_0) = 0$$

$$\Rightarrow \qquad h = -\frac{f(x_0)}{f'(x_0)} \tag{2}$$

$\therefore$ first approximation is

$$x_1 = x_0 + h$$

$$= x_0 - \frac{f(x_0)}{f'(x_0)}$$

To find a better approximation of x, we omit h^3 and higher powers of h.
Then, from (1), we get

$$f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)=0$$

$$\Rightarrow \quad f(x_0)+hf'(x_0)+\frac{1}{2}\left[-\frac{f(x_0)}{f'(x_0)}\right]^2 f''(x_0)=0 \qquad \text{[Using (2)]}$$

$$\Rightarrow \quad f(x_0)+hf'(x_0)+\frac{1}{2}\left[\frac{f(x_0)}{f'(x_0)}\right]^2 f''(x_0)=0$$

$$hf'(x_0)=-f(x_0)-\frac{1}{2}\left[\frac{f(x_0)}{f'(x_0)}\right]^2 f''(x_0)$$

$$h=-\frac{f(x_0)}{f'(x_0)}-\frac{1}{2}\frac{[f(x_0)]^2}{[f'(x_0)]^3}f''(x_0)$$

∴ first approximation is

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}-\frac{1}{2}\frac{[f(x_0)]^2}{[f'(x_0)]^3}f''(x_0)$$

Same way using x_1 in the place of x_0, we get the second approximation

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}-\frac{1}{2}\frac{[f(x_1)]^2}{[f'(x_1)]^3}f''(x_1) \text{ and so on.}$$

So, we get the iteration formula

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}-\frac{1}{2}\frac{[f(x_n)]^2}{[f'(x_n)]^3}f''(x_n), \quad n=0,1,2,\ldots$$

which is the iteration formula of Chebyshev.

Note:

(1) If the first and second derivatives are easily obtained, then this method is easy.

(2) The method requires three evaluations f, f', f'' at each iteration where as some of the earlier methods required only one evaluation.

(3) The rate of convergence of Chebyshev method is 3.

WORKED EXAMPLES

Example 1

Find the root of $x^4 - x - 10 = 0$ near $x = 2$, correct to four places of decimal, by Chebyshev's method.

Solution

The given equation is $x^4 - x - 10 = 0$ and $x_0 = 2$.

Let $f(x) = x^4 - x - 10, \quad f'(x) = 4x^3 - 1, \quad f''(x) = 12x^2$

By Chebyshev's method, the iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\frac{[f(x_n)]^2}{[f'(x_n)]^3} f''(x_n), \qquad n = 0,1,2,3,\ldots$$

First Iteration: Put $n=0$

$$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2}\frac{[f(x_0)]^2}{[f'(x_0)]^3} f''(x_0).$$

$$f(x_0) = f(2) = 2^4 - 2 - 10 = 16 - 12 = 4$$

$$f'(x_0) = f'(2) = 4.2^3 - 1 = 32 - 1 = 31$$

$$f''(x_0) = f''(2) = 12.2^2 = 48$$

$$\therefore \quad x_1 = 2 - \frac{4}{31} - \frac{1}{2}.\frac{4^2}{(31)^3}.48 = 2 - 0.1290 - 0.01289 = 1.8581$$

Second Iteration: Put $n=1$

$$\therefore \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} - \frac{1}{2}\frac{[f(x_1)]^2}{[f'(x_1)]^3} \times f''(x_1)$$

$$f(x_1) = f(1.8581) = (1.8581)^4 - 1.8581 - 10$$

$$= 11.9200 - 1.8581 - 10 = 0.0619$$

$$f'(x) = f'(1.8581) = 4(1.8581)^3 - 1 = 25.6606 - 1 = 24.6606$$

$$f''(x_1) = f''(1.8584) = 12(1.8581)^2 = 41.4304$$

$$\therefore \quad x_2 = 1.8581 - \frac{0.0619}{24.6606} - \frac{1}{2}\frac{(0.0619)^2}{(24.6606)^3}(41.4304)$$

$$= 1.8581 - 0.002510 - 0.00000529 = 1.8556$$

Third Iteration: Put $n=2$

$$\therefore \quad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} - \frac{1}{2}\frac{[f(x_2)]^2}{[f'(x_2)]^3} f''(x_2)$$

$$f(x_2) = f(1.8556) = (1.8556)^4 - 1.8552 - 10$$
$$= 11.8559799 - 11.8556 = 0.0003799$$

$$f'(x_2) = f'(1.8556) = 4(1.8556)^3 - 1.8556$$
$$= 25.5571889 - 1.8556 = 24.5572$$

$$f''(x_2) = f''(1.8556) = 12(1.8556)^2 = 41.3190$$

$$\therefore \quad x_3 = 1.8556 - \frac{0.0003799}{24.5572} - \frac{1}{2}\frac{(0.0003799)^2}{(24.5572)^3}(41.319)$$
$$= 1.8556 - 0.0000155 - 2.0134\times10^{-10} = 1.8556$$

Since $x_2 = x_3$ correct to 4 decimal places, the root is $x = \mathbf{1.8556}$ ■

Example 2

Solve $e^{-x} - \sin x = 0$ by Chebyshev's method and find the root to four places of decimals.

Solution

The given equation is

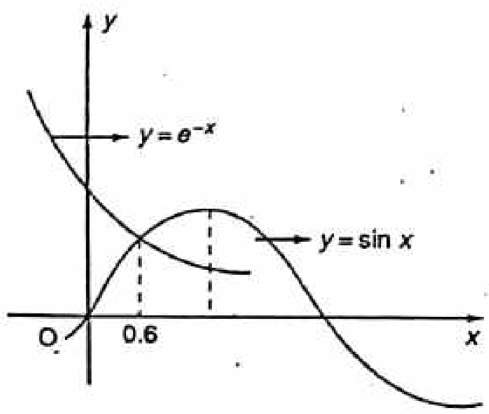
$$e^{-x} - \sin x = 0$$

Let $\quad f(x) = e^{-x} - \sin x.$

$$f(0) = e^{-0} - \sin 0 = 1 > 0$$

Now $\quad f(0.5) = e^{-0.5} - \sin(0.5) = 0.6065 - 0.4794 = 0.1271 > 0$

$$f(0.6) = e^{-0.6} - \sin(0.6) = 0.5488 - 0.5646 = -0.0158 < 0$$



$\therefore$ the root lies between 0.5 and 0.6 and it is near to 0.6
So, we take $x_0 = 0.6$
Chebyshev's iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\frac{[f(x_n)]^2}{[f''(x_n)]^3} f''(x), \quad n = 0,1,2,\ldots$$

Now $\quad f(x) = e^{-x} - \sin x$

$\therefore \quad f'(x) = -e^{-x} - \cos x$

and $\quad f''(x) = e^{-x} + \sin x$

First Iteration: Put $n = 0$

$$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2}\frac{[f(x_0)]^2}{[f'(x_0)]^3} f''(x_0)$$

Now
$$f(x_0) = f(0.6) = -0.0158$$
$$f'(x_0) = f'(0.6) = -e^{-0.6} - \cos(0.6)$$
$$= -0.5488 - 0.8253 = -1.3741$$
$$f''(x_0) = f''(0.6) = e^{-0.6} + \sin(0.6) = 0.5488 + 0.5646 = 1.1134$$

$$\therefore \quad x_1 = 0.6 - \frac{(-0.0158)}{(-1.3741)} - \frac{1}{2}\frac{(-0.0158)^2}{(-1.3741)^3}(1.1134)$$
$$= 0.6 - 0.011498 + 0.00005356 = 0.5886$$

Second Iteration: Put $n = 1$

$$\therefore \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} - \frac{1}{2}\frac{[f(x_1)]^2}{[f'(x_1)]^3} f''(x_1)$$

Now
$$f(x_1) = e^{-0.5886} - \sin(0.5886)$$
$$= 0.555103 - 0.555197 = 0$$

$\therefore$ So, we shall take $\quad x_2 = x_1 - 0 - 0 = x_1 \quad = 0.5886$

Hence the root is $\quad \mathbf{x = 0.5886}$ ■

Example 3

Find the root of $\cos x - x^2 - x = 0$ that lie between 0 and 1 by Chebyshev's method.

Solution

The given equation is $\cos x - x^2 - x = 0$

Let $\quad f(x) = \cos x - x^2 - x$

Given that the root lies between 0 and 1

$$\therefore \quad f(0) = \cos 0 - 0 - 0 = 1 > 0$$
$$f(1) = \cos 1 - 1 - 1$$
$$= 0.54030 - 2 = -1.4597 < 0$$

Hence the root is nearer to 0.

$$f(0.5) = \cos(0.5) - (0.5)^2 - 0.5$$
$$= 0.87758 - 0.25 - 0.5 = 0.12758 > 0$$
$$f(0.6) = 0.82533 - (0.6)^2 - 0.6$$
$$= 0.82533 - 0.36 - 0.6 = -0.13467 < 0$$

$\therefore$ the root lies between 0.5 and 0.6

Take $x_0 = 0.6$

Chebyshev's formula for iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\frac{[f(x_n)]^2}{[f'(x_n)]^3} f''(x_n), \qquad n = 0,1,2,3,\ldots$$

First Iteration: Put $n=0$

$$\therefore \qquad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2}\frac{[f(x_0)]^2}{[f'(x_0)]^3} f''(x_0)$$

$$f(x) = \cos x - x^2 - x$$
$$f'(x) = -\sin x - 2x - 1$$
$$f''(x) = -\cos x - 2$$

and $\qquad x_0 = 0.6$

$$f(x_0) = f(0.6) = -0.13467$$

$$\therefore \quad f'(x_0) = f'(0.6) = -\sin(0.6) - 2(0.6) - 1 = -0.5646 - 1.2 - 1 = -2.7646$$
$$f''(x_0) = f''(0.6) = -\cos(0.6) - 2 = -0.82533 - 2 = -2.82533$$

$$\therefore \qquad x_1 = 0.6 - \frac{-0.13467}{-2.7646} - \frac{1}{2}\frac{(-0.13467)^2}{(-2.7646)^3}(-2.82533)$$
$$= 0.6 - 0.0487 - 0.0012125 = 0.550087$$

Second Iteration: Put $n=1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} - \frac{1}{2}\frac{[f(x_1)]^2}{[f'(x_1)]^3} f''(x_1),$$

Now $\qquad f(x_1) = f(0.550087)$

$$= \cos(0.550087) - (0.550087)^2 - 0.550087$$
$$= 0.852479 - 0.3025957 - 0.550087$$
$$= 0.852479 - 0.8526827 = -0.0002037 = 0$$

$$\therefore \qquad x_2 = x_1 - 0 - 0 = x_1 \qquad = 0.550087$$

Hence the root is $\qquad \mathbf{x = 0.550087}$

Exercises 2.2

(1) Using Generalised Newton-Raphson method find the double root of $xe^{1-x}=1$ starting with $x_0=0$.

(2) Using Muller's method find the smallest positive root of $x^3-5x+1=0$ to four decimal places.
[**Hint:** Take $x_0=0$, $x_1=0,5$, $x_2=1$]

(3) Using Muller's method find the root of $x^3-3x-5=0$ which lies between 2 and 3.

[**Hint:** Take $x_0=2, x_2=3, x_1=\dfrac{x_0+x_2}{2}=2.5$]

(4) Using Mullers method find a root of $x^3-x-1=0$.
[**Hint:** Take $x_0=1$, $x_2=2$, $x_1=1.5$]

(5) Using Muller's method find the negative root of $x^3-4x+1=0$ that lies between -3 and -2.
[**Hint:** Take $x_0=-3$, $x_2=-2$, $x_1=-2.5$]

(6) Using Chebyshev's method find the root of $x^3-5x+1=0$.

(7) Using Chebyshev's method find the root of $3x-\cos x-1=0$.

(8) Find on approximate value of $\sqrt[3]{2}$ by Chebyshev's method.
[**Hint:** Let $x=\sqrt[3]{2} \Rightarrow x^3=2 \Rightarrow x^3-2=0$]

Answers 2.2

(1) 0.9998 (exact root is 1)	(2) 0.2016	(3) 2.2597
(4) 1.3247	(5) −2.1149	(6) 0.2016
(7) 0.6071	(8) 1.2599	

2.10 CONVERGENCE OF ITERATION METHODS

Definition 1: An iterative method is said to be of order of convergence p or rate of convergence p if p is the largest positive number for which there exists a constant $c>0$ such that $|\epsilon_{n+1}| \le c|\epsilon_n|^p$, where $\epsilon_n = x_n-\alpha$ and $\epsilon_{n-1}=x_{n-1}-\alpha$, where x_{n-1}, x_n are the $(n-1)^{th}$ and n^{th} approximations of the actual root α of $f(x)=0$.

So, ϵ_n is the error in the n^{th} iteration.

The constant c is called the asymptotic error and it depends on the derivatives of $f(x)$ at α.

When $p=1$, the convergence is said to be linear.

When $p=2$, the convergence is said to be quadratic.

1. Bisection method

In this method,

$$|x_n - \alpha| \le \left(\frac{1}{2}\right)^n (b-a)$$

$$\Rightarrow \quad |\epsilon_n| \le \left(\frac{1}{2}\right)^n (b-a)$$

$$\Rightarrow \quad |\epsilon_n| \le \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1} (b-a)$$

$$\Rightarrow \quad |\epsilon_n| \le \frac{1}{2}|\epsilon_{n-1}|$$

So, the convergence is linear.

2. Method of false position

In this method, we have

$$x_{n+1} = x_n + \frac{(x_n - x_{n-1})f(x_n)}{f(x_{n-1}) - f(x_n)}$$

$$= x_n - \frac{(x_n - x_{n-1})f(x_n)}{f(x_n) - f(x_{n-1})}$$

Put
$$\epsilon_n = x_n - \alpha \quad \Rightarrow x_n = \epsilon_n + \alpha$$
$$\epsilon_{n+1} = x_{n+1} - \alpha \quad \Rightarrow x_{n+1} = \epsilon_{n+1} + \alpha$$

and
$$x_{n-1} = \epsilon_{n-1} + \alpha.$$

$$\therefore \quad \epsilon_{n+1} + \alpha = \epsilon_n + \alpha - \frac{[\epsilon_n + \alpha - (\epsilon_{n-1} + \alpha)]f(\epsilon_n + \alpha)}{f(\alpha + \epsilon_n) - f(\alpha + \epsilon_{n-1})}$$

$$\Rightarrow \quad \epsilon_{n+1} = \epsilon_n - \frac{(\epsilon_n - \epsilon_{n-1})f(\alpha + \epsilon_n)}{f(\alpha + \epsilon_n) - f(\alpha + \epsilon_{n-1})}$$

Expanding $f(\alpha+\epsilon_n)$ and $f(\alpha+\epsilon_{n-1})$ by Taylor's formula about α and since $f(\alpha)=0$, and ϵ_n is small neglecting higher powers of ϵ_n, we get

$$\epsilon_{n+1} = c\,\epsilon_n \epsilon_{n-1}, \qquad \text{where } c = \left|\frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}\right|$$

But in regular falsi method if $f(x)=0$ has a root in (x_0, x_1), then either x_0 or x_1 is always fixed and the other point varies with the iteration.

If x_0 is fixed, then $\epsilon_{n+1} = c\,\epsilon_0 \epsilon_n$, where $c = \left|\frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}\right|$ and $\epsilon_0 = x_0 - \alpha$ is independent of n.

$\therefore \qquad |\epsilon_{n+1}| = k|\epsilon_n|$

Hence the Regular-falsi method has **linear rate of convergence or order 1**

3. **Secant method**

Suppose p is the order of convergence, then $\epsilon_{n+1} = k\,\epsilon_n^p$, where k and p are to be determined.

We have $\epsilon_n = k\,\epsilon_{n-1}^p \Rightarrow \epsilon_{n-1} = k^{-\frac{1}{p}}\,\epsilon_n^{\frac{1}{p}}$.

But $\epsilon_{n+1} = c\,\epsilon_n\epsilon_{n-1}$ from above

$$\therefore \qquad k\,\epsilon_n^p = c\,\epsilon_n\,k^{-\frac{1}{p}}\,\epsilon_n^{\frac{1}{p}}$$

$$\Rightarrow \qquad \epsilon_n^p = \left(ck^{-\frac{1}{p}-1}\right)\epsilon_n^{1+\frac{1}{p}}$$

Equating powers of ϵ_n, we get

$$p = 1 + \frac{1}{p}$$

$$\Rightarrow \qquad p^2 - p - 1 = 0$$

$$\therefore \qquad p = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Since p is positive, $\qquad p = \dfrac{1+\sqrt{5}}{2} \simeq 1.6$

So, the secant method has **rate of convergence of order 1.6**

4. **Newton-Raphson method**

In this method, the iteration formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Let α be a root $\therefore f(\alpha) = 0$

$$\epsilon_{n+1} = x_{n+1} - \alpha,\ \epsilon_n = x_n - \alpha$$

$$x_{n+1} = \epsilon_{n+1} + \alpha,\ x_n = \epsilon_n + \alpha$$

$$\epsilon_{n+1} + \alpha = \epsilon_n + \alpha - \frac{f(\alpha + \epsilon_n)}{f'(\alpha + \epsilon_n)}$$

$$\epsilon_{n+1} = \epsilon_n - \frac{\left[f(\alpha) + \epsilon_n f'(\alpha) + \frac{\epsilon_n^2}{2!} f''(\alpha) + \ldots\right]}{f'(\alpha) + \epsilon_n f''(\alpha) + \ldots} \quad \text{[using Taylor's series expansion]}.$$

$$= \epsilon_n - \frac{\left[\epsilon_n f'(\alpha) + \frac{\epsilon_n^2}{2!} f''(\alpha) + \ldots\right]}{f'(\alpha)\left[1 + \epsilon_n \frac{f''(\alpha)}{f'(\alpha)} + \ldots\right]} \qquad [\because f(\alpha) = 0]$$

$$= \epsilon_n - \frac{f'(\alpha)\left[\epsilon_n + \frac{\epsilon_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)} + \dots\right]}{f'(\alpha)\left[1 + \epsilon_n \frac{f''(\alpha)}{f'(\alpha)} + \dots\right]}$$

$$= \epsilon_n - \frac{\left[\epsilon_n + \frac{\epsilon_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)} + \dots\right]}{1 + \epsilon_n \frac{f''(\alpha)}{f'(\alpha)} + \dots}$$

$$= \epsilon_n - \left[\epsilon_n + \frac{\epsilon_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)} + \dots\right]\left[1 + \epsilon_n \frac{f''(\alpha)}{f'(\alpha)} + \dots\right]^{-1}$$

$$= \epsilon_n - \left[\epsilon_n + \frac{\epsilon_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)} + \dots\right]\left[1 - \epsilon_n \frac{f''(\alpha)}{f'(\alpha)} + \dots\right]$$

$$= \epsilon_n - \left[\epsilon_n - \frac{\epsilon_n^2 f''(\alpha)}{f'(\alpha)} + \frac{\epsilon_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)} + \text{terms containing higher powers of } \epsilon_n^3, \epsilon_n^4 \dots\right]$$

$$= \epsilon_n - \epsilon_n + \frac{1}{2}\epsilon_n^2 \frac{f''(\alpha)}{f'(\alpha)}, \qquad \text{neglecting higher powers of } \epsilon_n$$

$$= \frac{1}{2}\epsilon_n^2 \frac{f''(\alpha)}{f'(\alpha)}$$

$\therefore |\epsilon_{n+1}| = c|\epsilon_n|^2$, where $c = \left|\frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}\right|$

$\therefore p = 2$.

So, the order of convergence of Newton-Raphson method is 2.
That is it has **quadratic convergence or order 2.**

5. The Iteration method

Let $\qquad f(x) = 0$

Suppose it is rewritten as $x = \phi(x)$ and let α be a root, then $\alpha = \phi(\alpha)$
The iteration formula is

$$x_{n+1} = \phi(x_n), \quad n = 0, 1, 2, \dots$$

$\therefore$
$$x_{n+1} - \alpha = \phi(x_n) - \phi(\alpha)$$

$\Rightarrow$
$$x_{n+1} - \alpha = (x_n - \alpha)\frac{\phi(x_n) - \phi(\alpha)}{x_n - \alpha}$$

$$= (x_n - \alpha)\frac{\phi(\alpha) - \phi(x_n)}{\alpha - x_n}$$

By Lagrange's mean value theorem, we have

$$\phi'(\theta_n) = \frac{\phi(\alpha) - \phi(x_n)}{\alpha - x_n}, x_n < \theta_n < \alpha$$

$\therefore \quad x_{n+1} - \alpha = (x_n - \alpha)\phi'(\theta_n), x_n < \theta_n < \alpha$

$\Rightarrow \quad |\epsilon_{n+1}| = |\epsilon_n||\phi'(\theta_n)|$

Let k be the maximum value of $\phi'(x)$ in a neighbourhood I of α, then

$$|\phi'(x)| \le k \qquad \forall x \in I$$

$\therefore \quad |\epsilon_{n+1}| \le k|\epsilon_n|$

Here $p = 1$ and so **the order of convergence is 1.**

6. Condition for the convergence of Newton-Raphson method

The iteration formula for Newton-Raphson method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0,1,2,\ldots$$

$\therefore \quad x_{n+1} = \phi(x_n)$

$\Rightarrow \quad \phi(x) = x - \frac{f(x)}{f'(x)}$

$\therefore$

$$\phi'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2}$$

$$= 1 - \left[1 - \frac{f(x)f''(x)}{[f'(x)]^2}\right]$$

$$= \frac{f(x)f''(x)}{[f'(x)]^2}$$

We know that the condition for the convergence is

$|\phi'(x)| < 1$, in a neighbourhood I of the root α

$\Rightarrow \quad \left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| < 1$

$\Rightarrow \quad |f(x)f''(x)| \le |f'(x)|^2 \; \forall x \in I.$ ■

WORKED EXAMPLES

Example 1

Show that the modified Newton-Raphson method $x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)}$, gives a quadratic convergence when the equation $f(x) = 0$ has a double root in the neighbourhood of $x = \alpha$.

Solution

The Newton's modified iteration formula for double root α of $f(x) = 0$ is

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)}, \quad n = 0,1,2,3,...$$

Put
$$\epsilon_{n+1} = x_{n+1} - \alpha, \qquad \epsilon_n = x_n - \alpha$$

$\therefore$
$$x_{n+1} = \epsilon_{n+1} + \alpha, \qquad x_n = \epsilon_n + \alpha$$

$\therefore$
$$\epsilon_{n+1} + \alpha = \epsilon_n + \alpha - \frac{2f(\alpha + \epsilon_n)}{f'(\alpha + \epsilon_n)}$$

$\Rightarrow$
$$\epsilon_{n+1} = \epsilon_n - \frac{2f(\alpha + \epsilon_n)}{f'(\alpha + \epsilon_n)}$$

Using Taylor's series expansion, we get

$$\epsilon_{n+1} = \epsilon_n - \frac{2\left[f(\alpha) + \epsilon_n f'(\alpha) + \frac{\epsilon_n^2}{2!} f''(\alpha) + \frac{\epsilon_n^3}{3!} f'''(\alpha) + ...\right]}{f'(\alpha) + \epsilon_n f''(\alpha) + \frac{\epsilon_n^2}{2!} f'''(\alpha) + ...}$$

But α is a double root of $f(x) = 0$

$\therefore$
$$f(\alpha) = 0 \text{ and } f'(\alpha) = 0, \qquad \text{but} \qquad f''(\alpha) \neq 0$$

$\therefore$
$$\epsilon_{n+1} = \epsilon_n - \frac{2\left[\frac{\epsilon_n^2}{2} f''(\alpha) + \frac{\epsilon_n^3}{6} f'''(\alpha) + ...\right]}{\epsilon_n f''(\alpha) + \frac{\epsilon_n^2}{2} f'''(\alpha) + ...}$$

$$= \epsilon_n - \frac{\epsilon_n^2 f''(\alpha)\left[1 + \frac{\epsilon_n}{3} \frac{f'''(\alpha)}{f''(\alpha)} + ...\right]}{\epsilon_n f''(\alpha)\left[1 + \frac{\epsilon_n}{2} \frac{f'''(\alpha)}{f''(\alpha)} + ...\right]}$$

$$= \epsilon_n - \frac{\epsilon_n\left[1+\frac{\epsilon_n}{3}\frac{f'''(\alpha)}{f''(\alpha)}+\ldots\right]}{1+\frac{\epsilon_n}{2}\frac{f'''(\alpha)}{f''(\alpha)}+\ldots}$$

$$= \epsilon_n - \epsilon_n\left[1+\frac{\epsilon_n}{3}\frac{f'''(\alpha)}{f''(\alpha)}+\ldots\right]\left[1+\frac{\epsilon_n}{2}\frac{f'''(\alpha)}{f''(\alpha)}+\ldots\right]^{-1}$$

$$= \epsilon_n - \epsilon_n\left[1+\frac{\epsilon_n}{3}\frac{f''(\alpha)}{f''(\alpha)}+\ldots\right]\left[1-\frac{\epsilon_n}{2}\frac{f''(\alpha)}{f''(\alpha)}+\ldots\right]$$

$$= \epsilon_n - \epsilon_n\left[1-\frac{\epsilon_n}{2}\frac{f'''(\alpha)}{f''(\alpha)}+\frac{\epsilon_n}{3}\frac{f'''(\alpha)}{f''(\alpha)}+\ldots\right]$$

$$= \epsilon_n - \epsilon_n + \frac{1}{6}\epsilon_n^2\frac{f'''(\alpha)}{f''(\alpha)} + \text{higher powers of } \epsilon_n$$

$$= \frac{1}{6}\epsilon_n^2\frac{f'''(\alpha)}{f''(\alpha)}$$

$$\therefore \quad \epsilon_{n+1} = \frac{1}{6}\epsilon_n^2\frac{f'''(\alpha)}{f''(\alpha)}$$

$$\Rightarrow \quad |\epsilon_{n+1}| = c|\epsilon_n|^2, \text{ where } c = \frac{1}{6}\left|\frac{f'''(\alpha)}{f''(\alpha)}\right|$$

Here $p = 2$. So, the rate of convergence is quadratic. ■

Example 2

In an attempt to solve $x = 1.4\cos x$ by using iteration formula $x_{n+1} = 1.4\cos x_n$, it is found that for large n, x_n alternates between two values A and B.

(i) Calculate A and B correct to two decimal places.
(ii) Calculate the correct solution to four decimal places.

Solution

(i) Given equation is $x = 1.4\cos x$.

It is of the form $x = \phi(x)$, where $\phi(x) = 1.4\cos x$.

The iteration formula is $x_{n+1} = \phi(x_n)$, $\quad n = 0, 1, 2, 3, \ldots$

Take $x_0 = 0$

$$\phi'(x) = -1.4\sin x$$

$$\therefore \quad \phi'(x_0) = -1.4\sin 0 = 0 \quad \Rightarrow |\phi'(x_0)| < 1$$

$$x_1 = 1.4\cos 0 = 1.4$$
$$x_2 = \phi(x_1) = 1.4\cos 1.4 = 1.4(0.16997) = 0.238$$
$$x_3 = \phi(x_2) = 1.4\cos(0.238) = 1.4(0.9718) = 1.3605$$
$$x_4 = \phi(x_3) = 1.4\cos(1.3605) = 1.4(0.208749) = 0.2922$$
$$x_5 = \phi(x_4) = 1.4\cos(0.2922) = 1.4(0.9576) = 1.3407$$
$$x_6 = \phi(x_5) = 1.4\cos(1.3407) = 1.4(0.228071) = 0.3193$$
$$x_7 = \phi(x_6) = 1.4\cos(0.3193) = 1.4(0.94945) = 1.3292$$
$$x_8 = \phi(x_7) = 1.4\cos(1.3292) = 1.4(0.23925) = 0.3349$$
$$x_9 = \phi(x_8) = 1.4\cos(0.3349) = 1.4(0.94444) = 1.3222$$
$$x_{10} = \phi(x_9) = 1.4\cos(1.3222) = 1.4(0.24604) = 0.3444$$
$$x_{11} = \phi(x_{10}) = 1.4\cos(0.3444) = 1.4(0.941244) = 1.3178$$
$$x_{12} = \phi(x_{11}) = 1.4\cos(1.3178) = 1.4(0.250306) = 0.3504$$
$$x_{13} = \phi(x_{12}) = 1.4\cos(0.3504) = 1.4(0.93923) = 1.3149$$
$$x_{14} = \phi(x_{13}) = 1.4\cos(1.3149) = 1.4(0.253112) = 0.3544$$
$$x_{15} = \phi(x_{14}) = 1.4\cos(0.3544) = 1.4(0.93785) = 1.3130$$
$$x_{16} = \phi(x_{15}) = 1.4\cos(1.3130) = 1.4(0.25495) = 0.3569$$

We observe that the successive values fluctuate between 0.35 and 1.31.
For two decimal places the values are $A = 0.35$ and $B = 1.31$

(ii) **We shall find the root by Newton-Raphson method.**

Let $\quad f(x) = x - 1.4\cos x$

$\quad f'(x) = 1 + 1.4\sin x$

The iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots$$

Take $x_0 = 0.7$

First Iteration: Put $n = 0$

$$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.7 - \frac{f(0.7)}{f'(0.7)}$$

Now $\quad f(x_0) = f(0.7) = 0.7 - 1.4\cos(0.7) = -0.3708$

$\quad f'(x_0) = f'(0.7) = 1 + 1.4\sin(0.7) = 1.9019$

$\therefore$

$$x_1 = 0.7 - \frac{(-0.3708)}{1.9019} = 0.7 + 0.1950 = 0.895$$

Second Iteration: put $n = 1$

$\therefore$
$$x_2 = x_1 \frac{f(x_1)}{f'(x_1)} = 0.895 - \frac{f(0.895)}{f'(0.895)}$$

Now
$$f(x_1) = f(0.895) = 0.895 - 1.4\cos(0.895) = 0.895 - 0.8757 = 0.0193$$
$$f'(x_1) = f'(0.895) = 1 + 1.4\sin(0.895) = 1 + 1.0923 = 2.0923$$

$\therefore$
$$x_2 = 0.895 - \frac{0.0193}{2.0923} = 0.895 - 0.00922 = \mathbf{0.8858}$$

Third approximation: put $n = 2$

$\therefore$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.8858 - \frac{f(0.8858)}{f'(0.8858)}$$

Now
$$f(x_2) = f(0.8858)$$
$$= 0.8858 - 1.4\cos(0.8858) = 0.8858 - 0.8857 = 0.0001$$
$$f'(x_2) = 1 + 1.4\sin(0.8858) = 1 + 1.0842 = 2.0842$$
$$x_3 = 0.8858 - \frac{0.0001}{2.0842} = 0.8858 - 0.00004798 = 0.8858$$

Since $x_2 = x_3$, the root correct to four places of decimal, is $\boldsymbol{x = 0.8858}$ ■

NEWTON-RAPHSON METHOD FOR NON-LINEAR EQUATIONS IN TWO VARIABLES

Let $f(x,y) = 0$, $g(x,y) = 0$ be two given non-linear equations.

Let (x_0, y_0) be an initial approximation to the solution of the system.

$\therefore (x_0 + h, y_{\sigma} + k)$ is the solution of the system, where h and k are small

$\therefore f(x_0 + h, \ y_0 + k) = 0$ and $g(x_0 + h, \ y_0 + k) = 0$

Assuming the conditions for Taylor's theorem of function of two variables (that is f and g have continuous partial derivatives), we have

$$f(x_0 + h, \ y_0 + k) = f(x_0, y_0) + h\left(\frac{\partial f}{\partial x}\right)_0 + k\left(\frac{\partial f}{\partial y}\right)_0 + \ldots$$

and
$$g(x_0 + h, \ y_0 + k) = g(x_0, y_0) + h\left(\frac{\partial g}{\partial x}\right)_0 + k\left(\frac{\partial g}{\partial y}\right)_0 + \ldots$$

where $\left(\frac{\partial f}{\partial x}\right)_0 = \left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)}$, $\left(\frac{\partial f}{\partial y}\right)_0 = \left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0)}$

$\left(\frac{\partial g}{\partial x}\right)_0 = \left(\frac{\partial g}{\partial x}\right)_{(x_0,y_0)}$, $\left(\frac{\partial g}{\partial y}\right)_0 = \left(\frac{\partial g}{\partial y}\right)_{(x_0,y_0)}$

Since h and k are small, we neglect square and product of h and k terms in the expansion.

$\therefore$ we get

$$f(x_0,y_0)+h\left(\frac{\partial f}{\partial x}\right)_0+k\left(\frac{\partial f}{\partial y}\right)_0=0$$

$$g(x_0,y_0)+h\left(\frac{\partial g}{\partial x}\right)_0+k\left(\frac{\partial g}{\partial y}\right)_0=0$$

$\therefore$
$$\left(\frac{\partial f}{\partial x}\right)_0 h+\left(\frac{\partial f}{\partial y}\right)_0 k=-f(x_0,y_0) \tag{1}$$

$$\left(\frac{\partial g}{\partial x}\right)_0 h+\left(\frac{\partial g}{\partial y}\right)_0 k=-g(x_0,y_0) \tag{2}$$

Solving (1) and (2) by Cramer's rule, we get

$h=\frac{D_1}{D}$, $k=\frac{D_2}{D}$ where
$$D=\begin{vmatrix}\left(\frac{\partial f}{\partial x}\right)_0 & \left(\frac{\partial f}{\partial y}\right)_0\\ \left(\frac{\partial g}{\partial x}\right)_0 & \left(\frac{\partial g}{\partial y}\right)_0\end{vmatrix}$$

$$D_1=\begin{vmatrix}-f(x_0,y_0) & \left(\frac{\partial f}{\partial y}\right)_0\\ -g(x_0,y_0) & \left(\frac{\partial g}{\partial y}\right)_0\end{vmatrix},\quad D_2=\begin{vmatrix}\left(\frac{\partial f}{\partial x}\right)_0 & -f(x_0,y_0)\\ \left(\frac{\partial g}{\partial x}\right)_0 & -g(x_0,y_0)\end{vmatrix}$$

$\therefore$ the first approximation solution is (x_1,y_1) and is given by $x_1=x_0+h$, $y_1=y_0+h$.

Repeat the process with (x_1,y_1), find (x_2,y_2) and so on till two consecutive approximations coincide upto the desired degree of accuracy. ■

Example 1

Solve $x^3+2y^2=1$ and $5y^3+x^2-2xy=4$ assuming the initial solution $x=-0.6494$, $y=0.7981$.

Solution

The given equations are $x^3+2y^2=1$ and $5y^3+x^2-2xy=4$

$\Rightarrow$ $x^2+2y^2-1=0$ and $5y^3+x^2-2xy-4=0$

Let
$$f(x,y)=x^3+2y^2-1 \tag{1}$$

and $$g(x,y)=5y^3+x^2-2xy-4 \quad (2)$$

Given $x_0=-0.6494$, $y_0=0.7981$ as the initial approximation.

First approximation is $(x_1,y_1)=(x_0+h,\ y_0+k)$

where $h=\frac{D_1}{D}$, $k=\left|\frac{D_2}{D}\right|$, $D=\begin{vmatrix}\left(\frac{\partial f}{\partial x}\right)_0 & \left(\frac{\partial f}{\partial y}\right)_0\\ \left(\frac{\partial g}{\partial x}\right)_0 & \left(\frac{\partial g}{\partial y}\right)_0\end{vmatrix}$

$$D_1=\begin{vmatrix}-f(x_0,y_0) & \left(\frac{\partial f}{\partial y}\right)_0\\ -g(x_0,y_0) & \left(\frac{\partial g}{\partial y}\right)_0\end{vmatrix},\quad D_2=\begin{vmatrix}\left(\frac{\partial f}{\partial x}\right)_0 & -f(x_0,y_0)\\ \left(\frac{\partial g}{\partial x}\right)_0 & -g(x_0,y_0)\end{vmatrix}$$

Now $$f(x_0,y_0)=(-0.6494)^3+2(0.7981)^2-1$$
$$=-0.2738652+1.2739272-1=0.000062$$
$$g(x_0,y_0)=5(0.7981)^3+(-0.6494)^2-2(-0.6494)(0.7981)-4$$
$$=2.54180+0.42172+1.03657-4=0.0000959$$

Differentiating (1) and (2) partially w.r.to x and y respectively, we get

$$\frac{\partial f}{\partial x}=3x^2,\qquad \frac{\partial f}{\partial y}=4y$$
$$\frac{\partial g}{\partial x}=2x-2y,\qquad \frac{\partial g}{\partial y}=15y^2-2x$$

$$\therefore\quad \left(\frac{\partial f}{\partial x}\right)_0=3x_0^2=3(-0.6494)^2=1.26516$$
$$\left(\frac{\partial f}{\partial y}\right)_0=4y_0=4(0.7981)=3.1924$$
$$\left(\frac{\partial g}{\partial x}\right)_0=2x_0-2y_0=2(-0.6494)-2(0.7981)$$
$$=-1.2988-1.5962=-2.895$$
$$\left(\frac{\partial g}{\partial y}\right)_0=15y_0^2-2x_0$$
$$=15(0.7981)^2-2(-0.6494)$$
$$=9.55445+1.2988=10.85325$$

$$D=\begin{vmatrix}1.26516 & 3.1924\\ -2.895 & 10.85325\end{vmatrix}$$
$$=1.26516(10.85325)-(-2.895)(3.1924)$$
$$=13.73109777+9.241998=22.9731$$

$$D_1 = \begin{vmatrix} -0.000062 & 3.1924 \\ -0.0000959 & 10.85325 \end{vmatrix}$$

$$= (-0.000062)(10.85325) - (-0.0000959)(3.1924)$$

$$= -0.0006729 + 0.00030615 = -0.0003668$$

$$D_2 = \begin{vmatrix} 1.26516 & -0.000062 \\ -2.895 & -0.0000959 \end{vmatrix}$$

$$= 1.26516(-0.0000959) - (-2.895)(-0.000062)$$

$$= -0.0001213 - 0.00017949 = -0.0003$$

$\therefore$ $$h = \frac{D_1}{D} = -\frac{0.0003668}{22.9731} = -0.0000159$$

$$k = \frac{D_2}{D} = -\frac{0.0003}{22.9731} = -0.000013$$

$\therefore$ $$x_1 = -0.6494 - 0.0000159 = -0.649416$$

and $$y_1 = 0.7981 - 0.000013 = 0.798087$$

Since (x_1, y_1) is very close to (x_0, y_0), we take the solution as

$$\mathbf{x = -0.649416,\ y = 0.798087}$$ ■

Example 2

Find by Newton-Raphson method the solution of $x + 3\log_{10}x - y^2 = 0$, $2x^2 - xy - 5x + 1 = 0$ after two iterations starting with (3.4, 2.2).

Solution

The given equations are $x + 3\log_{10}x - y^2 = 0$ and $2x^2 - xy - 5x + 1 = 0$

Let $$f(x,y) = x + 3\log_{10}x - y^2 \quad (1)$$

$$g(x,y) = 2x^2 - xy - 5x + 1 \quad (2)$$

Given $$(x_0, y_0) = (3.4, 2.2)$$

First approximation is $(x_1, y_1) = (x_0 + h,\ y_0 + k)$

where $h = \frac{D_1}{D}$, $k = \frac{D_2}{D}$, $D = \begin{vmatrix} \left(\frac{\partial f}{\partial x}\right)_0 & \left(\frac{\partial f}{\partial y}\right)_0 \\ \left(\frac{\partial g}{\partial x}\right)_0 & \left(\frac{\partial g}{\partial y}\right)_0 \end{vmatrix}$

$$D_1 = \begin{vmatrix} -f(x_0, y_0) & \left(\frac{\partial f}{\partial y}\right)_0 \\ -g(x_0, y_0) & \left(\frac{\partial g}{\partial y}\right)_0 \end{vmatrix}, \quad D_2 = \begin{vmatrix} \left(\frac{\partial f}{\partial x}\right)_0 & -f(x_0, y_0) \\ \left(\frac{\partial g}{\partial x}\right)_0 & -g(x_0, y_0) \end{vmatrix}$$

Differentiating (1) and (2) patially w.r.to x and y respectively, we get

$$\frac{\partial f}{\partial x} = 1 + 3\log_{10}^{e}\left(\frac{1}{x}\right) = 1 + 3(0.43429)\frac{1}{x}$$

$$= 1 + (1.30288)\frac{1}{x}$$

$$\frac{\partial f}{\partial y} = -2y, \quad \frac{\partial g}{\partial x} = 4x - y - 5, \quad \frac{\partial g}{\partial y} = -x$$

$$f(x_0, y_0) = 3.4 + 3\log_{10} 3.4 - (2.2)^2$$

$$= 3.4 + 3(0.5314789) - 4.84$$

$$= 3.4 + 1.594436 - 4.84 = 0.154436$$

$$g(x_0, y_0) = 2(3.4)^2 - (3.4)(2.2) - 5(3.4) + 1$$

$$= 23.12 - 7.48 - 17 + 1 = -0.36$$

$$\left(\frac{\partial f}{\partial x}\right)_0 = 1 + 1.30288 \times \frac{1}{3.4} = 1 + 0.3832 = 1.3832$$

$$\left(\frac{\partial f}{\partial y}\right)_0 = -2(2.2) = -4.4$$

$$\left(\frac{\partial g}{\partial x}\right)_0 = 4(3.4) - 2.2 - 5 = 6.4 \quad \text{and} \quad \left(\frac{\partial g}{\partial y}\right)_0 = -3.4$$

$$D = \begin{vmatrix} 1.3832 & -4.4 \\ 6.4 & -3.4 \end{vmatrix}$$

$$= 1.3832(-3.4) - 6.4(-4.4) = -4.70288 + 28.16 = 23.45712$$

$$D_1 = \begin{vmatrix} -0.154436 & -4.4 \\ 0.36 & -3.4 \end{vmatrix}$$

$$= 0.52508 + 1.584 = 2.10908$$

$$D_2 = \begin{vmatrix} 1.3832 & -0.154436 \\ 6.4 & 0.36 \end{vmatrix}$$

$$= 0.497952 + 0.9883904 = 1.48634$$

$\therefore$ $$h = \frac{D_1}{D} = \frac{2.10908}{23.45712} = 0.08991$$

$$k = \frac{D_2}{D} = \frac{1.48634}{23.45712} = 0.06336$$

$\therefore$ $$x_1 = x_0 + h = 3.4 + 0.08991 = 3.48991$$

and $$y_1 = y_0 + k = 2.2 + 0.06336 = 2.26336$$

$\therefore$ $$\mathbf{(x_1, y_1) = (3.48991, 2.26336)}$$

Second approximation is $(x_2, y_2) = (x_1 + h, y_1 + k)$

where $h = \dfrac{D_1}{D}, \quad k = \dfrac{D_2}{D}, \quad D = \begin{vmatrix} \left(\dfrac{\partial f}{\partial x}\right)_1 & \left(\dfrac{\partial f}{\partial y}\right)_1 \\ \left(\dfrac{\partial g}{\partial x}\right)_1 & \left(\dfrac{\partial g}{\partial y}\right)_1 \end{vmatrix}$

$$D_1 = \begin{vmatrix} -f(x_1, y_1) & \left(\dfrac{\partial f}{\partial y}\right)_1 \\ -g(x_1, y_1) & \left(\dfrac{\partial g}{\partial y}\right)_1 \end{vmatrix}, \quad D_2 = \begin{vmatrix} \left(\dfrac{\partial f}{\partial x}\right)_1 & -f(x_1, y_1) \\ \left(\dfrac{\partial g}{\partial x}\right)_1 & -g(x_1, y_1) \end{vmatrix}$$

Now $\quad f(x_1, y_1) = x_1 + 3\log_{10} x_1 - y_1^2$

$$= 3.48991 + 3 \times \log_{10}(3.48991) - (2.26336)^2$$

$$= 3.48991 + 1.62844 - 5.12280 = -0.00445$$

$$g(x_1, y_1) = 2x_1^2 - x_1 y_1 - 5x_1 + 1$$

$$= 2(3.48991)^2 - (3.48991)(2.26336) - 5(3.48991) + 1$$

$$= 24.35894362 - 7.898922698 - 17.49955 + 1 = 0.01047$$

$$\left(\frac{\partial f}{\partial x}\right)_1 = 1 + 1.30288 \times \frac{1}{3.48991} = 1 + 0.37333 = 1.37333$$

$$\left(\frac{\partial f}{\partial y}\right)_1 = -2y_1 = -2(2.26336) = -4.52672$$

$$\left(\frac{\partial g}{\partial y}\right)_1 = 4x_1 - y_1 - 5$$

$$= 4(3.48991) - 2.26336 - 5 = 6.69628$$

$$\left(\frac{\partial g}{\partial y}\right)_1 = -x_1 = -3.48991$$

$\therefore$

$$D = \begin{vmatrix} 1.37333 & -4.52672 \\ 6.69628 & -3.48991 \end{vmatrix}$$

$$= -4.79280 + 30.31218 = 25.51938$$

$$D_1 = \begin{vmatrix} 0.00445 & -4.52672 \\ -0.01047 & -3.48991 \end{vmatrix}$$

$$= -0.015530099 - 0.047394758 = -0.06292$$

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$$D_2 = \begin{vmatrix} 1.37333 & 0.00445 \\ 6.69628 & -0.01047 \end{vmatrix}$$

$$= -0.014378765 - 0.029798446 = -0.04418$$

$$h = \frac{D_1}{D} = -\frac{0.06292}{25.51938} = -0.002466$$

$$k = \frac{D_2}{D} = -\frac{0.04418}{25.51938} = -0.0017312$$

$\therefore$ $x_2 = x_1 + h = 3.48991 - 0.002466 = 3.48744$

and $y_2 = y_1 + h = 2.26336 - 0.0017312 = 2.26163$

$\therefore$ $(x_2, y_2) = (3.48744, 2.26163)$

$\therefore$ the solution is $\boldsymbol{x = 3.4874}$ and $\boldsymbol{y = 2.2616}$ ■

Exercises 2.3

(1) The system of equations $y\cos(xy)+1=0$, $\sin(xy)+x-y=0$ has a solution near $x=1$, $y=2$. Find the solution correct to two decimal places by Newton-Raphson method.

(2) Find the solution of the equations $x+\tan y=0$, $e^x=y+2$ near $x=\frac{1}{2}$, $y=\frac{-1}{2}$ using Newton-Raphson method.

(3) Find the solution of the equation $x^2+y^2=1.12$ and $xy=0.23$ which is near $x=1$ and $y=0$, by Newton-Raphson method.

(4) Using Newton-Raphson method find a solution of $x^2-y^2=4$, $x^2+y^2=16$, near $x=3$ and $y=2$.

Answers 2.3

(1) $x=1.09$,	$y=1.94$	(2) $x=0.4564$,	$y=-0.4284$
(3) $x=1.035$,	$y=0.222$	(4) $x=3.162$,	$y=2.450$

SOLUTION OF POLYNOMIAL EQUATIONS

Horner's Method

1. Introduction

Horner's method is used to find a positive root of a polynomial equation $f(x)=0$. In finding the root, we will be using the techniques of diminishing the roots of $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$ by a fixed number and multiplying the roots by a fixed number.

(a) Diminishing the roots by a fixed number

Diminishing the roots by a fixed number h means putting $y = x - h \Rightarrow x = y + h$.
That is forming a transformed equation in y

Example 1

Diminish the roots of the equation $x^4-5x^3+7x^2-10x+5=0$ by 2.

Solution

Given equation is $$x^4-5x^3+7x^2-10x+5=0 \qquad (1)$$

To diminish the roots by 2, Put $y=x-2$ in (1)

$$\Rightarrow \qquad x=y+2$$

$\therefore$ the transformed equation is

$$(y+2)^4-5(y+2)^3+7(y+2)^2-10(y+2)+5=0$$

$$\Rightarrow y^4+4y^3.2+6y^2.2^2+4y.2^3+2^4-5(y^3+3y^2.2+3y.2^2+2^3)$$
$$+7(y^2+2y+4)-10y-20+5=0$$

$$\Rightarrow \qquad y^4+3y^3+y^2-10y-11=0 \qquad (2)$$

It can be re written in x as

$$x^4+3x^3+x^2-10x-11=0$$

This is the equation whose roots are the roots of (1) diminished by 2.

Alter: The coefficients of the new equation can also be obtained by finding the remainders when $f(x)$ is divided by $(x-2)$ successively.

We carryout this process as synthetic division by writing only the coefficients.

2	1	−5	7	−10	5
	0	2	−6	2	−16
	1	−3	1	−8	$\boxed{-11}$
	0	2	−2	−2	
	1	−1	−1	$\boxed{-10}$	
	0	2	2		
	1	1	$\boxed{1}$		
	0	2			
	1	$\boxed{3}$			

$\therefore$ the transformed equation is $x^4+3x^3+x^2-10x-11=0$

(b) Multiplying the roots of $f(x)=0$ by a fixed number.

Let us multiply the roots of the equation

$$x^4-5x^3+7x^2-10x+5=0 \qquad (1)$$

by a fixed number $m \neq 0$.

$\therefore$ Put $y=mx \qquad \Rightarrow x=\dfrac{y}{m}$

Substituting in (1), we get

$$\left(\frac{y}{m}\right)^4 - 5\left(\frac{y}{m}\right)^3 + 7\left(\frac{y}{m}\right)^2 - 10\frac{y}{m} + 5 = 0$$

$$\Rightarrow \quad y^4 - 5my^3 + 7m^2y - 10m^3y + 5m^4 = 0$$

The transformed equation is written as

$$x^4 - 5mx^3 + 7m^2x^2 - 10m^3y + 5m^4 = 0$$

This equation is obtained by multiplying the successive terms of (1) by $1, m, m^2, m^3, m^4$ respectively, starting with the highest degree term. ■

Example 2

Consider the equation

$$x^4 - 2x^2 + 5 = 0$$

$$\Rightarrow \quad x^4 + 0x^3 - 2x^2 + 0x + 5 = 0$$

Multiplying the roots by 2, the transformed equation is

1. $x^4 + 0\times2\times x^3 - 2\times2^2\times x^2 + 0\times2^3x + 5\times2^4 \quad = 0$

$$\Rightarrow \quad x^4 - 8x^2 + 80 = 0$$

2. Procedure to find the real root of the polynomial equation $f(x) = 0$ by Horner's method

Let $f(x) = 0$ be the given polynomial equation having a **positive root**.

Let a and $a+1$ be the positive integers such that $f(a)$ and $f(a+1)$ have opposite signs.

So, a root lies between a and $a + 1$. Then the root is of the form $a.\ a_1\ a_2\ a_3\ a_4 \ldots,$ where a_1, a_2, a_3, a_4 are non-negative integers to be determined.

Horner's method gives that root digit by digit upto any desired degree of accuracy.

Step 1: Diminish the roots of $f(x) = 0$ by a, then the resulting equation is written as $g_1(x) = 0$. Then the root of $g_1(x) = 0$ is $0.\ a_1\ a_2\ a_3 \ldots$

Multiply the roots of $g_1(x) = 0$ by 10.

The transformed equation is written as $f_1(x) = 0$ and the root of $f_1(x) = 0$ is $a_1.a_2\ a_3 \ldots$

Now by trial determine two Consecutive integers b and $b + 1$ such that $f(b)$ and $f(b+1)$ have opposite signs.

$\therefore$ a root of $f_1(x) = 0$ lies between b and $b + 1$

Hence b is the first decimal a_1 in the root. That is $\boldsymbol{a_1 = b}$

Step 2: Diminish the root of $f_1(x) = 0$ by b.

Denote the resulting equation as $g_2(x) = 0$. Then the root of $g_2(x) = 0$ is $0.\ a_2\ a_3\ a_4 \ldots$

Multiply the roots of $g_2(x) = 0$ by 10.

The transformed equation is taken as $f_2(x) = 0$ which has $a_2.\ a_3\ a_4 \ldots$ as a root.

By trial determine two consecutive integers c and $c + 1$ such that $f_2(c)$ and $f_2(c+1)$ have opposite signs.

$\therefore$ a root of $f_2(x) = 0$ lies between c and $c + 1$.

Hence c is the second decimal a_2 in the root. That is $a_2 = c$

Proceeding in this way we get the root of $f(x) = 0$ as $a.\ bc \ldots$

Note: If a negative root of $f(x) = 0$ is required, then we find the corresponding positive root of $f(-x) = 0$, by Horner's method. ■

WORKED EXAMPLES

Example 1

Find by Horner's method the root of $x^3 - 3x + 1 = 0$ which lies between 1 and 2, correct to 3 places of decimals.

Solution

Then given equation is $x^3 - 3x + 1 = 0$

Let $$f(x) = x^3 - 3x + 1$$

Since the root lies between 1 and 2, the root is $1.\ a_1\ a_2\ a_3\ a_4 \ldots$ where $a_1, a_2, a_3, \ldots$ are non-negative integens to be determined by Horner's method.

Step 1: Diminish the root of $f(x) = 0$ by 1

We do by synthetic division

1	1	0	–3	1
	0	1	1	–2
	1	1	–2	$\boxed{-1}$
	0	1	2	
	1	2	$\boxed{0}$	
	0	1		
	1	$\boxed{3}$		

The resulting equation is taken as $g_1(x) = 0$ with $0.\ a_1\ a_2\ a_3 \ldots.$ as a root.

$$\Rightarrow \quad x^3 + 3x^2 + 0x - 1 = 0$$

$$\Rightarrow \quad x^3 + 3x^2 - 1 = 0$$

Multiply the roots of $g_1(x) = 0$ by 10.

The transformed equation is $f_1(x) = 0$ with $a_1.a_2\ a_3 \ldots$ as a root

$$\Rightarrow \quad x^3 + 3.10x^2 + 0.10^2x - 1.10^3 = 0$$

$$\Rightarrow \quad x^3 + 30x^2 - 1000 = 0$$

By trial we find a root of $f_1(x) = 0$, Now $$f_1(x) = x^3 + 30x^2 - 1000$$

and $$f_1(5) = 5^3 + 30\times 5^2 - 1000$$
$$= 125 + 750 - 1000 = -125 < 0$$
$$f_1(6) = 6^3 + 30\times 6^2 - 1000$$
$$= 216^1 + 1080 - 1000 = 296 > 0$$

$\therefore$ a root of $f_1(x) = 0$ lies between 5 and 6.

So, we take the first decimal $\boldsymbol{a_1 = 5}$

Step 2: Diminish the root of $f_1(x) = 0$ by 5

5	1	30	0	−1000
	0	5	175	875
	1	35	175	$\boxed{-125}$
	0	5	200	
	1	40	$\boxed{375}$	
	0	5		
	1	$\boxed{45}$		

The resulting equation is taken as $g_2(x) = 0$ with $0.a_2\, a_3\ldots$ as a root.

$\Rightarrow$ $$x^3 + 45x^2 + 375x - 125 = 0$$

Multiply the roots of the equation $g_2(x) = 0$ by 10

The transformed equation is $f_2(x) = 0$ with $a_2.a_3\, a_4 \ldots$ as a root.

$\Rightarrow$ $$x^3 + 45\times 10\times x^2 + 375\times 10^2 \times x - 125\times 10^3 = 0$$

$\Rightarrow$ $$x^3 + 450x^2 + 37500x - 125000 = 0$$

Since the coefficients of the last two terms are very large compared to the other coefficients, the root is near $\dfrac{125000}{37500} = 3.3$

(The last coefficient divided by the coefficient of x ignoring the signs)

Now $$f_2(x) = x^3 + 450x^2 + 37500x - 125000$$

and $$f_2(3) = 3^3 + 450\times 3^2 + 37500\times 3 - 125000$$
$$= -8423 < 0$$
$$f_2(4) = 4^3 + 450\times 4^2 + 37500\times 4 - 125000$$
$$= 32264 > 0$$

$\therefore$ a root of $f_2(x) = 0$ lies between 3 and 4

So, the second decimal is $\boldsymbol{a_2 = 3}$

Step 3: Diminish the root of $f_2(x) = 0$ by 3.

3	1	450	37500	−12500
	0	3	1359	116577
	1	453	38859	−8423
	0	3	1368	
	1	456	40227	
	0	3		
	1	459		

The resulting equation is taken as $g_3(x) = 0$ with $0.a_3\, a_4 \ldots$ as a root.

$\Rightarrow$ $$x^3 + 459x + 40227x - 8423 = 0$$

Multiply the roots of $g_3(x) = 0$ by 10

The transformed equation is $f_3(x) = 0$ with $a_3.a_4\, a_5 \ldots$ as a root.

$\therefore$ $$x^3 + 459 \times 10 \times x^2 + 40227 \times 10^2 \times x - 8423 \times 10^3 = 0$$

$\Rightarrow$ $$x^3 + 4590x^2 + 4022700 - 8423000 = 0$$

Since the last two coefficients are very large compared to others coefficients, the roots is near $\dfrac{8423000}{4022700} = 2.09$

Now $$f_3(x) = x^3 + 4590x^2 + 4022700x - 8423000$$

and $$f_3(2) = 2^3 + 4590 \times 2^2 + 4022700 \times 2 - 8423000$$
$$= -359232 < 0$$

$$f_3(3) = 3^3 + 4590 \times 3^2 + 4022700 \times 3 - 8423000$$
$$= 3686437 > 0$$

$\therefore$ a root of $f_3(x) = 0$ lies between 2 and 3

So, the third decimal is $\boldsymbol{a_3 = 2}$

Step 4: Diminish the root of $f_3(x) = 0$ by 2

2	1	4590	4022700	−8423000
	0	2	9184	8063768
	1	4592	4031884	−359232
	0	2	9188	
	1	4594	4041072	
	0	2		
	1	4596		

The resulting equation is taken as $g_4(x) = 0$ with $0.a_4\, a_5 \ldots$ as a root.

$\therefore$ $$x^3 + 4596x^2 + 4041072x - 359232 = 0$$

Multiply the roots of $g_4(x)=0$ by 10

The transformed equation is $f_4(x)=0$ with $a_4 . a_5 \ldots$ as a root.

$$\therefore \quad x^3+4596\times 10\times x^2+4041072\times 10^2\times x-359232\times 10^3=0$$

$$\Rightarrow \quad x^3+45960x^2+404107200x-359232000=0$$

The root is near $\dfrac{359232000}{404107200}=0.8$

$$\therefore \quad a_4=0$$

$\therefore$ the root is 1.5320

So, the root correct to three decimal places is $\boldsymbol{x=1.532}$

Note:

(1) After one or two transformations when the coefficients of the last two terms become very large compared to the other coefficients, the root can be located by finding the ratio of the constant term by the coefficient of x (neglecting the sign).

(2) When the terms are verge large, few decimal places of the root can be taken from this ratio. ■

Example 2

Use Horner's method to find the root of $10x^3-15x+3=0$ which lies between 1 and 2, correct to three places of decimals.

Solution

The given equation is $\quad 10x^3-15x+3=0$

Let $\quad f(x)=10x^3-15x+3$

Since the root lies between 1 and 2 the root is $1.\,a_1\,a_2\,a_3\ldots$, where $a_1, a_2, a_3\ldots$ are to be determined by Horner's method.

Step 1: Diminish the root of $f(x)=0$ by 1, by synthetic division

$$\begin{array}{c|cccc}
1 & 10 & 0 & -15 & 3 \\
 & 0 & 10 & 10 & -5 \\
\hline
 & 10 & 10 & -5 & \boxed{-2} \\
 & 0 & 10 & 20 & \\
\hline
 & 10 & 20 & \boxed{15} & \\
 & 0 & 10 & & \\
\hline
 & 10 & \boxed{30} & &
\end{array}$$

The resulting equation is taken as $g_1(x)=0$ with $0.\,a_1\,a_2\,a_3\ldots$ as a root.

$$\therefore \quad 10x^3+30x^2+15x-2=0$$

Multiply the roots of the equation $g_1(x)=0$ by 10

The transformed equation is $f_1(x) = 0$ with $a_1, a_2, a_3, \ldots$ as a root

$$\Rightarrow \quad 10x^3 + 30 \times 10 \times x^2 + 15 \times 10^2 \times x - 2 \times 10^3 = 0$$

$$\Rightarrow \quad x^3 + 30x^2 + 150x - 200 = 0$$

Now $$f_1(x) = x^3 + 30x^2 + 150x - 200$$

and $$f_1(1) = 1 + 30 + 150 - 200 = -19 < 0$$

$$f_1(2) = 2^3 + 30 \times 2^2 + 150 \times 2 - 200 = 228 > 0$$

$\therefore$ a root of $f_1(x) = 0$ lies between 1 and 2.

So, the first decimal is $a_1 = 1$

Step 2: Diminish the root by 1

1	1	30	150	−200
	0	1	31	181
	1	31	181	−19
	0	1	32	
	1	32	213	
	0	1		
	1	33		

The resulting equation is $g_2(x) = 0$

$$\therefore \quad x^3 + 33x^2 + 213x - 19 = 0$$

Multiply the roots of $g_2(x) = 0$ by 10.

The transformed equation is $f_2(x) = 0$

$$\Rightarrow \quad x^3 + 33 \times 10 \times x^2 + 213 \times 10^2 \times x - 19 \times 10^3 = 0$$

$$\Rightarrow \quad x^3 + 330x^2 + 21300x - 19000 = 0$$

$$f_2(x) = x^3 + 330x^2 + 21300x - 19000$$

Since the coefficients of the last two terms are very large compared to others, the root is near $\frac{19000}{21300} \simeq 0.8$

Now $$f_2(0) = -19000 < 0$$

$$f_2(1) = 1 + 330 + 21300 - 19000 = 2631 > 0$$

$\therefore$ a root of $f_2(x) = 0$ lies between 0 and 1

So, the second decimal is $a_2 = 0$

Step 3: Diminishing the roots of $f_2(x) = 0$ by 0 will not change $f_2(x) = 0$ and the root is a decimal.

Multiply the roots by 10

The transformed equation is $f_3(x)=0$ with $a_3, a_4, a_5, \dots$ as a root.

$$\Rightarrow \quad x^3+330\times10\times x^2+21300\times10^2\times x-19000\times10^3=0$$

$$\Rightarrow \quad x^3+3300x^2+2130000x-19000000=0$$

$$f_3(x)=x^3+3300x^2+2130000x-19000000$$

Since the last two coefficients are very large compared to others, the root is near $\dfrac{19000000}{2130000}=8.9$

Now
$$f_3(8)=8^3+3300\times8^2+2130000\times8-19000000<0$$
$$f_3(9)=9^3+3300\times9^2+2130000\times9-19000000>0$$

$\therefore$ a root of $f_3(x)=0$ lies between 8 and 9.

So, the next decimal is $\boldsymbol{a_3=8}$

Step 4: Diminish the roots of $f_3(x)=0$ by 8

8	1	3300	2130000	−19000000
	0	8	26464	17251712
	1	3308	2156464	−1748288
	0	8	26528	
	1	3316	2182992	
	0	8		
	1	3324		

The resulting equation is $g_3(x)=0$

$$\Rightarrow \quad x^3+3324x^2+2182992x-1748288=0$$

Multiply the roots of $g_3(x)=0$ by 10.

The transformed equation is $f_4(x)=0$

$$\Rightarrow \quad x^3+3324\times10\times x^2+2182992\times10^2x-1748288\times10^3=0$$

$$\Rightarrow \quad x^3+33240x^2+218299200x-1748288000=0$$

Since the coefficients of last two terms are very large the root is near

$$\frac{1748288000}{218299200}=8.0026=8$$

$\therefore$ the next decimal is $\boldsymbol{a_4=8}$

$\therefore$ the root is 1.1088

So, correct to 3 decimal places, the root is $\boldsymbol{x=1.109}$ ■

Example 3

Find the negative root of $x^3-2x+5=0$ correct to two places of decimals.

Solution

The given equation is

$$x^3 - 2x + 5 = 0$$

Let $F(x) = x^3 - 2x + 5$

The negative root of $F(x) = 0$ is the positive root of

$$F(-x) = 0$$

$$\Rightarrow \quad (-x)^3 - 2(-x) + 5 = 0$$

$$\Rightarrow \quad -x^3 + 2x + 5 = 0$$

$$\Rightarrow \quad x^3 - 2x - 5 = 0$$

Let $\quad f(x) = x^3 - 2x - 5$

Now $\quad f(2) = 2^3 - 2\times 2 - 5 = 8 - 4 - 5 = -1 < 0$

$$f(3) = 3^3 - 2\times 3 - 5 = 27 - 6 - 5 = 16 > 0$$

$\therefore$ a root lies between 2 and 3

Let the root be $2.a_1a_2a_3\cdots$,where $a_1, a_2, a_3$are to be determined by Horner's method

Step 1: Diminish the roots of $f(x) = 0$ by 2

$$\begin{array}{c|cccc}
2 & 1 & 0 & -2 & -5 \\
 & 0 & 2 & 4 & 4 \\
\hline
 & 1 & 2 & 2 & \boxed{-1} \\
 & 0 & 2 & 8 & \\
\hline
 & 1 & 4 & \boxed{10} & \\
 & 0 & 2 & & \\
\hline
 & 1 & \boxed{6} & &
\end{array}$$

The resulting equation is $g_1(x) = 0$ with $0.\ a_1\ a_2\ a_3\ \ldots$ as a root.

$$\Rightarrow \quad x^3 + 6x^2 + 10x - 1 = 0$$

Multiply the roots of $g_1(x) = 0$ by 10

The transformed equation is $f_1(x) = 0$ with $a_1.\ a_2\ a_3\ a_4\ \ldots$ as a root.

$$\Rightarrow \quad x^3 + 6\times 10x^2 + 10\times 10^2 x - 1\times 10^3 = 0$$

$$\Rightarrow \quad x^3 + 60x^2 + 1000x - 1000 = 0$$

Now $\quad f_1(x) = x^3 + 60x^2 + 1000x - 1000$

$$f_1(0) = -1000 < 0$$

$$f_1(1) = 1 + 60 + 100 - 1000 - 61 > 0$$

$\therefore$ the root of $f_1(x) = 0$lies between 0 and 1

So, the root of $f_1(x) = 0$lies between 0 and 1

So, the first decimal is $\boldsymbol{a_1 = 0}$

Step 2: Diminishing the root by 0 will not change $f_1(x)=0$
So multiply the roots of the equation by 10
The transformed equation is $f_2(x)=0$ with $a_2.\ a_3\,a_4\ldots$ as a root.

$$x^3+60\times10x^2+1000\times10^2x-1000\times10^3=0$$
$$x^3+600x^2+100000x-1000000=0$$

The last coefficients, are larger than the other two coefficients, so the toot is near $\dfrac{1000000}{100000}=10$

Now
$$f_2(x)=x^3+6000x^2+100000x-1000000$$
$$f_2(9)=9^3+60009^2+100000\times9-1000000$$
$$=-51400<0$$
$$f_2(10)=10^3+600\times10^2+100000\times10-1000000$$
$$=61000>0$$

$\therefore$ a root of $f_2(x)=0$ lies between 9 and 10

So, the second decimal is $\boldsymbol{a_2=9}$

Step 3: Diminish the roots of $f_2(x)=0$ by 9

9	1	600	100000	–1000000
	0	9	5481	949329
	1	609	105481	–50671
	0	9	5562	
	1	618	111043	
	0	9		
	1	627		

The resulting equation is $g_3(x)=0$ with $0.\ a_3\,a_4\,a_5\ldots$ as a root.

$\Rightarrow$ $$x^3+627x^2+111043x-50671=0$$

Multiply the roots of $g_3(x)=0$ by 10
The transformed equation is $f_3(x)=0$ with $a_3.\ a_4\,a_5\ldots$ as a root.

$\Rightarrow$ $$x^3+627\times10x^2+111043\times10^2x-50671\times10^3=0$$

$\Rightarrow$ $$x^3+6270x^2+11104300x-50671000=0$$

Since the last two coefficients are very large compared to the first two coefficients, the root is near $\dfrac{50671000}{11104300}=4.56$

$$f_3(4)=4^3+6270\times4^2+11104300\times4-50671000$$
$$=-6153416<0$$

$$f_3(5) = 5^3 + 6270\times5^2 + 11104300\times5 - 50671000$$
$$= -5007675 > 0$$

$\therefore$ a root of $f_3(x) = 0$ lies between 4 and 5
So, the third decimal is $\alpha_3 = 9$

Hence the root is 2.094,
So, correct to two places of decimals the root is 2.09

$\therefore$ the negative root of the given equation is $\boldsymbol{x = -2.09}$ ■

Exercises 2.4

Obtain, by Horner's method the root of the following equations correct to three places of decimals,

(1) $x^3 - x - 9 = 0$

(2) $x^3 + 3x^2 - 12x - 11 = 0$

(3) $x^3 - 3x^2 + 2.5 = 0$ between 1 and 2.

(4) $x^3 - 2x^2 - 5 = 0$ between 2 and 3.

(5) $x^3 + x^2 + x - 100 = 0$ between 4 and 5.

(6) Find the negative root of $2x^3 - 3x + 6 = 0$ correct to 2 decimal places.

Answers 2.4

(1) 2.240 (2) 2.769 (3) 1.168 (4) 2.094 (5) 4.264 (6) −1.78

Graffe's Root-Squaring Method

This method is applicable only to polynomial equations with real coefficients and it is a direct method. Using Graffe's method we can find all the roots of the equation which may be real or complex.

The advantage of this method is that it does not require any prior information about the roots (like initial approximation etc.)

The basic idea of Graffe's method is to transform the given polynomial equation into another equation whose roots are very powers of roots of the given equation by root-squaring process The roots of the transformed equation will then be **separated** and so they can be found easily.

The roots of the transformed equation are said to be **separated** when the ratio of any root to the next larger is negligible compared to unity.

For example, let 2 and 3 be two of the roots of the given equation and let 2^{64} and 3^{64} be the corresponding roots of the transformed equation.

The ratio of the roots in the given equation is $\frac{2}{3}$ but the ratio of the roots in the transformed equation is $\frac{2^{64}}{3^{64}} < 0.01$ which is negligible.

That is the smaller root in the transformed equation is negligible in comparison with the larger one.

So 2^{64} and 3^{64} are separable.

1. The root squaring process

Suppose the polynomial equation $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots a_{n-1}x + a_n = 0$ has roots $\alpha_1, \alpha_2, \cdots, \alpha_n$. The method of transforming the given equation into another whose roots are $\alpha_1{}^2, \alpha_2{}^2, \cdots, \alpha_n{}^2$ is known as the root-squaring process.
For illustration, let us take the equation

$$f(x) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0 \tag{1}$$

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the roots
We shall rewrite the equation as

$$a_0x^4 + a_2x^2 + a_4 = -\left(a_1x^3 + a_3x\right)$$
$$= -x\left(a_1x^2 + a_3\right)$$

Squaring both sides, we get

$$\left(a_0x^4 + a_2x^2 + a_4\right)^2 = x^2\left(a_1x^2 + a_3x\right)^2$$

Put $y = x^2$, then the equation becomes

$$\left(a_0y^2 + a_2y + a_4\right)^2 = y\left(a_1y + a_3\right)^2$$

$$\Rightarrow \quad a_0^2y^4 + a_2^2y^2 + a^4 + 2a_0a_2y^3 + 2a_0a_4y^2 + 2a_2a_4y = y(a_1^2y^2 + 2a_1a_3y + a_3^2)$$
$$= a_1^2y^3 + 2a_1a_3y^2 + a_3^2y$$

$$\Rightarrow \quad a_0^2y^4 + (-a_1^2 + 2a_0a_2)y^3 + (a_2^2 + 2a_0a_4 - 2a_1a_3)y^2 + (-a_3^2 + 2a_2a_4)y + a_4^2 = 0 \tag{2}$$

$\therefore$ Let the transformed equations be $f_1(y) = 0$

Since $x = \alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the roots of $f(x) = 0$, the transformation $y = x^2$ means $\alpha_1^2, \alpha_2^2, \alpha_3^2, \alpha_4^2$ are the roots of $f_1(y) = 0$

i.e. $\quad f_1(x) = a_0^2x^4 + (-a_1^2 + 2a_0a_2)x^3 + (a_2^2 + 2a_0a_4 - 2a_1a_3)x^2 + (-a_3^2 + 2a_2a_4)x + a_4^2 = 0$

has roots $\alpha_1^2, \alpha_2^2, \alpha_3^2, \alpha_4^2$

Let $\quad f_1(x) = b_0x^4 + b_1x^3 + b_2x^2 + b_3x + b_4 = 0$

where

$$b_0 = a_1^2, \qquad b_1 = -a_1^2 + 2a_0a_2$$
$$b_2 = a_2^2 + 2a_0a_4 - 2a_1a_2, \qquad b_3 = -a_3^2 + 2a_2a_4 \text{ and } b_4 = a_4^2$$

In the same way squaring the roots of $f_1(x) = 0$, we get the transformed equation $f_2(x) = 0$ and so on.
We proceed till two successive equations are almost the same.
The transformations can also be schematically exhibited in the form of a table as shown below.

a_0	a_1	a_2	a_3	a_4
a_0^2	$-a_1^2$ $2a_0a_2$	a_2^2 $-2a_1a_3$ $2a_0a_4$	$-a_3^2$ $2a_2a_4$	a_4^2
b_0	b_1	b_2	b_3	b_4

This process can be extended to polynomial of any degree n.

Let $$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_{n-2}x^2 + a_{n-1}x + a_n = 0$$

and the root-squared equation is

$$f_1(x) = b_0x^n + b_1x^{n-1} + b_2x^{n-2} + b_{n-2}x^2 + b_{n-1}x + b_n = 0$$

a_0	a_1	a_2	a_3	a_4	$a_5 \cdots$
a_0^2	$-a_1^2$ $2a_0a_2$	a_2^2 $-2a_1a_3$ $2a_0a_4$	$-a_3^2$ $2a_2a_4$ $-2a_1a_5$ $2a_0a_6$	a_4^2 $-2a_3a_5$ $2a_2a_6$ $-2a_1a_7$	$-a_5^2 \cdots$ $2a_4a_6 \cdots$ $-2a_3a_7 \cdots$ $2a_2a_8 \cdots$
b_0	b_1	b_2	b_3	b_4	$b_5 \cdots$

The coefficients $b_0, b_1, b_2, b_3, b_4, \ldots$ of the transformed equations are the sums of the various columns.

They can be obtained according to the following rule.

1. First write the coefficients of the given equation.
   The first row numbers are the squares of the coefficients with alternating signs.
   i.e first + and second – and so on.
2. The numbers in the second row are doubled products.
   $2\,a_0\,a_2$ under a_1 is got by multiplying 2 with the two adjacent numbers a_0 and a_2 of a_1,
   $-2\,a_1\,a_3$ under a_2 is got similarly with alternative sign.
   So under a_3, the product is $2a_2\,a_4$ and so on.
3. The number in the third row are doubled products.
   $2\,a_0\,a_4$ under a_2 is got by multiplying 2 with the second adjacent numbers a_0 and a_4 of a_2.
   $-2\,a_1\,a_5$ under a_3 is similarly got with alternative sign and so on.

This process is continued

For illustration consider the equation $x^4 + 5x^3 + 3x^2 - 5x - 9 = 0$

Solution

The given equation is

$$x^4 + 5x^3 + 3x^2 - 5x - 9 = 0$$

Here $a_0 = 1$, $a_1 = 5$, $a_2 = 3$ $a_3 = -5$ $a_4 = -9$

We shall find the root-squared equation by the above scheme:

a_0	a_1	a_2	a_3	a_4
1	5	3	(–5)	(–9)
1	–25	9	–25	81
	2.1.3 = 6	–2.5(–5) = 50 2.1.(–9) = –18	2.3(–9) = –54	
1	–19	+41	–79	81
b_0	b_1	b_2	b_3	b_4

$\therefore$ the root-squared equation is $x^4-19x^3+41x^2-79x+81=0$

Aliter: $$x^4+5x^3+3x^2-5x-9=0$$

$\Rightarrow$ $$x^4+3x^2-9=-5x^3+5x$$

$\Rightarrow$ $$x^4+3x^2-9=-5x(x^2-1)$$

Squaring $$(x^4+3x^2-9)^2=25x^2(x^2-1)^2$$

Putting $y=x^2$, we get
$$(y^2+3y-9)^2=25y(y-1)^2$$
$$=25y(y^2-2y+1)$$

$\Rightarrow$ $$y^4+9y^2+81+6y^3-18y^2-54y=25y^3-50y^2+25y$$

$\Rightarrow$ $$y^4-19y^3+41y^2-79y+81=0$$

$\Rightarrow$ $$x^4-19x^3+41x^2-79x+81=0$$

Note: We observe that the direct method is easy for equation whose degree is ≤ 4.

For higher degree equations the table method would be easy.

2. Graffe's method for real and different roots

Let $a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-2}x^2+a_{n-2}x+a_n=0$ be the equation with real and different roots.

Let the roots be $\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n$ such that $|\alpha_1|>|\alpha_2|>|\alpha_3|>\cdots>|\alpha_n|$

From theory of equations in algebra the relation between the roots and coefficients are given by

$$\sum\alpha_i=\alpha_1+\alpha_2+\ldots+\alpha_n=-\frac{a_1}{a_0}$$

$$\sum_{1\leq i<j\leq n}\alpha_i\alpha_j=\alpha_1\alpha_2+\alpha_1\alpha_3+\ldots+\alpha_{n-1}\alpha_n=\frac{a_2}{a_0}$$

$$\sum_{1\leq i<j<k\leq n}\alpha_i\alpha_j\alpha_k=-\frac{a_3}{a_0}$$

$$\alpha_1\alpha_2\ldots\alpha_n=(-1)^n\frac{a_n}{a_0}$$

By repeated root-squaring, we shall assume that after m squarings, let the transformed equation be $b_0y^n + b_1y^{n-1} + b_2y^{n-2} + \cdots + b_{n-2}y^2 + b_{n-1}y + b_n = 0$ and let $\beta_1, \beta_2, \beta_3, \cdots, \beta_n$ be the roots

$$\therefore \qquad \beta_i = (\alpha_i)^{2^m}, \quad i = 1, 2, \ldots n$$

and β_i's are positive because β_i's are even powers of α_i.

Since $|\alpha_1| > |\alpha_2| > |\alpha_3| > \cdots > |\alpha_n|$, we have $\beta_1 \gg \beta_2 \gg \beta_3 \gg \ldots \gg \beta_n$ where $\gg$ means much greater than

$\therefore \dfrac{\beta_2}{\beta_1}, \dfrac{\beta_3}{\beta_1}, \dfrac{\beta_3}{\beta_2}, \ldots, \dfrac{\beta_n}{\beta_{n-1}}$, are negligible compared to unity.

By the relation between roots and coefficients we have,

$$\sum \beta_i = -\frac{b_1}{b_0}$$

$$\Rightarrow \qquad \beta_1 + \beta_2 + \beta_3 + \ldots + \beta_n = -\frac{b_1}{b_0}$$

$$\Rightarrow \qquad \beta_1\left(1 + \frac{\beta_2}{\beta_1} + \frac{\beta_3}{\beta_1} + \ldots + \frac{\beta_n}{\beta_1}\right) = -\frac{b_1}{b_0}$$

$$\sum \beta_i\beta_j = \frac{b_2}{b_0}$$

$$\Rightarrow \qquad \beta_1\beta_2 + \beta_1\beta_3 + \beta_1\beta_4 + \ldots = \frac{b_2}{b_0}$$

$$\Rightarrow \qquad \beta_1\beta_2\left(1 + \ldots\frac{\beta_3}{\beta_2} + \frac{\beta_4}{\beta_2} + \ldots\right) = \frac{b_2}{b_0}$$

Similarity

$$\beta_1\beta_2\beta_3\left(1 + \ldots\frac{\beta_4}{\beta_3} + \frac{\beta_5}{\beta_3} + \ldots\right) = -\frac{b_3}{b_0}$$

$$\vdots$$

$$\beta_1\beta_2\beta_3\cdots\beta_n = (-1)^n\frac{b_n}{b_0}$$

Since $\dfrac{\beta_2}{\beta_1}, \dfrac{\beta_3}{\beta_2}, \ldots$ are negligible compared to 1, these relations become

$$\beta_1 = -\frac{b_1}{b_0}, \qquad \beta_1\beta_2 = \frac{b_2}{b_0}, \qquad \beta_1\beta_2\beta_3 = -\frac{b_3}{b_0}, \ldots$$

$$\beta_1\beta_2\beta_3\cdots\beta_{n-1} = (-1)^{n-1}\frac{b_{n-1}}{b_0}$$

$$\beta_1\beta_2\beta_3\cdots\beta_n = (-1)^n\frac{b_n}{b_0}$$

$$\beta_1 = -\frac{b_1}{b_0},\ \beta_2 = -\frac{b_2}{b_1},\ \beta_3 = -\frac{b_3}{b_2},\cdots,\ \beta_{n-2} = -\frac{b_{n-2}}{b_{n-3}},\ \beta_{n-1} = -\frac{b_{n-1}}{b_{n-2}},\ \beta_n = -\frac{b_n}{b_{n-1}}$$

Since in the transformed equation $b_0, b_1, b_2, \ldots, b_n$ are known, β_i's are know n and

$$\beta_i = (\alpha_i^{2^m}) \quad \Rightarrow \alpha_i = \pm(\beta_i)^{\frac{1}{2^m}} \qquad (\because\ \beta_i \text{ is even power of } \alpha_i)$$

Hence α_i's are known for $i = 1, 2, 3, \ldots, n$.

Note:

(1) How many times root-squaring should be done cannot be decided in advance.
In practice, we continue the root squaring process until the doubled products in the second row of the table have no effect on the coefficients $b_0, b_1, \ldots, b_n$ of the next transformed equation.

(2) In numerical problems, the successive application of root squaring process greatly increases the magnitude of the coefficients. So we express the coefficients as a proper multiple of power of 10.

(3) To choose the proper sign of α_i, we substitute in the given equation and verify.

(4) To decide the number of positive roots and number of negative roots, we use Descarte's rule of sign. Hence, we can decide the sign of α_i.

"Descarte's rule of sign" says the number of positive roots of $f(x) = 0$ cannot exceed the number of changes of signs in the coefficients of $f(x)$ and the number of negative roots cannot exceed the number of changes of signs in the coefficients of $f(-x)$

For example: consider the equation $x^4 + 5x^3 + 3x^2 - 5x - 9 = 0$

Let
$$f(x) = x^4 + 5x^3 + 3x^2 - 5x - 9$$

Number of changes of sign is 1. So the number of positive roots is ≤ 1. i.e 0 or 1

$$f(-x) = x^4 - 5x^3 + 3x^2 + 5x - 9$$

Number of changes of sign is 3. So number of negative roots is ≤ 3

3. Finding a double root

After a certain number of root-squaring, if the magnitude of the coefficients bi is half the square of the magnitude of the corresponding coefficient in the previous equation, then it indicates that α_i is a double root of (1)

This double root can be found as below.

We have $\beta_i = -\frac{b_i}{b_{i-1}}$ and $\beta_{i+1} = -\frac{b_{i+1}}{b_i}$

Since for double root $\beta_{i+1} = \beta_i$, we get $\beta_i\ \beta_{i+1} = \beta_i . \beta_i = \beta_i^2$

$$\Rightarrow \quad \beta_i^2 = \frac{b_{i+1}}{b_{i-1}} \quad \Rightarrow \left(\alpha_i^{2^m}\right)^2 = \frac{b_{i+1}}{b_{i-1}}$$

which gives α_i with proper sign.

4. Finding complex roots

If the polynomial equation with real coefficients has complex roots, then it will occur in conjugate pairs.

ie. if $a+ib$ is a root, then $a-ib$ will also be a root.

When transforming an equation by root-squaring process, the presence of complex roots is inferred in two ways.

(1) The doubled products ($2a_0\, a_2$ etc) do not all di appear from the first row.

(2) The signs of some of the coefficients fluctuate as we continue with the transformation.

WORKED EXAMPLES

Example 1

Apply Graffe's method to find all the roots of the equation $x^3-6x^2+11x-6=0$.

Solution

The given equation is

$$x^3-6x^2+11x-6=0 \tag{1}$$

Let $$f(x)=x^3-6x^2+11x-6$$

The signs are + − + −

Since there are 3 changes of sign, the number of positive root is ≤ 3.

Now $$f(-x)=-x^3-6x^2-11x-6$$

The signs are − − − −

There is no change in sign.

So, there is no negative root.

Let $\alpha_1, \alpha_2, \alpha_3$ be the roots of (1)

$\therefore$
$$\Sigma\alpha_1=-\left(\frac{-6}{1}\right)=6>0$$

$$\Sigma\alpha_1\alpha_2=\frac{11}{1}=11>0$$

$$\alpha_1\,\alpha_2\,\alpha_3=-\frac{(-6)}{1}=6>0$$

Since $\Sigma\alpha_1$, $\Sigma\alpha_1\alpha_2$, and $\alpha_1\,\alpha_2\,\alpha_3$ are positive, all the roots are positive

We shall use Graffe's method to find the roots.

I Squaring
$$x^3-6x^2+11x-6=0$$

$\Rightarrow$ $$x^3+11x=6x^2+6$$

$\Rightarrow$ $$x(x^2+11)=6(x^2+1)$$

Squaring,

$$x^2(x^2+11)^2 = 36(x^2+1)^2$$

Put $y = x^2$, then we get

$$y(y+11)^2 = 36(y+1)^2$$

$\therefore$ $$y(y^2+22y+121) = 36(y^2+2y+1)$$

$\Rightarrow$ $$y^3+22y^2+121y = 36y^2+72y+36$$

$\Rightarrow$ $$y^3-14y^2+49y-36=0 \quad (2)$$

$\Rightarrow$

II Squaring

$$y^3+49y = 14y^2+36$$

$$y(y^2+49) = 2(7y^2+18)$$

$\Rightarrow$

Squaring

$$y^2(y^2+49)^2 = 2^2(7y^2+18)^2$$

Put $y^2 = z$, then we get

$\therefore$ $$z(z+49)^2 = 2^2(7z+18)^2$$

$\Rightarrow$ $$z(z^2+98y+2401) = 4(49z^2+252y+324)$$

$\Rightarrow$ $$z^3+98z^2+2401z = 196z^2+1008z+1296$$

$\Rightarrow$ $$z^3-98z^2+1393z-1296=0 \quad (3)$$

Now $z = y^2 = (x^2)^2 = x^4$

Since $\alpha_1, \alpha_2, \alpha_3$ are the roots of (1), $\alpha_1^4, \alpha_2^4, \alpha_3^4$ are the roots of (3)

Let $\beta_1, \beta_2, \beta_3$ be the roots of (3)

$\therefore$ $$\beta_1 = \alpha_1^4, \quad \beta_2 = \alpha_2^4, \quad \beta_3 = \alpha_3^4$$

Here $$b_0 = 1, \quad b_1 = -98, \quad b_2 = 1393, \quad b_3 = -1296$$

By Graffe's method,

$$\beta_1 = -\frac{b_1}{b_0}, \quad \beta_2 = -\frac{b_2}{b_1}, \quad \beta_3 = -\frac{b_3}{b_2}$$

$\therefore$ $$\alpha_1^4 = \beta_1 = -\frac{b_1}{b_0} = -(-98) = 98$$

$\Rightarrow$ $$\alpha_1 = \pm(98)^{\frac{1}{4}} = \pm 3.1463$$

$$\alpha_2^4 = \beta_2 = -\frac{b_2}{b_1} = \frac{1393}{98}$$

$\Rightarrow$ $$\alpha_2 = \pm\left(\frac{1393}{98}\right)^{\frac{1}{4}} = \pm(1.9417)$$

and $$\alpha_3^4 = \beta_3 = -\frac{b_3}{b_2} = \frac{1296}{1393}$$

$\Rightarrow$ $$\alpha_3 = \pm\left(\frac{1296}{1393}\right)^{\frac{1}{4}} = \pm 0.9821$$

Since all the roots are positive, we take $\alpha_1 = 3.1463$, $\alpha_2 = 1.9417$ and $\alpha_3 = 0.9821$.

Note: We find the actual roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$ are 1, 2, 3. ■

Example 2

Find all the roots of the equation $x^3 - 4x^2 + 5x - 2 = 0$ by Graffe's method, squaring thrice.

Solution

The given equation is $x^3 - 4x^2 + 5x - 2 = 0$ (1)

Let $$f(x) = x^3 - 4x^2 + 5x - 2$$

The signs are $+ - + -$

There are three changes of sign and so the number of positive roots is ≤ 3

Now $$f(-x) = -x^3 - 4x^2 - 5x - 2$$

The signs are $- - - -$

$\therefore$ no change in sign

$\therefore$ no negative root

Let $\alpha_1, \alpha_2, \alpha_3$ be the roots of (1), then

$$\sum \alpha_1 = -\left(\frac{-4}{1}\right) = 4 > 0$$

$$\sum \alpha_1\alpha_2 = \frac{5}{1} = 5 > 0$$

$$\alpha_1\alpha_2\alpha_3 = -\left(\frac{-2}{1}\right) = 2 > 0$$

$\therefore$ all the roots are positive and we shall find the roots by Graffe's method.

I Squaring

$$x^3 + 5x = 4x^2 + 2$$

$\Rightarrow$ $$x(x^2+5) = 2(2x^2+1)$$

Squaring we get

$$x^2(x^2+5)^2 = 4(2x^2+1)^2$$

put $y = x^2$, then we get

$$y(y+5)^2 = 4(2y+1)^2$$

$$y(y^2+10y+25)=4(4y^2+4y+1)$$

$\Rightarrow$ $$y^3+10y^2+25y=16y^2+16y+4$$

$\Rightarrow$ $$y^3-6y^2+9y-4=0$$

$\Rightarrow$

II Squaring

$$y^3+9y=6y^2+4$$

$$y(y^2+9)=2(3y^2+2)$$

$\Rightarrow$

Squaring, $$y^2(y^2+9)^2=4(3y^2+2)^2$$

Put $z=y^2$, then we get

$$z(z+9)^2=4(3z+2)^2$$

$$z(z^2+18z+81)=4(9z^2+12z+4)$$

$\Rightarrow$ $$z^3+18z^2+81z=36z^2+48z+16$$

Asad Ali

$\Rightarrow$ $$z^3-18z^2+33z-16=0$$

$\Rightarrow$

III Squaring

$$z^3+33z=18z^2+16$$

$$z(z^2+33)=2(9z^2+8)$$

$\Rightarrow$

Squaring, $$z^2(z^2+33)^2=4(9z^2+8)^2$$

Put $u=z^2$, then we get (2)

$$u(u+33)^2=4(9u+8)^2$$

$\Rightarrow$ $$u(u^2+66u+1089)=4(81u^2+144u+64)$$

$\Rightarrow$ $$u^3+66u^2+1089u=324u^2+576u+256$$

$\Rightarrow$ $$u^3-258u+513u-256=0$$

$\alpha_1, \alpha_2, \alpha_3$ are the roots of (1) and $\alpha_1^8, \alpha_2^8, \alpha_3^8$ are the roots of (2)

Let $\beta_1, \beta_2, \beta_3$ be the roots of (2)

$\therefore$ $$\beta_1=\alpha_1^8,\ \beta_2=\alpha_2^8,\ \beta_3=\alpha_3^8$$

Here $b_0=1$, $b_1=-258$, $b_2=513$, $b_3=-256$

By Graffe's method,

$$\beta_1=-\frac{b_1}{b_0},\quad \beta_2=-\frac{b_2}{b_1} \text{ and } \beta_3=-\frac{b_3}{b_2}$$

$\therefore$ $$\alpha_1^8=\beta_1=-\frac{b_1}{b_0}=-\frac{(-258)}{1}=258$$

$\Rightarrow$ $$\alpha_1=\pm(258)^{\frac{1}{8}}=\pm 2.0019$$

$$\alpha_2^8 = \beta_2 = -\frac{b_2}{b_1} = -\frac{513}{(-258)} = \frac{513}{258}$$

$\Rightarrow$
$$\alpha_2 = \pm\left(\frac{513}{258}\right)^{\frac{1}{8}} = \pm 1.0897$$

and
$$\alpha_3^8 = \beta_3 = -\frac{b_3}{b_1} = -\left(\frac{-256}{513}\right) = \frac{256}{513}$$

$\Rightarrow$
$$\alpha_3 = \pm\left(\frac{256}{513}\right)^{\frac{1}{8}} = \pm 0.09167$$

Since all the roots are positive,

$$\alpha_1 = 2.0019,\ \alpha_2 = 1.0897,\ \alpha_3 = 0.9167$$

Note: we find $\alpha_1 \simeq 2, \alpha_2 \simeq 1, \alpha_3 \simeq 1$ The actual roots are 2, 1, 1. ■

Example 3

Determine all the roots of the equation $12x^3 - 29x^2 + 23x - 6 = 0$ which are real and unequal by Graffe's method, correct to four significant figures.

Solution

The given equation is $12x^3 - 29x^2 + 23x - 6 = 0$ (1)

et
$$f(x) = 12x^3 - 29x^2 + 23x - 6$$
he signs are $+ - + -$
These are three changes of sign and so the number of positive roots is ≤ 3

Now
$$f(-x) = 12(-x)^3 - 29(-x)^2 + 23(-x) - 6$$
$$= -12x^3 - 29x^2 - 23x - 6$$

The signs are $- - - -$
There is no change in sign
$\therefore$ no negative root

$$\sum \alpha_1 = -\left(\frac{-29}{12}\right) = \frac{29}{12} > 0$$

$$\sum \alpha_1\alpha_2 = \frac{23}{12} > 0$$

$$\alpha_1\ \alpha_2\ \alpha_3 = -\left(\frac{-6}{12}\right) = \frac{6}{12} > 0$$

$\therefore$ all the roots are positive

Method I: work out as above and find the roots.

Method II: we shall tabulate the transformation as explained in the theory for illustration. We find the roots by Graffe's method.

Divide the coefficients of the equation (1) by 10 to reduce the magnitude of the coefficients
$\therefore$ the given equation becomes

$$1.2x^3 - 2.9x^2 + 2.3x - 0.6 = 0 \tag{2}$$

Let $\alpha_1, \alpha_2, \alpha_3$ be the roots of the given equation.

Let $\beta_1, \beta_2, \beta_3$ be the roots of the transformed equation with coefficients b_0, b_1, b_2, b_3.

$\therefore$ by Graffe's method

$$\beta_1 = -\frac{b_1}{b_0}, \quad \beta_2 = -\frac{b_2}{b_1}, \quad \beta_3 = -\frac{b_3}{b_2}$$

We find the b_i's by the following tabulating scheme

Root	x^3	x^2	x	Constant term
α	a_0 1.2	a_1 −2.9	a_2 2.3	a_3 −0.6
	1.44	−8.41 2(1.2)(2.3) = 5.52	5.29 (−2)(−2.9)(−0.6) = −3.48	−0.36
α^2	1.44	−2.89	1.81	−0.36
	2.0736	−8.3521 2(1.44)(1.81) = 5.2128	3.2761 −2(−2.89)(−0.36) = −2.0808	−0.1296
α^4	2.0736	−3.1393	1.1953	−0.1296
	4.2998	−9.8552 2(2.0736) (1.1953) = 4.9571	1.4287 −2(−3.1393) (−0.1296) = −0.8137	−0.0168
α^8	4.2998	−4.8981	0.615	−0.0168
	18.4883	−23.9914 2(4.2998) (0.615) = 5.2888	0.3782 −2(−4.8981) (−0.0168) = −0.1646	-2.8224×10^{-4}
α^{16}	18.4883	−18.7026	0.2136	-2.8224×10^{-4}
	341.8172	−349.7872 2(18.4883) (0.2136) =7.8982	0.456 −2(−18.7026) $(-2.8224)\times10^{-4}$ = −0.0106	-7.9659×10^{-8}
α^{32}	341.872 b_0	−341.889 b_1	0.035 b_2	-7.9659×10^{-8} b_3

Since $\alpha_1, \alpha_2, \alpha_3$ are the roots of given equation and $\beta_1, \beta_2, \beta_3$ are the roots of the transformed equation, we get

$$\beta_1 = \alpha_1^{32}, \quad \beta_2 = \alpha_2^{32}, \quad \beta_3 = \alpha_3^{32}$$

We have $b_0 = 341.8172$, $\quad b_1 = -341.889$, $\quad b_2 = 0.035$, $\quad b_3 = -7.9659 \times 10^{-8}$

$\therefore$
$$\alpha_1^{32} = \beta_1 = -\frac{b_1}{b_0} = -\frac{(-341.889)}{341.8172} = 1.0002$$

$\Rightarrow$
$$\alpha_1 = \pm(1.0002)^{\frac{1}{32}} = \pm 1.000006$$

$$\alpha_2^{32} = \beta_2 = -\frac{b_2}{b_1} = -\frac{0.035}{-(341.889)} = 0.0001$$

$\Rightarrow$
$$\alpha_1 = \pm(0.0001)^{\frac{1}{32}} = 0.7499$$

and
$$\alpha_3^{32} = \beta_3 = -\frac{b_3}{b_2} = -\frac{(-7.9659)}{0.035} = 0.0000023$$

$\Rightarrow$
$$\alpha_3 = \pm(0.0000023)^{\frac{1}{32}} = 0.6665$$

Since all the roots are positive, we take $\alpha_1 = 1$, $\quad \alpha_2 = 0.7499$, $\quad \alpha_3 = 0.6664$ ■

Example 4

Determine all the roots of the equation by Graffe's method.

$$\mathbf{x^4 - 10x^3 + 35x^2 - 50x + 24 = 0}$$

Solution

The given equation is

$$x^4 - 10x^3 + 35x^2 - 50x + 24 = 0 \qquad (1)$$

Let $\quad f(x) = x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$

The signs are + − + − +

Since there are four changes of sign, the number of positive root ≤ 4.

Now $\quad f(-x) = (-x)^4 - 10(-x)^3 + 35(-x)^2 - 50(-x) + 24$

$$= x^4 + 10x^3 + 35x^2 + 50x + 24$$

The signs are + + + +

There is no change of sign and so there is no negative root for (1)

So, all the roots are positive if all are real,

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the roots.

The $\Sigma\alpha_1 = 10 > 0$, $\Sigma\alpha_1\alpha_2 = 35 > 0$, $\Sigma\alpha_1\alpha_2\alpha_3 = 50 > 0$ and $\Sigma\alpha_1\alpha_2\alpha_3\alpha_4 = 24 > 0$

$\therefore \alpha_1, \alpha_2, \alpha_3, \alpha_4$ are positive.

We use Graffe's method to find the roots.

We shall find the roots by tabulating the transformations.

Let $\beta_1, \beta_2, \beta_3, \beta_4$ be the roots of the transformed equation.

By Graffe's method,

$$\beta_1 = -\frac{b_1}{b_0}, \quad \beta_2 = -\frac{b_2}{b_1}, \quad \beta_3 = -\frac{b_3}{b_2}, \text{ and } \quad \beta_4 = -\frac{b_4}{b_3}$$

where b_0, b_1, b_2, b_3, b_4 are the coefficients of the transformed equation.

Root		x^3	x^2	x	Constant term
α	a_0 1	a_1 -10	a_2 35	a_3 -50	a_4 24
	1	-100 $2a_0a_2 = 2\times 35$ $=70$	1225 $-2a_2a_3$ $= -2(-10)(-50)$ $= -1000$ $2a_0a_4$ $= 2.1.24 = 48$	-2500 $2a_2a_4$ $= 2\times 35\times 24$ $= 1680$	576
α^2	1	-30	273	-820	576
	1	-900 $2.1.273 = 546$	74529 $-2(-30)(-820) = -49200$ $2.1.576 = 1152$	-672400 $2\times 273\times 576$ $= 314496$	331776
α^4	1	-354 $= -0.0354\times 10^4$	26481 $= 0.2648\times 10^5$	-357904 $= -0.3579\times 10^6$	331776 $= 0.3318\times 10^6$
	1	-1.2532×10^5 $2.1(0.2648)\times 10^5$ $= 0.5296\times 10^5$	0.0701×10^{10} $-2\times(-0.0354)$ $\times 10^4(-0.3579)\times 10^6$ $= -0.0253\times 10^{10}$ $2.1(0.3318)\times 10^6$ $= 0.6636\times 10^6$	-12.8092×10^{10} $2(0.2648\times 10^5)$ (0.3318×10^6) $= 1.7572\times 10^{10}$	11.0091×10^{10}
α^8	1 b_0	-0.7236×10^5 b_1	0.0449×10^{10} b_2	-11.052×10^{10} b_3	11.0091×10^{10} b_4

Here $b_0 = 1$, $b_1 = -0.7236\times10^5$, $b_2 = 0.0449\times10^{10}$

$$b_3 = -11.052\times10^{10}, \quad b_4 = 11.0091\times10^{10}$$

Since $\beta_1, \beta_2, \beta_3, \beta_4$ are the roots of the transformed equation and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the roots of the given equation, we have

$$\beta_1 = \alpha_1^8, \quad \beta_2 = \alpha_2^8,$$
$$\beta_3 = \alpha_3^8 \text{ and } \beta_4 = \alpha^8$$

$$\therefore \quad \alpha_1^8 = \beta_1 = -\frac{b_1}{b_0} = -\frac{(-0.7236\times10^5)}{1} = 72360$$

$$\Rightarrow \quad \alpha_1 = \pm(72360)^{\frac{1}{8}} = \pm4.0498$$

$$\alpha_2^8 = \beta_2 = -\frac{b_2}{b_1} = \frac{-(0.0449\times10^{10})}{(-0.7236\times10^5)} = 6205.0857$$

$$\Rightarrow \quad \alpha_2 = \pm(6205.0857)^{\frac{1}{8}} = \pm2.9792$$

$$\alpha_3^8 = \beta_3 = -\frac{b_3}{b_2} = -\frac{(-11.052\times10^{10})}{0.0449\times10^{10}} = 246.1470$$

$$\alpha_3 = \pm(246.1470)^{\frac{1}{8}} = \pm1.9902$$

and
$$\alpha_4^8 = \beta_4 = -\frac{b_4}{b_3} = -\frac{11.0091\times10^{10}}{(-11.052)\times10^{10}} = 0.9961$$

$$\Rightarrow \quad \alpha_4 = \pm(0.9961)^{\frac{1}{8}} = \pm0.9995$$

Since the roots are positive, we take

$$\alpha_1 = 4.0498, \ \alpha_2 = 2.9792, \ \alpha_3 = 1.9902, \ \alpha_4 = 0.9995$$

Note: we find $\alpha_1 = 4, \quad \alpha_2 = 3, \quad \alpha_3 = 2, \quad \alpha_4 = 1$

It can be seen that 4,3,2,1 are the actual roots. ■

Exercises 2.5

Using Graffe's method find all the roots of the following equation.

(1) $x^3 - 8x^2 + 17x - 10 = 0$

(2) $x^3 - 3x + 1 = 0$

(3) $x^3 - 9x + 18x - 6 = 0$

Answers 2.5

(1) 5.0004, 2.0008, 0.9995 i.e approximately 5, 2, 1

(2) – 1.879, 1.532, 0.347

(3) 6.2902, 2.2942, 0.4158

Lin-Bairstow's Method

Bairstow's method is useful for finding the complex roots of a polynomial equation $f(x)=0$ with real coefficients.

We know that for a polynomial equation with real coefficients, complex roots occur in conjugate pairs.

That is, if $\alpha+i\beta$ is a root then $\alpha-i\beta$ will also be a root.

So, $x-(\alpha+i\beta)$ and $x-(\alpha-i\beta)$ are factors of $f(x)$.

$\Rightarrow$ $[x-(\alpha+i\beta)][x-(\alpha-i\beta)]$ is a factor of $f(x)$.

$\Rightarrow$ $(x-\alpha-i\beta)(x-\alpha+i\beta)$ is a factor of $f(x)$.

$\Rightarrow$ $(x-\alpha)^2+\beta^2=x^2-2\alpha x+(\alpha^2+\beta^2)$ is a factor of $f(x)$.

Thus each pair of complex roots corresponds to a quadractic factor of the form x^2+px+q.

Bairstow's method is used to find a quadractic factor of the form x^2+px+q by an iterative way. For the sake of simplicity we shall explain the method for a polynomial equation of fourth degree, which can be extended to polynomial equation of any degree n.

Consider the equation

$$a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0 \quad (1)$$

Let $$f(x)=a_0x^4+a_1x^3+a_2x^2+a_3x+a_4$$

Our aim is to find a factor of the form x^2+px+q.

When $f(x)$ is divided by x^2+px+q, the quotient will be a second degree polynomial of the form $b_0x^2+b_1x+b_2$ and the remainder will be a first degree polynomial of the form $Rx+S$.

Then $$f(x)=(x^2+px+q)(b_0x^2+b_1x+b_2)+Rx+S$$

$$\Rightarrow \quad a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=(x^2+px+q)(b_0x^2+b_1x+b_2)+Rx+S \quad (2)$$

Equating like coefficients in (2), we get

$$a_0=b_0,\quad a_1=b_0p+b_1 \qquad a_2=b_0q+b_1p+b_2,$$
$$a_3=b_1q+b_2p+R \quad \text{and} \quad a_4=b_2q+S$$

$$\Rightarrow \quad \begin{aligned} b_0&=a_0\\ b_1&=a_1-b_0p=a_1-pa_0\\ b_2&=a_2-b_1p-b_0q=a_2-p(a_1-pa_0)-qa_0\\ R&=a_3-b_1q+b_2p\\ &=a_3-p(a_2-pa_1+p^2a_0-qa_0)-q(a_1-pa_0)\end{aligned}$$

$$S = a_4 - b_2 q = a_4 - \left(a_2 - pa_1 + p^2 a_0 - qa_0\right)q \qquad (3)$$

where the ai's are known.

$\therefore$ R and S are non-linear functions of p and q.

$x^2 + px + q$ will be a factor of $f(x)$ if the remainder $Rx + S = 0 \Rightarrow R = 0$ and $S = 0$.

Thus, the problem of finding quadractic factor is reduced to find p and q such that

$$R(p,q) = 0 \text{ and } S(p,q) = 0 \qquad (4)$$

which are non-linear equations in p, q

The method used to solve these equations is similar to Newton-Raphson method for simultaneous equations.

Suppose (p_0, q_0) is an initial approximation for (p,q), then $(p_0 + \Delta p, q_0 + \Delta q)$ is the true solution.

$\therefore$
and
$$\left.\begin{aligned} R(p_0 + \Delta p,\ q_0 + \Delta q) = 0 \\ S(p_0 + \Delta p,\ q_0 + \Delta q) = 0 \end{aligned}\right\} \qquad (5)$$

By Taylor's series expansion, we get

$$R(p_0, q_0) + \frac{1}{1!}\left(\frac{\partial R}{\partial p}\Delta p + \frac{\partial R}{\partial q}\Delta q\right) + \ldots = 0$$

and

$$S(p_0, q_0) + \frac{1}{1!}\left(\frac{\partial S}{\partial p}\Delta p + \frac{\partial S}{\partial q}\Delta q\right) + \ldots = 0$$

Where the partial derivatives are evaluated at (p_0, q_0).

Assuming $\Delta p, \Delta q$ are small and neglecting other terms, we get

$$\frac{\partial R}{\partial p}\Delta p + \frac{\partial R}{\partial q}\Delta q = -R(p_0, q_0) \qquad (6)$$

and

$$\frac{\partial S}{\partial p}\Delta p + \frac{\partial S}{\partial q}\Delta q = -S(p_0, q_0) \qquad (7)$$

Treating as a system of linear equations in Δp, Δq and solving (6) and (7)

by Cramer's rule, we get

$$\Delta p = \frac{S\dfrac{\partial R}{\partial q} - R\dfrac{\partial S}{\partial q}}{\dfrac{\partial R}{\partial p}\cdot\dfrac{\partial S}{\partial q} - \dfrac{\partial R}{\partial q}\cdot\dfrac{\partial S}{\partial p}}$$

and

$$\Delta q = \frac{R\dfrac{\partial S}{\partial p} - S\dfrac{\partial R}{\partial p}}{\dfrac{\partial R}{\partial p}\cdot\dfrac{\partial S}{\partial q} - \dfrac{\partial R}{\partial q}\cdot\dfrac{\partial S}{\partial p}} \qquad (8)$$

where the functions and derivatives on the R.H.S are evaluated at (p_0, q_0)

Let (p_1, q_1) be the first approximation for (p, q), then

$$p_1 = p_0 + \Delta p, \quad q_1 = q_0 + \Delta q$$

Now using (p_1, q_1) we get the second approximation $p_2 = p_1 + \Delta p$, $q_2 = q_1 + \Delta q$ where Δp and Δq are evaluated at (p_1, q_1) and so on.

We stop at the stage when two successive approximations coincide upto the desired degree of accuracy.

Thus by iteration we find the best possible factor of $x^2 + px + q$.

The equation $x^2 + px + q = 0$ gives a pair of roots of $f(x) = 0$.

The roots will be complex if $p^2 - 4q < 0$, otherwise the roots will be real.

As in the case of Graeffe's method, here also we can exhibit the computations schematically in a table.

For the sake of uniformity of suffix, we assume $R = b_3$ and $S = b_4 + pb_3$.

Then the equations (3) can be rewritten as

$$b_0 = a_0, \quad b_1 = a_1 - pb_0, \quad b_2 = a_2 - pb_1 - qb_0$$
$$b_3 = a_3 + pb_2 - qb_1, \quad b_4 = a_4 - pb_3 - qb_2.$$

These equations can be written as a single recurrence relation

$$b_r = a_r \mp pb_{r-1} - q\,b_{r-2} \qquad r = 0,1,2,3,4 \tag{9}$$

where $\quad b_{-1} = 0, \quad b_{-2} = 0 \quad$ and $\quad b_0 = a_0.$

Differentiating (9) partially w.r.to p and q, we get

$$\therefore \quad \frac{\partial b_r}{\partial p} = -b_{r-1} - p.\frac{\partial b_{r-1}}{\partial p} - q.\frac{\partial b_{r-2}}{\partial p} \tag{10}$$

$$\frac{\partial b_0}{\partial p} = 0, \quad \frac{\partial b_{-1}}{\partial p} = 0, \quad \frac{\partial b_{-2}}{\partial p} = 0$$

and

$$\frac{\partial b_r}{\partial q} = -p\frac{\partial b_{r-1}}{\partial q} - q.\frac{\partial b_{r-2}}{\partial q} - b_{r-2} \tag{11}$$

$$\frac{\partial b_0}{\partial q} = 0, \quad \frac{\partial b_{-1}}{\partial q} = 0, \quad \frac{\partial b_{-2}}{\partial q} = 0$$

If r = 1,

$$\frac{\partial b_1}{\partial q} = -p\frac{\partial b_0}{\partial q} - q\frac{\partial b_{-1}}{\partial q} - b_{-1} = 0$$

$$\frac{\partial b_1}{\partial p} = -b_0 - p\frac{\partial b_0}{\partial p} - q\,\frac{\partial b_{-1}}{\partial p} = -a_0 - 0 - 0 = -a_0$$

$$\frac{\partial b_2}{\partial q} = -p\frac{\partial b_1}{\partial q} - q\,\frac{\partial b_0}{\partial q} - b_0 = 0 - 0 - a_0 = -a_0$$

$$\therefore \quad \frac{\partial b_0}{\partial p} = \frac{\partial b_1}{\partial q} \text{ and } \frac{\partial b_1}{\partial p} = \frac{\partial b_2}{\partial q}, \ldots$$

In general, $\dfrac{\partial b_r}{\partial p} = \dfrac{\partial b_{r+1}}{\partial q}$, $\quad r = 0,1,2,3,4$

We shall assume $\dfrac{\partial b_r}{\partial p} = \dfrac{\partial b_{r+1}}{\partial q} = -c_{r-1}$

$\therefore$ $\dfrac{\partial b_r}{\partial p} = -c_{r-1}$ and $\dfrac{\partial b_r}{\partial q} = -c_{r-2}$

$\therefore (10) \Rightarrow \quad -c_{r-1} = -b_{r-1} - p(-c_{r-2}) - q(-c_{r-3})$

$\Rightarrow \quad c_{r-1} = b_{r-1} - pc_{r-2} - qc_{r-3}$

and $(11) \Rightarrow \quad -c_{r-2} = -p(-c_{r-3}) - q(-c_{r-4}) - b_{r-2}$

$\Rightarrow \quad c_{r-2} = b_{r-2} - pc_{r-3} - qc_{r-4}$

These two equations can be rewritten as a single recurrence relation

$$c_r = b_r - pc_{r-1} - qc_{r-2}, \qquad r = 0,1,2,3$$

Where $\quad c_0 = b_0 (= a_0), \qquad c_{-1} = 0, \qquad c_{-2} = 0$

Since $\quad R = b_3, \quad \dfrac{\partial R}{\partial p} = \dfrac{\partial b_3}{\partial p} = -c_2$ and $\dfrac{\partial R}{\partial q} = \dfrac{\partial b_3}{\partial q} = -c_1$

Since $\quad S = b_4 + p b_3,$

$$\frac{\partial S}{\partial p} = \frac{\partial b_4}{\partial p} + p\frac{\partial b_3}{\partial p} + b_3 = -c_3 - pc_2 + b_3 = b_3 - c_3 - pc_2$$

and

$$\frac{\partial S}{\partial q} = \frac{\partial b_4}{\partial q} + p\frac{\partial b_3}{\partial q} = -c_2 - pc_1$$

Using these relations in (8), we get

$$\Delta p = \frac{(b_4 + pb_3)(-c_1) - b_3(-c_2 - pc_1)}{(-c_2)(-c_2 - pc_1) - (-c_1)(b_3 - c_3 - pc_2)}$$

$$= \frac{-b_4c_1 - pb_3c_1 + b_3c_2 + pb_3c_1}{c_2^2 + pc_1c_2 + b_3c_1 - c_1c_3 - pc_1c_2} = \frac{b_3c_2 - b_4c_1}{c_2^2 + c_1(b_3 - c_3)}$$

and

$$\Delta q = \frac{b_3(b_3 - c_3 - pc_2) - (b_4 + pb_3)(-c_2)}{c_2^2 + c_1(b_3 - c_3)}$$

$$= \frac{b_3^2 - b_3c_3 + b_4c_2}{c_2^2 + c_1(b_3 - c_3)} = \frac{b_4c_2 + b_3(b_3 - c_3)}{c_2^2 + c_1(b_3 - c_3)}$$

$\therefore$ the first approximation for p and q are $p_1 = p_0 + \Delta p$, $q_1 = q_0 + \Delta q$.

Same way with p_1, q_1 as initial values.

We get $p_2 = p_1 + \Delta p$, $q_2 = q_1 + \Delta q$ and so on.

We stop at the stage where two successive approximations are the same upto the desired degree of accuracy.

At each iteration Δp and Δq must be computed which are interms of b_r's and c_r's.

These can be computed by the following scheme, which is similar to synthetic division.

	a_0	a_1	a_2	a_3	a_4
$-p$	–	$-pb_0$	$-pb_1$	$-pb_2$	$-pb_3$
$-q$	–	–	$-qb_0$	$-qb_1$	$-qb_2$
	$b_0(=a_0)$	b_1	b_2	b_3	b_4
$-p$	–	$-pc_0$	$-pc_1$	$-pc_2$	–
$-q$	–	–	$-qc_0$	$-qc_1$	–
	$c_0(=a_0)$	c_1	c_2	c_3	

(i) The first row elements are the coefficients of the given equation. p, q are the current values of actual 'p' and 'q'.

(ii) Second row entries (from second column onwards) are obtained by multiplying b_0, b_1, b_2, b_3 by $-p$ respectively.

(iii) The third row elements (from third column onwards) are obtained by multiplying b_0, b_1, b_2 by $-q$. b_0, b_1, b_2, b_3, b_4 are the column totals.

(iv) Again proceed as above for the fifth and sixth rows (upto b_3), since the last c value is c_3.

This method can be adapted for the derivation of finding quadractic factor of an n^{th} degree equation.

$$a_0x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1}x + a_n = 0$$

The corresponding formula will be

$$\Delta p = \frac{b_{n-1}c_{n-2} - b_n c_{n-3}}{c_{n-2}^2 + c_{n-3}\left(b_{n-1} - c_{n-1}\right)}$$

and

$$\Delta q = \frac{b_n c_{n-2} + b_{n-1}\left(b_{n-1} - c_{n-1}\right)}{c_{n-2}^2 + c_{n-3}\left(b_{n-1} - c_{n-1}\right)}$$

The scheme for finding b's and c's are shown below.

	a_0	a_1	a_2	$\cdots$	a_{n-2}	a_{n-1}	a_n
$-p$	–	$-pb_0$	$-pb_3$	$\cdots$	$-pb_{n-3}$	$-pb_{n-2}$	$-pb_{n-1}$
$-q$	–	–	$-qb_0$	$\cdots$	$-qb_{n-4}$	$-qb_{n-3}$	$-qb_{n-2}$
	$b_0(=a_0)$	b_1	b_2	$\cdots$	b_{n-2}	b_{n-1}	b_n
$-p$	–	$-pc_0$	$-pc_1$	$\cdots$	$-pc_{n-3}$	$-pc_{n-2}$	–
$-q$	–	–	$-qc_0$	$\cdots$	$-qc_{n-4}$	$-c_{n-3}$	–
	$c_0(=a_0)$	c_1	c_2	$\cdots$	c_{n-2}	c_{n-1}	

When $n=4$, we get Δp and Δq as earlier.
For a polynomial of degree 3, put $n=3$ in Δp and Δq.

$$\therefore \qquad \Delta p = \frac{b_2c_1 - b_3c_0}{c_1^2 + c_0(b_2 - c_2)} \quad \text{and} \quad \Delta q = \frac{b_3c_1 + b_2(b_2 - c_2)}{c_1^2 + c_0(b_2 - c_2)}$$

b's and c's are obtained as above.

WORKED EXAMPLES

Example 1

Apply Bairstow's method to find a quadractic factor close to $x^2 + 2x + 2$ for the equation $x^4 - 3x^3 + 20x^2 + 44x + 54 = 0$. Do one step only.

Solution

The given equation is $\quad x^4 - 3x^3 + 20x^2 + 44x + 54 = 0$

Here $\quad a_0 = 1, a_1 = -3, a_2 = 20, a_3 = 44, a_4 = 54$

To find the quadractic factor close to $x^2 + 2x + 2$.

$$\therefore \qquad p_0 = 2,\ q_0 = 2.$$

By Bairstow's method,

$$\Delta p = \frac{b_3c_2 - b_4c_1}{c_2^2 + c_1(b_3 - c_3)} \quad \text{and} \quad \Delta q = \frac{b_4c_2 + b_3(b_3 - c_3)}{c_2^2 + c_1(b_3 - c_3)}$$

The b's and c's are computed by the following scheme.

	a_0	a_1	a_2	a_3	a_4
	1	–3	20	44	54
$-p_0 = -2$	–	–2	$(-2)(-5) = 10$	$(-2)(28) = -56$	$(-2)(-2) = 4$
$-q_0 = -2$	–	–	–2	$(-2)(-5) = 10$	$(-2)(28) = -56$
	$b_0 = 1$	$b_1 = -5$	$b_2 = 28$	$b_3 = -2$	$b_4 = 2$
$-p_0 = -2$	–	–2	$(-2)(-7) = 14$	$(-2)(40) = -80$	–
$-q_0 = -2$	–	–	$(-2)1 = -2$	$(-2)(-7) = 14$	–
	$c_0 = 1$	$c_1 = -7$	$c_2 = 40$	$c_3 = -68$	

$\therefore \qquad b_3 = -2, \quad b_4 = 2, \quad c_1 = -7, \quad c_2 = 40, \quad c_3 = -68$

$\therefore$
$$\Delta p = \frac{(-2)40 - 2(-7)}{40^2 + (-7)(-2+68)} = \frac{-66}{1138} = -0.058$$

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and
$$\Delta q = \frac{2\times 40 + (-2)(-2+68)}{40^2 + (-7)(-2+68)} = \frac{-52}{1138} = -0.04569$$

$\therefore$
$$p_1 = p_0 + \Delta p = 2 - 0.058 = 1.942$$
$$q_1 = q_0 + \Delta q = 2 - 0.04569 = 1.9543$$

$\therefore$ the quadractic factor is

$$\mathbf{x^2 + 1.942x + 1.9543}$$

■

Example 2

Apply Lin-Bairstow's method to find the quadractic factor of the equation $x^4 + 5x^3 + 3x^2 - 5x - 9 = 0$ close to $x^2 + 3x - 5$.

Solution

The given equation is $x^4 + 5x^3 + 3x^2 - 5x - 9 = 0$

Here $a_0 = 1, \quad a_1 = 5, \quad a_2 = 3, \quad a_3 = -5, \quad a_4 = -9$

Required quadractic factor is close to $x^2 + 3x - 5$

$\therefore \qquad p_0 = 3$, and $q_0 = -5$

By Bairstow's method,

$$\Delta p = \frac{b_3c_2 - b_4c_1}{c_2^2 + c_1(b_3 - c_3)} \quad \text{and} \quad \Delta q = \frac{b_4c_2 + b_3(b_3 - c_3)}{c_2^2 + c_1(b_3 - c_3)}$$

The b's and c's are computed by the following scheme.

I iteration

	a_0	a_1	a_2	a_3	a_4
	1	5	3	-5	-9
$-p=-3$	–	-3	$(-3)(2)=-6$	$(-3)(2)=-6$	$(-3)(-1)=3$
$-q=5$	–	–	$5\times1=5$	$5\times2=10$	$5\times2=10$
	$b_0=1$	$b_1=2$	$b_2=2$	$b_3=-1$	$b_4=4$
$-p=-3$	–	-3	$(-3)(-1)=3$	$(-3)\times10=-30$	–
$-q=5$	–	–	$5\times1=5$	$5(-1)=-5$	–
	$c_0=1$	$c_1=-1$	$c_2=10$	$c_3=-36$	

$\therefore$ $$b_3=-1,\quad b_4=4,\quad c_1=-1,\quad c_2=10,\quad c_3=-36$$

$\therefore$ $$\Delta p=\frac{(-1)10-4(-1)}{10^2+(-1)(-1+36)}=\frac{-10+4}{100-35}=\frac{-6}{65}=-0.0923$$

and $$\Delta q=\frac{4\times10+(-1)(-1+36)}{10^2+(-1)(-1+36)}=\frac{40-35}{65}=\frac{5}{65}=-0.0769$$

$\therefore$ **the first approximation is**

$$p_1=p_0+\Delta p=3-0.0923=2.9077$$
$$q_1=q_0+\Delta q=-5+0.0769=-4.9231$$

We repeat the process with the new p and q [ie with p_1 and q_1]

II iteration

	a_0	a_1	a_2	a_3	a_4
	1	5	3	-5	-9
$-p_1=-2.9077$	–	-2.9077	$(-2.9077)\times(2.0923)=-6.0838$	$(-2.9077)\times(1.8393)=-5.3481$	$(-2.9077)\times(-0.0475)=0.1381$
$-q_1=4.9231$	–	–	4.9231	$4.9231\times2.0923=10.3006$	$4.9231\times1.8393=9.0551$
	$b_0=1$	$b_1=2.0923$	$b_2=1.8393$	$b_3=-0.0475$	$b_4=0.1932$
$-p_1=-2.9077$	–	-2.9077	$-2.9077\times-0.8154=2.3709$	$(-2.9077)\times9.1333=-26.5569$	–
$-q_1=4.9231$	–	–	4.9231	$4.9231\times(-0.8454)=-4.0143$	–
	$c_0=1$	$c_1=-0.8154$	$c_2=9.1333$	$c_3=-30.6187$	

Here
$$b_3 = -0.0475, \quad b_4 = 0.1932$$
$$c_1 = -0.8154, \quad c_2 = 9.1333, \quad c_3 = -30.6187$$

$\therefore$
$$\Delta p = \frac{(-0.0475)(9.1333) - (0.1932)(-0.8154)}{(9.1333)^2 + (-0.8154)(-0.0475 + 30.6187)}$$
$$= \frac{-0.4338 + 0.1575}{83.4172 - 24.9278} = -\frac{0.2763}{58.4894} = -0.0047$$

and
$$\Delta q = \frac{0.1932 \times 9.1333 + (-0.0475)(-0.0475 + 30.6187)}{(9.1333)^2 + (-0.8154)(-0.0475 + 30.6187)}$$
$$= \frac{1.7646 - 1.4521}{58.4894} = \frac{0.3125}{58.4894} = 0.0053$$

$\therefore$ the second approximation is

$$p_2 = p_1 + \Delta p = 2.9077 + (-0.0047) = 2.903$$
$$q_2 = q_1 + \Delta q = -4.9231 + 0.0053 = -4.9178$$

Since the two approximations are correct to three significant figures, we take

$$p = 2.903, \quad q = -4.9178$$

$\therefore$ the quadractic factor is

$$\mathbf{x^2 + 2.903x - 4.9178.}$$ ■

Example 3

Find the complex roots of the equation $x^3 - 2x^2 + x - 2 = 0$ approximately by two iterations, by Bairstow's method.

Solution

The given equation is $x^3 - 2x^2 + x - 2 = 0$

Here $a_0 = 1, \quad a_1 = -2, \quad a_2 = 1, \quad a_3 = -2$

To find the complex roots, we have to find the corresponding quadractic factor $x^2 + px + q$.

We shall find it by Bairstow's iteration method starting with $p_0 = 0.5$ and $q_0 = 1$.

Putting $n = 3$ in the general formula, we have
By Bainstow's method,

$$\Delta p = \frac{b_2 c_1 - b_3 c_0}{c_1^2 + c_0 (b_2 - c_2)} \text{ and } \Delta q = \frac{b_3 c_1 + b_2 (b_2 - c_2)}{c_1^2 + c_0 (b_2 - c_2)}$$

The b's and c's are found by the following scheme.

I iteration

	a_0	a_1	a_2	a_3
	1	-2	1	-2
$-p_0 = -0.5$	–	-0.5	$(-0.5)(-2.5) = 1.25$	$(-0.5)(1.25) = -0.625$
$-q_0 = -1$	–	–	-1	$(-1)(-2.5) = 2.5$
	$b_0 = 1$	$b_1 = -2.5$	$b_2 = 1.25$	$b_3 = -0.125$
$-p_0 = -0.5$	–	-0.5	$(-0.5)(-3) = 1.5$	
$-q_0 = -1$	–	–	-1	–
	$c_0 = 1$	$c_1 = (-3)$	$c_2 = 1.75$	

Here $\quad b_2 = 1.25 \qquad b_3 = -0.125$

$c_0 = 1, \qquad c_1 = -3, \qquad c_2 = 1.75$

$$\therefore \quad \Delta p = \frac{(1.25)(-3)-(-0.125)\times 1}{(-3)^2+1(1.25-1.75)}$$

$$= \frac{-3.75+0.125}{9-0.5} = \frac{-3.625}{8.5} = -0.4265$$

and

$$\Delta q = \frac{(-0.125)(-3)+1.25(1.25-1.75)}{(-3)^2+1(-1.25-1.75)}$$

$$= \frac{0.375-0.625}{8.5} = \frac{0.25}{8.5} = 0.0294$$

$\therefore$ **the first approximation is**

$$p_1 = p_0 + \Delta p = 0.5 - 0.4265 = 0.0735$$
$$q_1 = q_0 + \Delta q = 1 - 0.0294 = 0.9706$$

We repeat the process with p_1 and q_1

II iteration

	a_0	a_1	a_2	a_3
	1	−2	1	−2
$-p_1 = -0.0735$	–	−0.0735	$(-0.0735)\times(-2.0735) = 0.1524$	$(-0.0735)\times(0.1818) = -0.0134$
$-q_1 = -0.9706$	–		$(-0.9706)\times 1 = -0.9706$	$(-0.9706)\times(-2.0735) = 2.0125$
	$b_0 = 1$	$b_1 = -2.0735$	$b_2 = 0.1818$	$b_3 = -0.0009$
$-p_1 = -0.0735$	–	0.0735	$(-0.735)\times(-2.147) = 0.1578$	–
$-q_1 = -0.9706$	–		$-0.9706\times 1 = -0.9706$	
	$c_0 = 1$	$c_1 = -2.147$	$c_2 = -0.631$	

Here $b_2 = 0.1818, \quad b_3 = -0.0009$

$c_0 = 1, \quad c_1 = -2.147, \quad c_2 = -0.631$

$\therefore$
$$\Delta p = \frac{(0.1818)(-2.147) - (-0.0009)\times 1}{(-2.147)^2 + 1(0.1818 + 0.631)}$$
$$= \frac{-0.3903 + 0.0009}{4.6096 + 0.8128} = -\frac{0.3894}{5.4224} = -0.0718$$

and
$$\Delta q = \frac{(-0.0009)(-2.147) - 0.1818(0.1818 + 0.631)}{(-2.147)^2 + 1(0.1818 + 0.631)}$$
$$= \frac{(0.0019 + 0.1478)}{5.4224} = \frac{0.1497}{5.4224} = 0.0276$$

$\therefore$ **the second approximation is**

$$p_2 = p_1 + \Delta p = 0.0735 - 0.0718 = 0.0017$$
$$q_2 = q_1 + \Delta q = 0.9706 + 0.0276 = 0.9982$$

We shall take $p = 0$ and $q = 1$

$\therefore$ the quadractic factor is $x^2 + 0x + 1 = x^2 + 1$

So, the complex roots are given by $x^2 + 1 = 0$

$\Rightarrow \qquad x^2 = -1$

$\Rightarrow \qquad x = \pm i \Rightarrow x = i, -i$

$\therefore$ the complex roots are $x = i$ and $x = -i$

Note: The actual root are 2, i, $-i$

Exercises 2.6

(1) Apply Bairstow's method to $x^4+5x^3+3x^2-5x-9=0$ to find a quadractic factor. Close to x^2-3x+5. Do one step only.

(2) Using Bairstow's method twice, find a quadractic factor of $x^4-5x^3+20x^2-40x+60=0$ given $p_0=4$ and $q_0=8$.

(3) Find the quadractic factor of $x^3+x^2-x+2=0$ doing two iterations of Bairstow's method with $p_0=-0.9$ and $q_0=0.9$.

(4) Using Bairstow's method twice extract a quadractic factor of the equation $x^4 - 3x^3 + 20x^2 + 44x + 54 = 0$ with $p_0 = 2, q_0 = 2$.

Answers 2.6

(1) $x^2 - 1.8549x + 1.7045$

(2) $x^2 - 3.8279x + 7.3064$

(3) $x^2 - x + 1$

(4) $x^2 + 1.9413x + 1.9538$

SHORT ANSWER QUESTIONS

1. How to reduce the number of iterations while finding the root of an equation by Regular falsi method?
2. State the formula to find a root of $f(x) = 0$ which lies in the interval [a, b] by Regular falsi method.
3. What is the condition for convergence in the fixed point iteration $x=f(x)$.
4. What is the order of convergence of the fixed point method?
5. To find the smallest positive root of $x^3-x-1=0$ by the iteration method, it should be written as $x=f(x)$. What is the suitable $f(x)$? Justify.
6. State Newton-Raphson iteration formula to find a root of $f(x) = 0$.
7. Newton-Raphson iteration method is also known as method of tangents. Why?
8. What is the order of convergence of Newton-Raphson method?
9. Write down the condition for convergence of Newton-Raphson method for $f(x) = 0$.

10. What are the merits of Newton's method of iteration?
11. Show that Newton-Raphson formula to find $\sqrt{a}$, $a > 0$, can be expressed in the form $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$, $n = 0,1,2,\ldots$
12. What is the order of convergence of secant method.
13. What is the order of convergence of Regula falsi method.
14. Derive Newton's algorithm for finding the P^{th} root of a positive integer N.
15. Prove that Newton's iterative formula for finding the reciprocal of N is $x_{n+1}=x_n(2-Nx_n)$.
16. When we should not use Newton-Raphson method?
17. How will you find the negative root of a polynomial equation by Horner's method?
18. Explain briefely the underlying principle in Graffe's root squaring method of solving a polynomial equation $f(x) = 0$.

Solution of System of Linear Algebraic Equations

INTRODUCTION

Very often we encounter simultaneous linear equations in various fields of science and engineering. We have seen the analysis of electronic circuits consisting of invariant elements ultimately depend on the solution of such equations by determinant and matrix methods. These methods become tedious for large systems of equations. To solve such equations there are numerical methods which are suitable for computations using computers. The methods of solution of linear algebraic equations may broadly be classified into two types, namely, direct methods and indirect methods.

Direct methods produce exact solution after a finite number of steps (disregarding round off errors). So direct method is also known as exact method.

Indirect methods give a sequence of approcimate solutions, which ultimately approach the actual solution. Iterative method is some times referred to as indirect method.

In this chapter, we also consider eigen value problems and method of factorisation or method of triangularisation to solve the system of equations.

DIRECT METHODS

Matrix Inverse Method

Consider the system of n linear equations in n variables $x_1, x_2, x_3, \ldots, x_n$

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\
\vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots & \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n &= b_n
\end{aligned} \tag{1}$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Then the system (1) can be written as a single matrix equation as

$$AX = B \tag{2}$$

If A is non-singular, then $|A| \neq 0$ and the inverse matrix A^{-1} exists and $A^{-1} = \dfrac{\text{adj } A}{|A|}$

where
$$\text{adj } A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^T$$

$A_{ij} = (-1)^{i+j} M_{ij}$ and M_{ij} is the minor of a_{ij} in $|A|$.

Pre multiplying (2) by A^{-1}, we get

$$\begin{aligned}
& A^{-1}(AX) = A^{-1}B \\
\Rightarrow \quad & (AA^{-1})X = A^{-1}B \\
\Rightarrow \quad & I\,X = A^{-1}B \\
\Rightarrow \quad & X = A^{-1}B \text{ is the solution of (1)}
\end{aligned}$$

WORKED EXAMPLES

Example 1

Solve the system of equations $2x + 3y + z = 9$; $x + 2y + 3z = 6$; $3x + y + 2z = 8$.

Solution

Given system is

$$2x + 3y + z = 9$$
$$x + 2y + 3z = 6$$
$$3x + y + 2z = 8$$

Here

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} \text{ and } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Now

$$|A| = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$
$$= 2(4-3) - 3(2-9) + 1(1-6) = 2 + 21 - 5 = 18 \neq 0$$

$\therefore A^{-1}$ exists and

$$A^{-1} = \frac{adj\ A}{|A|}$$

Now

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$A_{11} = (-1)^2 \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1.(4-3) = 1$$

$$A_{12} = (-1)^3 \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = -(2-9) = 7$$

$$A_{13} = (-1)^4 \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1 - 6 \quad = -5$$

$$A_{21} = (-1)^3 \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = -(6-1) = -5$$

$$A_{22} = (-1)^4 \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 4 - 3 \quad = 1$$

$$A_{23} = (-1)^5 \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = -(2-9) = 7$$

$$A_{31} = (-1)^4 \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} = 9 - 2 = 7$$

$$A_{32} = (-1)^5 \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = -(6-1) = -5$$

$$A_{33} = (-1)^6 \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4 - 3 = 1$$

$\therefore$
$$\text{adj } A = \begin{bmatrix} 1 & 7 & -5 \\ -5 & 1 & 7 \\ 7 & -5 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$$

$\therefore$
$$A^{-1} = \frac{1}{18}\begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$$

$\therefore$ the solution is $\quad X = A^{-1}B$

$\Rightarrow$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} 9 - 30 + 56 \\ 63 + 6 - 40 \\ -45 + 42 + 8 \end{bmatrix} = \begin{bmatrix} \frac{35}{18} \\ \frac{29}{18} \\ \frac{5}{18} \end{bmatrix}$$

$\therefore$ the solution is $\quad \mathbf{x = \frac{35}{18}, \quad y = \frac{29}{18}, \quad z = \frac{5}{18}}$ ■

Example 2

Solve $x + 2y + 3z = 10$; $2x - 3y + z = 1$ and $3x + y - 2z = 9$.

Solution

Given system is
$$x + 2y + 3z = 10$$
$$2x - 3y + z = 1$$
$$3x + y - 2z = 9$$

Here $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 1 \\ 3 & 1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 10 \\ 1 \\ 9 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\therefore$ the given system is $\quad AX = B$.

Now
$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -3 & 1 \\ 3 & 1 & -2 \end{vmatrix}$$
$$= 1(6-1) - 2(-4-3) + 3(2+9)$$
$$= 5 + 14 + 33 = 52 \neq 0$$

$\therefore A^{-1}$ exists and
$$A^{-1} = \frac{\text{adj } A}{|A|}$$

Now
$$\text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$A_{11} = (-1)^2 \begin{vmatrix} -3 & 1 \\ 1 & -2 \end{vmatrix} = 6 - 1 \quad = 5$$

$$A_{12} = (-1)^3 \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} = -(-4-3) = 7$$

$$A_{13} = (-1)^4 \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix} = 2 + 9 \quad = 11$$

$$A_{21} = (-1)^3 \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -(-4-3) = 7$$

$$A_{22} = (-1)^4 \begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix} = -2 - 9 \quad = -11$$

$$A_{23} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -(1-6) \quad = 5$$

$$A_{31} = (-1)^4 \begin{vmatrix} 2 & 3 \\ -3 & 1 \end{vmatrix} = 2 + 9 \quad = 11$$

$$A_{32} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -(1-6) \quad = 5$$

$$A_{33} = (-1)^6 \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = -3 - 4 \quad = -7$$

$\therefore$
$$\text{adj } A = \begin{bmatrix} 5 & 7 & 11 \\ 7 & -11 & 5 \\ 11 & 5 & -7 \end{bmatrix}^T = \begin{bmatrix} 5 & 7 & 11 \\ 7 & -11 & 5 \\ 11 & 5 & -7 \end{bmatrix}$$

$\therefore$
$$A^{-1} = \frac{1}{52} \begin{bmatrix} 5 & 7 & 11 \\ 7 & -11 & 5 \\ 11 & 5 & -7 \end{bmatrix}$$

$\therefore$ the solution is $\quad X = A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{52}\begin{bmatrix} 5 & 7 & 11 \\ 7 & -11 & 5 \\ 11 & 5 & -7 \end{bmatrix}\begin{bmatrix} 10 \\ 1 \\ 9 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{52}\begin{bmatrix} 50+7+99 \\ 70-11+45 \\ 110+5-63 \end{bmatrix} = \frac{1}{52}\begin{bmatrix} 156 \\ 104 \\ 52 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$\therefore$ the solution is $\mathbf{x = 3, \quad y = 2, \quad z = 1}$ ■

Gauss Elimination Method

It is a direct method. In this method the unknowns are eliminated in such a way the n equations in n unknowns are reduced to an equivalent upper triangular system which is then solved by back substitution. The method is explained below.

For simplicity, we shall consider a system of 3 equations in 3 unknowns x_1, x_2, x_3.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad (1)$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \quad (2)$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \quad (3)$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Then the system of equations can be written as a single matrix equation $AX = B$

The augmented matrix is $[A, B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & : & b_1 \\ a_{21} & a_{22} & a_{23} & : & b_2 \\ a_{31} & a_{32} & a_{33} & : & b_3 \end{bmatrix}$

First step is to eliminate x_1 from the 2nd and 3rd equation.

So we multiply (1) by $-\frac{a_{21}}{a_{11}}$ and add with (2).

If $a_{11} \neq 0$ then multiply (1) by $-\frac{a_{31}}{a_{11}}$ and add with (3).

Then x_1 will be eliminated from (2) and (3).

We will get an equivalent system.

$a_{11} \neq 0$ is called the first **pivot** and the equation (1) is called the **pivotal** equation.

In case $a_{11} = 0$, we interchange the equations and take the equation for which coefficient of $x_1 \neq 0$ as the first equation.

$\therefore$ the augmented matrix will become

$$[A,B] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & : & b_1 \\ 0 & a'_{22} & a'_{23} & : & b'_2 \\ 0 & a'_{32} & a'_{33} & : & b'_3 \end{bmatrix}$$

Now the second step is to eliminate x_2 from the new third equation (or third row in the matrix). For this $a'_{22} \neq 0$ is the new **pivot** or **second pivot**.

Multiply the second row by $-\frac{a'_{32}}{a_{22}}$ and add to third row.

So, at the end of second step

$$[A,B] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & : & b_1 \\ 0 & a'_{22} & a'_{23} & : & b'_2 \\ 0 & 0 & a''_{33} & : & b''_3 \end{bmatrix}$$

Thus the equations are reduced to an equivalent system of equations.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a'_{22}x_2 + a'_{23}x_3 &= b'_2 \\ a''_{33}x_3 &= b''_3 \end{aligned}$$

From third equation we get x_3, which can be substituted back in 2nd equation we get x_2 and then from the first equation we get x_1.

This elimination procedure is known as **Gauss elimination method.**

Partial pivoting: In the first step of elimination, the first column is searched for the numerically largest element and brought it as the first pivot by interchanging the rows. In the second step of elimination, the second column is searched for the numerically largest element among the remaining $(n-1)$ elements (leaving the first row) and it is brought as the 2nd pivot by interchange and so on. This procedure is continued and the coefficient matrix is reduced to an upper triangular matrix. This modified form of elimination is known as **partial pivoting**. This method ensures that errors are not propagated by large multipliers.

Complete pivoting: We search the coefficient matrix A for the numerically largest element and brought it as the first pivot, by suitable interchanges of rows and also interchange of position of the elements. At each step if we adopt this procedure the method is known as **complete pivoting**. But, this procedure is complicated and it does not improve accuracy appreciably.

WORKED EXAMPLES

Example 1

Solve by Gauss elimination method the equations $2x + y + 4z = 12$; $8x - 3y + 2z = 20$; $4x + 11y - z = 33$.

Solution

Given system is
$$2x + y + 4z = 12$$
$$8x - 3y + 2z = 20$$
$$4x + 11y - z = 33$$

The augmented matrix is

$$[A, B] = \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 8 & -3 & 2 & : & 20 \\ 4 & 11 & -1 & : & 33 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & -7 & -14 & : & -28 \\ 0 & 9 & -9 & : & 9 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + (-4)R_1 \\ R_3 \to R_3 + (-2)R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & -1 & : & 1 \end{bmatrix} \quad \begin{matrix} R_2 \to \left(-\frac{1}{7}\right)R_2 \\ R_3 \to \frac{1}{9}R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & -3 & : & -3 \end{bmatrix} \quad R_3 \to R_3 + (-1)R_2$$

$\therefore$ the equivalent reduced equations are

$$2x + y + 4z = 12$$
$$y + 2z = 4$$
$$-3z = -3 \quad \Rightarrow \quad z = 1$$

$\therefore$
$$y + 2 = 4 \quad \Rightarrow \quad y = 2$$
$$2x + 2 + 4 = 12 \quad \Rightarrow \quad 2x = 6 \quad \Rightarrow \quad x = 3$$

$\therefore$ the solution is $\mathbf{x = 3, \quad y = 2, \quad z = 1}$ ■

Example 2

Solve the equations $28x + 4y - z = 32$; $x + 3y + 10z = 24$; $2x + 17y + 4z = 35$ by Gauss elimination method.

Solution

Given system is
$$28x + 4y - z = 32$$
$$x + 3y + 10z = 24$$
$$2x + 17y + 4z = 35$$

We shall rearrange the equations as

$$x + 3y + 10z = 24$$
$$2x + 17y + 4z = 35$$
$$28x + 4y - z = 32$$

The augmented matrix is

$$[A, B] = \begin{bmatrix} 1 & 3 & 10 & : & 24 \\ 2 & 17 & 4 & : & 35 \\ 28 & 4 & -1 & : & 32 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 10 & : & 24 \\ 0 & 11 & -16 & : & -13 \\ 0 & -80 & -281 & : & -640 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + (-2)R_1 \\ R_3 \to R_3 + (-28)R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 10 & : & 24 \\ 0 & 1 & -\frac{16}{11} & : & -\frac{13}{11} \\ 0 & -80 & -281 & : & -640 \end{bmatrix} \quad R_2 \to \frac{1}{11} R_2$$

$$\sim \begin{bmatrix} 1 & 3 & 10 & : & 24 \\ 0 & 1 & -\frac{16}{11} & : & -\frac{13}{11} \\ 0 & 0 & \frac{-4371}{11} & : & -\frac{8080}{11} \end{bmatrix} \quad R_3 \to R_3 + 80R_2$$

$\therefore$ the equivalent equations are

$$x + 3y + 10z = 24$$
$$y - \frac{16}{11} z = \frac{-13}{11}$$
$$-\frac{4371}{11} z = -\frac{8080}{11} \quad \Rightarrow \quad z = \frac{8080}{4371} = 1.8485$$

$\therefore$ $$y = \frac{16}{11} \times 1.8485 - \frac{13}{11} = 1.5069$$

$\therefore$ $$x = 24 - 3y - 10z$$
$$= 24 - 3 \times 1.5069 - 10 \times 1.8485 = 24 - 4.5207 - 18.485 = 0.9943$$

$\therefore$ the solution is $\mathbf{x = 0.9943}$, $\mathbf{y = 1.5069}$, $\mathbf{z = 1.8485}$ ■

Example 3

Solve by Gauss elimination method the equations $x_1 + x_2 + x_3 + x_4 = 2$; $x_1 + x_2 + 3x_3 - 2x_4 = -6$; $2x_1 + 3x_2 - x_3 + 2x_4 = 7$; $x_1 + 2x_2 + x_3 - x_4 = -2$.

Solution

Given system is

$$x_1 + x_2 + x_3 + x_4 = 2$$
$$x_1 + x_2 + 3x_3 - 2x_4 = -6$$
$$2x_1 + 3x_2 - x_3 + 2x_4 = 7$$
$$x_1 + 2x_2 + x_3 - x_4 = -2$$

The augmented matrix is

$$[A, B] = \begin{bmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 1 & 1 & 3 & -2 & : & -6 \\ 2 & 3 & -1 & 2 & : & 7 \\ 1 & 2 & 1 & -1 & : & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 0 & 0 & 2 & -3 & : & -8 \\ 0 & 1 & -3 & 0 & : & 3 \\ 0 & 1 & 0 & -2 & : & -4 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + (-1)R_1 \\ R_3 \to R_3 + (-2)R_1 \\ R_4 \to R_4 + (-1)R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 0 & 1 & -3 & 0 & : & 3 \\ 0 & 0 & 2 & -3 & : & -8 \\ 0 & 1 & 0 & -2 & : & -4 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 0 & 1 & -3 & 0 & : & 3 \\ 0 & 0 & 2 & -3 & : & -8 \\ 0 & 0 & 3 & -2 & : & -7 \end{bmatrix} \quad R_4 \to R_4 + (-1)R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 0 & 1 & -3 & 0 & : & 3 \\ 0 & 0 & 1 & -\frac{3}{2} & : & -4 \\ 0 & 0 & 3 & -2 & : & -7 \end{bmatrix} \quad R_3 \to \frac{1}{2}R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 0 & 1 & -3 & 0 & : & 3 \\ 0 & 0 & 1 & -\frac{3}{2} & : & -4 \\ 0 & 0 & 0 & \frac{5}{2} & : & 5 \end{bmatrix} \quad R_4 \to R_4 + (-3)R_3$$

$\therefore$ the equivalent reduced equations are

$$x_1 + x_2 + x_3 + x_4 = 2$$
$$x_2 - 3x_3 = 3$$
$$x_3 - \frac{3}{2}x_4 = -4$$

$$\frac{5}{2}x_4 = 5 \quad \Rightarrow x_4 = 2$$

$\therefore$

$$x_3 - \frac{3}{2}\times 2 = -4 \quad \Rightarrow x_3 = -4+3 = -1$$

$$x_2 - 3(-1) = 3 \quad \Rightarrow x_2 = 0$$

and

$$x_1 - 1 + 2 = 2 \quad \Rightarrow x_1 = 1$$

$\therefore$ the solution is $\boldsymbol{x_1 = 1, \quad x_2 = 0, \quad x_3 = -1, \quad x_4 = 2}$ ■

Gauss-Jordan Method

This is a direct method which is a modification of Gauss elimination method. After eliminating one variable by Gauss elimination method, in the subsequent stages the elimination is performed not only in the equations below but also in the equations above. Thus the coefficient matrix is reduced to a diagonal matrix and hence the values of the unknowns are readily obtained. This modification is due to Jordan and hence it is known as Gauss-Jordan method.

Note: For large system of equations the number of algebraic operations involved in this method is more and so Gauss elimination method is preferred.

WORKED EXAMPLES

Example 1

Solve by Gauss-Jordan method the equations $x + y + z = 9$; $2x - 3y + 4z = 13$; $3x + 4y + 5z = 40$.

Solution

Given system is

$$\begin{aligned} x + y + z &= 9 \\ 2x - 3y + 4z &= 13 \\ 3x + 4y + 5z &= 40 \end{aligned}$$

The augmented matrix is

$$[A, B] = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & -3 & 4 & : & 13 \\ 3 & 4 & 5 & : & 40 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -5 & 2 & : & -5 \\ 0 & 1 & 2 & : & 13 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 + (-2)R_1 \\ R_3 \to R_3 + (-3)R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & 1 & 2 & : & 13 \\ 0 & -5 & 2 & : & -5 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & : & -4 \\ 0 & 1 & 2 & : & 13 \\ 0 & 0 & 12 & : & 60 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 - R_2 \\ \\ R_3 \to R_3 + 5R_2 \end{array}$$

$$\sim\begin{bmatrix} 1 & 0 & -1 & : & -4 \\ 0 & 1 & 2 & : & 13 \\ 0 & 0 & 1 & : & 5 \end{bmatrix} \quad R_3 \to \frac{1}{12}R_3$$

$$\sim\begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 + R_3 \\ \\ R_2 \to R_2 + (-2)R_3 \end{matrix}$$

$\therefore$ the solution is $x = 1, \quad y = 3, \quad z = 5$ ■

Example 2 .

Using Gauss-Jordan method solve $10x + y + z = 12;\ 2x + 10y + z = 13;\ x + y + 5z = 7$.

Solution

Given system is

$$\begin{aligned} 10x + y + z &= 12 \\ 2x + 10y + z &= 13 \\ x + y + 5z &= 7 \end{aligned}$$

Rearranging the equations, we get

$$\begin{aligned} x + y + 5z &= 7 \\ 2x + 10y + z &= 13 \\ 10x + y + z &= 12 \end{aligned}$$

The augmented matrix is

$$[A, B] = \begin{bmatrix} 1 & 1 & 5 & : & 7 \\ 2 & 10 & 1 & : & 13 \\ 10 & 1 & 1 & : & 12 \end{bmatrix}$$

$$\sim\begin{bmatrix} 1 & 1 & 5 & : & 7 \\ 0 & 8 & -9 & : & -1 \\ 0 & -9 & -49 & : & -58 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + (-2)R_1 \\ R_3 \to R_3 + (-10)R_1 \end{matrix}$$

$$\sim\begin{bmatrix} 1 & 1 & 5 & : & 7 \\ 0 & 1 & -\frac{9}{8} & : & -\frac{1}{8} \\ 0 & -9 & -49 & : & -58 \end{bmatrix} \quad R_2 \to \frac{1}{8}R_2$$

$$\sim\begin{bmatrix} 1 & 0 & \frac{49}{8} & : & \frac{57}{8} \\ 0 & 1 & -\frac{9}{8} & : & -\frac{1}{8} \\ 0 & 0 & \frac{-473}{8} & : & \frac{-473}{8} \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 - R_2 \\ \\ R_3 \to R_3 + 9R_2 \end{matrix}$$

$$\sim\begin{bmatrix} 1 & 0 & \frac{49}{8} & : & \frac{57}{8} \\ 0 & 1 & -\frac{9}{8} & : & -\frac{1}{8} \\ 0 & 0 & 1 & : & 1 \end{bmatrix} \quad R_3 \to -\frac{8}{473}R_3$$

$$\sim\begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & 1 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 + \left(\frac{-49}{8}\right)R_3 \\ R_2 \to R_2 + \frac{9}{8}R_3 \end{matrix}$$

$\therefore$ the solution is $x = 1,\quad y = 1,\quad z = 1$ ■

Example 3

Solve by using Gauss-Jordan method the system of equations $x + 2y + z = 3$, $2x + 3y + 3z = 10$, $3x - y + 2z = 13$.

Solution

Given system of equations is

$$x + 2y + z = 3$$
$$2x + 3y + 3z = 10$$
$$3x - y + 2z = 13$$

The augmented matrix is

$$[A, B] = \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 2 & 3 & 3 & : & 10 \\ 3 & -1 & 2 & : & 13 \end{bmatrix}$$

$$\sim\begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & -1 & 1 & : & 4 \\ 0 & -7 & -1 & : & 4 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim\begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & 1 & -1 & : & -4 \\ 0 & -7 & -1 & : & 4 \end{bmatrix} \quad R_2 \to (-1)R_2$$

$$\sim\begin{bmatrix} 1 & 0 & 3 & : & 11 \\ 0 & 1 & -1 & : & -4 \\ 0 & 0 & -8 & : & -24 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 + (-2)R_2 \\ R_3 \to R_3 + 7R_2 \end{matrix}$$

$$\sim\begin{bmatrix} 1 & 0 & 3 & : & 11 \\ 0 & 1 & -1 & : & -4 \\ 0 & 0 & 1 & : & 3 \end{bmatrix} \quad R_3 \to \left(-\frac{1}{8}\right)R_3$$

$$\sim\begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & -1 \\ 0 & 0 & 1 & : & 3 \end{bmatrix} \begin{array}{l} R_1 \to R_1 + (-3)R_3 \\ R_2 \to R_2 + R_3 \\ \end{array}$$

$\therefore$ the solution is $x = 2, \quad y = -1, \quad z = 3$ ■

Example 4

Solve the linear system $5x_1 + x_2 + x_3 + x_4 = 4$; $x_1 + 7x_2 + x_3 + x_4 = 12$; $x_1 + x_2 + 6x_3 + x_4 = -5$; $x_1 + x_2 + x_3 + 4x_4 = -6$ by Gauss-Jordan method.

Solution

Given system is
$$\begin{aligned} 5x_1 + x_2 + x_3 + x_4 &= 4 \\ x_1 + 7x_2 + x_3 + x_4 &= 12 \\ x_1 + x_2 + 6x_3 + x_4 &= -5 \\ x_1 + x_2 + x_3 + 4x_4 &= -6 \end{aligned}$$

Rearranging the system, we get
$$\begin{aligned} x_1 + x_2 + x_3 + 4x_4 &= -6 \\ x_1 + 7x_2 + x_3 + x_4 &= 12 \\ x_1 + x_2 + 6x_3 + x_4 &= -5 \\ 5x_1 + x_2 + x_3 + x_4 &= 4 \end{aligned}$$

The augmented matrix is

$$[A,B] = \begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 1 & 7 & 1 & 1 & : & 12 \\ 1 & 1 & 6 & 1 & : & -5 \\ 5 & 1 & 1 & 1 & : & 4 \end{bmatrix}$$

$$\sim\begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 0 & 6 & 0 & -3 & : & 18 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & -4 & -4 & -19 & : & 34 \end{bmatrix} \begin{array}{l} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 5R_1 \end{array}$$

$$\sim\begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 0 & 1 & 0 & -\frac{1}{2} & : & 3 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & -4 & -4 & -19 & : & 34 \end{bmatrix} \quad R_2 \to \frac{1}{6}R_2$$

$$\sim\begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 0 & 1 & 0 & -\frac{1}{2} & : & 3 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & 0 & -4 & -21 & : & 46 \end{bmatrix} \quad R_4 \to R_4 + 4R_2$$

$$\sim\begin{bmatrix} 1 & 0 & 1 & \frac{9}{2} & : & -9 \\ 0 & 1 & 0 & -\frac{1}{2} & : & 3 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & 0 & -4 & -21 & : & 46 \end{bmatrix} \quad R_1 \to R_1 - R_2$$

$$\sim\begin{bmatrix} 1 & 0 & 1 & \frac{9}{2} & : & -9 \\ 0 & 1 & 0 & -\frac{1}{2} & : & 3 \\ 0 & 0 & 1 & -\frac{3}{5} & : & \frac{1}{5} \\ 0 & 0 & -4 & -21 & : & 46 \end{bmatrix} \quad R_3 \to \frac{1}{5}R_3$$

$$\sim\begin{bmatrix} 1 & 0 & 0 & \frac{51}{10} & : & \frac{-46}{5} \\ 0 & 1 & 0 & -\frac{1}{2} & : & 3 \\ 0 & 0 & 1 & -\frac{3}{5} & : & \frac{1}{5} \\ 0 & 0 & 0 & -\frac{117}{5} & : & \frac{234}{5} \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 - R_3 \\ R_4 \to R_4 + 4R_3 \end{matrix}$$

$$\sim\begin{bmatrix} 1 & 0 & 0 & \frac{51}{10} & : & \frac{-46}{5} \\ 0 & 1 & 0 & -\frac{1}{2} & : & 3 \\ 0 & 0 & 1 & -\frac{3}{5} & : & \frac{1}{5} \\ 0 & 0 & 0 & 1 & : & -2 \end{bmatrix} \quad R_4 \to \frac{-5}{117}R_4$$

$$\sim\begin{bmatrix} 1 & 0 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & 0 & : & 2 \\ 0 & 0 & 1 & 0 & : & -1 \\ 0 & 0 & 0 & 1 & : & -2 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 - \frac{51}{10}R_4 \\ R_2 \to R_2 + \frac{1}{2}R_4 \\ R_3 \to R_3 + \frac{3}{5}R_4 \end{matrix}$$

∎

$\therefore$ the solution is $\boldsymbol{x_1 = 1}$, $\boldsymbol{x_2 = 2}$, $\boldsymbol{x_3 = -1}$, **and** $\boldsymbol{x_4 = -2}$

Matrix Inverse by Gauss-Jordan Method

We shall explain the method for 3 × 3 matrix.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

If A is non-singular, then there exists a 3 × 3 matrix $X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$

such that $AX = I$

$$\Rightarrow \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This equation is equivalent to the three equations below:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad (1)$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad (2)$$

and

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad (3)$$

Equation (1) is a system of linear equations. Solving by Jordan's method (or by Gauss elimination method) we get x_{11}, x_{21}, x_{31} and so the vector $\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix}$ is known. Similarly solving (2) and (3) we get the other columns of X and hence X is known. This matrix X is the inverse of A.

Now to solve the equation (1), we start with the augmented matrix $[A, I_1]$ where $I_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and transform by row operations so that A is reduced to unit matrix in Jordan's method, then we write the solution for x_{11}, x_{21}, x_{31} directly.

The same procedure is applied to solve (2) and (3) by writing $[A, I_2]$ and $[A, I_3]$.

In practice we will not do this individually and convert A into a unit matrix, but we start with $\left[A|I_1 I_2 I_3\right] = (A, I)$ and convert A into unit matrix by row operations and find X.

Working Rule

Consider the augmented matrix $[A, I]$, where I is the identity matrix of the same order as A. By row operations reduce A into a unit matrix, then correspondingly I will be changed into a matrix X. This matrix X is the inverse of A. It is advisable to change the pivot element to 1 before applying row operations at each step.

WORKED EXAMPLES

Example 1

Using Gauss-Jordan method, find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & 4 \end{bmatrix}$.

Solution

Given $$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & 4 \end{bmatrix} \qquad \text{To find } A^{-1}$$

Consider the augmented matrix

$$[A, I] = \begin{bmatrix} 1 & 1 & 3 & : & 1 & 0 & 0 \\ 1 & 3 & -3 & : & 0 & 1 & 0 \\ -2 & -4 & 4 & : & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & : & 1 & 0 & 0 \\ 0 & 2 & -6 & : & -1 & 1 & 0 \\ 0 & -2 & 10 & : & 2 & 0 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & : & 1 & 0 & 0 \\ 0 & 1 & -3 & : & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 5 & : & 1 & 0 & \frac{1}{2} \end{bmatrix} \quad \begin{matrix} R_2 \to \frac{1}{2}R_2 \\ R_3 \to \frac{1}{2}R_3 \end{matrix} \qquad \text{(The pivot 2 in } R_2 \text{ is reduced to 1)}$$

$$\sim \begin{bmatrix} 1 & 0 & 6 & : & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -3 & : & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 2 & : & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 + (-1)R_2 \\ \\ R_3 \to R_3 + R_2 \end{matrix}$$

$$\sim\begin{bmatrix} 1 & 0 & 6 & : & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -3 & : & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & : & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \quad R_3 \to \frac{1}{2}R_3 \qquad \text{(The pivot 2 in } R_3 \text{ is reduced to 1)}$$

$$\sim\begin{bmatrix} 1 & 0 & 0 & : & 0 & -2 & \frac{-3}{2} \\ 0 & 1 & 0 & : & \frac{1}{4} & \frac{5}{4} & \frac{3}{4} \\ 0 & 0 & 1 & : & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + 3R_3 \\ \\ R_1 \to R_1 + (-6)R_3 \end{matrix}$$

$\therefore$ the inverse matrix of A is
$$A^{-1} = \begin{bmatrix} 0 & -2 & -\frac{3}{2} \\ \frac{1}{4} & \frac{5}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

■

Example 2

Find the inverse of the matrix $A = \begin{bmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix}$ **by Gauss-Jordan method.**

Solution

Given
$$A = \begin{bmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix} \qquad \text{To find } A^{-1}$$

Consider the augmented matrix

$$[A, I] = \begin{bmatrix} 4 & 1 & 2 & : & 1 & 0 & 0 \\ 2 & 3 & -1 & : & 0 & 1 & 0 \\ 1 & -2 & 2 & : & 0 & 0 & 1 \end{bmatrix}$$

$$\sim\begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{2} & : & \frac{1}{4} & 0 & 0 \\ 2 & 3 & -1 & : & 0 & 1 & 0 \\ 1 & -2 & 2 & : & 0 & 0 & 1 \end{bmatrix} \quad R_1 \to \frac{1}{4}R_1 \qquad \text{(The pivot 4 in } R_1 \text{ is reduced to 1)}$$

$$\sim\begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{2} & : & \frac{1}{4} & 0 & 0 \\ 0 & \frac{5}{2} & -2 & : & -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{9}{4} & \frac{3}{2} & : & -\frac{1}{4} & 0 & 1 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 + (-2)R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim\begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{2} & : & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{-4}{5} & : & \frac{-1}{5} & \frac{2}{5} & 0 \\ 0 & \frac{-9}{4} & \frac{3}{2} & : & -\frac{1}{4} & 0 & 1 \end{bmatrix} \quad R_2 \to \frac{2}{5}R_2 \qquad \text{(The pivot } \frac{5}{2} \text{ in } R_2 \text{ is reduced to 1)}$$

$$\sim\begin{bmatrix} 1 & 0 & \frac{7}{10} & : & \frac{3}{10} & \frac{-1}{10} & 0 \\ 0 & 1 & \frac{-4}{5} & : & \frac{-1}{5} & \frac{2}{5} & 0 \\ 0 & 0 & -\frac{3}{10} & : & -\frac{7}{10} & \frac{9}{10} & 1 \end{bmatrix} \begin{matrix} R_1 \to R_1 + \left(-\frac{1}{4}\right)R_2 \\ \\ R_3 \to R_3 + \frac{9}{4}R_2 \end{matrix}$$

$$\sim\begin{bmatrix} 1 & 0 & \frac{7}{10} & : & \frac{3}{10} & \frac{-1}{10} & 0 \\ 0 & 1 & \frac{-4}{5} & : & \frac{-1}{5} & \frac{2}{5} & 0 \\ 0 & 0 & 1 & : & \frac{7}{3} & -3 & \frac{-10}{3} \end{bmatrix} \quad R_3 \to -\frac{10}{3}R_3 \text{ (The pivot } -\frac{3}{10} \text{ in } R_3 \text{ is reduced to 1)}$$

$$\sim\begin{bmatrix} 1 & 0 & 0 & : & \frac{-4}{3} & 2 & \frac{7}{3} \\ 0 & 1 & 0 & : & \frac{5}{3} & -2 & -\frac{8}{3} \\ 0 & 0 & 1 & : & \frac{7}{3} & -3 & \frac{-10}{3} \end{bmatrix} \begin{matrix} R_1 \to R_1 + \left(\frac{-7}{10}\right)R_3 \\ R_2 \to R_2 + \frac{4}{5}R_3 \\ \\ \end{matrix}$$

$\therefore$ the inverse of A is
$$A^{-1} = \begin{bmatrix} \frac{-4}{3} & 2 & \frac{7}{3} \\ \frac{5}{3} & -2 & -\frac{8}{3} \\ \frac{7}{3} & -3 & \frac{-10}{3} \end{bmatrix}$$

■

Example 3

Solve the system of equations $x+y+3z=4$; $x+3y-3z=2$; $-2x-4y-4z=8$ by finding the matrix inverse by Gauss-Jordan method.

Solution

The given system of equations is

$$x+y+3z=4$$
$$x+3y-3z=2$$
$$-2x-4y-4z=8$$

The coefficient matrix is

$$A=\begin{bmatrix}1 & 1 & 3\\ 1 & 3 & -3\\ -2 & -4 & -4\end{bmatrix},\quad B=\begin{bmatrix}4\\ 2\\ 8\end{bmatrix}\quad \text{and}\quad X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

$\therefore$ the system of equations is $AX=B \Rightarrow X=A^{-1}B$.

We find A^{-1} by the method of matrix inverse by Gauss-Jordan method.

Consider the augmented matrix

$$[A,I]=\begin{bmatrix}1 & 1 & 3 & : & 1 & 0 & 0\\ 1 & 3 & -3 & : & 0 & 1 & 0\\ -2 & -4 & -4 & : & 0 & 0 & 1\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 1 & 3 & : & 1 & 0 & 0\\ 0 & 2 & -6 & : & -1 & 1 & 0\\ 0 & -2 & 2 & : & 2 & 0 & 1\end{bmatrix}\quad \begin{matrix}R_2\to R_2-R_1\\ R_3\to R_3+2R_1\end{matrix}$$

$$\sim\begin{bmatrix}1 & 1 & 3 & : & 1 & 0 & 0\\ 0 & 1 & -3 & : & -\frac{1}{2} & \frac{1}{2} & 0\\ 0 & -1 & 1 & : & 1 & 0 & \frac{1}{2}\end{bmatrix}\quad \begin{matrix}R_2\to \frac{1}{2}R_2\\ R_3\to \frac{1}{2}R_3\end{matrix}$$

$$\sim\begin{bmatrix}1 & 0 & 4 & : & 2 & 0 & \frac{1}{2}\\ 0 & 1 & -3 & : & -\frac{1}{2} & \frac{1}{2} & 0\\ 0 & -1 & 1 & : & 1 & 0 & \frac{1}{2}\end{bmatrix}\quad R_1\to R_1+R_3$$

$$\sim\begin{bmatrix}1 & 0 & 4 & : & 2 & 0 & \frac{1}{2}\\ 0 & 1 & -3 & : & -\frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & -2 & : & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{bmatrix}\quad R_3\to R_3+R_2$$

$$\sim\left[\begin{array}{ccc:ccc} 1 & 0 & 4 & 2 & 0 & \frac{1}{2} \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right] \quad R_3 \to \left(-\frac{1}{2}\right)R_3$$

$$\sim\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - 4R_3 \\ R_2 \to R_2 + 3R_3 \end{array}$$

$\therefore$ the inverse of A is

$$A^{-1} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

$\therefore$

$$X = A^{-1}B$$

$\Rightarrow$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 12+12+12 \\ -5-\frac{1}{2}-6 \\ -1-\frac{1}{2}-2 \end{bmatrix} = \begin{bmatrix} 26 \\ -\frac{23}{2} \\ -\frac{7}{2} \end{bmatrix}$$

$\therefore$ the solution is $\boldsymbol{x = 26}$, $\boldsymbol{y = -\frac{23}{2}}$, $\boldsymbol{z = -\frac{7}{2}}$ ■

Example 4

Solve the system of equations $2x + y + 2z = 10$; $2x + 2y + z = 9$; $x + 2y + 2z = 11$ by finding the inverse by Gauss-Jordan method.

Solution

The given system of equations is

$$2x + y + 2z = 10$$
$$2x + 2y + z = 9$$
$$x + 2y + 2z = 11$$

The coefficient matrix is

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 10 \\ 9 \\ 11 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\therefore$ the system of equation is $\quad AX = B \quad \Rightarrow \quad X = A^{-1}B.$

We find A^{-1} by the method of matrix inverse by Gauss-Jordan method.

Consider the augmented matrix

$$[A, I] = \begin{bmatrix} 2 & 1 & 2 & : & 1 & 0 & 0 \\ 2 & 2 & 1 & : & 0 & 1 & 0 \\ 1 & 2 & 2 & : & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \frac{1}{2} & 1 & : & \frac{1}{2} & 0 & 0 \\ 2 & 2 & 1 & : & 0 & 1 & 0 \\ 1 & 2 & 2 & : & 0 & 0 & 1 \end{bmatrix} \quad R_1 \to \frac{1}{2}R_1$$

$$\sim \begin{bmatrix} 1 & \frac{1}{2} & 1 & : & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 & : & -1 & 1 & 0 \\ 0 & \frac{3}{2} & 1 & : & -\frac{1}{2} & 0 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{3}{2} & : & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & : & -1 & 1 & 0 \\ 0 & 0 & \frac{5}{2} & : & 1 & -\frac{3}{2} & 1 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 - \frac{1}{2}R_2 \\ R_3 \to R_3 - \frac{3}{2}R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{3}{2} & : & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & : & -1 & 1 & 0 \\ 0 & 0 & 1 & : & \frac{2}{5} & -\frac{3}{5} & \frac{2}{5} \end{bmatrix} \quad R_3 \to \frac{2}{5}R_3$$

$$\sim\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{2}{5} & \frac{2}{5} & -\frac{3}{5} \\ 0 & 1 & 0 & -\frac{3}{5} & \frac{2}{5} & \frac{2}{5} \\ 0 & 0 & 1 & \frac{2}{5} & -\frac{3}{5} & \frac{2}{5} \end{array}\right] \begin{array}{l} R_1 \to R_1 - \frac{3}{2}R_3 \\ R_2 \to R_2 + R_3 \\ \end{array}$$

$\therefore$ the inverse of A is

$$A^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

$\therefore$ $$X = A^{-1}B$$

$\Rightarrow$ $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{3}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 10 \\ 9 \\ 11 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{20}{5} + \frac{18}{5} - \frac{33}{5} \\ -\frac{30}{5} + \frac{18}{5} + \frac{22}{5} \\ \frac{20}{5} - \frac{27}{5} + \frac{22}{5} \end{bmatrix} = \begin{bmatrix} \frac{38-33}{5} \\ \frac{40-30}{5} \\ \frac{42-27}{5} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore$ the solution is $\boldsymbol{x = 1,\ y = 2,\ z = 3}$ ■

Exercises 3.1

(I) Solve by matrix inversion the following system of equations.

(1) $3x + y - z = 3;\ 2x - 8y + z = -5;\ x - 2y + 9z = 8$

(2) $2x + y + 4z = 12;\ 8x - 3y + 2z = 20;\ 4x + 11y - z = 33$

(3) $3x_1 + 2x_2 + 4x_3 = 7;\ 2x_1 + x_2 + x_3 = 4;\ x_1 + 3x_2 + 5x_3 = 2$

(4) $2x + 2y + 3z = -2;\ 2x + y + z = 0;\ x + 3y + 5z = -5$

(5) $x_1 + x_2 + 3x_3 = 4;\ x_1 + 3x_2 - 3x_3 = -14;\ x_1 + 2x_2 + 2x_3 = -1$

(II) Solve by Gauss-elimination method.

(6) $x+2y+z=4;\ 3x-y+2z=-3;\ x+2y+4z=7$

(7) $2x+y+4z=12;\ 8x-3y+2z=20;\ 4x+11y-z=33$

(8) $5x-2y+3z=18;\ x+7y-3z=-22;\ 2x-y+6z=22$

(9) $x+4y-z=-5;\ x+y-6z=-12;\ 3x-y-z=4$

(10) $10x-2y+3z=23;\ 2x+10y-5z=-33;\ 3x-4y+10z=41$

(11) $3x-y+2z=12;\ x+2y+3z=11;\ 2x-2y-z=2$

(12) $x_1-x_2+3x_3-3x_4=3;\ 2x_1+3x_2+x_3-11x_4=1;$
$5x_1-2x_2+5x_3-4x_4=5;\ 3x_1+4x_2-7x_3+2x_4=-7$

(III) Solve using Gauss-Jordan method.

(13) $x+2y+z=3;\ 2x+3y+3z=10;\ 3x-y+2z=13$

(14) $2x-3y+z=-1, x+4y+5z=25;\ 3x-4y+z=2$

(15) $10x+y+z=12;\ 2x+10y+z=13;\ x+y+5z=7$

(16) $5x_1+x_2+x_3+x_4=4;\ x_1+7x_2+x_3+x_4=12;$
$x_1+x_2+6x_3+x_4=-5; x_1+x_2+x_3+4x_4=-6$

(17) $10x-2y+3z=23;\ 2x+10y-5z=-33;\ 3x-y+10z=41$

(18) $3x+2y+7z=4;\ 2x+3y+z=5;\ 3x+4y+z=7$

(IV) Find the Matrix Inverse by Gauss-Jordan method.

(19) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ (20) $\begin{bmatrix} 2 & 2 & 6 \\ 2 & 6 & -6 \\ 4 & -8 & -8 \end{bmatrix}$

(21) $\begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ (22) $\begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}$

(23) $\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & 1 \end{bmatrix}$ (24) $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$

(25) Solve the system of linear equations $x+y+z=9,\ 2x-3y+4z=13,\ 3x+4y+5z=40,$ finding the inverse matrix by Gauss-Jordan method.

(26) Solve the system of equations $2x+y+z=10, 3x+2y+3z=18,\ x+4y+9z=16$, finding the inverse matrix by Gauss-Jordan method.

Answers 3.1

(I)

(1) $x = 1,\ y = 1,\ z = 1$

(2) $x = 3,\ y = 2,\ z = 1$

(3) $x_1 = \frac{9}{4},\ x_2 = -\frac{9}{8},\ x_3 = \frac{5}{8}$

(4) $x = 1, y = -2, z = 0$

(5) $x_1 = 1, x_2 = -3,\ x_3 = 2$

(II)

(6) $x = -1,\ y = 2,\ z = -1$

(7) $x = 3,\ y = 2,\ z = 1$

(8) $x = 1,\ y = -2,\ z = 3$

(9) $x = \frac{117}{71},\ y = -\frac{81}{71},\ z = \frac{148}{71}$

(10) $x = 1,\ y = -2,\ z = 3$

(11) $x = 3,\ y = 1,\ z = 2$

(12) $x_1 = x_2 = x_3 = 0, x_3 = 1$

(III)

(13) $x = 2,\ y = -1,\ z = 3$

(14) $x = 8.7,\ y = 5.7,\ z = -1.3$

(15) $x = 1,\ y = 1,\ z = 1$

(16) $x_1 = 1,\ x_2 = 2,\ x_3 = -1,\ x_4 = -2$

(17) $x = 1,\ y = -2,\ z = 3$

(18) $x = \frac{7}{8},\ y = \frac{9}{8},\ z = -\frac{1}{8}$

(IV)

(19) $A^{-1} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & \frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$

(20) $A^{-1} = \frac{1}{56}\begin{bmatrix} 12 & 4 & 6 \\ 1 & 5 & -3 \\ 5 & -3 & -1 \end{bmatrix}$

(21) $A^{-1} = \frac{1}{5}\begin{bmatrix} 2 & 2 & -3 \\ -2 & 2 & 2 \\ 2 & -3 & 2 \end{bmatrix}$

(22) $A^{-1} = \frac{1}{16}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

(23) $A^{-1} = \frac{1}{10}\begin{bmatrix} -5 & 2 & -16 \\ 0 & 2 & 4 \\ 5 & 0 & 10 \end{bmatrix}$

(24) $A^{-1} = \begin{bmatrix} -3 & \frac{5}{2} & -\frac{1}{2} \\ 12 & -\frac{17}{2} & \frac{3}{2} \\ -5 & \frac{7}{2} & -\frac{1}{2} \end{bmatrix}$

(25) $x = 1,\ y = 3,\ z = 5$

(26) $x = 7,\ y = -9,\ z = 5$

ITERATIVE METHODS

We have seen iteration is a successive approximation method. We start with an approximate solution and obtain a sequence of better approximations and stop at the step when two consecutive approximations coincide upto the desired degree of accuracy. The iterative method succeeds if the sequence of approximate solutions approach the actual root. For the iteration method to succeed, each equation of the system must possess one large coefficient and the large coefficient must be attached to different unknowns in the system. In other words, the coefficient matrix A is diagonally dominant.

The system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

will be solvable if $|a_1| \geq |b_1| + |c_1|, |b_2| \geq |a_2| + |c_2|$ and $|c_3| \geq |a_3| + |b_3|$

For example, the system of equations

$$2x + 8y - z = 6$$
$$6x + y + 2z = 8$$
$$-3x + y + 7z = 5$$

as such is not diagonally dominant. But it can be rearranged as

$$6x + y + 2z = 8$$
$$2x + 8y - z = 6$$
$$-3x + y + 7z = 5$$

This is a diagonally dominant system or the coefficient matrix

$$A = \begin{bmatrix} 6 & 1 & 2 \\ 2 & 8 & -1 \\ -3 & 1 & 7 \end{bmatrix} \text{ is diagnollay dominant.}$$

Gauss-Jacobi Method

Consider the system

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

We shall assume the system is diagonally dominant, ie. $|a_1| \geq |b_1| + |c_1|$, $|b_2| \geq |a_2| + |c_2|$ and $|c_3| \geq |a_3| + |b_3|$

The system can be rewritten as

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$
$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z)$$
$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

Let $x^{(0)}, y^{(0)}, z^{(0)}$ be the initial values of x, y, z.

Then I Iteration is

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1 y^{(0)} - c_1 z^{(0)})$$
$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2 x^{(0)} - c_2 z^{(0)})$$
$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3 x^{(0)} - b_3 y^{(0)})$$

II Iteration is

$$x^{(2)} = \frac{1}{a_1}(d_1 - b_1 y^{(1)} - c_1 z^{(1)})$$
$$y^{(2)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)} - c_2 z^{(1)})$$
$$z^{(2)} = \frac{1}{c_3}(d_3 - a_3 x^{(1)} - b_3 y^{(1)})$$

and proceed this way until we reach the solution.

Note: In the absence of better estimates for the initial values $x^{(0)}, y^{(0)}, z^{(0)}$, we take them as 0, 0, 0.

WORKED EXAMPLES

Example 1

Solve the system of equation correct to 3 decimal places using Jacobi method.

$$x + 17y - 2z = 48; \ 30x - 2y + 3z = 75; \ 2x + 2y + 18z = 30$$

Solution

The equations in diagonally dominant form is

$$30x - 2y + 3z = 75 \quad \Rightarrow x = \frac{1}{30}[75 + 2y - 3z]$$

$$x + 17y - 2z = 48 \quad \Rightarrow y = \frac{1}{17}[48 - x + 2z]$$

$$2x + 2y + 18z = 30 \quad \Rightarrow z = \frac{1}{18}[30 - 2x - 2y]$$

In Jacobi method for every iteration, the previous iteration values are used.

Initially take $x^{(0)} = 0,\ y^{(0)} = 0,\ z^{(0)} = 0$

I Iteration:

$$x^{(1)} = \frac{1}{30}[75 - 2y^{(0)} - 3z^{(0)}] = \frac{1}{30}(75) = 2.5$$

$$y^{(1)} = \frac{1}{17}[48 - x^{(0)} + 2z^{(0)}] = \frac{1}{7}(48) = 2.8235$$

$$z^{(1)} = \frac{1}{18}[30 - 2x^{(0)} - 2y^{(0)}] = \frac{30}{18} = 1.66667$$

II Iteration:

$$x^{(2)} = \frac{1}{30}[75 + 2y^{(1)} - 3z^{(1)}] = \frac{1}{30}[75 + 2(2.8235) - 3(1.66667)]$$
$$= 2.52157$$

$$y^{(2)} = \frac{1}{17}[48 - x^{(1)} + 2z^{(1)}] = \frac{1}{17}[48 - 2.5 + 2(1.66667)]$$
$$= 2.872549$$

$$z^{(2)} = \frac{1}{18}[30 - 2x^{(1)} - 2y^{(1)}] = \frac{1}{18}[30 - 2(2.5) - 2(2.8235)]$$
$$= 1.075167$$

III Iteration:

$$x^{(3)} = \frac{1}{30}[75 + 2y^{(2)} - 3z^{(2)}] = \frac{1}{30}[75 + 2(2.872549) - 3(1.075167)]$$
$$= 2.583987$$

$$y^{(3)} = \frac{1}{17}[48 - x^{(2)} + 2z^{(2)}] = \frac{1}{17}[48 - 2.52157 + 2(1.075167)]$$
$$= 2.801692$$

$$z^{(3)} = \frac{1}{18}[30 - 2x^{(2)} - 2y^{(2)}] = \frac{1}{18}[30 - 2(2.52157) - 2(2.872549)]$$
$$= 1.06732$$

IV Iteration:

$$x^{(4)} = \frac{1}{30}[75 + 2y^{(3)} - 3z^{(3)}] = \frac{1}{30}[75 + 2(2.80169) - 3(1.06732)]$$
$$= 2.58005$$

$$y^{(4)} = \frac{1}{17}[48 - x^{(3)} + 2z^{(3)}] = \frac{1}{17}[48 - 2.583987 + 2(1.06732)]$$
$$= 2.29710$$

$$z^{(4)} = \frac{1}{18}[30 - 2x^{(3)} - 2y^{(3)}] = \frac{1}{18}[30 - 2(2.583987) - 2(2.801692)]$$
$$= 1.06825$$

V Iteration:

$$x^{(5)} = \frac{1}{30}[75 + 2y^{(4)} - 3z^{(4)}] = \frac{1}{30}[75 + 2(2.79710) - 3(1.06825)]$$
$$= 2.57965$$

$$y^{(5)} = \frac{1}{17}[48 - x^{(4)} + 2z^{(4)}] = \frac{1}{17}[48 - 2.58005 + 2(1.06825)]$$
$$= 2.79745$$

$$z^{(5)} = \frac{1}{18}[30 - 2x^{(4)} - 2y^{(4)}] = \frac{1}{18}[30 - 2(2.58005) - 2(2.79710)]$$
$$= 1.06921$$

Iteration	x	y	z
1	2.5	2.8235	1.66667
2	2.52157	2.872549	1.075167
3	2.58398	2.801692	1.06732
4	2.58005	2.79710	1.06825
5	2.57965	2.79745	1.06921

$\therefore$ the solution correct to 3 decimal places is

$$\mathbf{x = 2.580,\ y = 2.797,\ z = 1.069}$$ ■

Gauss-Seidel Method

Gauss-Seidel method is a refinement of Gauss-Jacobi method.

Consider the system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

we shall assume the system is diagonally dominant,

ie. $|a_1| \geq |b_1| + |c_1|$, $|b_2| \geq |a_2| + |c_2|$ and $|c_3| \geq |a_3| + |b_3|$

The system can be written as

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

We start with the initial approximation $y^{(0)}, z^{(0)}$

I Iteration:

$$x^{(1)} = \frac{1}{a_1}\left(d_1 - b_1 y^{(0)} - c_1 z^{(0)}\right)$$

$$y^{(1)} = \frac{1}{b_2}\left(d_2 - a_2 x^{(1)} - c_2 z^{(0)}\right)$$

$$z^{(1)} = \frac{1}{c_3}\left(d_3 - a_3 x^{(1)} - b_3 y^{(1)}\right)$$

II Iteration:

$$x^{(2)} = \frac{1}{a_1}\left(d_1 - b_1 y^{(1)} - c_1 z^{(1)}\right)$$

$$y^{(2)} = \frac{1}{b_2}\left(d_2 - a_1 x^{(2)} - c_2 z^{(1)}\right)$$

$$z^{(2)} = \frac{1}{c_3}\left(d_3 - a_2 x^{(2)} - b_3 y^{(2)}\right)$$

At each iteration we use the latest available approximations for x, y, z. So, this method converges faster than the Jacobi method.

Note:

(1) It is found that the Gauss-Seidel iteration method is at least twice as fast as the Jacobi iteration method. So, we use Gauss-Seidel method when the method is not specified.

(2) Iteration is a self correcting method. Errors of computations made at a step will not affect the final answer, the number of iterations may be increased.

(3) In Gauss-seidel method if A is diagonally dominant, then the iteration scheme converges for any initial values of the variables.

WORKED EXAMPLES

Example 1

Solve the following equations by Gauss-Seidel method $27x + 6y - z = 85$; $x + y + 54z = 110$; $6x + 15y + 2z = 72$.

Solution

We note that the largest coefficient is attached to different variables in different equations. Also this largest coefficient is numerically larger than the sum of the other two coefficients. So, iteration method can be applied. Rearranging the equations in diagonally dominant form we have

$$27x+6y-z=85 \quad \Rightarrow x=\frac{1}{27}(85-6y+z)$$

$$6x+15y+2z=72 \quad \Rightarrow y=\frac{1}{15}(72-6x-2z)$$

$$x+y+54z=110 \Rightarrow z=\frac{1}{54}(110-x-y)$$

Initially take $\quad y^{(0)}=0,\ z^{(0)}=0$

I Iteration:

$$x^{(1)}=\frac{1}{27}\left[85-6y^{(0)}+z^{(0)}\right]=\frac{1}{27}\ [85]=3.14815$$

$$y^{(1)}=\frac{1}{15}\left[72-6x^{(1)}-2z^{(0)}\right]$$

$$=\frac{1}{15}\left[72-6(3.14815)-2(0)\right] \quad =3.54074$$

$$z^{(1)}=\frac{1}{54}\left[110-x^{(1)}-y^{(1)}\right]$$

$$=\frac{1}{54}\left[110-3.14815-3.54074\right]=1.91317$$

II Iteration:

$$x^{(2)}=\frac{1}{27}\left[85-6y^{(1)}+z^{(1)}\right]$$

$$=\frac{1}{27}\left[85-6(3.54074)+1.91317\right] \quad =2.432175$$

$$y^{(2)}=\frac{1}{15}\left[72-6x^{(2)}-2z^{(1)}\right]$$

$$=\frac{1}{15}\left[72-6(2.432175)-2(1.91317)\right]=3.572040$$

$$z^{(2)}=\frac{1}{54}\left[110-x^{(2)}-y^{(2)}\right]$$

$$=\frac{1}{54}\left[110-2.432175-3.572040\right] \quad =1.925848$$

III Iteration:

$$x^{(3)} = \frac{1}{27}\left[85 - 6y^{(2)} + z^{(2)}\right]$$

$$= \frac{1}{27}\left[85 - 6(3.572040) + 1.925848\right] \quad = 2.425689$$

$$y^{(3)} = \frac{1}{15}\left[72 - 6x^{(3)} - 2z^{(2)}\right]$$

$$= \frac{1}{15}\left[72 - 6(2.425689) - 2(1.925848)\right] = 3.572945$$

$$z^{(3)} = \frac{1}{54}\left[110 - x^{(3)} - y^{(3)}\right]$$

$$= \frac{1}{54}[110 - 2.425689 - 3.572945] \quad = 1.925951$$

IV Iteration:

$$x^{(4)} = \frac{1}{27}\left[85 - 6y^{(3)} + z^{(3)}\right]$$

$$= \frac{1}{27}\left[85 - 6(3.572945) + 1.925951\right] \quad = 2.42549$$

$$y^{(4)} = \frac{1}{15}\left[72 - 6x^{(4)} - 2z^{(3)}\right]$$

$$= \frac{1}{15}\left[72 - 6(2.42549) - 2(1.925951)\right] = 3.573011$$

$$z^{(4)} = \frac{1}{54}\left[110 - x^{(4)} - y^{(4)}\right]$$

$$= \frac{1}{54}[110 - 2.42549 - 3.573011] \quad = 1.925954$$

V Iteration:

$$x^{(5)} = \frac{1}{27}\left[85 - 6y^{(4)} + z^{(4)}\right]$$

$$= \frac{1}{27}\left[85 - 6(3.573011) + 1.925954\right] \quad = 2.425477$$

$$y^{(5)} = \frac{1}{15}\left[72 - 6x^{(5)} - 2z^{(4)}\right]$$

$$= \frac{1}{15}\left[72 - 6(2.425477) - 2(1.925954)\right] \quad = 3.573015$$

$$z^{(5)} = \frac{1}{54}\left[110 - x^{(5)} - y^{(5)}\right]$$

$$= \frac{1}{54}[110 - 2.425477 - 3.573015] \quad = 1.92595$$

Iteration	x	y	z
0	–	0	0
1	3.14815	3.54074	1.91317
2	2.432175	3.572040	1.925848
3	2.425689	3.572945	1.925951
4	2.42549	3.573011	1.925954
5	2.42548	3.573015	1.92595

From the table of values we notice that the 4th and 5th iterations coincide upto 4 places of decimals.

So, the solution is $x = \mathbf{2.42548}$, $y = \mathbf{3.57302}$, $z = \mathbf{1.92595}$ ∎

Example 2

Apply Gauss-Seidel method to solve the system of equations $20x + y - 2z = 17$; $3x + 20y - z = -18$; $2x - 3y + 20z = 25$.

Solution

We note that the largest coefficient is attached to different variables in the different equations. So the given equations are diagonally dominant.

$$20x + y - 2z = 17 \quad \Rightarrow x = \frac{1}{20}[17 - y + 2z]$$

$$3x + 20y - z = -18 \Rightarrow y = \frac{1}{20}[-18 - 3x + z]$$

$$2x - 3y + 20z = 25 \quad \Rightarrow z = \frac{1}{20}[25 - 2x + 3y]$$

Initially take $\quad y^{(0)} = 0,\ z^{(0)} = 0$

I Iteration:

$$x^{(1)} = \frac{1}{20}\left[17 - y^{(0)} + 2z^{(0)}\right] \qquad = \frac{17}{20} = 0.85$$

$$y^{(1)} = \frac{1}{20}\left[-18 - 3x^{(1)} + z^{(0)}\right] \qquad = \frac{1}{20}[-18 - 3(0.85) + 0] = -1.0275$$

$$z^{(1)} = \frac{1}{20}\left[25 - 2x^{(1)} + 3y^{(1)}\right]$$

$$= \frac{1}{20}[25 - 2(0.85) + (-1.0275)] = 1.010875$$

II Iteration:

$$x^{(2)} = \frac{1}{20}\left[17 - y^{(1)} + 2z^{(1)}\right]$$

$$= \frac{1}{20}\left[17 + 1.0275 + 2(1.010875)\right] \qquad = 1.00246$$

$$y^{(2)} = \frac{1}{20}\left[-18 - 3x^{(2)} + z^{(1)}\right]$$

$$= \frac{1}{20}\left[-18 - 3(1.00246) + 1.010875\right] \qquad = -0.999825$$

$$z^{(2)} = \frac{1}{20}\left[25 - 2x^{(2)} + 3y^{(2)}\right]$$

$$= \frac{1}{20}\left[25 - 2(1.00246) + 3(-0.999825)\right] = 0.99978$$

III Iteration:

$$x^{(3)} = \frac{1}{20}\left[17 - y^{(2)} + 2z^{(2)}\right]$$

$$= \frac{1}{20}\left[17 + 0.999825 + 2(0.99978)\right] \qquad = 0.999969$$

$$y^{(3)} = \frac{1}{20}\left[-18 - 3x^{(3)} + z^{(2)}\right]$$

$$= \frac{1}{20}\left[-18 - 3(0.999969) + (0.99978)\right] \qquad = -1.0000064$$

$$z^{(3)} = \frac{1}{20}\left[25 - 2x^{(3)} + 3y^{(3)}\right]$$

$$= \frac{1}{20}\left[25 - 2(0.999969) + 3(-1.0000064)\right] = 1.000002$$

Iteration	x	y	z
0	–	0	0
1	0.85	−1.0275	1.010875
2	1.00246	−0.999825	0.99978
3	0.999969	−1.0000064	1.000002

We shall take the solution as $\mathbf{x = 1, y = -1, z = 1}$

It is easily verified that it is the exact solution. ■

Example 3

Solve by Gauss-Seidel iteration the given system of equations starting with (0, 0, 0, 0) as solution. Do 5 iterations only. $4x_1 - x_2 - x_3 = 2;\ -x_1 + 4x_2 - x_4 = 2;\ -x_1 + 4x_3 - x_4 = 1;\ -x_2 + x_3 + 4x_4 = 1.$

Solution

The given equations are diagonally dominant.

$$\therefore \quad x_1 = \frac{1}{4}[2 + x_2 + x_3]$$

$$x_2 = \frac{1}{4}[2 + x_1 + x_4]$$

$$x_3 = \frac{1}{4}[1 + x_1 + x_4]$$

$$x_4 = \frac{1}{4}[1 + x_2 + x_3]$$

Initially, take $x_2^{(0)} = 0,\ x_3(0) = 0,\ x_4^{(0)} = 0$

I Iteration:

$$x_1^{(1)} = \left[2 + x_2^{(0)} + x_3^{(0)}\right] = \frac{1}{4}[2] = 0.5$$

$$x_2^{(1)} = \frac{1}{4}\left[2 + x_1^{(1)} + x_4^{(0)}\right] = \frac{1}{4}[2 + 0.5] = 0.625$$

$$x_3^{(1)} = \frac{1}{4}\left[1 + x_1^{(1)} + x_4^{(0)}\right] = \frac{1}{4}[1 + 0.5] = 0.375$$

$$x_4^{(1)} = \frac{1}{4}\left[1 + x_2^{(1)} + x_3^{(1)}\right] = \frac{1}{4}[1 + 0.625 + 0.375] = 0.5$$

II Iteration:

$$x_1^{(2)} = \frac{1}{4}\left[2 + x_2^{(1)} + x_3^{(1)}\right] = \frac{1}{4}[2 + 0.625 + 0.375] = 0.75$$

$$x_2^{(2)} = \frac{1}{4}\left[1 + x_1^{(2)} + x_4^{(1)}\right] = \frac{1}{4}[2 + 0.75 + 0.5] = 0.8125$$

$$x_3^{(2)} = \frac{1}{4}\left[1 + x_1^{(2)} + x_4^{(1)}\right] = \frac{1}{4}[1 + 0.75 + 5] = 0.5625$$

$$x_4^{(2)} = \frac{1}{4}\left[1 + x_2^{(2)} + x_3^{(2)}\right] = \frac{1}{4}[1 + 0.8125 + 0.5625] = 0.59375$$

III Iteration:

$$x_1^{(3)} = \frac{1}{4}\left[2 + x_2^{(2)} + x_3^{(2)}\right] = \frac{1}{4}[2 + 0.8125 + 0.5625] \quad = 0.84375$$

$$x_2^{(3)} = \frac{1}{4}\left[2 + x_1^{(3)} + x_4^{(2)}\right] = \frac{1}{4}[2 + 0.84375 + 0.59375] \quad = 0.859375$$

$$x_3^{(3)} = \frac{1}{4}\left[1 + x_1^{(3)} + x_4^{(2)}\right] = \frac{1}{4}[1 + 0.84375 + 0.59375] \quad = 0.609375$$

$$x_4^{(3)} = \frac{1}{4}\left[1 + x_2^{(3)} + x_3^{(3)}\right] = \frac{1}{4}[1 + 0.859375 + 0.609375] = 0.6171875$$

IV Iteration:

$$x_1^{(4)} = \frac{1}{4}\left[2 + x_2^{(3)} + x_3^{(3)}\right] = \frac{1}{4}[2 + 0.859375 + 0.609375] \quad = 0.8671875$$

$$x_2^{(4)} = \frac{1}{4}\left[2 + x_1^{(4)} + x_4^{(3)}\right] = \frac{1}{4}[2 + 0.8671875 + 0.6171875] \quad = 0.87109375$$

$$x_3^{(4)} = \frac{1}{4}\left[1 + x_1^{(4)} + x_4^{(3)}\right] = \frac{1}{4}[1 + 0.8671875 + 0.6171875] \quad = 0.62109375$$

$$x_4^{(4)} = \frac{1}{4}\left[1 + x_2^{(4)} + x_3^{(4)}\right] = \frac{1}{4}[1 + 0.87109375 + 0.62109375] = 0.623047$$

V Iteration:

$$x_1^{(5)} = \frac{1}{4}\left[2 + x_2^{(4)} + x_3^{(4)}\right] = \frac{1}{4}[2 + 0.87109375 + 0.62109375] = 0.873047$$

$$x_2^{(5)} = \frac{1}{4}\left[2 + x_1^{(5)} + x_4^{(4)}\right] = \frac{1}{4}[2 + 0.873047 + 0.623047] \quad = 0.874024$$

$$x_3^{(5)} = \frac{1}{4}\left[1 + x_1^{(5)} + x_4^{(4)}\right] = \frac{1}{4}[1 + 0.873047 + 0.623047] \quad = 0.624024$$

$$x_4^{(5)} = \frac{1}{4}\left[1 + x_2^{(5)} + x_3^{(5)}\right] = \frac{1}{4}[1 + 0.874024 + 0.624024] \quad = 0.624512$$

Iteration	x_1	x_2	x_3	x_4
1	0.5	0.625	0.375	0.5
2	0.75	0.8125	0.5625	0.59375
3	0.84375	0.859375	0.609375	0.6171875
4	0.8671875	0.87109375	0.624024	0.623047
5	0.873047	0.874024	0.624024	0.624512

$\therefore$ the solution is $\mathbf{x_1 = 0.873047,\ x_2 = 0.874024,\ x_3 = 0.624024,\ x_4 = 0.624512}$ ■

Exercises 3.2

Solve by Gauss-Seidel iteration method or Jacobi method

(1) $8x - y + z = 18, x + y - 3z = -6, 2x + 5y - 2z = 3$

(2) $4x + 2y + z = 14, x + 5y - z = 10, x + y + 8z = 20$

(3) $x + 3y + 52z = 173.61, x - 27y + 2z = 71.31, 41x - 2y + 3z = 65.46$, starting with $x = 1, y = -2, z = 3$

(4) $10x - 5y - 2z = 3, 4x - 10y + 3z = -3, x + 6y + 10z = -3$

(5) $8x - 3y + 2z = 20, 4x + 11y - z = 33, 6x + 3y + 12z = 35$

(6) $10x + 2y + z = 9, x + 10y - z = -22, 2x - 3y - 10z = -22$

(7) $20x + y - 2z = 17, 3x + 20y - z = -18, 2x - 3y + 20z = 25$

(8) $5x + 2y + z = 12, x + 4y + 2z = 15, x + 2y + 5z = 20$

(9) $8x - y + z = 18, 2x + 5y - 2z = 3, x + y - 3z = -6$

(10) $1.2x + 2.1y + 4.2z = 9.9,\ 5.3x + 6.1y + 4.7z = 21.6,\ 9.2x + 8.3y + z = 15.2$

Answers 3.2

(1) $x_1 = 2, y = 1, z = 3$
(2) $x = y = z = 2$
(3) $x = 1.23, y = -2.34, z = 3.45$
(4) $x = 0.3416, y = 0.2851, z = -0.50$
(5) $x = 3, y = 2, z = 1$
(6) $x = 1, y = -2, z = 3$
(7) $x = 1, y = -2, z = 1$
(8) $x = 1, y = 2, z = 3$
(9) $x = 2, y = 1, z = 3$
(10) $x = -13.223, y = 16.766, z = -2.306$

EIGEN VALUE PROBLEM

The problem of determining the eigen values and eigen vectors of a square matrix is known as **eigen value problem**. This problem occurs frequently in physical and engineering problems.

Definition: Let A be a $n \times n$ square matrix. A number λ is called an eigen value of A if there exists a non-zero $n \times 1$ column matrix X such that $AX = \lambda X$.

Then X is called an eigen vector of A corresponding to the eigen value λ.

Power Method

In practical applications, quite often, it is required to find the numerically largest eigen value. The power method is a simple iterative method which enables us to find the approximate value of the numerically largest eigen value of A. We shall discuss the method.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the distinct eigen values of A and let $|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|$. So λ_1 is the dominant eigen value of A. Let $X_1, X_2, \ldots, X_n$ be the corresponding eigen vectors.

Then $AX_i = \lambda_i X_i$, $i = 1,2,3,\ldots,n$.

Note that the X_i are n-component column vectors.

The method is applicable if the complete set of n eigen vectors are linearly independent. Then any vector $X^{(0)}$ in the space of the eigen vectors $X_1, X_2, \ldots, X_n$ can be written as

$$X^{(0)} = C_1X_1 + C_2X_2 + \ldots + C_nX_n,$$

where $C_1, C_2, \ldots C_n$ are constants.

Premultiplying by A, we get,

$$\begin{aligned} X^{(1)} = AX^{(0)} &= A(C_1X_1 + C_2X_2 + \ldots + C_nX_n) \\ &= C_1(AX_1) + C_2(AX_2) + \ldots + C_n(AX_n) \\ &= C_1(\lambda_1X_1) + C_2(\lambda_2X_2) + \ldots + C_n(\lambda_nX_n) \\ &= (C_1\lambda_1)X_1 + (C_2\lambda_2)X_2 + \ldots + (C_n\lambda_n)X_n \end{aligned}$$

Again applying A and simplifying we get

$$\begin{aligned} AX^{(1)} = A^2X^{(0)} &= (C_1\lambda_1)AX_1 + (C_2\lambda_2)AX_2 + \ldots + C_n\lambda_n(AX_n) \\ &= (C_1\lambda_1)\lambda_1X_1 + (C_2\lambda_2)\lambda_2X_2 + \ldots + (C_n\lambda_n)\lambda_nX_n \end{aligned}$$

$$\Rightarrow \qquad X^{(2)} = A^2X^{(0)} = C_1\lambda_1^2X_1 + C_2\lambda_2^2X_2 + \ldots + C_n\lambda_n^2X_n, \text{ where } X^{(2)} = AX^{(1)}.$$

Proceeding this way, after m such multipliers, we get

$$X^{(m)} = A^mX^{(0)} = C_1\lambda_1^mX_1 + C_2\lambda_2^mX_2 + \ldots + C_n\lambda_n^mX_n$$

Since λ_1 is numerically largest, taking out λ_1^m we get

$$X^{(m)} = \lambda_1^m\left[C_1X_1 + C_2\left(\frac{\lambda_2}{\lambda_1}\right)^m X_2 + \ldots + C_n\left(\frac{\lambda_n}{\lambda_1}\right)^m X_n\right]$$

Since $\left|\frac{\lambda_i}{\lambda_1}\right| < 1$ for all $i = 2,3,\ldots,n$, we get $\left|\frac{\lambda_i}{\lambda_1}\right|^m \to 0$ as $m \to \infty$

$\therefore$ as $m \to \infty$, $X^{(m)} \to \lambda_1^mC_1X_1$ if $C_1 \neq 0$ and so $X^{(m)}$ is a multiple of the eigen vector X_1.

The vector $C_1X_1 + C_2\left(\frac{\lambda_2}{\lambda_1}\right)^m X_2 + \ldots + C_m\left(\frac{\lambda_m}{\lambda_1}\right)^m X_n \to C_1X_1$;

which is an eigen vector corresponding to λ_1.

The eigen value λ_1 is obtained as the ratio of the corresponding components of $X^{(m+1)}$ and $X^{(m)}$

$$\lambda_1 = \lim_{m\to\infty}\frac{\left[X^{(m+1)}\right]_r}{\left[X^{(m)}\right]_r}, r = 1,2,3,\ldots,n$$

where the suffix r denotes the r^{th} component of the vector.

Note:

(1) If $|\lambda_2|$ is much smaller than $|\lambda_1|$, then the convergence will be faster.

(2) In order to keep the round off error in control, we normalise the vector (such that the largest element is 1) before premultiplying by A.

(3) If another eigen value is nearer to λ_1, then convergence will be very slow.

(4) Finding the largest eigen value when the roots are not different is beyond the scope of this book.

Working rule to find the largest eigen-value (numerically)

1. Let X_0 be the initial vector which is usually chosen as a vector with all components equal to 1 (i.e. normalised).

   ie. $X_0 = \begin{bmatrix}1\\1\\1\end{bmatrix}$. Sometimes we use $X_0 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ or $\begin{bmatrix}0\\0\\1\end{bmatrix}$

2. Form the product AX_0 and express it in the form $AX_0 = \lambda_1 X_1$, where X_1 is normalised by taking out the largest component λ_1.
3. Form $AX_1 = \lambda_2 X_2$, where X_2 is normalised in the same way and continue the process.
4. Thus we have a sequence of equations $AX_0 = \lambda_1 X_1, AX_1 = \lambda_2 X_2, AX_2 = \lambda_3 X_3, \ldots$ We stop at the stage where X_{r-1}, X_r are almost same.

Then λ_r is the largest eigen value and X_r is the corresponding eigen vector.

To find the numerically smallest eigen value of A

Method (1): Obtain the dominant eigen value λ_1 of A and the dominant eigen value λ_2 of $B = A - \lambda_1 I$. Then the numerically smallest eigen value of A is $\lambda_1 + \lambda_2$.

Method (2): Find the dominant eigen value λ of A^{-1} and then $\frac{1}{\lambda}$ is the smallest eigen value of A.

WORKED EXAMPLES

Example 1

Find the larger eigen value of the matrix $\begin{bmatrix}4 & 1\\1 & 3\end{bmatrix}$ and compare your result with the explicit solution of the characteristic equation.

Solution

Let $A = \begin{bmatrix}4 & 1\\1 & 3\end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} 4-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 - S_1\lambda + S_2 = 0$$

where S_1 = sum of the diagonal elements of A

$= 4 + 3 = 7$

$$S_2 = |A| = \begin{vmatrix} 4 & 1 \\ 1 & 3 \end{vmatrix} = 12 - 1 = 11$$

Characteristic equation is $\lambda^2 - 7\lambda + 11 = 0$

$$\Rightarrow \quad \lambda = \frac{7 \pm \sqrt{49-44}}{2} = \frac{7 \pm \sqrt{5}}{2}$$

$\therefore$ the largest eigen value is $\dfrac{7+\sqrt{5}}{2} = 4.6180$

We shall now find the largest eigen value by the iterative power method.

Take $X_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as the initial vector.

Then

$$AX_0 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4+1 \\ 1+3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} = 5\begin{bmatrix} 1 \\ \frac{4}{5} \end{bmatrix} = 5X_1$$

Now $$AX_1 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 4+0.8 \\ 1+2.4 \end{bmatrix} = \begin{bmatrix} 4.8 \\ 3.4 \end{bmatrix} = 4.8\begin{bmatrix} 1 \\ 0.7083 \end{bmatrix} = 4.8X_2$$

$$AX_2 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.7083 \end{bmatrix} = \begin{bmatrix} 4+0.7083 \\ 1+2.125 \end{bmatrix} = \begin{bmatrix} 4.7083 \\ 3.125 \end{bmatrix} = 4.7083\begin{bmatrix} 1 \\ 0.6637 \end{bmatrix} = 4.7083X_3$$

$$AX_3 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.6637 \end{bmatrix} = \begin{bmatrix} 4+0.6637 \\ 1+1.9911 \end{bmatrix} = \begin{bmatrix} 4.6637 \\ 2.9911 \end{bmatrix} = 4.6637\begin{bmatrix} 1 \\ 0.6414 \end{bmatrix} = 4.6637X_4$$

$$AX_4 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.6414 \end{bmatrix} = \begin{bmatrix} 4+0.6414 \\ 1+1.9242 \end{bmatrix} = \begin{bmatrix} 4.6414 \\ 2.9242 \end{bmatrix} = 4.6414\begin{bmatrix} 1 \\ 0.63 \end{bmatrix} = 4.6414X_5$$

$$AX_5 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.63 \end{bmatrix} = \begin{bmatrix} 4+0.63 \\ 1+1.89 \end{bmatrix} = \begin{bmatrix} 4.63 \\ 2.89 \end{bmatrix} = 4.63\begin{bmatrix} 1 \\ 0.6242 \end{bmatrix} = 4.63X_6$$

$$AX_6 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.6242 \end{bmatrix} = \begin{bmatrix} 4+0.6242 \\ 1+1.8726 \end{bmatrix} = \begin{bmatrix} 4.6242 \\ 2.8726 \end{bmatrix} = 4.6242\begin{bmatrix} 1 \\ 0.6212 \end{bmatrix} = 4.6242X_7$$

$$AX_7 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.6212 \end{bmatrix} = \begin{bmatrix} 4+0.6212 \\ 1+1.8636 \end{bmatrix} = \begin{bmatrix} 4.6212 \\ 2.8636 \end{bmatrix} = 4.6212\begin{bmatrix} 1 \\ 0.6197 \end{bmatrix} = 4.6212X_8$$

$$AX_8 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.6197 \end{bmatrix} = \begin{bmatrix} 4+0.697 \\ 1+1.8591 \end{bmatrix} = \begin{bmatrix} 4.6197 \\ 2.8591 \end{bmatrix} = 4.6197\begin{bmatrix} 1 \\ 0.6189 \end{bmatrix} = 4.6197X_9$$

$$AX_9 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.6189 \end{bmatrix} = \begin{bmatrix} 4+0.6189 \\ 1+1.8567 \end{bmatrix} = \begin{bmatrix} 4.6189 \\ 2.8567 \end{bmatrix} = 4.6189\begin{bmatrix} 1 \\ 0.6185 \end{bmatrix} = 4.6189X_{10}$$

$$AX_{10} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.6185 \end{bmatrix} = \begin{bmatrix} 4+0.6185 \\ 1+1.8555 \end{bmatrix} = \begin{bmatrix} 4.6185 \\ 2.8555 \end{bmatrix} = 4.6185\begin{bmatrix} 1 \\ 0.6183 \end{bmatrix} = 4.6185X_{11}$$

$$AX_{11} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.6183 \end{bmatrix} = \begin{bmatrix} 4+0.6183 \\ 1+1.8549 \end{bmatrix} = \begin{bmatrix} 4.6183 \\ 2.8549 \end{bmatrix} = 4.6183\begin{bmatrix} 1 \\ 0.6182 \end{bmatrix} = 4.6183X_{12}$$

$$AX_{12} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.6182 \end{bmatrix} = \begin{bmatrix} 4+0.6182 \\ 1+1.8546 \end{bmatrix} = \begin{bmatrix} 4.6182 \\ 2.8546 \end{bmatrix} \doteq 4.6182\begin{bmatrix} 1 \\ 0.6182 \end{bmatrix} = 4.6182X_{13}$$

We find the eigen value is **4.618** correct to 3 places of decimal and the eigen vector is $\begin{bmatrix} \mathbf{1} \\ \mathbf{0.618} \end{bmatrix}$

The largest eigen value due to explict method and power method are the same. ■

Example 2

Find the numerically largest eigen value of $A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ and the corresponding eigen vector.

Solution

Given $A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$. Take $X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ as the initial vector.

Then $$AX_0 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 25 \\ 1 \\ 2 \end{bmatrix} = 25\begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} = 25X_1$$

$$AX_1 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} = \begin{bmatrix} 25.2 \\ 1.12 \\ 1.18 \end{bmatrix} = 25.2\begin{bmatrix} 1 \\ 0.0444 \\ 0.0667 \end{bmatrix} = 25.2X$$

$$AX_2 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.0444 \\ 0.0667 \end{bmatrix} = \begin{bmatrix} 25.1778 \\ 1.1332 \\ 1.7337 \end{bmatrix} = 25.1778\begin{bmatrix} 1 \\ 0.0450 \\ 0.0688 \end{bmatrix} = 25.1778X_3$$

$$AX_3 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.0450 \\ 0.0688 \end{bmatrix} = \begin{bmatrix} 25.1826 \\ 1.135 \\ 1.7248 \end{bmatrix} = 25.1826\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix} = 25.1826X_4$$

$$AX_4 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix} = \begin{bmatrix} 25.1821 \\ 1.1353 \\ 1.7260 \end{bmatrix} = 25.1821\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix} = 25.1821X_5$$

where $$X_5 = \begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix}$$

Since $X_4 = X_5$, we stop the iteration and the largest eigen value is 25.1821 and the corresponding eigenvector is $$\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix}$$

Example 3

Find the smallest eigen value of $A = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix}$.

Solution

We shall use method (2) That is we shall find the largest eigen value of A^{-1}

Given $$A = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix}$$

then $$|A| = -15(-24+24) - 4(20-120) + 3(-40+240) = 400 + 600 = 1000 \neq 0$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{1000}\begin{bmatrix} 0 & 100 & 200 \\ -20 & -90 & 20 \\ 60 & 120 & 140 \end{bmatrix}^T$$

$$= \frac{1}{1000}\begin{bmatrix} 0 & -20 & 60 \\ 100 & -90 & 120 \\ 200 & 20 & 140 \end{bmatrix} = \begin{bmatrix} 0 & -0.02 & 0.06 \\ 0.1 & 0.09 & 0.12 \\ 0.2 & 0.02 & 0.14 \end{bmatrix}$$

Take $X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as the initial vector

Then $$A^{-1}X_0 = \begin{bmatrix} 0 & -0.02 & 0.06 \\ 0.1 & -0.09 & 0.12 \\ 0.2 & 0.02 & 0.14 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.04 \\ 0.13 \\ 0.36 \end{bmatrix} = 0.36\begin{bmatrix} 0.111 \\ 0.361 \\ 1 \end{bmatrix} = 0.36X_1$$

Now $$A^{-1}X_1 = \begin{bmatrix} 0 & -0.02 & 0.06 \\ 0.1 & -0.09 & 0.12 \\ 0.2 & 0.02 & 0.14 \end{bmatrix}\begin{bmatrix} 0.111 \\ 0.361 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.053 \\ 0.099 \\ 0.169 \end{bmatrix} = 0.169\begin{bmatrix} 0.314 \\ 0.586 \\ 1 \end{bmatrix} = 0.169X_2$$

$$A^{-1}X_2 = \begin{bmatrix} 0 & -0.02 & 0.06 \\ 0.1 & -0.09 & 0.12 \\ 0.2 & 0.02 & 0.14 \end{bmatrix}\begin{bmatrix} 0.314 \\ 0.586 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.048 \\ 0.099 \\ 0.215 \end{bmatrix} = 0.215\begin{bmatrix} 0.223 \\ 0.460 \\ 1 \end{bmatrix} = 0.215X_3$$

$$A^{-1}X_3 = \begin{bmatrix} 0 & -0.02 & 0.06 \\ 0.1 & -0.09 & 0.12 \\ 0.2 & 0.02 & 0.14 \end{bmatrix}\begin{bmatrix} 0.223 \\ 0.460 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.051 \\ 0.101 \\ 0.194 \end{bmatrix} = 0.194\begin{bmatrix} 0.263 \\ 0.515 \\ 1 \end{bmatrix} = 0.194X_4$$

$$A^{-1}X_4 = \begin{bmatrix} 0 & -0.02 & 0.06 \\ 0.1 & -0.09 & 0.12 \\ 0.2 & 0.02 & 0.14 \end{bmatrix}\begin{bmatrix} 0.263 \\ 0.515 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.050 \\ 0.100 \\ 0.203 \end{bmatrix} = 0.203\begin{bmatrix} 0.246 \\ 0.493 \\ 1 \end{bmatrix} = 0.203X_5$$

$$A^{-1}X_5 = \begin{bmatrix} 0 & -0.02 & 0.06 \\ 0.1 & -0.09 & 0.12 \\ 0.2 & 0.02 & 0.14 \end{bmatrix}\begin{bmatrix} 0.246 \\ 0.493 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.050 \\ 0.100 \\ 0.199 \end{bmatrix} = 0.199\begin{bmatrix} 0.251 \\ 0503 \\ 1 \end{bmatrix} = 0.199X_6$$

$$A^{-1}X_6 = \begin{bmatrix} 0 & -0.02 & 0.06 \\ 0.1 & -0.09 & 0.12 \\ 0.2 & 0.02 & 0.14 \end{bmatrix}\begin{bmatrix} 0.251 \\ 0.503 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.050 \\ 0.100 \\ 0.200 \end{bmatrix} = 0.2\begin{bmatrix} 0.25 \\ 0.50 \\ 1 \end{bmatrix} = 0.2X_7$$

$$A^{-1}X_7 = \begin{bmatrix} 0 & -0.02 & 0.06 \\ 0.1 & -0.09 & 0.12 \\ 0.2 & 0.02 & 0.14 \end{bmatrix}\begin{bmatrix} 0.25 \\ 0.50 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.050 \\ 0.100 \\ 0.200 \end{bmatrix} = 0.2\begin{bmatrix} 0.25 \\ 0.50 \\ 1 \end{bmatrix} = 0.2X_8$$

Since $X_7 = X_8$, we stop the iteration and take $\lambda = 0.2$ as the largest eigen value of A^{-1}

$\therefore$ the numerically smallest eigen value of A is $\frac{1}{0.2} = 5$ ■

Example 4

Using power method, find all the eigen values of $A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$.

Solution

Given $$A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

First we shall find the numerically largest eigen value.

Let $X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be the initial vector.

$$\text{Then } AX_0 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ -0.333 \\ 1 \end{bmatrix} = 6X_1$$

$$AX_1 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -0.333 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0.666 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 0.111 \\ 1 \end{bmatrix} = 6X_2$$

$$AX_2 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.111 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -0.222 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ -0.037 \\ 1 \end{bmatrix} = 6X_3$$

$$AX_3 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -0.037 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0.074 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 0.012 \\ 1 \end{bmatrix} = 6X_4$$

$$AX_4 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.012 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -0.024 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ -0.004 \\ 1 \end{bmatrix} = 6X_5$$

$$AX_5 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -0.004 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0.008 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 0.001 \\ 1 \end{bmatrix} = 6X_6$$

$$AX_6 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.001 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -0.002 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 6X_7$$

Approximating to 2 decimals -0.002 is taken as 0, the dominant eigen value of A is

$\lambda_1 = 6$ and the corresponding vector is $[1 \quad 0 \quad 1]^T$

To find the smallest eigen of A we shall use method (1). So, we form the matrix

$B = A - \lambda_1 I = A - 6I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ and find the numerically largest eigen value of B.

Take $Y_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as the initial vector.

Then $BY_0 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \\ 0 \end{bmatrix} = -8 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -8Y_1$

$$BY_1 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \\ 0 \end{bmatrix} = -8 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -8Y_2$$

Since $Y_1 = Y_2$, we stop the iteration and $\lambda_2 = -8$ as the dominant eigen value of B.

$\therefore$ the smallest eigen value of A is $\lambda_1 + \lambda_2 = 6 + (-8) = -2$

If λ is the third eigen value, then the eigen values of A are $6, -2, \lambda$

$\therefore 6 + (-2) + \lambda =$ sum of the diagonal elements of A

$\Rightarrow \quad 4 + \lambda = 5 + (-2) + 5 = 8 \quad \Rightarrow \quad \lambda = 4$

$\therefore$ the eigen value of A are **6, 4, −2.** ■

Exercises 3.3

(1) Obtain by power method, the numerically largest eigen value of the matrix $\begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}$.

(2) Obtain the largest eigen value and the corresponding eigen vector of the matrix $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

(3) By power method find the largest (numerically) eigen value of the matrix $\begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$. Also the corresponding eigen vector.

(4) Find the dominant eigen value and the corresponding eigen vector of the matrix $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$.

(5) Find the dominant eigen value and the corresponding eigen vector of the matrix $\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$.

(6) Find the dominant eigen value and eigen vector of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$ by power method and hence find all the eigen values.

Answers 3.3

(1) -20

(2) $3.41; \begin{bmatrix} -0.71 \\ +1 \\ -0.71 \end{bmatrix}$

(3) $7; [0.3 \quad 0.2 \quad 1]^T$

(4) $8; [1 \quad 0.5 \quad 1]^T$

(5) $11.66; [0.25 \quad 0.421 \quad 1]^T$

(6) $7; [0.56 \quad 0.18 \quad 1]^T$ and $1, -4$

Jacobi's Method to Find the Eigen Values of a Symmetric Matrix

Let A be a real symmetric matrix. We know that the eigen values of a real symmetric matrix are real and there exists an orthogonal matrix S such that $S^{-1}AS$ is a diagonal matrix D-whose diagonal elements are the eigen values of A.

But in Jacobi's method the matrix S is formed by a series of orthogonal transformations (or orthogonal matrices) $S_1, S_2, S_3 \ldots, S_n$ such that $S = S_1 S_2 S_3 \ldots S_n$. In this method we make all the off-diagonal elements of A as zero in a systematic way as below.

Let $A = [a_{ij}]$ be $n \times n$ matrix.

Among all the off-diagonal elements of A, choose the numerically largest element. Let a_{ik} be the element such that $|a_{ik}|$ is largest. Form a 2×2 submatrix with the elements $a_{ii}, a_{ik}, a_{ki}, a_{kk}$.

Let $A_1 = \begin{bmatrix} a_{ii} & a_{ik} \\ a_{ki} & a_{kk} \end{bmatrix}$.

Since A is symmetric $a_{ik} = a_{ki}$ and so A_1 is symmetric.

Let $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ be the orthogonal matrix which transforms A_1 into a diagonal matrix.

Now $R^{-1}A_1R = R^TA_1R$ $\qquad [\because R^{-1} = R^T]$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{ii} & a_{ik} \\ a_{ki} & a_{kk} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a_{ii}\cos\theta + a_{ki}\sin\theta & a_{ik}\cos\theta + a_{kk}\sin\theta \\ -a_{ii}\sin\theta + a_{ki}\cos\theta & -a_{ik}\sin\theta + a_{kk}\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a_{ii}\cos^2\theta + a_{ik}\sin 2\theta + a_{kk}\sin^2\theta & (a_{kk} - a_{ii})\sin\theta\cos\theta + a_{ik}\cos 2\theta \\ (a_{kk} - a_{ii})\sin\theta\cos\theta + a_{ik}\cos 2\theta & a_{ii}\sin^2\theta - a_{ik}\sin 2\theta + a_{kk}\cos^2\theta \end{bmatrix} \qquad (1)$$

Since this a diagonal matrix, we have

$$(a_{kk} - a_{ii})\sin\theta\cos\theta + a_{ik}\cos 2\theta = 0$$

$$\Rightarrow \qquad \frac{1}{2}(a_{ii} - a_{kk})\sin 2\theta = -a_{ik}\cos 2\theta$$

$$\Rightarrow \qquad \tan 2\theta = \frac{2a_{ik}}{a_{ii} - a_{kk}} \qquad \text{if } a_{ii} \neq a_{kk}$$

We shall choose the principal value of θ. ie. $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$

$$\therefore \qquad \theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{ik}}{a_{ii} - a_{kk}}\right) \qquad \text{if } a_{ii} \neq a_{kk} \qquad (2)$$

$$\text{If } a_{ii} = a_{kk}, \text{ then } \theta = \begin{cases} \dfrac{\pi}{4} & \text{if } a_{ik} > 0 \\ -\dfrac{\pi}{4} & \text{if } a_{ik} < 0 \end{cases} \tag{3}$$

When θ is given by (2) or (3) the off-diagonal elements of the matrix (1) are zero and the diagonal elements are simplified.

Geometrically, the matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ represents the transformation rotation in the plane. Thus by performing a two-dimenstional rotation, we make the a_{ik} element 0. Since $-\dfrac{\pi}{4} \le \theta \le \dfrac{\pi}{4}$, the rotation is the smallest.

Now coming back to the main problem, if a_{ik} is the numerically largest off-diagonal element, then the first rotation matrix S_1 is taken as

$$S_1 = \begin{array}{cc} & \begin{array}{ccccccccc} & & & & i^{th} & & k^{th} & & \end{array} \\ \begin{array}{c} \\ \\ \\ i^{th} \\ \\ k^{th} \\ \\ \\ \end{array} & \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ \vdots & & & & & & & & \\ 0 & 0 & 0 & \ldots & \cos\theta & \ldots & -\sin\theta & \ldots & 0 \\ \vdots & & & & & & & & \\ 0 & . & . & \ldots & \sin\theta & \ldots & \cos\theta & \ldots & 0 \\ \vdots & & & & & & & & \\ 0 & . & . & \ldots & 0 & 0\ldots & 0 & \ldots & 1 \end{bmatrix} \end{array}$$

where the elements in the positions (i, i), (i, k), (k, i), (k, k) are $\cos\theta, -\sin\theta, \sin\theta, \cos\theta$ and θ is found by (2) or (3) and all other elements are 1 or 0 as in a unit matrix.

Let $B_1 = S_1^T A S_1$

Next we find the largest off-diagonal element in B_1 and proceed as above with B_1 in the place of A.

Let S_2 be the second rotation matrix then

$$\begin{aligned} B_2 = S_2^T B_1 S_2 &= S_2^T S_1^T A S_1 S_2 \\ &= (S_1 S_2)^T A (S_1 S_2) \end{aligned}$$

Proceeding like this, we get

$$\begin{aligned} B_n &= (S_1 S_2 \ldots S_n)^T A (S_1 S_2 \ldots S_n) \\ &= S^T A S, \text{ where } S = S_1 S_2 \ldots S_n \end{aligned}$$

When n is large (ie. $n \to \infty$) B_n approaches a diagonal matrix whose diagonal elements are eigen values of A. The corresponding eigen vectors are the respective columns of $S = S_1\ S_2 \ldots S_n$.

Note:

(1) The minimal number of rotations (or transformations) requited to bring A into diagonal form is $\frac{(n-1)n}{2}$.

(2) The drawback of Jacobi's method is that the clement made 0 by a transformation may not remain 0 during the subsequent transformations.

(3) The value of θ must be checked with $\sin^2\theta + \cos^2\theta = 1$ for the sake of accuracy.

WORKED EXAMPLES

Example 1

Using Jacobi's method find the eigen values and eigen vectors of $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Solution

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

The largest off-diagonal element is $a_{12} = a_{21} = 2$

The rotation matrix $S_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Since $a_{11} = a_{22}$ and $a_{12} > 0$, we have $\theta = \frac{\pi}{4}$

$$\therefore \qquad S_1 = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Then the transformation (or rotation) gives

$$B_1 = S_1^T A S_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

This is a diagonal matrix. So, the eigenvalues are 3, −1 and the corresponding eigen vectors are the columns of S_1. ie. $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

Note: These eigen vectors are normalised. The eigen vectors are also given as $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ■

Example 2

Using Jacobi's method, find the eigen values and eigen vectors of $\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$.

Solution

Let $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$

The largest off-diagonal element is $a_{12} = a_{21} = 2$

And $a_{11} = 2, a_{22} = -1$. Here $a_{11} \neq a_{22}$

Let $S_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ be the rotation matrix.

Since $a_{11} \neq a_{22}, \tan 2\theta = \dfrac{2a_{12}}{a_{11} - a_{22}} = \dfrac{4}{2-(-1)} = \dfrac{4}{3}$

Here θ is not a familiar angle. So we proceed as below.

We know $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

$$\therefore \quad \frac{2\tan\theta}{1-\tan^2\theta} = \frac{4}{3}$$

$$\Rightarrow \quad 3\tan\theta = 2(1 - \tan^2\theta)$$

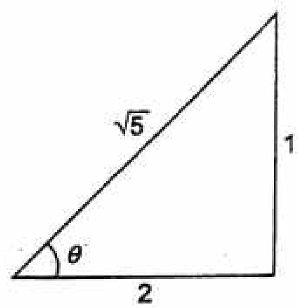
$$\Rightarrow \quad 2\tan^2\theta + 3\tan\theta - 2 = 0$$

$$\Rightarrow \quad (2\tan\theta - 1)(\tan\theta + 2) = 0$$

$$\Rightarrow \quad \tan\theta = \frac{1}{2} \text{ or } -2$$

Since θ is the smallest rotation, $\tan\theta = \dfrac{1}{2}$

$\therefore \sin\theta = \dfrac{1}{\sqrt{5}}, \ \cos\theta = \dfrac{2}{\sqrt{5}}$



$$\therefore \qquad S_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

Then the transformation $B_1 = S_1^T A S_1$

$$\Rightarrow \qquad B_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6}{\sqrt{5}} & \frac{3}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-4}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

Thus B_1 is diagonal matrix.
So the eigen values of A are 3, −2 and the corresponding eigen vectors are the columns of S_1.

$$\text{ie.} \qquad \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \text{ or } \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

■

Example 3

Using Jacobi's method find all the eigen values and eigen vectors of the matrix

$$\begin{bmatrix} \mathbf{1} & \sqrt{\mathbf{2}} & \mathbf{2} \\ \sqrt{\mathbf{2}} & \mathbf{3} & \sqrt{\mathbf{2}} \\ \mathbf{2} & \sqrt{\mathbf{2}} & \mathbf{1} \end{bmatrix}.$$

Solution

$$\text{Let } A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

The largest off-diagonal element is $a_{13} = a_{31}$ and $a_{11} = 1, a_{33} = 1$

The first rotation matrix S_1 will have the (1, 1), (1, 3), (3, 1), (3, 3) positions $\cos\theta, -\sin\theta, \sin\theta, \cos\theta$ and other places 0 or 1 as in a unit matrix.

$$\therefore \qquad S_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

Since $a_{11} = a_{33}$ and $a_{13} > 0$, $\theta = \dfrac{\pi}{4}$

$$\therefore \qquad S_1 = \begin{bmatrix} \cos\frac{\pi}{4} & 0 & -\sin\frac{\pi}{4} \\ 0 & 1 & 0 \\ \sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\therefore$ the first transformation or rotation gives

$$B_1 = S_1^T A S_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{2}} & 2 & \frac{3}{\sqrt{2}} \\ \sqrt{2} & 3 & \sqrt{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

This is not a diagonal matrix. So we make a second rotation or transformation.

The largest off-diagonal element in B_1 is $a_{12} = a_{21} = 2$ and $a_{11} = 3, a_{22} = 3$

The second rotation matrix is $S_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Since $a_{11} = a_{22}$ and $a_{12} > 0, \theta = \dfrac{\pi}{4}$

$$\therefore \qquad S_2 = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

e second transformation gives

$$B_2 = S_2^T B_1 S_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{\sqrt{2}} & \frac{5}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \text{ which is a diagonal matrix.}$$

. the eigen values are 5, 1, −1.

Now
$$S = S_1 S_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

∴ the corresponding eigen vectors are

$$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ \sqrt{2} \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

■

Example 4

Using Jacobi's method, find all the eigen values and eigen vectors of the matrix $\begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$.

Solution

Let $A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

The largest off-diagonal element is $a_{13} = a_{31} = 1$ and $a_{11} = 5, a_{33} = 5$

The first rotation matrix $S_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

Since $a_{11} = a_{33}, a_{13} > 0$, we have $\theta = \frac{\pi}{4}$

$$\therefore \qquad S_1 = \begin{bmatrix} \cos\frac{\pi}{4} & 0 & -\sin\frac{\pi}{4} \\ 0 & 1 & 0 \\ \sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\therefore$ the first transformation matrix gives

$$B_1 = S_1^T A S_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6}{\sqrt{2}} & 0 & \frac{6}{\sqrt{2}} \\ 0 & -2 & 0 \\ \frac{-4}{\sqrt{2}} & 0 & \frac{4}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Thus by a single transformation, A is reduced to diagonal form.

$\therefore$ The eigen values are 6, −2, 4 and the eigen vectors are the columns of S_1

ie. $$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Exercises 3.4

By Jacobi's method find all the eigen values and the corresponding eigen vectors of the matrices.

(1) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ (2) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (3) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

(4) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (5) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Answers 3.4

(1) $5,1,1;\ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ (2) $1,1,3;\ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ (3) $-3,-3,5;\ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

(4) $2,2,8;\ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ (5) $1,1,4;\ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

METHOD OF FACTORISATION OR METHOD OF TRIANGULARISATION

Let $AX = B$ represent a system of linear equations. We factorise A as $A = LU$, where L is a lower triangular matrix and U is an upper triangular matrix.
For simplicity we consider A as 3×3 matrix.

$$\therefore \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Let $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Then $A = LU$

Equating the elements in $A=LU$, we will get 9 equations in 12 unknowns (6 lij's and 6 uij's) So, it is not possible to get the unique solution.

In order to get unique solution, we have to fix values for 3 unknowns, so that there are 9 equations in 9 unknowns.

To obtain unique solution

Doolittle chose $l_{11} = 1,\ l_{22} = 1,\ l_{33} = 1$ ie L is chosen as unit lower triangular matrix and U is an upper triangular matrix.

Crout chose $u_{11} = 1,\ u_{22} = 1,\ u_{33} = 1$ so that U is unit upper triangular matrix and L is a lower triangular matrix.

But the LU Decomposition exists if the principal minors of the matrix A are non-singular. That is

$$a_{11} \neq 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0 \text{ and } |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

Such a factorisation is unique. However, it is only a sufficient condition.
This method is also known as LU Decomposition method.

Doolittle's Method

Consider the equations

$$a_{11}\ x_1 + a_{12}\ x_2 + a_{13}\ x_3 = b_1$$
$$a_{21}\ x_1 + a_{22}\ x_2 + a_{23}\ x_3 = b_2$$
$$a_{31}\ x_1 + a_{32}\ x_2 + a_{33}\ x_3 = b_3$$

Here $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\therefore$ the given system is $AX = B$.

Let $A = LU$, where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Then $LUX = B$.

Put $UX = Y$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, then $LY = B$

Since L and B are known, Y is known.
Substitute in $UX = Y$.
Then U and Y are known and so X is known.
Substitute in $UX = Y$
Then U and Y are known and so X is known.

Note:

(1) In this method, to find the elements of L and U, we equate the corresponding row elements of A and LU.

(2) A^{-1} can also be determined, since $A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$

WORKED EXAMPLES

Example 1

Solve the linear system of equations $2x+y+4z=12$, $8x-3y+2z=20$, $4x+11y-z=33$ by factorisation method.

Solution

The given system of equations is

$$2x+y+4z=12$$
$$8x-3y+2z=20$$
$$4x+11y-z=33$$

Here
$$A=\begin{bmatrix}2 & 1 & 4\\ 8 & -3 & 2\\ 4 & 11 & -1\end{bmatrix},\quad B=\begin{bmatrix}12\\20\\33\end{bmatrix} \text{ and } X=\begin{bmatrix}x\\y\\z\end{bmatrix}$$

$\therefore$ the given system is $AX = B$.

We solve the system by Doolittle's method $\therefore$ we take $A = LU$

where
$$L=\begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix} \text{ and } U=\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$$

$\therefore$
$$\begin{bmatrix}2 & 1 & 4\\ 8 & -3 & 2\\ 4 & 11 & -1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$$

$\Rightarrow$
$$\begin{bmatrix}2 & 1 & 4\\ 8 & -3 & 2\\ 4 & 11 & -1\end{bmatrix}=\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23}\\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix}$$

Equating the elements of I rows, we get $u_{11}=2,\ u_{12}=1,\ u_{13}=4$

Equating the elements of II rows, we get

$$l_{21}u_{11} = 8 \quad \Rightarrow \quad l_{21}\, 2 = 8 \quad \Rightarrow l_{21} = 4$$

$$l_{21}u_{12} + u_{22} = -3 \quad \Rightarrow 4\, 1 + u_{22} = -3 \quad \Rightarrow u_{22} = -7$$

$$l_{21}u_{13} + u_{23} = 2 \quad \Rightarrow 4\, 4 + u_{23} = 2 \quad \Rightarrow u_{23} = -14$$

Equating the elements of III rows, we get

$$l_{31}u_{11} = 4 \quad \Rightarrow l_{31} 2 = 4 \quad \Rightarrow l_{31} = 2$$

$$l_{31}u_{12} + l_{32}u_{22} = 11 \Rightarrow 2\,1 + l_{32}(-7) = 11 \Rightarrow l_{32} = -\frac{9}{7}$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = -1$$

$$\Rightarrow \quad 2.4 + \left(-\frac{9}{7}\right)(-14) + u_{33} = -1$$

$$\Rightarrow \quad 8 + 18 + u_{33} = -1 \quad \Rightarrow u_{33} = -27$$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & -\frac{9}{7} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 2 & 1 & 4 \\ 0 & -7 & -14 \\ 0 & 0 & -27 \end{bmatrix}$$

$$\therefore \quad AX = B \Rightarrow LUX = B$$

Put $UX = Y$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$\therefore \quad LY = B$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & -\frac{9}{7} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 20 \\ 33 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} y_1 \\ 4y_1 + y_2 \\ 2y_1 - \frac{9}{7}y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 20 \\ 33 \end{bmatrix}$$

$$\therefore \quad y_1 = 12,$$

$$4y_1 + y_2 = 20 \quad \Rightarrow y_2 = 20 - 4 \times 12 = 20 - 48 = -28$$

and $$2y_1 - \frac{9}{7}y_2 + y_3 = 33$$

$\Rightarrow$ $$y_3 = 33 - 2y_1 + \frac{9}{7}y_2$$

$$= 33 - 2\times 12 + \frac{9}{7}(-28) = 33 - 24 - 36 = -27$$

$\therefore$ $$Y = \begin{bmatrix} 12 \\ -28 \\ -27 \end{bmatrix}$$

Now $$UX = Y$$

$\Rightarrow$ $$\begin{bmatrix} 2 & 1 & 4 \\ 0 & -7 & -14 \\ 0 & 0 & -27 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ -28 \\ -27 \end{bmatrix}$$

$$\begin{bmatrix} 2x + y + 4z \\ -7y - 14z \\ -27z \end{bmatrix} = \begin{bmatrix} 12 \\ -28 \\ -27 \end{bmatrix}$$

$\therefore$ $$-27z = -27 \quad \Rightarrow z = 1$$

$$-7y - 14z = -28$$

$\Rightarrow$ $$-7y = -28 + 14z = -28 + 14 = -14 \Rightarrow y = 2$$

and $$2x + y + 4z = 12$$

$\Rightarrow$ $$2x = 12 - y - 4z = 12 - 2 - 4\times 1 = 6$$

$\therefore$ $$x = 3$$

$\therefore$ the solution is $\mathbf{x = 3, \ y = 2, \ z = 1}$ ■

Example 2

Factorise the matrix $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ and hence solve the system of equations.

$2x + 3y + z = 9,\ x + 2y + 3z = 6,\ 3x + y + 2z = 8$

Solution

Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

We factorize A using Doolittle's method. $\therefore$ we take $A = LU$, where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

$$\therefore \quad \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

Equating elements of first rows, we get

$$u_{11} = 2, \ u_{12} = 3, \ u_{13} = 1$$

Equating elements of second rows, we get

$$l_{21}u_{11} = 1 \ \Rightarrow l_{21}.2 = 1 \ \Rightarrow l_{21} = \frac{1}{2}$$

$$l_{21}u_{12} + u_{22} = 2$$

$$\Rightarrow \quad \frac{1}{2}.3 + u_{22} = 2 \ \Rightarrow u_{22} = 2 - \frac{3}{2} = \frac{1}{2}$$

$$l_{21}u_{13} + u_{23} = 3 \ \Rightarrow \frac{1}{2}.1 + u_{23} = 3 \ \Rightarrow u_{23} = 3 - \frac{1}{2} = \frac{5}{2}$$

Equating elements of III rows, we get

$$l_{31}u_{11} = 3 \quad \Rightarrow l_{31}.2 = 3 \quad \Rightarrow l_{31} = \frac{3}{2}$$

$$l_{31}u_{12} + l_{32}u_{22} = 1 \quad \Rightarrow \frac{3}{2}.3 + l_{32}.\frac{1}{2} = 1 \ \Rightarrow \frac{1}{2}l_{32} = 1 - \frac{9}{2} = -\frac{7}{2} \quad \Rightarrow l_{32} = -7$$

and

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2$$

$$\Rightarrow \quad \frac{3}{2}.1 + (-7)\frac{5}{2} + u_{33} = 2$$

$$\Rightarrow \quad \frac{3 - 25}{2} + u_{33} = 2 \ \Rightarrow \ u_{33} = 2 + 16 = 18$$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}$$

Given system is $2x + 3y + z = 9$

$$x + 2y + 3z = 6$$

$$3x + y + 2z = 8$$

The coefficient matrix is $= \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

which is same as A

$\therefore$ the matrix equation is $AX = B$, where $B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Now $AX = B \Rightarrow LUX = B$

Put $Y = UX$, then $LY = B$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$\therefore \qquad LY = B$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ \frac{y_1}{2} + y_2 \\ \frac{3}{2}y_1 - 7y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$$\Rightarrow \qquad y_1 = 9$$

$$\frac{y_1}{2} + y_2 = 6 \Rightarrow y_2 = 6 - \frac{y_1}{2} = 6 - \frac{9}{2} = \frac{3}{2}$$

$$\frac{3}{2}y_1 - 7y_2 + y_3 = 8$$

$$\Rightarrow \qquad y_3 = 8 - \frac{3}{2}y_1 + 7y_2$$

$$= 8 - \frac{3}{2} \times 9 + 7 \times \frac{3}{2} = 8 - \frac{27}{2} + \frac{21}{2} = 8 - \frac{6}{2} = 8 - 3 = 5$$

$$\therefore \qquad Y = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}$$

Now $$UX = Y$$

$$\Rightarrow \qquad \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} 2x + 3y + z \\ \frac{y}{2} + \frac{5}{2}z \\ 18z \end{bmatrix} = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}$$

$$\therefore \qquad 2x + 3y + z = 9$$

$$\frac{y}{2} + \frac{5z}{2} = \frac{3}{2}$$

$$\Rightarrow \qquad y + 5z = 3$$

$$18z = 5 \quad \Rightarrow \quad z = \frac{5}{18}$$

$$\therefore \qquad y = 3 - 5z = 3 - \frac{5 \times 5}{18} = \frac{54 - 25}{18} = \frac{29}{18}$$

and $$2x = 9 - 3y - z$$

$$= 9 - 3 \times \frac{29}{18} - \frac{5}{18} = \frac{162 - 87 - 5}{18} = \frac{70}{18}$$

$$\therefore \qquad x = \frac{35}{18}$$

$\therefore$ the solution is $$\mathbf{x = \frac{35}{18},\ y = \frac{29}{18},\ z = \frac{5}{18}}$$ ■

Example 3

(a) Find the matrices L and R such that $LR = A$, where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \quad R = \begin{bmatrix} r_{11} & r_{12} & r_{12} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$

(b) A system of equations is given by $AX = B$, where $X = \{x_1, x_2, x_3\}$ $B = \{14, 18, 20\}$. Rewrite the equation $AX = B$ as $LZ = B$ and $RX = Z$ where $Z = \{z_1, z_2, z_3\}$ and for X, determining Z first, the matrices L, R, A being as given in (a).

Solution

(a) Given

$$A=\begin{bmatrix}1&2&3\\2&5&2\\3&1&5\end{bmatrix},\quad L=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix},\quad \text{and}\quad R=\begin{bmatrix}r_{11}&r_{12}&r_{13}\\0&r_{22}&r_{23}\\0&0&r_{32}\end{bmatrix}$$

By Doolittle's method, we factorize A and we take $A= LR$

$$\therefore\quad \begin{bmatrix}1&2&3\\2&5&2\\3&1&5\end{bmatrix}=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}\begin{bmatrix}r_{11}&r_{12}&r_{13}\\0&r_{22}&r_{23}\\0&0&r_{33}\end{bmatrix}$$

$$=\begin{bmatrix}r_{11}&r_{12}&r_{13}\\l_{21}r_{11}&l_{21}r_{12}+r_{22}&l_{21}r_{13}+r_{23}\\l_{31}r_{11}&l_{31}r_{12}+l_{32}r_{22}&l_{31}r_{13}+l_{32}r_{23}+r_{33}\end{bmatrix}$$

Equating the corresponding elements row wise on both sides, we get

$$r_{11}=1,\qquad r_{12}=2,\qquad r_{13}=3$$

$$l_{21}r_{11}=2\ \Rightarrow\ l_{21}.1=2\ \Rightarrow l_{21}=2$$

$$l_{21}r_{12}+r_{22}=5\ \Rightarrow\ 2.2+r_{22}=5\ \Rightarrow r_{22}=1$$

$$l_{21}r_{13}+l_{23}=2\ \Rightarrow\ 2\times3+r_{23}=2\ \Rightarrow r_{23}=-4$$

$$l_{31}r_{11}=3\ \Rightarrow\ l_{31}.1=3\ \Rightarrow l_{31}=3$$

$$l_{31}r_{12}+l_{32}r_{22}=1$$

$\Rightarrow$ $$l_{31}r_{12}+l_{32}r_{22}.1=1\ \Rightarrow\ l_{32}=1-6=-5$$

and $$l_{31}r_{13}+l_{32}r_{23}+r_{33}=5$$

$\Rightarrow$ $$3\times3+(-5)\times(-4)+r_{33}=5\ \Rightarrow\ r_{33}=5-29=-24$$

$$\therefore\quad L=\begin{bmatrix}1&0&0\\2&1&0\\3&-5&1\end{bmatrix}\quad\text{and}\quad R=\begin{bmatrix}1&2&3\\0&1&-4\\0&0&-24\end{bmatrix}\qquad(3)$$

(b) Given $AX=B$, $LZ=B$ and $RX=Z$

where $$A=\begin{bmatrix}1&2&3\\2&5&2\\3&1&5\end{bmatrix},$$ L and R are given by (3),

$$B=\begin{bmatrix}14\\18\\20\end{bmatrix},\quad X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\quad\text{and}\quad Z=\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix}$$

$\therefore \qquad LZ = B$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 18 \\ 20 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} z_1 \\ 2z_1 + z_2 \\ 3z_1 - 5z_2 + z_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 18 \\ 20 \end{bmatrix}$$

$$\therefore \quad z_1 = 14$$
$$2z_1 + z_2 = 18$$
$$3z_1 - 2z_2 + z_3 = 20$$

$$\therefore \quad z_2 = 18 - 2z_1 = 18 - 2 \times 14 = 18 - 28 = -10$$

and
$$z_3 = 20 - 3z_1 + 5z_2$$
$$= 20 - 3 \times 14 + 5 \times (-10) = 20 - 42 - 50 = -72$$

$$\therefore \quad Z = \begin{bmatrix} 14 \\ -10 \\ -72 \end{bmatrix}$$

Now $\qquad RX = Z$

$$\Rightarrow \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -24 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -10 \\ -72 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ x_2 - 4x_3 \\ -24x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -10 \\ -72 \end{bmatrix}$$

$$\therefore \quad x_1 + 2x_2 + 3x_3 = 14$$
$$x_2 - 4x_3 = -10$$
$$-24x_3 = -72 \quad \Rightarrow x_3 = 3$$
$$x_2 = -10 + 4x_3 = -10 + 4 \times 3 = -10 + 12 = 2$$

and
$$x_1 = 14 - 2x_2 - 3x_3 = 14 - 2 \times 2 - 3 \times 3 = 14 - 4 - 9 = 1$$

$\therefore$ the solution is $\quad \boldsymbol{x_1 = 1,\ x_2 = 2,\ x_3 = 3}$ ■

Example 4

Solve the system of equations by Doolittle's method.

$$\begin{aligned} 10x_1 - 7x_2 + 3x_3 + 5x_4 &= 6 & \qquad -6x_1 + 8x_2 - 4x_3 - 4x_4 &= 5 \\ 3x_1 + x_2 + 4x_3 + 4x_4 &= 2 & \qquad 5x_1 - 9x_2 - 2x_3 + 4x_4 &= 7 \end{aligned}$$

Solution

The given system of equations is

$$10x_1 - 7x_2 + 3x_3 + 5x_4 = 6$$
$$-6x_1 + 8x_2 - 4x_3 - 4x_4 = 5$$
$$3x_1 + x_2 + 4x_3 + 4x_4 = 2$$
$$5x_1 - 9x_2 - 2x_3 + 4x_4 = 7$$

Here

$$A = \begin{bmatrix} 10 & -7 & 3 & 5 \\ -6 & 8 & -1 & -4 \\ 3 & 1 & 4 & 11 \\ 5 & -9 & -2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ 5 \\ 2 \\ 7 \end{bmatrix}, \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$\therefore$ the given system is $AX = B$ (1)
By Doolittle's method, we take $A = LU$

where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix}, \quad \text{and} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

$\therefore$

$$AX = B \Rightarrow LUX = B$$

Now $\qquad A = LU$

$$\Rightarrow \begin{bmatrix} 10 & -7 & 3 & 5 \\ -6 & 8 & -1 & -4 \\ 3 & 1 & 4 & 11 \\ 5 & -9 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + l_{22}u_{23} + u_{33} & l_{21}u_{14} + l_{42}u_{24} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} & l_{31}u_{14} + l_{32}u_{24} + u_{34} \\ l_{41}u_{11} & l_{41}u_{12} + l_{42}u_{22} & l_{41}u_{13} + l_{42}u_{23} + l_{43}u_{33} & l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34} + u_{44} \end{bmatrix}$$

Equating, the corresponding elements of I rows, we get

$$u_{11} = 10, \quad u_{12} = -7, \quad u_{13} = 3, \quad u_{14} = 4$$

Equating the corresponding elements of second rows, we get

$$l_{21}u_{11} = -6 \quad \Rightarrow l_{21} \times 10 = -6 \qquad \Rightarrow l_{21} = -\frac{6}{8} = -0.6$$

$$l_{21}u_{12} + u_{22} = 8$$

$$\Rightarrow \quad (-0.6)(-7) + u_{22} = 8 \quad \Rightarrow \quad u_{22} = 8 - 4.2 = 3.8$$

$$l_{21}u_{13} + u_{23} = -1$$

$$\Rightarrow \quad (-0.6)(3) + u_{23} = -1 \quad \Rightarrow \quad u_{23} = -1 + 1.8 = 0.8$$

$$l_{21}u_{14} + u_{24} = -4$$

$$\Rightarrow \quad (-0.6)(5) + u_{24} = -4 \quad \Rightarrow \quad u_{24} = -4 + 3 = -1$$

Equating the corresponding elements of third rows, we get

$$l_{31}u_{11} = 3 \quad \Rightarrow l_{31} \times 10 = 3 \quad \Rightarrow l_{31} = \frac{3}{10} = 0.3$$

$$l_{31}u_{12} + l_{32}u_{22} = 1$$

$$\Rightarrow \quad (0.3)(-7) + l_{32}u_{22} = 1 \quad \Rightarrow 3.8 l_{32} = 3.1 \quad \Rightarrow l_{32} = \frac{3.1}{3.8} = 0.8158$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 4$$

$$\Rightarrow \quad (0.3)(3) + (0.8158)(0.8) + u_{33} = 4 \quad \Rightarrow u_{33} = 4 - 0.9 - 0.65264 = 2.4474$$

$$l_{31}u_{14} + l_{32}u_{24} + u_{34} = 11$$

$$\Rightarrow \quad (0.3)(5) + (0.8158)(-1) + u_{34} = 11 \quad \Rightarrow u_{34} = 11 - 1.5 + 0.8158 = 10.3158$$

Equating the corresponding elements of fourth rows, we get

$$l_{41}u_{11} = 5 \quad \Rightarrow l_{41} \times 10 = 5 \quad \Rightarrow l_{41} = \frac{5}{10} = 0.5$$

$$l_{41}u_{12} + l_{42}u_{22} = -9$$

$$\Rightarrow \quad (0.5)(-7) + l_{42}(3.8) = -9$$

$$3.8 l_{42} = -5.5 \Rightarrow l_{42} = -\frac{5.5}{3.8} = -1.4474$$

$$l_{41}u_{13} + l_{42}u_{23} + l_{43}u_{33} = -2$$

$$\Rightarrow (0.5)(3) + (-1.4474)(0.8) + l_{43}(2.4474) = -2$$

$$\Rightarrow \quad 1.5 - 1.15792 + 2.4474 l_{43} = -2$$

$$\Rightarrow \quad 2.4474 l_{43} = -2 - 0.34208 = -2.3421$$

$$l_{43} = -\frac{2.3421}{2.4474} = -0.9570$$

and $\quad l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34} + u_{44} = 4$

$$\Rightarrow (0.5)5 + (-1.4474)(-1) + (-0.9570)(10.3158) + u_{44} = 4$$

$$\Rightarrow \quad 2.5 + 1.4474 - 9.87222 + u_{44} = 4$$

$$\Rightarrow \quad -5.92482 + u_{44} = 4 \Rightarrow u_{44} = 9.9248$$

Hence $l_{21} = -0.6$, $\quad l_{31} = 0.3$, $\quad l_{32} = 0.8158$

$l_{41} = 0.5$, $\quad l_{42} = -1.4474$, $\quad l_{43} = -0.9570$

$u_{11} = 10$, $\quad u_{12} = -7$, $\quad u_{13} = 3$, $u_{14} = 5$

$u_{22} = 3.8$, $\quad u_{23} = 0.8$, $\quad u_{24} = -1$

$u_{33} = 2.4474$, $\quad u_{34} = 10.3158$ $\quad u_{44} = 9.9248$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.6 & 1 & 0 & 0 \\ 0.3 & 0.8158 & 1 & 0 \\ 0.5 & -1.4474 & -0.9570 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3.8 & 0.8 & -1 \\ 0 & 0 & 2.4474 & 10.3158 \\ 0 & 0 & 0 & 9.9248 \end{bmatrix}$$

$$AX = B \Rightarrow LUX = B$$

Put $UX = Y$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ $\quad \therefore \quad LY = B$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.6 & 1 & 0 & 0 \\ 0.3 & 0.8158 & 1 & 0 \\ 0.5 & -1.4474 & -0.9570 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 2 \\ 7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ -0.6y_1 + y_2 \\ 0.3y_1 + 0.8158y_2 + y_3 \\ 0.5y_1 - 1.4474y_2 - 0.9570y_3 + y_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 2 \\ 7 \end{bmatrix}$$

$$\therefore \quad y_1 = 6$$

$$-0.6y_1 + y_2 = 5$$

$$\Rightarrow \quad y_2 = 5 + 0.6y_1 = 5 + 0.6 \times 6 = 5 + 3.6 = 8.6$$

$$0.3y_1 + 0.8158y_2 + y_3 = 2$$

$$\Rightarrow \quad y_3 = 2 - 0.3y_1 - 0.8158y_2$$
$$= 2 - (0.3)(6) - (0.8158)(8.6) = 2 - 1.8 - 7.01588 = -6.81588$$

and $0.5y_1 - 1.4474y_2 - 0.9570y_3 + y_4 = 7$

$$\Rightarrow \quad y_4 = 7 - 0.5y_1 + 1.4474y_2 + 0.9570y_3$$

$$\Rightarrow \quad y_4 = 7 - (0.5)(6) + (1.4474)(8.6) + (0.9570)(-6.81588)$$
$$= 7 - 3 + 12.44764 - 6.522797 = 9.9248$$

Now $\quad UX = Y$

$$\Rightarrow \begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3.8 & 0.8 & -1 \\ 0 & 0 & 2.4474 & 10.3158 \\ 0 & 0 & 0 & 9.9248 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8.6 \\ -6.91588 \\ 9.9248 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 10x_1 - 7x_2 + 3x_3 + 5x_4 \\ 3.8x_2 + 0.8x_3 - x_4 \\ 2.4474x_3 + 10.3158x_4 \\ 9.9248x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8.6 \\ -6.91588 \\ 9.9248 \end{bmatrix}$$

$$\therefore \quad 10x_1 - 7x_2 + 3x_3 + 5x_4 = 6$$

$$3.8x_2 + 0.8x_3 - x_4 = 8.6$$

$$2.4474x_3 + 10.3158x_4 = -6.81588$$

and

$$9.9248x_4 = 9.9248 \qquad \Rightarrow x_4 = 1$$

$$2.4474x_3 = -6.81588 - 10.3158x_4$$

$$= -6.81588 - 10.3158 \quad = -17.13168$$

$$\therefore \quad x_3 = -\frac{17.13168}{2.4474} = -6.9999 = -7$$

$$3.8x_2 = 8.6 - 0.8x_3 + x_4$$

$$= 8.6 - 0.8(-7) + 1 = 8.6 + 5.6 + 1 = 15.2$$

$$\therefore \quad x_2 = \frac{15.2}{3.8} = 4$$

$$10x_1 = 6 + 7x_2 - 3x_3 - 5x_4$$

$$= 6 + 7 \times 4 - 3 \times (-7) - 5.1 \ = 6 + 28 + 21 - 5 = 50$$

$$\therefore \quad x_1 = \frac{50}{10} = 5$$

$\therefore$ the solution is $\mathbf{x_1 = 5,\ x_2 = 4,\ x_3 = -7,\ x_4 = 1}$

Crout's Method

Consider the system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{31}x_2 + a_{33}x_3 = b_3$$

Now

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{ and } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\therefore$ the system of equations is $AX = B$

In Crout's method we factorise A as $A = LU$, where L is taken as lower triangular matrix and U is taken as unit upper triangular matrix.

That is $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ and $U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore \qquad AX = B \quad \Rightarrow LUX = B$$

Let $UX = Y$, so that $LY = B$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Since L and B are known, Y is known.
Substituting Y in $UX = Y$, we can find X, since U is known.
Thus, we can find the solution of the given system of equations. ■

Note:

In Crout's method to find $l_{11}, l_{21}, l_{22}, l_{31}, l_{32}\ l_{33}$ and u_{12}, u_{13}, u_{23}, we equate the corresponding elements of the columns of A and LU respectively.

That is to find the elements of L and U, we equate the corresponding elements of columns of A and LU.

WORKED EXAMPLES

Example 1

Factorise $A = \begin{bmatrix} 2 & -2 & +4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}$ **by Crout's method.**

Solution

Given $A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}$

By Crout's method, we take $A = LU$,

where $$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

Equating elements of I Columns, we get

$$l_{11} = 2, \qquad l_{21} = 2, \qquad l_{31} = -1$$

Equating elements of II Column, we get

$$l_{11}u_{12} = -2 \quad \Rightarrow \quad 2 \times u_{12} = -2 \quad \Rightarrow \quad u_{12} = -1$$
$$l_{21}u_{12} + l_{22} = 3 \quad \Rightarrow \quad 2 \times (-1) + l_{22} = 3 \quad \Rightarrow -2 + l_{22} = 3 \quad \Rightarrow l_{22} = 5$$
$$l_{31}u_{12} + l_{32} = -1 \quad \Rightarrow \quad (-1)(-1) + l_{32} = -1 \quad \Rightarrow \quad 1 + l_{32} = -1 \quad \Rightarrow l_{32} = 0$$

Equating elements of III column, we get

$$l_{11}u_{13} = 4 \quad \Rightarrow \quad 2 \times u_{13} = 4 \quad \Rightarrow \quad u_{13} = 2$$
$$l_{21}u_{13} + l_{22}u_{23} = 2 \quad \Rightarrow 2 \times 2 + 5 \times u_{23} = 2 \quad \Rightarrow \quad 5u_{23} = -2 \quad \Rightarrow u_{23} = -\frac{2}{5}$$
$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = -1$$
$$\Rightarrow (-1)(2) + 0 \times u_{23} + l_{33} = -1 \quad \Rightarrow -2 + l_{33} = -1 \quad \Rightarrow \quad l_{33} = 1$$

$$\therefore \quad L = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 5 & 0 \\ -1 & 0 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 5 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

■

Example 2

Solve the system of equations $2x - 6y + 8z = 24$; $5x + 4y - 3z = 2$; $3x + y + 2z = 16$ using Crout's method.

Solution

The given system of equations is

$$2x - 6y + 8z = 24$$
$$5x + 4y - 3z = 2$$
$$3x + y + 2z = 16$$

Here $$A = \begin{bmatrix} 2 & -6 & 8 \\ 5 & 4 & -3 \\ 3 & 1 & 2 \end{bmatrix}, \ B = \begin{bmatrix} 24 \\ 2 \\ 16 \end{bmatrix} \text{ and } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\therefore$ the given system is $AX = B$.

By Crout's method, we take $A = LU$,

where $$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 2 & -6 & 8 \\ 5 & 4 & -3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

Equating the corresponding elements of I column, we get

$$l_{11} = 2,\ l_{21} = 5,\ l_{31} = 3$$

Equating the corresponding elements of II columns, we get

$$l_{11}u_{12} = -6 \quad \Rightarrow \quad 2.u_{12} = -6 \Rightarrow u_{12} = -3$$

$$l_{21}u_{12} + l_{22} = 4$$

$$\Rightarrow \quad 5(-3) + l_{22} = 4 \quad \Rightarrow -15 + l_{22} = 4 \quad \Rightarrow \quad l_{22} = 19$$

$$l_{31}u_{12} + l_{32} = 1$$

$$\Rightarrow \quad 3(-3) + l_{32} = 1 \quad \Rightarrow \quad -9 + l_{32} = 1 \quad \Rightarrow \quad l_{32} = 10$$

Equating the corresponding elements of third column, we get

$$l_{13}u_{13} = 8 \quad \Rightarrow \quad 2u_{13} = 8 \quad \Rightarrow u_{13} = 4$$

$$l_{21}u_{13} + l_{22}u_{23} = -3$$

$$\Rightarrow \quad 5 \times 4 + 19u_{23} = -3$$

$$\Rightarrow \quad 19u_{23} = -23 \quad \Rightarrow \quad u_{23} = -\frac{23}{19} = -1.2105$$

$$\therefore \quad l_{31}u_{13} + l_{32}u_{23} + l_{33} = 2$$

$$\Rightarrow \quad 3 \times 4 + 10 \times \left(\frac{-23}{19}\right) + l_{33} = 2$$

$$\Rightarrow \quad l_{33} = -10 + \frac{230}{19} = \frac{-190 + 230}{19} = \frac{40}{19}$$

$$\therefore \quad L = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 19 & 0 \\ 3 & 10 & \frac{40}{19} \end{bmatrix}, \text{ and } U = \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & \frac{-23}{19} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad AX = B \Rightarrow LUX = B.$$

Put $Y = UX$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$\therefore \quad LY = B$$

$$\Rightarrow \quad \begin{bmatrix} 2 & 0 & 0 \\ 5 & 19 & 0 \\ 3 & 10 & \frac{40}{19} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 2 \\ 16 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2y_1 \\ 5y_1 + 19y_2 \\ 3y_1 + 10y_2 + \frac{40}{19}y_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 2 \\ 16 \end{bmatrix}$$

$$\therefore \quad 2y_1 = 24 \qquad \Rightarrow y_1 = 12$$

$$5y_1 + 19y_2 = 2$$

$$19y_2 = 2 - 5 \times 12 = 2 - 60 = -58 \qquad \Rightarrow y_2 = -\frac{58}{19}$$

and $$3y_1 + 10y_2 + \frac{40}{19}y_3 = 16$$

$$\Rightarrow \quad \frac{40}{19}y_3 = 16 - 3 \times 12 - 10\left(-\frac{58}{19}\right)$$

$$= 16 - 36 + \frac{580}{19} = -20 + \frac{580}{19} = \frac{-380 + 580}{19} = \frac{200}{19}$$

$$\therefore \quad y_3 = \frac{200}{19} \times \frac{19}{40} = 5$$

$$\therefore \quad Y = \begin{bmatrix} 12 \\ -\frac{58}{19} \\ 5 \end{bmatrix}$$

Now $$UX = Y$$

$$\Rightarrow \quad \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & \frac{-23}{19} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ -\frac{58}{19} \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x - 3y + 4z \\ y - \frac{23}{19}z \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ -\frac{58}{19} \\ 5 \end{bmatrix}$$

$\therefore \qquad x-3y+4z=12$

$$y-\frac{23}{19}z=\frac{-58}{19} \quad \text{and} \quad z=5$$

$$\therefore \qquad y=\frac{23}{19}\times 5-\frac{58}{19}=\frac{115}{19}-\frac{58}{19}=\frac{57}{19}=3$$

and $\qquad x=12+3y-4z=12+3\times 3-4\times 5=12+9-20=1$ ■

$\therefore$ the solution is $\boldsymbol{x=1, \; y=3, \; z=5}$

Example 3

Solve by Crout's method, the system of equations $10x+y+z=12$; $2x+10y+z=13$; $2x+2y+10z=14$.

Solution

The given system of equations is

$$10x+y+z=12$$
$$2x+10y+z=13$$
$$2x+2y+10z=14$$

Here
$$A=\begin{bmatrix}10 & 1 & 1\\ 2 & 10 & 1\\ 2 & 2 & 10\end{bmatrix}, \quad B=\begin{bmatrix}12\\ 13\\ 14\end{bmatrix} \text{ and } X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

$\therefore$ the given system is $AX=B$

By Crout's method, we take $A=LU$

where
$$L=\begin{bmatrix}l_{11} & 0 & 0\\ l_{21} & l_{22} & 0\\ l_{27} & l_{32} & l_{33}\end{bmatrix} \text{ and } U=\begin{bmatrix}1 & u_{12} & u_{13}\\ 0 & 1 & u_{23}\\ 0 & 0 & 1\end{bmatrix}$$

$$\therefore \quad \begin{bmatrix}10 & 1 & 1\\ 2 & 10 & 1\\ 2 & 2 & 10\end{bmatrix}=\begin{bmatrix}l_{11} & 0 & 0\\ l_{21} & l_{22} & 0\\ l_{31} & l_{32} & l_{33}\end{bmatrix}\begin{bmatrix}1 & u_{12} & u_{13}\\ 0 & 1 & u_{23}\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}l_{11} & l_{11}u_{12} & l_{11}u_{13}\\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23}\\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33}\end{bmatrix}$$

Equating the corresponding elements of the first columns, we get

$$l_{11}=10, \quad l_{21}=2, \quad l_{31}=2$$

Equating the corresponding elements of the second columns, we get

$$l_{11}u_{12} = 1 \quad \Rightarrow \quad 10 \times u_{12} = 1 \quad \Rightarrow u_{12} = \frac{1}{10}$$

$$l_{21}u_{12} + l_{22} = 10$$

$$\Rightarrow \quad 2 \times \frac{1}{10} + l_{22} = 10 \quad \Rightarrow \quad l_{22} = 10 - \frac{2}{10} = 10 - \frac{1}{5} = \frac{49}{5}$$

$$l_{31}u_{12} + l_{32} = 2 \quad \Rightarrow \quad 2 \times \frac{1}{10} + l_{32} = 2 \quad \Rightarrow l_{32} = 2 - \frac{1}{5} = \frac{9}{5}$$

Equating the corresponding elements of third columns, we get

$$l_{11}u_{13} = 1 \quad \Rightarrow \quad 10u_{13} = 1 \quad \Rightarrow \quad u_{13} = \frac{1}{10}$$

$$l_{21}\, u_{13} + l_{22}\, u_{23} = 1$$

$$\Rightarrow \quad 2 \times \frac{1}{10} + \frac{49}{5} u_{23} = 1 \quad \Rightarrow \quad \frac{49}{5} u_{23} = 1 - \frac{1}{5} = \frac{4}{5} \quad \Rightarrow \quad u_{23} = \frac{4}{49}$$

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 10$$

$$\Rightarrow \quad 2 \times \frac{1}{10} + \frac{9}{5} \times \frac{4}{49} + l_{33} = 10$$

$$\Rightarrow \quad l_{33} = 10 - \frac{1}{5} - \frac{36}{5 \times 49} = \frac{49}{5} - \frac{36}{5 \times 49} = \frac{2365}{5 \times 49} = \frac{473}{19}$$

$$\therefore \quad L = \begin{bmatrix} 10 & 0 & 0 \\ 2 & \frac{49}{5} & 0 \\ 2 & \frac{9}{5} & \frac{473}{49} \end{bmatrix}, \quad U = \begin{bmatrix} 1 & \frac{1}{10} & \frac{1}{10} \\ 0 & 1 & \frac{4}{49} \\ 0 & 0 & 1 \end{bmatrix}$$

Now $\quad AX = B \Rightarrow LUX = B$

Put $UX = Y$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad \therefore LY = B$

$$\Rightarrow \quad \begin{bmatrix} 10 & 0 & 0 \\ 2 & \frac{49}{5} & 0 \\ 2 & \frac{9}{5} & \frac{473}{49} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 14 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 10y_1 \\ 2y_1 + \frac{49}{5} y_2 \\ 2y_1 + \frac{9}{5} y_2 + \frac{473}{49} y_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 14 \end{bmatrix}$$

$$\therefore \quad 10y_1 = 12 \quad \Rightarrow y_1 = \frac{6}{5}$$

$$2y_1 + \frac{49}{5}y_2 = 13$$

$$\Rightarrow \quad \frac{49}{5}y_2 = 13 - 2\times\frac{6}{5} = \frac{65-12}{5} = \frac{53}{5} \Rightarrow y_2 = \frac{53}{49}$$

$$2y_1 + \frac{9}{5}y_2 + \frac{473}{49}y_3 = 14$$

$$\Rightarrow \quad \frac{473}{49}y_3 = 14 - 2\times\frac{6}{5} - \frac{9}{5}\times\frac{53}{49} = \frac{58}{5} - \frac{477}{5\times 49} = \frac{2365}{5\times 49} = \frac{473}{49}$$

$$\therefore \quad y_3 = 1$$

$$\therefore \quad Y = \begin{bmatrix} \frac{6}{5} \\ \frac{53}{49} \\ 1 \end{bmatrix}$$

Now $\quad UX = Y$

$$\Rightarrow \quad \begin{bmatrix} 1 & \frac{1}{10} & \frac{1}{10} \\ 0 & 1 & \frac{4}{49} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ \frac{53}{49} \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x + \frac{y}{10} + \frac{z}{10} \\ y + \frac{4}{49}z \\ z \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ \frac{53}{49} \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad x + \frac{y}{10} + \frac{z}{10} = \frac{6}{5}$$

$$y + \frac{4}{49}z = \frac{53}{49} \quad \text{and} \quad z = 1$$

$$\therefore \quad y = \frac{53}{49} - \frac{4}{49}z = \frac{53}{49} - \frac{4}{49} = \frac{49}{49} = 1 \quad \Rightarrow y = 1$$

$$x+\frac{y}{10}+\frac{z}{10}=\frac{6}{5}$$

$$\Rightarrow \qquad x=\frac{6}{5}-\frac{1}{10}-\frac{1}{10}=\frac{12-1-1}{10}=\frac{10}{10}=1$$

$\therefore$ the solution is $\quad \boldsymbol{x = y = z = 1}$ ■

Cholesky Decomposition

We have seen the triangular factorisation of a square matrix by Doolittle's method and Crout's method. Cholesky decomposition is slightly different from the two methods.

Cholesky method is commonly used in signal processing.

In Cholesky method, to factorise the square matrix, the matrix should be symmetric and positive definite.

A symmetric matrix is positive definite if all the principal minors are positive.

That is if
$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The principal minors are

$$D_1=a_{11}>0 \quad D_2=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}>0 \quad \text{and} \quad D_3=|A|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{33} & a_{33} \end{vmatrix}>0$$

Definition: Let A be a symmetric matrix and positive definite. Then A can be decomposed as $A = LL^T$, where L is a lower triangular matrix with digonal elements positive.
L and L^T are called cholesky factors of A.

Application of Cholesky decomposition to solve the system of linear equations

Consider the system of equations
$$a_{11}\,x+a_{12}\,y+a_{13}\,z=b_1$$
$$a_{21}\,x+a_{22}\,y+a_{23}\,z=b_2$$
$$a_{31}\,x+a_{32}\,y+a_{33}\,z=b_3$$

Let
$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},\ B=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{ and } X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\therefore$ the system of equations is $AX = B$.

By Cholesky decomposition, let $A = LL^T$

where
$$L=\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix},\ L^T=\begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} \text{ and } l_{ii}>0$$

$\therefore\ LL^T X = B,$

Put $Y = L^T X$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ $\therefore\ LY = B$

Since L and B are known, Y is known.

$\therefore\ L^T X = Y$ gives X. ■

WORKED EXAMPLES

Example 1

Obtain Cholesky factorisation of the matrix $\begin{bmatrix} 4 & 8 & 12 \\ 8 & 20 & 20 \\ 12 & 20 & 41 \end{bmatrix}$.

Solution

Let
$$A = \begin{bmatrix} 4 & 8 & 12 \\ 8 & 20 & 20 \\ 12 & 20 & 41 \end{bmatrix}$$

The elements equidistant from the main diagonal are the same and hence A is symmetrix,
The principal minors are

$$D_1 = |4| > 0, \qquad D_2 = \begin{vmatrix} 4 & 8 \\ 8 & 20 \end{vmatrix} = 80 - 64 = 16 > 0$$

$$D_3 = |A| = \begin{bmatrix} 4 & 8 & 12 \\ 8 & 20 & 20 \\ 12 & 20 & 41 \end{bmatrix}$$

$$= 4(820 - 400) - 8(308 - 240) + 12(160 - 240)$$

$$= 1680 - 704 - 960 = 16 > 0$$

All the principal minors are positive and so A is positive definite.
So, we can use cholesky method to factorize A. We take $A = LL^T$

where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$, $l_{ii} > 0$ and $L^T = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$

$$\therefore \quad \begin{bmatrix} 4 & 8 & 12 \\ 8 & 20 & 20 \\ 12 & 20 & 41 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Equating the corresponding elements of I columns, we get

$$l_{11}^2 = 4 \Rightarrow l_{11} = 2$$
$$l_{21}l_{11} = 8 \Rightarrow l_{21} \times 2 = 8 \Rightarrow l_{21} = 4$$
$$l_{31}l_{11} = 12 \Rightarrow l_{31} \times 2 = 12 \Rightarrow l_{31} = 6$$

Equating the corresponding elements of second columns

$$l_{21}^2 + l_{22}^2 = 20 \Rightarrow 16 + l_{22}^2 = 20 \Rightarrow l_{22}^2 = 4 \Rightarrow l_{22} = 2$$

$$l_{31}l_{21} + l_{32}l_{22} = 20$$
$$\Rightarrow \quad 6 \times 4 + l_{32} \times 2 = 20 \Rightarrow 2l_{32} = 20 - 24 = -4 \Rightarrow l_{32} = -2$$

Equating the corresponding elements of third columns, we get

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 41$$
$$\Rightarrow \quad 36 + 4 + l_{33}^2 = 41 \Rightarrow l_{33}^2 = 41 - 40 = 1 \Rightarrow l_{33} = 1$$

$$\therefore \quad L = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ 6 & -2 & 1 \end{bmatrix} \text{ and } L^T = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ 6 & -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ 6 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

■

Example 2

Find Cholesky's factorisation to the matrix $\begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$. Hence solve the equations $x + 2y + 6z = 5$; $2x + 5y + 15z = 2$; $6x + 15y + 46z = 6$.

Solution

Let
$$A = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$$

A is symmetric and principal minors are $D_1 = 1 > 0,\ D_2 = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5 - 4 = 1 > 0$

and $D_3 = |A| = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{vmatrix} = 1(230-225) - 2(92-90) + 6(30-30) = 5 - 4 = 1 > 0$

$\therefore$ A is symmetric and positive definite:

So, we use cholesky's method to factorize A. We take $A = LL^T$,

where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$, $\quad L^T = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$, $l_{ii} > 0$

$$\therefore \quad \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Equating the corresponding elements of first columns, we get

$$\begin{aligned} l_{11}^2 &= 1 \quad \Rightarrow \quad l_{11} = 1 \\ l_{21}l_{11} &= 2 \quad \Rightarrow \quad l_{21}.1 = 2 \quad \Rightarrow \quad l_{21} = 2 \\ l_{31}l_{11} &= 6 \quad \Rightarrow \quad l_{31} \times 1 = 6 \quad \Rightarrow \quad l_{31} = 6 \end{aligned}$$

Equating the corresponding elements of second columns, we get

$$l_{21}^2 + l_{22}^2 = 5 \;\Rightarrow\; 2^2 + l_{22}^2 = 5 \;\Rightarrow\; l_{22}^2 = 1 \;\Rightarrow\; l_{22} = 1$$

and $\quad l_{31}l_{21} + l_{32}l_{22} = 15$

$\Rightarrow \quad 6 \times 2 + l_{32} \times 1 = 15 \Rightarrow l_{32} = 15 - 12 \Rightarrow l_{32} = 3$

Equating the corresponding elements of third columns, we get

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 46 \;\Rightarrow\; 6^2 + 3^2 + l_{33}^2 = 46 \;\Rightarrow\; l_{33}^2 = 46 - 36 - 9 = 1$$
$$\Rightarrow\; l_{33} = 1$$

$$\therefore \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix} \text{ and } L^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

The given system of equations is

$$\begin{aligned} x + 2y + 6z &= 5 \\ 2x + 5y + 15z &= 2 \\ 6x + 15y + 46z &= 6 \end{aligned}$$

The coefficient matrix is

$$\begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$$

This is same as the given matrix A.

$\therefore$ the given system of equations is $AX = B$, where $B = \begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\therefore \qquad AX = B \;\Rightarrow\; LL^T X = B$$

Put $L^T X = Y$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ $\qquad \therefore LY = B$

$$\Rightarrow \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} y_1 \\ 2y_1 + y_2 \\ 6y_1 + 3y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}$$

$$\therefore \qquad y_1 = 5$$
$$2y_1 + y_2 = 2$$
$$\Rightarrow \qquad y_2 = 2 - 2y_1 = 2 - 2 \times 5 = 2 - 10 = -8$$

and $\qquad 6y_1 + 3y_2 + y_3 = 6$

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$\Rightarrow \qquad y_3 = 6 - 6y_1 - 3y_2$

$\Rightarrow \qquad y_3 = 6 - 6 \times 5 - 3(-8) = 6 - 30 + 24 = 0$

$$\therefore \qquad Y = \begin{bmatrix} 5 \\ -8 \\ 0 \end{bmatrix}$$

$$\therefore \qquad L^T X = Y$$

$$\Rightarrow \qquad \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \\ 0 \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} x + 2y + 6z \\ y + 3z \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \\ 0 \end{bmatrix}$$

$\Rightarrow$

$$x + 2y + 6z = 5$$
$$y + 3z = -8 \quad \text{and} \quad z = 0$$

$$\therefore \qquad y = -8 - 3z = -8 - 0 = -8$$
$$x = 5 - 2y - 6z = 5 - 2(-8) - 6 \times 0 = 5 + 16 = 21$$

$\therefore$ the solution is $x = 21,\ y = -8,\ z = 0$ ■

Example 3

Obtain the Cholesky decomposition of $A = \begin{bmatrix} 9 & 6 & 12 \\ 6 & 13 & 11 \\ 12 & 11 & 26 \end{bmatrix}$ **and hence solve the system.**

$AX = b$ **where** $X = [x, y, z]^T$ **and** $b = [174 \ \ 236 \ \ 308]^T$.

Solution

Given $A = \begin{bmatrix} 9 & 6 & 12 \\ 6 & 13 & 11 \\ 12 & 11 & 26 \end{bmatrix}$ and it is symmetric

The principal minors are $D_1 = 9 > 0,\ D_2 = \begin{vmatrix} 9 & 6 \\ 6 & 13 \end{vmatrix} = 117 - 36 = 81 > 0$

$$D_3 = |A| = \begin{vmatrix} 9 & 6 & 12 \\ 6 & 13 & 11 \\ 12 & 11 & 26 \end{vmatrix}$$
$$= 9(338-121) - 6(156-132) + 12(66-156)$$
$$= 1953 - 144 + 12(-90) = 729 > 0$$

$\therefore$ A is symmetric and positive definite.
So, we can use Cholesky's decomposition to factorize A. We take $A = LL^T$

where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$, $L^T = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$, $l_{ii} > 0$

$$\therefore \quad \begin{bmatrix} 9 & 6 & 12 \\ 6 & 13 & 11 \\ 12 & 11 & 26 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Equating Corresponding elements of first columns, we get

$$l_{11}^2 = 9 \quad \Rightarrow \quad l_{11} = 3$$
$$l_{21}l_{11} = 6 \quad \Rightarrow l_{21} \times 3 = 6 \quad \Rightarrow l_{21} = 2$$
$$l_{31}l_{11} = 12 \quad \Rightarrow l_{31} \times 3 = 12 \quad \Rightarrow l_{31} = 4$$

Equating corresponding elements of second columns, we get

$$l_{21}^2 + l_{22}^2 = 13$$
$$\Rightarrow \quad 2^2 + l_{22}^2 = 13 \quad \Rightarrow \quad l_{22}^2 = 13 - 4 = 9 \quad \Rightarrow \quad l_{22} = 3$$
$$l_{31}l_{21} + l_{32}l_{22} = 11$$
$$\Rightarrow \quad 4 \times 3 + 3l_{32} = 11 \quad \Rightarrow \quad l_{32} = 11 - 8 = 3 \quad \Rightarrow \quad l_{32} = 1$$

Equating the corresponding elements of third columns, we get

$$\Rightarrow \quad l_{31}^2 + l_{32}^2 + l_{33}^2 = 26$$
$$\Rightarrow \quad 4^2 + 1^2 + l_{33}^2 = 26 \quad \Rightarrow l_{33}^2 = 26 - 17 = 9 \quad \Rightarrow l_{33} = 3$$

$$\therefore \qquad L=\begin{bmatrix}3&0&0\\2&3&0\\4&1&3\end{bmatrix} \text{ and } L^T=\begin{bmatrix}3&2&4\\0&3&1\\0&0&3\end{bmatrix}$$

Given $AX = b$, where $X = [x, y, z]^T$ and $b = [174 \; 236 \; 308]^T$

Now $AX = b \Rightarrow LL^T X = b$

Put $L^T X = Y$, where $Y = \begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$ $\qquad \therefore \; LY = b$

$$\Rightarrow \qquad \begin{bmatrix}3&0&0\\2&3&0\\4&1&3\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}174\\236\\308\end{bmatrix}$$

$$\begin{bmatrix}3y_1\\2y_1+3y_2\\4y_1+y_2+3y_3\end{bmatrix}=\begin{bmatrix}174\\236\\308\end{bmatrix}$$

$$\Rightarrow \qquad 3y_1 = 174$$

$$2y_1 + 3y_2 = 236$$

$$4y_1 + y_2 + 3y_3 = 308$$

$$3y_1 = 174 \;\Rightarrow\; y_1 = \frac{174}{3} = 58$$

$$3y_2 = 236 - 2y_1 = 236 - 2\times 58 = 236 - 116 = 120$$

$$\therefore \qquad y_2 = 40$$

and $$3y_3 = 308 - 4y_1 - y_2$$

$$= 308 - 4\times 58 - 40 = 308 - 232 - 40 = 36$$

$$\Rightarrow \qquad y_3 = \frac{36}{3} = 12$$

$$\therefore \qquad Y = \begin{bmatrix}58\\40\\12\end{bmatrix}$$

Now $$L^T X = Y$$

$$\Rightarrow \begin{bmatrix} 3 & 2 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 58 \\ 40 \\ 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3x+2y+4z \\ 3y+z \\ 3z \end{bmatrix} = \begin{bmatrix} 58 \\ 40 \\ 12 \end{bmatrix}$$

$$\therefore \quad \begin{aligned} 3x+2y+4z &= 58 \\ 3y+z &= 40 \\ 3z &= 12 \quad \Rightarrow z = 4 \end{aligned}$$

$$\therefore \quad 3y = 40 - z = 40 - 4 = 36 \qquad \Rightarrow y = \frac{36}{3} = 12$$

and $$\begin{aligned} 3x &= 58 - 2y - 4z \\ &= 58 - 2\times 12 - 4\times 4 = 58 - 24 - 16 = 18 \end{aligned}$$

$$\therefore \quad x = \frac{18}{3} = 6$$

$\therefore$ the solution is $\boldsymbol{x = 6,\ y = 12,\ z = 4}$ ■

Note: In L, if the diagonal elements are not unity, then also we can factorize the given matrix A.

Example 4

Let $A = \begin{bmatrix} 3 & 12 & 9 \\ 2 & 10 & 12 \\ 1 & 12 & 2 \end{bmatrix}$, then find two triangular matrices L (lower triangular) and U (upper triangular) such that $A = LU$, using the diagonal elements of L as 3, 1, 5. Hence find A^{-1}.

Solution

Given $A = \begin{bmatrix} 3 & 12 & 9 \\ 2 & 10 & 12 \\ 1 & 12 & 2 \end{bmatrix}$

Let $A = LU$, where $L = \begin{bmatrix} 3 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 5 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

$$\therefore \qquad A = LU$$

$$\Rightarrow \qquad \begin{bmatrix} 3 & 12 & 9 \\ 2 & 10 & 12 \\ 1 & 12 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 5 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 3u_{11} & 3u_{12} & 3u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + 5u_{33} \end{bmatrix}$$

Equating the corresponding elements of each row, we get

$$3u_{11} = 3 \qquad \Rightarrow u_{11} = 1, 3u_{12} = 12 \qquad \Rightarrow u_{12} = 4$$

$$3u_{13} = 9 \qquad \Rightarrow u_{13} = 3$$

$$l_{21}u_{11} = 2 \qquad \Rightarrow l_{21} \times 1 = 2 \qquad \Rightarrow l_{21} = 2$$

$$l_{21}u_{12} + u_{22} = 10 \qquad \Rightarrow 2 \times 4 + u_{22} = 10 \qquad \Rightarrow u_{22} = 10 - 8 = 2$$

$$l_{21}u_{13} + u_{23} = 12 \qquad \Rightarrow 2 \times 3 + u_{23} = 12 \qquad \Rightarrow u_{23} = 12 - 6 = 6$$

$$l_{31}u_{11} = 1 \qquad \Rightarrow l_{31}.1 = 1 \qquad \Rightarrow l_{32} = 1$$

$$l_{31}u_{12} + l_{32}u_{22} = 12 \qquad \Rightarrow 1 \times 4 + l_{32} \times 2 = 12 \qquad \Rightarrow 2l_{32} = 12 - 4 = 8 \Rightarrow l_{32} = 4$$

$$l_{31}u_{13} + l_{32}u_{23} + 5u_{33} = 2$$

$$1 \times 3 + 4 \times 6 + 5u_{33} = 2 \qquad \Rightarrow 5u_{33} = 2 - 27 = -25 \Rightarrow u_{33} = -5$$

$$\therefore \qquad L = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 4 & 5 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & -5 \end{bmatrix}$$

Since $A = LU$, then $A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$, where $U^{-1} = \dfrac{\text{adj}U}{|U|}$ and $L^{-1} = \dfrac{\text{adj}L}{|L|}$.

$$|U| = \begin{vmatrix} 1 & 4 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & -5 \end{vmatrix} = 1(-10) = -10$$

$$\text{adj}U = \begin{bmatrix} -10 & 0 & 0 \\ 20 & -5 & 0 \\ 18 & -6 & 2 \end{bmatrix}^T = \begin{bmatrix} -10 & 20 & 18 \\ 0 & -5 & -6 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore \qquad U^{-1} = \frac{-1}{10}\begin{bmatrix} -10 & 20 & 18 \\ 0 & -5 & -6 \\ 0 & 0 & 2 \end{bmatrix}$$

$$|L| = \begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 4 & 5 \end{vmatrix} = 3(5 - 0) = 15$$

$$\text{adj } L = \begin{vmatrix} 5 & -10 & 7 \\ 0 & 15 & -12 \\ 0 & 0 & 3 \end{vmatrix}^T = \begin{vmatrix} 5 & 0 & 0 \\ -10 & 15 & 0 \\ 7 & -12 & 3 \end{vmatrix}$$

$$L^{-1} = \frac{1}{15}\begin{vmatrix} 5 & 0 & 0 \\ -10 & 15 & 0 \\ 7 & -12 & 3 \end{vmatrix}$$

$$A^{-1} = -\frac{1}{10}\begin{bmatrix} -10 & 20 & 18 \\ 0 & -5 & -6 \\ 0 & 0 & 2 \end{bmatrix} \cdot \frac{1}{15}\begin{bmatrix} 5 & 0 & 0 \\ -10 & 15 & 0 \\ 7 & -12 & 3 \end{bmatrix}$$

$$= -\frac{1}{150}\begin{bmatrix} -50-200+126 & 300+216 & 54 \\ 50-42 & -75+72 & -18 \\ 14 & -24 & 6 \end{bmatrix} = -\frac{1}{150}\begin{bmatrix} -124 & 84 & 54 \\ 8 & -3 & -18 \\ 14 & -24 & 6 \end{bmatrix}$$ ■

Exercises 3.5

(1) Determine the LU factorisation of the matrix $\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$ by Doolittle's method.

(2) Determine the LU factorisation of the matrix $\begin{bmatrix} 2 & -6 & 10 \\ 1 & 5 & 1 \\ -1 & 15 & -5 \end{bmatrix}$ by Doolittle's method.

(I) Solve by using Doolittle's method the following system equations.

(3) $2x - 3y + 10z = 3;\ x - 4y - 2z = -20;\ 5x + 2y + z = -12$

(4) $2x + y + 3z = 13;\ x + 5y + z = 14;\ 3x + y + 4z = 17$

(5) $x_1 + 3x_2 + x_3 = 3;\ x_1 + 4x_2 + 2x_3 = 3;\ x_1 + 2x_2 - 3x_3 = 6$

(6) $x_1 + 2x_2 + x_3 = 8;\ 2x_1 + 3x_2 + 4x_3 = 20;\ 4x_1 + 3x_2 + 2x_3 = 16$

(7) Find the LU factorisation of the matrix $\begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & 1 \end{bmatrix}$ by Doolittle's method and hence solve the system of equations $2x + y + 4z = 12;\ 8x - 3y + 2z = 20;\ 4x + 11y - z = 6$.

(8) Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$, find the LU factorisation of A by Doolittle's method and hence solve the system of equations $AX = B$ where $X = [x_1\ x_2\ x_3]^T$, $B = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

(II) Solve by using Crout's method the following system of equations.

(9) $3x+2y+7z=4;\ 2x+3y+z=5;\ 3x+4y+z=7.$

(10) $2x+3y+z=9;\ x+2y+3z=6;\ 3x+y+2z=8$

(11) $2x+2y+z=12;\ 3x+2y+2z=8;\ 5x+10y-8z=10$

(12) Find the inverse of $A=\begin{bmatrix}3 & 2 & 1\\ 2 & 3 & 2\\ 1 & 2 & 2\end{bmatrix}$ using crout's method.

(13) Find the crout's factorisation of the matrix $A=\begin{bmatrix}1 & 1 & 1\\ 4 & 3 & -1\\ 3 & 5 & 5\end{bmatrix}$.

(III) Solve by using Cholesky's method

(14) $x+2y+2z=5;\ 2x+8y+22z=6;\ 3x+22y+82z=-10$

(15) $x_1-x_2=1;\ -x_1+4x_2-x_3=0;\ -x_2+4x_3=0$

(16) $4x_1+2x_2+4x_3=10;\ 2x_1+2x_2+3x_3+2x_4=18$
$4x_1+2x_2+6x_3+3x_4=30;\ 2x_2+3x_3+9x_4=61$

(17) Find the Cholesky decomposition of the matrix $A=\begin{bmatrix}4 & 2 & -2\\ 2 & 10 & 2\\ -2 & 2 & 5\end{bmatrix}$ and hence solve the system $AX=B$, where $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}2\\ 28\\ 17\end{bmatrix}$.

(18) Find the Cholesky factorisation of the matrix $A=\begin{bmatrix}1 & 4 & 5\\ 4 & 20 & 32\\ 5 & 32 & 70\end{bmatrix}$ and hence solve the system $AX=B$, where $X=[x\,y\,z]^T$ and $B=[16\ \ 84\ \ 149]^T$.

(19) Find the Cholesky decomposition for the matrix $\begin{bmatrix}9 & -3\\ -3 & 2\end{bmatrix}$.

Answers 3.5

(1) $L=\begin{bmatrix}1 & 0 & 0\\ \frac{7}{5} & 1 & 0\\ \frac{3}{5} & \frac{41}{19} & 1\end{bmatrix}$ and $U=\begin{bmatrix}5 & -2 & 1\\ 0 & \frac{19}{5} & -\frac{32}{5}\\ 0 & 0 & \frac{327}{19}\end{bmatrix}$

(2) $L=\begin{bmatrix}1 & 0 & 0\\ \frac{1}{2} & 1 & 0\\ -\frac{1}{2} & \frac{3}{2} & 1\end{bmatrix}$ and $U=\begin{bmatrix}2 & -6 & 10\\ 0 & 8 & -4\\ 0 & 0 & 0\end{bmatrix}$

(3) $x=-4,\ y=3,\ z=2$

(4) $x=1,\ y=2,\ z=3$

(5) $x_1=1, x_2=1, x_3=-1$

(6) $x_1=1, x_2=2, x_3=3$

(7) $L=\begin{bmatrix}1 & 0 & 0\\ 4 & 1 & 0\\ 2 & \frac{-9}{7} & 1\end{bmatrix},\quad U=\begin{bmatrix}2 & 1 & 4\\ 0 & -7 & -14\\ 0 & 0 & -27\end{bmatrix},\quad x=2, y=0, z=2$

(8) $L=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 3 & 1\end{bmatrix},\quad U=\begin{bmatrix}1 & 2 & 3\\ 0 & -1 & -4\\ 0 & 0 & 7\end{bmatrix},\quad x_1=-4, x_2=4, x_3=-1$

(9) $x=\frac{7}{8},\ y=\frac{9}{8},\ z=-\frac{1}{8}$

(10) $x=\frac{35}{18},\ y=\frac{29}{18},\ z=\frac{5}{18}$

(11) $x=-\frac{51}{4},\ y=\frac{115}{8},\ z=\frac{35}{4}$

(12) $L=\begin{bmatrix}3 & 0 & 0\\ 2 & \frac{5}{3} & 0\\ 1 & \frac{4}{3} & \frac{3}{5}\end{bmatrix}, U=\begin{bmatrix}1 & 3 & 3\\ 0 & 1 & \frac{4}{5}\\ 0 & 0 & 1\end{bmatrix},\ A^{-1}=\frac{1}{3}\begin{bmatrix}2 & -2 & 1\\ -2 & 5 & -4\end{bmatrix}$

(13) $L=\begin{bmatrix}1 & 0 & 0\\ 4 & -1 & 0\\ 3 & 2 & -10\end{bmatrix}, U=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 5\\ 0 & 0 & 1\end{bmatrix}$

(14) $x=2,\ y=3,\ z=-1$

(15) $x_1=\frac{16}{56},\ x_2=\frac{1}{4},\ x_3=\frac{1}{56}$

(16) $x_1=3,\ x_2=-1,\ x_3=0,\ x_4=7$

(17) $L=\begin{bmatrix}2 & 0 & 0\\ 1 & 3 & 0\\ -1 & 1 & \sqrt{3}\end{bmatrix}, L^T=\begin{bmatrix}2 & 1 & -1\\ 0 & 3 & 1\\ 0 & 0 & \sqrt{3}\end{bmatrix}$ and $x=1,\ y=2,\ z=3$

(18) $L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{bmatrix}, L^T = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$ and $x = 3,\ y = 2,\ z = 1$

(19) $L = \begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix}, L^T = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$

SHORT ANSWER QUESTIONS

1. What are direct methods?
2. What are indirect methods?
3. Give two direct methods to solve a system of linear equations.
4. Give two indirect methods to solve a system of linear equations.
5. Using Gauss elimination method solve $x + y = 2,\ 2x + 3y = 5$.
6. Explain briefly Gauss-Seidel method of solving simultaneous linear equations.
7. Solve the linear system $x_1 - 4x_2 = -2,\ 3x_1 + x_2 = 7$ by Gauss-Jordan method.
8. What is the condition for convergence of Gauss-Seidel method?
9. Why Gauss-Seidel methd is better than Jacobi's-iterative method?
10. Compare Gauss elimination and Gauss-Seidel method?
11. If we start with zero values for $x,\ y,\ z$ while solving $10x + y + z = 12$, $x + 10y + z = 12$, $x + y + 10z = 12$ by Gauss-Seidel iteration, find the values for $x,\ y,\ z$ after one iteration.
12. Find the inverse of $A = \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$ by Gauss-Jordan method.
13. Find the first iteration values of x, y, z satisfying $28x + 4y - z = 32$, $x + 3y + 10z = 24$ and $2x + 17y + 4z = 35$ by Gauss-Seidel method.
14. Given $5x - 2y = 10, 3x + 4y = 12$ find the inverse of the coefficient matrix.
15. Solve $x + y = 2, 2x - 3y = -1$ by Jordan's method.
16. Write down the sufficient conditions for the solution of the linear system $Ax = B$ by LU method of Doolittle or Crout.
17. Write down the sufficient Condition for the solution of the linear system $Ax = B$ by Cholesky's method.
18. Factorize $\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ by Doolittle's method.
19. Factorize $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ by Crout's method.

20. Factorize $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ by Choleskey's method.

21. What is the drawback of Jacobi's method in finding the eigen values of a symmetric matrix?

22. What is the minimum number of rotations required to bring the symmetric matrix A of order n in to diagonal form.

23. Determine the largest eigen value and the corresponding eigen vector of the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, correct to two decimal places using power method.

24. Write down all possible types of initial vectors to determine the largest eigen value and the corresponding eigen vector of a matrix of size 2×2.

Polynomial Interpolation

INTRODUCTION

Let $y = f(x)$ be a function of x. Suppose we are given a set of value of y, say $y_0, y_1, y_2, \ldots y_i, \ldots, y_n$, corresponding to given values of x, say x_0, x_1, x_2, ..., x_i, ..., x_n. The values of x are called arguments and values of y are called entries.

Interpolation may be defined as the process of finding the values of a function $f(x)$ for any intermediate value of the argument x between x_0 and x_n. This process has been described by **Thiele as "the art of reading between the lines of a table"**.

In the broader sense interpolation is the process of replacing the unknown function or complicated function $f(x)$ by a simpler function $\phi(x)$ which assumes the same values of y for the given values of x. The function $\phi(x)$ is called the **interpolating function** or **interpolating formula.** In many engineering applications this function is called a **smoothing function**.

A desirable characteristic of interpolating function is that it must be simple. Since polynomials are the simplest of the functions, we usually choose $\phi(x)$ to be a polynomial and so it is called **interpolating polynomial**. Nearly all the standard formulae of interpolation are polynomials. In case the given values indicate that the function is periodic, we represent it by a finite trigonometric series. But we consider here only polynomial interpolation.

The process of representing by a polynomial is justified by the Weierstrass theorem: **"Every function $f(x)$, which is continuous in an interval (a,b) can be represented in that interval, to any desired degree of accuracy by a polynomial $\phi(x)$".**

Some of the important interpolating polynomial formulae are

(1) Newton's forward and backward formulae
(2) Newton's divided difference formula
(3) Lagrange's formula
(4) The central difference formulae

(a) Gauss forward and backward formulae
(b) Stirling's formula
(c) Everrett's formula
(d) Bessel's formula etc

All these formulae are expressed interms of difference operators of various types. So we shall first introduce the different difference operators.

Extrapolation: The process of obtaining the value of a tabulated function outside the interval of the given values of the arguments is called **extrapolation** or **prediction**.

FINITE DIFFERENCE OPERATORS

When the changes in the independent variable or argument of a function is discrete, the infinitesimal calculus cannot be applied to study such functions. But the calculus of finite differences enable us to study such functions. The calculus of finite differences form the basis of many processes and is used in the derivation of many formulae in numerical analysis.

Forward Difference Operator Δ

Let $y_0, y_1, y_2, \ldots, y_n$ be the values of the function $y = f(x)$ at equally spaced arguments $x_0, x_1, x_2, \ldots, x_n$ respectively.
Let the equal space or interval be h.

Then $x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1} = h$.

Consider the differences $y_1 - y_0, y_2 - y_1, y_3 - y_2, \ldots, y_n - y_{n-1}$
We denote these as

$$\Delta y_0 = y_1 - y_0\ , \quad \Delta y_1 = y_2 - y_1, \ldots\ , \quad \Delta y_{n-1} = y_n - y_{n-1}$$

and are called **first order forward differences** or **first differences and Δ is called the forward difference operator.**
The difference of first order differences are called second order differences.

They are
$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0, \qquad \Delta^2 y_1 = \Delta y_2 - \Delta y_1$$
$$\Delta^2 y_2 = \Delta y_3 - \Delta y_2, \ldots, \quad \Delta^2 y_{n-1} = \Delta y_n - \Delta y_{n-1}$$

Similarly we can find 3rd, 4th, ..., nth order differences.
Thus

$$\Delta^n y_i = \Delta^{n-1} y_{i+1} - \Delta^{n-1} y_i, \; i = 1, 2, 3, \ldots n$$

If $y_i = f(x_i)$, then $\Delta y_i = y_{i+1} - y_i$

or
$$\Delta f(x_i) = f(x_i + h) - f(x_i)$$

$\therefore$
$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0			
		Δy_0		
x_1	y_1		$\Delta^2 y_0$	
		Δy_1		$\Delta^3 y_0$
x_2	y_2		$\Delta^2 y_1$	
		Δy_2		
x_3	y_3			

The first entry y_0 is called the leading term and the differences $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ in the diagonal through y_0 are called the leading differences.

Backward Difference Operator ∇ (Read as Del or Nebla)

The differences $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are denoted by $\nabla y_1, \nabla y_2, \ldots, \nabla y_n$ and are called first order **backward differences. The operator ∇ is called backward difference operator**.

$\therefore\ \nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \ldots, \nabla y_n = y_n - y_{n-1}$

The differences

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1,\ \nabla^2 y_3 = \nabla y_3 - \nabla y_2, \ldots,\ \nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$$

are called second order backward differences and so on.

$$\nabla y_i = y_i - y_{i-1},\quad \nabla^2 y_i = \nabla y_i - \nabla y_{i-1},\quad \nabla^3 y_i = \nabla^2 y_i - \nabla^2 y_{i-1},\quad \ldots,\quad \nabla^n y_i = \nabla^{n-1} y_i - \nabla^{n-1} y_{i-1}$$

The backward difference table is

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
		∇y_1			
x_1	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		∇y_3		$\nabla^3 y_4$	
x_3	y_3		$\nabla^2 y_4$		
		∇y_4			
x_4	y_4				

Prove that $\nabla^n y_n = \Delta^n y_0$

Prove:

We have
$$\Delta y_0 = y_1 - y_0$$
$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$
$$= y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

and
$$\Delta y_1 = y_1 - y_0 = \Delta y_0$$
$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$$
$$= y_2 - y_1 - (y_1 - y_0) = \Delta^2 y_0$$

$\therefore$
$$\nabla y_1 = \Delta y_0$$

and
$$\nabla^2 y_2 = \Delta^2 y_0$$

Similarly
$$\nabla^3 y_3 = \Delta^3 y_0, \cdots, \nabla^n y_n = \Delta^n y_0$$

Shift Operator or Displacement Operator *E*

The operator E is defined by $Ef(x) = f(x+h)$, where h is the interval of differencing. That is E shifts $f(x)$ to next higher value $f(x+h)$.

In other words $Ey_r = y_{r+1}$, where $y_r = f(x_r)$ and $y_{r+1} = f(x_{r+1})$

$$E^2 f(x) = E[Ef(x)] = E[f(x+h)] = f(x+2h)$$
$$E^3 f(x) = E[E^2 f(x)] = E[f(x+2h)] = f(x+3h)$$
$$\vdots$$
$$E^n f(x) = f(x+nh)$$

The operator E^{-1} is defined as $E^{-1}f(x) = f(x-h)$, where h is the interval of differencing.

That is E^{-1} shifts $f(x)$ to preceeding lower value $f(x-h)$.

In other words, $E^{-1}y_r = y_{r-1}$ where $y_r = f(x_r)$ and $y_{r-1} = f(x_{r-1})$

$$E^{-2} f(x) = E^{-1}\left[E^{-1} f(x)\right] = E^{-1}\left(f(x-h)\right) = f(x-2h)$$
$$E^{-3} f(x) = E^{-1}\left[E^{-2} f(x)\right] = E^{-1} f(x-2h) = f(x-3h)$$
$$\vdots$$
$$E^{-n} f(x) = f(x-nh)$$

Note: $\Delta^\circ = 1$ and $E^\circ = 1$, where 1 is the identity operator

Relation Between the Operators E, Δ, ∇

1. **Prove that $E = 1 + \Delta$,** where 1 is the identity operator.

Proof:

We have
$$\Delta f(x) = f(x+h) - f(x)$$
and
$$Ef(x) = f(x+h)$$
$$(1+\Delta)f(x) = 1.f(x) + \Delta f(x)$$
$$= \cancel{f(x)} + f(x+h) - \cancel{f(x)}$$
$$= f(x+h)$$
$$= Ef(x)$$
$\therefore$
$$Ef(x) = (1+\Delta)f(x) \text{ for any } f(x)$$
$\therefore$
$$E = 1 + \Delta$$
Similarly
$$E^2 = (1+\Delta)^2,$$
$$E^3 = (1+\Delta)^3, \ldots, E^n = (1+\Delta)^n$$

Note: The method of writing the operator equality is known as the method of separation of symbols. However it should be remembered that operators cannot really stand alone and the operand $f(x)$ is always understood.

2. **Prove that $E^{-1} = 1 - \Delta$**

Proof:

We have
$$\nabla f(x) = f(x) - f(x-h)$$
and
$$E^{-1}f(x) = f(x-h)$$
Now
$$(1-\nabla)f(x) = 1f(x) - \nabla f(x)$$
$$= f(x) - \{f(x) - f(x-h)\}$$
$$= f(x-h)$$
$$= E^{-1}f(x)$$
$\therefore$
$$E^{-1}f(x) = (1-\nabla)f(x) \text{ for any } f(x)$$
$\therefore$
$$\boxed{E^{-1} = 1 - \nabla}$$

Properties of Δ and *E*

1. **Δ is linear operator**

   ie. $\Delta[f(x)+g(x)]=\Delta[f(x)]+\Delta[g(x)]$

   and $\Delta[Cf(x)]=C\Delta[f(x)]$, where C is a constant.

2. ***E* is a linear operator**

   ie. $E[f(x)+g(x)]=E[f(x)]+E[g(x)]$

   and $E[Cf(x)]=CE[f(x)]$

3. **Index laws**

   If m and n are positive integers, then

$$\Delta^m[\Delta^n f(x)]=\Delta^{m+n}[f(x)]=\Delta^n[\Delta^m f(x)]$$

$\therefore$
$$\Delta^m.\Delta^n=\Delta^{m+n}$$

Similarly
$$E^m.E^n=E^{m+n}$$

Δ^{-n} is the inverse operator of Δ^n

E^{-n} is the inverse operator of E^n

4. **Prove that $\Delta\nabla=\nabla\Delta$**

Proof:

We have
$$\Delta f(x)=f(x+h)-f(x)$$
$$\nabla f(x)=f(x)-f(x-h)$$

$\therefore$
$$\begin{aligned}(\Delta\nabla)f(x)&=\Delta[\nabla f(x)]\\&=\Delta[f(x)-f(x-h)]\\&=\Delta f(x)-\Delta f(x-h)\\&=f(x+h)-f(x)-[f(x)-f(x-h)]\end{aligned}$$

$\Rightarrow$
$$(\Delta\nabla)f(x)=f(x+h)-2f(x)+f(x-h)$$

and
$$\begin{aligned}(\nabla\Delta)f(x)&=\nabla[\Delta f(x)]\\&=\nabla[f(x+h)-f(x)]\\&=\nabla f(x+h)-\nabla f(x)\end{aligned}$$

$$= f(x+h) - f(x) - [f(x) - f(x-h)]$$
$$= f(x+h) - 2f(x) + f(x-h)$$

$\therefore$ $$(\Delta\nabla)f(x) = (\nabla\Delta)f(x) \text{ for any } f(x)$$

$\therefore$ $$\Delta\nabla = \nabla\Delta$$

5. **Prove that $\Delta E = E\Delta$**

Proof:

We have $$\Delta f(x) = f(x+h) - f(x)$$

and $$Ef(x) = f(x+h)$$

Now $$(\Delta E)f(x) = \Delta[Ef(x)]$$
$$= \Delta[f(x+h)]$$
$$= f(x+2h) - f(x+h)$$

and $$(E\Delta)f(x) = E[\Delta f(x)]$$
$$= E[f(x+h) - f(x)]$$
$$= E[f(x+h)] - E[f(x)]$$
$$= f(x+2h) - f(x+h)$$

$\therefore$ $$(\Delta E)f(x) = (E\Delta)f(x) \quad \text{for any } f(x)$$

$\therefore$ $$\Delta E = E\Delta$$

6. **Prove that $\underline{\Delta E^{-1} = E^{-1}\Delta}$**

Proof:

We have $$\Delta f(x) = f(x+h) - f(x)$$
$$E^{-1}f(x) = f(x-h)$$

Now $$[\Delta E^{-1}]f(x) = \Delta[E^{-1}f(x)]$$
$$= \Delta[f(x-h)]$$
$$= f(x) - f(x-h)$$

and $$[E^{-1}\Delta]f(x) = E^{-1}[\Delta f(x)]$$
$$= E^{-1}[f(x+h) - f(x)]$$
$$= E^{-1}[f(x+h)] - E^{-1}[f(x)]$$
$$= f(x) - f(x-h)$$

$\therefore$ $$(\Delta E^{-1})f(x) = (E^{-1}\Delta)f(x) \text{ for any } f(x)$$

$\Rightarrow$ $$\Delta E^{-1} = E^{-1}\Delta$$

Because of this property, we shall write each side $= \dfrac{\Delta}{E}$

Theorem 4.1

The n^{th} differences of a polynomial of n^{th} degree are constant.

ie. If $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ is an n^{th} degree polynomial, then

$\Delta^n f(x) = a_0 n! h^n =$ constant, where h is the interval of differencing.

Note:

(1) $\Delta^{n+1} f(x) = 0$. More generally, $\Delta^r f(x) = 0 \forall r \geq n+1$.

(2) The converse of this theorem is also true.
That is, if the n^{th} differences of a function tabulated at equally spaced arguments are constants, then the function is a polynomial of n^{th} degree.
The Converse is of great use in practical situations. If in a difference table the n^{th} differences are constant or nearly so (since rounding off errors may prevent them being exactly equal), then the function may be represented by a polynomial of n^{th} degree by a suitable interpolation formula.

7. **The Central difference operator**
The Central difference operator δ is defined by $\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$.

Now
$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$
$$= E^{\frac{1}{2}} f(x) - E^{-\frac{1}{2}} f(x)$$

$\Rightarrow$
$$\delta f(x) = \left[E^{1/2} - E^{-1/2}\right] f(x)$$

$\therefore$
$$\delta = E^{1/2} - E^{-1/2}$$

8. The averaging operator or the mean value operator μ is defined by

$$\mu[f(x)] = \frac{1}{2}\left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)\right]$$

Now
$$\mu[f(x)] = \frac{1}{2}\left[f\left(x + \frac{h}{2}\right)\right] + f\left(x - \frac{h}{2}\right)\right]$$

$$= \frac{1}{2}\left[E^{1/2} f(x) + E^{-1/2} f(x)\right]$$

$\Rightarrow$
$$\mu[f(x)] = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] f(x)$$

$$\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$$

9. **Prove that $\Delta = \nabla E = \delta E^{1/2}$**

Proof:

We have
$$\Delta f(x) = f(x+h) - f(x)$$
$$\nabla f(x) = f(x) - f(x-h)$$
$$Ef(x) = f(x+h)$$
$$E^{1/2} f(x) = f\left(x + \frac{h}{2}\right)$$
$$(\nabla E) f(x) = \nabla\left[Ef(x)\right]$$
$$= \nabla\left[f(x+h)\right]$$
$\Rightarrow$
$$(\nabla E) f(x) = f(x+h) - f(x)$$
$$= \Delta f(x)$$
$\therefore$
$$\nabla E f(x) = \Delta f(x) \text{ for any } f(x)$$
$\therefore$
$$\nabla E = \Delta$$
Now
$$\left(\delta E^{1/2}\right) f(x) = \delta\left[E^{1/2}\left(f(x)\right)\right]$$
$$= \delta\left[f\left(x+\frac{h}{2}\right)\right]$$
$$= f\left(x + \frac{h}{2} + \frac{h}{2}\right) - f\left(x + \frac{h}{2} - \frac{h}{2}\right)$$
$$= f(x+h) - f(x)$$
$$= \Delta f(x)$$
$$\left(\delta E^{1/2}\right) f(x) = \Delta f(x) \text{ for any } f(x)$$
$$\delta E^{1/2} = \Delta$$
$$\Delta = \nabla E = \delta E^{1/2}$$

10. **Prove that $\mu = \sqrt{1 + \frac{1}{4}\delta^2}$**

Proof:

We have
$$\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] \tag{1}$$
and
$$\delta = E^{1/2} - E^{-1/2}$$
Now
$$1 + \frac{1}{4}\delta^2 = 1 + \frac{1}{4}\left[E^{1/2} - E^{-1/2}\right]^2$$
$$= \frac{1}{4}\left[4 + \left(E - 2E^{1/2}E^{-1/2} + E^{-1}\right)\right]$$
$$= \frac{1}{4}\left[4 + E - 2 + E^{-1}\right]$$

$$= \frac{1}{4}\left[E + 2 - E^{-1}\right]$$

$$= \frac{1}{4}\left[E^{1/2} + E^{-1/2}\right]^2$$

$$\therefore \quad \sqrt{1 + \frac{1}{4}\delta^2} = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$$

$$\therefore \quad \mu = \sqrt{1 + \frac{1}{4}\delta^2} \qquad \text{[using (1)]}$$

11. Relation between differentical operator and difference operator

We know $Df(x) = \frac{d}{dx} f(x)$,

$$D^2 f(x) = \frac{d^2}{dx^2} f(x), \quad D^3 f(x) = \frac{d^3}{dx^3} f(x), \ldots$$

We have Taylor's series for $f(x)$ as

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots$$

$$\Rightarrow \quad Ef(x) = f(x) + \frac{h}{1!} Df(x) + \frac{h^2}{2!} D^2 f(x) + \frac{h^3}{3!} D^3 f(x) + \ldots$$

$$= \left(1 + \frac{h}{1!} D + \frac{h^2}{2!} D^2 + \frac{h^3}{3!} D^3 + \ldots\right) f(x)$$

$$\Rightarrow \quad Ef(x) = e^{hD} f(x) \quad \Rightarrow \mathbf{E = e^{hD}}$$

But $\qquad E = 1 + \Delta$

$\therefore \qquad E = e^{hD}$ or $1 + \Delta = e^{hD}$

WORKED EXAMPLES

Example 1

If $f(x) = \sin x$, then show that $\Delta^2 f(x) = k\, E f(x)$, where k is a constant.

Solution

Given $\quad f(x) = \sin x$

We have $Ef(x) = E \sin x = \sin(x+h)$

and $\quad \Delta f(x) = f(x+h) - f(x)$

$\qquad = \sin(x+h) - \sin x \qquad$ where h is the intereal of differencing.

$$= 2\cos\left(\frac{x+h+x}{2}\right).\sin\left(\frac{x+h-x}{2}\right)$$

$$= 2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$$

$\therefore$
$$\Delta^2 f(x) = \Delta f(x+h) - \Delta f(x)$$

$$= 2\cos\left(x+h+\frac{h}{2}\right)\sin\frac{h}{2} - 2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$$

$$= 2\sin\frac{h}{2}\left[\cos\left(x+\frac{3h}{2}\right) - \cos\left(x+\frac{h}{2}\right)\right]$$

$$= 2\sin\frac{h}{2}\left[(-2)\sin\left(\frac{x+\frac{3h}{2}+x+\frac{h}{2}}{2}\right)\times\sin\left(\frac{x+\frac{3h}{2}-x-\frac{h}{2}}{2}\right)\right]$$

$$= -4\sin\frac{h}{2}\sin(x+h)\sin\frac{h}{2}$$

$$= -4\sin^2\frac{h}{2}\sin(x+h)$$

$$= kE\sin x, \qquad \text{where } k = -4\sin^2\frac{h}{2}$$

$\Rightarrow$
$$\Delta^2 f(x) = kEf(x)$$

■

Example 2

Prove that $\Delta \log f(x) = \log\left[1+\frac{\Delta f(x)}{f(x)}\right]$.

Solution
$$L.H.S = \Delta \log f(x)$$

$$= \log f(x+h) - \log f(x)$$

$$= \log\left[\frac{f(x+h)}{f(x)}\right]$$

$$= \log\frac{Ef(x)}{f(x)}$$

$$= \log\left[\frac{(1+\Delta)f(x)}{f(x)}\right]$$

$$= \log\left[\frac{f(x)+\Delta f(x)}{f(x)}\right]$$

$$= \log\left(1+\frac{\Delta f(x)}{f(x)}\right)$$

$$= R.H.S$$

■

Example 3

Test whether $\left(\frac{\Delta^2}{E}\right)x^3$ and $\frac{\Delta^2 x^3}{Ex^3}$ are equal.

Solution

$$\left(\frac{\Delta^2}{E}\right)x^3 = \left[\frac{(E-1)^2}{E}\right]x^3 \quad [\because \Delta = E-1]$$
$$= \left[\frac{E^2-2E+1}{E}\right]x^3$$
$$= \left(E-2+E^{-1}\right)x^3$$
$$= E\left(x^3\right)-2x^3+E^{-1}\left(x^3\right)$$
$$= (x+h)^3-2x^3+(x-h)^3, \text{ (where } h \text{ is the length of the interval)}$$
$$= x^3+3x^2h+3xh^2+h^3-2x^3+x^3-3x^2h+3xh^2-h^3$$
$$= 6xh^2$$

$$\Delta^2x^3 = (E-1)^2x^3$$
$$= \left(E^2-2E+1\right)x^3$$
$$= E^2\left(x^3\right)-2E\left(x^3\right)+x^3$$
$$= (x+2h)^3-2(x+h)^3+x^3$$
$$= x^3+6x^2h+12xh^2+8h^3-2\left(x^3+3x^2h+3xh^2+h^3\right)+x^3$$

$\Rightarrow$
$$\Delta^2x^3 = 6xh^2+6h^3 = 6h^2(x+h)$$
$$Ex^3 = (x+h)^3$$

$\therefore$
$$\frac{\Delta^2x^3}{Ex^3} = \frac{6h^2(x+h)}{(x+h)^3} = \frac{6h^2}{(x+h)^2} \qquad (2)$$

■

$\therefore$ from (1) and (2), we get $\left(\frac{\Delta^2}{E}\right)x^3 \neq \frac{\Delta^2x^3}{Ex^3}$

Example 4

Find the value of $\Delta^2\left[\frac{a^{2x}+a^{4x}}{\left(a^2-1\right)^2}\right]$, taking $h=1$.

Solution

Let $f(x) = \frac{a^{2x}+a^{4x}}{\left(a^2-1\right)^2}$

$$\therefore \quad \Delta^2 f(x) = (E-1)^2 f(x)$$
$$= (E^2 - 2E + 1) f(x)$$
$$= E^2 f(x) - 2Ef(x) + f(x)$$
$$= f(x+2) - 2Ef(x+1) + f(x)$$
$$= \frac{a^{2(x+2)} + a^{4(x+2)}}{(a^2-1)^2} - 2\frac{\left(a^{2(x+1)} + a^{4(x+1)}\right)}{(a^2-1)^2} + \frac{a^{2x} + a^{4x}}{(a^2-1)^2}$$
$$= \frac{1}{(a^2-1)^2}\left[a^4.a^{2x} + a^8.a^{4x} - 2\left(a^2.a^{2x} + a^4.a^{4x}\right) + a^{2x} + a^{4x}\right]$$
$$= \frac{1}{(a^2-1)^2}\left[a^{2x}\left(a^4 - 2a^2 + 1\right) + a^{4x}\left(a^8 - 2a^4 + 1\right)\right]$$
$$= \frac{1}{(a^2-1)^2}\left[a^{2x}\left(a^2-1\right)^2 + a^{4x}\left(a^4-1\right)^2\right]$$
$$= \frac{1}{(a^2-1)^2}\left[a^{2x}\left(a^2-1\right)^2 + a^{4x}\left(a^2-1\right)^2\left(a^2+1\right)^2\right]$$
$$= \frac{(a^2-1)^2}{(a^2-1)^2}\left[a^{2x} + a^{4x}\left(a^2+1\right)^2\right] = a^{2x} + \left(a^2+1\right)^2 a^{4x} \qquad \blacksquare$$

Example 5

Prove that $\left(\frac{\Delta^2}{E}\right)e^x \frac{Ee^x}{\Delta^2 e^x} = e^x.$

Solution $\quad L.H.S = \left(\frac{\Delta^2}{E}\right)e^x \frac{Ee^x}{\Delta^2 e^x}$

$$\left(\frac{\Delta^2}{E}\right)e^x = \frac{(E-1)^2}{E}e^x$$
$$= \frac{\left(E^2 - 2E + 1\right)}{E}e^x$$
$$= \left[E - 2 + E^{-1}\right]e^x$$
$$= Ee^x - 2e^x + E^{-1}e^x$$
$$= e^{x+h} - 2e^x + e^{x-h} \qquad \text{[where } h \text{ is the interval of differencing]}$$
$$= e^x\left[e^h - 2 + e^{-h}\right]$$
$$= e^x\left[e^h - 2 + \frac{1}{e^h}\right] \qquad (1)$$

$$= \frac{e^x}{e^h}\left[e^{2h} - 2e^h + 1\right]$$

$$\Rightarrow \qquad \left(\frac{\Delta^2}{E}\right)e^x = \frac{e^x}{e^h}\left(e^h - 1\right)^2$$

and
$$\frac{Ee^x}{\Delta^2 e^x} = \frac{e^{x+h}}{(E-1)^2 e^x}$$

$$= \frac{e^{x+h}}{\left(E^2 - 2E + 1\right)e^x}$$

$$= \frac{e^{x+h}}{E^2e^x - 2Ee^x + e^x}$$

$$= \frac{e^{x+h}}{e^{x+2h} - 2e^{x+h} + e^x}$$

$$= \frac{e^{x+h}}{e^x\left[e^{2h} - 2e^h + 1\right]}$$

$$\Rightarrow \qquad \frac{Ee^x}{\Delta^2 e^x} = \frac{e^h}{\left(e^h - 1\right)^2} \qquad (2)$$

From (1) and (2),

$$L.H.S = \frac{e^x}{e^h}\left(e^h - 1\right)^2 \frac{e^h}{\left(e^h - 1\right)^2} = e^x = R.H.S$$ ■

Example 6

Evaluate $\Delta^{10}(1-x)(1-2x^2)(1-3x^3)(1-4x^4)$ if the interval of differencing is 2.

Solution

Let
$$f(x) = (1-x)(1-2x^2)(1-3x^3)(1-4x^4)$$

Given $h = 2$

We know that for a polynomial of n^{th} degree $\Delta^n f(x) = a_0 n! h^n$, where h is the interval of differencing and a_0 is coefficient of x^n.
When $f(x)$ is expanded as a polynomial in x, the leading term is

$$(-1)(-2)(-3)(-4)\cdot x \cdot x^2 \cdot x^3 \cdot x^4 = 24x^{10}$$

Here
$$a_0 = 24, \quad n = 10, \quad h = 2$$

$$\therefore \qquad \Delta^{10} f(x) = 24 \times 10! \times 2^{10}$$ ■

Example 7

Prove the following with usual notation.

(i) $hD = \log_e(1+\Delta)$

(ii) $hD = -\log_e(1-\nabla)$

(iii) $hD = \sin h^{-1}(\mu\delta)$

Solution

(i) We know that

$$E = 1+\Delta$$

and

$$E = e^{hD}$$

$\therefore$

$$\log_e E = \log_e e^{hD}$$

$\Rightarrow$

$$\log_e(1+\Delta) = hD\log_e e$$

$\Rightarrow$

$$hD = \log(1+\Delta)$$

(ii) we have

$$hD = \log_e(1+\Delta)$$

$\Rightarrow$

$$hD = \log_e E$$

$$= -\log_e E^{-1}$$

$$= -\log_e(1-\nabla)$$

(iii) We have

$$\mu = \frac{1}{2}\left[E^{1/2}+E^{-1/2}\right] \quad \text{and} \quad \delta = E^{1/2}-E^{-1/2}$$

$\therefore$

$$\mu\delta = \frac{1}{2}\left(E^{1/2}+E^{-1/2}\right)\left(E^{1/2}-E^{-1/2}\right)$$

$$= \frac{1}{2}\left[E-E^{-1}\right]$$

$$= \frac{1}{2}\left[e^{hD}-e^{-hD}\right]$$

$$= \sin hD \qquad \left[\because \sin h\theta = \frac{e^{\theta}-e^{-\theta}}{2}\right]$$

$\therefore$

$$hD = \sin h^{-1}(\mu\,\delta)$$

■

Example 8

Prove that $1+\delta^2\mu^2 = \left(1+\frac{1}{2}\delta^2\right)^2$.

Solution

We know that $\delta = E^{1/2}-E^{-1/2}$ and $\mu = \frac{1}{2}\left[E^{1/2}+E^{-1/2}\right]$

$$\therefore \qquad \delta\mu = \left(E^{1/2} - E^{-1/2}\right)\frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$$

$$= \frac{1}{2}\left(E - E^{-1}\right)$$

$$\therefore \qquad 1 + \delta^2\mu^2 = 1 + \frac{1}{4}\left(E - E^{-1}\right)^2$$

$$= 1 + \frac{1}{4}\left[E^2 - 2EE^{-1} + E^{-2}\right]$$

$$= \frac{1}{4}\left[4 + E^2 - 2 + E^{-2}\right]$$

$$= \frac{1}{4}\left[E^2 + 2 + E^{-2}\right]$$

$$\Rightarrow \qquad 1 + \delta^2\mu^2 = \frac{1}{4}\left[E + E^{-1}\right]^2 \qquad (1)$$

Now

$$1 + \frac{1}{2}\delta^2 = 1 + \frac{1}{2}\left[E^{1/2} - E^{-1/2}\right]^2$$

$$= 1 + \frac{1}{2}\left[E - 2 + E^{-1}\right]$$

$$= \frac{2 + E - 2 + E^{-1}}{2}$$

$$= \frac{E + E^{-1}}{2}$$

$$\therefore \qquad \left(1 + \frac{1}{2}\delta^2\right)^2 = \frac{1}{4}\left(E + E^{-1}\right)^2 \qquad (2)$$

From (1) and (2), we get

$$1 + \delta^2\mu^2 = \left(1 + \frac{1}{2}\delta^2\right)^2$$ ■

Example 9

Using the method of separation of symbols, prove that

$$u_0 + \frac{u_1}{1!}x + \frac{u_2}{2!}x^2 + \frac{u_3}{3!}x^3 + \ldots = e^x\left[u_0 + x\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \ldots\right].$$

Solution

$$L.H.S = u_0 + \frac{u_1}{1!}x + \frac{u_2}{2!}x^2 + \frac{u_3}{3!}x^3 + \ldots$$

We know

$$E^r u_0 = u_r$$

$\therefore$
$$E u_0 = u_1$$
$$E^2 u_0 = u_2, E^3 u_0 = u_3, \ldots$$

$\therefore$
$$L.H.S = u_0 + \frac{x}{1!}E u_0 + \frac{x^2}{2!}E^2 u_0 + \frac{x^3}{3!}E^3 u_0 + \ldots$$
$$= \left(1 + \frac{x}{1!}E + \frac{x^2}{2!}E^2 + \frac{x^3}{3!}E^3 + \ldots\right) u_0$$
$$= e^{xE} u_0$$
$$= e^{x(1+\Delta)} u_0$$
$$= e^x . e^{x\Delta} u_0$$
$$= e^x \left[1 + \frac{x\Delta}{1!} + \frac{x^2\Delta^2}{2!} + \frac{x^3\Delta^3}{3!} + \ldots\right] u_0$$
$$= e^x \left[u_0 + \frac{x}{1!}\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \frac{x^3}{3!}\Delta^3 u_0 + \ldots\right]$$
$$= R.H.S$$

■

Example 10

Using the method of separation of symbols, prove that

$$u_1 x + u_2 x^2 + u_3 x^3 + \ldots = \frac{x}{1-x} u_1 + \frac{x^2}{(1-x)^2}\Delta u_1 + \frac{x^3}{(1-x)^3}\Delta^2 u_1 + \ldots$$

Solution

$$L.H.S = u_1 x + u_2 x^2 + u_3 x^3 + \ldots$$
$$= u_1 x + x^2 E u_1 + x^3 E^2 u_1 + \ldots$$
$$= \left(x + x^2 E + x^3 E^2 + \ldots\right) u_1$$
$$= x\left(1 + xE + x^2 E^2 + \ldots\right) u_1$$
$$= x(1 - xE)^{-1} u_1$$
$$= \frac{x}{(1-xE)} u_1$$
$$= \frac{x}{1 - x(1+\Delta)} u_1 \qquad [\because E = 1 + \Delta]$$
$$= \frac{x}{1 - x - x\Delta} u_1$$

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$$= \frac{x}{(1-x)\left(1-\frac{x\Delta}{1-x}\right)}u_1$$

$$= \frac{x}{1-x}\left(1-\frac{x\Delta}{1-x}\right)^{-1} u_1$$

$$= \frac{x}{1-x}\left[1+\frac{x}{1-x}\Delta+\frac{x^2}{(1-x)^2}\Delta^2+\frac{x^3}{(1-x)^3}\Delta^3+\dots\right]u_1$$

$$= \frac{x}{1-x}\left[u_1+\frac{x}{1-x}\Delta u_1+\frac{x^2}{(1-x)^2}\Delta^2 u_1+\frac{x^3}{(1-x)^3}\Delta^3 u_1+\dots\right]$$

$$= \frac{x}{1-x}u_1+\frac{x^2}{(1-x)^2}\Delta u_1+\frac{x^3}{(1-x)^3}\Delta^2 u_1+\dots$$

$$= R.H.S$$

■

Example 11

Using the method of separation of symbols prove that

$$(u_1-u_0)-x(u_2-u_1)+x^2(u_3-u_2)-\dots$$
$$= \frac{\Delta u_0}{1+x}-\frac{x}{(1+x)^2}\Delta^2 u_0+\frac{x^2}{(1+x)^3}\Delta^3 u_0-\dots$$

Solution

$$L.H.S = (u_1-u_0)-x(u_2-u_1)+x^2(u_3-u_2)-\dots$$
$$= \Delta u_0 - x\Delta u_1 + x^2\Delta u_2 - x^3\Delta u_3+\dots$$

$$R.H.S = \frac{1}{1+x}\Delta u_0-\frac{x}{(1+x)^2}\Delta^2 u_0+\frac{x^2}{(1+x)^3}\Delta^3 u_0-\dots$$

$$= \frac{1}{x}\left[\frac{x}{1+x}\Delta u_0-\frac{x^2}{1+x^2}\Delta^2 u_0+\frac{x^3}{(1+x)^3}\Delta^3 u_0-\dots\right]$$

$$= \frac{1}{x}\left[\frac{x}{1+x}\Delta-\frac{x^2}{1+x^2}\Delta^2+\frac{x^3}{(1+x)^3}\Delta^3-\dots\right]u_0$$

$$= \frac{1}{x}\left[\frac{\frac{x}{1+x}\Delta}{1+\frac{x\Delta}{1+x}}\right]u_0$$

$$= \frac{1}{x}\left[\frac{\left(\frac{x}{1+x}\Delta\right)}{\frac{1+x+x\Delta}{1+x}}\right]u_0$$

$$= \frac{\Delta}{1+x(1+\Delta)} u_0$$
$$= \frac{\Delta}{1+xE} u_0$$
$$= (1+xE)^{-1} \Delta u_0$$
$$= (1 - xE + x^2E^2 - x^3E^3 + \cdots)\Delta u_0$$
$$= \Delta u_0 - xE\Delta u_0 + x^2E^2\Delta u_0 - x^3E^3\Delta u_0 + \ldots$$
$$= \Delta u_0 - x\Delta E u_0 + x^2\Delta E^2 u_0 - x^3\Delta E^3 u_0 + \ldots \qquad [\because E^r\Delta = \Delta E^r]$$
$$= \Delta u_0 - \Delta u_1 + x^2\Delta u_2 - x^3\Delta u_3 + \ldots$$
$$\therefore \quad L.H.S = R.H.S$$

Example 12

Show that

$$u_x - u_{x+1} + u_{x+2} - u_{x+3} + \ldots = \frac{1}{2}\left[u_{x-\frac{1}{2}} - \frac{1}{8}\Delta^2 u_{x-\frac{3}{2}} + \frac{1.3}{2!}\frac{1}{8^2}\Delta^4 u_{x-\frac{5}{2}} - \ldots\right].$$

Solution

$$R.H.S = \frac{1}{2}\left[u_{x-\frac{1}{2}} - \frac{1}{8}\Delta^2 u_{x-\frac{3}{2}} + \frac{1.3}{2!}\frac{1}{8^2}\Delta^4 u_{x-\frac{5}{2}} - \ldots\right]$$
$$= \frac{1}{2}\left[E^{-1/2}u_x - \frac{1}{8}\Delta^2 E^{-\frac{3}{2}}u_x + \frac{1.3}{2!}\frac{1}{8^2}\Delta^4 E^{-\frac{5}{2}}u_x - \ldots\right]$$
$$= \frac{1}{2}\left[E^{-1/2} - \frac{1}{8}\Delta^2 E^{-3/2} + \frac{1.3}{2!}\frac{1}{8^2}\Delta^4 E^{-5/2} - \ldots\right]u_x$$
$$= \frac{1}{2}\left[1 - \frac{1}{8}\Delta^2 E^{-1} + \frac{1.3}{2!}\frac{1}{8^2}\Delta^4 E^{-2} - \ldots\right]E^{-1/2}u_x$$
$$= \frac{1}{2}\left[1 - \frac{1}{2}\left(\frac{\Delta^2 E^{-1}}{4}\right) + \frac{1.3}{2!}\frac{1}{4}\left(\frac{\Delta^2 E^{-1}}{4}\right)^2 - \ldots\right]E^{-1/2}u_x$$
$$= \frac{1}{2}\left[1 + \frac{\Delta^2 E^{-1}}{4}\right]^{-1/2} E^{-1/2}u_x$$
$$= \frac{1}{2}\left[1 + \frac{\Delta^2}{4E}\right]^{-1/2} E^{-1/2}u_x$$
$$= \frac{1}{2}\left[\frac{4E + \Delta^2}{4E}\right]^{-1/2} E^{-1/2}u_x$$
$$= \frac{1}{2}\left[\frac{4(1+\Delta) + \Delta^2}{4E}\right]^{-1/2} E^{-1/2}u_x$$

$$= \frac{1}{2}\left[\frac{(2+\Delta)^2}{4E}\right]^{-1/2} E^{-1/2}u_x$$

$$= \frac{1}{2}\frac{\left[(1+(1+\Delta))^2\right]^{-\frac{1}{2}}}{4^{-1/2}.E^{-1/2}} E^{-1/2}u_x$$

$$= \frac{1}{2}\frac{\left[(1+E)^2\right]^{-1/2}}{2^{-1}} E^{1/2}E^{-1/2}u_x$$

$$= (1+E)^{-1}u_x$$

$$= (1-E+E^2-E^3+E^4-\ldots)u_x$$

$$= u_x - Eu_x + E^2u_x - E^3u_x + \ldots$$

$$= u_x - u_{x+1} + u_{x+2} - u_{x+3} + \ldots$$

$$= L.H.S$$ ■

Example 13

If $u_1 = (12-x)(4+x)$, $u_2 = (5-x)(4-x)$, $u_3 = (x+18)(x+6)$ and $u_4 = 94$. Obtain the values of x, assuming the second differences are constant.

Solution

Given

$$u_1 = (12-x)(4+x)$$
$$u_2 = (5-x)(4-x)$$
$$u_3 = (x+18)(x+6)$$

and $$u_4 = 94.$$

$\therefore$ the arguments are 1, 2, 3, 4. Since the second differences are constants, third differences are zero.

$\therefore$ $$\Delta^3 u_1 = 0$$

$\Rightarrow$ $$(E-1)^3 u_1 = 0$$

$\Rightarrow$ $$(E^3-3E^2+3E-1)u_1 = 0$$

$\Rightarrow$ $$E^3u_1 - 3E^2u_1 + 3Eu_1 - u_1 = 0$$

$\Rightarrow$ $$u_4 - 3u_3 + 3u_2 - u_1 = 0$$

$\Rightarrow$ $$94 - 3(x+18)(x+6) + 3(5-x)(4-x) - (12-x)(4+x) = 0$$

$\Rightarrow$ $$94 - 3(x^2+24x+108) + 3(20-9x+x^2) - (48+8x-x^2) = 0$$

$\Rightarrow$ $$x^2 - 107x - 218 = 0$$

$\Rightarrow$ $$(x-109)(x+2) = 0$$

$\Rightarrow$ $$\mathbf{x = -2 \text{ or } 109}$$ ■

Example 14

If $f(0)=1$, $f(1)+f(2)=10$, $f(3)+f(4)+f(5)=65$ find the value of $f(4)$, assuming the second differences are constant.

Solution

Given the arguments are $0, 1, 2, 3, 4, 5$ and $f(0)=1$, $f(1)+f(2)=10$, $f(3)+f(4)+f(5)=65$ and the second differences are constant.

We form the difference table.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
0	$f(0)=1$		
		$f(1)-1$	
1	$f(1)$		$f(2)-f(1)-(f(1)-1)$ $=f(2)-2f(1)+1$
		$f(2)-f(1)$	
2	$f(2)$		$f(3)-f(2)-(f(2)-f(1))$ $=f(3)-2f(2)+f(1)$
		$f(3)-f(2)$	
3	$f(3)$		$f(4)-f(3)-(f(3)-f(2))$ $=f(4)-2f(3)+f(2)$
		$f(4)-f(3)$	
4	$f(4)$		$f(5)-f(4)-(f(4)+f(3))$ $=f(5)-2f(4)+f(3)$
		$f(5)-f(4)$	
5	$f(5)$		

Since the second differences are constant, say k, we have

$$f(2)-2f(1)+1=k \tag{1}$$

$$f(3)-2f(2)+f(1)=k \tag{2}$$

$$f(4)-2f(3)+f(2)=k \tag{3}$$

$$f(5)-2f(4)+f(3)=k \tag{4}$$

$(1) \Rightarrow$ $$3f(2)-2(f(1)+f(2))+1=k$$

$\Rightarrow$ $$3f(2)-2\times 10+1=k \tag{5}$$

$\Rightarrow$ $$3f(2)=k+19$$

$(2)+(1) \Rightarrow$ $$f(3)-f(2)-f(1)+1=2k$$

$\Rightarrow$ $$f(3)-[f(1)+f(2)]+1=2k$$

$\Rightarrow$ $$f(3)-10+1=2k \tag{6}$$

$\Rightarrow$ $$f(3)=2k+9$$

$\Rightarrow$

Substituting in (3), we get $$f(4)-2(2k+9)+f(2)=k$$

$\Rightarrow$ $$f(4)+f(2)=5k+18$$

$\Rightarrow$ $$3f(4)+3f(2)=15k+54 \quad [\text{using}(5)]$$

$\Rightarrow$ $$3f(4)+k+19=15k+54 \tag{7}$$

$\Rightarrow$ $$3f(4)=14k+35$$

$\Rightarrow$

$(4)\times 3 \Rightarrow$ $$3f(5)-6f(4)+3f(3)=3k$$

$\Rightarrow$ $$3f(5)-2(14k+35)+3(2k+9)=3k$$

$\Rightarrow$ $$3f(5)-28k-70+6k+27=3k \tag{8}$$

$\Rightarrow$ $$3f(5)=25k+43$$

$\Rightarrow$

Given $$f(3)+f(4)+f(5)=65$$

$\therefore$ $$3f(3)+3f(4)+3f(5)=195$$

$\Rightarrow$ $$3(2k+9)+14k+35+25k+43=195$$

$\Rightarrow$ $$6k+27+14k+35+25k+43=195$$

$\Rightarrow$ $$45k=195-105=90 \Rightarrow k=2$$

$\therefore$ $$3f(4)=14\times 2+35=63 \Rightarrow \boldsymbol{f(4)=21} \quad \blacksquare$$

Example 15

Estimate the missing value in the table.

x	0	1	2	3	4
$y=f(x)$	1	3	9	-	81

Solution

Since four values are given, the fourth differences are zero.

$$\Delta^4 f(x)=0 \quad \forall x$$

Let a be the missing value
We from the difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	1				
		2			
1	3		4		
		6		$a-19$	
2	9		$a-15$		$124-4a$
		$a-9$		$105-3a$	
3	a		$90-2a$		
		$81-a$			
4	81				

Since $\Delta^4 f(x) = 0$, we get

$\Rightarrow$ $$124-4a=0 \Rightarrow 4a=124 \Rightarrow a=31$$

$\therefore$ **the missing value is 31**

Aliter:

Since four values are given, the fourth differences are zero.

$\therefore$ $$\Delta^4 f(x)=0 \quad \forall x$$

$\Rightarrow$ $$(E-1)^4 f(x)=0 \quad \forall x$$

$\Rightarrow$ $$\left(E^4-4E^3+6E^2-4E+1\right)f(x)=0 \quad \forall x$$

$\Rightarrow$ $$E^4 f(x)-4E^3 f(x)+6E^2 f(x)-4Ef(x)+f(x)=0$$

$\Rightarrow$ $$f(x+4)-4f(x+3)+6f(x+2)-4f(x+1)+f(x)=0 \quad \forall x$$

Put $x=0$

$\therefore$ $$f(4)-4f(3)+6f(2)-4f(1)+f(0)=0$$

$\Rightarrow$ $$81-4f(3)+6\times 9-4\times 3+1=0$$

$\Rightarrow$ $$4f(3)=124$$

$\Rightarrow$ $$f(3)=\frac{124}{4}=\mathbf{31}$$ ■

Example 16

Obtain the missing values in the table.

x	1	2	3	4	5	6	7	8
$f(x)$	1	8	-	64	-	216	343	512

Solution

Since six values are given, the sixth differences are zero.

$$\Delta^6 f(x) = 0 \quad \forall x$$

$\therefore$

$$(E-1)^6 f(x) = 0 \quad \forall x$$

$\Rightarrow$

$$(E^6 - 6C_1E^5 + 6C_2E^4 - 6C_3E^3 + 6C_4E^2 - 6C_5E + 6C_6) f(x) = 0 \quad \forall x$$

$\Rightarrow$

$$(E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1) f(x) = 0 \quad \forall x$$

$\Rightarrow$

$$E^6 f(x) - 6E^5 f(x) + 15E^4 f(x) - 20E^3 f(x) + 15E^2 f(x) - 6Ef(x)$$

$$+15E^2 f(x) - 6Ef(x) + f(x) = 0 \quad \forall x = 1,2,\ldots$$

$\Rightarrow$

$$f(x+6) - 6f(x+5) + 15f(x+4)$$

$$-20f(x+3) + 15f(x+2) - 6f(x+1) + f(x) = 0 \quad \forall x = 1,2,\ldots \tag{1}$$

Put $x = 1$ in (1), we get

$\therefore$ $$f(7) - 6f(6) + 15f(5) - 20f(4) + 15f(3) - 6f(2) + f(1) = 0$$

$\Rightarrow$ $$343 - 6\times216 + 15f(5) - 20\times64 + 15f(3) - 6\times8 + 1 = 0$$

$\Rightarrow$ $$15f(5) + 15f(3) = 2280$$

$\Rightarrow$ $$f(5) + f(3) = 152 \tag{2}$$

Put $x = 2$ in (1), we get

$$f(8) - 6f(7) + 15f(6) - 20f(5) + 15f(4) - 6f(3) + f(2) = 0$$

$\Rightarrow$ $$512 - 6\times343 + 15\times216 - 20f(5) + 15\times64 - 6f(3) + 8 = 0$$

$\Rightarrow$ $$20f(5) + 6f(3) = 2662$$

$\Rightarrow$ $$10f(5) + 3f(3) = 1331 \tag{3}$$

(2) $\times$ (3) $\Rightarrow$ $$3f(5) + 3f(3) = 456$$

Subtracting, we get

$$7f(5) = 875 \Rightarrow f(5) = \frac{875}{7} = 125$$

(2) $\Rightarrow$ $$f(3) = 152 - 125 = 27$$

$\therefore$ $$f(3) = 27, f(5) = 125$$

■

Exercises 4.1

(1) Evaluate (a) $\Delta^2 e^x$ (b) $\Delta \tan x$ (c) $\Delta^2 \cos 2x$, taking h as the interval of differencing.

(2) Taking interval of differencing as h, show that

(a) $\Delta\left(\tan^{-1}x\right)=\tan^{-1}\left(\frac{h}{1+hx+x^2}\right)$ (b) $\Delta\cos(ax+b)=2\sin\left(\frac{ah}{2}\right)\cos\left(ax+b+\frac{ah+\pi}{2}\right)$

(c) $\Delta\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\Delta f(x)-f(x)\Delta g(x)}{g(x)g(x+h)}$ (d) $\Delta\left[\frac{2^x}{(x+1)1}\right]=-x\frac{2^x}{(x+2)!}$

(3) Show that $\left(\frac{\Delta^2}{E}\right)\sin(x+h)+\frac{\Delta^2\sin(x+h)}{E\sin(x+h)}=-4\sin^2\frac{h}{2}[\sin(x+h)+1]$

(4) Prove the following with usual notation

(a) $(1+\Delta)(1-\nabla)=1$ (b) $\Delta\nabla=\nabla\Delta=\delta^2$

(c) $\Delta=\mu\delta+\frac{1}{2}\delta^2$ (d) $\delta=2\sin\left(\frac{hD}{2}\right)$ and $\mu=\cos\left(\frac{hD}{2}\right)$

(5) Using method of separation of symbols, prove the following.

(a) $u_0+u_1+u_2+\ldots+u_n={}^{(n+1)}C_1u_0+{}^{n+1}C_1\Delta u_0+{}^{(n+1)}C_2\Delta^2u_0+\ldots+{}^{(n+1)}C_{n+1}\Delta^nu_0$

(b) $u_x=u_{x-1}+\Delta u_{n-2}+\Delta^2u_{n-3}+\ldots+\Delta^{n-1}u_{x-n}+\Delta^nu_{x-n}$

$\left[\text{ie. to prove } u_x-\Delta^nu_{x-n}=u_{x-1}+\Delta u_{x-2}+\Delta^2u_{x-3}+\ldots+\Delta^{n-1}u_{x-n}\right]$

(c) $u_{x+\frac{1}{2}}=\frac{1}{2}[u_x+u_{x+1}]-\frac{1}{16}\left(\Delta^2u_x+\Delta^2u_{x+1}\right)$ if $\Delta^3u_x=0$

(d) $\Delta x^m-\frac{1}{2}\Delta^2x^m+\frac{1.3}{2.4}\Delta^3x^m-\frac{1.3.5}{2.4.6}\Delta^4x^m+\ldots=\left(x+\frac{1}{2}\right)^m-\left(x-\frac{1}{2}\right)^m$, where $h=1$

(6) Find the missing values from the table.

x	0	1	2	3	4
y_x	1	2	4	–	16

(7) Find the missing values from the table.

x	0	1	2	3	4	5	6
$f(x)$	–4	–2	–	–	220	546	1148

(8) Assuming y_x as a polynomial of degree 4, compute the missing values from the following table.

x	0	1	2	3	4	5	6	7
y_x	0	1	2	1	0	–	–	–

Answers 4.1

(6) $y_3=8.25$ (7) $u_2=12, u_3=68$ (8) $y_5=5, y_6=26, y_7=77$

Factorial Polynomial

The product of the form $x(x-h)(x-2h)\cdots(x-(r-1)h)$ starting with x and the successive factors decrease by a constant is called factorial polynomial of degree r and it is denoted by $[x]^r$ or $x^{(r)}$, where r is a positive integer

If $h = 1$, then $[x]^r = x(x-1)(x-2)\cdots(x-r+1)$

we shall use this factorial polynomial in our discussions.

Differences of $[x]^r$

$$\begin{aligned}\Delta[x]^r &= [x+1]^r - [x]^r \\ &= (x+1)x(x-1)(x-2)\ldots(x-r+2) - x(x-1)(x-2)\cdots(x-r+1) \\ &= x(x-1)(x-2)\ldots(x-r+2)[x+1-(x-r+1)] \\ &= x(x-1)(x-2)\ldots(x-r+2)^r = r[x]^{r-1}\end{aligned}$$

similarly $\quad \Delta^2[x]^r = r(r-1)[x]^{r-2}$

$$\vdots$$

$$\Delta^r[x]^r = r!$$

Note:

(1) Since $\quad \Delta[x]^r = r[x]^{r-1}$

it is analogous to differentiation $\quad \dfrac{d}{dx}(x^r) = rx^{r-1}$,

When $r = 0$, $[x]^0 = 1$

(2) $$\Delta^{-1}\left[\Delta[x]^r\right] = \Delta^{-1}\left[r[x]^{r-1}\right]$$

$\Rightarrow$ $$[x]^r = \Delta^{-1}\left[r[x]^{r-1}\right] = r\Delta^{-1}\left([x]^{r-1}\right)$$

$\Rightarrow$ $$\Delta^{-1}\left\{[x]^{r-1}\right\} = \frac{1}{r}[x]^r$$

So inverse operator Δ^{-1} behaves like integration for factorial polynomial.

Thus $$\Delta^{-1}\left([x]^r\right) = \frac{[x]^{r+1}}{r+1}$$

Because of this special property, it is convenient to represent a polynomial in terms of factorial polynomial.

Example 1

Represent $x^3 + 3x^2 + 5x + 12$ as a factorial polynomial.

Solution

Let $x^3 + 3x^2 + 12x + 12 = x(x-1)(x-2) + a_1 x(x-1) + a_2 x + a_3$
where a_1, a_2, a_3 are constants.

Put $x = 0$, then $a_3 = 12.$

Put $x = 1$, then $a_2 + a_3 = 1 + 3 + 12 + 12$

$\Rightarrow$ $a_2 + 12 = 16 + 1 \quad \Rightarrow \quad a_2 = 16$

Put $x = 2$, then

$$a_1\, 2\,(2-1) + a_2\, 2 + a_3 = 8 + 12 + 24 + 12$$

$$\Rightarrow \quad 2a_1 = 56 - 2a_2 - a_3$$

$$= 56 - 2 \times 16 - 12 \quad \Rightarrow \quad a_1 = 6$$

$$\therefore \quad x^3 + 3x^2 + 12x + 12 = [x]^3 + 6[x]^2 + 16[x] + 12$$

Aliter:

By Synthetic division, we can find the coefficients a_1, a_2, a_3.

0	1	3	12	12
	0	0	0	0
1	1	3	12	$\boxed{12}$
	0	1	4	
2	1	4	$\boxed{16}$	
	0	2		
	1	$\boxed{6}$		

$$\therefore \quad x^3 + 3x^2 + 12\,x + 12 = [x]^3 + 6[x]^2 + 16[x] + 12$$ ■

Example 2

Represent $x^4 - 12x^3 + 42x^2 - 30x + 9$ and its successive differences in factorial notation.

Solution

Let $f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$

We shall use synthetic division method to express $f(x)$ in factorial notation.

0	1	−12	42	−30	9
	0	0	0	0	0
1	1	−12	42	−30	$\boxed{9}$
	0	1	−11	31	
2	1	−11	31	$\boxed{1}$	
	0	2	−18		
3	1	−9	$\boxed{13}$		
	0	3			
	1	$\boxed{-6}$			

$$\therefore \quad f(x) = [x]^4 - 6[x]^3 + 13[x]^2 + [x] + 9$$

$$\Delta f(x) = \Delta[x]^4 - 6\,\Delta[x]^3 + 13\,\Delta[x]^2 + \Delta[x] + 9$$

$$= 4[x]^3 - 6.3[x]^2 + 13.2[x] + 1 + 0$$

$$= 4[x]^3 - 18[x]^2 + 26[x] + 1$$

$$\Delta^2 f(x) = \Delta[\Delta f(x)]$$
$$= \Delta[4[x]^3 - 18[x]^2 + 26[x] + 1]$$
$$= 4\Delta[x]^3 - 18\Delta[x]^2 + 26\Delta[x] + \Delta(1)$$
$$= 4.3[x]^2 - 18.2[x] + 26.1 = 12[x]^2 - 36[x] + 26$$
$$\Delta^3 f(x) = \Delta[\Delta^2 f(x)]$$
$$= \Delta\left[12[x]^2 - 36[x] + 26\right]$$
$$= 12\Delta[x]^2 - 36\Delta[x] + \Delta(26) = 24[x] - 36$$
$$\Delta^4 f(x) = \Delta[\Delta^3 f(x)]$$
$$= \Delta[24[x] - 36]$$
$$= 24\Delta[x] - \Delta[36] = 24.1 = 24$$

and higher differences are zero ■

INTERPOLATION WITH EQUALLY SPACED ARGUMENTS OR INTERPOLATION WITH EQUAL INTERVALS

Let the function $y = f(x)$ take values $y_0, y_1, y_2, \ldots, y_n$ for equidistant values of the arguments $x_0, x_1, x_2, \ldots, x_n$. Let the equal interval be h ie. $x_1 - x_0 = x_2 - x_1 = \ldots x_n - x_{n-1} = h$. Then $x_n = x_0 + nh$. We have to find the interpolating polynomial $\phi(x)$ which represents $f(x)$ in the interval $x_0 \le x \le x_0 + nh$

Since $(n + 1)$ values of the function are known, we can assume $\phi(x)$ to be a polynomial of n^{th} degree and it is determined uniquely. $\phi(x) = f(x)$ at $x = x_0, x_1, \ldots, x_n$ and approximately equal in intermediate points. Hence we write $f(x)$ itself in the place of $\phi(x)$.

Newton's Forward Formula for Interpolation

Newton's forward formula is

$$y = f(x) = f(x_0 + hu)$$
$$= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$
$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \ldots + \frac{u(u-1)\ldots(u-\overline{n-1})}{n!}\Delta^n y_0$$

where $u = \dfrac{x - x_0}{h}$

Proof: Let $y = f(x)$ be a function which takes values $y_0, y_1, y_2, \ldots, y_n$ at equally spaced arguments $x_0, x_1, x_2, \ldots, x_n$. Let h be the interval so that $x_r = x_0 + rh$. Let $\phi(x)$ be the interpolating polynomial of the n^{th} degree which may be written in the form

$$\phi(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \ldots$$
$$+ a_n(x-x_0)(x-x_1)\ldots(x-x_{n-1}) \qquad (1)$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants to be determined such that

$$\phi(x_0) = y_0,\ \phi(x_1) = y_1, \ldots, \phi(x_n) = y_n.$$

Put $x = x_0, x_1, x_2, \ldots, x_n$ successively in (1)

we get

$$\phi(x_0) = a_0 \Rightarrow a_0 = y_0$$

$$\phi(x_1) = a_0 + a_1(x_1 - x_0)$$

$$\Rightarrow \quad y_1 = y_0 + a_1 h \quad \Rightarrow a_1 = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$$

$$\phi(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$\Rightarrow \quad y_2 = y_0 + \frac{y_1 - y_0}{h}.2h + a_2.2h.h$$

$$\Rightarrow \quad 2h^2 a_2 = y_2 - 2y_1 + y_0 = \Delta^2 y_0$$

$$\Rightarrow \quad a_2 = \frac{\Delta^2 y_0}{2h^2} = \frac{\Delta^2 y_0}{2!h^2}$$

Similarly, $a_3 = \dfrac{\Delta^3 y_0}{3!h^3}$ and $a_4 = \dfrac{\Delta^4 y_0}{4!h^4}, \ldots, a_n = \dfrac{\Delta^n y_0}{n!h^n}$

$\therefore$ (1) becomes
$$\phi(x) = f(x) = y_0 + (x-x_0)\frac{\Delta y_0}{1!h} + (x-x_0)(x-x_1)\frac{\Delta^2 y_0}{2!h^2}$$
$$+(x-x_0)(x-x_1)(x-x_2)\frac{\Delta^3 y_0}{3!h^3} + \ldots$$
$$+(x-x_0)(x-x_1)\ldots(x-x_{n-1})\frac{\Delta^n y_0}{n!h^n} \qquad (2)$$

This is Newton's forward formula interms of x.

Now let $u = \dfrac{x - x_0}{h} \quad \Rightarrow x = x_0 + hu$

Now
$$x - x_1 = x_0 + hu - (x_0 + h) = h(u-1)$$
$$x - x_2 = x_0 + hu - (x_0 + 2h) = h(u-2)$$
$$\vdots$$
$$x - x_{n-1} = h\left(u - \overline{(n-1)}\right)$$

Hence $$f(x_0+hu) = y_0 + hu\frac{\Delta y_0}{1!h} + hu.h(u-1)\frac{\Delta^2 y_0}{2!h^2}$$
$$+hu.h(u-1)h(u-2).\frac{\Delta^3 y_0}{3!h^3} + \ldots + huh(u-1)\ldots h(u-\overline{n-1})\frac{\Delta^n y_0}{n!h^n}$$

$\Rightarrow$ $$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$
$$+\ldots+\frac{u(u-1)(u-2)\ldots(u-\overline{n-1})}{n!}\Delta^n y_0$$

Remark:

(1) Newton forward formula is also known as **Newton-Gregory formula for forward interpolation.**

(2) The formula involves y_0 and its leading differences. ie. it involves y_0 and the values of the function to the right of y_0. Hence it is called forward interpolation formula.

(3) The formula is used for interpolating the values of y near the beginning of a set of tabular values.

(4) It can also be used for extrapolating values of y at a short distance on the left side of x_0 (ie. $x < x_0$ and x is close to x_0).

Newton's Backward Formula for Interpolation

Newton's backward formula is

$$y = f(x) = f(x_n + hv)$$
$$= y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \ldots + \frac{v(v+1)(v+2)\ldots(v+n-1)}{n!}\nabla^n y_n$$

where $v = \dfrac{x-x_n}{h}$.

Proof: Let $y = f(x)$ be a function which takes values $y_0, y_1, y_2, \ldots, y_n$ for equally spaced arguments $x_0, x_1, x_2, \ldots, x_n$

where $x_1 - x_0 = x_2 - x_1 = \ldots x_n - x_{n-1} = h$.

$\therefore$ $$x_n = x_0 + nh$$

Let $\phi(x)$ be the interpolating polynomial of n^{th} degree which may be written as

$$\phi(x) = a_0 + a_1(x-x_n) + a_2(x-x_n)(x-x_{n-1})$$
$$+a_3(x-x_n)(x-x_{n-1})(x-x_{n-2}) + \ldots + a_n(x-x_n)(x-x_{n-1})\ldots(x-x_1) \quad (1)$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants to be determined in such a way that

$$\phi(x_n) = y_n, \quad \phi(x_{n-1}) = y_{n-1}, \ldots, \phi(x_0) = y_0$$

Substituting $x_n, x_{n-1}, \ldots, x_1, x_0$ in succession in (1) we get

$$\phi(x_n) = a_0 \qquad \Rightarrow a_0 = y_n$$

$$\phi(x_{n-1}) = a_0 + a_1(x_{n-1} - x_n)$$

$$\Rightarrow \quad y_{n-1} = y_n + a_1(-h) = y_n - a_1 h$$

$$\Rightarrow \quad a_1 = \frac{y_n - y_{n-1}}{h} = \frac{\nabla y_n}{h}$$

$$\phi(x_{n-2}) = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

$$\Rightarrow \quad y_{n-2} = y_n + \frac{y_n + y_{n-1}}{h}(-2h) + a_2(-2h)(-h)$$

$$= y_n - 2(y_n - y_{n-1}) + 2h^2 a_2$$

$$\Rightarrow \quad 2h^2 a_2 = y_n - 2y_{n-1} + y_{n-2} = \nabla^2 y_n$$

$$\therefore \quad a_2 = \frac{\nabla^2 y_n}{2!h^2}$$

Similarly, $$a_3 = \frac{\nabla^3 y_n}{3!h^3}, a_4 = \frac{\nabla^4 y_n}{4!h^4}, \ldots, \qquad a_n = \frac{\nabla^n y_n}{n!h^n}$$

Substituting in (1), we get

$$\phi(x) = f(x) = y_n + (x - x_n)\frac{\nabla y_n}{1!h} + (x - x_n)(x - x_{n-1})\frac{\nabla^2 y_n}{2!h^2}$$
$$+ (x - x_n)(x - x_{n-1})(x - x_{n-2})\frac{\nabla^3 y_n}{3!h^3} + \ldots + (x - x_n)(x - x_{n-1})\ldots(x - x_1)\frac{\nabla^n y_n}{n!h^n}$$

Put $$\frac{x - x_n}{h} = v \Rightarrow x = x_n + hv$$

$$\therefore \quad x - x_{n-1} = x_n + hv - x_{n-1} = hv + x_n - x_{n-1} = hv + h = h(v+1)$$
$$x - x_{n-2} = x_n + hv - x_{n-2} = hv + x_n - x_{n-2} = hv + 2h = h(v+2)$$
$$\vdots$$
$$x - x_1 = h(v + \overline{n-1})$$

$$\therefore \quad y = f(x_n + hv) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n$$
$$+ \frac{v(v+1)(v+2)(v+3)}{4}\nabla^4 y_n + \ldots + \frac{v(v+1)(v+2)+\ldots+(v+\overline{n-1})}{n!}\nabla^n y_n$$

Remark:

(1) This formula is also known as Newton-Gregory's backward formula.

(2) Since the formula involves y_n and the values of the function to the left of it, it is called backward formula.

(3) It is used for interpolating values of y near the end of a set of tabular values.

(4) It can also be used for extrapolating values of y at a short distance on the right of x_n (ie. $x > x_n$ and x is close to x_n).

Note: If the tabulated function is a polynomial then for any value of x, both the forward and backward formula of Newton will give the exact value of the function whether it is interpolation or extrapolation.

WORKED EXAMPLES

Example 1

Using Newton's forward interpolation formula find the cubic polynomial which takes the following values.

x	0	1	2	3
$f(x)$	1	2	1	10

Evaluate $f(4)$.

Solution

We use Newton's forward formula to find the polynomial in x.
Newton's forward formula is

$$f(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

where $u = \dfrac{x - x_0}{h}$.

Here $x_0 = 0$, $h = 1$ $\therefore$ $u = \dfrac{x-0}{1} = x$

Now we form the forward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_0 = 0$	$y_0 = 1$			
		$2 - 1 = 1$		
$x_1 = 1$	$y_1 = 2$		$-1 - 1 = -2$	
		$1 - 2 = -1$		$10 - (-2) = 12$
$x_2 = 2$	$y_2 = 1$		$9 - (-1) = 10$	
		$10 - 1 = 9$		
$x_3 = 3$	$y_3 = 10$			

$y_0 = 1,\quad \Delta y_0 = 1,\quad \Delta^2 y_0 = -2,\quad \Delta^3 y_0 = 12$

$$f(x) = 1 + \frac{u}{1!}1 + \frac{u(u-1)}{2!}(-2) + \frac{u(u-1)(u-2)}{3!}\times 12$$

$$= 1 + u - u(u-1) + 2u(u^2 - 3u + 2)$$

$$= 1 + u - u^2 + u + 2u^3 - 6u^2 + 4u$$

$$= 2u^3 - 7u^2 + 6u + 1$$

$\therefore$ $$f(x) = 2x^3 - 7x^2 + 6x + 1 \qquad [\because u = x]$$

When $x = 4$, $f(4) = 2\times 4^3 - 7\times 4^2 + 6\times 4 + 1 = 128 - 112 + 24 + 1 = \mathbf{41}$ ∎

Example 2

A third degree polynomial passes through the points (0, −1), (1, 1), (2, 1) and (3, −2). Using Newton's forward formula, find the polynomial. Hence find the value at 1.5.

Solution

We use Newton's forward formula to find the polynomial passing through (0, −1) (1, 1), (2, 1) and (3, −2).

Newton's forward formula is

$$f(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{2!}\Delta^3 y_0 + \dots$$

where $\quad u = \dfrac{x - x_0}{h}.$

Here $\quad x_0 = 0, \quad h = 1 \quad \therefore \quad u = \dfrac{x-0}{1} = x \qquad (1)$

Now we form the forward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_0 = 0$	$\mathbf{y_0 = -1}$			
		$\mathbf{1-(-1) = 2}$		
$x_1 = 1$	$y_1 = 1$		$\mathbf{0-2=-2}$	
		$1-1=0$		$\mathbf{-3-(-2)=-1}$
$x_1 = 2$	$y_2 = 1$		$-3-0=-3$	
		$-2-1=-3$		
$x_3 = 3$	$y_3 = -2$			

$y_0 = -1, \ \Delta y_0 = 2, \ \Delta^2 y_0 = -2, \ \Delta^3 y_0 = -1$

Substituting in (1), we get

$$\therefore \quad f(x) = -1 + \frac{u}{1!}2 + \frac{u(u-1)}{2!}(-2) + \frac{u(u-1)(u-2)}{3!}(-1)$$

$$= -1 + 2u - u^2 + u - \frac{1}{6}\left(u^3 - 3u^2 + 2u\right)$$

$$= \frac{1}{6}\left[-6 + 12u - 6u^2 + 6u - u^3 + 3u^2 - 2u\right]$$

$$= \frac{1}{6}\left[-u^3 - 3u^2 + 16u - 6\right]$$

$$= -\frac{1}{6}\left[u^3 + 3u^2 - 16u + 6\right]$$

Since $u = x$, the polynomial is $f(x) = -\frac{1}{6}\left[x^3 + 3x^2 - 16x + 6\right]$

When $x = 1.5$, $y = f(1.5) = -\frac{1}{6}\left[(1.5)^3 + 3(1.5)^2 - 16(1.5) + 6\right]$

$$= -\frac{1}{6}[3.375 + 6.75 - 24 + 6]$$

$$= -\frac{1}{6}(-7.875) = \mathbf{1.3125}$$

Example 3

The population of a city in Census taken once in 10 years is given below. Estimate the population in the year 1955.

Year	**1951**	**1961**	**1971**	**1981**
Population in thousands	**35**	**42**	**58**	**84**

Solution

Let us denote year as x, and population as y.

Let $y = f(x)$

$x = 1955$ is near the beginning of the table. So we use Newton's forward difference formula to find y.

Newton's forward formula is

$$y = f(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots \qquad (1)$$

where $u = \frac{x - x_0}{h}$. Here $x_0 = 1951$, $h = 10$ $\therefore$ $u = \frac{x - 1951}{10}$

When $x = 1955$, $u = \frac{1955 - 1951}{10} = \frac{4}{10} = 0.4$

Now we form the forward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1951	**35**			
		7		
1961	42		**9**	
		16		**1**
1971	58		10	
		26		
1981	84			

$\therefore$ $y_0 = 35$, $\Delta y_0 = 7$, $\Delta^2 y_0 = 9$, $\Delta^3 y_0 = 1$

and when $x = 1955$, $u = 0.4$

Substituting in (1), we get

$$y = f(1955) = 35 + 0.4\times 7 + \frac{(0.4)(0.4-1)}{2!}\times 9 + \frac{(0.4)(0.4-1)(0.4-2)}{3!}\times 1$$
$$= 35 + 2.8 - 1.08 + 0.064 = \mathbf{36.784}$$

■

Example 4

From the data given below find the number of students whose weight is between 60 to 70.

Weight in lbs:	**0–40**	**40–60**	**60–80**	**80–100**	**100–120**
No. of students	**250**	**120**	**100**	**70**	**50**

Solution

Let weight be denoted by x and number of students be denoted by y
Let $y = f(x)$
We use Newton's forward formula to find y when x lies between 60–70.
We rewrite the table as cumulative table showing the number of students less than x lbs.
Newton's forward formula is

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \ldots \quad (1)$$

where $u = \frac{x-x_0}{h}$.

Here $x_0 = 40$, $\quad h = 20 \quad \therefore \quad u = \frac{x-40}{20}$

We shall find y when $x = 70$

$\therefore$ when $x = 70$, $\quad u = \frac{70-40}{20} = \frac{30}{20} = 1.5$

Now we form the forward table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
below 40	**250**				
		120			
below 60	370		**–20**		
		100		**–10**	
below 80	470		–30		**20**
		70		10	
below 100	540		–20		
		50			
below 120	590				

Hence $y_0 = 250$, $\Delta y_0 = 120$, $\Delta^2 y_0 = -20$, $\Delta^3 y_0 = -10$

$\Delta^4 y_0 = 20$ and when $x = 70$, $u = 1.5$

Substituting in (1), we get $y = 250 + 1.5 \times 120 + \frac{(1.5)(1.5-1)}{2!}(-20)$

$$+\frac{(1.5)(1.5-1)(1.5-2)}{3!}(-10)$$

$$+\frac{(1.5)(1.5-1)(1.5-2)(1.5-3)}{4!} \times 20$$

$$= 250 + 180 - 7.5 + 0.625 + 0.46875 \approx 423.59 = 424$$

No. of students whose weight is below 70 is 424

$\therefore$ no. of students whose weight is between 60 – 70 is = 424 – 370 = **54**

Example 5

If

x	0	1	2	3	4	5
y	27	32	25	36	32	41

then find approximately the value of y when $x = -0.5$.

Solution

Let $y = f(x)$. To find y when $x = -0.5$.
$x = -0.5$ is outside the table value and is near the beginning of the table.
$\therefore$ It is extrapolation. So, we use Newton's forward formula to find y when $x = -0.5$
Newton's forward formula is

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$
$$+\frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \dots \quad (1)$$

where $u = \frac{x - x_0}{h}$. Here $x_0 = 0$, $h = 1$ $\therefore$ $u = \frac{x-0}{1} = x$

When $x = -0.5$, $u = -0.5$

Now we form the difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	**27**					
		5				
1	32		**−12**			
		−7		**30**		
2	25		18		**−63**	
		11		−33		**124**
3	36		−15		61	
		−4		28		
4	32		13			
		9				
5	41					

Hence $y_0 = 27,\quad \Delta y_0 = 5,\quad \Delta^2 y_0 = -12,\quad \Delta^3 y_0 = 30,\quad \Delta^4 y_0 = -63,\quad \Delta^5 y_0 = 124$

and when $x = -0.5,\quad u = -0.5$

Substituting in (1), we get

$$f(-0.5) = 27 + (-0.5)\times 5 \frac{(-0.5)(-0.5-1)}{2!}(-12)$$
$$+\frac{(-0.5)(-0.5-1)(-0.5-2)}{3!}\times 30$$
$$+\frac{(-0.5)(-0.5-1)(-0.5-2)((-0.5)-3)}{4!}\times(-63)$$
$$+\frac{(-0.5)(-0.5-1)(-0.5-2)(-0.5-3)(-0.5-4)}{5!}\times(124)$$
$$= 27 - 2.5 - 4.5 - 9.375 - 17.2265 - 30.5156 = -37.1171 = \mathbf{-37.12}$$ ■

Example 6

The following table gives melting point of an alloy of zinc and lead, θ is the temperature and x is the percentage of lead. Using Newton's interpolation formula find θ when $x = 84$.

x	40	50	60	70	80	90
θ	184	204	226	250	276	304

Solution

Let $y = f(x)$, where $y = \theta$,

$x = 84$ is near the end of the table.

$\therefore$ we use Newton's backward formula to find θ when $x = 84$.

Newton's backward formula is

$$y = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n$$
$$+\frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n + \ldots + \frac{v(v+1)(v+2)+\ldots+(v+n-1)}{n!}\nabla^n y_n \qquad (1)$$

where $v = \frac{x-x_n}{h}$. Here $x_n = 90$, $h = 10$, $\therefore \; v = \frac{x-90}{10}$

When $x = 84$, $v = \frac{84-90}{10} = \frac{-6}{10} = -0.6$

Now we form the table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
40	184			
		20	2	
50	204			
		22		0
60	226		2	
		24		0
70	250		2	
		26		**0**
80	276		**2**	
		28		
90	**304**			

Hence $y_n = 304$, $\nabla y_n = 28$ $\nabla^2 y_n = 2$, $\nabla^3 y_n = 0$

and when $x = 84$, $v = -0.6$

Substituting in (1) we get

$$y = 304 + (-0.6)\times 28 + \frac{(-0.6)(-0.6+1)}{2!}\times 2$$
$$= 304 - 16.8 - 0.24$$
$$= 286.96 = 287 \text{ approximately}$$

$\therefore \qquad \theta = \mathbf{287}$ ■

Example 7

If l_x represents the number of people living at age x in a life table, find l_{47}, given $l_{20} = 512$, $l_{30} = 439$, $l_{40} = 346$ and $l_{50} = 243$.

Solution

Let $y = l_x$.
$x = 47$ is near the end of the table values. So, we use Newton's backward formula to find l_{47}.
Newton's backward formula is

$$y = l_x = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n + \ldots + \frac{v(v+1)(v+2)+\ldots+(v+n-1)}{n!}\nabla^n y_n \qquad (1)$$

where $v = \frac{x - x_n}{h}$. Here $x_n = 50, \quad h = 10 \quad \therefore \; v = \frac{x-50}{10}$

When $x = 47, \quad v = \frac{47-50}{10} = \frac{-3}{10} = -0.3$

Now we form the difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	512			
		–73		
30	439		–20	
		–93		**10**
40	346		**–10**	
		–103		
50	**243**			

$\therefore \qquad y_n = 243, \quad \nabla y_n = -103, \quad \nabla^2 y_n = -10, \quad \nabla^3 y_n = 10$

and when $x = 47$, $v = -0.3$
Substituting in (1), we get

$$y = l_{47} = 243 + (-0.3)(-103) + \frac{(-0.3)(-0.3+1)}{2!}(-10) + \frac{(-0.3)(-0.3+1)(-0.3+2)}{3!} \times 10$$
$$= 243 + 30.9 + 1.05 - 0.595$$
$$= 274.355$$

Hence the number of people expected to live at the age of 47 is **274**. ■

Example 8

A function y is given by the following table. Estimate the value of y when $x = 5$.

x	0	1	2	3	4
y	79	91	105	116	127

Solution

Let $y = f(x)$.

We want to find y when $x = 5$, which is outside the end of the table and hence extrapolation. So, we use Newton's backward formula to find y when $x = 5$.

Newton's backward formula is

$$y = f(x) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n$$
$$+\frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n + \ldots + \frac{v(v+1)(v+2)+\ldots+(v+n-1)}{n!}\nabla^n y_n \quad (1)$$

where $v = \frac{x - x_n}{h}$. Here $x_n = 4$, $h = 1$ $\quad\therefore\quad v = \frac{x-4}{1} = x - 4$

When $x = 5$, $\quad v = 5 - 4 = 1$

Now we form the difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	79				
		12			
1	91		2		
		14		-5	
2	105		-3		**8**
		11		**3**	
3	116		**0**		
		11			
4	**127**				

Here

$$y_n = 127,\ \nabla y_n = 11,\ \nabla^2 y_n = 0,\ \nabla^3 y_n = 3$$

$$\nabla^4 y_n = 8 \quad \text{and} \quad \text{when } x = 5, \quad v = 1$$

Substituting in (1), we get

$$y = f(1) = 127 + 1.11 + \frac{1.(1+1)}{2!}\times 0 + \frac{1(1+1)(1+2)}{3!}\times 3 + \frac{1.2.3.4}{4!}\times 8$$
$$= 127 + 11 + 3 + 8 = \mathbf{149}$$

■

Example 9

The following are data from the steam table.

Temperature °C	**140**	**150**	**160**	**170**	**180**
Pressure kg/cm²	**3.685**	**4.854**	**6.302**	**8.076**	**10.225**

Find the pressure at temperature 142°C and 175°C.

Solution

Let temperature be x°C and pressure be y kg/cm²
Let $y = f(x)$
Since x = 142°C is near the beginning of the table, we use Newton's forward formula to find y.
Newton's forward formula is

$$y = y_0 = \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \ldots \quad (1)$$

where $u = \dfrac{x - x_0}{h}$. Here $x = 140$, $h = 10$ $\therefore$ $u = \dfrac{x-140}{10}$

When $x = 142°\text{C}$, $u = \dfrac{142-140}{10} = \dfrac{2}{10} = 0.2$

We now form the forward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 140$	$\mathbf{y_0 = 3.685}$				
		1.169			
$x_1 = 150$	$y_1 = 4.854$		**0.279**		
		1.448		**0.047**	
$x_2 = 160$	$y_2 = 6.302$		0.326		**0.002**
		1.774		**0.049**	
$x_3 = 170$	$y_3 = 8.076$		**0.375**		
		2.149			
$x_4 = 180$	$\mathbf{y_4 = 10.225}$				

Here $y_0 = 3.685$, $\Delta y_0 = 1.169$, $\Delta^2 y_0 = 0.279$, $\Delta^3 y_0 = 0.047$, $\Delta^4 y_0 = 0.002$

When temperature is x = 142°C, then pressure

$$y = 3.685 + \frac{0.2}{1!}\times 1.169 + \frac{0.2(0.2-1)}{2!}\times 0.279 + \frac{0.2(0.2-1)(0.2-2)}{3!}\times 0.047 + \frac{0.2(0.2-1)(0.2-2)(0.2-3)}{4!}\times 0.002$$

$$= 3.685 + 0.2\times 1.169 - \frac{0.2\times 0.8}{2}\times 0.279 + \frac{0.2\times 0.8\times 1.8}{6}\times 0.047 - \frac{0.2\times 0.8\times 1.8\times 2.8\times 0.047}{24}$$

$$= 3.685 + 0.2338 - 0.0223 + 0.002256 - 0.0000672$$

= **3.899 to 3** decimal places.

(ii) Since $x = 175°C$ is near the end of the table, we use Newton's backward formula.

Newton's backward formula is

$$y = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n$$
$$+\frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n + \ldots + \frac{v(v+1)(v+2)+\ldots+(v+n-1)}{n!}\nabla^n y_n \quad (1)$$

where $\quad v = \frac{x - x_n}{h} = \frac{x-180}{10}$

When $x = 175$, $\quad v = \frac{175-180}{10} = -\frac{5}{10} = -0.5$

From the difference table, we have

$$y_n = 10.225, \quad \nabla y_n = 2.149, \quad \nabla^2 y_n = 0.375, \quad \nabla^3 y_n = 0.049, \quad \nabla^4 y_n = 0.002$$

$\therefore$ $$\text{Prssure } y = 10.225 + \frac{(-0.5)}{1!}2.149 + \frac{(-0.5)(-0.5+1)}{2!}\times 0.375$$
$$+\frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}\times 0.049$$
$$+\frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{4!}\times 0.002$$
$$= 10.225 - 0.5\times 2.149 - \frac{0.5\times 0.5\times 0.375}{2}$$
$$-\frac{0.5\times 0.5\times 1.5\times 0.049}{6} - \frac{0.5\times 0.5\times 1.5\times 2.5\times .002}{24}$$
$$= 10.225 - 1.0745 - 0.046875 - 0.00030625 - 0.000078125$$
$$= \mathbf{9.10048}$$

■

Example 10

Estimate the value of $f(22)$ and $f(42)$ from the following data:

x	20	25	30	35	40	45
$f(x)$	354	332	291	260	231	204

Solution

We find $f(22)$ using Newton's forward formula.
Newton's forward formula is

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$
$$+\frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \ldots \quad (1)$$

where $u = \frac{x - x_0}{h}$. Here $x_0 = 20$, $h = 5$ $\therefore$ $u = \frac{x-20}{5}$

When $x = 22$, $u = \frac{22-20}{5} = \frac{2}{5} = 0.4$

From forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
20	**354**					
		−22				
25	332		**−19**			
		−41		**29**		
30	291		+10		**−37**	
		−31		−8		**+45**
35	260		+2		**+8**	
		−29		**0**		
40	231		**+2**			
		−27				
45	**204**					

$$y_0 = 354, \quad \Delta y_0 = -22, \quad \Delta^2 y_0 = -19, \quad \Delta^3 y_0 = 29, \quad \Delta^4 y_0 = -37, \quad \Delta^5 y_0 = 45$$

$$\begin{aligned} f(22) &= 354 + 0.4(-22) + \frac{(0.4)(0.4-1)}{2!}(-19) \\ &\quad + \frac{(0.4)(0.4-1)(0.4-2)}{3!} \times 29 + \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{4!} \times (-37) \\ &\quad + \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)(0.4-4)}{5!} \times (45) \\ &= 354 - 8.8 + 2.28 + 1.856 + 1.5392 + 1.3478 = \mathbf{352.22} \end{aligned}$$

To find y when $x=42$

Given $y = f(x)$

$x = 42$ is near the end of the table. So, we use Newton's backward formula to find $f(42)$.

Newton's backward formula is

$$\begin{aligned} y &= y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n \\ &\quad + \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n + \ldots + \frac{v(v+1)(v+2)+\ldots+(v+n-1)}{n!}\nabla^n y_n \end{aligned} \quad (1)$$

where $v = \frac{x - x_n}{h}$. Here $x_n = 45$, $h = 5$ $\therefore$ $v = \frac{x-45}{5}$

When $x = 42$, $\qquad v = \dfrac{42-45}{5} = \dfrac{-3}{5} = -0.6.$

$y_n = 204, \qquad \nabla y_n = -27, \qquad \nabla^2 y_n = 2, \qquad \nabla^3 y_n = 0, \qquad \nabla^4 y_n = 8, \qquad \nabla^5 y_n = 45$

and when $x = 42$, $\qquad v = -0.6$

Substituting in (1) we get

$$f(42) = 204 + (-0.6)(-27) + \frac{(-0.6)(-0.6+1)}{2!} \times 2$$
$$+ \frac{1}{3!}(-0.6)(-0.6+1)(-0.6+2) \times 0 + \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{4!} \times 8$$
$$+ \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)(-0.6+4)}{5!} \times 45$$
$$= 204 + 16.2 - 0.24 - 0.2688 - 1.028 = \mathbf{218.66}$$

■

$f(42) = 218.66 = 219$ approximately

Exercises 4.2

(1) Use Newton's backward formula to find a polynomial of degree 3 for the data $f(-0.75) = -0.0781250$, $f(-0.5) = -0.024750$, $f(-0.25) = 0.33493750$, $f(0) = 1.10100$. Hence find $f\left(-\dfrac{1}{3}\right)$.

(2) From the following table, find the value of tan 45° 15'

x^0	45	46	47	48	49	50
tan x^0	1.0	1.03553	1.07237	1.11061	1.15037	1.19175

(3) Find $f(2.5)$ using Newton's forward difference formula for the given data.

x	1	2	3	4	5
$y = f(x)$	0	1	8	27	64

(4) Using Newton's backward difference formula find the area of a circle of diameter 98 from the given table of diameter and area of a circle.

Diameter	80	85	90	95	100
Area	5026	5674	6362	7088	7854

(5) Find u_9, given that $u_0 = 1$, $u_1 = 11$, $u_2 = 21$, $u_3 = 28$, $u_4 = 29$.

(6) A function $f(x)$ is given by the following table. Estimate $f(0.2)$ by any appropriate formula.

x	0	1	2	3	4	5	6
$f(x)$	176	185	194	203	212	220	229

(7) Find the value of $f(x)$ when $x = 32$, given the following table:

x	30	35	40	45	50
$f(x)$	15.9	14.9	14.1	13.3	12.5

(8) Find the number of students from the following data who secured marks not more than 45.

Marks	30–40	40–50	50–60	60–70	70–80
No of students	35	48	70	40	22

(9) The following are the annual premiums charge by the L.I.C of India for a policy of Rs. 1000. Calculate the premium payable at the age of 22.

Age in year	20	25	30	35	40
Premium Rs.	23	26	30	35	42

(10) Calculate the value of sin 33° 13′ 30″ from the following table of sines.

$x°$	30	31	32	33	34
$\sin x°$	0.5000	0.5150	0.5290	0.5446	0.5592

(11) The table gives the distance in nautical miles of the visible horizon for the given height in feet above the earths surface. Find the values of y when $x = 218'$ and $x = 410'$.

x	100	150	200	250	300	350	400
y	10.63	13.03	15.04	16.81	18.42	19.90	21.27

(12) Find u_6 given that

u_0	u_1	u_2	u_3	u_4	u_5
25	25	22	18	15	5

(13) By appropriate formula estimate the population for the year 2006, given

Year:	2001	2002	2003	2004	2005
Population in thousands:	251	279	319	383	483

(14) A function y is given by the following table. Estimate the value of y when $x = 5$.

x	0	1	2	3	4
y	79	91	105	116	127

(15) Two variables x and y have the following related values

x	0	10	20	30	40	50	60
y	2501	2795	2838	3030	3050	3381	3888

Find y when $x = 54$.

(16) Given

x	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
$y = f(x)$	0.34375	0.87616	1.47697	2.17408	3.00139	4	5.21941	6.71872

Find $f(1.18)$.

Answers 4.2

(1) $y = x^3 + 4.001x^2 + 4.002x + 1.101,\ f\left(\frac{-1}{3}\right) = 0.1745$

(2) 1.00876 (3) 3.4 (4) 7543 (5) −161
(6) 177.67 (7) 15.45 (8) 51 (9) 24.04
(10) 0.54788 (11) 15.697, 21.53 (12) 20 (13) 631
(14) 149 (15) 3674.97 (16) 6.39328

CENTRAL DIFFERENCE INTERPOLATION FORMULAE

We have seen that Newton's formward formula is suitable to interpolate near the beginning of a table of values and Newton's backward formula is suitable to interpolate near the end of the table with equally spaced arguments.

For interpolating near the middle of the table the central difference formulae are better suited than others. The central difference formulae employ the difference lying as nearly as possible on a horizontal line through y_0 near the centre.

We will discuss the following central difference formulae.

(1) Gauss's forward formula
(2) Gauss's backward formula
(3) Stirling's formula
(4) Bessel's formula
(5) Everett's formula.

Of these Bessel's and Stirlings are the most important ones.

Central difference table

Let $y = f(x)$ be the function. Let $y_0, y_1, y_2, y_3 \dots, y_n$ be the values of y corresponding to $x_0, x_1 = x_0 + h\ x_2 = x_0 + 2h, x_3 = x_0 + 3h, \dots, x_n = x_0 + nh$ and $y_{-1}, y_{-2}, y_{-3} \dots$ be the values corresponding to $x_0 - h, x_0 - 2h, x_0 - 3h, \dots$

Then, the difference table with these values on either side of x_0 is given by the table below.

Put $u = \dfrac{x - x_0}{h}$, where x_0 is the origin.

The values of u corresponding to

$x_0 - 3h,\quad x_0 - 2h,\quad x_0 - h,\quad x_0,\quad x_0 + h,\quad x_0 + 2h,\quad x_0 + 3h, \dots$

are respectively $u = \dfrac{x_0 - 3h - x_0}{h} = -3,\qquad u = \dfrac{x_0 - 2h - x_0}{h} = -2$

$$u = \frac{x_0 - h - x_0}{h} = -1, \qquad u = \frac{x_0 - x_0}{h} = 0, \qquad u = \frac{x_0 + h - x_0}{h} = 1$$

$$u = \frac{x_0 + h - x_0}{h} = 1, \qquad u = \frac{x_0 + 2h - x_0}{h} = 2, \qquad u = \frac{x_0 + 3h - x_0}{h} = 3 \dots$$

x	$u = \dfrac{x - x_0}{h}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_0-3h	-3	y_{-3}						
			Δy_{-3}					
x_0-2h	-2	y_{-2}		$\Delta^2 y_{-3}$				
			Δy_{-2}		$\Delta^3 y_{-3}$			
x_0-h	-1	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
			Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	0	$\boxed{y_0}$		$\boxed{\Delta^2 y_{-1}}$		$\boxed{\Delta^4 y_{-2}}$		$\boxed{\Delta^6 y_{-2}}$
			$\boxed{\Delta y_0}$		$\boxed{\Delta^3 y_{-1}}$		$\boxed{\Delta^5 y_{-2}}$	
x_0-h	1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
			Δy_1		$\Delta^3 y_0$			
x_0-2h	2	y_2		$\Delta^2 y_1$				
			Δy_2					
x_0-3h	3	y_3	$\vdots$					

Gauss's Forward Formula for Interpolation

Gauss's forward Interpolation formula is

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \ldots$$

Proof: We derive Gauss's forward difference interpolation formula from Newton's forward formula and using even differences on the horizontal line through y_0 and odd differences on the horizontal line between y_0 and y_1 from the central difference table.

Newton's forward formula is

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \tag{1}$$

where $u = \dfrac{x - x_0}{h}$

From the Central difference table, we have

$$\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$$

$\Rightarrow$ $$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$

$$\Delta^3 y_0 - \Delta^3 y_{-1} = \Delta^4 y_{-1}$$

$\Rightarrow$ $$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$

Similarly, $$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1}$$

Also $$\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$$

$\Rightarrow$ $$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$

Similarly, $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ and so on.

Substituting for $\Delta^2 y_0, \Delta^3 y_0, \ldots,$ in (1) we get

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\left[\Delta^2 y_{-1} + \Delta^3 y_{-1}\right]$$

$$+ \frac{u(u-1)(u-2)}{3!}\left[\Delta^3 y_{-1} + \Delta^4 y_{-1}\right] + \frac{u(u-1)(u-2)(u-3)}{4!}\left[\Delta^4 y_{-1} + \Delta^5 y_{-1}\right] + \ldots$$

$$= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \left[\frac{u(u-1)}{2!} + \frac{u(u-1)(u-2)}{3!}\right]\Delta^3 y_{-1}$$

$$+ \left[\frac{u(u-1)(u-2)}{3!} + \frac{u(u-1)(u-2)(u-3)}{4!}\right]\Delta^4 y_{-1} + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^5 y_{-1} + \ldots$$

$$\Rightarrow y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)}{2!}\left[1+\frac{u-2}{3}\right]\Delta^3 y_{-1}$$

$$+\frac{u(u-1)(u-2)}{3!}\left[1+\frac{u-3}{4}\right]\left(\Delta^4 y_{-2}+\Delta^5 y_{-2}\right)+\frac{u(u-1)(u-2)(u-3)}{4!}\Delta^5 y_{-1}+\dots$$

$$= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u-2)(u+1)}{4!}\Delta^4 y_{-2}+\dots$$

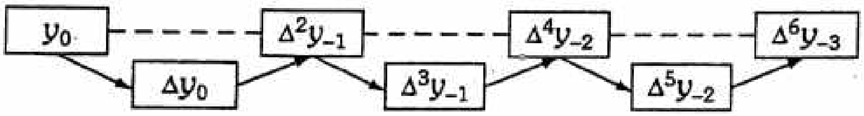
$$= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2}+\dots$$

If the symbol $\binom{u}{r} = \frac{u(u-1)(u-2)\dots(u-r+1)}{r!}$, then the formula can be written as

$$y_u = y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1} + \binom{u+1}{4}\Delta^4 y_{-2} + \binom{u+2}{5}\Delta^5 y_{-2} + \dots$$

Note:

(1) Gauss's forward formula involves even differences on the central line and odd differences below the central line as



(2) Gauss's forward formula is mainly used to interpolate if

$$0 \le u \le \frac{1}{2}.$$

(3) Since it is measured from the origin forwardly, this formula is called forward formula.

WORKED EXAMPLES

Example 1

Use Gauss's forward formula to find the value of y when $x = 2.7$ from the following table.

x	1.5	2	2.5	3	3.5	4
y	37.9	246.2	409.3	537.2	636.3	715.9

Solution

We use Gauss's forward formula to find y when $x = 2.7$, since it is near the middle of the given table.

Gauss's forward formula is

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2}+\dots$$

where $u = \frac{x-x_0}{h}$ and x_0 is the origin. Here $h = 0.5$

We take $x_0 = 2.5$, since $x = 2.7$ lies is between $x = 2.5$ and $x = 3$

$$\therefore \qquad u = \frac{x - 2.5}{0.5}$$

When $x = 2.7$,
$$u = \frac{2.7 - 2.5}{0.5} = 0.4$$

We shall form the forward difference table.

x	$u = \frac{x-2.5}{0.5}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.5	−2	37.9					
			208.3				
2	−1	246.2		−45.2			
			163.1		10		
2.5	0	**409.3**		**−35.2**		**−3.6**	
			127.9		**6.4**		**6.5**
3	1	537.2		−28.8		2.9	
			99.1		9.3		
3.5	2	636.3		−19.5			
			79.6				
4	3	715.9					

$$y_0 = 409.3, \quad \Delta y_0 = 127.9, \quad \Delta^2 y_{-1} = -35.2$$
$$\Delta^3 y_{-1} = 6.4, \quad \Delta^4 y_{-2} = -3.6, \quad \Delta^5 y_{-2} = 6.5$$

Substituting in (1), we get

$$y_u = 409.3 + u \times 127.9 + \frac{u(u-1)}{2!}(-35.2) + \frac{(u+1)u(u-1)}{3!}(6.4)$$
$$+ \frac{(u+1)u(u-1)(u-2)}{4!}(-3.6) + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}(6.5)$$

When $u = 0.4$

$$y_u = 409.3 + (0.4)(127.9) + \frac{(0.4)(0.4-1)}{2!}(-35.2) + \frac{(0.4+1)(0.4)(0.4-1)}{3!}(6.4) + \frac{(0.4+1)(0.4)(0.4-1)(0.4-2)}{4!}(-3.6) + \frac{(0.4+2)(0.4+1)(0.4)(0.4-1)(0.4-2)}{5!}\times(6.5)$$

$$= 409.3 + 51.16 + 4.224 - 0.3584 - 0.08064 + 0.069888 = 464.309$$

When $x = 2.7$, $y = \mathbf{464.31}$ ∎

Example 2

Using Gauss's forward formula find y_{30}, given that

$$y_{21} = 18.4708, \quad y_{25} = 17.8144,$$
$$y_{29} = 17.1070, \quad y_{33} = 16.3432, \quad y_{37} = 15.5154.$$

Solution

The given values of x are 21, 25, 29, 33, 37 and the corresponding y values are

$$y_{21} = 18.4708, \quad y_{25} = 17.8144,$$
$$y_{29} = 17.1070, \quad y_{33} = 16.3432, \quad y_{37} = 15.5154$$

$x = 30$ is near the middle of the table.

So, we use Gauss's forward formula to find y when $x = 30$ and formula is

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-2} + \dots \quad (1)$$

where $u = \dfrac{x - x_0}{h}$ and x_0 is the origin.

Here $h = 4$ and take $x_0 = 29$, since $x = 30$ lies between $x = 29$ and $x = 33$.

$$\therefore \quad u = \frac{x-29}{4}$$

When $x = 30$, $\quad u = \dfrac{30-29}{4} = \dfrac{1}{4} = 0.25$

Now we shall form the central difference table.

x	$u = \frac{x-29}{4}$	y_u	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
21	–2	18.4708				
			–0.6564			
25	–1	17.8144		–0.0510		
			–0.7074		–0.0054	
29	0	$\boxed{17.1017}$		$\boxed{-0.0564}$		$\boxed{-0.0022}$
			$\boxed{-0.7638}$		$\boxed{-0.0076}$	
33	1	16.3432		–0.0640		
			–0.8278			
37	2	15.5154				

$$y_0 = 17.1070, \qquad \Delta y_0 = -0.7638, \qquad \Delta^2 y_{-1} = -0.0564$$

$$\Delta^3 y_{-1} = -0.0076, \qquad \Delta^4 y_{-2} = -0.0022$$

Substituting in (1), we get

$$y_u = 17.1070 + u(-0.7638) + \frac{u(u-1)}{2!}(-0.0564)$$
$$+ \frac{(u+1)u(u-1)}{3!}(-0.0076) + \frac{(u+1)u(u-1)(u-2)}{4!}(-0.0022)$$

When $u = 0.25$,

$$y_u = 17.1070 + (0.25)(-0.7638) + \frac{(0.25)(0.25-1)}{2!}(-0.0564)$$
$$+ \frac{(0.25+1)(0.25)(0.25-1)}{3!}(-0.0076) + \frac{(0.25+1)(0.25)(0.25-1)(0.25-2)}{4!}(-0.0022)$$

$$= 17.1070 - 0.19095 + 0.0052875 + 0.000296875 - 0.000037598$$

$$= 16.92159678$$

$\therefore \qquad \mathbf{y_{30} = 16.9216}$ ■

Example 3

Use Gauss's forward interpolation formula to find log sin 0°16'8.5" from the given table.

x	**0°16'7"**	**0°16'8"**	**0°16'9"**	**0°16'10"**
$\log_e \sin x$	**7.67100**	**7.67145**	**7.67190**	**7.67235**

Solution

We use Gauss's forward formula to find $\log_e \sin 0°16'\,8.5"$, since $x = 0°16'\,8.5"$ is near the middle of the table.
Gauss's forward formula is

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \ldots \quad (1)$$

where $u = \dfrac{x - x_0}{h}$ and x_0 is the origin

Here $h = 1"$ and take $x_0 = 0°\,16'\,8"$, since $x = 0°\,16'\,8.5"$ lies between $x = 0°\,16'\,8"$ and $x = 0°\,16'\,9"$.

$$\therefore \qquad u = \frac{x - 0°16'8.5"}{1°} = x - 0°16'8.5"$$

When $x = 0°\,16'\,8.5"$, $\qquad u = 0°\,16'\,8.5" - 0°\,16'\,88" = 0.5$

We shall form the central difference table.

x	$u = \dfrac{x - 0°\,16'\,8''}{0.5}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$
$0°\,10'\,7''$	-1	7.67100			
			0.00045		
$0°\,16'\,8''$	0	$\boxed{7.67145}$		$\boxed{0}$	
			$\boxed{0.00045}$		$\boxed{0}$
$0°\,16'\,9''$	1	7.67190		0	
			0.00045		
$0°\,16'\,10''$	2	7.67235			

$y_0 = 7.67145,\qquad \Delta y_0 = 0.00045,\qquad \Delta^2 y_{-1} = 0,\qquad \Delta^3 y_{-1} = 0$

Substituting in (1), we get

$$y_u = 7.67145 + u(0.00045)$$

When $u = 0.5$,

$$y_u = 7.67145 + (0.5)(0.00045) = 7.67145 + 0.000225 = 7.67175$$

$\therefore$ **$\log_e \sin 0^\circ\, 16'8.5'' = 7.671675$** ■

Example 4

Estimate the value of $f(3.5)$ from the following data.

x	0	1	2	3	4	5	6
$f(x)$	176	185	194	203	212	220	229

Solution

We use Gauss's forward formula to find $f(3.5)$, since $x = 3.5$ is near the middle of the table. Gauss's forward formula is

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1}$$

$$+\frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-2} + \dots \qquad (2)$$

where $u = \frac{x - x_0}{h}$ and x_0 is the origin.

Here $h = 1$ and take $x_0 = 3$, since $x = 3.5$ lies between $x = 3$ and $x = 4$.

$$\therefore \qquad u = \frac{x-3}{1} = x - 3.$$

When $x = 3.5$, $\qquad u = 3.5 - 3 = 0.5$

We shall form the central difference table.

x	$u = x - 3$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0	−3	176						
			9					
1	−2	185		0				
			9		0			
2	−1	194		0		0		
			9		0		−1	
3	0	$\boxed{203}$		$\boxed{0}$		$\boxed{-1}$		$\boxed{5}$
			$\boxed{9}$		$\boxed{-1}$		$\boxed{4}$	
4	1	212		−1		3		
			8		2			
5	2	220		1				
			9					
6	3	229						

$y_0 = 203, \quad \Delta y_0 = 0, \quad \Delta^2 y_{-1} = 0$

$\Delta^3 y_{-1} = -1, \quad \Delta^4 y_{-2} = -1, \quad \Delta^5 y_{-2} = 4, \quad \Delta^6 y_{-3} = 5$

Substituting in (1), we get

$$y = 203 + u \times 9 + \frac{u(u-1)}{2!} \times 0 + \frac{(u+1)u(u-1)}{3!}(-1) + \frac{(u+1)u(u-1)(u-2)}{4!}(-1)$$

$$+ \frac{(u+2)(u+1)u(u-1)(u-2)}{5!} \times 4 + \frac{(u+2)(u+1)u(u-1)(u-2)(u-3)}{6!} \times 5$$

When $u = 0.5$,

$$y = 203 + 0.5 \times 9 + \frac{(0.5+1)(0.5)(0.5-1)}{6}(-1)$$

$$+\frac{(0.5+1)(0.5)(0.5-1)(0.5-2)}{24}(-1)$$

$$+\frac{(0.5+2)(0.5+1)(0.5)(0.5-1)(0.5-2)}{120}\times 4$$

$$+\frac{(0.5+2)(0.5+1)(0.5)(0.5-1)(0.5-2)(0.5-3)}{720}\times 5$$

$$y = 203 + 4.5 + 0.0625 - 0.0234375 + 0.046875 - 0.0244141$$

$$= 207.5615234 = 207.56$$

$\therefore$ $f(3.5) = \mathbf{207.56}$ ■

Example 5

Interpolate by means of Gauss's forward formula the value of $f(32)$, given that $f(25) = 0.2707$, $f(30) = 0.3027$, $f(35) = 0.3386$, $f(40) = 0.3794$.

Solution

The given values of the arguments x are 25, 30, 35, 40 and the corresponding values of y are $f(25) = 0.2707$, $f(30) = 0.3027$, $f(35 = 0.3386)$, $f(40) = 0.3794$. $x = 32$ is near the middle of the table.

So, we use Gauss's forward formula to find $f(32)$.

Gauss's forward formula is

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1}$$

$$+\frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \dots \qquad (1)$$

where $u = \frac{x - x_0}{h}$ and x_0 is the origin.

Here $h = 5$ and take $x_0 = 30$, since $x = 32$ lies between $x = 30$ and $x = 35$.

$\therefore$ $$u = \frac{x-30}{5}$$

When $x = 32$, $$u = \frac{32-30}{5} = \frac{2}{5} = 0.4$$

We shall form the central difference table.

x	$u = \frac{x-30}{5}$	y_u	Δy	$\Delta^2 y$	$\Delta^3 y$
25	-1	0.2707			
			0.0320		
30	0	**0.3027**		**0.0039**	
			0.0359		**0.0010**
35	1	0.3386		0.0049	
			0.0408		
40	2	0.3794			

$$y_0 = 0.3027, \quad \Delta y_0 = 0.0359, \quad \Delta^2 y_{-1} = 0.0039, \quad \Delta^3 y_{-1} = 0.0010$$

Substituting in (1), we get

$$y_u = 0.3027 + u(0.0359) + \frac{u(u-1)}{2!}(0.0039) + \frac{(u+1)u(u-1)}{3!}(0.0010)$$

When $\quad u = 0.4$

$$y_u = 0.3027 + (0.4)(0.0359) + \frac{(0.4)(0.4-1)}{2}(0.0039)$$

$$+ \frac{(0.4+1)(0.4)(0.4-1)}{6}(0.0010)$$

$$= 0.3027 + 0.01436 - 0.000468 - 0.000056 = 0.316536 = 0.3165 \quad \blacksquare$$

$\therefore \quad$ **$f(32) = 0.3165$**

4.3.2 Gauss's Backward Formula for Interpolation

Gauss's Backward formula for Interpolation is

$$y_u = y_0 + u\Delta y_{-1} + \frac{(u+1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2} + \dots$$

Proof: We derive Gauss's backward difference formula from Newton's forward formula by using even differences along the central line and odd differences above the central line. Newton's forward formula is

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots \quad (1)$$

where $u = \dfrac{x - x_0}{h}$ and x_0 is the origin.

We have $\Delta y_0 - \Delta y_{-1} = \Delta^2 y_{-1}$

$\Rightarrow$ $\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$

.Similarly $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$

$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$ and so on.

Also $\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$

$\Rightarrow$ $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$

$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ and so on.

Substituting Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$ in (1) we get

$$y = y_0 + u\left[\Delta y_{-1} + \Delta^2 y_{-1}\right] + \frac{u(u-1)}{2!}\left(\Delta^2 y_{-1} + \Delta^3 y_{-1}\right)$$
$$+ \frac{u(u-1)(u-2)}{3!}\left[\Delta^3 y_{-1} + \Delta^4 y_{-1}\right] + \frac{u(u-1)(u-2)(u-3)}{4!}\left[\Delta^4 y_{-1} + \Delta^5 y_{-1}\right] + \ldots$$

$$= y_0 + u\ \Delta y_{-1} + \left[u + \frac{u(u-1)}{2!}\right]\Delta^2 y_{-1}$$
$$+ \left[\frac{u(u-1)}{2!} + \frac{u(u-1)(u-2)}{3!}\right]\Delta^3 y_{-1} + \left[\frac{u(u-1)(u-2)}{3!} + \frac{u(u-1)(u-2)(u-3)}{4!}\right]\Delta^4 y_{-1} + \ldots$$

$$= y_0 + u\ \Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u-2)(u+1)}{4!}\Delta^4 y_{-1} + \ldots$$

$$= y_0 + u\ \Delta y_{-1} + \frac{(u+1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\left[\Delta^3 y_{-2} + \Delta^4 y_{-2}\right]$$
$$+ \frac{(u+1)(u)(u-1)(u-2)}{4!}\left[\Delta^4 y_{-2} + \Delta^5 y_{-2}\right]$$

$$= y_0 + u\ \Delta y_{-1} + \frac{(u+1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2}$$
$$+ \left[\frac{(u+1)u(u-1)}{3!} + \frac{(u+1)u(u-1)(u-2)}{4!}\right]\Delta^4 y_{-2} + \ldots$$

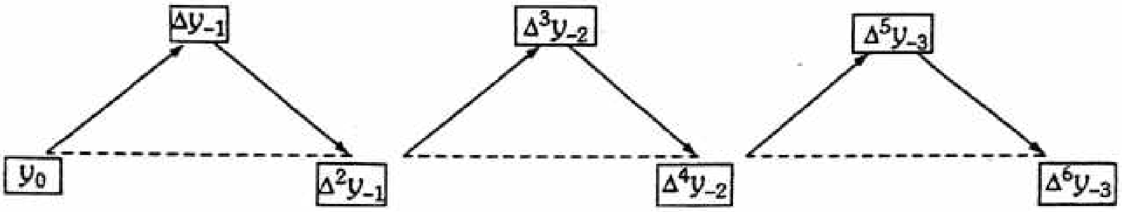
$$= y_0 + u\,\Delta y_{-1} + \frac{(u+1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} + \frac{(u+2)(u+1)(u)(u-1)}{4!}\Delta^4 y_{-2} + \ldots$$

$$\Rightarrow \qquad y_u = y_0 + \binom{u}{1}\Delta y_{-1} + \binom{u+1}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-2} + \binom{u+2}{4}\Delta^4 y_{-2} + \ldots$$

where $\binom{u}{r} = \dfrac{u(u-1)(u-2)\ldots(u-r+1)}{r!}$

Note:

(1) Gauss's backward formula uses the differences on the indicated path



(2) Gauss's backward formula is used for interpolation if $-1 < u < 0$.
Because of this reason, Gauss's backward formula is some times known as Gauss's formula for negative interpolation.

WORKED EXAMPLES

Example 1

Given that $\sqrt{12500} = 111.80339$, $\sqrt{12510} = 111.848111$, $\sqrt{12520} = 111.892806$, $\sqrt{12530} = 111.937483$. Show by Gauss's backward formula $\sqrt{12516} = 111.874930$.

Solution

Given values of x are 12500, 12510, 12520, 12530.

Required the value of $\sqrt{12516}$ by Gauss's backward formula.

Let $y = \sqrt{x}$ Then to find $y = \sqrt{12516}$

Gauss's backward formula is

$$y_u = y_0 + u\,\Delta y_{-1} + \frac{(u+1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2} + \ldots \quad (1)$$

where $u = \dfrac{x - x_0}{h}$ and x_0 is the origin.

Here $h = 10$ and take $x_0 = 12520$, since $x = 12516$ lies between $x = 12510$ and $x = 12520$.

$$\therefore \qquad u = \frac{x - 12520}{10}$$

When $x = 12516$, $u = \dfrac{12516 - 12520}{10} = -\dfrac{4}{10} = -0.4$

We shall form the central difference table.

x	$u = \frac{x - 12520}{10}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$
12500	−2	111.803399			
			0.044712		
12510	−1	111.848111		−0.000017	
			0.044695		**−0.000001**
12510	0	**111.892806**		**0.000018**	
			0.044677		
12530	1	111.937483			

$$y_0 = 111.892806, \quad \Delta y_{-1} = 0.044695$$
$$\Delta^2 y_{-1} = -0.000018, \quad \Delta^3 y_{-2} = -0.000001$$

Substituting in (1), we get

$$y = 111.892806 + u(0.044695) + \frac{(u+1)u}{2!}(-0.000018) + \frac{(u+1)u(u-1)}{3!}(-0.000001)$$

When $u = -0.4$,

$$y = 111.892806 + (-0.4)(0.044695) + \frac{(-0.4+1)(-0.4)}{2}(-0.000018) + \frac{(-0.4+1)(-0.4)(-0.4-1)}{6}(-0.000001)$$
$$= 111.892806 - 0.017878 + 0.00000216 - 0.000000056$$
$$= 111.8749301$$

$$\therefore \quad \sqrt{1251} = \mathbf{111.87493}$$

■

Example 2

Given the following data

x	1.72	1.73	1.74	1.75	1.76
e^{-x}	0.17907	0.17728	0.17552	0.17377	0.17204

Find the value of $e^{-1.735}$ using Gauss's backward formula.

Solution

Required the value of $y = e^{-x}$ when $x = 1.735$ by Gauss's backward formula

$$y = y_0 + u\Delta y_0 + \frac{(u+1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2}$$

$$+\frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \ldots \quad (1)$$

Where $u = \dfrac{x - x_0}{h}$ and x_0 is the origin.

Here $h = 0.01$ and take $x_0 = 1.74$, since $x = 1.735$ lies between $x = 1.73$ and $x = 1.74$.

$\therefore$ $$u = \frac{x - 1.74}{0.01}$$

When $x = 1.735$, $u = \dfrac{1.735 - 1.74}{0.01} = -\dfrac{0.005}{0.01} = -0.5$

We shall form the central difference table.

x	$u = \dfrac{x-1.74}{0.01}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.72	−2	0.17907				
			−0.0179			
1.73	−1	0.17728		0.00003		
			0.00176		**−0.00002**	
1.74	0	**0.17552**		**0.00001**		**0.00003**
			−0.00175		0.00001	
1.75	1	0.17377		0.00002		
			−0.00173			
1.76	2	0.17204				

$$y_0 = 0.17552 \qquad \Delta y_{-1} = -0.00176 \qquad \Delta^2 y_{-1} = 0.00001$$
$$\Delta^3 y_{-2} = -0.00002 \qquad \Delta^4 y_{-2} = 0.00003$$

Substituting in (1) we get

$$y = 0.17552 + u(-0.00176) + \frac{(u+1)u}{2!}(0.00001)$$
$$+ \frac{(u+1)u(u-1)}{3!}(-0.00002) + \frac{(u+2)(u+1)u(u-1)}{4!}(0.00003)$$

When $u = -0.5$ we get

$$y = 0.17552 + (-0.5)(-0.00176) + \frac{(-0.5+1)(-0.5)}{2!}(0.00001)$$
$$+ \frac{(-0.5+1)(-0.5)(-0.5-1)}{3!}(-0.00002) + \frac{(-0.5+2)(-0.5+1)(-0.5)(-0.5-1)}{4!}(0.00003)$$
$$= 0.17552 + 0.00088 - 0.00000125 - 0.00000125 + 0.000000703$$
$$= 0.176398203 = 0.176398$$

■

$$e^{-1.735} = \mathbf{0.176398}$$

Example 3

Using Gauss's backward formula estimate the number of persons earnings wages between Rs. 60 and 70 from the following data.

Wages (Rs.)	Below 40	40–60	60–80	80–100	100–120
No of person's (in thousands)	250	120	100	70	50

Solution

We take the given intervals as 20–40, 40–60, 60–80, 80–100, 100–120.

Let the middle points of the intervals be the x-values and numbers of persons as the corresponding y values

Values of x	30	50	70	90	110
y	250	120	100	70	50

To find the number of persons earning wages between Rs 60 and 70, we use Gauss's backward formula

Gauss's backward formula is

$$y_u = y_0 + \Delta y_{-1} + \frac{(u+1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2} + \dots \quad (1)$$

where $u = \dfrac{x - x_0}{h}$ and x_0 is the origin.

Here $h = 20$

For the interval 60–70, middle value is $x = 65$

$\therefore$ take $x_0 = 70$, since $x = 65$ lies between $x = 50$ and $x = 70$

$\therefore$ $$u = \frac{x-70}{20}$$

When $x = 65$, $$u = \frac{65-70}{20} = -\frac{5}{20} = -0.25$$

We shall form the central difference table.

x	$u = \dfrac{x-70}{20}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
30	−2	250				
			−130			
50	−1	120		110		
			$\boxed{-20}$		$\boxed{-120}$	
70	0	$\boxed{100}$		$\boxed{-10}$		$\boxed{140}$
			−30			
90	1	70		10	20	
			−20			
110	2	50				

$$y_0 = 100, \quad \Delta y_{-1} = -20, \quad \Delta^2 y_{-1} = -10, \quad \Delta^3 y_{-2} = -120, \quad \Delta^4 y_{-2} = 140$$

Substituting in (1), we get

$$y = 100 + u(-20) + \frac{(u+1)u}{2!}(-10) + \frac{(u+1)u(u-1)}{3!}(-120) + \frac{(u+2)(u+1)u(u-1)}{4!}(140)$$

When $u = -0.25$, we get

$$y = 100 + (-0.25)(-20) + \frac{(-0.25+1)(-0.25)}{2!}(-10)$$
$$+ \frac{(-0.25+1)(-0.25)(-0.25-1)}{3!}(-120) + \frac{(-0.25+2)(-0.25+1)(-0.25)(-0.25-1)}{4!}(140)$$
$$= 100 + 5 + 0.9375 - 4.6875 + 2.392578 = 103.642578 = 103.6428$$

$\therefore$ the number of persons in the wage range 60 to 70 is equal to **104.** ■

Exercises 4.3

(1) Using Gauss's forward formula find $f(3.5)$, given that

x	2	3	4	5
$f(x)$	2.426	3.454	4.784	6.986

(2) Given that

Year	1930	1932	1934	1936	1938	1940
Population (in thousand)	12	16	21	27	32	40

Using Newton-Gauss's forward formula find the population of the town in 1935.

(3) Using Gauss forward formula find the value of $y_{28.3}$ given that $y_{26} = 0.038462$, $y_{27} = 0.037037$ $y_{28} = 0.035714$, $y_{29} = 0.034483$ $y_{30} = 0.33333$.

(4) Given $\sin 25° \, 41' \, 40'' = 0.433572$, $\sin 25° \, 42' \, 0' = 0433659$, $\sin 25° \, 42' \, 20'' = 0.433746$, $\sin 25° \, 42' \, 40'' = 0.433834$. Find the value of $\sin 25° \, 42' \, 10''$ by Gauss's forward formula.

(5) If $u_0 = 11, u_1 = 7, u_2 = 6, u_3 = 16$ find $u_{2.5}$ approximately.

(6) Given $f(25) = 2707, f(30) = 3027, f(35) = 3386, f(40) = 3794$, find the value of $f(32)$ by Gauss's backward formula.

(7) The specific gravities of Zinc sulphate solutions of different concentrations at 15°C are given below

Concentration (%):	10	12	14	16	18	20	22
Specific gravity:	1.059	1.073	1.085	1.097	1.110	1.124	1.137

Find the specific gravity of 15.8% solution at 15°C, using Gauss's Backward formula.

(8) Interpolate by means of Gauss's backward formula the sales of a concern for the year 1997 given that

Year:	1962	1972	1982	1992	2002	2012
Sales (in cores of Rs.)	12	15	20	27	39	52

(9) Find the value of $\cos 51°\,42'$ by Gauss's backward formula, given that

x:	50°	51°	52°	53°	54°
cos x:	0.6428	0.6293	0.6152	0.6018	0.5878

Answers 4.3

(1) 4.034 (2) 24.046875 (3) 0.0353355
(4) 0.43 3703 (5) 7.6 (6) 3165.36
(7) 1.0958 (8) 32.625 crores of Rs. (9) 0.6198

Stirling's Formula for Interpolation

Stirling's formula is

$$y_u = y_0 + \frac{u}{1!}\left[\frac{\Delta y_0 + \Delta y_1}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-2} + \frac{u(u^2-1^2)}{3!}\left[\frac{\Delta^2 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$
$$+\frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \frac{u(u^2-1^2)(u^2-2^2)}{5!}\left[\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right] + \dots$$

Proof: Stirling's formula is derived by taking the average Gauss's forward and backward formulae.

Gauss's forward formula is

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1}$$
$$+\frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-2} + \dots \quad (1)$$

Gauss's backward formula is

$$y_u = y_0 + u\Delta y_{-1} + \frac{(u+1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2}$$
$$+\frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-3} + \dots \quad (2)$$

Adding (1) and (2), we get

$$2y_u = 2y_0 + u\left[\Delta y_0 + \Delta y_{-1}\right] + \frac{u}{2!}[u-1+u+1]\Delta^2 y_{-1}$$
$$+\frac{(u+1)u(u-1)}{3!}\left[\Delta^3 y_{-1} + \Delta^3 y_{-2}\right] + \frac{(u+1)u(u-1)}{4!}[u-2+u+2]\Delta^4 y_{-2}$$
$$+\frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\left[\Delta^5 y_{-2} + \Delta^5 y_{-3}\right] + \dots$$

$$= 2y_0 + u\left[\Delta y_0 + \Delta y_{-1}\right] + \frac{u}{2!}2u\,\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\left(\Delta^3 y_{-1} + \Delta^3 y_{-2}\right)$$

$$+ \frac{(u+1)u(u-1)}{4!}2u\,\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\left[\Delta^5 y_{-2} + \Delta^5 y_{-3}\right] + \dots$$

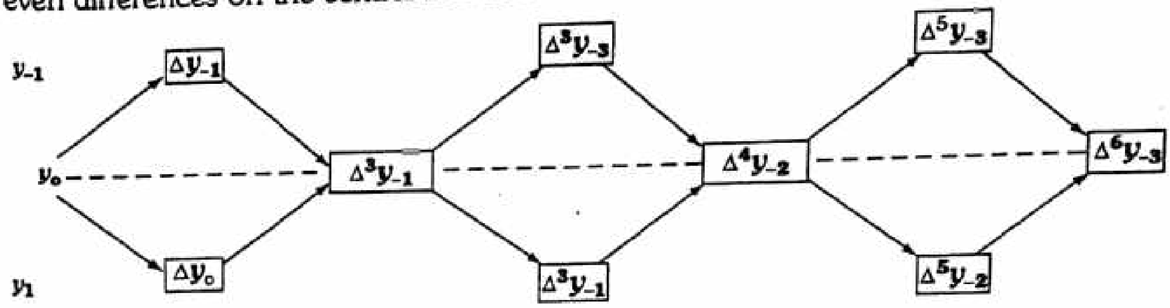
$$y_u = y_0 + \frac{u}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$

$$+ \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \frac{u(u^2-1^2)(u^2-2^2)}{5!}\left[\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right] + \dots$$

This is called Stirling's formula or Newton-Stirling's formula.

Note:

(1) Stirling's formula involves average of odd difference above and below the central line and even differences on the central line as shown.



(2) Stirling's formula gives fairly accurate results when $-\frac{1}{4} \le u \le \frac{1}{4}$, though this formula can be used if $-\frac{1}{2} < u < \frac{1}{2}$.

WORKED EXAMPLES

Example 1

Given

$\theta°$	0°	5°	10°	15°	20°	25°	30°
$\tan\theta°$	0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

Find the value of $\tan 16°$, using Stirling's formula.

Solution

Let $\theta° = x°$

Given $y = \tan\theta° = \tan x°$

To find the value of $y = \tan 16°$.

We use Stirling's formula to find y when $x = 16$

Stirling's formula is

$$y_u = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \ldots$$

where $u = \dfrac{x - x_0}{h}$ and x_0 is the origin.

Here $h = 5°$ and take $x_0 = 15°$, since $x = 16°$ lies between $x = 15°$ and $x = 20°$

$\therefore$ $$u = \frac{x - 15°}{5°}$$

When $x = 16°$, $$u = \frac{16° - 15°}{5°} = \frac{1}{5} = 0.2$$

We shall form the central difference table.

$x° = \theta°$	$u = \frac{x-15}{5}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0	−3	0						
			0.0875					
5	−2	0.0875		0.0013				
			0.0888		0.0015			
10	−1	0.1763		0.0028		0.0002		
			0.0916		**0.0017**		**0.0002**	
15	0	**0.2679**		**0.0045**		**0**		**0.0011**
			0.0961		**0.0017**		**0.0009**	
20	1	0.3640		0.0062		0.0009		
			0.1023		0.0026			
25	2	0.4663		0.0088				
			0.1111					
30	3	0.5774						

$y_0 = 0.2679, \quad \Delta y_{-1} = 0.0916, \quad \Delta y_0 = 0.0961, \quad \Delta^2 y_{-1} = 0.0045,$

$\Delta^3 y_{-2} = 0.0017, \quad \Delta^3 y_{-1} = 0.0017, \quad \Delta^4 y_{-2} = 0, \quad \Delta^5 y_{-3} = -0.0002,$

$\Delta^5 y_{-2} = 0.0009 \quad \Delta^6 y_{-3} = 0.0011.$

Substituting in (1), we get

$$y_u = 0.2679 + \frac{u}{2}[0.0961 + 0.0916] + \frac{u^2}{2!}(0.0045)$$
$$+ \frac{u(u^2-1^2)}{3!}\frac{(0.0017+0.0017)}{2} + \frac{u^2(u^2-1^2)}{4!} \times 0$$
$$+ \frac{u(u^2-1^2)(u^2-2^2)}{5!}\left[\frac{-0.0002+0.0009}{2}\right] + \frac{u^2(u^2-1^2)(u^2-2^2)}{6!}(0.0011)$$

When $u = 0.2$, we get

$$y_u = 0.2679 + \frac{0.2}{2}(0.1877) + \frac{(0.2)^2}{2}(0.0045) + \frac{(0.2)(0.2^2-1)}{6}(0.0017)$$
$$+ \frac{(0.2)(0.2^2-1)(0.2^2-4)}{120}(0.00035) + \frac{(0.2)^2(0.2^2-1)(0.2^2-4)}{720}(0.0011)$$
$$= 0.2679 + 0.01877 + 0.00009 - 0.0000544 + 0.0000022176 + 0.00000023232$$
$$= 0.2867, \text{ Correct to 4 places.}$$

$\therefore \quad$ **tan 16° = 0.2867** ■

Example 2

The following table gives the values of the probability integral $f(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx$. Find the value o the integral when $x = 0.5437$.

x	0.51	0.52	0.53	0.54	0.55	0.56
$f(x)$	0.52924	0.53789	0.54646	0.55494	0.56332	0.57162

Solution

Required to find $f(x)$ when $x = 0.5437$, by using Stirling's formula.
Stirling's formula is

$$y_u = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\frac{\left[\Delta^3 y_{-1} + \Delta^3 y_{-2}\right]}{2}$$
$$+ \frac{u^2(u^2-1^2)}{4!}\left[\Delta^4 y_{-2}\right] + u\frac{(u^2-1^2)(u^2-2^2)}{5!}\frac{\left[\Delta^5 y_{-2} + \Delta^5 y_{-3}\right]}{2} + \dots \quad (1)$$

where $u = \frac{x - x_0}{h}$ and x_0 is the origin.

Here $h = 0.01$ and take $x_0 = 0.54$, since $x = 0.5437$ lies between $x = 0.54$ and $x = 0.55$,

$$\therefore \qquad u = \frac{x - 0.54}{0.01}$$

When $x = 0.5437$, $u = \dfrac{0.5437 - 0.54}{0.01} = \dfrac{0.0037}{0.01} = 0.37$

We shall form the central difference table.

x	$u = \dfrac{x-0.54}{0.01}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0.51	−3	0.52924					
			0.00865				
0.52	−2	0.53789		−0.00008			
			0.00857		−0.00001		
0.53	−1	0.54646		−0.00009		0	
			0.00848		**0.00001**		**0.00003**
0.54	0	**0.55494**		**−0.00010**		**0.00003**	
			0.00838		**0.00002**		
0.55	1	0.56332		−0.00008			
			0.00830				
0.56	2	0.57162					

$y_0 = 0.55494$, $\quad \Delta y_0 = 0.00838$, $\quad \Delta y_{-1} = 0.00848$, $\quad \Delta^2 y_{-1} = -0.00010$,

$\Delta^3 y_{-1} = 0.00002$, $\quad \Delta^3 y_{-2} = -0.00001$, $\quad \Delta^4 y_{-2} = 0.00003$, $\quad \Delta^5 y_{-3} = 0.00003$, $\quad \Delta^5 y_{-2} = 0$

Substituting in (1), we get

$$y_u = 0.55494 + u\frac{(0.00838 + 0.00848)}{2} + \frac{u^2}{2!}(-0.00010)$$

$$+ \frac{u(u^2 - 1^2)}{3!}\frac{[0.00002 + (-0.00001)]}{2} + \frac{u^2(u^2 - 1^2)}{4!}(0.00003)$$

$$+ \frac{u(u^2 - 1^2)(u^2 - 2^2)}{5!}\frac{[0 + 0.00003]}{2}$$

When $u = 0.37$, we get

$$y_u = 0.55494 + 0.37(0.00843) + \frac{(0.37)^2}{2}(-0.00010)$$

$$+\frac{0.37(0.37^2-1^2)}{6}0.000005 + \frac{(0.37)^2(0.37^2-1^2)}{24}(0.00003)$$

$$+\frac{(0.37)\{(0.37)^2-1^2\}\{(0.37)^2-2^2\}}{120}(0.000015)$$

$$= 0.55494 + 0.0031191 - 0.000006845 - 0.0000002661225$$

$$-0.000000147698 + 0.000000154209$$

$$= 0.558051995 = 0.55805$$

when $\quad \boldsymbol{x = 0.5437} \quad \boldsymbol{y = 0.55805}$ ■

Example 3

Using Stirling's formula find the annual net premium at the age of 25 from the table of annual net premium given below.

Age	20	24	28	32
Premium	0.01427	0.01581	0.01772	0.01996

Solution

Let us denote age by x and premium as y.

Required, the net premium at the age of 25 using Stirling's formula.

That is to find y when $x = 25$.

Stirling's formula is

$$y_u = y_0 + u\frac{(\Delta y_0 + \Delta y_{-1})}{2} + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\frac{[\Delta^3 y_{-1} + \Delta^3 y_{-2}]}{2} + \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \dots \quad (1)$$

where $u = \dfrac{x - x_0}{h}$ and x_0 is the origin.

Here $h = 4$ and take $x_0 = 24$, since $x = 25$ lies between $x = 24$ and $x = 28$.

$$\therefore \qquad u = \frac{x-24}{4}$$

When $x = 25$, $\qquad u = \dfrac{25-24}{4} = \dfrac{1}{4} = 0.25$

We shall form the central difference table.

x	$u = \frac{x-24}{4}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	−1	0.01427			
			0.00154		
24	0	**0.01581**		**0.00037**	
			0.00191		**−0.00004**
28	1	0.01772		−0.00033	
			0.00224		
32	2	0.01996			

Here $h = 4$ and take $x_0 = 24$, since $x = 25$ lies between $x = 24$ and $x = 28$

$$y_0 = 0.01581, \quad \Delta y_0 = 0.00191, \quad \Delta y_{-1} = 0.00154,$$
$$\Delta^2 y_{-1} = 0.00037, \quad \Delta^3 y_{-1} = -0.00004$$

Substituting in (1), we get

$$y = 0.01581 + u\left[\frac{0.00191 + 0.00154}{2}\right]$$
$$+ \frac{u^2}{2!}(0.00037) + \frac{u(u^2 - 1^2)}{3!}\left[\frac{-0.00004}{2}\right]$$

When $x = 0.25$

$$y = 0.01581 + 0.25(0.001725) + \frac{(0.25)^2}{2}(0.00037)$$
$$+ \frac{(0.25)(0.25^2 - 1^2)}{6}(-0.00002)$$
$$= 0.01581 + 0.00043125 + 0.0000115625 + 0.00000078125$$
$$= 0.016253593 = 0.01625 \text{ Correct to 5 places of decimals}$$

∴ net Premium when x **= 25 is 0.01625** ■

Example 4

Find the value of $f(1.63)$ and $f(1.67)$ from the following table using Stirling's formula.

x	1.50	1.60	1.70	1.80	1.90
$f(x)$	17.609	20.412	23.045	25.527	27.875

Solution

Let $y = f(x)$

Required the values of y when $x = 1.63$ and $x = 1.67$ by using Stirling's formula.

Stirling's formula is

$$y = y_u + u\frac{(\Delta y_0 + \Delta y_{-1})}{2} + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \dots \quad (1)$$

where $u = \dfrac{x - x_0}{h}$ and x_0 is the origin.

Here $h = 0.10$ and take the orgin as $x_0 = 1.60$, since $x = 1.63$ lies between 1.60 and 1.70

$$\therefore \qquad u = \frac{x - 1.60}{0.10}$$

When $x = 1.63$, $u = \dfrac{1.63 - 1.60}{0.10} = \dfrac{0.03}{0.10} = 0.3$

We shall form the central difference table when $x_0 = 1.60$

x	$u = \dfrac{x-1.60}{0.1}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.50	−1	17.609				
			2.803			
1.60	0	**20.412**		**−0.170**		
			2.633		**0.019**	
1.70	1	23.045		−0.151		−0.002
			2.482		0.017	
1.80	2	25.527		−0.134		
			2.348			
1.90	3	27.875				

$$y_0 = 20.412, \quad \Delta y_0 = 2.633, \quad \Delta y_{-1} = 2.803$$
$$\Delta^2 y_{-1} = -0.170, \quad \Delta^3 y_{-1} = 0.019 \quad \text{and} \quad u = 0.3$$

Substituting in (1), we get

$$y = f(1.63) = 20.412 + (0.3)\frac{[2.633 + 2.803]}{2}$$
$$+\frac{(0.3)^2}{2!}(-0.170) + \frac{(0.3)(0.3^2 - 1)}{3!}\frac{(0.019)}{2}$$
$$= 20.412 + 0.8154 - 0.00765 - 0.000432 = \mathbf{21.21932}$$

When $x = 1.67$, $u = \dfrac{1.67 - 1.60}{0.10} = \dfrac{0.07}{0.10} = 0.7 > \frac{1}{2}$

We use Stirlings formula if $-\frac{1}{2} < u < \frac{1}{2}$ only. So we cannot take the origin as $x = 1.60$ to find y when $x = 1.67$, since $0.7 > \frac{1}{2}$.

So, we change the origin to $x = 1.70$, to find y when $x = 1.67$

$$\therefore \qquad u = \frac{x - 1.70}{0.10}$$

When $x = 1.67$, $u = \dfrac{1.67 - 1.70}{0.10} = -0.3$ [-0.3 lies is the interval $-\dfrac{1}{2} < u < \dfrac{1}{2}$]

When $x_0 = 1.70$, the difference table is

x	$u = \dfrac{x - 1.70}{0.1}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.50	–2	17.609				
			2.803			
1.60	–1	20.412		–0.170		
			$\boxed{2.633}$		$\boxed{0.019}$	
1.70	0	$\boxed{23.045}$		$\boxed{-0.151}$		$\boxed{-0.002}$
			$\boxed{2.482}$		$\boxed{0.017}$	
1.80	1	25.527		–0.134		
			2.348			
1.90	2	27.875				

$$y_0 = 23.045, \quad \Delta y_0 = 2.482, \quad \Delta y_{-1} = 2.633, \quad \Delta^2 y_{-1} = -0.151,$$
$$\Delta^3 y_{-1} = 0.017, \quad \Delta^3 y_{-2} = 0.019 \quad \Delta^4 y_{-2} = -0.002 \text{ and } \quad u = -0.3.$$

Substituting in (1), we get

$$y_u = 23.045 + (-0.3)\frac{[2.482 + 2.633]}{2} + \frac{(-0.3)^2}{2!}(-0.151)$$

$$+\frac{(-0.3)\left[(-0.3)^2 - 1^2\right]}{3!}\frac{[0.017 + 0.019]}{2} + \frac{(-0.3)^2\left((-0.3)^2 - 1^2\right)}{4!}(-0.002)$$

$$\Rightarrow \quad f(1.67) = 23.045 - 0.76725 - 0.006795 + 0.000819 + 0.000006825$$

$$\Rightarrow \quad f(1.67) = 22.27178083 = 22.27178$$

$$\therefore \quad f(1.63) = 21.21932 \text{ and } f(1.67) = \mathbf{22.27178}$$

Bessel's Formula for Interpolation

Bessel's formula is

$$y_u = y_0 + y_1 + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left[\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right] + \frac{\left(u - \frac{1}{2}\right)u(u-1)}{3!}\Delta^3 y_{-1}$$
$$+\frac{(u+1)u(u-1)(u-2)}{4!}\left[\frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2}\right] + \ldots$$

Proof: We derive the Bessel's formula using Gauss's forward and backward formula.

Gauss's forward formula is

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1}$$
$$+\frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-2} + \ldots \tag{1}$$

Gauss's backward formula is

$$y_u = y_0 + u\Delta y_{-1} + \frac{(u+1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2}$$
$$+\frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-3} + \ldots \tag{2}$$

Shifting the origin of u from 0 to 1, that is replace u by $u-1$ in backward formula, we get

$$y_u = y_1 + (u-1)\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_{-1}$$

$$+\frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-1} + \frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\Delta^5 y_{-2} + \dots \quad (3)$$

Adding (1) and (3) we get

$$2y_u = y_0 + y_1 + (u+u-1)\Delta y_0 + \frac{u(u-1)}{2!}\left[\Delta^2 y_{-1} + \Delta^2 y_0\right]$$
$$+\frac{u(u-1)}{3!}[u+1+u-2]\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\left[\Delta^4 y_{-2} + \Delta^4 y_{-1}\right]$$
$$+\frac{(u+1)u(u-1)(u-2)}{5!}[u+2+u-3]\Delta^5 y_{-2} + \dots$$

$$2y_u = (y_0 + y_1) + (2u-1)\Delta y_0 + \frac{u(u-1)}{2!}\left[\Delta^2 y_{-1} + \Delta^2 y_0\right] + \frac{u(u-1)}{3!}[2u-1]\Delta^3 y_{-1}$$
$$+\frac{(u+1)u(u-1)(u-2)}{4!}\left[\Delta^4 y_{-2} + \Delta^4 y_{-1}\right] + \frac{(u+1)u(u-1)(u-2)}{5!}[2u-1]\Delta^5 y_{-2} + \dots$$

$$\therefore \quad y_u = \frac{y_0 + y_1}{2} + (u - ½)\Delta y_0 + \frac{u(u-1)}{2!}\frac{\left[\Delta^2 y_0 + \Delta^2 y_{-1}\right]}{2} + \frac{(u-½)u(u-1)}{3!}\Delta^3 y_{-1}$$
$$+\frac{(u+1)u(u-1)(u-2)}{4!}\left[\frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2}\right] + \frac{(u-½)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-2} + \dots$$

This is Bessel's formula.

If we put $v = u - \frac{1}{2}$, then $u = v + \frac{1}{2}$ and Bessel's formula reduces to a more symmetrical form.

$$y = \frac{y_0 + y_1}{2} + v\,\Delta y_0 + \frac{\left(v^2 - \frac{1}{4}\right)}{2!}\frac{\left[\Delta^2 y_{-1} + \Delta^2 y_0\right]}{2}$$
$$+\frac{v\left(v^2 - \frac{1}{4}\right)}{3!}\Delta^3 y_{-1} + \frac{\left(v^2 - \frac{1}{4}\right)\left(v^2 - \frac{9}{4}\right)}{4!}\frac{\left[\Delta^4 y_{-2} + \Delta^4 y_{-1}\right]}{2} + \dots$$

Note:

(1) Bessel's formula employs odd differences below the central line and the means of even differences on and below the central line.

(2) If $0 < u < 1$ Bessel's formula is preferable. However it will give more accurate result when interpolating near the middle of the interval $\frac{1}{4} < u < \frac{3}{4}$.

When $u = \frac{1}{2}, 0, 1$ Bessel's formula gives best results with minimum number of terms.

(3) When $u = \frac{1}{2}$, we get the special case of Bessel's formula.

$$y = \frac{1}{2}[y_0 + y_1] - \frac{1}{8}\left[\Delta^2 y_{-1} + \Delta^2 y_0\right] + \frac{3}{128}\left[\Delta^4 y_{-2} + \Delta^4 y_{-1}\right] + \ldots$$

This is called formula for interpolating to halves.

WORKED EXAMPLES

Example 1

Apply Bessel's formula to find $\log_{10} 3375$, given $\log_{10} 310 = 2.49137$, $\log_{10} 320 = 2.250515$, $\log_{10} 330 = 2.51851$, $\log_{10} 340 = 2.53148$, $\log_{10} 350 = 2.54407$, $\log_{10} 360 = 2.55630$.

Solution

Required the value of $\log_{10} 3375$ using Bessel's formula,
Bessel's formula is

$$y = \frac{y_0 + y_1}{2} + \left(u - \tfrac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\frac{\left[\Delta^2 y_0 + \Delta^2 y_{-1}\right]}{2}$$
$$+ \frac{\left(u - \frac{1}{2}\right)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\frac{\left[\Delta^4 y_{-1} + \Delta^4 y_{-2}\right]}{2}$$
$$+ \frac{\left(u - \frac{1}{2}\right)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-2} + \ldots \qquad (1)$$

where $u = \frac{x - x_0}{h}$ and x_0 is the origin.

The given function is $y = \log_{10} x$ and the x values are 310, 320, 330, 340, 350, 360.

$$\begin{aligned}\log_{10} 3375 &= \log 337.5 \times 10 \\ &= \log_{10} 10 + \log_{10} 337.5 \\ &= 1 + \log_{10} 337.5\end{aligned}$$

Here $h = 10$ and take the origin as $x_0 = 330$, since $x = 337.5$ lies between $x = 330$ and $x = 340$

$\therefore$ $$u = \frac{x - 330}{h}$$

When $x = 337.5$, $$u = \frac{337.5 - 330}{10} = \frac{7.5}{10} = 0.75$$

We shall form the difference table.

x	$u = \frac{x - 330}{10}$	$y = \frac{10gx}{10}$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
310	−2	2.49137					
			0.01378				
320	−1	2.50515		−0.00042			
			0.01336		0.00003		
330	0	$\boxed{2.51851}$		$\boxed{-0.00039}$		$\boxed{-0.00002}$	
			$\boxed{0.01297}$		$\boxed{0.00001}$		$\boxed{0.00003}$
340	1	$\boxed{2.53148}$		$\boxed{-0.00038}$		$\boxed{0.00001}$	
			0.01259		0.00002		
350	2	2.54407		−0.00036			
			0.01223				
360	3	2.55630					

$y_0 = 2.51851$, $y_1 = 2.53148$, $\Delta y_0 = 0.01297$, $\Delta^2 y_0 = -0.00038$, $\Delta^2 y_{-1} = -0.00039$, $\Delta^3 y_{-1} = 0.00001$, $\Delta^4 y_{-1} = -0.00001$, $\Delta^4 y_{-2} = 0.00002$, $\Delta^5 y_{-2} = 0.00003$ and $u = 0.75$

Substituting in (1), we get

$$y = \frac{2.51851 + 2.53148}{2} + (0.75 - \tfrac{1}{2})(0.01297)$$
$$+ \frac{(0.75)(0.75-1)}{2!}\frac{[-0.00039 - 0.00038]}{2} + \frac{(0.75-1/2)(0.75)(0.75-1)}{3!}(0.00001)$$
$$+ \frac{(0.75+1)(0.75)(0.75-1)(0.75-2)}{4!}\frac{[-0.00002 + 0.000001]}{2}$$
$$+ \frac{(0.75-1/2)(0.75+1)(0.75)(0.75-1)(0.75-2)}{5!}(6.00003)$$
$$= 2.524995 + 0.0032425 + 0.0000360375 - 0.000000078125$$
$$- 0.000000085449 + 0.0000000256347$$
$$= 2.528273$$

$\therefore \quad \log_{10} 337.5 = 2.528273$

$\therefore \quad \log_{10} 3375 = 1 + \log 337.5$

$\qquad = 1 + 2.528273 = 3.528273.$

$\Rightarrow \quad \mathbf{\log_{10} 3375 = 3.52827}$ ■

Example 2

The area A of a circle and diameter d is given by the following table.

d	80	85	90	95	100
Area A	5026	5674	6362	7088	7854

Find the area when the diameter is 91.

Solution

Let us denote the diameter d as x and area A as y.
Required, the area y when $x = 91$.
We use Bessel's formula.
Bessel's formula is

$$y = \frac{y_0 + y_1}{2} + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right]$$
$$+ \frac{\left(u - \frac{1}{2}\right)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\frac{\left[\Delta^4 y_{-1} + \Delta^4 y_{-2}\right]}{2} + \ldots \qquad (1)$$

where $u = \dfrac{x - x_0}{h}$ and x_0 is the origin.

Here $h = 5$ and take the origin as $x_0 = 90$, since $x = 91$ lies between $x = 90$ and $x = 95$.

$$\therefore \qquad u = \frac{x-90}{5}$$

When $x = 91$, $\quad u = \frac{91-90}{5} = \frac{1}{5} = 0.2$

We shall form the central difference table.

x	$u = \frac{x-90}{5}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
80	−2	5026				
			648			
85	−1	5674		40		
			688		−2	
90	0	$\boxed{6362}$		$\boxed{38}$		$\boxed{4}$
			$\boxed{726}$		$\boxed{2}$	
95	1	$\boxed{7088}$		$\boxed{40}$		
			766			
100	2	7854				

$y_0 = 6362$, $\quad y_1 = 7088$, $\quad \Delta y_0 = 726$, $\quad \Delta^2 y_0 = 40$,
$\Delta^2 y_{-1} = 38$, $\quad \Delta^3 y_{-1} = 2$, $\quad \Delta^4 y_{-2} = 4$, and $u = 0.2$

Substituting in (1), we get

$$y = \frac{6262+7088}{2} + \left(0.2 - \frac{1}{2}\right)(726) + \frac{0.2(0.2-1)}{2!}\left[\frac{38+40}{2}\right]$$

$$+ \frac{\left(0.2-\frac{1}{2}\right)(0.2)(0.2-1)}{3!}(2) + \frac{(0.2+1)(0.2)(0.2-1)(0.2-2)}{4!}\left[\frac{4}{2}\right]$$

$$= 6725 - 217.8 - 3.12 + 0.016 + 0.0288 = 6504.1248$$

$\therefore$ **area = 6504.1248 sq. unit** ■

Example 3

Given $y_0, y_1, y_2, y_3, y_4,$ and fifth differences are constant.

Prove that $y_{5/2} = \frac{1}{2}c + \frac{25(c-b)+3(a-c)}{256}$

Where $a = y_0 + y_5,\ b = y_1 + y_4,\ c = y_2 + y_3.$

Solution

Given the values $y_0, y_1, y_2, y_3, y_4, y_5$ are such that the fifth differences are constant.

$\therefore$ 6^{th}, 7^{th} ... differences are zero.

Also given, $a = y_0 + y_5$, $b = y_1 + y_4$ and $c = y_2 + y_3$ (1)

Bessel's formula for $u = \frac{1}{2}$ is

$$y_{1/2} = \frac{y_0 + y_1}{2} - \frac{1}{8}\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2} + \frac{3}{128}\frac{\left(\Delta^4 y_{-1} + \Delta^4 y_{-2}\right)}{2} + \dots$$

Now shift the origin to 2, then $\frac{1}{2} \to \frac{5}{2}, 0 \to 2, 1 \to 3$ and so on.

$$\therefore \quad y_{5/2} = \frac{y_2 + y_3}{2} - \frac{1}{8}\frac{\Delta^2 y_2 + \Delta^2 y_1}{2} + \frac{3}{128}\frac{\left(\Delta^4 y_1 + \Delta^4 y_0\right)}{2}$$

But

$$\Delta^2 y_2 = (E-1)^2 y_2 = \left(E^2 - 2E + 1\right) y_2 = y_4 - 2y_3 + y_2$$

$$\Delta^2 y_1 = (E-1)^2 y_1 = \left(E^2 - 2E + 1\right) y_1 = y_3 - 2y_2 + y_1$$

$$\begin{aligned}\Delta^4 y_1 &= (E-1)^4 y_1 \\ &= \left(E^4 - 4E^3 + 6E^2 - 4E + 1\right) y_1 \\ &= E^4 y_1 - 4E^3 y_1 + 6E^2 y_1 - 4E y_1 + y_1 = y_5 - 4y_4 + 6y_3 - 4y_2 + y_1\end{aligned}$$

Similarly,

$$\Delta^4 y_0 = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

$$\begin{aligned}\therefore \quad y_{5/2} &= \frac{y_2 + y_3}{2} - \frac{1}{16}\left[y_4 - 2y_3 + y_2 + y_3 - 2y_2 + y_1\right] \\ &\quad + \frac{3}{256}\left[y_5 - 4y_4 + 6y_3 - 4y_2 + y_1 + y_4 - 4y_3 + 6y_2 - 4y_1 + y_0\right] \\ &= \frac{y_2 + y_3}{2} - \frac{1}{16}\left[y_4 - y_3 - y_2 + y_1\right] + \frac{3}{256}\left[y_5 - 3y_4 + 2y_3 + 2y_2 - 3y_1 + y_0\right]\end{aligned}$$

$$=\frac{c}{2}-\frac{1}{16}[y_1+y_4-(y_2+y_3)]+\frac{3}{256}[y_0+y_5-3(y_1+y_4)+2(y_2+y_3)]$$
$$=\frac{c}{2}-\frac{1}{16}[b-c]+\frac{3}{256}[a-3b+2c]$$
$$=\frac{c}{2}+\frac{1}{256}[-16(b-c)+3(a-3b+2c)]$$
$$=\frac{c}{2}+\frac{1}{256}[-16b+16c+3a-9b+6c]$$
$$=\frac{c}{2}+\frac{1}{256}[3a-25b+22c]$$
$$=\frac{c}{2}+\frac{1}{256}[3(a-c)-25b+25c]=\frac{c}{2}+\frac{1}{256}[3(a-c)-25(b-c)]$$

Example 4

Using Bessel's interpolation formula show that $y_{x+\frac{1}{2}}=\frac{1}{2}(y_x+y_{x+1})-\frac{1}{16}(\Delta^2y_{-1}+\Delta^2y_x)$ assuming suitable level of approximation.

Solution

Bessels formula is

$$y_u=\frac{y_0+y_1}{2}+(u-\tfrac{1}{2})\Delta y_0+\frac{u(u-1)}{2!}\frac{[\Delta^2y_{-1}+\Delta^2y_0]}{2}+\frac{(u-\frac{1}{2})u(u-1)}{3!}\Delta^3y_{-1}+\frac{(u+1)u(u-1)(u-2)}{4!}\frac{[\Delta^4y_{-2}+\Delta^4y_{-1}]}{2}+\ldots \quad (1)$$

Replacing u by x in (1) we get

$$y_x=\frac{y_0+y_1}{2}+(x-\tfrac{1}{2})\Delta y_0+\frac{x(x-1)}{2!}\frac{[\Delta^2y_{-1}+\Delta^2y_0]}{2}+\frac{\left(x-\frac{1}{2}\right)x(x-1)}{3!}\Delta^3y_{-1}+\frac{(x+1)x(x-1)(x-2)}{4!}\frac{[\Delta^4y_{-2}+\Delta^4y_{-1}]}{2}$$

Assuming the fourth order and higher differences are very small, neglecting them and putting $x=\frac{1}{2}$ we get

$$y_{\frac{1}{2}}=\frac{y_0+y_1}{2}+(\tfrac{1}{2}-\tfrac{1}{2})\Delta y_0+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\frac{[\Delta^2y_{-1}+\Delta^2y_0]}{2}+\frac{(\frac{1}{2}-\frac{1}{2})\frac{1}{2}(\frac{1}{2}-1)}{3!}\Delta^3y_{-1}$$
$$=\frac{y_0+y_1}{2}-\frac{1}{16}[\Delta^2y_{-1}+\Delta^2y_0]$$

Now shifting the origin to x,

$$y_{x+\frac{1}{2}}=\frac{1}{2}[y_x+y_{x+1}]-\frac{1}{16}[\Delta^2y_{x-1}+\Delta^2y_x]$$ ■

Laplace-Everett Formula for Interpolation

Laplace-Everett's formula for interpolation is

$$y_u = v y_0 + \frac{v(v^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{v(v^2-1^2)(v^2-2^2)}{5!}\Delta^4 y_{-2} + \dots$$

$$+ u y_1 + \frac{u(u^2-1^2)}{3!}\Delta^2 y_0 + \frac{u(u^2-1^2)(u^2-2^2)}{5!}\Delta^4 y_{-1} + \dots$$

where $v = 1-u$.

Proof: Laplace-Everett's formula is obtained from Gauss's forward formula by replacing the odd differences interms of even differences

Gauss forward formula is

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-2} + \dots \qquad (1)$$

Now $\Delta y_0 = y_1 - y_0$, $\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}$, $\Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2}$ and so on

Substituting in (1), we get

$$y_u = y_0 + u(y_1 - y_0) + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\left[\Delta^2 y_0 - \Delta^2 y_{-1}\right]$$

$$+ \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}(\Delta^4 y_{-1} - \Delta^4 y_{-2}) + \dots$$

$$= (1-u)y_0 + uy_1 + \left[\frac{u(u-1)}{2!} - \frac{(u+1)u(u-1)}{3!}\right]\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^2 y_0$$

$$+ \left[\frac{(u+1)u(u-1)(u-2)}{4!} - \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\right]\Delta^4 y_{-2}$$

$$+ \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^4 y_{-1} + \dots$$

$$= (1-u)y_0 + uy_1 + \frac{u(u-1)}{3!}\left[3-(u+1)\right]\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^2 y_0$$

$$+ \frac{(u+1)u(u-1)(u-2)}{4!} - \left[5-(u+2)\right]\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^4 y_{-1} + \dots$$

$$= (1-u)y_0 + uy_1 + \frac{u(u-1)(u-2)}{3!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^2 y_0$$

$$- \frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^4 y_{-1} + \ldots$$

$$\therefore \quad y_u = (1-u)y_0 + uy_1 - \frac{u(u-1)(u-2)}{3!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\Delta^2 y_0$$

$$- \frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\Delta^4 y_{-2} + \frac{u(u^2-1^2)(u^2-2^2)}{5!}\Delta^4 y_{-1} + \ldots$$

This formula is usually written in a more convenient form by putting $u = 1 - v$ in the terms with negative sign.

$$y = (1-u)y_0 + uy_1 - \frac{(1-v)(1-v-1)(1-v-2)}{3!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\Delta^2 y_0$$

$$+ \frac{(1-v+1)(1-v)(1-v-1)(1-v-2)(1-v-3)}{5!}\Delta^4 y_{-2} + \frac{u(u^2-1^2)(u^2-2^2)}{5!}\Delta^4 y_{-1} + \ldots$$

$$= (1-u)y_0 + uy_1 - \frac{u(u^2-1^2)}{3!}\Delta^2 y_0 + \frac{u(u^2-1^2)(u^2-2^2)}{5!}\Delta^4 y_{-1}$$

$$+ \frac{(1-v)(-v)(-v-1)}{3!}\Delta^2 y_{-1} + \frac{(-v+2)(1-v)(-v)(-v-1)(-v-2)}{5!}\Delta^4 y_{-2} + \ldots$$

$$= (1-u)y_0 + uy_1 - \frac{u(u^2-1^2)}{3!}\Delta^2 y_0 + \frac{u(u^2-1^2)(u^2-2^2)}{5!}\Delta^4 y_{-1}$$

$$+ \frac{v(1-v^2)}{3!}\Delta^2 y_{-1} + \frac{v(v^2-1^2)(v^2-2^2)}{5!}\Delta^4 y_{-2} + \ldots$$

$$\therefore \quad y = vy_0 + \frac{v(v^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{v(v^2-1^2)(v^2-2^2)}{5!}\Delta^4 y_{-2} + \ldots$$

$$+ uy_1 + \frac{u(u^2-1^2)}{3!}\Delta^2 y_0 + \frac{u(u^2-1^2)(u^2-2^2)}{5!}\Delta^4 y_{-1} + \ldots$$

where $v = 1 - u$

Note:

(1) This formula involves the even differences on and below the central line.

(2) This formula is used when u lies between 0 and 1, but more accurate results will be obtained when $u \le \frac{3}{4}$ or when $v \ge \frac{1}{4}$ or vice-versa.

WORKED EXAMPLES

Example 1

From the following table find $f(34)$ using Everett's formula.

x	20	25	30	35	40
$f(x)$	11.4699	12.7834	13.7648	14.4982	15.0463

Solution

Everett's formula is

$$y_v = v y_0 + \frac{v(v^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{v(v^2-1^2)(v^2-2^2)}{5!}\Delta^4 y_{-2} + \ldots$$

$$+ u y_1 + \frac{u(u^2-1^2)}{3!}\Delta^2 y_0 + \frac{u(u^2-1^2)(u^2-2^2)}{5!}\Delta^4 y_{-1} + \ldots \quad (1)$$

where $v = 1-u$ and $u = \frac{x-x_0}{h}$ and x_0 is the origin.

Here $h=5$ and take $x_0 = 30$, since $x = 34$ lies between $x = 30$ and $x = 35$.

$\therefore$ $$u = \frac{x-30}{5}$$

When $x = 34$, $$u = \frac{34-30}{5} = \frac{4}{5} = 0.8$$

$\therefore$ $$v = 1-u = 1-0.8 = 0.2$$

We shall form the difference table.

x	$u=\frac{x-30}{5}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
20	–2	11.4699				
			1.3135			
25	–1	12.7834		–0.3321		
			0.9814		0.0841	
30	0	$\boxed{\mathbf{13.7648}}$		$\boxed{\mathbf{-0.248}}$		$\boxed{\mathbf{-0.0214}}$
			0.7334		0.0627	
35	1	$\boxed{\mathbf{14.4982}}$		$\boxed{\mathbf{-0.1853}}$		
			0.5481			
40	2	15.0463				

$$y_0 = 13.7648,\ y_1 = 14.4982,\ \Delta^2 y_{-1} = -0.248,\ \Delta^2 y_0 = -0.1863$$

$$\Delta^4 y_{-2} = -0.0214,\quad u = 0.8 \quad \text{and} \qquad v = 0.2$$

Substituting in (1), we get

$$\begin{aligned} y &= (0.2)(13.7648) + \frac{0.2\left[(0.2)^2 - 1^2\right]}{3!}(-0.248) \\ &\quad + \frac{(0.2)\left[(0.2)^2 - 1^2\right]\left[(0.2)^2 - 2^2\right]}{5!}(-0.0214) \\ &\quad + (0.8)(14.4982) + \frac{(0.8)\left[(0.8)^2 - 1^2\right]}{3!}[-0.1853] \\ &= 2.75296 + 0.007936 - 0.0001356 + 11.59856 + 0.0088944 \\ &= 14.3682148 \end{aligned}$$

$\therefore$ when $\boldsymbol{x = 34}$, $\boldsymbol{y = 14.3682}$ ■

Example 2

Apply Everett's formula to obtain y_{25} given $y_{20} = 2854$, $y_{24} = 3162$, $y_{28} = 3544$, $y_{32} = 3992$.

Solution

Everett's formula is

$$y = vy_0 + \frac{v(v^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{v(v^2-1^2)(v^2-2^2)}{5!}\Delta^4 y_{-2} + \dots$$
$$+uy_1 + \frac{u(u^2-1^2)}{3!}\Delta^2 y_0 + \frac{u(u^2-1^2)(u^2-2^2)}{5!}\Delta^4 y_{-1} + \dots$$

where $v = 1-u$, $u = \frac{x-x_0}{h}$ and x_0 is the origin.

The given values of x are 20, 24, 28, 32.
Required the value of y when $x = 25$

Here $h = 4$ and take $x_0 = 24$, since $x = 25$ lies between $x = 24$ and $x = 28$.

$\therefore$ $u = \frac{x-24}{4}$

When $x = 25$, $u = \frac{25-24}{4} = \frac{1}{4} = 0.25$

$\therefore$ $v = 1-u = 1-0.25 = 0.75$

We shall form the difference table.

x	$u = \frac{x-24}{4}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	−1	2854			
			308		
24	0	$\boxed{3162}$		$\boxed{74}$	
			382		−8
28	1	$\boxed{3544}$		$\boxed{66}$	
			448		
32	2	3992			

$$y_0 = 3162,\quad y_1 = 3544,\quad \Delta^2 y_{-1} = 74$$
$$\Delta^2 y_0 = 66,\quad u = 0.25 \text{ and } v = 0.75$$

Substituting in (1), we get

$$y = 0.75(3162) + \frac{(0.75)\left[(0.75)^2 - 1^2\right]}{3!}(74) + 0.25(3544) + \frac{(0.25)\left[(0.25)^2 - 1^2\right]}{3!}(66)$$
$$= 2371.5 - 4.04688 + 886 - 2.57813$$
$$= 3250.875$$

$\therefore$ when $\quad x = 25, \quad y = 3250.875$

Example 3

If the third differences of u_x are constant, show that

$$u_x = x u_1 + \frac{x(x^2-1)}{3!}\Delta^2 u_0 + y u_0 + \frac{y(y^2-1)}{3!}\Delta^2 u_{-1}$$

where $y = 1 - x$.

Solution

We shall prove the result using Everett's formula.
Since the third differences are constants the fourth and higher differences are zero.
Everett's formula is

$$y = v y_0 + \frac{v(v^2-1^2)}{3!}\Delta^2 y_{-1} + u y_1 + \frac{u(u^2-1^2)}{3!}\Delta^2 y_0$$

where $u = \frac{x - x_0}{h}$, and $v = 1 - u$.

Here $\quad y = u_x$ and $y = 1 - x \quad \therefore u = x \quad$ and $\quad v = y$
$\therefore$ the formula becomes

$$u_x = y u_0 + \frac{y(y^2-1)}{3!}\Delta^2 u_{-1} + x u_1 + \frac{x(x^2-1)}{3!}\Delta^2 u_0$$

Exercises 4.4

(1) Using Stirlings formula find $f(1.22)$ given

x	1	1.1	1.2	1.3	1.4
$f(x)$	0.841	0.891	0.932	0.963	0.985

(2) Using Sitrling's formula find u_x given

x	1	6	11	16	21
u_x	831	723	592	430	392

(3) Using Stirling's formula find y_{35} given $y_{20} = 512, y_{30} = 439, y_{46} = 346, y_{50} = 243$, where y_x represents the number of persons at age x years in a life table.

(4) Using Stirling's formula find the value $(1.02125)^{50}$ given that

x	1.01	1.015	1.02	1.025	1.03
x^{50}	1.64463	2.10524	2.69159	3.43711	4.38391

(5) Apply Stirling's formula to find the value of $f(5.44)$ given that

x	3.4	4.4	5.4	6.4	7.4	8.4
$f(x)$	1156	1936	2916	4096	5476	7056

(6) Using Bessel's formula to find the value of y when $x = 15$, given that

x	10	12	14	16	18	20
y	51.21	60.24	75.32	96.02	119.78	151.45

(7) Using Bessel's formula find $f(28)$ given

x	15	20	25	30	35	40
$f(x)$	10.3797	12.4622	14.0939	15.3725	16.3742	17.1591

(8) Use Bessel's formula to obtain y_{25} given $y_{20} = 2854, y_{24} = 3162, y_{28} = 3544, y_{32} = 3992$.

(9) Given

x	20	24	28	32
$f(x)$	24	32	35	40

Find $f(25)$ using Bessel's formula.

(10) If

x	30	35	40	45	50	55	60
u_x	771	862	1001	1224	1572	2123	2983

then find u_{42} by Everett's method.

(11) Use Everett's formula to obtain $f(1.15)$, given that $f(1) = 1, f(1.10) = 1.049$, $f(1.20) = 1.096, f(1.30) = 1.140$.

Answers 4.4

(1) 0.934 (2) 673.58 (3) 395 (4) 2.86155
(5) 2959.36 (6) 85.189 (7) 15.0818 (8) 3250.8715
(9) 32.945 (10) 1431.5 (11) 1.073

INTERPOLATION WITH UNEQUAL INTERVALS

Newton's forward and backward formulae can be applied when the arguments are equally spaced. When the arguments are unequally spaced, we use Lagrange's interpolation formula.

Lagrange's Interpolation Formula L/Q

Theorem 4.2

Let $y_0, y_1, y_2, \ldots, y_n$ be the entries corresponding to the arguments $x_0, x_1, x_2, \ldots, x_n$ which are not necessarily equally spaced, then Lagrange's interpolation formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}y_1$$

$$+\frac{(x-x_0)(x-x_1)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)\cdots(x_2-x_n)}y_2+\ldots+\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}y_n$$

Proof: Let $y_0, y_1, y_2, \ldots, y_n$ be the entries corresponding to the arguments $x_0, x_1, x_2, \ldots, x_n$ not necessarily equally spaced, then the interpolating polynomial $\phi(x)$ for $f(x)$ is of degree n and let

$$f(x) = a_0(x-x_1)(x-x_2)\ldots(x-x_n) + a_1(x-x_0)(x-x_2)\ldots(x-x_n)$$
$$+a_2(x-x_0)(x-x_1)\ldots(x-x_n)+\cdots+a_n(x-x_0)(x-x_1)\ldots(x-x_{n-1})$$

where $a_1, a_2, \ldots, a_n$ are constants to be determined such that

$$\phi(x_0) = y_0, \qquad \phi(x_1) = y_1, \qquad \phi(x_2) = y_2, \ldots, \qquad \phi(x_n) = y_n$$

When $x = x_0$

$$\phi(x_0) = a_0(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)$$

$\Rightarrow$
$$y_0 = a_0(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)$$

$\Rightarrow$
$$a_0 = \frac{y_0}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}$$

When $x = x_1$

$$\phi(x_1) = a_1(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)$$

$\Rightarrow$
$$y_1 = a_1(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)$$

$\Rightarrow$
$$a_1 = \frac{y_1}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}$$

Similarly, when $x = x_2, x_3, \ldots, x_n$, we get

$$a_2 = \frac{y_2}{(x_2 - x_0)(x_2 - x_1)\ldots(x_2 - x_n)}$$

$$a_3 = \frac{y_3}{(x_3 - x_0)(x_3 - x_1)\ldots(x_3 - x_n)}$$

$$\vdots$$

$$a_n = \frac{y_n}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})}$$

$$\therefore \quad \phi(x) = \frac{(x - x_1)(x - x_2)\cdots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2)\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} y_1$$
$$+ \frac{(x - x_0)(x - x_1)\ldots(x - x_n)}{(x_2 - x_0)(x_2 - x_1)\ldots(x_2 - x_n)} y_2 + \ldots + \frac{(x - x_0)(x - x_1)\ldots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})} y_n \qquad (1)$$

In the place of $\phi(x)$, we can write $f(x)$

$$\therefore \quad y = f(x) = \frac{(x - x_1)(x - x_2)\cdots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2)\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} y_1$$
$$+ \frac{(x - x_0)(x - x_1)\ldots(x - x_n)}{(x_2 - x_0)(x_2 - x_1)\ldots(x_2 - x_n)} y_2 + \ldots + \frac{(x - x_0)(x - x_1)\ldots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})} y_n$$

WORKED EXAMPLES

Example 1

Find $f(x)$ as a polynomial in x from the given data and find $f(8)$.

x	3	7	9	10
$f(x)$	168	120	172	63

Solution

Given the values of x and y are

$$x_0 = 3,\ x_1 = 7,\ x_2 = 9,\ x_3 = 10 \text{ and } y_0 = 168,\ y_1 = 120,\ y_2 = 72,\ y_3 = 63$$

The values of x are not equally spaced. So we use Lagrange's formula to find $y = f(x)$
Lagrange's formula for a set of four pairs of values is

$$\Rightarrow \quad y = f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} y_1$$
$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} y_3$$

$$=\frac{(x-7)(x-9)(x-10)}{(3-7)(3-9)(3-10)}\times 168+\frac{(x-3)(x-9)(x-10)}{(7-3)(7-9)(7-10)}\times 120$$

$$+\frac{(x-3)(x-7)(x-10)}{(9-3)(9-7)(9-10)}\times 72+\frac{(x-3)(x-7)(x-9)}{(10-3)(10-7)(10-9)}\times 63$$

$$=\frac{168\left[x^3-(7+9+10)x^2+(63+70+90)x-630\right]}{(-4)(-6)(-7)}$$

$$+\frac{120\left[x^3-(3+9+10)x^2+(27+30+90)x-270\right]}{4(-2)(-3)}$$

$$+\frac{72\left[x^3-(3+7+10)x^2+(21+30+70)x-210\right]}{6\times 2\times(-1)}$$

$$+\frac{63\left[x^3-(3+7+9)x^2+(21+27+63)x-189\right]}{7\times 3\times 1}$$

$$=-\left(x^3-26x^2+223x-630\right)+5\left(x^3-22x^2+147x-270\right)$$

$$-6\left(x^3-20x^2+121x-210\right)+3\left(x^3-19x^2+111x-189\right)$$

$$f(x)=x^3-21x^2+119x-27$$

$$f(8)=8^3-21\times 8^2+119\times 8-27$$

$$=512-1344+952-27=\mathbf{93}$$

Note: $(x-a)(x-b)(x-c)=x^3-(a+b+c)x^2+(ab+bc+ca)x-abc$

Example 2

Find the polynomial $f(x)$ by using Lagrange's formula and hence find $f(3)$ for the following values of x and y.

x	0	1	2	5
y	2	3	12	147

and hence find $f(3)$.

Solution

Given values of x are $x_0=0,\quad x_1=1,\quad x_2=2,\quad x_3=5$

and y are $y_0=2,\quad y_1=3,\quad y_2=12,\quad y_3=47$

The values of x are not equally spaced so we use Lagrange's formula to find $y=f(x)$
Lagrange's formula for a set of four pairs of values is

$$y=f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$=\frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}\times 2+\frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}\times 3$$

$$+\frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}\times 12+\frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}\times 147$$

$$=\frac{2\left[x^3-(1+2+5)x^2+(2+5+10)x-10\right]}{(-1)(-2)(-5)}$$

$$+\frac{3\left[x\left(x^2-7x+10\right)\right]}{1(-1)(-4)}+\frac{12\left[x\left(x^2-6x+5\right)\right]}{2.1(-3)}+\frac{147\left[x\left(x^2-3x+2\right)\right]}{5.4.3}$$

$$=-\frac{1}{5}\left[x^3-8x^2+17x-10\right]$$

$$+\frac{3}{4}\left[x^3-7x^2+10x\right]-2\left[x^3-6x^2+5x\right]$$

$$+\frac{147}{60}\left(x^3-3x^2+2x\right)$$

$$=\frac{1}{60}\left[-12x^3+96x^2-240x+120+45x^3-315x^2+450x-120x^3\right.$$

$$\left.+720x^2-600x+147x^3-441x^2+294x\right]$$

$$=\frac{1}{60}\left[60x^3+60x^2-60x+120\right]$$

$$\Rightarrow \qquad f(x)=x^3+x^2-x+2$$

$$f(3)=3^3+3^2-3+2=\mathbf{35}$$

Example 3

Using Lagrange's formula prove that

$$\boldsymbol{y_1=y_3-0.3\left(y_5-y_{-3}\right)+0.2\left(y_{-3}-y_{-5}\right).}$$

Solution

From the given equation, the values of x involved are $x_0=-5,\ x_1=-3,\ x_2=3,\ x_3=5$ and the corresponding y values are $y_0=y_{-5},\ y_1=y_{-3},\ y_2=y_3,\ y_3=y_5$

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The values of x are not equally spaced.

So, we use Lagrange's formula to find $y = f(x)$

Lagrange's formula for a set of four pairs of values is

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$= \frac{(x+3)(x-3)(x-5)}{(-5+3)(-5-3)(-5-5)}y_{-5} + \frac{(x+5)(x-3)(x-5)}{(-3+5)(-3-3)(-3-5)}y_{-3}$$

$$+\frac{(x+5)(x+3)(x-5)}{(3+5)(3+3)(3-5)}y_3 + \frac{(x+5)(x+3)(x-3)}{(5+5)(5+3)(5-3)}y_5$$

When $x = 1$, we get

$$y_1 = f(1) = \frac{(1+3)(1-3)(1-5)}{(-2)(-8)(-10)}y_{-5} + \frac{(1+5)(1-3)(1-5)}{2(-6)(-8)}y_{-3}$$

$$+\frac{(1+5)(1+3)(1-5)}{8\times 6\times(-2)}y_3 + \frac{(1+5)(1+3)(1-3)}{10\times 8\times 2}y_5$$

$$= -\frac{4(-2)(-4)}{2\times 8\times 10}y_{-5} + \frac{6(-2)(-4)}{2\times 6\times 8}y_{-3} + \frac{6\times 4\times(-4)}{8\times 6\times(-2)}y_3 + \frac{6\times 4\times(-2)}{10\times 8\times 2}y_5$$

$$= -0.2y_{-5} + 0.5y_{-3} + y_3 - 0.3y_5$$

$$\Rightarrow \quad \boldsymbol{y_1 = y_3 - 0.3(y_5 - y_{-3}) + 0.2(y_{-3} - y_{-5})}$$

Example 4

Given the values

x	5	7	11	13	17
$y = x\,f(x)$	150	392	1452	2366	5202

Evaluate $f(9)$ using Lagrange's formula.

Solution

Given the values of x and y are $x_0 = 5,\quad x_1 = 7,\quad x_2 = 11,\quad x_3 = 13,\quad x_4 = 17$

and $y_0 = 150,\ y_1 = 392,\ y_2 = 1452,\ y_3 = 2366,\ y_4 = 5202$

The values of x are not equally spaced, so we use Lagrange's formula to find $y = f(x)$. Lagrange's formula for a set of four pairs of values is

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}y_3$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}y_4$$

$$y = \frac{(x-7)(x-11)(x-13)(x-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 + \frac{(x-5)(x-11)(x-13)(x-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392$$

$$+ \frac{(x-5)(x-7)(x-13)(x-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452 + \frac{(x-5)(x-7)(x-11)(x-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366$$

$$+ \frac{(x-5)(x-7)(x-11)(x-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202$$

When $x = 9$,

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(-2)(-6)(-8)(-12)} \times 150 + \frac{(9-5)(9-11)(9-13)(9-17)}{2(-4)(-6)(-10)} \times 392$$

$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{6 \times 4 \times (-2)(-6)} \times 1452 + \frac{(9-5)(9-7)(9-11)(9-17)}{8 \times 6 \times 2 \times (-4)} \times 2366$$

$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{12 \times 10 \times 6 \times 4} \times 5202$$

$$= \frac{2 \times (-2)(-4)(-8)}{2 \times 6 \times 8 \times 12} \times 150 - \frac{4 \times (-2)(-4)(-8)}{2 \times 4 \times 6 \times 10} \times 392$$

$$+ \frac{4 \times 2 \times (-4)(-8)}{6 \times 4 \times 2 \times 6} \times 1452 - \frac{4 \times 2(-2)(-8)}{8 \times 6 \times 2 \times 4} \times 2366 + \frac{4 \times 2 \times (-2)(-4)}{12 \times 10 \times 6 \times 4} \times 5202$$

$$= -16.67 + 209.0667 + 1290.67 - 788.67 + 115.6$$

$$= 809.9967$$

Exercises 4.5

(1) Using Lagrange's formula fit a polynomial to the data and find $f(5)$.

x	-1	1	2
$f(x)$	7	5	15

(2) Using Lagrange's formula find $y\ (10)$ given that $y\ (5) = 12,\ y\ (6) = 13,\ y\ (9) = 14,$ $y\,(11) = 16.$

(3) Using Lagrange's interpolation formula calculate the profit in the year 2000 from the following data:

Year	1997	1999	2001	2002
Profit in Lakhs Rs.	43	65	159	248

(4) Apply Lagrange's formula to find $f(5)$ and $f(6)$ given that $f(1) = 2,\ f(2) = 4,\ f(3) = 8,$ $f(4) = 16,$ and $f(7) = 128$

(5) Find the form of the function u_x given that $u_0 = 9,\ u_1 = 12,\ u_4 = 69,\ u_5 = 124$

(6) Use Lagrange's formula to find $f(5)$ from the following data:

x	2	3	4	6	7
$f(x)$	1	5	13	61	125

(7) Find the form of u_x, give that $u_0 = 8,\ u_1 = 11,\ u_4 = 13,\ u_5 = 123,$

$$u_x = \frac{1}{60}\left[10x^3 + 240x^2 - 70x + 480\right]$$

(8) Find u_5 given that $u_1 = 4,\ u_2 = 7,\ u_4 = 13$ and $u_7 = 30$

(9) The following table gives the premium payable at the ages in years completed. Interpolate the premium payable at the age of 35 completed.

Age completed in years:	25	30	40	60
Premium in Rs.	50	55	70	95

(10) From the following table find the value of y when $x = 10$

x	5	6	9	11
y	12	13	14	16

Answers 4.5

(1) $f(x) = \frac{1}{3}[11x^2 - 3x + 7],\quad f(5) = 87$ 89

(2) $y(10) = 14.666$

(3) Rs 100 lakhs

(4) $f(5) = 39.43, f(6) = 66.5$

(5) $u_x = x^3 - x^2 + 3x + 9$

(6) $f(5) = 28.6$

(7) $u_x = \frac{1}{6}(x^3 + 24x^2 - 7x + 48)$

(8) 17.06

(9) 61.5

(10) 14.67

Divided Differences

In forward and backward differences for equally spaced arguments we considered only differences of the entries. But in divided differences, we divide this difference by difference of the corresponding arguments.

Let a function $y = f(x)$ take values $f(x_0), f(x_1), f(x_2), \ldots, f(x_n)$ corresponding to the arguments $x_0, x_1, x_2, \ldots, x_n$, not necessarily equally spaced.

The first divided difference for the arguments x_0, x_1 is defined as $\dfrac{f(x_1)-f(x_0)}{x_1-x_0}$.

It is denoted by $f(x_0,x_1)$ or $\underset{x_1}{\Delta} f(x_0)$ or $[x_0,x_1]$

We shall denote,

$$\underset{x_1}{\Delta} f(x_0)=\frac{f(x_1)-f(x_0)}{x_1-x_0}$$

Similarly,

$$\underset{x_2}{\Delta} f(x_1)=\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

$$\underset{x_3}{\Delta} f(x_2)=\frac{f(x_3)-f(x_2)}{x_3-x_2}$$

$$\vdots$$

and

$$\underset{x_n}{\Delta} f(x_{n-1})=\frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}$$

The second divided difference for the arguments x_0, x_1, x_2 is defined as

$$\underset{x_1x_2}{\Delta^2} f(x_0)=\frac{\underset{x_2}{\Delta} f(x_1)-\underset{x_1}{\Delta} f(x_0)}{x_2-x_0}$$

The second difference for x_1, x_2, x_3 is

$$\underset{x_2x_3}{\Delta^2} f(x_1)=\frac{\underset{x_3}{\Delta} f(x_2)-\underset{x_2}{\Delta} f(x_1)}{x_3-x_1}$$

$$\vdots$$

The third divided difference for x_0, x_1, x_2, x_3 is

$$\underset{x_1x_2x_3}{\Delta^3} f(x_0)=\frac{\underset{x_2x_3}{\Delta^2} f(x_1)-\underset{x_1x_2}{\Delta^2} f(x_0)}{x_3-x_0}$$

The third difference for x_1, x_2, x_3, x_4 is

$$\underset{x_2x_3x_4}{\Delta^3} f(x_1)=\frac{\underset{x_3x_4}{\Delta^2} f(x_2)-\underset{x_2x_3}{\Delta^2} f(x_1)}{x_4-x_1} \text{ and so on.}$$

Properties of divided differences

1. The divided differences are symmetric functions of their arguments.
   For example

$$\underset{x_1}{\Delta} f(x_0) = \underset{x_0}{\Delta} f(x_1)$$

$$\underset{x_1 x_2}{\Delta^2} f(x_0) = \underset{x_0 x_2}{\Delta^2} f(x_1) = \underset{x_0 x_1}{\Delta^2} f(x_2)$$

2. The n^{th} divided differences of a polynomial of degree n are constants.
3. $\Delta[f(x)+g(x)] = \Delta f(x) + \Delta g(x)$

   and $\Delta[cf(x)] = c\Delta f(x)$, c is constant.
   This shows that Δ is a linear operator.
4. The n^{th} divided difference is a quotient of two determinants, each of order $n+1$.
   We have

$$\underset{x_1}{\Delta} f(x_0) = \frac{f(x_1)-f(x_0)}{x_1 - x_0}$$

$$= \frac{\begin{vmatrix} f(x_1) & f(x_0) \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & x_0 \\ 1 & 1 \end{vmatrix}}$$

$\therefore \underset{x_1}{\Delta} f(x_0)$ is a quotient of two determinant of same order 2.

Similarly we can write the other divided differences as a quotient of determinants of the same order.

WORKED EXAMPLES

Example 1

For the function $\frac{1}{x}$, prove that the third divided difference with arguments a, b, c, d is equal to $-\frac{1}{abcd}$.

Solution

Given $\qquad f(x) = \frac{1}{x}$

$$\underset{b}{\Delta} f(a) = \frac{f(b)-f(a)}{b-a} = \frac{\frac{1}{b}-\frac{1}{a}}{b-a} = -\frac{1}{ab}$$

Similarly $\qquad \underset{c}{\Delta} f(b) = -\frac{1}{bc}, \underset{d}{\Delta} f(c) = -\frac{1}{cd}$

Now $\qquad \underset{bc}{\Delta^2} f(a) = \frac{\underset{c}{\Delta} f(b) - \underset{b}{\Delta} f(a)}{c-a} = \frac{-\frac{1}{bc}+\frac{1}{ab}}{c-a} = \frac{-a+c}{abc(c-a)} = \frac{1}{abc}$

$\therefore$

$$\underset{cd}{\Delta^2} f(b) = \frac{1}{bcd}$$

$$\underset{bcd}{\Delta^3} f(a) = \frac{\underset{cd}{\Delta^2} f(b) - \underset{bc}{\Delta^2} f(a)}{d-a} = \frac{\frac{1}{bcd} - \frac{1}{abc}}{d-a} = \frac{a-d}{abcd(d-a)} = -\frac{1}{abcd}$$

Example 2

Prove that $\underset{yz}{\Delta^2} x^3 = x+y+z.$

Solution

Given the function $f(x) = x^3$ and the arguments are x, y, z

$\therefore$

$$\underset{y}{\Delta} f(x) = \frac{f(y)-f(x)}{y-x} = \frac{y^3-x^3}{y-x} = \frac{(y-x)(x^2+xy+y^2)}{y-x} = x^2+xy+y^2$$

Similarly,

$$\underset{z}{\Delta} f(y) = y^2+yz+z^2$$

Now

$$\underset{yz}{\Delta^2} f(x) = \frac{\underset{z}{\Delta} f(y) - \underset{y}{\Delta} f(x)}{z-x} = \frac{y^2+yz+z^2-(x^2+xy+y^2)}{z-x} = \frac{z^2-x^2+yz-xy}{z-x}$$

$$= \frac{(z+x)(z-x)+y(z-x)}{z-x} = x+y+z$$

Divided difference table

A divided difference table can be constructed using same principle as for ordinary difference tables. It is a diagonal difference table as illustrated by the following example.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	2			
		$\frac{3-2}{1-0}=1$		
1	3		$\frac{9-1}{2-0}=4$	
		$\frac{12-3}{2-1}=9$		$\frac{9-4}{5-0}=1$
2	12		$\frac{45-9}{5-1}=9$	
		$\frac{147-12}{5-2}=45$		
5	147			

Newton's General Interpolation Formula or Newton's Divided Difference Formula for Interpolation

Let $f(x_0), f(x_1), \ldots, f(x_n)$ be the given values of the function $f(x)$ corresponding to unequally spaced arguments $x_0, x_1, \ldots, x_n$.
Newton's general interpolation formula is

$$\begin{aligned} f(x) = f(x_0) &+ (x-x_0)\underset{x_1}{\Delta} f(x_0) + (x-x_0)(x-x_1)\underset{x_1x_2}{\Delta^2} f(x_0) \\ &+ (x-x_0)(x-x_1)(x-x_2)\underset{x_1x_2x_3}{\Delta^3} f(x_0) + \ldots \\ &+ (x-x_0)(x-x_1)\cdots(x-x_{n-1})\underset{x_1x_2,\ldots,x_n}{\Delta^n} f(x_0) \end{aligned}$$

Proof: Corresponding to the given unequally arguments $x_0, x_1, \ldots, x_n$, the values of the function $f(x)$ are $f(x_0), f(x_1), \ldots, f(x_n)$

Let x be the argument for which the value is required.
Then for the arguments x_0, x

$$\underset{x_0}{\Delta} f(x) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\therefore \quad f(x) = f(x_0) + (x-x_0)\underset{x_0}{\Delta} f(x) \tag{1}$$

Again for the arguments x_0, x_1, x

$$\underset{x_0x_1}{\Delta^2} f(x) = \frac{\underset{x_0}{\Delta} f(x) - \underset{x_1}{\Delta} f(x_0)}{x - x_1}$$

$$\therefore \quad \underset{x_0}{\Delta} f(x) = \underset{x_1}{\Delta} f(x_0) + (x-x)\underset{x_0x_1}{\Delta^2} f(x)$$

Substituting in (1), we get

$$\begin{aligned} f(x) &= f(x_0) + (x-x_0)\left[\underset{x_1}{\Delta} f(x_0) + (x-x_1)\underset{x_0x_1}{\Delta^2} f(x)\right] \\ &= f(x_0) + (x-x_0)\underset{x_1}{\Delta} f(x_0) + (x-x_0)(x-x_1)\underset{x_0x_1}{\Delta^2} f(x) \end{aligned} \tag{2}$$

Now for the arguments x_0, x_1, x_2, x

$$\underset{x_0x_1x_2}{\Delta^3} f(x) = \frac{\underset{x_0x_1}{\Delta^2} f(x) - \underset{x_1x_2}{\Delta^2} f(x_0)}{x - x_2}$$

$$\Rightarrow \quad \underset{x_0x_1}{\Delta^2} f(x) = \underset{x_1x_2}{\Delta^2} f(x_0) + (x-x_2)\underset{x_0x_1x_2}{\Delta^3} f(x)$$

Substituting in (2) we get,

$$f(x) = f(x_0) + (x - x_0) \underset{x_1}{\Delta} f(x_0)$$
$$+ (x - x_0)(x - x_1)\left[\underset{x_1 x_2}{\Delta^2} f(x_0) + (x - x_2) \underset{x_1 x_2 x_3}{\Delta^3} f(x)\right]$$
$$= f(x_0) + (x - x_0) \underset{x_1}{\Delta} f(x_0) + (x - x_0)(x - x_1) \underset{x_1 x_2}{\Delta^2} f(x_0)$$
$$+ (x - x_0)(x - x_1)(x - x_2) \underset{x_0 x_1 x_2}{\Delta^3} f(x) \qquad (3)$$

Proceeding in this way we obtain

$$f(x) = f(x_0) + (x - x_0) \underset{x_1}{\Delta} f(x_0) + (x - x_0)(x - x_1) \underset{x_1 x_2}{\Delta^2} f(x_0)$$
$$+ (x - x_0)(x - x_1)(x - x_2) \underset{x_1 x_2 x_3}{\Delta^3} f(x_0) + \ldots$$
$$+ (x - x_0)(x - x_1) \ldots (x - x_{n-1}) \underset{x_1 x_2 \ldots x_n}{\Delta^n} f(x_0) + R_n(x)$$

where $R_n(x)$ is the remainder term.

$$R_n(x) = (x - x_0)(x - x_1) \ldots (x - x_n) \underset{x_0 x_1 \ldots x_n}{\Delta^{n+1}} f(x)$$

Let the interpolating polynomial be $\phi(x)$

If we put $$\phi(x) = f(x_0) + (x - x_0) \underset{x_1}{\Delta} f(x_0) + (x - x_0)(x - x_1) \underset{x_1 x_2}{\Delta^2} f(x_0)$$
$$+ (x - x_0)(x - x_1) \ldots (x - x_{n-1}) \underset{x_1 x_2 \ldots x_n}{\Delta^n} f(x_0)$$

then $$f(x) = \phi(x) + R_n(x)$$

We see that $f(x_i) = \phi(x_i) \forall i = 0, 1, 2, \cdots n$, because $R_n(x_i) = 0$

Hence the interpolation formula is $\phi(x)$ or

$$f(x) = f(x_0) + (x - x_0) \underset{x_1}{\Delta} f(x_0) + (x - x_0)(x - x_1) \underset{x_1 x_2}{\Delta^2} f(x_0)$$
$$+ (x - x_0)(x - x_1) \ldots (x - x_{n-1}) \underset{x_1 x_2 \ldots x_n}{\Delta^n} f(x_0)$$ ■

Note: Using the relation between divided differences and ordinary differences from the general formula, we can deduce Newton's forward and backward difference formulae for interpolation.

WORKED EXAMPLES

Example 1

Construct the divided difference table for the following data and find the value of $f(2)$.

x	4	5	7	10	11	12
$f(x)$	50	102	296	800	1010	1224

Solution

Given value of x are $x_0 = 4$, $x_1 = 5$, $x_2 = 7$, $x_3 = 10$, $x_4 = 11$, $x_5 = 12$

The values of x are unequally spaced, so we use Newton's divided difference formula to find $f(x)$ when $x = 2$.

Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0)\underset{x_1}{\Delta} f(x_0) + (x - x_0)(x - x_1)\underset{x_1 x_2}{\Delta^2} f(x_0)$$
$$+ (x - x_0)(x - x_1)(x - x_2)\underset{x_1 x_2 x_3}{\Delta^3} f(x_0) + \ldots \tag{1}$$

We form the divided difference table.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
4	50					
		$\frac{102-50}{5-4} = 52$				
5	102		$\frac{97-52}{7-4} = 15$			
		$\frac{296-102}{7-5} = 97$		$\frac{14.2-15}{10-4} = -0.133$		
7	296		$\frac{168-97}{10-5} = 14.2$		$\frac{-0.617+0.133}{11-4} = -0.069$	
		$\frac{800-296}{10-7} = 168$		$\frac{10.5-14.2}{11-5} = -0.617$		$\frac{-0.158+0.069}{12-4} = -0.011$
10	800		$\frac{210+168}{11-7} = 10.5$		$\frac{-1.7+0.617}{12-5} = -0.158$	
		$\frac{1010-800}{11-10} = 210$		$\frac{2-10.5}{12-7} = -1.7$		
11	1010		$\frac{214-210}{12-10} = 2$			
		$\frac{1224-1010}{12-11} = 214$				
12	1224					

(1) becomes

$$f(x) = 50 + (x-4)52 + (x-4)(x-5)15 + (x-4)(x-5)(x-7)(-0.133)$$

$$+ (x-4)(x-5)(x-7)(x-10)(-0.069) + (x-4)(x-5)(x-7)(x-10)(x-11)(-0.011)$$

when $x = 2$,

$$f(2) = 50 + (2-4)52 + (2-4)(2-5)15 + (2-4)(2-5)(2-7)(-0.133)$$

$$+ (2-4)(2-5)(2-7)(2-10)(-0.069) + (2-4)(2-5)(2-7)(2-10)(2-11)(-0.011)$$

$$= 50 - 102 + 90 + 3.99 - 16.56 + 23.76 = \mathbf{49.19}$$

■

Example 2

By using Newton's divided difference formula find $f(8)$, given

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Also find $f(6)$, $f(9)$, $f(15)$.

Solution

Given the values of x are $x_0 = 4,\ x_1 = 5,\ x_2 = 7,\ x_3 = 10,\ x_4 = 11,\ x_5 = 13$

The values of x are unequally spaced, so, we use Newton's divided difference formula to find $f(x)$ when $x = 8$.

Newton's divided difference formula is

$$f(x) = f(x_0) + \underset{x_1}{\Delta} f(x_0) + (x-x_0)(x-x_1)\underset{x_1,x_2}{\Delta^2} f(x_0)$$

$$+ (x-x_0)(x-x_1)(x-x_2)\underset{x_1,x_2,x_3}{\Delta^3} f(x_0) + \dots$$

$$+ (x-x_0)(x-x_1)\dots(x-x_{n-1})\underset{x_1,x_2,x_3\dots x_n}{\Delta^n} f(x_0) \qquad (1)$$

We form the divided difference table.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
		$\dfrac{100-48}{5-4}=52$			
5	100		$\dfrac{97-52}{7-4}=15$		
		$\dfrac{294-100}{7-5}=97$		$\dfrac{21-15}{10-4}=1$	
7	294		$\dfrac{202-97}{10-5}=21$		$\dfrac{1-1}{11-4}=0$
		$\dfrac{900-294}{10-7}=202$		$\dfrac{27-21}{11-5}=1$	
10	900		$\dfrac{310-202}{11-7}=27$		$\dfrac{1-1}{13-5}=0$
		$\dfrac{1210-900}{11-10}=310$		$\dfrac{33-27}{13-7}=1$	
11	1210		$\dfrac{409-310}{13-10}=33$		
		$\dfrac{2028-1210}{13-11}=409$			
12	2028				

(1) becomes

$$f(x) = 48 + (x-4)\times 52 + (x-4)(x-5)\times 15 + (x-4)(x-5)(x-7)\times 1$$

[$\because$ all higher order differences are zero]

When $x = 8$, $f(8) = 48 + (8-4)\times 52 + (8-4)(8-5)\times 15 + (8-4)(8-5)(8-7)$

$= 48 + 208 + 180 + 12 = \mathbf{448}$

When $x = 6$, $f(6) = 48 + (6-4)\times 52 + (6-4)(6-5)\times 15 + (6-4)(6-5)(6-7)$

$= 48 + 104 + 30 - 2 = \mathbf{180}$

When $x = 9$, $f(9) = 48 + (9-4)\times 52 + (9-4)(9-5)\times 15 + (9-4)(9-5)(9-7)$

$= 48 + 260 + 300 + 40 = \mathbf{648}$

When $x = 15$

$$x = 15,\ f(15) = 48 + (15-4)\times 52 + (15-4)(15-5)\times 15 + (15-4)(15-5)(15-7)$$
$$= 48 + 572 + 1650 + 880 = \mathbf{3150}$$

■

Example 3

Given the data

x	0	1	2	5
$f(x)$	2	3	12	147

Find the cubic function of x, using Newton's divided difference formula, and hence find $f(2)$.

Solution

Given $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 5$

The values of x are unequally spaced. We use Newton's divided difference formula to find $f(x)$

Newton's divided difference formula is

$$f(x) = f(x_0) + (x-x_0)\underset{x_1}{\Delta} f(x_0) + (x-x_0)(x-x_1)\underset{x_1x_2}{\Delta^2} f(x_0)$$
$$+ (x-x_0)(x-x_1)(x-x_2)\underset{x_1x_2x_3}{\Delta^3} f(x_0) + \ldots$$
$$+ (x-x_0)(x-x_1)\ldots(x-x_{n-1})\underset{x_1x_2x_3\ldots x_n}{\Delta^n} f(x_0) \quad (1)$$

We form the divided difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	2			
		$\frac{3-2}{1-0} = 1$		
1	3		$\frac{9-1}{2-0} = 4$	
		$\frac{12-3}{2-1} = 9$		$\frac{9-4}{5-0} = 1$
2	12		$\frac{45-9}{5-1} = 9$	
		$\frac{147-12}{5-2} = 45$		
5	147			

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$\therefore$ (1) becomes

$$f(x) = 2 + (x-0)\times 1 + (x-0)(x-1)\times 4 + (x-0)(x-1)(x-2).1$$
$$= 2 + x + 4x(x-1) + x(x-1)(x-2)$$
$$= 2 + x + 4x^2 - 4x + x(x^2 - 3x + 2)$$
$$= 2 + x + 4x^2 - 4x + x^3 - 3x^2 + 2x$$
$$= x^3 + x^2 - x + 2$$

When $x = 2$, $f(2) = 2^3 + 2^2 - 2 + 2 = 8 + 4 = \mathbf{12}$ ∎

Example 4

Find the third divided difference with arguments 2, 4, 9, 10 of the function $f(x) = x^3 - 2x$.

Solution

Given the values of x are $x_0 = 2$, $x_1 = 4$, $x_2 = 9$, $x_3 = 10$

and $f(x) = x^3 - 2x$

$$f(2) = 2^3 - 2\times 2 = 8 - 4 = 4$$
$$f(4) = 4^3 - 2\times 4 = 64 - 8 = 56$$
$$f(9) = 9^3 - 2\times 9 = 729 - 18 = 711$$
$$f(10) = 10^3 - 2\times 10 = 1000 - 20 = 980$$

We form the divided difference table to find the divided differences.

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
2	**4**			
		$\frac{\mathbf{56-4}}{\mathbf{4-2}} = \mathbf{26}$		
4	56		$\frac{\mathbf{131-26}}{\mathbf{9-2}} = \mathbf{15}$	
		$\frac{711-56}{9-4} = 131$		$\frac{\mathbf{23-15}}{\mathbf{10-2}} = \mathbf{1}$
9	711		$\frac{269-131}{10-4} = 23$	
		$\frac{980-711}{10-9} = 269$		
10	980			

The third divided difference is $\Delta^3 f(x) = 1$ ∎

Example 5

Given the following data, find $f(x)$ as a polynomial in x.

x	0	2	3	4	7	9
$f(x)$	4	26	58	112	466	922

Solution

Given $x_0 = 0,\ x_1 = 2,\ x_2 = 3,\ x_3 = 4,\ x_4 = 7,\ x_5 = 9$

The values of x are unequally spaced. So, we find $f(x)$ using Newton's divided difference formula.

$$\begin{aligned} f(x) &= f(x_0) + (x-x_0)\underset{x_1}{\Delta} f(x_0) + (x-x_0)(x-x_1)\underset{x_1x_2}{\Delta^2} f(x_0) \\ &+ (x-x_0)(x-x_1)(x-x_2)\underset{x_1x_2x_3}{\Delta^3} f(x_0) + \ldots \\ &+ (x-x_0)(x-x_1)\ldots(x-x_{n-1})\underset{x_1x_2x_3\ldots x_n}{\Delta^n} f(x_0) \end{aligned} \tag{1}$$

We form the divided difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	4				
		$\frac{26-4}{2-0} = 11$			
2	26		$\frac{32-11}{3-0} = 7$		
		$\frac{58-26}{3-2} = 32$		$\frac{11-7}{4-0} = 1$	
3	58		$\frac{54-32}{4-2} = 11$		0
		$\frac{112-58}{4-3} = 54$		$\frac{16-11}{7-2} = 1$	
4	112		$\frac{118-54}{7-3} = 16$		0
		$\frac{466-112}{7-4} = 118$		$\frac{22-16}{9-3} = 1$	
7	466		$\frac{228-118}{9-4} = 22$		
		$\frac{922-466}{9-7} = 228$			
9	922				

$\therefore$ (1) becomes $\quad f(x) = 4 + (x-0)11 + (x-0)(x-2).7 + (x-0)(x-2)(x-3).1$

[$\because$ all higher powers are zero]

$$= 4 + 11x + 7x^2 - 14x + x(x-2)(x-3)$$

$$= 4 + 11x + 7x^2 - 14x + x\left(x^2 - 5x + 6\right) = x^3 + 2x^2 + 3x + 4 \qquad \blacksquare$$

Example 6

If $f(0) = 0$, $f(1) = 0$, $f(2) = -12$, $f(4) = 0$, $f(5) = 600$, $f(7) = 7308$, find a polynomial that satisfies this data using Newton's divided difference formula. Hence find $f(6)$.

Solution

Given the values of $f(x)$ are

$$f(0) = 0,\ f(1) = 0,\ f(2) = -12;\ f(4) = 0,\ f(5) = 600,\ f(7) = 7308$$

$$\therefore \quad x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 4,\ x_4 = 5,\ x_5 = 7$$

The values of x are unequally spaced. We use Newton's divided difference formula to find $f(x)$. We form the divided difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0	0					
		$\frac{0-0}{1-0} = 0$				
1	0		$\frac{-12-0}{2-0} = -6$			
		$\frac{-12-0}{2-1} = -12$		$\frac{6+6}{4-0} = 3$		
2	-12		$\frac{6+12}{4-1} = 6$		$\frac{48-3}{5-0} = 9$	
		$\frac{0+12}{4-2} = 6$		$\frac{198-6}{5-1} = 48$		$\frac{16-9}{7-0} = 1$
4	0		$\frac{600-6}{5-2} = 198$		$\frac{144-48}{7-1} = 16$	
		$\frac{600-0}{5-4} = 600$		$\frac{918-198}{7-2} = 144$		
5	600		$\frac{3354-600}{7-4} = 918$			
		$\frac{7308-600}{7-5} = 3354$				
7	7308					

$\therefore$ Newton's divided difference formula is

$$f(x)=f(x_0)+(x-x_0)\underset{x_1}{\Delta}f(x_0)+(x-x_0)(x-x_1)\underset{x_1x_2}{\Delta^2}f(x_0)$$
$$+(x-x_0)(x-x_1)(x-x_2)\underset{x_1x_2x_3}{\Delta^3}f(x_0)+\ldots$$
$$+(x-x_0)(x-x_1)\ldots(x-x_{n-1})\underset{x_1x_2x_3\ldots x_n}{\Delta^n}f(x_0) \quad (1)$$

$\therefore$
$$f(x)=0+(x-0)\times 0+(x-0)(x-1)\times(-6)$$
$$+(x-0)(x-1)(x-2)\times 3+(x-0)(x-1)(x-2)(x-4)\times 9$$
$$+(x-0)(x-1)(x-2)(x-4)(x-5)\times 1$$
$$=-6x(x-1)+3x(x-1)(x-2)+9x(x-1)(x-2)(x-4)$$
$$+x(x-1)(x-2)(x-4)(x-5)$$
$$=x(x-1)[-6+3(x-2)+9(x-2)(x-4)+(x-2)(x-4)(x-5)] \text{ when}$$
$$=(x^2-x)[-6+3x-6+9(x^2-6x+8)+x^3-11x^2+38x-40]$$
$$=(x^2-x)[x^3-2x^2-13x+20]$$
$$=x^5-2x^4-13x^3+20x^2-x^4+2x^3+13x^2-20x$$
$$=x^5-3x^4-11x^3+33x^2-20x$$

When $x=6$, $f(6)=6^5-3\times 6^4-11\times 6^3+33\times 6^2-20\times 6$
$$=7776-3888-2376+1188-120=2580$$

Aliter:

Given $f(0)=0, f(1)=0, f(2)=-12, f(4)=0$
$f(5)=600, f(7)=7308.$

Since 6 values of $f(x)$ are given, $f(x)$ is polynomial of degree 5.

Since $f(0)=0$, $f(1)=0$, $f(4)=9$,

$f(x)=0$ when $x=0$, $x=1$ and $x=4$.

So, by factor theorem $x-0$, $x-1$, $x-4$ are factors of $f(x)$.

$\therefore f(x)=x(x-1)(x-4)g(x)$, where $g(x)$ is a polynomial of degree 2.

We shall find this polynomial $g(x)$ by using the other three values

$f(2)=-12, f(15)=600, f(7)=7308.$

When $x=2, f(2)=2(2-1)(2-4)g(2) \quad \Rightarrow -4g(2)=-12 \quad \Rightarrow g(2)=3$

When $x=5, f(5)=5(5-1)(5-4)g(5) \quad \Rightarrow 20g(5)=600 \quad \Rightarrow g(5)=30$

When $x=7, f(7)=7(7-1)(7-4)g(7) \quad \Rightarrow 7.6.3g(7)=7308 \quad \Rightarrow g(7)=58$

We shall find $g(x)$ by using Newton's divided difference formula

	x	$g(x)$	$\Delta g(x)$	$\Delta^2 g(x)$
x_0	2	3		
			$\frac{30-3}{5-2}=9$	
x_1	5	30		$\frac{14-9}{7-2}=1$
			$\frac{50-30}{7-5}=14$	
x_2	7	58		

By Newton's divided difference formula

$$g(x)=g(x_0)+(x-x_0)\underset{x_1}{\Delta} g(x_0)+(x-x_0)(x-x_1)\underset{x_1x_2}{\Delta^2} g(x_0)$$
$$=3+(x-2)\times 9+(x-2)(x-5)\times 1$$
$$=3+9x-18+x^2-7x+10$$
$$=x^2+2x-5$$

$\therefore$ the polynomial $f(x)=x(x-1)(x-4)(x^2+2x-5)$

Now

$$f(6)=6(6-1)(6-4)(6^2+2\times 6-5)=60(43)=2580$$ ■

Exercises 4.6

(1) Given the following data:

x	3	7	9	10
$f(x)$	168	120	72	63

Calculate $f(6)$.

(2) Using Newton's divided difference formula, find $u(3)$ given $u(1)=-26, u(2)=12, u(4)=256, u(6)=844$.

(3) Find $f(10)$ using Newton's divided difference formula given that

x	11	17	21	23
$f(x)$	14646	83526	194486	279845

(4) Find the function $f(x)$ from the following table:

x	0	1	4	5
$f(x)$	8	11	78	123

(5) Given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$ and $\log_{10} 661 = 2.8202$, find by divided difference the value of $\log_{10} 656$.

(6) Find the equation of the curve passing through the points (–1, –21), (1, 15), (2, 12), (3, 3). Find also $f(0)$.

(7) Find the cubic function from the following table:

x	0	1	3	4
$f(x)$	1	4	40	85

(8) Given $y_0 = -12$, $y_1 = 0$, $y_3 = 6$, $y_4 = 12$, find y_2 using Newton's divided difference formula.

Answers 4.6

(1) 147 (2) 100 (3) 130198

(4) $x^3 - x^2 + 3x + 8$ (5) 2.8168 (6) $f(x) = x^3 - 9x^2 + 17x + 6$; $f(0) = 6$

(7) $f(x) = x^3 + x^2 + x + 1$ (8) $y_2 = 4$

ERRORS IN INTERPOLATION FORMULAE

We state some results without proof

1. **Rolle's Theorem:** If the real function f is (i) Continuous on the closed interval $[a, b]$ (ii) differentiable in the open interval (a, b) and (iii) $f(a) = f(b)$, then there exists at least one point c in (a, b) such that $f'(c) = 0$

Corollary: If $f(x)$ is a polynomial of degree n and if $f(a) = f(b) = 0$, then there is at least one $c \in (a, b)$ such that $f'(c) = 0$

From this it follows that between two roots of $f(x) = 0$, there is a root of $f'(x) = 0$.

In other words, the real roots of $f'(x) = 0$ separate the real roots of the equation $f(x) = 0$.

More generally, if $f(x)$ is a polynomial of degree n with a and b as the smallest and largest roots of $f(x) = 0$, then

$$f'(x) = 0 \text{ has } (n-1) \text{ roots in } (a, b)$$

$$f''(x) = 0 \text{ has } (n-2) \text{ roots in } (a, b)$$

$$\vdots$$

$$f^{n-1}(x) = 0 \text{ has at least one root in } (a, b).$$

9.5.1 Remainder Term in Interpolation Formulae

Let $f(x)$ be a function defined at $(n+1)$ points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ and let $f(x)$ be continuous and it has continuous derivatives of all orders.

Let $p(x)$ be a polynomial of degree not exceeding n such that $p(x_i) = y_i$, $i = 0, 1, 2, \ldots, n$, be an approximation for $f(x)$

Let $f(x) - p(x) = g(x)$ (1)

Since $f(x_i) = y_i = p(x_i)$ for $i = 0, 1, 2, \ldots, n$, we get

$$f(x_i) - p(x_i) = 0$$

$\Rightarrow$ $g(x_i) = 0, \qquad i = 0, 1, 2, \ldots, n$

$\therefore$ $x_0, x_1, x_2, \ldots, x_n$ are roots of $g(x) = 0$

Hence $g(x) = h(x)(x - x_0)(x - x_1)\ldots(x - x_n)$

$\Rightarrow$ $f(x) - p(x) = h(x)(x - x_0)(x - x_1)\ldots(x - x_n)$, where $h(x)$ is to be determined.

$$\therefore \quad f(x) = p(x) + h(x)(x - x_0)(x - x_1)\ldots(x - x_n) \tag{2}$$

Now consider the function

$$F(t) = f(t) - p(t) - h(x)(t - x_0)(t - x_1)\ldots(t - x_n) \tag{3}$$

$\therefore$ $F(x_i) = 0$ for $t = x_0, x_1, x_2, \ldots x_n$

Since $f(x_i) - p(x_i) = 0$ and

$h(x)(x_i - x_0)(x_i - x_1)\ldots(x_i - x_n) = 0$ for $i = 0, 1, 2, \ldots, n$ and $F(t) = 0$ for $t = x$ by (2).

Hence we find $F(t) = 0$ at $n + 2$ points $x, x_0, x_1, x_2, \ldots, x_n$.

Also $F(t)$ is continuous and differentiable n times in $[x_0, x_n]$, where x_0 is the least root and x_n is the largest root.

So, by Rolle's theorem,

$F'(t)$ has at least $(n + 1)$ roots in $[x_0, x_n]$.

$F''(t)$ has at least n roots in $[x_0, x_n]$.

$\vdots$

$F^{(n+1)}(t)$ has at least one root, say $t = t_0$, in (x_0, x_n)

ie $F^{(n+1)}(t_0) = 0, \quad x_0 < t_0 < x_n$.

But $F^{(n+1)}(t) = f^{(n+1)}(t) - h(x)(n + 1)!$.

Since $p(x)$ is a polynomial of degree n, $p^{(n+1)}(t) = 0$.

Since $F^{(n+1)}(t_0) = 0$, we get $f^{(n+1)}(t_0) - h(x)(n + 1)! = 0$

$$\Rightarrow \quad h(x) = \frac{f^{(n+1)}(t_0)}{(n + 1)!}$$

$$\therefore \quad f(x) = p(x) + \frac{f^{(n+1)}(t_0)}{(n + 1)!}(x - x_0)(x - x_1)\ldots(x - x_n)$$

$$\Rightarrow \qquad f(x)-p(x)=\frac{f^{(n+1)}(t_0)}{(n+1)!}(x-x_0)(x-x_1)\dots(x-x_n)$$

Since $f(x)-p(x)$ is difference between the given function and the polynomial at any point x, it represents the **error** comitted in approximating the given function $f(x)$ by the polynomial $p(x)$

Hence the
$$\text{error} = R_n$$
$$=\frac{f^{(n+1)}(t_0)}{(n+1)!}(x-x_0)(x-x_1)\dots(x-x_n),\ x_0<t_0<x.$$

This error $f(x)-p(x)$ is called the **truncation error** for $x\in[x_0,x_n]$

Remainder term in Newton's forward formula

Let $x_0, x_1, x_2, \dots, x_n$ be equally spaced arguments with interval h.

Then
$$\frac{x-x_0}{h}=u \Rightarrow x-x_0=hu$$

$\therefore$
$$x-x_1=x-(x_0+h)=x-x_0-h=hu-h=h(u-1)$$

and
$$x-x_2=h(u-2),\dots,x-x_n=h(u-n)$$

$\therefore$
$$R_n=\frac{f^{(n+1)}(t_0)}{(n+1)!}h^{n+1}u(u-1)(u-2)\dots(u-n),\ x_0<t_0<x_n.$$

Suppose the analytical form of $f(x)$ is not known, then we cannot find the derivative $f^{(n+1)}(t_0)$

$\therefore$ we replace $f^{(n+1)}(t_0)$ by difference,

we have the formula
$$\Delta^n f(x)=(\Delta x)^n f^{(n)}(x+\theta n\Delta x),$$

$\Rightarrow$
$$\lim_{\Delta x\to 0}\frac{\Delta^n f(x)}{(\Delta x)^n}=f^{(n)}(x).$$

Put $x=x_0,\ \Delta x=h$

$\therefore$
$$f^{(n)}(x_0+\theta nh)=\frac{\Delta^n f(x_0)}{h^n}$$

Take $t_0=x_0+\theta nh$, then
$$f^{(n)}(t_0)=\frac{\Delta^n f(x_0)}{h^n}$$

$\therefore$
$$f^{(n+1)}(t_0)=\frac{\Delta^{n+1}f(x_0)}{h^{n+1}}=\frac{\Delta^{n+1}y_0}{h^{n+1}}$$

$\therefore$
$$R_n=\frac{\Delta^{n+1}y_0}{(n+1)!}u(u-1)(u-2)\dots(u-n)$$

Similarly we can find the error or remainder term in all the interpolation formulae.

We shall list them.

(1) Newton's forward formula

(a) $R_n = \frac{h^{n+1} f^{(n+1)}(t_0)}{(n+1)!} u(u-1)(u-2)\ldots(u-n)$

(b) $R_n = \frac{\Delta^{n+1} y_0}{(n+1)!} u(u-1)\ldots(u-n)$, where $u = \frac{x-x_0}{h}$

(2) Newton's backward formula

(a) $R_n = \frac{h^{n+1} f^{(n+1)}(t_0)}{(n+1)!} u(u+1)(u+2)\ldots(u+n)$

(b) $R_n = \frac{\nabla^{n+1} y_n}{(n+1)!} u(u+1)(u+2)\ldots(u+n)$, where $u = \frac{x-x_n}{h}$

(3) Stirling's formula

(a) $R_n = \frac{h^{2n+1} f^{(2n+1)}(t_0)}{(2n+1)!} u(u^2-1^2)(u^2-2^2)\ldots(u^2-n^2)$

$= \frac{\Delta^{2n+1} y_{-n-1} + \Delta^{2n+1} y_{-n}}{2(2n+1)!} u(u^2-1^2)(u^2-2^2)\ldots(u^2-n^2)$

(4) Bessells formula in terms of v

(a) $R_n = \frac{h^{2n+2} f^{(2n+2)}(t_0)}{(2n+2)!} (v^2 - \frac{1}{4})(v^2 - \frac{9}{4})\ldots\left(v^2 - \frac{(2n+1)^2}{4}\right)$

(b) $R_n = \frac{\Delta^{2n+2} y_{-n-1} + \Delta^{2n+2} y_{-n}}{2(2n+2)!} (v^2 - \frac{1}{4})(v^2 - \frac{9}{4})\ldots\left(v^2 - \frac{(2n+1)^2}{4}\right)$

(5) Lagrange's formula

$R_n = \frac{f^{n+1}(t_0)}{(n+1)!}(x-x_0)(x-x_1)\ldots(x-x_n).$

WORKED EXAMPLES

Example 1

Find the error in the computation of sin 52° by using Newton's forward formula, given that

x	45°	50°	55°	60°
$y = \sin x$	0.7071	0.7660	0.8192	0.8660

Solution

The error in Newton's forward formula is

$$R_n = \frac{\Delta^{n+1} y_0}{(n+1)!} u(u-1)(u-2)\ldots(u-n)$$

where $$u = \frac{x - x_0}{h} = \frac{x - 45}{5}$$

We form the difference table.

x	$u = \frac{x - x_0}{h}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45°	0	0.7071			
			0.0589		
50°	1	0.7660		−0.0057	
			0.0532		**−0.0007**
55°	2	0.8192		−0.0064	
			0.0468		
60°	3	0.8660			

Here $x = 52$, $n + 1 = 3 \Rightarrow n = 2$ (highest difference is third order)

$\therefore$ $$u = \frac{x - 45}{5}$$

When $x = 52$, $$u = \frac{52 - 45}{5} = \frac{7}{5} = 1.4$$

$\therefore$ $$R_2 = \frac{\Delta^3 y_0}{3!} u(u-1)(u-2)$$

$\Rightarrow$ $$R_2 = \frac{\Delta^3 y_0}{3!}(1.4)(1.4-1)(1.4-2) = \frac{-0.007}{6}(1.4)(0.4)(0.6) = -0.000039$$ ■

Example 2

The values of e^{-x} for certain equidistant values of x are given below

x	1.72	1.73	1.74	1.75	1.76
e^{-x}	0.1791	0.1773	0.1755	0.1738	0.1720

Find the error in computing $e^{-1.735}$ by Stirling formula.

Solution

The error in Stirling's formula is

$$R_n = \frac{\Delta^{2n+1} y_{-n-1} + \Delta^{2n+1} y_{-n}}{2(2n+1)!} u(u^2 - 1^2)(u^2 - 2^2)\ldots(u^2 - n^2)$$

where $u = \frac{x - x_0}{h}$ and x_0 is the origin. Here $x_0 = 1.73$, $h = 0.01$

When $x = 1.735$, $$u = \frac{1.735 - 1.73}{0.01} = 0.5$$

x	$u=\frac{x-x_0}{h}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1.72	−1	0.1791			
			−0.0018		
1.73	0	0.1773		0	
			−0.0018		**0.0001 = 10⁻⁴**
1.74	1	0.1755		0.0001	
			−0.0017		**−0.0002**
1.75	2	0.1738		−0.0001	
			−0.0018		
1.76	3	0.1720			

Here $2n+1=3 \quad \Rightarrow 2n=2 \quad \Rightarrow n=1.$

$$\therefore \quad R_1=\frac{\Delta^3 y_{-2}+\Delta^3 y_{-1}}{2\times 3!}u(u^2-1^2)$$

$$=\frac{10^{-4}}{2\times 6}0.5(0.5^2-1)=-0.000003125$$

■

Example 3

The following table gives certain values of $\log_{10}x$. Using Lagrange's interpolation formula find the value of log 323.5. Also find the error.

x	321	322.8	324.2	325
$\log_{10}x$	2.50651	2.50893	2.51081	2.51188

Solution

The given values of x are $x_0=321$, $x_1=322.8$, $x_2=324.2$, $x_3=325$
Using Lagrange's formula, it can be seen that log 323.5 = 2.50987
We shall now compute the error.

$$R_n=\frac{f^{n+1}(t_0)}{(n+1)!}(x-x_0)(x-x_1)\ldots(x-x_n),\quad x_0<t_0<x_n$$

Here $n+1=4 \quad \Rightarrow n=3$

$$\therefore \quad R_3=\frac{f^4(t_0)}{(4)!}(x-x_0)(x-x_1)(x-x_2)(x-x_3)$$

$$=\frac{f^4(t_0)}{(24)}(323.5-321)(323.5-322.8)(323.5-324.2)(323.5-325)$$

$$=\frac{f^4(t_0)}{(24)}(1.5)(0.7)(-0.7)(-1.5)$$

$$=\frac{f^4(t_0)}{(24)}(1.8375),\quad 321<t_0<325$$

But $\quad f(x)=\log_{10}x=\log_e x\cdot\log_{10}e$

$$\therefore \quad f'(x) = \log_{10} e\left(\frac{1}{x}\right), \qquad f''(x) = \log_{10} e\left(-\frac{1}{x^2}\right)$$

$$f'''(x) = \log_{10} e\left(\frac{2}{x^3}\right), \qquad f^4(x) = \log_{10} e\left(-\frac{6}{x^4}\right)$$

$$\therefore \quad f^4(t_0) = -\frac{6}{(321)^4} \times 0.43429 \qquad (\text{Taking } t_0 = 321)$$

$$= -2.4542 \times 10^{-10} = -0.24542 \times 10^{-11} = -0.00000000002454$$

$$\therefore \quad R_3 = -\frac{2.4542 \times 10^{-10}(1.8375)}{24}$$

$$= -0.00000000001879$$

This shows that the error is less than 1 in the 11th place. ■

INTERPOLATION WITH A CUBIC SPLINE

Introduction

The interpolation formulae we have seen so far represent a single polynomial passing through the points (or nodes) in a given interval. If the number of points is more, then the interpolating polynomial for many functions will be of higher degree, which tend to oscillate more and more between nodes as the degree increases. Hence the values computed using these interpolating polynomials will be very rough, except the case where the given set of points is for a polynomial function. This drawback is overcome by the **method of splines** which was introduced by I.J. Schoenberg in 1946.

Instead of using a single high-degree interpolating polynomial in an interval $a \le x \le b$, we subdivide the interval into a number of subintervals and in each subinterval we use a lower degree polynomial and join them together to get an interpolating function, called **spline.** Thus **spline interpolation is a piece wise polynomial interpolation**. Though splines can be of any degree, **cubic splines** are the most popular ones. The name 'spline is borrowed from the **draftman's spline**, a device which is an elastic rod used for drawing a smooth curve through a set of points.

Cubic Spline Interpolation

Let $y = f(x)$ be the given function on the interval $a \le x \le b$.
Divide $[a, b]$ into n subintervals by the points $a = x_0, x_1, \ldots, x_n = b$

where $$x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1} = h$$

Let $S(x)$ be the cubic spline that approximates $f(x)$ such that

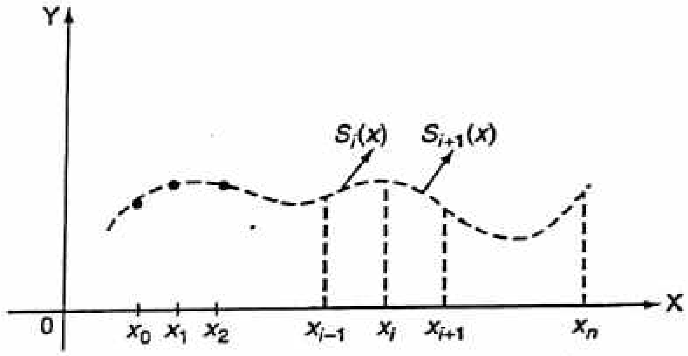
(i) $S(x_i) = f(x_i) = y_i$ for $i = 1, 2, 3, \ldots, n$

(ii) $S(x) = S_i(x)$ is a third degree polynomial in each subinterval $[x_{i-1}, x_i]$ $i = 1, 2, 3, \ldots, n$

(iii) $S(x), S'(x)$ and $S''(x)$ are continuous on (a, b)

ie. $S_i(x_i) = S_{i+1}(x_i), S_i'(x_i) = S_{i+1}'(x_i), S_i''(x_i) = S_{i+1}''(x_i), i = 1,2,3,\ldots,n-1.$

ie. at the joining point x_i, the successive cubics have the same slope and same curvature.



In each interval $[x_{i-1}, x_i]$, $S(x) = S_i(x)$ is a cubic polynomial, and so $S''(x) = S_i''(x)$ is linear. By Lagrange's formula, for the arguments x_{i-1}, x_i, x, we have

$$S_i''(x) = \frac{(x - x_i)}{(x_{i-1} - x_i)} S_i''(x_{i-1}) + \frac{(x - x_{i-1})}{(x_i - x_{i-1})} S_i''(x_i)$$

Since $x_i - x_{i-1} = h$ and $S_i''(x_{i-1}) = S''(x_{i-1})$ and $S_i''(x_i) = S''(x_i)$,

$$S_i''(x) = \frac{1}{h}\left[-(x - x_i)S''(x_{i-1}) + (x - x_{i-1})S''(x_i)\right]$$

Let $S_i''(x_i) = M_i, i = 1,2,\ldots,n$ and $M_0 = S_i''(x_0)$

$$\therefore \quad S_i''(x) = \frac{1}{h}\left[-(x - x_i)M_{i-1} + (x - x_{i-1})M_i\right]$$

Integrating twice w.r.to x, we get

$$S_i(x) = \frac{1}{h}\left[\frac{-(x - x_i)^3}{2.3} M_{i-1} + \frac{(x - x_{i-1})^3}{2.3} M_i\right] + c_1 x + c_2 \tag{2}$$

where c_1 and c_2 are arbitrary constants.

To find c_1 and c_2 use the hypothesis $S_i(x_{i-1}) = y_{i-1}$ and $S_i(x_i) = y_i$

Put $x = x_{i-1}$ in (2)

$$\therefore \quad S_i(x_{i-1}) = \frac{1}{h}\left[\frac{-(x_{i-1} - x_i)^3}{6} M_{i-1} + 0\right] + c_1 x_{i-1} + c_2$$

$$\Rightarrow \quad y_{i-1} = \frac{-1}{6h}(-h)^3 M_{i-1} + c_1 x_{i-1} + c_2$$

$$\Rightarrow \quad y_{i-1} = \frac{h^2}{6} M_{i-1} + c_1 x_{i-1} + c_2 \tag{3}$$

Put $x = x_i$ in (2)

$$\therefore \quad S_i(x_i) = \frac{1}{h}\left[0 + \frac{(x_i - x_{i-1})^3}{6}M_i\right] + c_1 x_i + c_2$$

$$\Rightarrow \quad y_i = \frac{1}{h}\cdot\frac{h^3}{6}M_i + c_1 x_i + c_2$$

$$\Rightarrow \quad y_i = \frac{h^2}{6}M_i + c_1 x_i + c_2 \qquad (4)$$

$$(4)-(3) \Rightarrow y_i - y_{i-1} = \frac{h^2}{6}[M_i - M_{i-1}] + c_1(x_i - x_{i-1})$$

$$= \frac{h^2}{6}[M_i - M_{i-1}] + c_1 h$$

$$\Rightarrow \quad c_1 = \frac{1}{h}\left[(y_i - y_{i-1}) - \frac{h^2}{6}(M_i - M_{i-1})\right]$$

$$= \frac{1}{h}(y_i - y_{i-1}) - \frac{h}{6}(M_i - M_{i-1})$$

$$(3)\times x_i \Rightarrow \quad x_i y_{i-1} = \frac{h^2}{6}x_i M_{i-1} + c_1 x_i x_{i-1} + c_2 x_i$$

$$(4)\times x_{i-1} \Rightarrow \quad x_{i-1} y_i = \frac{h^2}{6}x_{i-1} M_i + c_1 x_i x_{i-1} + c_2 x_{i-1}$$

Subtracting, we get

$$x_{i-1}y_i - x_i y_{i-1} = \frac{h^2}{6}[x_{i-1}M_i - x_i M_{i-1}] + c_2(x_{i-1} - x_i)$$

$$= \frac{h^2}{6}[x_{i-1}M_i - x_i M_{i-1}] - hc_2$$

$$\therefore \quad c_2 h = \frac{h^2}{6}[x_{i-1}M_i - x_i M_{i-1}] - (x_{i-1}y_i - x_i y_{i-1})$$

$$\Rightarrow \quad c_2 = \frac{h}{6}[x_{i-1}M_i - x_i M_{i-1}] - \frac{1}{h}(x_{i-1}y_i - x_i y_{i-1})$$

$$\therefore (2) \Rightarrow \quad S_i(x) = \frac{1}{h}\left[-\frac{1}{6}(x - x_i)^3 M_{i-1} + \frac{(x - x_{i-1})^3}{6}M_i\right] + \left[\frac{y_i - y_{i-1}}{h} - \frac{h}{6}(M_i - M_{i-1})\right]x$$
$$+ \frac{h}{6}[x_{i-1}M_i - x_i M_{i-1}] - \frac{1}{h}[x_{i-1}y_i - x_i y_{i-1}]$$

$$= M_{i-1}\left[-\frac{1}{6h}(x-x_i)^3+\frac{hx}{6}-\frac{hx_i}{6}\right]+M_i\left[\frac{1}{6h}(x-x_{i-1})^3-\frac{hx}{6}+\frac{hx_{i-1}}{6}\right]$$

$$+\frac{1}{h}(y_i-y_{i-1})x-\frac{1}{h}[x_{i-1}y_i-x_iy_{i-1}]$$

$$S_i(x)=M_{i-1}\left[-\frac{1}{6h}(x-x_i)^3+\frac{h}{6}(x-x_i)\right]+M_i\left[\frac{1}{6h}(x-x_{i-1})^3-\frac{h}{6}(x-x_{i-1})\right]$$

$$+\frac{1}{h}(x_i-x)y_{i-1}+\frac{1}{h}(x-x_{i-1})y_i \text{ in } x_{i-1}\le x\le x_i \tag{5}$$

where M_{i-1} and M_i are unknown quantities to be determined by using the continuity of $S_i'(x)$ at the point x_i

$\Rightarrow \quad S_i'(x_i-\epsilon)=S_{i+1}'(x_i+\epsilon)$ as $\epsilon\to 0$

ie. the left and right limits at x_i are equal.

Now differentiating (5) w.r.to x, we get

$$S_i'(x)=M_{i-1}\left[\frac{-1}{2h}(x-x_i)^2+\frac{h}{6}\right]+M_i\left[\frac{1}{2h}(x-x_{i-1})^2-\frac{h}{6}\right]+\frac{y_i-y_{i-1}}{h},\quad x_{i-1}<x<x_i$$

$$\Rightarrow \quad S_i'(x)=\frac{-(x-x_i)^2}{2h}M_{i-1}+\frac{(x-x_{i-1})^2}{2h}M_i-(M_i-M_{i-1})\frac{h}{6}+\frac{(y_i-y_{i-1})}{h},\quad x_{i-1}<x<x_i \tag{6}$$

Replacing i by $i+1$, we get

$$S_{i+1}'(x)=\frac{-(x-x_{i+1})^2}{2h}M_i+\frac{(x-x_i)^2}{2h}M_{i+1}-(M_{i+1}-M_i)\frac{h}{6}+\frac{(y_{i+1}-y_i)}{h},\quad x_i<x<x_{i+1} \tag{7}$$

From (6),
$$S_i'(x_i-\epsilon)=\frac{-(x_i-\epsilon-x_i)^2}{2h}M_{i-1}+\frac{(x_i-\epsilon-x_{i-1})^2}{2h}M_i \tag{8}$$

$$-(M_i-M_{i-1})\frac{h}{6}+\frac{(y_i-y_{i-1})}{h}$$

$$=\frac{-\epsilon^2}{2h}M_{i-1}+\frac{(h-\epsilon)^2}{2h}M_i-(M_i-M_{i-1})\frac{h}{6}+\frac{y_i-y_{i-1}}{h}$$

From (7),
$$S_i'(x_i-\epsilon)=\frac{-(x_i-\epsilon-x_{i+1})^2}{2h}M_i+\frac{(x_i-\epsilon-x_i)^2}{2h}M_{i+1}$$

$$-(M_{i+1}-M_i)\frac{h}{6}+\frac{y_{i+1}-y_i}{h}$$

$$=\frac{-(-h+\epsilon)^2}{2h}M_i+\frac{\epsilon^2}{2h}M_{i+1}-(M_{i+1}-M_i)\frac{h}{6}+\frac{y_{i+1}-y_i}{h} \tag{9}$$

Since $S_i'(x_i-\epsilon)=S'_{i+1}(x_i+\epsilon)$ as $\epsilon\to 0$,

we get from (8) and (9), (putting $\epsilon=0$ in the RHS)

$$\frac{h}{2}M_i-(M_i-M_{i-1})\frac{h}{6}+\frac{y_i-y_{i-1}}{h}=-\frac{h}{2}M_i-(M_{i+1}-M_i)\frac{h}{6}+\frac{y_{i+1}-y_i}{h}$$

$$\Rightarrow \quad hM_i+(M_{i+1}-M_i)\frac{h}{6}-(M_i-M_{i-1})\frac{h}{6}=\frac{y_{i+1}-y_i}{h}-\frac{(y_i-y_{i-1})}{h}$$

$$\Rightarrow \quad \frac{h}{6}[M_{i+1}+4M_i+M_{i-1}]=\frac{1}{h}[y_{i+1}-2y_i+y_{i-1}]$$

$$\Rightarrow \quad M_{i+1}+4M_i+M_{i-1}=\frac{6}{h^2}[y_{i+1}-2y_i+y_{i-1}] \quad in \ x_{i-1}\le x\le x_i \qquad (10)$$

This is true for all $i=1,2,\ldots,n-1$

Equations (10) give a system of $(n-1)$ linear equations in $n+1$ unknowns $M_0, M_1, M_2, \ldots, M_n$.

To solve for these unknowns we need two more equations. These two conditions may be taken in different ways.

We usually assume that outside the interval (x_0, x_n), $S(x)$ is flat or a straight line, then $S''(x)=0$.

$$\therefore \quad S''(x_0)=S_1''(x_0)=M_0=0 \text{ and } S''(x_n)=S_n''(x_n)=M_n=0$$

Thus we can solve for $M_0, M_1, \ldots, M_n$ and these values are substituted in (5) to get the cubic spline $S(x)$.

Note:

(1) The end conditions $M_0=0, M_n=0$ are called natural conditions because they yield a **natural spline**. It is called a natural spline because the draftman's spline behaves like this.

(2) Functions with abrupt local changes can be better approximated by cubic splines.

Working Rule: Given the table of values for $y=f(x)$

x	x_0	x_1	...	x_n
y	y_0	y_1	...	y_n

The cubic spline approximation $S(x)$ for these points is $S(x)=S_i(x)$ in the interval $x_{i-1}\le x\le x_i$, $i=1,2,\ldots,n$ is

$$S_i(x)=\frac{1}{6h}\left[-(x-x_i)^3M_{i-1}+(x-x_{i-1})^3M_i\right]$$
$$-\frac{(x-x_i)}{h}\left[y_{i-1}-\frac{h^2}{6}M_{i-1}\right]+\frac{(x-x_{i-1})}{h}\left[y_i-\frac{h^2}{6}M_i\right]$$

where $M_{i-1}+4M_i+M_{i+1}=\frac{6}{h^2}[y_{i-1}-2y_i+y_{i+1}]$, $i=1,2,\ldots,n-1$ and $M_0=0, M_n=0$

WORKED EXAMPLES

Example 1

Obtain the cubic spline approximation for the function $y = f(x)$ from the following data, given that $y''_0 = y''_3 = 0$.

x	-1	0	1	2
y	-1	1	3	35

Solution

The given values of x and y are

$$x_0 = -1, \quad x_1 = 0, \quad x_2 = 1, \quad x_3 = 2$$

and $$y_0 = -1, \quad y_1 = 1, \quad y_2 = 3, \quad y_3 = 35$$

The values of x are equally spaced with $h = 1$. Here $n = 3$.

The cubic spline for the interval $x_{i-1} \le x \le x_i, \quad i = 1, 2, 3$

is $$S_i(x) = \frac{1}{6h}\left[-(x-x_i)^3 M_{i-1} + (x-x_{i-1})^3 M_i\right]$$
$$-\frac{(x-x_i)}{h}\left[y_{i-1} - \frac{h^2}{6}M_{i-1}\right] + \frac{(x-x_{i-1})}{h}\left[y_i - \frac{h^2}{6}M_i\right] \quad (1)$$

where $M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}],\ i = 1, 2 \qquad [\because n-1 = 3-1 = 2] \quad (2)$

Given $M_0 = y''_0 = 0$ and $M_3 = y''_3 = 0$

Put i = 1 in (2), we get, $M_0 + 4M_1 + M_2 = 6[y_0 - 2y_1 + y_2]$

$\Rightarrow$ $4M_1 + M_2 = 6[-1-2(1)+3] = 0 \qquad \Rightarrow M_2 = -4M_1$

Put i = 2 in (2), we get, $M_1 + 4M_2 + M_3 = 6[y_1 - 2y_2 + y_3]$

$$M_1 + 4M_2 = 6[1-2(3)+35]$$
$$M_1 + 4M_2 = 180$$

$\therefore$ $M_1 + 4(-4M_1) = 180$

$\Rightarrow$ $-15M_1 = 180 \Rightarrow M_1 = -12,$

$\therefore$ $M_2 = 48$.

The cubic spline for $x_{i-1} \le x \le x_i$ is

$$S_i(x) = \frac{1}{6}\left[-(x-x_i)^3 M_{i-1} + (x-x_{i-1})^3 M_i\right]$$
$$-(x-x_i)\left[y_{i-1} - \frac{1}{6}M_{i-1}\right] + (x-x_{i-1})\left[y_i - \frac{1}{6}M_i\right] \quad (3)$$

Put $i = 1$ in (1), then the interval $x_0 \le x \le x_1$ is $-1 \le x \le 0$

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Then the cubic spline for $-1 \le x \le 0$ is

$$S_1(x) = \frac{1}{6}\left[-(x-x_1)^3 M_0 + (x-x_0)^3 M_1\right]$$
$$-(x-x_1)\left[y_0 - \frac{1}{6}M_0\right] + (x-x_0)\left[y_1 - \frac{1}{6}M_1\right]$$
$$= \frac{1}{6}\left[-(x-1)^3 .0 + (x+1)^3(-12)\right]$$
$$-(x-0)[-1-0] + (x+1)\left[1 - \frac{1}{6}(-12)\right]$$
$$= -2(x+1)^3 + x + 3(x+1)$$
$$= -2\left[x^3 + 3x^2 + 3x + 1\right] + x + 3x + 3$$
$$= -2x^3 - 6x^2 - 2x + 1$$

Put $i = 2$ in (1), then the interval $x_1 \le x \le x_2$ is $0 \le x \le 1$.

Then the cubic spline for $0 \le x \le 1$ is

$$S_2(x) = \frac{1}{6}\left[-(x-x_2)^3 M_1 + (x-x_1)^3 M_2\right]$$
$$-(x-x_2)\left[y_1 - \frac{1}{6}M_1\right] + (x-x_1)\left[y_2 - \frac{1}{6}M_2\right]$$
$$= \frac{1}{6}\left[-(x-1)^3(-12) + (x-0)^3 .48\right]$$
$$-(x-1)\left[1 - \frac{1}{6}(-12)\right] + (x-0)\left[3 - \frac{1}{6}(48)\right]$$
$$= 2(x-1)^3 + 8x^3 - 3(x-1) - 5x$$
$$= 2[x^3 - 3x^2 + 3x - 1] + 8x^3 - 3x + 3 - 5x$$
$$= 10x^3 - 6x^2 - 2x + 1$$

Put $i = 3$ in (1), then the interval $x_2 \le x \le x_3$ is $1 \le x \le 2$

Then the cubic spline for $1 \le x \le 2$ is

$$S_3(x) = \frac{1}{6}\left[-(x-x_3)^3 M_2 + (x-x_2)^3 M_3\right]$$
$$-(x-x_3)\left[y_2 - \frac{1}{6}M_2\right] + (x-x_2)\left[y_3 - \frac{1}{6}M_3\right]$$
$$= \frac{1}{6}\left[-(x-2)^3(48) + 0\right] - (x-2)\left[3 - \frac{1}{6}(48)\right] + (x-1)[35-0]$$
$$= -8(x-2)^3 + 5(x-2) + 35(x-1)$$
$$= -8\left[x^3 - 6x^2 + 12x - 8\right] + 40x - 45$$
$$= -8x^3 + 8x^2 - 56x + 19$$

$\therefore$ the cubic spline approximation for the given function is

$$S(x)=\begin{cases} S_1(x)=-2x^3-6x^2-2x+1, & -1\le x\le 0\\ S_2(x)=10x^3-6x^2-2x+1, & 0\le x\le 1\\ S_3(x)=-8x^3+8x^2-56x+19, & 1\le x\le 2\end{cases}$$ ■

Example 2

Find the cubic spline approximation for the function $f(x)$ given by the data:

x	0	1	2	3
$f(x)$	1	2	33	244

With $M_0 = 0 = M_3$. Hence estimate the value of $f(2.5)$, $f(1.5)$.

Solution

Let $y = f(x)$

The given the values of x and y are

$$x_0 = 0,\quad x_1 = 1,\quad x_2 = 2,\quad x_3 = 3$$
$$y_0 = 1,\quad y_1 = 2,\quad y_2 = 33,\quad y_3 = 244$$

The values of x are equally spaced with $h = 1$, Here $n = 3$.

The cubic spline for the interval $x_{i-1}\le x\le x_i$, $i = 1, 2, 3$ is

$$S_i(x)=\frac{1}{6h}\left[-(x-x_i)^3M_{i-1}+(x-x_{i-1})^3M_i\right]$$
$$-\frac{(x-x_i)}{h}\left[y_{i-1}-\frac{h^2}{6}M_{i-1}\right]+\frac{(x-x_{i-1})}{h}\left[y_i-\frac{h^2}{6}M_i\right]$$
$$=\frac{1}{6}\left[-(x-x_i)^3M_{i-1}+(x-x_{i-1})^3M_i\right]$$
$$-(x-x_i)\left[y_{i-1}-\frac{1}{6}M_{i-1}\right]+(x-x_{i-1})\left[y_i-\frac{1}{6}M_i\right] \quad (1)$$

where $$M_{i-1}+4M_i+M_{i+1}=\frac{6}{h^2}\left[y_{i-1}-2y_i+y_{i+1}\right],\ i=1,2 \quad (2)$$

and $$M_0=0,\ M_3=0$$

Put $i = 1$ in (2),

$\therefore$ $$M_0+4M_1+M_2=6\left[y_0-2y_1+y_2\right]$$

$\Rightarrow$ $$4M_1+M_2=6\left[1-2(2)+33\right]=180$$

Put $i = 2$ in (2),

$\therefore$ $$M_1+4M_2+M_3=6\left[y_1-2y_2+y_3\right]$$

$\Rightarrow$ $$M_1+4M_2=6\left[2-2(33)+244\right]=1080$$

Solving we get $M_1=-24$, $M_2=276$

Put $i = 1$ in (1), then the interval $x_0\le x\le x_1$ is $0\le x\le 1$

Then the cubic spline for the interval $0 \le x \le 1$ is

$$S_1(x) = \frac{1}{6}\left[-(x-x_1)^3 M_0 + (x-x_0)^3 M_1\right] - (x-x_1)\left[y_0 - \frac{1}{6}M_0\right] + (x-x_0)\left[y_1 - \frac{1}{6}M_1\right]$$

$$= \frac{1}{6}\left[-(x-1)^3 .0 + (x-0)^3(-24)\right] - (x-1)\left[1 - \frac{1}{6}.0\right] + (x-0)\left[2 - \frac{1}{6}(-24)\right]$$

$$= -4x^3 - (x-1) + 6x$$

$$= -4x^3 + 5x + 1$$

Put $i = 2$ in (1), then the interval $x_1 \le x \le x_2$ is $1 \le x \le 2$

Then the cubic spline for the interval $1 \le x \le 2$ is

$$S_2(x) = \frac{1}{6}\left[-(x-x_2)^3 M_1 + (x-x_1)^3 M_2\right] - (x-x_2)\left[y_1 - \frac{1}{6}M_1\right] + (x-x_1)\left[y_2 - \frac{1}{6}M_2\right]$$

$$= \frac{1}{6}\left[-(x-2)^3(-24) + (x-1)^3(276)\right] - (x-2)\left[2 - \frac{1}{6}(-24)\right] + (x-1)\left[33 - \frac{1}{6}(276)\right]$$

$$= 4(x-2)^3 + 46(x-1)^3 - 6(x-2) - 13(x-1)$$

$$= 4(x^3 - 6x^2 + 12x - 8) + 46(x^3 - 3x^2 + 3x - 1) - 19x + 25$$

$$= 50x^3 - 162x^2 + 167x - 53$$

Put $i = 3$ in (1), then the interval $x_2 \le x \le x_3$ is $2 \le x \le 3$.

Then the cubic spline for $2 \le x \le 3$ is

$$S_3(x) = \frac{1}{6}\left[-(x-x_3)^3 M_2 + (x-x_2)^3 M_3\right] - (x-x_3)\left[y_2 - \frac{1}{6}M_2\right] + (x-x_2)\left[y_3 - \frac{1}{6}M_3\right]$$

$$= \frac{1}{6}\left[-(x-3)^3(276) + 0\right] - (x-3)\left[33 - \frac{1}{6}(276)\right] + (x-2)\left[244 - \frac{1}{6}(0)\right]$$

$$= -46(x-3)^3 + 13(x-3) + 244(x-2)$$

$$= -46(x^3 - 9x^2 + 27x - 27) + 257x - 527$$

$$= -46x^3 + 414x^2 - 985x + 715$$

$\therefore$ the cubic spline is

$$S(x)=\begin{cases} S_1(x)=-4x^3+5x+1, & 0\le x\le 1 \\ S_2(x)=50x^3-162x^2+167x-53, & 1\le x\le 2 \\ S_3(x)=-46x^3+414x^2-985x+715, & 2\le x\le 3 \end{cases}$$

When $x=2.5$,

$$f(2.5)=S_3(2.5)=-46(2.5)^3+414(2.5)^2-985(2.5)+715$$
$$=\mathbf{121.25}$$

When $x=1.5$,

$$f(1.5)=S_2(1.5)=50(1.5)^3-162(1.5)^2+167(1.5)-53$$
$$=\mathbf{1.75}$$

■

Example 3

The following values of x an y are given

x	1	2	3	4
y	1	2	5	11

Find the cubic splines and evaluate $y(1.5)$ and $y'(3)$.

Solution

Given the values of x and y are

$$x_0=1,\quad x_1=2,\quad x_2=3,\quad x_3=4$$
$$y_0=1,\quad y_1=2,\quad y_3=5,\quad y_3=11$$

The values of x are equally spaced with $h=1$. Here $n=3$

The cubic spline for the interval $x_{i-1}\le x\le x_i$, $i=1,2,3$ is

$$S_i(x)=\frac{1}{6h}\left[-(x-x_i)^3M_{i-1}+(x-x_{i-1})^3M_i\right]$$
$$-\frac{(x-x_i)}{h}\left[y_{i-1}-\frac{h^2}{6}M_{i-1}\right]+\frac{(x-x_{i-1})}{h}\left[y_i-\frac{h^2}{6}M_i\right] \quad (1)$$

$$S_i(x)=\frac{1}{6}\left[-(x-x_i)^3M_{i-1}+(x-x_{i-1})^3M_i\right]$$
$$-(x-x_i)\left[y_{i-1}-\frac{1}{6}M_{i-1}\right]+(x-x_{i-1})\left[y_i-\frac{1}{6}M_i\right] \quad (2)$$

where $\quad M_{i-1}+4M_i+M_{i+1}=6\left[y_{i-1}-2y_i+y_{i+1}\right], i=1,2,$

We assume the conditions of natural spline $M_0=0, M_3=0$

Putting $i = 1$ in (2) we get,

$$M_0 + 4M_1 + M_2 = 6[y_0 - 2y_1 + y_2]$$

$$\Rightarrow \quad 4M_1 + M_2 = 6[1 - 2(2) + 5] = 12 \qquad (3)$$

Putting $i = 2$ in (2) we get,

$$M_1 + 4M_2 + M_3 = 6[y_1 - 2y_2 + y_3]$$

$$\Rightarrow \quad M_1 + 4M_2 = 6[2 - 2(5) + 11] = 18 \qquad (4)$$

Solving (3) and (4), we get $M_1 = 2, M_2 = 4$

Now Putting $i = 1$ in (1), then the interval $x_0 \le x \le x_1$ is $1 \le x \le 2$.

Then the cubic spline for the interval $1 \le x \le 2$ is

$$S_1(x) = \frac{1}{6}\left[-(x - x_1)^3 M_0 + (x - x_0)^3 M_1\right] - (x - x_1)\left[y_0 - \frac{1}{6}M_0\right] + (x - x_0)\left[y_1 - \frac{1}{6}M_1\right]$$

$$= \frac{1}{6}\left[-(x-2)^3.0 + (x-1)^3(2)\right] - (x-2)\left[1 - \frac{1}{6}.0\right] + (x-1)\left[2 - \frac{1}{6}(2)\right]$$

$$= \frac{1}{3}(x-1)^3 - (x-2) + \frac{5}{3}(x-1)$$

$$= \frac{1}{3}\left[x^3 - 3x^2 + 3x - 1 - 3(x-2) + 5(x-1)\right]$$

$$= \frac{1}{3}\left[x^3 - 3x^2 + 5x\right]$$

Putting $i = 2$ in (1), then the interval $x_1 \le x \le x_2$ is $2 \le x \le 3$.

Then the cubic spline for the interval $2 \le x \le 3$ is

$$S_2(x) = \frac{1}{6}\left[-(x - x_2)^3 M_1 + (x - x_1)^3 M_2\right]$$

$$- (x - x_2)\left[y_1 - \frac{1}{6}M_1\right] + (x - x_1)\left[y_2 - \frac{1}{6}M_2\right]$$

$$= \frac{1}{6}\left[-(x-3)^3(2) + (x-2)^3(4)\right]$$

$$- (x-3)\left[2 - \frac{1}{6}(2)\right] + (x-2)\left[5 - \frac{1}{6}(4)\right]$$

$$= \frac{1}{3}\left[-(x-3)^3 + 2(x-2)^3\right] - \frac{5}{3}(x-3) + \frac{13}{3}(x-2)$$

$$= \frac{1}{3}\left[-(x^3 - 9x^2 + 27x - 27) + 2(x^3 - 6x^2 + 12x - 8) - 5(x-3) + 13(x-2)\right]$$

$$= \frac{1}{3}\left[x^3 - 3x^2 + 5x\right]$$

Putting $i = 3$ in (1), then the interval $x_2 \le x \le x_3$ is $3 \le x \le 4$.

Then the cubic spline for $3 \le x \le 4$ is

$$S_3(x) = \frac{1}{6}\left[-(x-x_3)^3 M_2 + (x-x_2)^3 M_3\right]$$
$$-(x-x_3)\left[y_2 - \frac{1}{6}M_2\right] + (x-x_2)\left[y_3 - \frac{1}{6}M_3\right]$$
$$= \frac{1}{6}\left[-(x-4)^3(4)+0\right] - (x-4)\left[5-\frac{1}{6}(4)\right] + (x-3)\left[11-\frac{1}{6}(0)\right]$$
$$= -\frac{2}{3}(x-4) - \frac{13}{3}(x-4) + 11(x-3)$$
$$= \frac{1}{3}\left[-2(x^3-12x^2+48x-64) - 13(x-4) + 33(x-3)\right]$$
$$= \frac{1}{3}\left[-2x^3+24x^2-76x+81\right]$$

$\therefore$ the cubic spline is

$$S(x) = \begin{cases} S_1(x) = \frac{1}{3}\left[x^3-3x^2+5x\right], & 1 \le x \le 2 \\ S_2(x) = \frac{1}{3}\left[x^3-3x^2+5x\right], & 2 \le x \le 3 \\ S_3(x) = \frac{1}{3}\left[-2x^3+24x^2-76x+81\right], & 3 \le x \le 4 \end{cases}$$

When $x = 1.5$,

$$y = S_1(1.5) = \frac{1}{3}\left[(1.5)^3 - 3(1.5)^2 + 5(1.5)\right] = \mathbf{1.375}$$

Differentiating $S(x)$ w.r.to x, we get

$$S'(x) = \begin{cases} \frac{1}{3}\left[3x^2-6x+5\right], & 1 \le x \le 2 \\ \frac{1}{3}\left[3x^2-6x+5\right], & 2 \le x \le 3 \\ \frac{1}{3}\left[-6x^2+48x-76\right], & 3 \le x \le 4 \end{cases}$$

The derivates at the joining points should exist for the spline.

$\therefore y'(3) = S'(3)$ can be obtained from [2, 3] or [3, 4] and they should be equal.

So we can find from [2, 3], $y'(3) = \frac{1}{3}\left(3.3^2 - 6.3 + 5\right) = \frac{14}{3}$

Note:

If we find from [3, 4], $y'(3) = \frac{1}{3}[-6.3^2 + 48.3 + 76] = \frac{14}{3}$

Exercises 4.7

(1) Given the following table, find $f(2.5)$ using cubic spline functions.

i	0	1	2	3
x_i	1	2	3	4
$f(x_i)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$

(2) Find a natural cubic spline to the data, given

x:	1	2	3	4
y:	1	5	11	8

Hence find $y(1.5)$ and $y'(2)$.

(3) Find the natural cubic spline valid for the interval [3, 4] for the data

x:	1	2	3	4
y:	3	10	29	65

and hence find the value of $y(3.2)$.

(4) Find the natural cubic spline for the data

x	0	2	4	6
$f(x)$	4	0	4	80

(5) Find the cubic spline interpolation for the data

x:	0	2	4	6
$f(x)$:	1	9	41	41

Given $M_0 = 0, M_3 = -12$.

Answers 4.7

(1) $$S(x)=\begin{cases}\frac{1}{150}\left[3x^3-9x^2-19x+100\right], & 0\le x\le 1\\ \frac{1}{300}\left[-5x^3+48x^2-170x+288\right], & 1\le x\le 2\\ \frac{1}{300}\left[-x^3+12x^2-62x+52\right], & 2\le x\le 3\end{cases}$$

(2) $$y=S(x)=\begin{cases}\frac{1}{15}\left[17x^3-51x^2+94x-45\right], & 1\le x\le 2\\ \frac{1}{15}\left[-55x^3+381x^2-770x+531\right], & 2\le x\le 3\\ \frac{1}{15}\left[38x^3-456x^2+1741x-1980\right], & 3\le x\le 4\end{cases}$$

$y(1.5)=2.575,\ y'(2)=6.267$

(3) $S(x) = S_3(x) = \frac{1}{15}\left[-56x^3 + 672x^2 - 2092x + 2175\right]$

$y(3.2) = 35.1248$

(4) $S(x) = \begin{cases} -x^3 + x^2 + 4, & 0 \le x \le 2 \\ 5x^3 - 35x^2 + 72x - 44, & 2 \le x \le 4 \\ -11x^3 + 157x^2 - 696x + 1108, & 4 \le x \le 6 \end{cases}$

(5) $S(x) = \begin{cases} x^3 + 1, & 0 \le x \le 2 \\ -2x^3 + 18x^2 - 36x + 25, & 2 \le x \le 4 \\ -6x^2 + 60x - 103, & 4 \le x \le 6 \end{cases}$

SHORT ANSWER QUESTIONS

1. When Newton's forward interpolation formula is used ?
2. When Newton's backward formula for interpolation is used ?
3. State Newton's backward formula for interpolation.
4. State Newton's forward formula for interpolation.
5. What do you mean by interpolation?
6. Given $u_1 = 1, u_3 = 17, u_4 = 43, u_5 = 89$, find the value of u_2.
7. A third degree polynomial passes through (0, –1), (1, 1), (2, 1) and (3, –2). Find its value at $x = 4$.
8. Given $f(0) = -1$, $f(1) = 1$ and $f(2) = 4$, find the root of the Newton's interpolating polynomial equation $f(x) = 0$.
9. Form the divided difference table for the following data:

x	2	5	10
y	5	29	109

10. Find the second degree polynomial through the points (0, 2), (2, 1), (1, 0).
11. What is the Lagrange's formula to find y, if $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ are given?
12. What are the advantages of Lagrange's formula over Newton's formula ?
13. Write down the Lagrange's interpolating polynomial for the data $(x_i, f(x_i))$, $i = 0,1,2,3$.
14. Find the second degree polynomial fitting the data:

x	1	2	4
y	4	5	13

15. Show that the second divided difference $[x_0, x_1, x_2]$ is independent of the arguments.
16. Find the second divided differences with arguments a, b, c of the function $\frac{1}{x}$.

17. Form the divided difference table for the data (0, 1), (1, 4) (3, 40) and (4, 85).

18. Form the divided difference table

x	–1	1	2	4
y	–1	5	23	119

19. Find $y(l)$, given

x	0	3	4
y	12	6	8

20. Define a cubic spline $S(x)$ which is commonly used for interpolation.
21. State the properties of cubic spline.
22. For cubic splines what are the $4n$ conditions required to evaluate the unknowns.
23. Find the divided differences of $f(x) = x3 + x + 2$ for the arguments 1, 3, 6.

Inverse Interpolation

INTRODUCTION

In Chapter 4, using interpolation methods, we found the value of the entry y for an intermediate value of the argument x, from a given table of value of x and y.

Sometimes we have to find the value of x for a given values of y not in the table. This reverse process is known as **inverse interpolation.**

Thus **inverse interpolation** is defined as the process of finding the value of the argument corresponding to a given value of the function lying between two tabulated functional values.

In this chapter, we use the following methods to find the value of the argument x for a given value of y.

(1) Lagrange's inverse interpolation formula
(2) Successive approximation method or iteration method
(3) Reversion of series method

LAGRANGE'S INVERSE INTERPOLATION FORMULA

Whether the arguments are equally spaced or not, the entries are always unequally spaced. So, Lagrange's formula is the natural choice, because lagrange's formula is merely a relation between two variables, either of which may be chosen as the independent variable.

Given a set of values $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, treating y as the independent variable and x as the dependent variable we have the inverse form of Lagrange's formula

$$x = \frac{(y-y_1)(y-y_2)\dots(y-y_n)}{(y_0-y_1)(y_0-y_2)\dots(y_0-y_n)}x_0 + \frac{(y-y_0)(y-y_2)\dots(y-y_n)}{(y_1-y_0)(y_1-y_2)\dots(y_1-y_n)}x_1$$

$$+\frac{(y-y_0)(y-y_1)\dots(y-y_n)}{(y_2-y_0)(y_2-y_1)\dots(y_2-y_n)}x_2+\dots+\frac{(y-y_0)(y-y_1)\dots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\dots(y_n-y_{n-1})}x_n$$

WORKED EXAMPLES

Example 1

If

x	2	3	5
U_x	113	286	613

find correct to two decimals the value of x for which $U_x = 1001$.

Solution

Let $y = U_x$

Given $x_0 = 2,\quad x_1 = 3,\quad x_2 = 5$

and $y_0 = 113,\quad y_1 = 286,\quad y_2 = 613$

Required the value of x when $y = 1001$

Lagrange's inverse interpolation formula is

$$x = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)}x_0 + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)}x_1 + \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)}x_2$$

$$= \frac{(y-286)(y-613)}{(113-286)(113-613)}\times 2 + \frac{(y-113)(y-613)}{(286-113)(286-613)}\times 3 + \frac{(y-113)(y-286)}{(613-113)(613-286)}\times 5$$

$$= \frac{(y-286)(y-613)}{(-173)(-500)}\times 2 + \frac{(y-113)(y-613)}{(173)(-327)}\times 3 + \frac{(y-113)(y-286)}{500\times 327}\times 5$$

When $y = 1001$, we get

$$x = \frac{(1001-286)(1001-613)}{173\times 500}\times 2 + \frac{(1001-113)(1001-613)}{173(-327)}\times 3$$

$$+\frac{(1001-113)(1001-286)}{500\times 327}\times 5$$

$$= \frac{715 \times 388}{173 \times 500} \times 2 - \frac{888 \times 388}{173 \times 327} \times 3 + \frac{888 \times 715}{500 \times 327} \times 5$$

$$= 6.4143 - 18.2714 + 19.4165 \;= 7.5594$$

$\therefore$ the value of x correct to 2 places of decimals is **7.56** ■

Example 2

Use Lagrange's formula inversely to obtain the value of t when $A = 85$ from the following table.

t	2	5	8	14
A	94.8	87.9	81.3	68.7

Solution

Let us take $t = x$ and $A = y$

$\therefore$ $x_0 = 2, \quad x_1 = 5, \quad x_2 = 8, \quad x_3 = 14$

and $y_0 = 94.8, \quad y_1 = 87.9, \quad y_2 = 81.3, \quad y_3 = 68.7$

Lagrange's inverse interpolation formula is

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}x_1$$

$$+\frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)}x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}x_3$$

$$= \frac{(y-87.9)(y-81.3)(y-68.7)}{(94.8-87.9)(94.8-81.3)(94.8-68.7)} \times 2$$

$$+\frac{(y-94.8)(y-81.3)(y-68.7)}{(87.9-94.8)(87.9-81.3)(87.9-68.7)} \times 5$$

$$+\frac{(y-94.8)(y-87.9)(y-68.7)}{(81.3-94.8)(81.3-87.9)(81.3-68.7)} \times 8$$

$$+\frac{(y-94.8)(y-87.9)(y-81.3)}{(68.7-94.8)(68.7-87.9)(68.7-81.3)} \times 14$$

When $y = 85$, we get

$$x = \frac{(85-87.9)(85-81.3)(85-68.7)}{(94.8-87.9)(94.8-81.3)(94.8-68.7)} \times 2$$
$$+\frac{(85-94.8)(85-81.3)(85-68.7)}{(87.9-94.8)(87.9-81.3)(87.9-68.7)} \times 5$$
$$+\frac{(85-94.8)(85-87.9)(85-68.7)}{(81.3-94.8)(81.3-87.9)(81.3-68.7)} \times 8$$
$$+\frac{(85-94.8)(85-87.9)(85-81.3)}{(68.7-94.8)(68.7-87.9)(68.7-81.3)} \times 14$$
$$= \frac{(-2.9)(3.7)(16.3) \times 2}{(6.9)(13.5)(26.1)} + \frac{(-9.8)(3.7)(16.3) \times 5}{(-6.9)(6.6)(19.2)}$$
$$+\frac{(-9.8)(-2.9)(16.3) \times 8}{(-13.5)(-6.6)(12.6)} + \frac{(-9.8)(-2.9)(3.7) \times 14}{(-26.1)(-19.2)(-12.6)}$$
$$= -0.1439 + 3.3798 + 3.3011 - 0.2332 = 6.3038$$

■

$\therefore$ when $A = 85$, $\boldsymbol{t = 6.3038}$

Example 3

Apply Lagrange's formula inversely to obtain the root of the equation $f(x) = 0$ given that $f(30) = -30$, $f(34) = -13$, $f(38) = 3$ and $f(42) = 18$.

Solution

Let $y = f(x)$

The given values of x are $x_0 = 30$, $x_1 = 34$, $x_2 = 38$, $x_3 = 42$

and the corresponding values of y are $y_0 = -30$, $y_1 = -13$, $y_2 = 3$, $y_3 = 18$

Required the value of x when $y = 0$

Lagrange's inverse interpolation formula is

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}x_1$$
$$+\frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)}x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}x_3$$
$$= \frac{(y+13)(y-3)(y-18)}{(-30+13)(-30-3)(-30-18)} \times 30 + \frac{(y+30)(y-3)(y-18)}{(-13+30)(-13-3)(-13-18)} \times 34$$
$$+\frac{(y+30)(y+13)(y-18)}{(3+30)(3+13)(3-18)} \times 38 + \frac{(y+30)(y+13)(y-3)}{(18+30)(18+13)(18-3)} \times 42$$

When $y=0$, we get

$$x=\frac{(13)(-3)(-18)}{(-17)(-33)(-48)}\times 30+\frac{(30)(-3)(-18)}{(17)(-16)(-31)}\times 34+\frac{(30)(13)(-18)}{(33)(16)(-15)}\times 38+\frac{(30)(13)(-3)}{(48)(31)(15)}\times 42$$

$$=-0.7821+6.5323+33.6818-2.2016=37.23$$

$\therefore$ the root of the equation $f(x)=0$ is $\mathbf{x=37.23}$ ■

Exercises 5.1

(1) Applying Lagrange's formula inversely find x when $f(x)=19$, given $f(0)=0$, $f(1)=1, f(2)=20$.

(2) In the following table h denote the height (in feet) above sea level and p, the barometric pressure

h	0	2753	4763	6942	1053
p	30	27	27	23	20

Find the height when the pressure is 32 by Lagrange's inverse interpolation formula.

(3) If $f(0)$ = 16.35, $f(5)$ = 14.88, $f(10)$ = 13.59, $f(15)$ = 12.46, find x when $f(x)$ = 14 by Lagrange's method.

(4) Apply Lagrange's formula inversely to find the value of x when $f(x)$ = 15 from the given data.

x	5	6	9	11
$f(x)$	12	13	14	16

(5) By inverse interpolation formula find x when cosh 1.285 from the given table.

x	0.736	0.737	0.738	0.739	0.740
$y=\cosh x$	1.2833	1.2841	1.2849	1.2859	1.2865

Answers 5.1

(1) 2.8 (2) 1691 feet below sea level (3) 8.337 (4) 11.5 (5) 0.73811

SUCCESSIVE APPROXIMATION METHOD OR ITERATION METHOD

In Lagrange's inverse interpolation formula each term is important and omission of any term will lead to serious errors. So this method is unsuitable when the number of arguments (and entries) is large as the computations become very tedious.

When the arguments are equally spaced, we have a simpler method based on successive approximation or iteration. It consists in using Newton's forward interpolation formula or any one of the central difference formulae.

We shall illustrate the method with Newton's forward difference formula

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \dots$$

where $u = \dfrac{x - x_0}{h}$

$$\Rightarrow \quad u\Delta y_0 = y - y_0 - \frac{u(u-1)}{2!}\Delta^2 y_0 - \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 - \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 - \dots$$

$$\Rightarrow \quad u = \frac{y - y_0}{\Delta y_0} - \frac{u(u-1)}{2!}\frac{\Delta^2 y_0}{\Delta y_0} - \frac{u(u-1)(u-2)}{3!}\frac{\Delta^3 y_0}{\Delta y_0} - \frac{u(u-1)(u-2)(u-3)}{4!}\frac{\Delta^4 y_0}{\Delta y_0} - \dots \qquad (1)$$

To find the first approximation for u, we neglect the second and higher order differences.

$$\therefore \quad u_1 = \frac{y - y_0}{\Delta y_0}$$

To find the second approximation, put $u = u_1$ on the R.H.S of (1)

$$\therefore \quad u_2 = \frac{y - y_0}{\Delta y_0} - \frac{u_1(u_1-1)}{2!}\frac{\Delta^2 y_0}{\Delta y_0} - \frac{u_1(u_1-1)(u_1-2)}{3!}\frac{\Delta^3 y_0}{\Delta y_0}$$
$$- \frac{u_1(u_1-1)(u_1-2)(u_1-3)}{4!}\frac{\Delta^4 y_0}{\Delta y_0} - \dots$$

To find the third approximation put $u = u_2$ in the R.H.S of (1)

$$\therefore \quad u_3 = \frac{y - y_0}{\Delta y_0} - \frac{u_2(u_2-1)}{2!}\frac{\Delta^2 y_0}{\Delta y_0} - \frac{u_2(u_2-1)(u_2-2)}{3!}\frac{\Delta^3 y_0}{\Delta y_0}$$
$$- \frac{u_2(u_2-1)(u_2-2)(u_2-3)}{4!}\frac{\Delta^4 y_0}{\Delta y_0} + \dots$$

This process is repeated till two successive approximations coincide up to the desired degree of accuracy.

WORKED EXAMPLES

Example 1

Find x when $y = 0.2$ from the following table:

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
y	0.141	0.158	0.176	0.194	0.213	0.231	0.249	0.268	0.287

Solution

From the given table of values of x and y, we find that value of $y = 0.2$ occurs for a value of x lying between 0.3 and 0.4

So, we choose the origin as $x_0 = 0.3$. Here $h = 0.1$

$$\therefore \quad u = \frac{x - x_0}{h} = \frac{x - 0.3}{0.1}$$

We form the difference table.

x	$u = \frac{x-0.3}{0.1}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	−3	0.141				
			0.017			
0.1	−2	0.158		0.001		
			0.018		−0.001	
0.2	−1	0.176		0		0.002
			0.018		0.001	
0.3	0	**0.194**		0.001		−0.003
			0.019		−0.002	
0.4	1	0.213		**−0.001**		0.003
			0.018		**0.001**	
0.5	2	0.231		0		**0**
			0.018		0.001	
0.6	3	0.249		0.001		−0.002
			0.019		−0.001	
0.7	4	0.268		0		
			0.019			
0.8	5	0.287				

Applying Newton's formula, we get

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots$$

$$= 0.194 + u(0.019) + \frac{u(u-1)}{2!}(-0.001) + \frac{u(u-1)(u-2)}{3!}(0.001)$$

$$\therefore \quad 0.019u = y - 0.194 + \frac{u(u-1)}{2}(0.001) - \frac{u(u-1)(u-2)}{6}(0.001)$$

$$\Rightarrow \quad u = \frac{y-0.194}{0.019} + \frac{u(u-1)}{2}\frac{(0.001)}{0.019} - \frac{u(u-1)(u-2)}{6}\frac{(0.001)}{0.019}$$

When $y = 0.2$, we get

$$\Rightarrow \quad u = \frac{0.2-0.194}{0.019} + u(u-1)(0.0263) - u(u-1)(u-2)(0.00877)$$

$$u = 0.3158 + 0.0263\, u(u-1) - 0.00877\, u(u-1)(u-2) \qquad (1)$$

First approximation value of u is

$$u_1 = \mathbf{0.3158}$$

Substituting in the R.H.S of (1), the second approximation is

$$u_2 = 0.3158 + 0.0263\, u_1(u_1-1) - 0.00877\, u_1(u_1-1)(u_1-2)$$

$$u_2 = 0.3158 + 0.0263\,(0.3158)\,(0.3158-1)$$
$$- 0.00877\,(0.3158)\,(0.3158-1)\,(0.3158-2)$$
$$= 0.3158 - 0.00568 - 0.00319 = \mathbf{0.3069}$$

Third approximation is

$$u_3 = 0.3158 + 0.0263\, u_2(u_2-1) - 0.00877\, u_2(u_2-1)\,(u_2-2)$$

$$\Rightarrow \quad u_3 = 0.3158 + 0.0263\,(0.3069)\,(0.3069-1)$$
$$- 0.00877\,(0.3069)\,(0.3069-1)\,(0.3069-2)$$
$$= 0.3158 - 0.00559 - 0.003158 = \mathbf{0.3070}$$

Fourth approximation is

$$u_4 = 0.3158 + 0.0263\, u_3(u_3-1) - 0.00877\, u_3\,(u_3-1)\,(u_3-2)$$
$$= 0.3158 + 0.0263\,(0.3070)\,(0.3070-1)$$
$$- 0.00877\,(0.0370)\,(0.3070-1)(0.3070-2)$$
$$= 0.3158 + 0.005596 - 0.003159 = \mathbf{0.3070}$$

Since u_3 and u_4 are same for 3 places; we take $u = 0.3070$

$$\therefore \quad \frac{x-0.3}{0.1} = 0.3070$$

$$\Rightarrow \quad x - 0.3 = 0.3070 \times 0.1 = 0.03070$$

$$\Rightarrow \quad x = 0.3 + 0.0307 = \mathbf{0.3307}$$

Example 2

Solve the equation $x = \log_{10}x$, given the following data.

x	1.35	1.36	1.37	1.38
$\log_{10}x$	0.1303	0.1335	0.1367	0.1399

Solution

Given $$x = 10\log_{10}x$$

$\Rightarrow$ $$x - \log_{10}x = 0$$

Let $$y = x - 10\log_{10}x$$

Hence the given table becomes

x	1.35	1.36	1.37	1.38
y	0.047	0.025	0.003	−0.019

It can be seen from the table that the value $y = 0$ corresponds to a value of x between 1.37 and 1.38

So we take the origin as $x_n = 1.38$ and apply Newton's backward formula.

$\therefore$ $$y = y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \dots$$

where $$v = \frac{x - x_n}{h}$$

Here $$x_n = 1.38 \text{ and } h = 0.01$$

$\therefore$ $$v = \frac{x - 1.38}{0.01}$$

We form the difference table.

x	$v = \frac{x-1.38}{0.01}$	y	Δy	$\Delta^2 y$
1.35	−3	0.047		
			−0.022	
1.36	−2	0.025		0
			−0.022	
1.37	−1	0.003		$\boxed{0}$
			$\boxed{-0.022}$	
1.38	0	$\boxed{-0.019}$		

$\therefore \qquad y = -0.019 + v(-0.022)$

When $y = 0$, we get

$$0 = -0.019 - 0.022v$$

$\Rightarrow \qquad 0.022v = -0.019$

$\Rightarrow \qquad v = -\frac{0.019}{0.022} = -0.8636$

$\Rightarrow \qquad \frac{x - 1.38}{0.01} = -0.8636$

$\Rightarrow \qquad x - 1.38 = -0.8636 \times 0.01$

$\Rightarrow \qquad x = 1.38 - 0.008636 = 1.371364.$

$\therefore$ the root correct to four decimal places is $x = \mathbf{1.3714}$ ■

Example 3

Determine the positive root of $x^3 + x^2 + x - 100 = 0$ by inverse interpolation.

Solution

Let $\qquad f(x) = x^3 + x^2 + x - 100$

We find $\qquad f(4) = 4^3 + 4^2 + 4 - 100$

$$= 64 + 16 + 4 - 100 = -16 < 0$$

$$f(4.5) = (4.5)^3 + (4.5)^2 + 4 - 100$$

$$= 15.875 > 0$$

So, the root is between 4 and 4.5
We take the values of x as 4, 4.1, 4.2, 4.3, 4.4
We shall form the table as below with $y = f(x)$

x	4	4.1	4.2	4.3	4.4
$y = f(x)$	−16	−10.169	−4.072	2.297	8.944

It can be seen that the root lies between 4.2 and 4.3
We shall take the origin as $x_0 = 4.2$ and use Stirling's formula to find the root.
Stirling's formula is

$$y = y_0 + \left[\frac{u(\Delta y_0 + \Delta y_{-1})}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2 - 1^2)}{3!}\left[\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2}\right] + \ldots$$

where $\qquad u = \frac{x - x_0}{h} = \frac{x - 4.2}{0.1}$

We shall form the difference table.

x	$u=\dfrac{x-4.2}{0.1}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
4	−2	−16				
			5.831			
4.1	−1	−10.169		0.266		
			$\boxed{6.097}$		$\boxed{0.006}$	
4.2	0	$\boxed{-4.072}$		$\boxed{0.272}$		$\boxed{0}$
			$\boxed{6.369}$		$\boxed{0.006}$	
4.3	1	2.297		0.278		
			6.647			
4.4	2	8.944				

$$\therefore \quad y=-4.072+\frac{u[6.369+6.097]}{2}+\frac{u^2}{2}(0.272)+\frac{u(u^2-1)}{6}\left[\frac{0.006+0.006}{2}\right]$$

When $y = 0$, we get

$$0 = -4.072 + 6.233\,u + 0.136\,u^2 + 0.001\,u\,(u^2-1)$$

$$\Rightarrow \quad 6.233\,u = 4.072 - 0.136\,u^2 - 0.001\,u\,(u^2-1)$$

$$\Rightarrow \quad u=\frac{4.072}{6.233}-\frac{0.136}{6.233}u^2-\frac{0.001}{6.233}u(u^2-1)$$

$$\Rightarrow \quad u = 0.6533 - 0.0218\,u^2 - 0.00016\,u(u^2-1)$$

First approximation is

$$u_1 = \mathbf{0.6533}$$

Second approximation is

$$\begin{aligned} u_2 &= 0.6533-0.0218u_1^2-0.00016\,u_1(u_1^2-1) \\ &= 0.6533-0.0218(0.6533)^2-0.00016(0.6533)[(0.6533)^2-1] \\ &= 0.6533-0.0093+0.0000599 \\ &= \mathbf{0.6439} \end{aligned}$$

Third approximation is

$$u_3 = 0.6533 - 0.0218\, u_2^2 - 0.00016\, u_2(u_2^2 - 1)$$

$$= 0.6533 - 0.0218(0.6439)^2 - 0.00016(0.6439)\left[(0.6439)^2 - 1\right]$$

$$= 0.6533 - 0.009038 + 0.00006031$$

$$= \mathbf{0.6442}$$

Fourth approximation is

$$u_4 = 0.6533 - 0.0218\, u_3^2 - 0.00016\, u_3(u_3^2 - 1)$$

$$= 0.6533 - 0.0218(0.6442)^2 - 0.00016(0.6442)\left[(0.6442)^2 - 1\right]$$

$$= 0.6533 - 0.009047 + 0.00006030$$

$$= \mathbf{0.6442}$$

Since u_3 and u_4 coincide upto 4 decimal places, $u = 0.6442$

$$\Rightarrow \quad \frac{x - 4.2}{0.1} = 0.6442$$

$$\Rightarrow \quad x - 4.2 = 0.6442 \times 0.1$$

$$\Rightarrow \quad x = 4.2 + 0.06442 = 4.2644$$

$\therefore$ the root correct to four decimal places is $\mathbf{x = 4.2644}$ ■

Example 4

Find the value of x if $\sqrt[3]{x} = 3.756$ given the following table.

x	50	52	54	56
$\sqrt[3]{x}$	3.684	3.732	3.779	3.825

Solution

Let $y = \sqrt[3]{x}$

Required the value of x if $\sqrt[3]{x} = 3.756$. That is to find x when $y = 3.756$

From the given table, it is obvious that 3.756 lies between $x = 52$ and $x = 54$.

So, we choose the origin as $x_0 = 52$ and use Stirling's formula to find the value of x.

Stirling's formula is

$$y = y_0 + \frac{u(\Delta y_0 + \Delta y_{-1})}{2} + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2 - 1^2)}{3!}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2} + \ldots$$

where $\quad u = \dfrac{x - x_0}{h} = \dfrac{x - 52}{2}$

We shall form the difference table

x	$u = \frac{x-52}{2}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$
50	−1	3.684			
			$\boxed{0.048}$		
52	0	$\boxed{3.732}$		$\boxed{-0.001}$	
			$\boxed{0.047}$		$\boxed{0}$
54	1	3.779		−0.001	
			0.046		
56	2	3.825			

$$\therefore \quad y = 3.732 + \frac{u[0.047+0.048]}{2} + \frac{u^2}{2}(-0.001)$$

$$= 3.732 + 0.0475\,u - 0.0005u^2$$

$$\Rightarrow \quad 0.0475\,u = y - 3.732 + 0.0005u^2$$

When $y = 3.756$, we get

$$0.0475u = 3.756 - 3.732 + 0.0005u^2$$

$$= 0.024 + 0.0005u^2$$

$$\Rightarrow \quad u = \frac{0.024}{0.0475} + \frac{0.0005}{0.0475}u^2$$

$$= 0.5053 + 0.01053\,u^2$$

First approximation is

$$u_1 = 0.5053$$

Second approximation is

$$u_2 = 0.5053 + 0.01053u_1^2$$

$$= 0.5053 + 0.01053\,(0.5053)^2$$

$$= 0.5053 + 0.002689$$

$$= 0.5080$$

Third approximation is

$$u_3 = 0.5053 + 0.01053u_2^2$$

$$= 0.5053 + 0.01053\,(0.5080)^2$$

$$= 0.5053 + 0.0027174$$

$$= 0.5080$$

Since u_2 and u_3 coincide upto four decimal places,

$$u = 0.5080$$

$$\frac{x-52}{2} = 0.5080$$

$$\Rightarrow \quad x - 52 = 0.5080 \times 2$$

$$\Rightarrow \quad x = 52 + 1.0160 = 53.0160 = 53$$

$\therefore$ if $\sqrt[3]{x} = 3.756$, then the value of $\boldsymbol{x = 53}$

Exercises 5.2

(1) Find the solution of the equation $x^2 - 6x - 11 = 0$ between 3 and 4, by iteration.

(2) Given the following table of values of probability integral $I(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx$, find the value of x for which the integral is equal to $\frac{1}{2}$, by iteration.

x	0.45	0.46	0.47	0.48	0.49	0.50
I(x)	0.475	0.484	0.493	0.502	0.511	0.520

(3) The following table gives the values of sinh x for certain equidistance values if x.

x	4.80	4.81	4.82	4.83	4.84
sinh x	60.7511	61.3617	61.9785	62.6015	63.2307

Find the value of x when $\sinh x = 62$.

(4) Find the real root of the equation $x^3 + x - 5 = 0$, correct to two places of decimals.

(5) Find the real root of the equation $x^3 - 2x - 5 = 0$, correct to two places of decimals.

(6) Find the positive root of the equation $x^3 - 5x + 3 = 0$.

Answers 5.2

(1) 3.091 (2) 0.4769 (3) 4.8203 (4) 1.52 (5) 2.09 (6) 0.6566

REVERSION OF SERIES METHOD

All the interpolation formula we have seen so far are expressed as a series containing powers of u, where $u = \frac{x - x_0}{h}$.

In other words, they are all power series in u of the form

$$y_u = a_0 + a_1 u + a_2 u^2 + a_3 u^3 + \ldots$$

where $a_0, a_1, a_2, \ldots$ are known constants

We know that any convergent power series can be reverted.

Now
$$\frac{y_u - a_0}{a_1} = u + \frac{a_2}{a_1} u^2 + \frac{a_3}{a_1} u^2 + \ldots$$

Put
$$v = \frac{y_u - a_0}{a_1}, \text{ then}$$

$$v = u + b_2 u^2 + b_3 u^3 + \ldots$$

where $$b_2 = \frac{a_2}{a_1},\ b_3 = \frac{a_3}{a_1}\ldots$$

We shall express u as a power series in v.

Let $$u = c_0 + c_1 v + c_2 v^2 + c_3 v^3 + \ldots$$

Substituting for v, we get

$$u = c_0 + c_1(u + b_2u^2 + b_3u^3 + \ldots) + c_2(u + b_2u^2 + b_3u^3 + \ldots)^2 + c_3(u + b_2u^2 + b_3u^3 + \ldots)^3 + \ldots$$

$$\Rightarrow \quad u = c_0 + c_1u + (c_1b_2 + c_2)u^2 + (c_1b_3 + 2c_2b_2 + c_3)u^3 + (c_1b_4 + c_2b_2{}^2 + 2c_2b_3 + 3c_3b_2 + c_4)u^4 + \ldots$$

Equating like coefficients of u on both sides, we get $c_0 = 0$, $c_1 = 1$,

$$c_1b_2 + c_2 = 0 \quad \Rightarrow c_2 = -c_1b_2 = -\frac{a_2}{a_1}$$

$$c_1b_3 + 2c_2b_2 + c_3 = 0$$

$$\Rightarrow \quad c_3 = -c_1b_3 - 2c_2b_2 = -b_3 - 2c_2b_2 = -\left(\frac{a_3}{a_1}\right) - 2\left(-\frac{a_2}{a_1}\right)\cdot\left(\frac{a_2}{a_1}\right) = -\frac{a_3}{a_1} + 2\left(\frac{a_2}{a_1}\right)^2$$

$$c_1b_4 + c_2b_2^2 + 2c_2b_3 + 3c_3b_2 + c_4 = 0$$

$$\Rightarrow \quad c_4 = -c_1b_4 - c_2b_2^2 - 2c_2b_3 - 3c_3b_2$$

$$= -\frac{a_4}{a_1} + \left(\frac{a_2}{a_1}\right)^3 - 2\left(-\frac{a_2}{a_1}\right)\left(\frac{a_3}{a_1}\right) - 3\left\{-\frac{a_3}{a_1} + 2\left(\frac{a_2}{a_1}\right)^2\right\}\frac{a_2}{a_1}$$

$$= -\frac{a_4}{a_1} + \left(\frac{a_2}{a_1}\right)^3 + \frac{2a_2a_3}{a_1^2} + \frac{3a_2a_3}{a_1^2} - 6\left(\frac{a_2}{a_1}\right)^3$$

$$\Rightarrow \quad c_4 = -\frac{a_4}{a_1} - \frac{5a_2a_3}{a_1^2} - 5\left(\frac{a_2}{a_1}\right)^3 \text{ and so on.}$$

Hence $$u = v - \left(\frac{a_2}{a_1}\right)v^2 + \left\{-\frac{a_3}{a_1} + 2\left(\frac{a_2}{a_1}\right)^2\right\}v^3 + \left\{-\frac{a_4}{a_1} - 5\frac{a_2a_3}{a_1^2} - 5\left(\frac{a_2}{a_1}\right)^3\right\}v^4 + \ldots$$

where v is known, once $f(u)$ is known.

WORKED EXAMPLES

Example 1

Using the method of reversion of series of Newton's formula of interpolation find x, for $y = 3000$ from the following data.

x	10	15	20
$y = f(x)$	1754	2648	3564

Solution

We have to use Newton's formula to find x when $y = 3000$.

Choose the origin as $x_0 = 10$. Here $h = 5$

$$\therefore \quad u = \frac{x - x_0}{h} = \frac{x-10}{5}$$

We form the difference table.

x	$u = \frac{x-10}{5}$	y_u	Δy	$\Delta^2 y$
10	0	$\boxed{1754}$		
			$\boxed{894}$	
15	1	2648		$\boxed{222}$
			916	
20	2	3564		

Newton's formula is

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0$$

$$= 1754 + 894u + \frac{u^2 - u}{2} \times 22$$

$$= 1754 + 894u + 11u^2 - 11u$$

$$= 1754 + 883u + 11u^2$$

$$\therefore \quad a_0 = 1754,\ a_1 = 883,\ a_2 = 11$$

Now
$$v = \frac{y_u - a_0}{a_1}$$

Given $y = 3000$

$$\therefore \quad v = \frac{3000 - 1754}{883} = 1.411$$

Now
$$u = c_1 v + c_2 v^2$$

where
$$c_1 = 1, \quad c_2 = -\frac{a_2}{a_1} = -\frac{11}{883} = -0.1246$$

$\therefore$ $$u = 1.411 - 0.01246 \times (1.411)^2$$
$$= 1.3862$$

$\Rightarrow$ $$\frac{x-10}{5} = 1.3862$$

$\Rightarrow$ $$x = 10 + 5 \times 1.3862 = 16.931$$

$\therefore$ when $\boldsymbol{y = 3000,\ x = 16.931}$ ■

Example 2

The equation $x^2 - 6x - 11 = 0$ has a root between 3 and 4 Obtain it by inverse interpolation.

Solution

Let $y = x^2 - 6x - 11$

Since the root lies between 3 and 4, we form a table and use reversely.

The table is

x	2.8	2.9	3	3.1	3.2	3.3
y	–5.848	–4.011	–2	0.191	2.568	5.137

The value of $y = 0$ for a value of x between 3 and 3.1 and it is closer to 3.1.

We choose the origin as $x_0 = 3.1$. Here $h = 0.1$

$\therefore$ $$u = \frac{x - x_0}{h} = \frac{x - 3.1}{0.1}$$

We form the difference table.

x	$u = \dfrac{x-3.1}{0.1}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$
2.8	–3	–5.848			
			1.837		
2.9	–2	–4.011		0.174	
			2.011		0.006
3	–1	–2		0.180	
			2.191		0.006
3.1	0	$\boxed{0.191}$		0.186	
			$\boxed{2.377}$		0.006
3.2	1	2.568		$\boxed{0.192}$	
			2.569		
3.3	2	5.137			

We shall apply Newton's forward formula and find the value of x when $y = 0$
Newton's forward formula is

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

$$= 0.191 + u(2.377) + \frac{(u^2-u)}{2!}(0.192)$$

$$= 0.191 + 2.377u + (u^2-u)(0.096)$$

$$= 0.191 + 2.281u + 0.096\,u^2$$

Here $a_0 = 0.191$, $a_1 = 2.281$ $a_2 = 0.096$

$$v = \frac{y_u - a_0}{a_1} = \frac{y_u - 0.191}{2.281}$$

Since $y = 0$,

$$v = \frac{0-0.191}{2.281} = -0.0837$$

Now $$u = c_1 v + c_2 v^2,$$

where $c_1 = 1$ and $c_2 = -\frac{a_2}{a_1}$

$$= -\frac{0.096}{2.281} = -0.0421$$

$\therefore$ $$u = v - 0.0421v^2$$

$$= -0.0837 - 0.0421(-0.0837)^2$$

$$= -0.08399$$

$\Rightarrow$ $$\frac{x-3.1}{0.1} = -0.08399$$

$\Rightarrow$ $$x = 3.1 - 0.08399 \times 0.1$$

$$= 3.1 - 0.008399$$

$\Rightarrow$ $$x = 3.0916$$

$\therefore$ the root of the equation is $\mathbf{x = 3.0916}$ ■

Example 3

From the data given below, determine correct to four decimals the value of x for which $f(x)$ is equal to 0.5.

x	0.45	0.46	0.47	0.48	0.49	0.5
$f(x)$	0.47548	0.48466	0.49375	0.50275	0.51167	0.5205

Solution

Let $y = f(x)$

From the given table, we observe that when $y = f(x) = \frac{1}{2}$, the value of x is between 0.47 and 0.48 and is nearer to 0.48

$\therefore$ choose the origin as $x_0 = 0.48$ Here $h = 0.01$

$\therefore$ $u = \frac{x - x_0}{h} = \frac{x - 0.48}{0.01}$

We use Stirling's formula to find x when $y = \frac{1}{2}$
Stirling formula is

$$y_u = y_0 + \frac{u(\Delta y_0 + \Delta y_{-1})}{2!} + \frac{u^2}{2}\Delta^2 y_{-1}$$
$$+ \frac{u(u^2 - 1^2)}{3!}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2}$$
$$+ \frac{u^2(u^2 - 1^2)}{4!}\Delta^4 y_{-2} + \ldots$$

We form the difference table.

x	$u = \frac{x-0.48}{0.01}$	$y = y_u$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.45	−3	0.47548				
			0.00918			
0.46	−2	0.48466		−0.00009		
			0.00909		0	
0.47	−1	0.49375		−0.00009		0.00001
			$\boxed{0.00900}$		$\boxed{0.00001}$	
0.48	0	$\boxed{0.50275}$		$\boxed{0.00008}$		$\boxed{-0.00002}$
			$\boxed{0.00892}$		$\boxed{-0.00001}$	
0.49	1	0.51167		−0.00009		
			0.00883			
0.50	2	0.52050				

$\therefore$

$$y_u = 0.50275 + \frac{u[0.00892 + 0.00900]}{2} + \frac{u^2}{2}(-0.00008)$$
$$+ \frac{u(u^2-1)}{6}\frac{(-0.00001+0.00001)}{2} + \frac{u^2(u^2-1)}{24}(-0.00002)$$
$$= 0.50275 + 0.00896u - 0.00004u^2 - 0.0000008333(u^4 - u^2)$$
$$= 0.50275 + 0.00896u - 0.00003917u^2 - 0.0000008333u^4$$

$\therefore$ $a_0 = 0.50275,\quad a_1 = 0.00896,\quad a_2 = -0.00003917,\quad a_3 = 0,$
$a_4 = -0.0000008333.$

Given $$y_u = \frac{1}{2} = 0.5$$

$\therefore$ $$v = \frac{y_u - a_0}{a_1} = \frac{0.5 - 0.50275}{0.00896} = \mathbf{-0.3069}$$

Now $$u = c_1v + c_2v^2 + c_3v^3 + c_4v^4$$

where $$c_1 = 1,\ c_2 = -\frac{a_2}{a_1},\quad c_3 = -\frac{a_3}{a_1} + 2\left(\frac{a_2}{a_1}\right)^2$$

$\therefore$ $$c_2 = -\frac{(-0.00003917)}{0.00896} = 0.00437$$

$$c_3 = 0 + 2(0.00437)^2 = 0.00003819$$

and $$c_4 = -\frac{a_4}{a_1} + 5\frac{a_2a_3}{a_1^2} - 5\left(\frac{a_2}{a_1}\right)^3$$
$$= -\frac{(-0000008333)}{0.00896} + 0 - 5(0.00437)^3$$
$$= 0.00009297 - 0.000000417 = 0.00009255$$

$\therefore$ $$u = v + 0.00437v^2 + 0.0003819v^3 + 0.00009255v^4$$

$\Rightarrow$ $$u = -0.3069 + 0.00437(-0.3069)^2$$
$$+ 0.0003819(-0.3069)^3 + 0.00009255(-0.3069)^4 = -0.306499$$

$\Rightarrow$ $$\frac{x - 0.48}{0.01} = -0.306499$$

$\Rightarrow$ $$x = 0.48 + 0.01(-0.306499)$$
$$= 0.48 - 0.00306499$$
$$= 0.48 - 0.00306 = 0.4769 \text{ to four places.}$$

$\therefore$ when $f(x) = \frac{1}{2}$, $\quad \mathbf{x = 0.4769}$ ■

Exercises 5.3

(1) Estimate the value of x when $f(x) = 2$, given the following table:

x	4.80	4.81	4.82	4.83	4.84
$f(x)$	0.7511	1.3617	1.9785	2.6015	3.2307

(2) Find x when $u_x = 0.163$, given

x	80	82	84	86	88
u_x	0.134	0.154	0.176	0.200	0.227

(3) Using inverse interpolation find the root of $x^3 - 3x - 7 = 0$ which lies between 2 and 3.

Answers 5.3

(1) 4.8202 (2) 82.8 (3) 2.42599

SHORT ANSWER QUESTIONS

1. What is inverse interpolation?
2. Write the Lagrange's inverse interpolation formula for a set of n observations $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.
3. Write the Lagrange's inverse interpolation formula to find x if $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ are given.
4. Find the value of x when $y = 1.5$ from the following data.

x	0	1	2
y	2	0	1

5. Find x when $y = 5$, from the following data.

x	1	2	3
y	1	8	27

6. Using Lagrange's inverse interpolation formula find x when $y = 10$ from the following data.

x	12	13
y	5	6

7. Given $f(0) = -1$, $f(1) = 1$, $f(2) = 4$ find the value of x when $y = 5$ by using Lagrange's formula inversely.

Numerical Differentiation

INTRODUCTION

Let a function $y = f(x)$ be given by a table of values (x_i, y_i), $i = 1, 2, 3, \ldots, n$. The process of finding the derivative $\frac{dy}{dx}$ for some particular value of $x = x'$ is called **numerical differentiation**. This process consists of approximating the function by a suitable interpolation formula and then differentiating it.

If the values of x are equally spaced, to find derivative near the beginning of the table, we use Newton's forward difference formula and to find derivative near the end of the table, we use Newton's backward difference formula. If the arguments are unequally spaced, we use the divided difference formula to find the derivative.

Remark on numerical differentiation: Numerical differentiation should be used with caution. If the differences of some order are constant, polynomial approximation for $y(x)$ is quite accurate and we can use the method. Otherwise, derivative will be rough value with errors of considerable magnitude.

In this chapter, we discuss maxima and minima of functions represented by a table of values, using numerical differentiation technique.

NUMERICAL DIFFERENTIATION

Derivative Using Newton's Forward Difference Interpolating Formula

Given (x_r, y_r), $r = 0, 1, 2, 3, \ldots, n$, where $x_r = x_0 + rh$,

Newton's forward difference interpolation formula is

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \ldots \quad (1)$$

where $u = \dfrac{x - x_0}{h}$ (2)

y is a function of u and u is a function of x.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{h}\cdot\frac{dy}{du} \qquad \text{[by Chain rule and from (2)]}$$

Differentiating (1) w.r.to u, we get

$$\frac{dy}{du} = \Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2 - 6u + 2}{6}\right)\Delta^3 y_0 + \left(\frac{4u^3 - 18u^2 + 22u - 6}{24}\right)\Delta^4 y_0 + \ldots$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2 - 6u + 2}{6}\right)\Delta^3 y_0 + \left(\frac{4u^3 - 18u^2 + 22u - 6}{24}\right)\Delta^4 y_0 + \ldots\right] \quad (3)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx}$$

$$= \frac{1}{h}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \left(\frac{6u^2 - 18u + 11}{12}\right)\Delta^4 y_0 + \ldots\right]\cdot\frac{1}{h}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \left(\frac{6u^2 - 18u + 11}{12}\right)\Delta^4 y_0 + \cdots\right]$$

At $x = x_0$, $u = \dfrac{x_0 - x_0}{h} = 0$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 + \ldots\right]$$

and $$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \ldots\right]$$

Similarly we can find the derivative of higher orders, such as $\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}$ etc.

Derivative Using Newton's Backward Difference Interpolating Formula

Given $(x_i, y_i), i = 0, 1, 2, 3, \ldots, n$ where $x_r = x_0 + rh$,
Newton's backward interpolation formula is

$$y = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n + \ldots \quad (1)$$

where $$v = \frac{x - x_n}{h}$$

$\therefore$ y is a function of v and v is a function of x.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{dx} = \frac{1}{h}\frac{dy}{dv}$$

Now differentiating (1) w.r.to v, we get

$$\frac{dy}{dv} = \nabla y_n + \left(\frac{2v+1}{2!}\right)\nabla^2 y_n + \left(\frac{3v^2+6v+2}{3!}\right)\nabla^3 y_n + \left(\frac{4v^3+18v^2+22v+6}{4!}\right)\nabla^4 y_n + \ldots$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \left(\frac{2v+1}{2}\right)\nabla^2 y_n + \left(\frac{3v^2+6v+2}{6}\right)\nabla^3 y_n + \left(\frac{4v^3+18v^2+22v+6}{24}\right)\nabla^4 y_n + \ldots\right]$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dv}\left(\frac{dy}{dx}\right)\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (v+1)\nabla^3 y_n + \left(\frac{6v^2+18v+11}{12}\right)\nabla^4 y_n + \ldots\right]$$

At $x = x_n,\ v = \frac{x_n - x_n}{h} = 0$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \ldots\right]$$

$$\text{and } \left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \ldots\right]$$

Similarly we can find the derivatives of higher orders, such as $\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}$, etc.

WORKED EXAMPLES

Example 1

Find $y'(x)$ from the table given below and hence find $y'(0)$ and $y''(0)$.

x	0	1	2	3	4
y	4	8	15	7	6

Solution

Given values of x are equally spaced and $x_0 = 0$ is in the beginning of the table.
So, we use Newton's forward formula to find $y'(x), y'(0)$ and $y''(0)$
By Newton's forward formula,

$$y'(x) = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2 - 6u + 2}{6}\right)\Delta^3 y_0 + \left(\frac{4u^3 - 18u^2 + 22u - 6}{24}\right)\Delta^4 y_0 + \ldots\right]$$

where $u = \frac{x - x_0}{h}$. Here $x_0 = 0,\ h = 1$

$$\therefore \quad u = x$$

$$\therefore \quad y'(x) = \Delta y_0 + \left(\frac{2x-1}{2}\right)\Delta^2 y_0 + \left(\frac{3x^2 - 6x + 2}{6}\right)\Delta^3 y_0 + \frac{4x^3 - 18x^2 + 22x - 6}{24}\Delta^4 y_0 + \ldots$$

Now we form the forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	**4**				
		4			
1	8		**3**		
		7		**−18**	
2	15		−15		**40**
		−8		22	
3	7		7		
		−1			
4	6				

$$\therefore \quad y'(x) = 4 + \left(\frac{2x-1}{2}\right)3 + \left(\frac{3x^2-6x+2}{6}\right)(-18) + \left(\frac{4x^3-18x^2+22x-6}{24}\right)(40)$$

$$= 4 + 3x - \frac{3}{2} - 9x^2 + 18x - 6 + \frac{20}{3}x^3 - 30x^2 + \frac{110}{3}x - 10$$

$$= \frac{20}{3}x^3 - 39x^2 + \left(21 + \frac{110}{3}\right)x - \left(12 + \frac{3}{2}\right)$$

$$\Rightarrow \quad y'(x) = \frac{20}{3}x^3 - 39x^2 + \frac{173}{3}x - \frac{27}{2}$$

$$\therefore \quad y''(x) = 20x^2 - 78\,x + \frac{173}{3}$$

At $x = 0$, $y'(0) = \frac{-27}{2} = -13.5$ and $y''(0) = \frac{173}{3} = 57.7$ ■

Example 2

Find $y'(x)$ given

x	0	1	2	3	4
$y(x)$	1	1	15	40	85

Hence find $y'(x)$ at $x = 0.5$.

Solution

The values of x are equally spaced and $x = 0.5$ is near the beginning of the table. So, we use Newton's forward formula to find $y'(0.5)$

$$y'(x) = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{3}\right)\Delta^3 y_0 + \left(\frac{4u^3-18u^2+22u-6}{24}\right)\Delta^4 y_0 + \ldots\right]$$

where $u = \dfrac{x-x_0}{h}$. Here $x_0 = 0, h = 1$ $\therefore u = x$

$\therefore$
$$y'(x) = \Delta y_0 + \left(\frac{2x-1}{2}\right)\Delta^2 y_0 + \left(\frac{3x^2-6x+2}{6}\right)\Delta^3 y_0 + \left(\frac{4x^3-18x^2+22x-6}{24}\right)\Delta^4 y_0 + \ldots \quad (1)$$

We form the forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		0			
1	1		**14**		
		14		**-3**	
2	15		11		**12**
		25		9	
3	40		20		
		45			
4	85				

Substituting the values of Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$, $\Delta^4 y_0$ in (1), we get,

$$y'(x) = 0 + \left(\frac{2x-1}{2}\right)(14) + \left(\frac{3x^2-6x+2}{6}\right)(-3) + \left(\frac{4x^3-18x^2+22x-6}{24}\right)(12)$$

$$= 7(2x-1) - \frac{1}{2}(3x^2-6x+2) + 2x^3 - 9x^2 + 11x - 3$$

$$= 2x^3 - \frac{21}{2}x^2 + 28x - 11 = 2x^3 - 10.5x^2 + 28x - 11$$

At $x = 0.5$,

$$y'(0.5) = 2(0.5)^3 - 10.5(0.5)^2 + 28(0.5) - 11$$
$$= 0.250 - 2.625 + 14 - 11 = \mathbf{0.625}$$

■

Example 3

Find sec 31° from the following data:

$\theta°$	31	32	33	34
tan θ	0.6008	0.6249	0.6494	0.6745

Solution

The values of θ are equally spaced and $\theta° = 31°$ is the beginning value of the table. So, we find sec 31° using Newton's forward formula. Let $y = \tan\,\theta$, θ is radian.

Newton's forward formula is

$$\frac{dy}{d\theta} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \left(\frac{4u^3-18u^2+22u-6}{24}\right)\Delta^4 y_0 + \ldots\right]$$

where $u = \dfrac{\theta - \theta_0}{h}$. Here $x_0 = 31$, $h = 1° = \dfrac{\pi}{180}$ radian $\therefore u = \dfrac{\theta - 31°}{1} = \theta - 31°$

When $\theta = 31°, u = 0$

$$\therefore \quad \frac{dy}{d\theta} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right]$$

Now we form the forward difference table.

$\theta°$	$y = \tan\theta$	Δy	$\Delta^2 y$	$\Delta^3 y$
31	0.6008			
		0.0241		
32	0.6249		**0.0004**	
		0.0245		**0.0002**
33	0.6494		0.0006	
		0.0251		
34	0.6745			

Substituting the values of Δy, $\Delta^2 y$, $\Delta^3 y$ in (1), we get

$$\therefore \quad \left(\frac{dy}{d\theta}\right)_{\theta=31} = \frac{180}{\pi}\left[0.0241 - \frac{1}{2}(0.0004) + \frac{1}{3}(0.0002)\right]$$

$$= \frac{180}{\pi}[0.0241 - 0.0002 + 0.000067]$$

$$= \frac{180}{\pi}(0.024147) = 1.3835$$

But $\quad y = \tan\theta$

$$\therefore \quad \frac{dy}{d\theta} = \sec^2\theta \Rightarrow \sec 31° = \sqrt{1.3835} = \mathbf{1.176}$$

■

Example 4

Find the value of cos 1.74 using the values given in the table below:

X	1.70	1.74	1.78	1.82	1.86
sin x	0.9916	0.9857	0.9781	0.9691	0.9584

Solution

The values of x are equally spaced and $x = 1.74$ is near the beginning of the table. So, we use Newton's forward formula to find cos 1.74.

Let $y = \sin x$

By Newton's forward formula,

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \left(\frac{4u^3-18u^2+22u-6}{24}\right)\Delta^4 y_0 + \ldots\right]$$

where $u = \dfrac{x - x_0}{h}$. $\quad$ Here $x_0 = 1.70$, $\quad h = 0.04 \quad \therefore u = \dfrac{x - 1.70}{0.04}$

When $x = 1.74$, $\quad u = \dfrac{1.74 - 1.70}{0.04} = \dfrac{0.04}{0.04} = 1$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{1}{2}\Delta^2 y_0 - \frac{1}{6}\Delta^3 y_0 + \frac{1}{12}\Delta^4 y_0 + \ldots\right] \qquad (1)$$

We now form the forward difference table

x	$y = \sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.70	0.9916				
		−0.0059			
1.74	0.9857		**−0.0017**		
		−0.0076		**0.0003**	
1.78	0.9781		−0.0014		**−0.0006**
		−0.0090		−0.0003	
1.82	0.9691		−0.0017		
		−0.0107			
1.86	0.9584				

Substituting the values of Δy, $\Delta^2 y$, $\Delta^3 y$, $\Delta^4 y$ in (1), we get,

$$\left(\frac{dy}{dx}\right)_{x=1.74} = \frac{1}{0.04}\left[-0.0059 + \frac{1}{2}(-0.0017) - \frac{1}{6}(0.0003) + \frac{1}{12}(-0.0006)\right]$$

$$= 25[-0.0059 - 0.00085 - 0.00005 - 0.00005]$$

$$= -0.17125$$

we have $\quad y = \sin x$

$$\therefore \quad \frac{dy}{dx} = \cos x \Rightarrow \cos 1.74 = \left(\frac{dy}{dx}\right)_{x=1.74} = \mathbf{-0.17125}$$

■

Example 5

Find the first derivative of $f(x)$ at $x = 0.4$ from the following table:

x	0.1	0.2	0.3	0.4
$f(x)$	1.10517	1.22140	1.34986	1.49182

Solution

The values of x are equally spaced and $x = 0.4$ is the end value of the table. To find $f'(x)$ at $x = 0.4$, we use Newton's backward formula.

By Newton's backward formula,

$$f'(x) = \frac{1}{h}\left[\nabla y_n + \left(\frac{2v+1}{2}\right)\nabla^2 y_n + \left(\frac{3v^2+6v+2}{6}\right)\nabla^3 y_n + \left(\frac{4v^3+18v^2+22v+6}{24}\right)\nabla^4 y_n + \ldots\right]$$

where $\quad v = \frac{x - x_n}{h}$. $\quad$ Here $x = 0.4$, $h = 0.1$ $\therefore v = \frac{x - 0.4}{0.1}$

When $x = 0.4$, $\quad v = \frac{0.4 - 0.4}{0.1} = 0$

$$f'(0.4) = \frac{1}{0.1}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots\right]$$

Now we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.1	1.10517			
		0.11623		
0.2	1.22140		0.01223	
		0.12846		**0.00127**
0.3	1.34986		**0.01350**	
		0.14196		
0.4	1.49182			

$$\therefore \quad f'(0.4) = \frac{1}{0.1}\left[0.14196 + \frac{1}{2}(0.01350) + \frac{1}{3}(0.00127)\right]$$

$$= 10[0.14196 + 0.00675 + 0.00042] = \mathbf{1.4913}$$ ■

Example 6

A jet fighter's position on an aircraft carrier's runway was timed during landing.

t, (sec)	**1.0**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**
y, (m)	**7.989**	**8.403**	**8.781**	**9.129**	**9.451**	**9.750**	**10.031**

where y is the distance from the end of the carrier. Estimate velocity $\frac{dy}{dt}$ and acceleration $\frac{d^2y}{dt^2}$ at (i) $t = 1.1$, (ii) $t = 1.6$ using numerical differentiation.

Solution

The values of t are equally spaced and $t = 1.1$ is near the beginning of the table. So, we use Newton's forward formula to find $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ at $t = 1.1$.

By Newton's forward formula,

$$\frac{dy}{dt}=\frac{1}{h}\left[\Delta y_0+\left(\frac{2u-1}{2}\right)\Delta^2 y_0+\left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0+\left(\frac{4u^3-18u^2+22u-6}{24}\right)\Delta^4 y_0+\dots\right]$$

where $u=\frac{t-t_0}{h}$. Here $t_0=1, h=0.1 \quad \therefore u=\frac{t-1}{0.1}$

When $t=1.1,\ u=\frac{1.1-1}{0.1}=1$

$$\frac{dy}{dt}=\frac{1}{h}\left[\Delta y_0+\frac{1}{2}\Delta^2 y_0-\frac{1}{6}\Delta^3 y_0+\frac{1}{12}\Delta^4 y_0+\dots\right] \quad (1)$$

and

$$\frac{d^2y}{dt^2}=\frac{1}{h^2}\left[\Delta^2 y_0+(u-1)\Delta^3 y_0+\frac{6u^2-18u+11}{12}\Delta^4 y_0+\dots\right]$$

$$\frac{d^2y}{dt^2}=\frac{1}{h^2}\left[\Delta^2 y_0-\frac{1}{12}\Delta^4 y_0+\dots\right] \quad (2)$$

We form the difference table

t	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1	7.989						
		0.414					
1.1	8.403		**−0.036**				
		0.378		**0.006**			
1.2	8.781		−0.030		**−0.002**		
		0.348		0.004		**0.001**	
1.3	9.129		−0.026		−0.001		**0.002**
		0.322		0.003		**0.003**	
1.4	9.451		−0.023		**0.002**		
		0.299		**0.005**			
1.5	9.750		**−0.018**				
		0.281					
1.6	10.031						

Substituting the values of Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$, $\Delta^4 y_0$ in (1) and (2) we get

$$\left(\frac{dy}{dt}\right)_{t=1.1}=\frac{1}{0.1}\left[0.414+\frac{1}{2}(-0.036)-\frac{1}{6}(0.006)+\frac{1}{12}(-0.002)\right] \text{ (omitting higher powers)}$$

$$=10[0.414-0.018-0.001-0.00017]$$

$$=10[0.39483]=\mathbf{3.9483}$$

and
$$\left(\frac{d^2y}{dt^2}\right)_{t=1.1}=\frac{1}{(0.1)^2}\left[-0.036-\frac{1}{12}(-0.002)\right]$$

$$=100[-0.036+0.000167]=100[-0.0358]=\mathbf{-3.583}$$

(ii) $t=1.6$ is the end value of the given table.
So we use Newton's backward formula for derivative.
By Newton's backward formula

$$\frac{dy}{dt}=\frac{1}{h}\left[\nabla y_n+\left(\frac{2v+1}{2}\right)\nabla^2 y_n+\left(\frac{3v^2+6v+2}{6}\right)\nabla^3 y_n+\left(\frac{4v^3+18v^2+22v+6}{24}\right)\nabla^4 y_n+\ldots\right]$$

where $v=\dfrac{t-t_n}{h}$. Here $t_n=1.6$, $h=0.1$

When $t=1.6$, $v=\dfrac{1.6-1.6}{0.1}=0$

$$\therefore \left(\frac{dy}{dt}\right)_{t=1.6}=\frac{1}{0.1}\left[\nabla y_n+\frac{1}{2}\nabla^2 y_n+\frac{1}{3}\nabla^3 y_n+\frac{1}{4}\nabla^4 y_n+\frac{1}{5}\nabla^5 y_n+\frac{1}{6}\nabla^6 y_n\right]$$

$$=10\left[0.281+\frac{1}{2}(-0.018)+\frac{1}{3}(0.005)+\frac{1}{4}(0.002)+\frac{1}{5}(0.003)+\frac{1}{6}(0.002)\right]$$

$$=10(0.281-0.009+0.00167+0.0005+0.0006+0.00033)=\mathbf{2.751}$$

we know
$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n}=\frac{1}{h^2}\left[\nabla^2 y_n+\nabla^3 y_n+\frac{11}{12}\nabla^4 y_n+\ldots\right]$$

$$\therefore \left(\frac{d^2y}{dt^2}\right)_{t=1.6}=\frac{1}{(0.1)^2}\left[-0.018+0.005+\frac{11}{12}(0.002)\right]$$

$$=100[-0.0112]=\mathbf{-1.12}$$

■

Example 7

Find $f'(10)$ from the following data:

x	3	5	11	27	34
$f(x)$	−13	23	899	17315	35606

Solution

The values of x are not equally spaced. So, we use Newton's divided difference formula to find $f(x)$ and then we find $f'(10)$

Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0)\underset{x_1}{\Delta} f(x_0) + (x - x_0)(x - x_1)\underset{x_1x_2}{\Delta^2} f(x_0)$$
$$+ (x - x_0)(x - x_1)(x - x_2)\underset{x_1x_2x_3}{\Delta^3} f(x_0) + \ldots \quad (1)$$

We form the divided difference table

x	y	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
3	-13				
		$\mathbf{\dfrac{23+13}{5-3} = 18}$			
5	23		$\mathbf{\dfrac{146-18}{11-3} = 16}$		
		$\dfrac{899-23}{11-5} = 146$		$\mathbf{\dfrac{40-16}{27-3} = 1}$	
11	899		$\dfrac{1026-146}{27-5} = 40$		**0**
		$\dfrac{17315-899}{27-11} = 1026$		$\dfrac{69-40}{34-5} = 1$	
27	17315		$\dfrac{2613-1026}{34-11} = 69$		
		$\dfrac{36606-17315}{34-27} = 2613$			
34	35606				

Substituting the values of $f(x_0)$, $\Delta f(x_0)$, $\Delta^2 f(x_0)$, $\Delta^3 f(x_0)$ in (1) we get

$$f(x) = -13 + (x-3)\times 18 + (x-3)(x-5)\times 16 + (x-3)(x-5)(x-11)\times 1$$
$$= -13 + 18x - 54 + 16(x^2 - 8x + 15) + x^3 - 19x^2 + 103x - 165$$

$\Rightarrow \qquad f(x) = x^3 - 3x^2 - 7x + 8$

$\therefore \qquad f'(x) = 3x^2 - 6x - 7$

When $x = 10$, $f'(10) = 3\times 100 - 6\times 10 - 7 = \mathbf{233}$ ■

Example 8

Using the following data find $f(x)$ as polynomial in x and hence find $f'(4)$, $f''(4)$.

x	0	1	2	5
$f(x)$	2	3	12	147

Solution

The values of x are not equally spaced and so, we use Newton's divided difference formula to find $f(x)$ and then we find $f'(4)$, $f''(4)$

Newton's divided difference formula is

$$f(x)=f(x_0)+(x-x_0)\underset{x_1}{\Delta} f(x_0)+(x-x_0)(x-x_1)\underset{x_1x_2}{\Delta^2} f(x_0).$$
$$+(x-x_0)(x-x_1)(x-x_2)\underset{x_1x_2x_3}{\Delta^3} f(x_0)+\dots \quad (1)$$

Now we form the divided difference table

x	y	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	2			
		$\frac{3-2}{1-0}=1$		
1	3		$\frac{9-1}{2-0}=4$	
		$\frac{12-3}{2-1}=9$		$\frac{9-4}{5-0}=1$
2	12		$\frac{45-9}{5-1}=9$	
		$\frac{147-12}{5-2}=45$		
5	147			

Substituting the values of $f(x_0)$, $\Delta f(x_0)$, $\Delta^2 f(x_0)$, $\Delta^3 f(x_0)$ in (1), we get

$$f(x)=2+(x-0)\times1+(x-0)(x-1)\times4+(x-0)(x-1)(x-2)\times1$$

$\Rightarrow$ $$f(x)=2+x+4x^2-4x+x^3-3x^2+2x$$

$\Rightarrow$ $$f(x)=x^3+x^2-x+2$$

Differentiating w.r.to x,

$$f'(x) = 3x^2 + 2x - 1,\ f''(x) = 6x + 2$$

When $x = 4$,

$$f'(4) = 3 \times 4^2 + 2 \times 4 = 56,\ f''(4) = 6 \times 4 + 2 = 26$$

■

Exercises 6.1

(1) Find the value of $f'(x)$ and hence $f'(0)$ from the following table:

x	0	1	2	3	4	5
y	0	0.25	0	2.25	16	56.25

(2) The following table gives the velocity of a body at t. Find its derivative at $t = 1.1$

t	1.0	1.1	1.2	1.3	1.4
v	43.1	47.7	52.1	56.4	60.8

(3) Find $y'(0)$ and $y''(0)$ from the following table:

x	0	1	2	3	4	5
y	4	8	15	7	6	2

(4) Find the first and second derivatives at $x = 1.5$ if

x	1.5	2	2.5	3	3.5	4
$f(x)$	3.375	7.000	13.625	24.000	38.875	59.000

(5) Find $f'(x), f''(x)$ of the function $f(x)$, given by the following table at the point $x = 1.1$

x	1	1.2	1.4	1.6	1.8	2
$f(x)$	0	1.28	0.544	1.296	2.432	4

(6) Given the following table of values of x and y.

x	1	1.05	1.10	1.15	1.20	1.25	1.30
y	1	1.0247	1.0488	1.0723	1.0954	1.1180	1.1401

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at (a) $x = 1$, and (b) $x = 1.25$

(7) Find the derivative of $f(x)$ at $x = 0.4$ from the following table:

x	0.1	0.2	0.3	0.4
$f(x)$	1.10517	1.22140	1.34986	1.49182

(8) Given the following table:

x	1.96	1.98	2	2.02	2.04
y	0.7825	0.7739	0.7651	0.7563	0.7473

Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ at $x = 2.03$.

(9) Find the values of $f'(3)$ and $f''(3)$ from the following data, using the method Newton's divided difference formula for interpolation.

x .	1	3	5	7	9
$f(x)$	85.3	74.3	67.0	60.5	54.3

(10) Find the values of $f'(4)$ and $f''(4)$ from the following data using Newton's divided difference formula:

x	1	2	4	8	10
$f(x)$	0	1	5	21	27

(11) Find the values of $y'(x)$ at $x = 2$, $x = 5$ from the following data:

x	0	1	3	6
$f(x)$	18	10	−18	40

(12) Find the values of $y'(x)$ and $y''(x)$ at (i) $x = 51$ and (ii) $x = 55$ from the following data:

x	50	51	52	53	54	55	56
y	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

(13) Find the values of $f'(23)$ and $f''(23)$ from the following data:

x	15	17	19	21	23	25
$f(x)$	3.873	4.123	4.359	4.583	4.796	5

(14) Find the first and second derivatives of $y = \sqrt{x}$ at (i) $x = 15$ and (ii) $x = 25$ from the following data:

x	15	17	19	21	23	25
$y = \sqrt{x}$	3.873	4.123	4.359	4.583	4.796	5.000

(15) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 51$ from the following data:

x	50	60	70	80	90
y	19.96	36.65	58.81	77.21	94.61

Answers 6.1

(1) $f'(x) = 4x^3 - 12x^2 + 8x,\ f'(0) = 0$

(2) 45.16

(3) −27.9, 117.67

(4) $f'(1.5) = 4.75,\ f''(1.5) = 9$

(5) $f'(x) = 0.63,\ f''(x) = 6.6$

(6) (a) 0.5005, −0.2732;
(b) 0.4473, −0.1583

(7) 1.4913

(8) $\frac{dy}{dx} = -0.06,\ \frac{d^2y}{dx^2} = 0.5$

(9) $f'(3) = -4.3167,\ f''(3) = 0.7917$

(10) 0.58, 1.029

(11) −15.444, 28.889

(12) (i) 0.02425, −0.0003
(ii) 0.02305, −0.0003

(13) (i) 0.1040 (ii) −0.0023

(14) (i) 0.1292, −0.0046,
(ii) 0.1004, −0.0029

(15) $\frac{dy}{dx} = 1.0313;\ \frac{d^2y}{dx^2} = 0.2303$

MAXIMA AND MINIMA OF TABULATED FUNCTION

In calculus, for a differentiable function f, local maximum and minimum are obtained by solving $f'(x) = 0$ and testing the sign of $f''(x)$ for the solutions of $f'(x) = 0$.

If $f''(x) < 0$ for a solution x_1, then $f(x)$ has a maximum at x_1.

If $f''(x) > 0$ for a solution x_2, then $f(x)$ has a minimum at x_2.

In the case of tabulated data, we represent it by a suitable interpolation formula and proceed as above.

Suppose the data give rise to a constant column in the difference table, we represent by Newton's forward formula.

Otherwise, we have to choose a suitable value as the origin and represent by an appropriate interpolation formula.

Suppose the given values increase up to x_1 and then decrease, then in the neighbourhood of x_1 maximum occurs.

Choose x_1 as the origin and $u = \frac{x - x_1}{h}$. Since u is small higher powers of u may be neglected and so we may take the formula in u upto third or fourth degree and proceed as above to get the maximum or minimum value of the function.

WORKED EXAMPLES

Example 1

Find the maximum and minimum values of y tabulated below:

x	-2	-1	0	1	2	3	4
y	2	-0.25	0	-0.25	2	15.75	56

Solution

The given values of x and y are

$x_0 = -2,\quad x_1 = -1,\quad x_2 = 0,\quad x_3 = 1,\quad x_4 = 2,\quad x_5 = 3,\quad x_6 = 4$

$y_0 = 2,\quad y_1 = -0.25,\quad y_2 = 0,\quad y_3 = -0.25,\quad y_4 = 2,\quad y_5 = 15.75,\quad y_6 = 4$

Let us form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
-2	2					
		-2.25				
-1	-0.25		2.50			
		0.25		-3		
0	0		-0.50		6	
		-0.25		3		
1	-0.25		2.50		6	
		2.25		9		
2	2		11.50		6	
		13.75		15		
3	15.75		26.50			
		40.25				
4	56					

Since the fourth differences are constant, the given function is a polynomial of degree 4 in x
We shall represent the function by Newton's forward formula

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \ldots$$

where $u = \dfrac{x - x_0}{h}$

Choose the origin as $x_0 = 0$, $h = 1 \quad \therefore u = x$

$$\therefore \quad y = y_0 + x\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 y_0 + \frac{x(x-1)(x-2)(x-3)}{4!}\Delta^4 y_0$$

$$= 0 + x(-0.25) + \frac{x(x-1)}{2}(2.5) + \frac{x(x-1)(x-2)}{6}\times 9 + \frac{x(x-1)(x-2)(x-3)}{24}\times 6$$

$$\Rightarrow \quad y = -0.25x + 1.25(x^2 - x) + 1.5(x^3 - 3x^2 + 2x) + 0.25(x^4 - 6x^3 + 11x^2 - 6x)$$

$$\Rightarrow \quad y = \frac{x^4}{4} - \frac{x^2}{2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{4}\times 4x^3 - \frac{1}{2}\times 2x = x^3 - x \qquad (1)$$

$$\frac{d^2y}{dx^2} = 3x^3 - 1$$

For maximum or minimum $\frac{dy}{dx} = 0$

$$\Rightarrow \quad x^3 - x = 0$$

$$\Rightarrow \quad x(x^2 - 1) = 0$$

$$\Rightarrow \quad x = 0 \text{ or } x^2 - 1 = 0$$

$$\Rightarrow \quad x = 0 \text{ or } x = \pm 1$$

When $x = 0$, $\frac{d^2y}{dx^2} = 0 - 1 = -1 < 0$

$\therefore$ y is maximum when $x = 0$ and the maximum value = **0**

When $x = -1$, $\frac{d^2y}{dx^2} = 3(-1)^2 - 1 = 3 - 1 = 2 > 0$

$\therefore$ y is minimum when $x = -1$ and the minimum value = **−0.25**

When $x = 1$, $\frac{d^2y}{dx^2} = 3\times 1^2 - 1 = 3 - 1 = 2 > 0$

$\therefore$ when $x = 1$, y is minimum and the minimum value = **−0.25** ■

Example 2

For what values of x is the following tabulated function a minimum?

x	3	4	5	6	7	8
$f(x)$	−205	−240	−259	−262	−250	−224

Solution

we observe from the given table that the functional values keep on decreasing upto -262 (ie upto $x = 6$) and then increases

So, the minimum occurs in the neighbourhood of $x = 6$

$\therefore$ choose the origin as $x_0 = 6$. Here $h = 1$

$$\therefore \qquad u = \frac{x - x_0}{h} = x - 6$$

Since the origin 6 is near the middle of the table, a central difference formula is used

$\therefore$ we use Stirling's formula for the given data.

Now we form the difference table

x	$u = x - 6$	$f(x) = y_u$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
3	-3	-205					
			-35				
4	-2	-240		16			
			-19		0		
5	-1	-259		16		-1	
			$\boxed{-3}$		$\boxed{-1}$		1
6	0	$\boxed{-262}$		$\boxed{15}$		$\boxed{0}$	
			$\boxed{12}$		$\boxed{-1}$		
7	1	-250		14			
			26				
8	2	-224					

Stirling's formula in u is

$$y_u = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$

$$+ \frac{u^2(u^2_{-1})}{4!}\Delta^4 y_{-2} + \ldots$$

$$\Rightarrow \qquad y_u = -262 + \frac{u(12-3)}{2} + \frac{u^2}{2} \times 15 + \frac{u(u^2-1)}{6}\left(\frac{-1-1}{2}\right) + \frac{u^2(u^2-1)}{24} \times 0$$

$$= -262 + \frac{9}{2}u + \frac{15}{2}u^2 - \frac{1}{6}(u^3 - u)$$

$$= -\frac{1}{6}u^3 + \frac{15}{2}u^2 + \left(\frac{9}{2} + \frac{1}{6}\right)u - 262$$

$$= -\frac{1}{6}u^3 + \frac{15}{2}u^2 + \frac{14}{3}u - 262$$

Differentiating w.r. to u, we get

$$\frac{dy_u}{du} = -\frac{3}{6}u^2 + \frac{15}{2} \times 2u + \frac{14}{3}$$
$$= -\frac{1}{2}u^2 + 15u + \frac{14}{3}$$
$$\frac{d^2y_u}{du^2} = -u + 15$$

For minimum of $f(x)$, $\quad f'(x) = 0$

$\Rightarrow \quad \frac{dy_u}{du} = 0 \quad \left[\because f(x) = y_u, f'(x) = \frac{d}{dx}(y_u) = \frac{d}{u}(y_u)\frac{du}{dx} = \frac{1}{h}\frac{dy_u}{du}\right]$

$\Rightarrow \quad -\frac{1}{2}u^2 + 15u + \frac{14}{3} = 0.$

$\Rightarrow \quad 3u^2 - 90u - 28 = 0$

$\Rightarrow$
$$u = \frac{90 \pm \sqrt{90^2 - 4 \times 3(-28)}}{2.3}$$
$$= \frac{90 \pm \sqrt{8100 + 336}}{6}$$
$$= \frac{90 \pm 91.8477}{6}$$
$$= \frac{181.8477}{6} \text{ or } \frac{-1.8477}{6}$$
$$= 30.30795 \text{ or } -0.30795$$

When $u = 30.30795$, $\quad x - 6 = 30, 30795$

$\Rightarrow \quad x = 36.30795$

which is outside the table and so we reject it.

$\therefore \quad u = -0.30795$ gives the minimum

$\therefore \quad x - 6 = -0.30795$

$\Rightarrow \quad x = -0.30795 + 6 = 5.6921$

$\therefore$ the function is minimum when $\boldsymbol{x = 5.6921}$

Note: When $u = -0.3075$, $\frac{d^2y_u}{du^2} = 0.3075 + 15 = 15.3075 > 0$

$\therefore$ y is minimum when $u = -3075$

$\Rightarrow \quad x = 5.6921$ ■

Example 3

Find the appropriate maximum value of $f(x)$ given the following table.

x	1	2	3	4	5	6	7
$f(x)$	4774	4968	5104	5183	5208	5181	5104

Solution

we observe from the given table that the functional values keep on increasing up to 5208 ie up to $x = 5$ and then decreases.

So, the maximum occurs in the neighbourhood of $x = 5$

Choose the origen as $x_0 = 5$. Here $h = 1$

$$\therefore \qquad u = \frac{x - x_0}{h} = x - 5$$

We can use Stirling's formula or Bessel's formula

We use Bessel's formula.

$$y = \frac{y_0 + y_1}{2} + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right]$$

$$+ \frac{\left(u - \frac{1}{2}\right)u(u-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(u+1)u(u-1)(u-2)}{4!}\left[\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right] + \ldots$$

Now we form the difference table

x	$u = x - 5$	$f(x) = y_x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1	-4	4774					
			194				
2	-3	4968		-58			
			136				
3	-2	5104		-57		2	
			79		3		
4	-1	5183		-54		-1	
			25		2		
5	0	$\boxed{5208}$		$\boxed{-52}$		$\boxed{0}$	
			$\boxed{-27}$		$\boxed{2}$		
6	1	$\boxed{5181}$		$\boxed{-50}$			
			-77				
7	2	5104					

$$\therefore \quad y_u = \frac{5208+5181}{2} + \left(u-\frac{1}{2}\right)(-27) + \frac{(u^2-u)}{2}\left(\frac{-52-50}{2}\right) + \frac{\left(u-\frac{1}{2}\right)(u)(u-1)}{6} \times 2$$

$$= 5184.5 - 27\left(u-\frac{1}{2}\right) - 25.5(u^2-u) + \frac{1}{3}\left(u^3 - \frac{3}{2}u^2 - \frac{1}{2}u\right)$$

$$\Rightarrow \quad y_u = \frac{1}{3}u^3 - 26u^2 - 1.6667u + 5208 \qquad (1)$$

$$\therefore \quad \frac{dy_u}{du} = u^2 - 52u - 1.6667$$

$$\frac{dy_u}{du^2} = 2u - 52$$

For a maximum or minimum of $f(x)$, $f'(x) = 0$

$$\Rightarrow \quad \frac{dy_u}{du} = 0 \quad \left[\because f'(x) = \frac{d}{dx}(y_u) = \frac{d}{du}(y_u).\frac{d_u}{d_x} = \frac{1}{h}\frac{dy_u}{du}\right]$$

$$\Rightarrow \quad u^2 - 52u - 1.6667 = 0$$

$$\Rightarrow \quad u = \frac{52 \pm \sqrt{52^2 - 4(-1.6667)}}{2}$$

$$= \frac{52 \pm \sqrt{2704 + 6.6668}}{2}$$

$$= \frac{52 \pm 52.641}{2}$$

$$= 52.032 \text{ or } -0.0321$$

But $u = 52.032$ is outside the table and so we reject it

$$\therefore \quad u = -0.0321$$

$$\Rightarrow \quad x - 5 = -0.321$$

$$\Rightarrow \quad x = 5 - 0.0321 = 4.9679$$

To find the maximum value $f(4.9679)$, put $u = -0.0321$ in (1)

$\therefore$ maximum value $= \frac{1}{3}(-0.0321)^3 - 26\,(-0.0321)^2 - 1.6667(-0.0321) + 5208$

$= -0.00001103 - 0.02679066 + 0.05350107 + 5208$

$= \mathbf{5208.0267}$ ■

Example 4

Find the maximum and minimum values of $f(x)$ tabulate below.

x	0	0.2	0.4	0.6	0.8	1	1.2	1.4
$f(x)$	1.23	0.32	–0.35	–0.05	0.08	1.25	1.01	0

Solution

Since maximum and minimum values are required, we shall first form the difference table to see any difference column has constant values.

Now we form the difference table

x	$u=\dfrac{x-0.4}{0.2}$	$f(x)=y_u$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	–2	1.23				
			–0.91			
0.2	–1	0.32		0.24	0.73	
			$\boxed{-0.67}$		$\boxed{0.73}$	
0.4	0	$\boxed{-0.35}$		$\boxed{0.97}$		$\boxed{-1.87}$
			$\boxed{0.30}$		$\boxed{-1.14}$	
0.6	1	–0.05		–0.17		2.35
			0.13		1.21	
0.8	2	0.08		1004		–3.66
			1.17		–2.45	
1	3	1.25		–1.41		3.09
			–0.24			
1.2	4	1.01				
			–1.01			
14	5	0				

Since the values are not equal in the last column, we find the maximum and the minimum separately using stirling's formula.

We observe that the functional values decreases upto -0.35 ie. upto $x = 0.4$ and then increases.

So, the minimum occurs in the neighbourhood of $x = 0.4$

So, we use the origin as $x_0 = 0.4$. Here $h = 0.2$

$$\therefore \qquad u = \frac{x - x_0}{h} = \frac{x - 0.4}{0.2}$$

By Stirling's formula,

$$y_u = y_0 + u\left[\frac{\Delta y_0 + \Delta_{y-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

$$= -0.35 + u\left[\frac{0.30 - 0.67}{2}\right] + \frac{u^2}{2}[0.97] + \frac{u(u^2-1)}{6}\left[\frac{0.73 - 1.14}{2}\right] + \frac{u^2(u^2-1)}{24}(-1.87)$$

$$= -0.35 - 0.185u + 0.485u^2 - 0.0342(u^3 - u) - 0.0779(u^4 - u^2)$$

$$y_u = -0.0779u^4 - 0.0342u^3 + 0.5629u^2 - 0.1508u - 0.35 \qquad (1)$$

Differentiating (1) w.r. to u, we get

$$\frac{dy_u}{du} = -0.0779 \times 4u^3 - 0.0342 \times 3u^2 + 0.5629 \times 2u - 0.1508$$

$$= -0.3116u^3 - 0.1026u^2 + 1.1258u - 0.1508$$

$$\frac{d^2y_u}{du^2} = -0.3116 \times 3u^2 - 0.1026 \times 2u + 1.1258$$

$$= -0.9348u^2 - 0.2052u + 1.1258$$

For maximum or minimum $f'(x) = 0 \Rightarrow \frac{dy}{du} = 0$

$$\Rightarrow \qquad -0.3116 \quad u^3 - 0.1026u^2 + 1.1258u - 01508 = 0$$

we shall find the value of u by successive approximation method

$$\therefore \qquad 1.1258u = 0.1508 + 0.1026u^2 + 0.3116u^3$$

$$\Rightarrow \qquad u = \frac{0.1508}{1.1258} + \frac{0.1026}{1.1258}u^2 + \frac{0.3116}{1.1258}u^3$$

$$= 0.1339 + 0.0911u^2 + 0.2768u^3$$

First approximation is

$$u_1 = \mathbf{0.1339}$$

Second approximation is

$$\begin{aligned} u_2 &= 0.1339 + 0.0911u_1^2 + 0.2768u_1^3 \\ &= 0.1339 + 0.0911(0.1339)^2 + 0.2768(0.1339)^3 \\ &= 0.1339 + 0.001633 + 0.000665 = \mathbf{0.1362} \end{aligned}$$

Third approximation is

$$\begin{aligned} u_3 &= 0.1339 + 0.0911u_2^2 + 0.2768u_2^3 \\ &= 0.1339 + 0.0911\ (0.1362)^2 + 0.2768(0.1362)^3 \\ &= 0.1339 + 0.00169 + 0.0006994 = \mathbf{0.1363} \end{aligned}$$

Fourth approximation is

$$\begin{aligned} u_4 &= 0.1339 + 0.0911u_3^2 + 0.2768u_3^3 \\ &= 0.1339 + 0.0911(0.1363)^2 + 0.2768(0.1363)^3 \\ &= 0.1339 + 0.001692 + 0.000700896 \end{aligned}$$

$$\Rightarrow \quad u_4 = \mathbf{0.1363}$$

Since u_3 u_4 for four places of decimals,

we take $\quad u = 0.1363$

$$\Rightarrow \quad \frac{x-0.4}{0.2} = 0.1363$$

$$\Rightarrow \quad x - 0.4 = 0.2\ (0.1363) = 0.02726$$

$$\Rightarrow \quad x = 0.4 + 0.0273 = \mathbf{0.4273}$$

when $x = 0.4273$, y is minimum

substituting $u = 0.1363$ in (1), we get

$$\begin{aligned} \text{the minimum value} &= -0.779\ (0.1363)^4 - 0.0342\ (0.1363)^3 + 0.5629\ (0.1363)^2 \\ &\quad -0.1508\ (0.1363) - 0.35 \\ &= -0.00002688 - 0.0000866 + 0.01045738 - 0.02055404 - 0.35 \\ &= -0.3602 \end{aligned}$$

$\therefore$ maximum value **−0.3602**

Now we shall find the maximum value.

We observe that values increase upto 1.25 ie upto $x = 1$, and then decrease

So, the maximum occurs in the neighbourhood of $x = 1$

Choose the origin as $x_0 = 1$. Here $h = 0.2$

$$\therefore \quad u = \frac{x-1}{0.2}$$

Now we form the difference table

x	$u=\dfrac{x-1}{0.2}$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	–5	1.23				
			–0.91			
0.2	–4	0.32		0.24		
			–0.67		0.73	
0.4	–3	–0.35		0.97		–1.87
			0.30		–1.14	
0.6	–2	–0.05		–0.17		
			0.13		1.21	
0.8	–1	0.08		1.04		–3.66
			$\boxed{1.17}$		$\boxed{-2.45}$	
1	0	$\boxed{1.25}$		$\boxed{-1.41}$		$\boxed{3.09}$
			$\boxed{-0.24}$		$\boxed{0.65}$	
1.2	1	1.01		–0.77		
			1.01			
14	2	0				

We use Stirling's formula to find u.
Stirling's formula is

$$y_u = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!} - \left(\frac{\Delta^2 y_{-1} + \Delta^2 y_{-2}}{2}\right) + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2}$$

$$y_u = 1.25 + u\left[\frac{1.17-0.24}{2}\right] + \frac{u^2}{2}(-1.41) + \frac{u(u^2-1)}{6}\left[\frac{-2.45+0.65}{2}\right] + \frac{u^2(u^2-1)}{24}(3.09)$$

$$= 1.25 + 0.465u - 0.705u^2 - 0.15(u^3 - u) + 0.1288(u^4 - u^2)$$

$$\Rightarrow \quad y_u = 0.1288u^4 - 0.15u^3 - 0.8338u^2 + 0.615u + 1.25 \qquad (2)$$

$$\frac{dy_u}{du} = 0.1288 \times 4u^3 - 0.15 \times 3u^2 - 0.8338 \times 2u + 0.615$$

$$= 05152u^3 - 0.45u^2 - 1.6676u + 0.615$$

For maximum

$$f'(x) = 0$$

$\Rightarrow$

$$\frac{dy_u}{du} = 0$$

$\Rightarrow$

$$0.5152u^3 - 0.45u^2 - 1.6676u + 0.615 = 0$$

We find u by successive approximation method.

$$1.6676u = 0.615 + 0.5152u^3 - 0.45u^2$$

$\Rightarrow$

$$u = \frac{0.615}{1.6676} + \frac{0.5152}{1.6676}u^3 - \frac{0.45}{1.6676}u^2$$

$$= 0.36879 + 0.30895u^3 - 0.2698u^2$$

First approximation is

$$u_1 = \mathbf{0.36879}$$

Second approximation is

$$u_2 = 0.36879 + 0.30895u_1^3 - 0.2698u_1^2$$

$$= 0.36879 + 0.30895(0.36879)^3 - 0.2698(0.36879)^2$$

$$= 0.36879 + 0.015796 - 0.03669 = 0.347596 = \mathbf{0.3476}$$

Third approximation is $u_3 = 0.36879 + 0.30895u_2^3 - 0.2698u_1^2$

$$u_3 = 0.36879 + 0.30895(0.3476)^3 - 0.2698(0.3476)^2]$$

$$= 0.36879 + 0.01296 - 0.0326 = \mathbf{03492}$$

Fourth approximation is $u_4 = 0.36879 + 0.30895u_3^3 - 0.2698u_3^2$

$$u_4 = 0.36879 + 0.30895(0.3492)^3 - 0.2698(0.3492)^2$$

$$= 0.36879 + 0.013156 - 0.03290 = \mathbf{0.3490}$$

Fifth approximation is $u_5 = 0.36879 + 0.30895u_4^3 - 0.2698u_4^2$

$$u_5 = 0.36879 + 0.30895(0.3490)^3 - 0.2698(0.3490)^2$$

$$= 0.36879 + 0.01313 - 0.03290 = \mathbf{0.34902}$$

Since $u_4 = u_5$ for four place of decimals, we take

$$u = 0.3490$$

$\Rightarrow$

$$\frac{x-1}{0.2} = 0.3490$$

$\Rightarrow$

$$x = 1 + 0.3490 \times (0.2) = 1.0698 = \mathbf{1.07}$$

Substituting $u = 0.349$ in (2), we get the maximum value.

The maximum value $= 0.1288\,(0.349)^4 - 0.15(0.349)^3 - 0.8338\,(0.349)^2$

$$+ 0.615(0.349) + 1.25$$

$$= 0.001911 - 0.006376 - 0.10156 + 0.21464 + 1.25$$

$$= \mathbf{1.3586}$$

■

Exercises 6.2

(1) Find the minimum value of $f(x)$ from the following table.

x	0	1	2	3	4	5
$f(x)$	58	43	40	45	52	60

(2) For what value of x, the following tabulated function of x is minimum?

x	0.2	0.3	0.4	0.5	0.6	0.7
y	0.918	0.898	0.887	886	0.894	0.909

(3) For what value of x is the following tabulated function maximum?

x	0.2	0.3	0.4	0.5	0.6	0.7
y	8.3	9.4	8.7	6.4	2.4	4.6

(4) From the table below determine the value of x for which the function is a maximum.

x	3	4	5	6	7	8
y	208	240	259	262	250	224

(5) Find the maximum and minimum values of the function from the following table.

x	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7
$f(x)$	−11	−12.106	−13.008	−13.62	−14.104	−14.250	−14.096	3.018

[**Hint:** Third difference will be constant, use Newton's forward difference formula]

(6) Find the maximum and minimum values of y tabulated below.

x	0	1	3	4	5
y	0	0.25	0.10	−0.20	3

Answers 6.2

(1) Minimum at $x = 1.7889$
Minimum value = 39.8027

(3) Maximum at $x = 0.308$

(5) Maximum at $x = 1$
Maximum value = 17
Minimum at = 1.5
Minimum value = −14.25

(2) Minimum at $x = 0.496$
Minimum value = 0.8852

(4) Maximum at $x = 5.695$

(6) Maximum at $x = 1.0728$
Maximum value = 0.2561
Minimum at $x = 3.4363$
Minimum value = −0.4.894

SHORT ANSWER QUESTIONS

1. Find $\frac{dy}{dx}$ at $x = 1$ from the following table.

x	1	2	3	4
y	1	8	27	64

2. State Newton's formula to find $f'(x)$ using the forward differences.
3. If $f(x) = a^x, a \neq 0$, is given for $x = 0, 0.5, 1, \ldots$, then show by numerical differentiation that $f'(0) = 4\sqrt{a} - a - 3$.
4. Write down the formula for $(y')x = x_n$ using Newton's backward difference formula.
5. For a function given by the table find $\frac{dy}{dx}$ at $x = 3$.

x	3	4	5
y	36	73	134

6. The following data gives the velocity of a particle for 15 seconds at an interval of 5 seconds. Find the initial acceleration using the entire data.

Time t (secs)	0	5	10	15
Velocity v (m/sec) v	0	3	14	69

7. The following data gives the corresponding values of pressure and specific volume of a super heated steam.

V	2	4	6	8
P	105	43.7	25.3	16.7

Find the rate of change of pressure w.r.to volume when $V = 2$.

8. For the data

x	0	3	4
y	12	6	8

Find $\frac{dy}{dx}$ at $x = 1$.

9. If u_6=1.556, u_7=1.690, u_9=1.908 find $\frac{d}{dx}(u_x)$ when $x = 8$.
10. Assuming Bessel's interpolation formula, prove that

$$\frac{dy_x}{dx} = \Delta y_{x-\frac{1}{2}} - \frac{1}{24}\Delta^3 y_{x-\frac{3}{2}} + \ldots$$

11. Assuming Bessel's formula prove that

$$\frac{d^2 y_x}{dx^2} = \frac{1}{2}\left[\Delta^2 y_{x-\frac{3}{2}} + \Delta^3 y_{x-\frac{1}{2}}\right] + \ldots$$

Numerical Integration

INTRODUCTION

We know that $\int_a^b f(x)dx$ can be evaluated if there exists a differentiable function F such that $F'(x) = f(x)$ in (a, b). However, in applications we come across integrals of the form $\int_a^b e^{-\frac{x^2}{2}} dx$ which cannot be evaluated in closed form or analytical form. Sometimes $f(x)$ is given by a set of recorded values

$$(x_i, f(x_i)),\ i = 1, 2, \ldots n.$$

In such cases we use the method of numerical integration.

Numerical Integration is the process of evaluating a definite integral $\int_a^b f(x)dx$ from a set of tabulated values of the integrand $f(x)$, which is not known or complicated. This process is known as **mechanical quadrature.**

Geometrically, $\int_a^b f(x)dx$ represents the area bounded by the curve $y = f(x)$ and the x-axis between $x = a$ and $x = b$.

The process of Numerical integration is carried out by first approximating the integrand $f(x)$ by an interpolating polynomial $\phi(x)$ and then integrating between the given limits. Thus $\int_a^b f(x)dx$ is approximately equal to $\int_a^b \phi(x)dx$.

We shall now derive a general quadrature formula for equidistant ordinates based on Newton's forward difference formula.

A GENERAL QUADRATURE FORMULA OR NEWTON-COTES QUADRATURE FORMULA

Let $\int_a^b f(x)dx$ be the integral to be evaluated.

Divide $[a,b]$ into n subintervals each of equal length $h = \dfrac{b-a}{n}$.

Let $a = x_0, x_1, x_2, \ldots, x_n = b$ be the points of division of $[a,b]$ so that

$$x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \ldots = x_n - x_{n-1} = b.$$

$$\therefore \quad x_1 = x_0 + h, \quad x_2 = x_0 + 2h, \quad x_3 = x_0 + 3h, \ldots, x_n = x_0 + nh = b.$$

Let $y_0, y_1, y_2, \ldots, y_n$ be the values of the function $y = f(x)$ at the points $x_0, x_1, x_2, \ldots, x_n$ respectively.

Newton's forward difference formula for this data (x_i, y_i), $i = 0, 1, 2, \ldots n$ is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

where $\quad u = \dfrac{x - x_0}{h} \qquad \therefore du = \dfrac{1}{h}dx \qquad \Rightarrow dx = h\,du$

When $\quad x = x_0, \quad u = 0$ and when $x = x_0 + nh$, $u = n$

$$\therefore \int_{x=x_0}^{x=x_0+nh} f(x)dx = \int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 \right.$$

$$\left. + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \ldots \right] h\,du$$

$$= h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u^2 - u}{2!}\Delta^2 y_0 + \frac{u^3 - 3u^2 + 2u}{3!}\Delta^3 y_0 \right.$$

$$\left. + \frac{u^4 - 6u^3 + 11u^2 - 6u}{4!}\Delta^4 y_0 + \ldots \right] du$$

$$= h\left[y_0 u + \frac{u^2}{2}\Delta y_0 + \frac{1}{2!}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)\Delta^2 y_0 + \frac{1}{3!}\left\{\frac{u^4}{4} - 3\frac{u^3}{3} + 2\frac{u^2}{2}\right\}\Delta^3 y_0 \right.$$

$$\left. + \frac{1}{4!}\left\{\frac{u^5}{5} - 6\frac{u^4}{4} + 11\frac{u^3}{3} - 6\frac{u^2}{2}\right\}\Delta^4 y_0 + \cdots \right]_0^n$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2!}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{3!}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0\right.$$

$$\left. + \frac{1}{4!}\left(\frac{n^5}{5} - 3\frac{n^4}{2} + \frac{11}{3}n^3 - 3n^2\right)\Delta^4 y_0 + \frac{1}{5!}\left(\frac{n^6}{6} - 2n^5 + \frac{35}{4}n^4 - \frac{50}{3}n^3 + 12n^2\right)\Delta^5 y_0 + \ldots\right] \quad (1)$$

From this general formula we deduce some important quadrature formulae by putting $n = 1, 2, 3, 4, 5, 6$.

TRAPEZOIDAL RULE

The trapezoidal rule is

$$\int_a^b y\,dx = h\left[\frac{y_0 + y_n}{2} + y_1 + y_2 + y_3 + \ldots + y_{n-1}\right]$$

Newton-Cotes formula is

$$\int_{a=x_0}^{b=x_0+nh} f(x)dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2!}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{3!}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0\right.$$

$$+ \frac{1}{4!}\left(\frac{n^5}{5} - 3\frac{n^4}{2} + 11n^3 - 3n^2\right)\Delta^4 y_0$$

$$\left. + \frac{1}{5!}\left(\frac{n^6}{6} - 2n^5 + \frac{35}{4}n^4 - \frac{50}{3}n^3 + 12n^2\right)\Delta^5 y_0 + \ldots\right] \quad (1)$$

Put $n = 1$ in (1), then we get $\int_{x_0}^{x_1=x_0+h} y\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$

Because in this case, we have only two points x_0 and $x_1 = x_0 + h$ of x and the values y_0 and y_1 of y, the second and higher differences are zero

$$\int_{x_0}^{x_1=x_0+h} y\,dx = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] \qquad [\because \Delta y_0 = y_1 - y_0]$$

$$= \frac{h}{2}[y_0 + y_1]$$

Similarly

$$\int_{x_1=x_0+h}^{x_2=x_0+2h} y\,dx = \frac{h}{2}[y_1 + y_2]$$

$$\int_{x_0+2h}^{x_0+3h} y\,dx = \frac{h}{2}[y_2 + y_3], \ldots$$

$$\int_{x_0+(n-1)h}^{x_0+nh} y\,dx = \frac{h}{2}[y_{n-1} + y_n]$$

$$\therefore \quad \int_{x_0}^{x_0+nh} y\,dx = \int_{x_0}^{x_0+h} y\,dx + \int_{x_0+h}^{x_0+2h} y\,dx + \int_{x_0+2h}^{x_0+3h} y\,dx + \ldots + \int_{x_0+(n-1)h}^{x_0+nh} y\,dx$$

$$\Rightarrow \quad \int_{x_0}^{x_0+nh} y\,dx = \frac{h}{2}[y_0+y_1] + \frac{h}{2}[y_1+y_2] + \frac{h}{2}[y_2+y_3] + \ldots + \frac{h}{2}[y_{n-1}+y_n]$$

$$= \frac{h}{2}[y_0 + y_n + 2(y_1+y_2+y_3+\ldots+y_{n-1})]$$

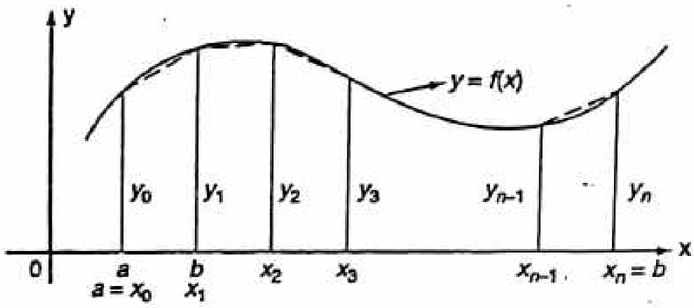
$$\Rightarrow \quad \int_a^b y\,dx = h\left[\frac{(y_0+y_n)}{2} + y_1 + y_2 + y_3 + \ldots + y_{n-1}\right]$$

This is known as trapezoidal rule. This is also known as general form of **Trapezoidal rule or composite trapezoidal rule.**

Geometrical Meaning

The integral $\int_{x_0}^{x_n = x_0+nh} y\,dx$ represents the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = x_0$ and $x = x_n$.

In trapezoidal rule the curve $y = f(x)$ is replaced by n straight line segments joining the points (x_0, y_0) and (x_1, y_1); (x_1, y_1) and (x_2, y_2),...; (x_{n-1}, y_{n-1}) and (x_n, y_n).
The sum of the areas of these trapeziums is taken as the area under the curve.
Hence the name Trapezoidal rule.



WORKED EXAMPLES

Example 1

Dividing the range into 10 equal parts, find the approximate value of $\int_0^\pi \sin x\,dx$ by Trapezoidal rule.

Solution

Given $\int_0^\pi \sin x\,dx$.

Here $\therefore y = \sin x dx$. The interval is $[0, \pi]$

The interval $[0, \pi]$ is divided into 10 equal parts.

$$h=\frac{\pi-0}{10}=\frac{\pi}{10}$$

$\therefore$ The values of x are $x_0=0, x_1=\frac{\pi}{10}, x_2=\frac{2\pi}{10}, x_3=\frac{3\pi}{10}, x_4=\frac{4\pi}{10}, x_5=\frac{5\pi}{2}, \ldots, x_{10}=\pi$.

We shall find the values of $y=\sin x$ at these points $x_0, x_1, \ldots, x_{10}$.

Values of x	0	$\frac{\pi}{10}$	$\frac{2\pi}{10}$	$\frac{3\pi}{10}$	$\frac{4\pi}{10}$	$\frac{5\pi}{10}$	$\frac{6\pi}{10}$	$\frac{7\pi}{10}$	$\frac{8\pi}{10}$	$\frac{9\pi}{10}$	π
Values of $y=\sin x$	0 y_0	0.3090 y_1	0.5878 y_2	0.8090 y_3	0.9511 y_4	1 y_5	0.9511 y_6	0.8090 y_7	0.5878 y_8	0.3090 y_9	0 y_{10}

By Trapezoidal rule,

$$\therefore \int_0^{\pi}\sin x\,dx=h\left[\frac{y_0+y_{10}}{2}+(y_1+y_2+y_3+y_4+y_5+y_6+y_7+y_8+y_9)\right]$$

$$=\frac{\pi}{10}[0+0.3090+0.5878+0.8090+0.9511+1+0.9511+0.8090+0.5878+0.3090]$$

$$=\frac{\pi}{10}[2(0.3090+0.5878+0.8090+0.9511)+1]$$

$$=\pi(0.63138)=\mathbf{1.9835}$$

■

Example 2

Evaluate $\int_0^2 \frac{dx}{x^2+4}$ using trapezoidal rule taking $h=0.25$.

Solution

Given $\int_0^2 \frac{dx}{x^2+4}$

Here $y=\frac{1}{x^2+4}$, $h=0.25$ $\therefore n=\frac{2}{0.25}=8$

$\therefore$ the ponits are $x_0=0$, $x_1=0.25$, $x_2=0.5$, $x_3=0.75$,
$x_4=1$, $x_5=1.25$, $x_6=1.5$, $x_7=1.75$, $x_8=2$,

We shall find the values of y and they are given by the table below.

x	0	0.25	0.5	0.75	1	1.25	1.50	1.75	2
$y=\frac{1}{x^2+4}$	$\frac{1}{4}$ y_0	0.2462 y_1	0.2353 y_2	0.2192 y_3	0.2 y_4	0.1798 y_5	0.16 y_6	0.1416 y_7	0.1250 y_8

By Trapezoidal rule,

$$\int_0^2 \frac{dx}{x^2+4} = h\left[\frac{y_0+y_8}{2} + y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7\right]$$

$$= 0.25\left[\frac{0.25+0.125}{2} + 0.2462 + 0.2353 + 0.2192 + 0.2 + 0.1798 + 0.16 + 0.1416\right]$$

$$= 0.25\left[\frac{0.375}{2} + 1.3821\right] = \mathbf{0.3924}$$ ■

Example 3

A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and it is in the table below. Using trapezoidal rule find the velocity of the rocket at t = 80 secs.

***t* secs**	**0**	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**
***f* : cm/sec²**	**30**	**31.63**	**33.34**	**35.47**	**37.75**	**40.33**	**43.25**	**46.69**	**40.67**

Solution

Let v and a be the velocity and acceleration at time t

Given $a = f(t)$ cm/sec²

$\therefore \quad \frac{dv}{dt} = f(t) \quad \Rightarrow v = \int_0^t f(t)\,dt$

We use trapezoidal rule to find the velocity v, when $t = 80$ secs

$$\therefore \quad v = \int_0^{80} f(t)\,dt$$

They given table is

t secs	0	10	20	30	40	50	60	70	80
$y = f(t)$ cm/sec²	30 y_0	31.63 y_1	33.34 y_2	35.47 y_3	37.75 y_4	40.33 y_5	43.25 y_6	46.69 y_7	40.67 y_8

Here $n = 8$, and $h = 10$

By Trapezoidal rule,

$$\int_0^{80} f(t)dt = h\left[\frac{y_0+y_8}{2} + y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7\right]$$

$$= 10\left[\frac{30+40.67}{2} + 31.63 + 33.34 + 35.47 + 37.75 + 40.33 + 43.25 + 46.69\right]$$

$$= 10[35.335 + 268.46] = 3037.95$$ ■

$\therefore$ when $t = 80$ secs, the velocity is $\boldsymbol{v = 3037.95}$ **cm/sec**

Example 4

Evaluate $\int_4^{5.2} \log_e x\,dx$ using the following table by trapezoidal rule.

x	4	4.2	4.4	4.6	4.8	5	5.2
$\log_e x$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

Solution

The given table is

x	4	4.2	4.4	4.6	4.8	5	5.2
$\log_e x$	1.3863 y_0	1.4351 y_1	1. 4816 y_2	1. 5261 y_3	1. 5686 y_4	1.6094 y_5	1.6487 y_6

Here $n = 6, \quad h = 0.2$

By Trapezoidal rule,

$$\int_4^{5.2} \log_e x\ dx = h\left[\frac{y_0 + y_6}{2} + (y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= 0.2\left[\frac{1.3863 + 1.6487}{2} + 1.4351 + 1.4816 + 1.5261 + 1.5686 + 1.6094\right]$$

$$= 0.2[1.5175 + 7.6208] = 0.2(9.1383) = \mathbf{1.82766}$$ ■

Example 5

Evaluate $\int_0^1 e^{-x^2} dx$ dividing the range into 4 equal parts by Trapezoidal rule.

Solution

Given $\int_0^1 e^{-x^2} dx$

Here $y = e^{-x^2}$. Interval is $[0,1]$

The interval $[0,1]$ is divided into 4 equal parts.

$$\therefore \quad h = \frac{1-0}{4} = \frac{1}{4} = 0.25$$

$\therefore$ the points are $x_0 = 0, \quad x_1 = 0.25, \quad x_2 = 0.5, \quad x_3 = 0.75, \quad x_4 = 1.$

We find the values of $y = e^{-x^2}$ at these points and they are given by the table below.

x	0	0.25	0.5	0.75	1
x^2	0	0.0625	0.25	0.5625	1
$y = e^{-x^2}$	1 y_0	0.9394 y_1	0.7788 y_2	0.5698 y_3	0.3679 y_4

By Trapezoidal rule,

$$\int_0^1 e^{-x^2}\,dx = h\left[\frac{y_0+y_4}{2}+(y_1+y_2+y_3)\right]$$

$$= 0.25\left[\frac{1+0.3679}{2}+0.9394+0.7788+0.5698\right]$$

$$= 0.25[0.68395+2.288] = 0.25(2.97195) = \mathbf{0.743}$$

SIMPSON'S RULE OR SIMPSON'S $\frac{1}{3}$ RULE

Simpson's rule is $\int_a^b y\,dx = \frac{h}{3}[(y_0+y_n)]+4(y_1+y_3+y_5+\dots+y_{n-1})$

$$+2(y_2+y_4+\dots+y_{n-2})$$

Proof: In the general quadrature formula put $n=2$

Then the interval is x_0 to x_0+2h. So, the points are x_0, x_0+h, x_0+2h and the values of y are y_0, y_1, y_2 respectively.

Since three values are given, third and higher differences are zero

$$\therefore \quad \int_{x_0}^{x_0+2h} y\,dx = h\left[2y_0+\frac{2^2}{2}\Delta y_0+\frac{1}{2!}\left(\frac{2^3}{3}-\frac{2^2}{2}\right)\Delta^2 y_0\right]$$

Now

$$\Delta y_0 = y_1 - y_0,\ \Delta y_1 = y_2 - y_1$$

$$\Delta^2 y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\therefore \quad \int_{x_0}^{x_0+2h} = h\left[2y_0+2(y_1-y_0)+\frac{1}{2}\cdot\frac{2}{3}(y_2-2y_1+y_0)\right]$$

$$= h\left[2y_1+\frac{1}{3}(y_2-2y_1+y_0)\right]$$

$$= \frac{h}{3}[6y_1+y_2-2y_1+y_0]$$

$$\Rightarrow \quad \int_{x_0}^{x_0+2h} y\,dx = \frac{h}{3}[y_0+4y_1+y_2]$$

Similarly,

$$\int_{x_0+2h}^{x_0+4h} y\,dx = \frac{h}{3}[y_2+4y_3+y_4]$$

$$\int_{x_0+4h}^{x_0+6h} y\,dx = \frac{h}{3}[y_4+4y_5+y_6]$$

$$\vdots$$

$$\int_{x_0+(n-2)h}^{x_0+nh} y\,dx = \frac{h}{3}[y_{n-2}+4y_{n-1}+y_n]$$

Adding, we get

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})]$$

where n is a multiple of 2.
That is the number of sub-intervals should be 2, 4, 6, 8, ...

This formula is called **Simpson's rule** or **Simpson's $\frac{1}{3}$ rule.**

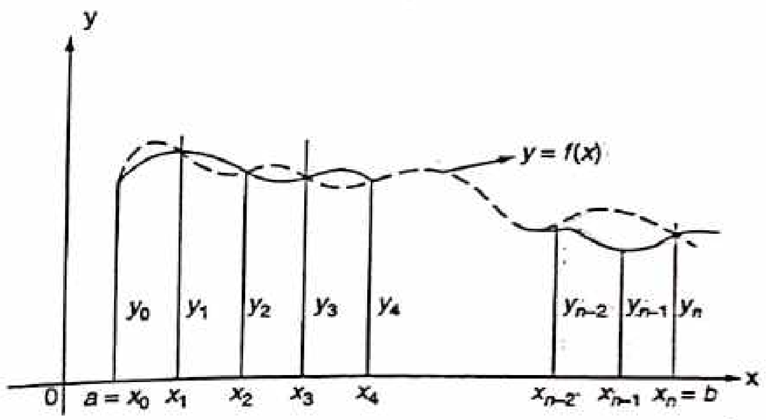
It is also known as composite Simpson's $\frac{1}{3}$ rule.

Geometrical Meaning

In Simpson's method in each interval $(x_0, x_0 + 2h), (x_0 + 2h, x_0 + 4h), (x_0 + 4h, x_0 + 6h) \dots ((x_0 + (n-2)h)x_0 + nh)$ where n is even, the curve $y = f(x)$ is replaced by a second degree polynomial or parabola.

That is the curve $y = f(x)$ is replaced by $\frac{n}{2}$ arcs of parabola. So, the area under the curve $y = f(x)$ is taken as sum of the areas under the $\frac{n}{2}$ parabolic arcs.

The dotted line is the graph of $y = f(x)$ and the thick lines are various parabolae.



WORKED EXAMPLES

Example 1

Evaluate $\int_2^{10} \frac{dx}{1+x}$ by dividing the interval into eight equal parts and hence find the approximate value of $\log_e \frac{11}{3}$.

Solution

$$\int_2^{10} \frac{dx}{1+x}.$$

Given

$$y = \frac{1}{1+x}$$

Here

The interval is [2, 10].

We divide the interval into eight equal parts $\therefore h = \frac{10-2}{8} = \frac{8}{8} = 1$

So, we use Simpson's $\frac{1}{3}$ rule to find $\int_2^{10} \frac{dx}{1+x}$.

The points are $x_0 = 2, x_1 = 3, x_2 = 4, x_3 = 5, x_4 = 6, x_5 = 7, x_6 = 8, x_7 = 9, x_8 = 10$.

The values of y at these pernts are given by the table below.

x	2	3	4	5	6	7	8	9	10
$y = \frac{1}{1+x}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{11}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

By Simpson's formula,

$$\int_2^{10} \frac{1}{1+x} dx = \frac{h}{3}[y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{1}{3}\left[\frac{1}{3} + \frac{1}{11} + 4\left(\frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10}\right) + 2\left(\frac{1}{5} + \frac{1}{7} + \frac{1}{9}\right)\right]$$

$$= \frac{1}{3}[0.3333 + 0.0909 + 4(0.25 + 0.1667 + 0.1250 + 0.1)$$

$$+ 2(0.2 + 0.1429 + 0.1111)] = \frac{1}{3}(3.8990) = \mathbf{1.2997}$$

By direct integration,

$$\int_2^{10} \frac{1}{1+x} dx = [\log_e(1+x)]_2^{10} = \log_e 11 - \log_e 3 = \log_e \frac{11}{3}$$

$$\therefore \qquad \log_e\left(\frac{11}{3}\right) = 1.2997$$

Note: But the value of $\log_e\left(\frac{11}{3}\right) = 1.29928$ correct upto 5 places.

So, we find Simpson's formula gives the value correctly to 3 decimal places. ■

Example 2

A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and it is in the table below. Using Simpson's $\frac{1}{3}$ rule, find the velocity of the rocket at $t = 80$ secs.

t secs	0	10	20	30	40	50	60	70	80
f : cm/sec^2	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	40.67

Solution

Let v be the velocity of the rocket at t secs,

$$\therefore \qquad v = \int_0^t f(t)\,dt$$

When $\qquad t = 80\,\text{secs}, \quad v = \int_0^{80} f(t)\,dt.$

Given table is

t secs	0	10	20	30	40	50	60	70	80
$y = f(t)$ cm/sec^2	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	40.67
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Here $n=8$, $\quad h=10$

By Simpson's $\frac{1}{3}$ rule,

$$v = \int_0^{80} f(t)dt = \frac{h}{3}[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{10}{3}[(30 + 40.67) + 4(31.63 + 35.47 + 40.33 + 46.69) + 2(33.34 + 37.75 + 43.25)]$$

$$= \frac{10}{3}[70.67 + 616.48 + 228.68] = 3052.77$$

$\therefore$ when $t = 80$ secs, the velocity is $\boldsymbol{v} = \mathbf{3052.77}$ **cm/sec** ■

Example 3

Use Simpson's $\frac{1}{3}$ rule to find $\int_0^{0.6} e^{-x^2 dx}$ by taking seven ordinates.

Solution

Given $\qquad \int_0^{0.6} e^{-x^2 dx}.$

Here $\qquad y = e^{-x^2}$

The interval is [0, 0.6] and divide the interval into six equal parts and so there will be seven ordinates.

$$\therefore \qquad h = \frac{0.6 - 0}{6} = 0.1$$

The points are $x_0 = 0,\ x_1 = 0.1,\ x_2 = 0.2,\ x_3 = 0.3,\ x_4 = 0.5,\ x_5 = 0.5,\ x_6 = 0.6.$

Now we shall find the values of $y = e^{-x^2}$ at these points are given by the table below.

x	0	0.1	0.2	0.3	0.4	0.5	0.6
x^2	0	0.01	0.04	0.09	0.16	0.25	0.36
$y = e^{-x^2}$	1 y_0	0.9900 y_1	0.9608 y_2	0.9139 y_3	0.8521 y_4	0.7788 y_5	0.6977 y_6

By Simpson's $\frac{1}{3}$ rule,

$$\int_0^{0.6} e^{-x^2}\,dx = \frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{0.1}{3}[(1 + 0.6977) + 4(0.9900 + 0.9139 + 0.7788) + 2(0.9608 + 0.8521)]$$

$$= \frac{0.1}{3}[1.6977 + 10.7308 + 3.6258] = \frac{0.1}{3}(16.0543) = \mathbf{0.5351}$$

■

Example 4

The following table gives the velocity u of a particle at time t:

t (secs)	0	2	4	6	8	10	12
u (m/sec)	4	6	16	34	60	94	136

Find the distance moved by the particle in 12 secs and also the acceleration at $t = 2$ secs.

Solution

Let s be the distance moved at time t.

Let v and a be the velocity and acceleration at time t.

We know that velocity $v = \frac{ds}{dt}$ and acceleration $a = \frac{dv}{dt}$

$$s = \int_0^t v\,dt$$

When $t = 12$ secs, the distance travelled $s = \int_0^{12} v\,dt = \int_0^{12} y\,dt$, where $y = v$

The values of y are given by the table.

t	0	2	4	6	8	10	12
$y = v$	4 y_0	6 y_1	16 y_2	34 y_3	60 y_4	94 y_5	136 y_6

By Simpson's $\frac{1}{3}$ rule,

$$s = \int_0^{12} v\,dt = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{2}{3}[(4 + 136) + 4(6 + 34 + 94) + 2(16 + 60)]$$

$$= \frac{2}{3}[140 + 536 + 152] = 552 \text{ metres}$$

$\therefore$ when $t = 12$, distance moved is $\mathbf{s = 552}$ **meters**

We want acceleration when $t = 2$ sec.

$$\Rightarrow \qquad a = \left(\frac{dv}{dt}\right)_{t=2}$$

ie. we want to find the derivative of v.

So, we form the difference table.

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$
0	**4**			
		2		
2	6		**8**	
		10		**0**
4	16		8	
		18		0
6	34		8	
		26		0
8	60		8	
		34		0
10	94		8	
		42		
12	136			

Newton's forward formula is

$$v = v_0 + u\Delta v_0 + \frac{u(u-1)}{2}\Delta^2 v_0 + \dots$$

where $u = \frac{t - t_0}{h}$, $\qquad t_0 = 0, \quad h = 2$

When $t = 3$, $\qquad u = \frac{2-0}{2} = 1$

$$\therefore \qquad \frac{dv}{dt} = \frac{1}{2}\left[\Delta v_0 + \frac{1}{2}(2u-1)\Delta^2 v_0\right] \qquad \left[\because \frac{dv}{dt} = \frac{dv}{du}\cdot\frac{du}{dt} = \frac{1}{2}\frac{dv}{du}\right]$$

$$\therefore \qquad \left(\frac{dv}{dt}\right)_{t=2} = \frac{1}{2}\left[2 + \frac{1}{2}(2.1 - 1).8\right] = \frac{1}{2}[2 + 4] = 3 \text{ m/sec}^2$$

$\therefore$ when $t = 3$ secs, the acceleration is $\mathbf{a = 3}$ **m/sec²**

Example 5

Evaluate $\int_0^6 [f(x)]^2\,dx$ using Simpson's $\frac{1}{3}$ rule, given that

x	0	1	2	3	4	5	6
$f(x)$	1	0	1	4	9	16	25

Solution

Let $$I = \int_0^6 [f(x)]^2$$

Here $$y = [f(x)]^2$$

The number of intervals is 6 and $h = 1$ $\quad \therefore n = 6$, a multiple of y_2.

So, we can use Simpson's $\frac{1}{3}$ rule

The points are $x_0 = 0,\quad x_1 = 1,\quad x_2 = 2,\quad x_3 = 3,\quad x_4 = 4,\quad x_5 = 5,\quad x_6 = 6$

The values of y are given by the table below.

x	0	1	2	3	4	5	6
$f(x)$	1	0	1	4	9	16	25
$y = [f(x)]^2$	1	0	1	16	81	256	625
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's rule,

$$\int_0^6 [f(x)]^2\,dx = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3}[(1 + 625) + 4(0 + 16 + 256) + 2(1 + 81)]$$

$$= \frac{1}{3}[626 + 1088 + 164] = \frac{1}{3}(1878) = \mathbf{626}$$

■

Example 6

A solid of revolution is formed by rotating about the x-axis, the area between the x-axis and the lines $x = 0$, $x = 1$ and a curve through the points with the following coordinates.

x	0.0	0.25	0.50	0.75	1
y	1	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed using Simpson's rule.

Solution

The volume v of solid of revolution about the x-axis, the area between the curve $y = f(x)$, the x-axis and the lines $x = 0$ and $x = 1$ is $\int_0^1 \pi y^2 dx$.

The interval [0, 1] is divided into four equal parts with $h = 0.25$

$\therefore n = 4$, a multiple of 2.

So, we can use Simpson's $\frac{1}{3}$ rule to find the required volume $\int_0^1 \pi y^2 dx$.

The points are $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$

The values of y^2 are given by the table below.

x	0	0.25	0.5	0.75	1
y	1	0.9896	0.9589	0.9089	0.8415
y^2	1 y_0^2	0.9793 y_1^2	0.9195 y_2^2	0.8261 y_3^2	0.7081 y_4^2

By Simpson's rule,

$$\begin{aligned}\int_0^1 \pi y^2 dx &= \pi \frac{h}{3}[(y_0^2 + y_4^2) + 4(y_1^2 + y_3^2) + 2y_2^2]\\ &= \pi \frac{(0.25)}{3}[(1 + 0.7081) + 4(0.9793 + 0.8261) + 2(0.9195)]\\ &= \pi \frac{(0.25)}{3}[1.7081 + 7.2216 + 1.839]\\ &= \pi \frac{(0.25)}{3}(10.7687) = 0.8974\pi = \mathbf{2.8193}\end{aligned}$$

$\therefore$ volume $v =$ **2.8193 cubic units**

7.4 SIMPSON'S $\frac{3}{8}$ RULE

Simpson's $\frac{3}{8}$ rule is

$$\int_a^b y\,dx = \frac{3h}{8}[y_0 + y_n + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \ldots + y_{n-3})]$$

Proof: We derive the formula using the general quadrative formula.
In the general quadrature formula, put $n = 3$
Then the interval is $(x_0, x_0 + 3h)$ and the four values of y are y_0, y_1, y_2, y_3.
So, the fourth and higher differences are zero

$$\therefore \qquad \int_{x_0}^{x_0+3h} y\,dx = h\left[3y_0 + \frac{3^2}{2}\Delta y_0 + \frac{1}{2!}\left(\frac{3^3}{3} - \frac{3^2}{2}\right)\Delta^2 y_0 + \frac{1}{3!}\left(\frac{3^4}{4} - 3^3 + 3^2\right)\Delta^3 y_0\right]$$

Now $\qquad \Delta y_0 = y_1 - y_0, \quad \Delta y_1 = y_2 - y_1, \quad \Delta y_2 = y_3 - y_2$

$$\Delta^2 y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\Delta^2 y_1 = y_3 - y_2 - (y_2 - y_1) = y_3 - 2y_2 + y_1$$

$$\Delta^3 y_0 = y_3 - 2y_2 + y_1 - (y_2 - 2y_1 + y_0) = y_3 - 3y_2 + 3y_1 - y_0$$

$$\therefore \qquad \int_{x_0}^{x_0+3h} y\,dx = h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{1}{2}\cdot\frac{9}{2}(y_2 - 2y_1 + y_0) + \frac{9}{6}\left(\frac{9}{4} - 2\right)(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= h\left[3y_0 + \frac{9}{2}y_1 - \frac{9}{2}y_0 + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{3}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= h\left[3y_0 - \frac{9}{2}y_0 + \frac{9}{4}y_0 - \frac{3}{8}y_0 + \frac{9}{2}y_1 - \frac{9}{2}y_1 + \frac{9}{8}y_1 + \frac{9}{4}y_2 - \frac{9}{8}y_2 + \frac{3}{8}y_3\right]$$

$$= h\left[\frac{(24 - 36 + 18 - 3)}{8}y_0 + \frac{9}{8}y_1 + \frac{9}{8}y_2 + \frac{3}{8}y_3\right]$$

$$= h\left[\frac{3}{8}y_0 + \frac{9}{8}y_1 + \frac{9}{8}y_2 + \frac{3}{8}y_3\right]$$

$$\Rightarrow \qquad \int_{x_0}^{x_0+3h} y\,dx = \frac{3}{8}h[y_0 + 3y_1 + 3y_2 + y_3]$$

Similarly,

$$\int_{x_0+3h}^{x_0+6h} y\,dx = \frac{3}{8}h[y_3 + 3y_4 + 3y_5 + y_6]$$

$$\int_{x_0+6h}^{x_0+9h} y\,dx = \frac{3}{8}h[y_6 + 3y_7 + 3y_8 + y_9]$$

$$\vdots$$

$$\int_{x_0+(n-3)h}^{x_0+nh} y\,dx = \frac{3}{8}h[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

Adding we get

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3}{8}h[y_0 + y_n + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \ldots + y_{n-3})]$$

where n is a multiple of 3.

This formula is called Simpson's $\frac{3}{8}$ rule.

This formula is also known as Composite Simpson's $\frac{3}{8}$ rule.

Note: For applying Simpson's $\frac{3}{8}$ rule, the number of sub-intervals of $[a,b]=[x_0,x_0+nh]$ should be a multiple of 3. Usually we divide into 6 or 9 sub-intervals.

Geometrical Meaning

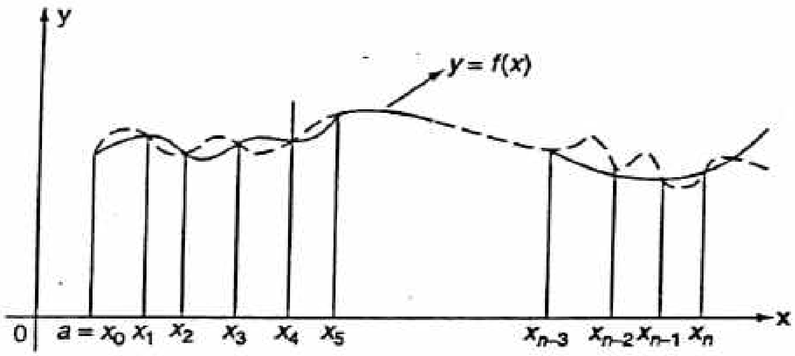
In Simpson's $\frac{3}{8}$ rule, in each interval,

$(x_0, x_0+3h),(x_0+3h, x_0+6h),(x_0+6h, x_0+9h),\dots\ (x_0+(n-3)h, x_0+nh)$

Where n is a multiple of 3, the curve $y=f(x)$ is replaced by $\frac{n}{3}$ are of a cubic polynomials.

So, the area under the curve $y=f(x)$ is taken as the sum of the areas under the $\frac{n}{3}$ arcs of the cubic polynomial.

The dotted line is the curve $y=f(x)$ and the thick lines are various arcs of the cubic polynomials.



WORKED EXAMPLES

Example 1

Evaluate $\int_0^1 \frac{dx}{1+x}$ taking 7 ordinates by applying Simpson's $\frac{3}{8}$ rule. Deduce the value of $\log_e 2$.

Solution

Let $$I=\int_0^1 \frac{dx}{1+x}$$

Here $$y=\frac{1}{1+x}.$$

Since the number of ordinates is 7, the number of subintervals is 6.

$\therefore$ $$h=\frac{1-0}{6}=\frac{1}{6}.$$

$\therefore$ the points are $x_0 = 0$, $x_1 = \frac{1}{6}$, $x_2 = \frac{1}{3}$, $x_3 = \frac{1}{2}$, $x_4 = \frac{2}{3}$, $x_5 = \frac{5}{6}$, $x_6 = 1$.

The values of y are $y_0, y_1, y_2, y_3, y_4, y_5, y_6$ and they are given by the table below.

x	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$y = \frac{1}{1+x}$	1 y_0	0.8571 y_1	0.75 y_2	0.6667 y_3	0.6 y_4	0.5455 y_5	0.5 y_6

By Simpson's $\frac{3}{8}$ rule,

$$I = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8} \cdot \frac{1}{6}[(1 + 0.5) + 3((0.8571) + 0.75 + 0.6 + 0.5455) + 2(0.6667)]$$

$$= \frac{1}{16}[1.5 + 8.2578 + 1.3334] = \mathbf{0.6932}$$

But

$$\int_0^1 \frac{1}{1+x}dx = [\log(1+x)]_0^1 = \log_e 2$$

$\therefore$ $$\mathbf{\log_e 2 = 0.6932}$$ ■

Example 2

A curve is drawn to pass through the points given by the following table.

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1

Using Simpson's $\frac{3}{8}$ rule, estimate the area bounded the curve, $y = f(x)$, the x-axis and the lines $x = 1$, $x = 4$.

Solution

Required area is $A = \int_1^4 f(x)\,dx$

The given table is

x	1	1.5	2	2.5	3	3.5	4
y	2 y_0	2.4 y_1	2.7 y_2	2.8 y_3	3 y_4	2.6 y_5	2.1 y_6

Number of intervals is 6 and $h = 0.5$

$\therefore$ $n = 6$, a multiple of 3.

So, we can use Simpson's $\frac{3}{8}$ rule to find the required area bounded by $y = f(x)$, the x-axis and the lines $x = 1$, $x = 4$.

By Simpson's $\frac{3}{8}$ rule,

$$A = \int_1^4 y\,dx = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8}(0.5)[2 + 2.1 + 3(2.4 + 2.7 + 3 + 2.6) + 2(2.8)]$$

$$= \frac{1.5}{8}[4.1 + 32.1 + 5.6] = \frac{1.5}{8}(41.8) = 7.8375$$

$\therefore$ area is $\mathbf{A = 7.8375}$ ■

Example 3

A river is 60 ft wide. The depth y feet at a distance x ft from one bank is given by the following table.

x	0	10	20	30	40	50	60
y	0	4	7	9	12	15	14

Find approximately the area of cross-section.

Solution

The area of the cross-section is $A = \int_0^{60} y\,dx$.

The given table is

x	0	10	20	30	40	50	60
y	0	4	7	9	12	15	14
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Number of intervals is 6 and $h = 10$

$\therefore n = 6$, a multiple of 3.

So, we use Simpson's $\frac{3}{8}$ rule to find the area of the cross-section.

By Simpson's $\frac{3}{8}$ rule,

$$\int_0^{60} y\,dx = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8} \times 10[(0 + 14) + 3(4 + 7 + 12 + 15) + 2 \times 9]$$

$$= \frac{15}{4}[14 + 114 + 18] = \frac{15}{4}[146] = 547.5 \text{ sq.ft}$$ ■

$\therefore$ area of the cross-section is $\mathbf{A = 547.5}$ **sq.ft**

Example 4

The velocity v of a particle at a distance s from a point on its path is given by the table

s.ft	0	10	20	30	40	50	60
v ft/sec	47	58	64	65	61	52	38

Estimate the time taken to travel 60 ft by using Simpson's $\frac{1}{3}$ rule. Compare the result with Simpson's $\frac{3}{8}$ rule.

Solution

Let s be the distance travelled and v be the velocity at time t secs.

We know that velocity $\quad v = \frac{ds}{dt}$

$$\Rightarrow \quad dt = \frac{1}{v} ds$$

$$\therefore \quad \int_0^t dt = \int_0^s \frac{1}{v} ds$$

$$t = \int_0^s y\, ds, \qquad \text{where } y = \frac{1}{v}$$

$\therefore$ the time taken to travel $s = 60$ ft is $\int_0^{60} y\, ds$. Let $I = \int_0^{60} y\, ds$

We shall form the table of values of y.

s	0	10	20	30	40	50	60
$y = \frac{1}{v}$	0.0213 y_0	0.0172 y_1	0.0156 y_2	0.0154 y_3	0.0164 y_4	0.0192 y_5	0.0263 y_6

Here $h = 10$ and $n = 6$, even

By Simpson's $\frac{1}{3}$ rule,

$$I = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

Here $h = 10$,

$$\therefore \quad I = \frac{10}{3}[(0.0213+0.0263)+4(0.0172+0.0154+0.0192)+2(0.0156+0.0164)]$$

$$= \frac{10}{3}[0.0476+0.2072+0.064] = \mathbf{1.0627}$$

By Simpson's $\frac{3}{8}$ rule,

$$I = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3\times 10}{8}[(0.0213 + 0.0263) + 3(0.0172 + 0.0156 + 0.0164 + 0.0192) + 2(0.0154)]$$

$$= \frac{15}{4}[0.0476 + 0.236] = \mathbf{1.0635}$$

∎

$\therefore$ the time taken by the particle to travel 60 ft by the two methods are equal, correct to 2 places of decimals.

Example 5

Evaluate $\int_0^6 \frac{1}{1+x}dx$ by using (i) direct integration (ii) Trapezoidal rule (iii) Simpson's $\frac{1}{3}$ rule (iv) Simpson's $\frac{3}{8}$ rule.

Solution

Let $$I = \int_0^6 \frac{1}{1+x}dx$$

Here $$y = \frac{1}{1+x}$$

Let $n = 6$. $$\therefore h = \frac{6-0}{6} = 1$$

The parts are $x_0 = 0,\quad x_1 = 1,\quad x_2 = 2,\quad x_3 = 3,\quad x_4 = 4,\quad x_5 = 5,\quad x_6 = 6$

The values of y are $y_0, y_1, y_2, y_3, y_4, y_5, y_6$ and they are given by the table below

x	0	1	2	3	4	5	6
$y = \frac{1}{1+x}$	1	0.5	0.3333	0.25	0.2	0.1667	0.1428
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(ii) Trapezoidal rule

By Trapezoidal rule,

$$I = \int_0^6 \frac{dx}{1+x} = h\left[\frac{y_0 + y_6}{2} + y_1 + y_2 + y_3 + y_4 + y_5\right]$$

$$= \left[\frac{1 + 0.1428}{2} + 0.5 + 0.3333 + 0.25 + 0.2 + 0.1667\right]$$

$$= 0.5714 + 1.45 = \mathbf{2.0214}$$

(iii) Simpson's $\frac{1}{3}$ rule

By Simpson's $\frac{1}{3}$ rule,

$$I = \int_0^6 \frac{dx}{1+x} = \frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3}[1 + 0.1428 + 4(0.5 + 0.25 + 0.1667) + 2(0.3333 + 0.2)]$$

$$= \frac{1}{3}[1.1428 + 3.6668 + 1.0666]$$

$$= \frac{1}{3}[5.8762] = \mathbf{1.9587}$$

(iv) Simpson's $\frac{3}{8}$ rule

By Simpson's $\frac{3}{8}$ rule,

$$I = \int_0^6 \frac{dx}{1+x} = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3\times1}{8}[1 + 0.1428 + 3(0.5 + 0.3333 + 0.2 + 0.1667) + 2\times0.25]$$

$$= \frac{3}{8}[1.1428 + 3(1.2) + 0.5] = \frac{3}{8}[5.2428] = \mathbf{1.9661}$$

(i) By direct integration

$$I = \int_0^6 \frac{1}{1+x}dx = [\log_e(1+x)]_0^6$$

$$= \log_e(1+6) - \log_e(1+0)$$

$$= \log_e 7 - \log_e 1 = \log_e 7 = \mathbf{1.9459}$$

■

7.5 BOOLE'S RULE

Boole's rule is

$$\int_a^b y dx = \frac{2h}{45}[7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7$$
$$+ 14y_8 + \ldots + 14y_{n-4} + 32y_{n-3} + 12y_{n-2} + 32y_{n-1} + 7y_n]$$

Proof: We derive the formula using the general quadrature formula

In the general quadrature formula, put $\underline{n=4}$, then the interval is (x_0, x_0+4h) and the five values of y are y_0, y_1, y_2, y_3, y_4.

So, the fifth and higher differences are zero.

60 20 1200

$$\therefore \int_{x_0}^{x_0+4h} y\,dx = h\left[4y_0 + \frac{4^2}{2}\Delta y_0 + \frac{1}{2!}\left(\frac{4^3}{3} - \frac{4^2}{2}\right)\Delta^2 y_0 + \frac{1}{3!}\left(\frac{4^4}{4} - 4^3 + 4^2\right)\Delta^3 y_0\right.$$

$$\left. + \frac{1}{4!}\left(\frac{4^5}{5} - \frac{3}{2}4^4 + \frac{11}{3}4^3 - 3.4^2\right)\Delta^4 y_0\right]$$

$$= h\left[4y_0 + 8\Delta y_0 + \frac{20}{3}\Delta^2 y_0 + \frac{8}{3}\Delta^3 y_0 + \frac{28}{90}\Delta^4 y_0\right]$$

Now

$$\Delta y_0 = y_1 - y_0, \quad \Delta y_1 = y_2 - y_1, \quad \Delta y_2 = y_3 - y_2, \quad \Delta y_3 = y_4 - y_3$$

$$\Delta^2 y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\Delta^2 y_1 = y_3 - y_2 - (y_2 - y_1) = y_3 - 2y_2 + y_1$$

$$\Delta^2 y_2 = y_4 - y_3 - (y_3 - y_2) = y_4 - 2y_3 + y_2$$

$$\Delta^3 y_0 = y_3 - 2y_2 + y_1 - (y_2 - 2y_1 + y_0) = y_3 - 3y_2 + 3y_1 - y_0$$

$$\Delta^3 y_1 = y_4 - 2y_3 + y_2 - (y_3 - 2y_2 + y_1) = y_4 - 3y_3 + 3y_2 - y_1$$

$$\Delta^4 y_0 = y_4 - 3y_3 + 3y_2 - y_1 - (y_3 - 3y_2 + 3y_1 - y_0)$$

$$= y_4 - 4y_3 + 4y_2 - 4y_1 + y_0 \cdots$$

$$\therefore \int_{x_0}^{x_0+4h} y\,dx = h\left[4y_0 + 8(y_1 - y_0) + \frac{20}{3}(y_2 - 2y_1 + y_0) + \frac{8}{3}(y_3 - 3y_2 + 3y_1 - y_0)\right.$$

$$\left. + \frac{28}{90}(y_4 - 4y_3 + 4y_2 - 4y_1 + y_0)\right]$$

$$= 4h\left[y_0 + 2(y_1 - y_0) + \frac{5}{3}(y_2 - 2y_1 + y_0) + \frac{2}{3}(y_3 - 3y_2 + 3y_1 - y_0)\right.$$

$$\left. + \frac{7}{90}(y_4 - 4y_3 + 4y_2 - 4y_1 + y_0)\right]$$

$$= 4h\left[\frac{7}{90}y_0 + \frac{16}{45}y_1 + \frac{6}{45}y_2 + \frac{16}{45}y_3 + \frac{7}{10}y_4\right]$$

$$= \frac{4h}{90}[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4]$$

$$= \frac{2h}{45}[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4]$$

Similarly,

$$\int_{x_0+4h}^{x_0+8h} y\,dx = \frac{2h}{45}[7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8]$$

$$\int_{x_0+8h}^{x_0+12h} y\,dx = \frac{2h}{45}[7y_8 + 32y_9 + 12y_{10} + 32y_{11} + 7y_{12}]$$

$$\vdots$$

$$\int_{x_0+(n-4)h}^{x_0+nh} y\,dx = \frac{2h}{45}[7y_{n-4} + 32y_{n-3} + 12y_{n-2} + 32y_{n-1} + 7y_n]$$

Adding all these integrals, we get

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{2h}{45}[7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 14y_8$$
$$+ \dots + 14y_{n-4} + 32y_{n-3} + 12y_{n-2} + 32y_{n-1} + 7y_n]$$

where n is a multiple of 4

ie the number of sub-intervals should be 4, 8, 12, ...

This formula is called Boole's formula.

This formula is also known as composite Boole's formula.

WORKED EXAMPLES

Example 1

Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Boole's rule with $h = \frac{1}{4}$.

Solution

Let $\qquad I = \int_0^1 \frac{dx}{1+x^2}$

Here $\qquad y = \frac{1}{1+x^2}$ and $\qquad h = \frac{1}{4} = 0.25$

$\therefore$ number of sub-intervals is 4.

So, we can use Boole's rule.

The points are $x_0 = 0$, $\quad x_1 = 0.25$, $\quad x_2 = 0.5$, $\quad x_3 = 0.75$, $\quad x_4 = 1$

The values of y are y_0, y_1, y_2, y_3, y_4 and they are given by the table below

x	0	0.25	0.5	0.75	1
x^2	0	0.0625	0.25	0.5625	1
$y = \frac{1}{1+x^2}$	1	0.9412	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4

By Boole's rule,

$$I = \int_0^1 \frac{dx}{1+x^2} = \frac{2h}{45}[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4]$$

$$= \frac{2}{45} \times \frac{1}{4}[7 \times 1 + 32(0.9412) + 12(0.8) + 32(0.64) + 7(0.5)]$$

$$= \frac{1}{90}[7 + 30.1184 + 9.6 + 20.48 + 3.5]$$

$$= \frac{1}{90}[70.6984] = \mathbf{0.7855}$$

■

Example 2

Use Boole's rule to estimate approximately the area of the cross-section if the river is 80 ft wide, the depth *d* ft at a distance *x* from one bank being given by the following table.

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

Solution

Required area is $\int_0^{80} y\,dx$, where $y = d$.

Given table is

x	0	10	20	30	40	50	60	70	80
$y = d$	0	4	7	9	12	15	14	8	3
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Number of interval is 8

$\therefore n = 8$, a multiple of 4 and $h = 10$

So, we can use Boole's rule to find the required area.

By Boole's rule,

$$\int_0^{80} y\,dx = \frac{2h}{45}[7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8]$$

$$= \frac{2\times10}{45}[7\times0+32\times4+12\times7+32\times9+14\times12+32\times15+12\times14+32\times8+7\times3]$$

$$= \frac{4}{9}[128+84+288+168+480+168+256+21]$$

$$= \frac{4}{9}\times1593 = 708 \text{ sq.ft}$$

∴ the area of the cross–section is **708 sq.ft**

Example 3

The velocity v of a particle at distance x from a point on its path is given below

x (metre)	**0**	**10**	**20**	**30**	**40**
v (m/metre)	**45**	**60**	**65**	**64**	**42**

Find the time taken to cover the distance of 40 meters.

Solution

Let v be the velocity at time t.

Since x is given as the distance covered by a particle at time t, we know the velocity is

$$v = \frac{dx}{dt}$$

$$\Rightarrow \qquad dt = \frac{1}{v}dx$$

$$\therefore \qquad t = \int_0^x \frac{1}{v}dx = \int_0^x y\,dx, \qquad \text{where } y = \frac{1}{v}$$

∴ the time taken to travel $x = 40$ is $\int_0^{40} y\,dx$.

The values of y are given by the table below

x	0	10	20	30	40
v	45	60	65	64	42
$y = \frac{1}{v}$	0.0222 y_0	0.0167 y_1	0.0154 y_2	0.0156 y_3	0.0238 y_4

The number of interval is 4 and $h = 10$ $\quad \therefore n = 4$

So, we can use Boole's rule to find $\int_0^{40} y\,dx$.

By Boole's rule,

$$\int_0^{40} y\,dx = \frac{2h}{45}[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4]$$
$$= \frac{2\times 10}{45}[7\times 0.0222 + 32(0.0167) + 12(0.0154) + 32(0.0156) + 7(0.0238)]$$
$$= \frac{4}{9}[0.1554 + 0.5344 + 0.1848 + 0.4992 + 0.1666]$$
$$= \frac{4}{9}(1.5404) = 0.6846$$

$\therefore$ the time taken to travel the distance 40 metres 0.6846 minutes

$= 0.6648 \times 60$ secs $=$ **41.076 secs**

WEDDLE'S RULE

Weddle's rule is

$$\int_a^b y\,dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8$$
$$+ 6y_9 + y_{10} + 5y_{11} + 2y_{12} + \ldots + y_n]$$

Proof: We derive the formula using the general quadrature formula. In the General Quadrature formula put $n = 6$, then the interval is $(x_0, x_0 + 6h)$ is and the seven values $y_0, y_1, y_2, y_3, y_4, y_5, y_6$ of y are involved.

So, the seventh difference and higher difference are zero.

$$\therefore \quad \int_{x_0}^{x_0+6h} y\,dx = h\left[6y_0 + \frac{6^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{6^3}{3} - \frac{6^2}{2}\right)\Delta^2 y_0\right.$$
$$+ \frac{1}{3!}\left(\frac{6^4}{4} - 6^3 + 6^2\right)\Delta^3 y_0$$
$$+ \frac{1}{4!}\left(\frac{6^5}{5} - \frac{3}{2}\times 6^4 + \frac{11}{3}\times 6^3 - 3\times 6^2\right)\Delta^4 y_0$$
$$+ \frac{1}{5!}\left(\frac{6^6}{6} - 2\times 6^5 + \frac{35}{4}\times 6^4 - \frac{50}{3}\times 6^3 + 12\times 6^2\right)\Delta^5 y_0$$
$$\left. + \frac{1}{6!}\left(\frac{6^7}{7} - 15\frac{6^6}{6} + 85\frac{6^5}{5} - 225\frac{6^4}{4} + 274\frac{6^3}{3} - 120\frac{6^2}{2}\right)\Delta^6 y_0\right]$$
$$= h\left[6y_0 + 18\,\Delta y_0 + \underline{27\,\Delta^2 y_0} + 24\,\Delta^3 y_0 + \frac{123}{10}\,\Delta^4 y_0 + \frac{33}{10}\,\Delta^5 y_0 + \frac{41}{140}\,\Delta^6 y_0\right]$$

To rewrite the formula conveniently, replace $\frac{41}{140}\Delta^6 y_0$ by $\frac{3}{10}\Delta^6 y_0$, as the error involved is negligible for small values of h.

$$\int_{x_0}^{x_0+6h} y\,dx = h\left[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{3}{10}\Delta^6 y_0\right]$$

$$= h\Big[6y_0 + 18(y_1 - y_0) + 27(y_2 - 2y_1 + y_0)$$

$$+ 24(y_3 - 3y_2 + 3y_1 - y_0) + \frac{123}{10}(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0)$$

$$+ \frac{33}{10}(y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0)$$

$$+ \frac{33}{10}(y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0)\Big]$$

$$\Rightarrow \qquad \int_{x_0}^{x_0+6h} y\,dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

Similarly,

$$\int_{x_0+6h}^{x_0+12h} y\,dx = \frac{3h}{10}[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}]$$

$$\int_{x_0+12h}^{x_0+18h} y\,dx = \frac{3h}{10}[y_{12} + 5y_{13} + y_{14} + 6y_{15} + y_{16} + 5y_{17} + y_{18}]$$

$$\vdots$$

$$\int_{x_0+(n-6)h}^{x_0+nh} y\,dx = \frac{3h}{10}[y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n]$$

Adding all these integrals, we get

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{10}\Big[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6$$

$$+ 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + 2y_{12} + \ldots + y_n\Big]$$

where n is the number of intervals and it is a multiple of 6.
That is the number of sub-intervals should be 6, 12, 18, ...
This is known as Weddle's rule.

WORKED EXAMPLES

Example 1

Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using Weddle's rule.

Solution

Let $\qquad I = \int_0^6 \frac{dx}{1+x^2} \qquad (1)$

Here $\qquad y = \frac{1}{1+x^2}$

Divide the interval [0, 6] into 6 equal intervals.
$\therefore$ the number of sub-intervals is 6, a multiple of 6.

$\therefore \quad n = 6, \quad$ and $\quad h = \frac{6-0}{6} = 1$

So, we can use Weddle's rule to find (1)

The points are $x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 3,\ x_4 = 4,\ x_5 = 5,\ x_6 = 6$

The values of y are $y_0, y_1, y_2, y_3, y_4, y_5, y_6$ and they are given by the table below

x	0	1	2	3	4	5	6
$y = \frac{1}{1+x^2}$	1 y_0	0.5 y_1	0.2 y_2	0.1 y_3	0.0588 y_4	0.0385 y_5	0.0270 y_6

By Weddle's rule,

$$I = \int_0^6 \frac{dx}{1+x^2} = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$$= \frac{3\times 1}{10}[1 + 5(0.5) + 0.2 + 6(0.1) + 0.0588 + 5(0.0385) + 0.0270]$$

$$= \frac{3}{10}[1 + 2.5 + 0.2 + 0.6 + 0.0588 + 0.1925 + 0.0270]$$

$$= \frac{3}{10}(4.5783) = \mathbf{1.37349}$$

■

Example 2

A curve is drawn to pass through the points given by the following table.

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1

Using Weddle's rule, estimate the area bounded by the curve, the x-axis and the lines $x = 1$, $x = 4$.

Solution

The area bounded by the curve $y = f(x)$, the x-axis and the lines $x = 1, x = 4$ is $\int_1^4 y\,dx$.

The interval [1, 4] is divided into 6 equal parts with $h = 0.5$

$\therefore n = 6$, a multiple of 6

So, we can use Weddle's rule to find $\int_1^4 y\,dx$.

The points are $x_0 = 1,\quad x_1 = 1.5,\quad x_2 = 2,\quad x_3 = 2.5,\quad x_4 = 3,\quad x_5 = 3.5,\quad x_6 = 4$

The values of y are $y_0, y_1, y_2, y_3, y_4, y_5, y_6$ and they are given by the table below.

x	1	1.5	2	2.5	3	3.5	4
y	2 y_0	2.4 y_1	2.7 y_2	2.8 y_3	3 y_4	2.6 y_5	2.1 y_6

By Weddle's rule,

$$\int_1^4 y\,dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$$= \frac{3(0.5)}{10}[2 + 5(2.4) + 2.7 + 6(2.8) + 3 + 5(2.6) + 2.1]$$

$$= \frac{1.5}{10}[2 + 12 + 2.7 + 16.8 + 3 + 13 + 2.1]$$

$$= \frac{1.5}{10}[51.6] = 7.74$$

∴ the area is **7.74 sq. units**

Example 3

Use Weddle's rule to evaluate the approximate value of $\int_4^{5.2} \log_e x\,dx$, given that

x	4.0	4.2	4.4	4.6	4.8	5	5.2
$\log_e x$	1.3863	1.4351	1.4816	1.5260	1.5686	1.6094	1.6486

Compare it with exact value.

Solution

Let $$I = \int_4^{5.2} \log_e x\,dx \tag{1}$$

Here $y = \log_e x$

The interval is (4, 5.2) and it is divided into 6 equal parts with $h = 0.2$

∴ $n = 6$,

So, we can use Weddle's rule to find (1).

The points are $x_0 = 4$, $x_1 = 4.2$, $x_2 = 4.4$, $x_3 = 4.6$, $x_4 = 4.8$, $x_5 = 5$, $x_6 = 5.2$

The values of y are $y_0, y_1, y_2, y_3, y_4, y_5, y_6$ and they are given by the table below.

x	4	4.2	4.4	4.6	4.8	5	5.2
$y = \log_e x$	1.3863 y_0	1.4351 y_1	1.4816 y_2	1.5260 y_3	1.5686 y_4	1.6094 y_5	1.6486 y_6

By Weddle's rule,

$$I = \int_4^{5.2} \log_e x\,dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$$= \frac{3(0.2)}{10}[1.3863 + 5(1.4351) + 1.4816 + 6(1.5260)$$
$$+ 1.5686 + 5(1.6094) + 1.6486]$$

$$=\frac{0.6}{10}[1.3863+7.1755+1.4816+9.156+1.5686+8.047+1.6486]$$

$$=\frac{0.6}{10}[30.4636]=\mathbf{1.8278}$$

By direct integration,

$$I=\int_4^{5.2}\log_e x\,dx=\left[\log_e x.\,x\right]_4^{5.2}-\int_4^{5.2}\frac{1}{x}.x\,dx$$

$$=5.2\log_e 5.2-4\log_e 4-\int_4^{5.2}dx$$

$$=5.2\log_e 5.2-4\log_e 4-[x]_4^{5.2}$$

$$=5.2\log_e 5.2-4\log_e 4-(5.2-4)$$

$$=8.5730-5.5452-5.2+4=1.8278$$

$\therefore$ the actual value is 1.8278

By Weddle's rule, the value is 1.8278

$\therefore$ error is zero.

Example 4

The velocity of the particle at a distance *s* from a point on its path is given by the table below.

s (metres)	0	10	20	30	40	50	60
v (m/sec)	47	58	64	65	61	52	38

Estimate the time taken to travel 60 meters by (i) Simpson's rule

(ii) Simpson's $\frac{3}{8}$ rule

(iii) Weddle rule.

Solution

Let s be the distance travelled and v be the velocity at time t.

We know the velocity $\qquad v=\dfrac{ds}{dt}$

$\Rightarrow \qquad dt=\dfrac{1}{v}ds$

$\therefore \qquad \displaystyle\int_0^t dt=\int_0^s\frac{1}{v}ds \quad\Rightarrow\quad t=\int_0^s y\,ds,$ where $y=\dfrac{1}{v}$

$\therefore$ the time taken to travel 60 ft is $=\displaystyle\int_0^{60} y\,ds.$

The values of y are $y_0, y_1, y_2, y_3, y_4, y_5, y_6$ and they are given by the table below

s	0	10	20	30	40	50	60
v	47	58	64	65	61	52	38
$y = \frac{1}{v}$	0.0213	0.0172	0.0156	0.0154	0.0164	0.0192	0.0263
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Number of intervals is 6 with $h = 10$

(i) Simpson's $\frac{1}{3}$ rule,

$$\int_0^{60} y ds = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{10}{3}[(0.0213 + 0.0263) + 4(0.0172 + 0.0154 + 0.0192) + 2(0.0156 + 0.0164)]$$

$$= \frac{10}{3}[0.0476 + 0.2072 + 0.064] = \mathbf{1.0627}$$

(ii) By Simpson's $\frac{3}{8}$ rule,

$$\int_0^{60} y dx = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3\times 10}{8}[(0.0213 + 0.0263) + 3(0.0172 + 0.0156 + 0.0164 + 0.0192) + 2(0.0154)]$$

$$= \frac{15}{4}[0.0476 + 0.236] = \mathbf{1.0635}$$

(iii) By Weddle's rule,

Since $n = 6$, we can use weddle's formula.

$$\therefore \int_0^{60} y\, ds = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_2 + y_4 + 5y_5 + y_6]$$

$$= \frac{3\times 10}{10}[0.0213 + 5(0.0172) + 0.0156 + 6(0.0154) + 0.0164 + 5(0.0192) + 0.0263]$$

$$= 3[0.0213 + 0.086 + 0.0156 + 0.0924 + 0.0164 + 0.096 + 0.0263]$$

$$= 3[0.354] = \mathbf{1.062}$$

$\therefore$ by the three methods, the time taken to travel the distance is equal, correct to two places of decimal. ■

ERROR IN NUMERICAL INTEGRATION FORMULAE

Error in Trapezoidal Rule

Trapezoidal rule is

$$\int_a^b f(x)dx = h\left[\frac{y_0 + y_n}{2} + y_1 + y_2 + \ldots + y_{n-1}\right]$$

where $y_0, y_1, \ldots, y_n$ are the values of $y = f(x)$ at $x_0, x_1, \ldots, x_n$

where $x_\eta = x_0 + \eta h$ and $h = \frac{b-a}{n}$ (1)

Taylor's series expansion for $f(x)$ about x_0 is

$$f(x) = f(x_0) + \frac{x - x_0}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \cdots$$

$$\therefore \int_{x_0}^{x_1} f(x)dx = \int_{x_0}^{x_1}\left[f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \cdots\right]dx$$

$$= \left[f(x_0)x + \frac{(x-x_0)^2}{2.1!} f'(x_0) + \frac{(x-x_0)^3}{3.2!} f''(x_0) + \cdots\right]_{x_0}^{x_1} \quad (2)$$

$$= f(x_0)(x_1 - x_0) + \frac{(x_1-x_0)^2}{2!} f'(x_0) + \frac{(x_1-x_0)^3}{3!} f''(x_0) + \cdots$$

$$\int_{x_0}^{x_1} f(x)dx = hy_0 + \frac{h^2}{2!} y'_0 + \frac{h^3}{3!} y''_0 + \ldots \quad (3)$$

where $h = x_1 - x_0$, $y'_0 = f'(x_0)$, $y''_0 = f''(x_0)$

Area of the trapezium in (x_0, x_1) is $\frac{h}{2}[y_0 + y_1] = A_0$ (4)

$\Rightarrow$

$$A_0 = \frac{h}{2}[y_0 + y_1]$$

Put $x = x1$ in (1) we get $f(x_1) = f(x_0) + \frac{(x_1-x_0)}{1!} f'(x_0) + \frac{(x_1-x_0)^2}{2!} f''(x_0) + \cdots$

$\Rightarrow$

$$y_1 = y_0 + hy'_0 + \frac{h^2}{2!} y''_0 + \ldots$$

$\therefore$

$$A_0 = \frac{h}{2}\left[y_0 + y_0 + hy'_0 + \frac{h^2}{2!} y''_0 + \ldots\right]$$

$$= h y \;\; h^2 \;\; h^3$$

$$\int_{x_0}^{x_1} f(x)dx - A_0 = h^3 y_0''\left(\frac{1}{6}-\frac{1}{4}\right)+\ldots\ldots = -\frac{h^3}{12}y_0''+\ldots.$$

$\therefore$ The principal part of error in (x_0, x_1) is $= \frac{-h^3}{12}y_0''$

Similarly principal part of error in (x_1, x_2) is $= \frac{-h^3}{12}y_1''$, etc.

error in (x_{n-1}, x_n) is $\frac{-h^3}{12}y_{n-1}''$

$\therefore$ total error E is given by $E = \frac{-h^3}{12}(y_0''+y_1''+\ldots y_{n-1}'')$

If $M = Max\{|y_0''|, |y_1''|, \ldots, |y_{n-1}''|\}$, then $|E| < \frac{nh^3}{12} = \frac{(b-a)h^2}{12}M$ $\left(\because h = \frac{b-a}{n}\right)$

Hence the total error in Trapezoidal rule is of order h^2

8 4 2

Error in Simpson's Rule

Simpson's rule is

$$\int_a^b f(x)dx = \frac{h}{3}[(y_0+y_n)+4(y_1+y_3+y_5+\ldots+y_{n-1})$$
$$+2(y_2+y_4+y_6+\ldots+y_{n-2})]$$

where $h = \frac{b-a}{n}$, n is even, $y_0, y_1, y_2, \ldots, y_n$ are the values of $y = f(x)$ at $x_0, x_1, x_2, \ldots, x_n$

Taylor's series expansion of $f(x)$ about x_0 is

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!}f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0)+\ldots$$

$\therefore$

$$\int_{x_0}^{x_2} f(x)\,dx = \int_{x_0}^{x_2}\left[f(x_0)+\frac{(x-x_0)}{1!}f'(x_0)+\frac{(x-x_0)^2}{2!}f''(x_0)+\ldots\right]dx \qquad (1)$$

$$= \left[xf(x_0)+\frac{(x-x_0)^2}{2.1!}f'(x_0)+\frac{(x-x_0)^3}{3.2!}f''(x_0)+\ldots\right]_{x_0}^{x_2}$$

$$=(x_2-x_0)f(x_0)+\frac{(x_2-x_0)^2}{2!}f'(x_0)+\frac{(x_2-x_0)^3}{3!}f''(x_0)+\ldots$$

$$= 2hy_0 + \frac{(2h)^2}{2!}y_0' + \frac{(2h)^3}{3!}y_0'' + \frac{(2h)^4}{4!}y_0''' + \ldots$$

$= 2hy_0 + 2h^2y_0' + \frac{4}{3}h^3y_0'' + \frac{2}{3}h^4y_0''' + \frac{4}{15}y_0^{iv}$

But by Simpson's rule, area in $[x_0, x_2]$ is $A_0 = \frac{h}{3}[y_0+4y_1+y_2]$ (2)

Putting $x = x_1$ in (1), we have

$$f(x_1) = f(x_0) + \frac{(x_1 - x_0)}{1!} f'(x_0) + \frac{(x_1 - x_0)^2}{2!} f''(x_0) + \ldots$$

$$\Rightarrow \qquad y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \ldots$$

Putting $x = x_2$ in (1), we have

$$f(x_2) = f(x_0) + \frac{(x_2 - x_0)}{1!} f'(x_0) + \frac{(x_2 - x_0)^2}{2!} f''(x_0) + \ldots$$

$$\Rightarrow \qquad y_2 = y_0 + \frac{2h}{1!} y_0' + \frac{(2h)^2}{2!} y_0'' + \ldots$$

Substituting for y_1 and y_2 in (3) we get

$$A_0 = \frac{h}{3}\left[y_0 + 4\left(y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \ldots \right) + \left(y_0 + \frac{2h}{1!} y_0' + \frac{4h^2}{2!} y_0'' + \ldots \right) \right]$$

$$= 2hy_0 + 2h^2 y_0' + \frac{4h^3}{3} y_0'' + \frac{2h^4}{3} y_0^{(3)} + \frac{5h^5}{18} y_0^{(4)} + \ldots \tag{4}$$

$$\therefore \int_{x_0}^{x_2} f(x)\,dx - A_0 = \left(\frac{4}{15} - \frac{5}{18} \right) h^5 y_0^{(4)} + \ldots = \frac{-h^5}{90} y_0^{(4)} + \ldots$$

Omitting h^6 and higher powers, the principal part of error in $[x_0, x_2]$ is $= -\frac{h^5}{90} y^{(4)}$ and so on.

the total error $= -\frac{h^5}{90}\left\{ y_0^{(4)} + y_1^{(4)} + \ldots \right\}$

nh^5

$$\therefore \qquad |E| < \frac{nh^5}{90} M, \qquad \text{where } M = \max\left\{ y_0^{(4)} + y_1^{(4)} + \ldots \right\}$$

$$\Rightarrow \qquad |E| < \frac{(b-a)}{180} h^4 M \qquad \left[\because h = \frac{b-a}{2n} \right]$$

In the same way, we can find the error in the quadrature formulae. We shall indicate the principal error in each formula

Error in Simpson's $\frac{3}{8}$ rule

The principal error in the interval $[x_0, x_3]$ is $-\frac{3}{80} h^5 \; y_0^{(4)}$

Error in Boole's rule

The principal error in the interval $[x_0, x_4]$ is $-\frac{8\,h^7}{945} y_0^{(6)}$

Error in Weddle's rule

The principal error in the interval $[x_0, x_6]$ is $-\frac{h^7}{140} y_0^6$

Exercises 7.1

(1) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using

(i) Trapezoidal rule (ii) Simpson's $\frac{1}{3}$ rule with $h=\frac{1}{4}$

(iii) Simpson's $\frac{3}{8}$ rule with $h=\frac{1}{6}$

(2) Evaluate $\int_4^{5.4} \log_e x\,dx$ using the following data.

x	4	4.2	4.4	4.6	4.8	5	5.2
$\log_e x$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

(i) by Trapezoidal rule (ii) by Simpson's rule

(iii) by Simpson's $\frac{3}{8}$ rule

(3) Using the following table evaluate $\int_0^2 y\,dx$ by (a) Trapezoidal rule (b) Simpson's rule.

x	0	0.5	1.0	1.5	2.0
y	0.399	0.352	0.242	0.129	0.054

(4) Evaluate $\int_0^1 e^{-x}dx$, taking $i = 0.1$ by 9

(a) Trapezoidal rule (b) Simpson's rule

(5) Evaluate $\int_{-3}^{3} x^4 dx$ using

(i) Trapezoidal rule (ii) Simpson's rule

(6) Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by

(i) Trapezoidal rule (ii) Simpson's rule

(7) Evaluate $\int_0^{0.6} e^{-x^2} dx$ by Simpson's rule with $h = 0.1$.

(8) Evaluate $\int_0^1 \frac{x^2}{1+x^2} dx$ using Simpson's rule with $h = 0.25$ and hence reduce the value of $\log_e 3\sqrt{2}$.

(9) Evaluate $\int_{0.2}^{1.4} (\sin x - \log_e x + e^x)dx$.

(10) Using Simpson's rule with $h = 0.2$. Compare the results by integration.

(11) Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by Simpson's $\frac{3}{8}$ rule.

(12) Evaluate $\int_4^{5.2} \log_e^x dx$ by Simpson's $\frac{3}{8}$ rule.

(13) Evaluate $\int_0^{\frac{\pi}{2}} e^{\sin x} dx$ by Boole's rule.

(14) Evaluate $\int_{0.5}^{0.7} x^{1/2} e^{-x} dx$ by Boole's rule.

(15) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by Boole's rule with 13 ordinates.

(16) The following gives the velocity v of a particle at time t.

t in hrs	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
velocity v km/hr	6	7.5	8	9	8.5	10.5	9.5	7	6

Find the distance travelled in 2 hours.

(17) Evaluate using Weddle's rule

(i) $\int_{0.2}^{1.4} (\sin x - \log_e x + e^x) dx$

(ii) $\int_0^{\pi/2} \sqrt{1 - 0.162 \sin^2 x}\, dx \quad \left[h = \frac{\pi}{2}\right]$

(18) Evaluate $\int_0^1 \frac{dx}{1+x}$ by Weddle's rule with 7 ordinates and reduce the value of $\log_e 2$.

(19) A curve is drawn to pass through the points given by the following table

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1

using Weddle's rule, estimate the area bounded by the curve, the x-axis and the lines $x = 1$, $x = 4$.

Answers 7.1

(1) 0.7854, 0.7854, 0.7854
(2) 1.82765, 1.82784, 1.82785
(3) 0.475, 0.477
(4) 0.6686; 0.6321
(5) 115, 98
(6) 1.4108; 1.3662
(7) 0.5351
(8) 0.3108
(9) 4.05214
(10) 1.3571
(11) 1.3571
(12) 1.8278
(13) 3.1067
(14) 0.0549
(15) 0.8236
(16) 16.68 km
(17) (i) 4.05145 (ii) 0.5051
(18) 0.6932
(19) 3.032

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ROMBERG'S METHOD FOR INTEGRATION

We have seen different formulae for numerical integration obtained by the finite difference methods. Of the three formulae trapezoidal rule, Simpson's $\frac{1}{3}$ rule and Simpson's $\frac{3}{8}$ rule, Simpson's $\frac{1}{3}$ rule gives the best result.

A method due to L.F Richarson known as **Richardson's deferred approach to the limit** provides a method to improve the accuracy of the approximate values of definite integrals obtained by finite difference methods. **This technique is known as Romberg's integration.**

Romberg's Integration Formula Based on Trapezoidal Rule

Let $I = \int_a^b f(x)\,dx$

Let I_1, I_2 be two values of I by trapezoidal rule with subintervals of width h_1, h_2 and let E_1, E_2 be their corresponding errors. Let $y = f(x)$

Then $E_1 = -\frac{1}{12}(b-a)h_1^2 y''(x_\alpha)$

and $E_2 = -\frac{1}{12}(b-a)h_2^2 y''(x_\beta)$

Since $y''(x_\alpha)$, $y''(x_\beta)$ are both largest values of $y''(x)$, we can assume $y''(x_\alpha)$ and $y''(x_\beta)$ are nearly equal.

$$\therefore \qquad \frac{E_1}{E_2} = \frac{h_1^2}{h_2^2} \Rightarrow \frac{E_1}{E_2 - E_1} = \frac{h_1^2}{h_2^2 - h_1^2}$$

But $I = I_1 + E_1$ and also $I = I_2 + E_2$

$$\therefore \qquad I_1 + E_1 = I_2 + E_2 \Rightarrow E_2 - E_1 = I_1 - I_2$$

$$\therefore \qquad E_1 = \frac{h_1^2}{h_2^2 - h_1^2}(E_2 - E_1) \Rightarrow E_1 = \frac{h_1^2}{h_2^2 - h_1^2}(I_1 - I_2)$$

$$\therefore \qquad I = I_1 + \frac{h_1^2}{h_2^2 - h_1^2}(I_1 - I_2)$$

$$\Rightarrow \qquad I = \frac{I_1 h_2^2 - I_2 h_1^2}{h_2^2 - h_1^2}$$

which is a better approximation of I than I_1 and I_2.

This method is called **Richardson's deferred approach to the limit.**

To evaluate I systematically, we take $h_1 = h$, $\qquad h_2 = \frac{h}{2}$

$$\therefore \qquad I = \frac{I_1 \frac{h^2}{4} - I_2 h^2}{\frac{h}{4} - h^2}$$

$$= \frac{\frac{h^2}{4}[I_1 - 4I_2]}{\frac{-3h^2}{4}} = \frac{4I_2 - I_1}{3}$$

$$\Rightarrow \qquad I = I_2 + \frac{(I_2 - I_1)}{3} \qquad (1)$$

This is known as Romberg's formula.

The formula (1) is obtained by trapezoidal rule twice with width of intervals h and $\frac{h}{2}$.

We again apply trapezoidal rule with width $\frac{h}{4}$ and apply several times halving the intervals.

Each time the error is reduced by a factor $\frac{1}{4}$.

Thus we have a sequence of values of I.

$I_1, I_2, I_3, I_4, \ldots$ are the values with width $h, \frac{h}{2}, \frac{h}{4}, \frac{h}{8}, \ldots$

Applying (1) for the pairs I_1, I_2; I_2, I_3; I_3, I_4; ... , we get $A_1, A_2, A_3, \ldots$. which are improved values.

Again applying (1) to the pairs

$$A_1, A_2; \quad A_2, A_3; \quad A_3, A_4; \quad \ldots$$

We get

$$B_1, B_2, B_3, \quad \ldots.$$

This systematic refinement of Richardson's method is called **Romberg method** or **Romberg integration.**

WORKED EXAMPLES

Example 1

Evaluate $\int_0^1 \frac{dx}{1+x}$ correct to 3 places using Romberg's method and hence find the value of $\log_e 2$.

Solution

Let $$I = \int_0^1 \frac{dx}{1+x}$$

Here $$y = f(x) = \frac{1}{1+x}$$

Trapezoidal rule is

$$\int_0^1 \frac{1}{1+x}dx = h\left[\left(\frac{y_0+y_n}{2}\right)+(y_1+y_2+\ldots+y_{n-1})\right]$$

We shall find the value of the integral when $h = 0.5, 0.25, 0.125$

When $h = 0.5$, the values of y are given by the table below

x	0	0.5	1
y	1 y_0	0.6667 y_1	0.5 y_2

$$\therefore \qquad I_1 = h\left[\frac{y_0+y_2}{2}+y_1\right] = 0.5\left[\frac{1+0.5}{2}+0.6667\right] = 0.7084$$

When $h = \frac{1}{2}(0.5) = 0.25,$ the values of y are given by the table below

x	0	0.25	0.5	0.75	1
y	1 y_0	0.8 y_1	0.6667 y_2	0.5714 y_3	0.5 y_4

$$\therefore \qquad I_2 = h\left[\frac{(y_0+y_4)}{2}+(y_1+y_2+y_3)\right]$$

$$= 0.25\left[\frac{(1+0.5)}{2}+(0.8+0.6667+0.5714)\right]$$

$$= 0.25[2.7881] = 0.6970$$

When $h = \frac{1}{2}(0.25) = 0.125,$ the values of y are given by the table below

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	1 y_0	0.8889 y_1	0.8 y_2	0.7273 y_3	0.6667 y_4	0.6154 y_5	0.5714 y_6	0.5333 y_7	0.5 y_8

$$I_3 = h\left[\left(\frac{y_0+y_8}{2}\right)+(y_1+y_2+y_3+y_4+y_5+y_6+y_7)\right]$$

$$= 0.125\left[\left(\frac{1+0.5}{2}\right)+(0.8889+0.8+0.7273+0.6667+0.6154+0.5714+0.53333)\right]$$

$$= 0.125[5.553] = 0.6941$$

We shall now use Romberg's formula for I_1, I_2

$$I = I_2 + \frac{(I_2-I_1)}{3} \quad = 0.6970 + \frac{(0.6970-0.7084)}{3}$$

$$= 0.6932 = (A_1)$$

Again for I_2 and I_3, we have

$$I = I_3 + \frac{(I_3 - I_2)}{3} = 0.6941 + \frac{(0.6941 - 0.6970)}{3}$$
$$= 0.6931 = (A_2)$$

Now again applying Romberg's method with $A_1 = I_1, A_2 = I_2$ we get

$$I = 0.6931 + \frac{(0.6931 - 0.6932)}{3} = 0.6931$$

$$\therefore \quad \int_0^1 \frac{1}{1+x}dx = 0.6931$$

By direct integration,

$$\int_0^1 \frac{1}{1+x}dx = [\log(1+x)]_0^1 = \log_e 2 - \log_e 1 = \log_e 2$$

$$\therefore \quad \log_e 2 = 0.6931$$

Note: We find using calculator $\log_e 2 = 0.69314718$.
So, the answer by Romberg's method is correct to 4 decimal places. ∎

Example 2

Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using Romberg's method correct to 4 decimal places. Hence deduce an approximate value of π.

Solution

Let $$I = \int_0^1 \frac{dx}{1+x^2}.$$

Here $$y = \frac{1}{1+x^2}$$

Trapezoidal rule is $\int_0^1 y\,dx = h\left[\frac{(y_0 + y_n)}{2} + (y_1 + y_2 + \ldots + y_{n-1})\right]$

We shall find the value of I by trapezoidal rule for $h = 0.5$, 0.25, and 0.125
When $h = 0.5$, then the values of y are given by the table below

x	0	0.5	1
y	1	0.8	0.5
	y_0	y_1	y_2

Then $I = h\left[\frac{y_0 + y_2}{2} + y_1\right] = 0.5\left[\frac{1.5}{2} + 0.8\right] = 0.775$

When $h = \frac{1}{2}(0.5) = 0.25$, the values of y are given by the table below

x	0	0.25	0.5	0.75	1
y	1	0.9412	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4

Then

$$I_2 = h\left[\frac{y_0 + y_4}{2} + y_1 + y_2 + y_3\right]$$

$$= 0.25\left[\frac{1+5}{2} + 0.9412 + 0.8 + 0.64\right] = 0.7828$$

When $h = \frac{1}{4}(0.5) = 0.125$, the values of y are given by the table below.

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	1	0.9846	0.9412	0.8767	0.8	0.7191	0.64	0.5664	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Then

$$I_3 = h\left[\frac{y_0 + y_8}{2} + y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7\right]$$

$$= 0.125\left[\frac{1+5}{2} + 0.9486 + 0.9412 + 0.8767 + 0.8 + 0.7191 + 0.64 + 0.5664\right]$$

$$= 0.7848$$

We shall now use Romberg's formula $I = I_2 + \frac{(I_2 - I_1)}{3}$ for the pairs I_1, I_2.

$$\therefore \qquad I = 0.7828 + \frac{(0.7828 - 0.775)}{3} = 0.7854$$

and for the pair I_2, I_3

$$I = I_2 + \frac{(I_3 - I_2)}{3} = 0.7848 + \frac{(0.7848 - 0.7828)}{3}$$

$$I = 0.78546$$

So, for 4 decimals the 2 improvements coincide.

$\therefore$ the value of $I = 0.7854$

$$\therefore \qquad \int_0^1 \frac{dx}{1+x^2} = 0.7854$$

By direct integration, $\displaystyle\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}$

$\therefore \qquad \frac{\pi}{4} = 0.7854 \qquad \Rightarrow \pi = 4(0.7854) = \mathbf{3.1416}$

Note: From calculator $\pi = 3.141592654$.
So, the value is correct to 4 places. ■

Example 3

Evaluate $\int_0^{\frac{\pi}{4}} \cos^2 x\, dx$ using Romberg's formula.

Solution

Let $\qquad I = \int_0^{\frac{\pi}{4}} \cos^2 x\, dx.$

Let $\qquad y = \cos^2 x = \frac{1+\cos 2x}{2} = 0.5 + \frac{\cos 2x}{2}$

Trapezoidal rule is $\int_0^{\frac{\pi}{4}} \cos^2 x\, dx$

$$= h\left[\frac{y_0 + y_n}{2} + (y_1 + y_2 + y_3 + \ldots + y_{n-1})\right]$$

We shall find the values of the integral when $h = \frac{\pi}{8}, \frac{\pi}{16}, \frac{\pi}{32}$.

When $h = \frac{\pi}{8}$, the values of y are given by the table below

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$
y	1 y_0	0.8536 y_1	0.5 y_2

$$\therefore \qquad I_1 = h\left[\frac{y_0 + y_2}{2} + y_1\right]$$

$$= \frac{\pi}{8}\left[\frac{1+0.5}{2} + 0.8536\right]$$

$$= \frac{\pi}{8}[1.6036]$$

$$= 3.1416 \times 1.6036 = 0.6297$$

When $h = \frac{\pi}{16}$, the values of y are given by the table below.

x	0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{4\pi}{16}$	$\frac{4\pi}{16} = \frac{\pi}{4}$
y	1 y_0	0.9619 y_1	0.8536 y_2	0.6913 y_3	0.5 y_4

$$\therefore \quad I_1 = h\left[\frac{y_0 + y_4}{2} + y_1 + y_2 + y_3\right]$$

$$= \frac{\pi}{16}\left[\frac{1+0.5}{2} + 0.9619 + 0.8536 + 0.6913\right]$$

$$= \frac{3.1416}{16}[3.2568] \quad = 0.6395$$

When $h = \frac{\pi}{32}$, the values of y are given by the table below.

x	0	$\frac{\pi}{32}$	$\frac{\pi}{16}$	$\frac{3\pi}{32}$	$\frac{\pi}{8}$	$\frac{5\pi}{32}$	$\frac{6\pi}{32}$	$\frac{7\pi}{32}$	$\frac{\pi}{4}$
$y = \cos^2 x$	1	0.9904	0.9619	0.9157	0.8536	0.7778	0.6913	0.5975	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

$$\therefore \quad I_2 = h\left[\frac{y_0 + y_8}{2} + y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7\right]$$

$$= \frac{\pi}{32}\left[\frac{1+0.5}{2} + 0.9904 + 0.9619 + 0.9157 + 0.8536 + 0.7778 + 0.6913 + 0.5975\right]$$

$$= \frac{3.1416}{32}(6.5382) = 0.6419$$

Applying Romberg's formula for the pairs I_1, I_2 and I_2, I_3. We get

$$I = I_2 + \frac{I_2 - I_1}{3}$$

$$= 0.6395 + \frac{0.6395 - 0.6297}{3} = 0.6428 (= A_1)$$

Again for the pairs I_2, I_3, we get

$$I = I_3 + \frac{I_3 - I_2}{3}$$

$$= 0.6419 + \frac{(0.6419 - 0.6395)}{3} = 0.6427 (= A_2)$$

Again applying Romberg's formula for $A_1 = I_1, A_2 = I_2$, we get

$$I = I_2 + \frac{I_2 - I_1}{3}$$

$$= 0.6427 + \frac{(0.6427 - 0.6428)}{3} = 0.64266 = 0.6427$$

$$\therefore \quad \int_0^{\frac{\pi}{4}} \cos^2 x \, dx = 0.6427$$

Note: By direct integration

$$\int_0^{\frac{\pi}{4}} \cos^2 x \, dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} (1+\cos 2x)\, dx$$

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{\sin\frac{\pi}{2}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{2}\right] = \frac{\pi}{8} + 0.25 = 0.3927 + 0.25 = 0.6427$$

So, we find that the value by Romberg's method is correct. ■

Example 4

Evaluate $\int_0^2 \frac{dx}{x^2+4}$ using Romberg's method.

Solution

Let $$I = \int_0^2 \frac{dx}{x^2+4}$$

Here $$y = \frac{1}{x^2+4}$$

Trapezoidal rule is

$$\int_0^2 \frac{dx}{x^2+4} = h\left[\frac{y_0+y_n}{2} + y_1 + y_2 + y_3 + \ldots y_{n-1}\right]$$

When $h = 1, \frac{1}{2}, \frac{1}{4}$, we shall find the values of the integral.

When $h = 1$, the values of y are given by the table below.

x	0	1	2
$y = \frac{1}{x^2+4}$	0.25 y_0	0.2 y_1	0.1250 y_2

$\therefore$ $$I_1 = h\left[\frac{y_0+y_2}{2} + y_1\right]$$

$$= 1\left[\frac{0.25+0.125}{2} + 0.2\right] = 0.1875 + 0.2 = 0.3875$$

When $h = \frac{1}{2} = 0.5$, the values of y are given by the table below.

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x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$y = \frac{1}{x^2+4}$	0.25 y_0	0.2353 y_1	0.2 y_2	0.16 y_3	0.125 y_4

$$I_2 = h\left[\frac{y_0+y_4}{2} + y_1 + y_2 + y_3\right]$$
$$= 0.5\left[\frac{0.25+0.125}{2} + 0.2353 + 0.2 + 0.16\right]$$
$$= 0.5[0.1875 + 0.5953] = 0.5[0.7828] = 0.3914$$

When $h = \frac{1}{4}$, the values of y are given by the table below

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
$y = \frac{1}{x^2+4}$	0.25 y_0	0.2462 y_1	0.2353 y_2	0.2192 y_3	0.2 y_4	0.1798 y_5	0.16 y_6	0.1416 y_7	0.125 y_8

$$I_3 = h\left[\frac{y_0+y_8}{2} + y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7\right]$$
$$= 0.25\left[\frac{0.25+0.125}{2} + 0.2462 + 0.2353 + 0.2192 + 0.2 + 0.1798 + 0.16 + 0.1416\right]$$
$$= 0.25[0.1875 + 1.3821] = 0.3924$$

$\therefore \qquad I_1 = 0.3875, \quad I_2 = 0.3914, \quad I_3 = 0.3924$

We shall now use Romberg's formula for I_1, I_2

$$I = I_2 + \frac{I_2 - I_1}{3}$$
$$= 0.3914 + \frac{0.3914 - 0.3875}{3}$$
$$= 0.3914 + \frac{0.39}{3}$$
$$= 0.3914 + 0.0013 = 0.3927 = (A_1)$$

Again applying Romberg's method with I_2, I_3, we get

$$I = I_3 + \frac{I_3 - I_2}{3}$$

$$= 0.3924 + \frac{0.3924 - 0.3914}{3}$$

$$= 0.3924 + \frac{0.0010}{3} = 0.3924 + 0.0003 = 0.3927 = (A_2)$$

Since $A_1 = A_2$ upto 4 places of decimals.

We shall take $\quad I = \mathbf{0.3927}$

Note:

By direct integration $\int_0^2 \frac{dx}{x^2+4} = \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^2$

$$= \frac{1}{2}\left[\tan^{-1}1 - \tan^{-1}0\right]$$

$$= \frac{1}{2}\frac{\pi}{4} = \frac{\pi}{8} = \frac{3.1416}{8} = 0.3927$$

So, we find that the value by Romberg's method is correct.

Romberg Integration Formula Based on Simpson's Rule

Let $I = \int_a^b f(x)\,dx$.

Let I_1, I_2 be two values of I by Simpson's $\frac{1}{3}$ rule with sub-intervals width h_1, h_2 and let E_1, E_2 be their corresponding errors.

Then $$E_1 = -\frac{(b-a)}{180} h_1^4 y^{(4)}(\alpha)$$

and $$E_2 = -\frac{(b-a)}{180} h_2^4 y^{(4)}(\beta)$$

where $y^{(4)}(\alpha)$ and $y^{(4)}(\beta)$ are largest values of $y^{(4)}(x)$,

We assume them to be nearly equal.

$\therefore$ $$\frac{E_1}{E_2} = \frac{h_1^4}{h_2^4} \qquad \Rightarrow E_1 = \frac{h_1^4}{h_2^4} E_2$$

Suppose $h_1 = h$ and $h_2 = \frac{h}{2}$, then $E_1 = \frac{h^4}{\frac{h^4}{16}} E_2 = 16E_2$

But $I = I_1 + E_1$ and $I = I_2 + E_2$

$\Rightarrow$ $I = I_1 + 16E_2$

$\therefore$ $I_1 + 16E_2 = I_2 + E_2$

$\Rightarrow$ $I_2 - I_1 = 15E_2$

$\Rightarrow$ $E_2 = \frac{I_2 - I_1}{15}$

$\therefore$
$$I = I_2 + \frac{I_2 - I_1}{15} \tag{1}$$

This is Romberg's formula using Simpson's rule.

Again applying Simpson's rule with $\frac{h}{4}, \frac{h}{8}, \cdots$, we get a sequence of values of I.

Let $I_1, I_2, I_3, I_4, \ldots$ be the values of the integral with width $h, \frac{h}{2}, \frac{h}{4}, \frac{h}{8}, \cdots$

Applying the formula (1) for the pairs $I_1, I_2;\ I_2, I_3;\ I_3, I_4; \ldots,$

we get $A_1, A_2, A_3, \ldots,$ which are improved values of I.

Again applying (1) to the pairs $A_1, A_2;\ A_2, A_3;\ A_3, A_4; \ldots,$

we get $B_1, B_2, B_3, \cdots$ which are still improved values.

This method is called Romberg's method.

WORKED EXAMPLES

Example 1

Find the value of $\int_0^1 \left(1 + \frac{\sin x}{x}\right) dx$ correct to 4 decimal places using Simpson's rule and Romberg's integration.

Solution

Let
$$I = \int_0^1 \left(1 + \frac{\sin x}{x}\right) dx$$

Here
$$y = f(x) = 1 + \frac{\sin x}{x}$$

Simpson's $\frac{1}{3}$ rule is

$$\int_0^1 y\,dx = \frac{h}{3}[y_0 + y_n + 4(y_1 + y_3 + y_5 + \ldots y_{n-1}) + 2(y_2 + y_4 + y_6 + \ldots y_{n-2})]$$

where n is even

We shall find the values of the integral with $h = 0.5,\quad 0.25,\quad 0.125$

When $h = 0.5$, number of intervals is 2 $\quad\therefore n = 2$

Since $\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$, the values of y are given by the table below.

x	0	0.5	1
$y = 1 + \frac{\sin x}{x}$	2 y_0	1.95885 y_1	1.84147 y_2

Simpson's formula is

$$I_1 = \frac{h}{3}[y_0 + y_2 + 4y_1]$$

$$= \frac{0.5}{3}[2 + 1.84147 + 4 \times 1.95885]$$

$$= \frac{0.5}{3}[3.84147 + 7.8354]$$

$\Rightarrow$ $$I_1 = \frac{0.5}{3} \times 11.67687 = \mathbf{1.946145}$$

When $\quad h = 0.25, \qquad n = 4$, even

The values of y are given by the table below.

x	0	0.25	0.5	0.75	1
$y = 1 + \frac{\sin x}{x}$	2 y_0	1.98962 y_1	1.95885 y_2	1.90885 y_3	1.84147 y_4

$\therefore$ $$I_2 = \frac{h}{3}[y_0 + y_4 + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{0.25}{3}[2 + 1.84147 + 4 \times (1.98962 + 1.90885) + 2(1.95885)]$$

$$= \frac{0.25}{3}[3.84147 + 15.59388 + 3.9177]$$

$\Rightarrow$ $$I_2 = \frac{0.25}{3}(23.35305) = \mathbf{1.946088}$$

When $h = 0.125, \qquad n = 8$, even

The values of y are given by the table below.

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
$y = 1 + \frac{\sin x}{x}$	2 y_0	1.99740 y_1	1.98962 y_2	1.97673 y_3	1.95885 y_4	1.93616 y_5	1.90885 y_6	1.87719 y_7	1.84147 y_8

$$\therefore \quad I_3 = \frac{h}{3}[y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_3 + y_6)]$$

$$= \frac{0.125}{3}[2 + 1.84147 + 4(1.99740 + 1.97673 + 1.93616 + 1.87719)$$

$$+ 2(1.98962 + 1.95885 + 1.90885)]$$

$$\Rightarrow \quad I_3 = \frac{0.125}{3}[3.84147 + 31.14992 + 11.71464] = \frac{0.125}{3}(46.70603) = \mathbf{1.94608}$$

We now use Romberg's formula for the pairs I_1, I_2 and I_2, I_3

$$\therefore \quad I = I_2 + \frac{I_2 - I_1}{15}$$

$$= 1.946088 + \frac{1}{15}(1.946088 - 1.946145)$$

$$= 1.946088 - 0.0000038 = 1.946083 (= A_1)$$

Also $$I = I_3 + \frac{I_3 - I_2}{15}$$

$$= 1.94608 + \frac{1.94608 - 1.946088}{15}$$

$$= 1.94608 - 0.000008$$

$$= 1.94608 (= A_2)$$ (neglecting second terms as 6 zeros after the decimal)

Since $A_1 = A_2$ upto five places of decimals, we take

$$I = 1.94608$$

So, the value I,corrected to 4 places of decimal is **1.9461.** ■

Example 2

A certain curve is given by the following points

x	1	2	3	4	5	6	7	8	9
y	0.2	0.7	1	1.3	1.5	1.7	1.9	2.1	2.3

Using Simpson's rule and Romberg's formula, find the volume generated by revolving the area bounded by this curve, the x-axis and the ordinates $x = 1$ and $x = 9$ about the x-axis.

Solution

Required volume $$I = \int_1^9 \pi y^2 dx$$

Let us evaluate this integral by Simpson's rule with $h = 4, 2, 1$

When $h = 4$, $n = 2$, even. The values of y^2 are given by the table below.

x	1	5	9
y	0.2	1.5	2.3
y^2	0.04 y_0	2.25 y_1	5.29 y_2

By Simpson's formula,

$$I_1 = \frac{h}{3}[(y_0 + y_2) + 4y_1]$$

$$= \frac{4}{3}[(0.04 + 5.29) + 4(2.25)]$$

$$\Rightarrow \quad I_1 = \frac{3}{3}[5.33 + 9] = \frac{4}{3}(14.33) = \mathbf{19.106667}$$

When $h = \frac{4}{2} = 2$, $\quad n = 4$, even

The values of y^2 are given by the table below.

x	1	3	5	7	9
y	0.2	1	1.5	1.9	2.3
y^2	0.04 y_0	1 y_1	2.25 y_2	3.61 y_3	5.29 y_4

By Simpson's formula,

$$I_2 = \frac{h}{3}[y_0 + y_4 + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{2}{3}[0.04 + 5.29 + 4(1 + 3.61) + 2(2.25)]$$

$$\Rightarrow \quad I_2 = \frac{2}{3}[5.33 + 18.44 + 4.5] = \frac{2}{3}(28.27) = \mathbf{18.846667}$$

When $\quad h = \frac{1}{2}(2) = 1$, $\quad n = 8$, even

The values of y^2 are given by the table below.

x	1	2	3	4	5	6	7	8	9
y	0.2	0.7	1	1.3	1.5	1.7	1.9	2.1	2.3
y^2	0.04 y_0	0.49 y_1	1 y_2	1.69 y_3	2.25 y_4	2.89 y_5	3.61 y_6	4.41 y_7	5.29 y_8

By Simpson's formula,

$$I_3 = \frac{h}{3}\left[y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right]$$

$$= \frac{1}{3}[(0.04 + 5.29) + 4(0.49 + 1.69 + 2.89 + 4.41) + 2(1 + 2.25 + 3.61)]$$

$$\Rightarrow \quad I_3 = \frac{1}{3}[5.33 + 37.92 + 13.72] = \frac{1}{3}(56.97) = \mathbf{18.99}$$

Applying Romberg's formula for the pairs I_1, I_2 and I_2, I_3 we get

$$I = I_2 + \frac{I_2 - I_1}{15}$$

$$= 18.846667 + \frac{18.846667 - 19.106667}{15}$$

$$= 18.846667 - 0.0173333 = 18.82933 \ (= A_1)$$

Also

$$I = I_3 + \frac{I_3 - I_2}{15}$$

$$= 18.99 + \frac{18.99 - 18.846667}{15}$$

$$= 18.99 + 0.0095555 = 18.99956 \ (= A_2)$$

Applying Romberg's formula for A_1, A_2

$$I = A_2 + \frac{A_2 - A_1}{15}$$

$$= 18.99956 + \frac{18.99956 - 18.82933}{15}$$

$$\therefore \quad = 18.99956 + 0.01134867 = 19.0109087$$

$$\text{volume} = \pi \times I$$

$$= \pi \times 19.0109087 = \mathbf{59.7245 \text{ cubic units}}$$

Example 3

Evaluate $\int_0^2 \frac{dx}{x^2+4}$ using Simpson's rule and Romberg's method.

Solution

Let

$$I = \int_0^2 \frac{dx}{x^2+4}$$

Here

$$y = \frac{1}{x^2+4}$$

Simpson's rule is

$$I=\int_0^2 \frac{dx}{x^2+4}=\frac{h}{3}[(y_0+y_n)+4(y_1+y_3+y_5+\ldots y_{n-1})$$
$$+2(y_2+y_4+y_6+\ldots+y_{n-2})]$$

Let us evaluate the integral with $h=1,\quad \frac{1}{2},\quad \frac{1}{4}$.

When $h=1$, $\quad n=2$, even

The values of y are given by

x	0	1	2
x^2	0	1	4
$y=\frac{1}{x^2+4}$	0.25 y_0	0.2 y_1	0.125 y_2

By Simpson's rule,

$$I_1=\frac{h}{3}[(y_0+y_2)+4y_1]$$
$$=\frac{1}{3}[0.25+0.125+4(0.2)]$$
$$=\frac{1}{3}[0.375+0.8]$$

$\Rightarrow$
$$I_1=\frac{1}{3}[1.175]=0.39166=\mathbf{0.3917}$$

When $h=\frac{1}{2}$, $\quad n=4$, even

The values of y are given by the table below.

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
x^2	0	0.25	1	2.25	4
$y=\frac{1}{x^2+4}$	0.25 y_0	0.2353 y_1	0.2 y_2	0.16 y_3	0.125 y_4

By Simpson's rule,

$$I_2=\frac{h}{3}[(y_0+y_4)+4(y_1+y_3)+2y_2]$$
$$=\frac{0.5}{3}[0.25+0.125+4(0.2353+0.16)+2(0.2)]$$

$\Rightarrow$
$$I_2=\frac{0.5}{3}[0.375+1.5812+0.4]=\frac{0.5}{3}(2.3562)=\mathbf{0.3927}$$

When $h=\frac{1}{4}$, $\quad n=8$, even

The values of y are given by the table below.

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
$y=\dfrac{1}{x^2+4}$	0.25	0.2462	0.2353	0.2192	0.2	0.1798	0.16	0.1416	0.125
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

By Simpson's rule,

$$
\begin{aligned}
I_2 &= \frac{h}{3}[(y_0+y_8)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)] \\
&= \frac{0.25}{3}[0.25+0.125+4(0.2462+0.2192+0.1798+0.1416) \\
&\qquad +2(0.2353+0.2+0.16)] \\
&= \frac{0.25}{3}[0.375+3.1472+1.1906]
\end{aligned}
$$

$$
\Rightarrow \qquad I_3 = \frac{0.25}{3}(4.7128) = \mathbf{0.39273}
$$

Now we use Romberg's formula for the pairs I_2, I_2 and I_2, I_3.

$$
\begin{aligned}
I &= I_2 + \frac{I_2 - I_1}{15} \\
&= 0.3927 + \frac{0.3927-0.3917}{15} \\
&= 0.3927 + 0.0000667 = 0.3927667 \ (= A_1)
\end{aligned}
$$

Also

$$
\begin{aligned}
I &= I_3 + \frac{I_3 - I_2}{15} \\
&= 0.39273 + \frac{0.39273-0.3927}{15} \\
&= 0.39273 + 0.000002 = 0.392732 \ (= A_2)
\end{aligned}
$$

Now applying Romberg's formula with $A_1 = I_1$ and $A_2 = I_2$, we get

$$
\begin{aligned}
I &= I_2 + \frac{I_2 - I_1}{15} \\
&= 0.392732 + \frac{0.392732-0.392766}{15}
\end{aligned}
$$

$$
\Rightarrow \qquad I = 0.392732 + 0.0000023 = 0.3927297 = \mathbf{0.39273} \qquad \blacksquare
$$

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TWO AND THREE POINT GAUSSIAN QUADRATURE FORMULAE

Introduction

To evaluate the integral $\int_a^b f(x)\,dx$, we have developed various approximate formulae, namely trapezoidal rule, Simpson's $\frac{1}{3}$ rule and Simpson's $\frac{3}{8}$ rule. In these formulae we used the values of the function $f(x)$ at equally spaced values of argument x. Here the values of x are predetermined. Gaussian quadrature formula uses the same number of values of $f(x)$, but the values of x are not equally spaced and choosing the points suitably we can find an improved estimate of the integral.

Any finite interval $[a, b]$ can be transformed into $[-1, 1]$ by the transformation $x = \frac{b-a}{2}t + \frac{b+a}{2}$

$$\therefore \qquad y = f(x) = f\left[\left(\frac{b-a}{2}\right)t + \frac{b+a}{2}\right] = \phi(t)$$

and
$$\int_a^b f(x)\,dx = \left(\frac{b-a}{2}\right)\int_{-1}^{1} \phi(t)\,dt \qquad \left[\because dx = \frac{b-a}{2}dt\right]$$

So, we consider the integral in the form $\int_{-1}^{1} f(x)\,dx$

The general n-point Gaussian quadrature formula assumes an approximation of the form $\int_{-1}^{1} f(x)\,dx = \lambda_1 f(x_1) + \lambda_2 f(x_2) + \ldots + \lambda_n f(x_n) \ldots (1)$, where $x_1, x_2, \ldots, x_n$ are the points of division of the interval $[-1, 1]$, which are known as nodes and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are known as the weights. Since there are $2n$ unknowns in the relation (1), $2n$ relations between the variables $x_i, \lambda_i, i = 1, 2, \ldots, n$ are necessary and the formula is exact for polynomials of degree $\le 2n-1$ Formulae based on this idea are called **Gaussian quadrature formulae.**

This method is applicable only when $f(x)$ is known explicitly so that the values of the function can be evaluated at any given value of x.

The quadrature formula is derived assuming $x_1, x_2, \ldots, x_n$ are the roots of the equation $P_n(x) = 0$, where $P_n(x)$ is the Legendre polynomial of degree n given by Rodrigue's formula

$$P_n(x) = \frac{1}{2^n . n!}\frac{d^n}{dx^n}(x^2 - 1)^n$$

We also assume $f(x)$ can be expanded as a power series of degree $2n-1$ in $(-1, 1)$ so that

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{2n-1} x^{2n-1}$$

We shall derive the quadrature formulae when $n = 2$ and $n = 3$ which are called the **two point quadrature formula** and **three point quadrature formula** respectively.

Two Point Gaussian Quadrature Formula

When $n = 2$, the two point quadrature formula is

$$\int_{-1}^{1} f(x)dx = \lambda_1 f(x_1) + \lambda_2 f(x_2) \qquad (1)$$

Where x_1 and x_2 are the roots of $P_2(x) = 0$

$$\Rightarrow \qquad \frac{1}{2^2.2!}\frac{d^2}{dx^2}(x^2-1)^2 = 0$$

$$\Rightarrow \qquad \frac{d^2}{dx^2}(x^4 - 2x^2 + 1) = 0$$

$$\Rightarrow \qquad 12x^2 - 2.2 = 0$$

$$\Rightarrow \qquad 3x^2 - 1 = 0 \qquad \Rightarrow x = \pm\frac{1}{\sqrt{3}}$$

$$\therefore \qquad x_1 = -\frac{1}{\sqrt{3}}, \quad x_2 = \frac{1}{\sqrt{3}}$$

Since $n = 2$, $2n-1 = 3$. So $f(x)$ is a polynomial of degree 3.

Let $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$$\therefore \qquad \int_{-1}^{1} f(x)dx = \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

$$\Rightarrow \int_{-1}^{1}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right)dx = \lambda_1\left[a_0 + a_1\left(\frac{-1}{\sqrt{3}}\right) + a_2\left(-\frac{1}{\sqrt{3}}\right)^2 + a_3\left(\frac{-1}{\sqrt{3}}\right)^3\right]$$
$$+ \lambda_2\left[a_0 + a_1\left(\frac{1}{\sqrt{3}}\right) + a_2\left(\frac{1}{\sqrt{3}}\right)^2 + a_3\left(\frac{1}{\sqrt{3}}\right)^3\right]$$

$$\Rightarrow \left[a_0 x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + a_3\frac{x^4}{4}\right]_{-1}^{1} = a_0(\lambda_1 + \lambda_2) + a_1\left(\frac{-1}{\sqrt{3}}\lambda_1 + \frac{1}{\sqrt{3}}\lambda_2\right)$$
$$+ a_2\left(\frac{1}{3}\lambda_1 + \frac{1}{3}\lambda_2\right) + a_3\left(\frac{-1}{3\sqrt{3}}\lambda_1 + \frac{1}{3\sqrt{3}}\lambda_2\right)$$

$$\Rightarrow \quad 2a_0 + \frac{2}{3}a_2 = a_0(\lambda_1 + \lambda_2) - \frac{1}{\sqrt{3}}a_1(\lambda_1 - \lambda_2) + \frac{a_2}{3}(\lambda_1 + \lambda_2) - \frac{a_3}{\sqrt{3}}(\lambda_1 - \lambda_2)$$

∴ Equating like coefficients we get

$$\lambda_1 + \lambda_2 = 2, \ (\lambda_1 - \lambda_2) = 0 \quad \Rightarrow \lambda_1 = \lambda_2$$

$$\therefore \quad \lambda_1 = \lambda_2 = 1$$

The two point quadrature formula is

$$\int_{-1}^{1} f(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Note: It is also known as two point Gauss-Legendre formula. It gives good estimate with two functional values.

Three Point Gaussian Quadrature Formula

The three point formula is

$$\int_{-1}^{1} f(x)dx = \lambda_1 f(x_1) + \lambda_2 f(x_2) + \lambda_3 f(x_3) \tag{1}$$

where x_1, x_2, x_3 are roots of $P_3(x) = 0$, where $P_3(x)$ is the Legendre polynomial of degree 3.

$$P_3(x) = 0$$

$$\frac{1}{2^3 3!}\frac{d^3}{dx^3}(x^2 - 1)^3 = 0$$

$$\Rightarrow \quad \frac{d^3}{dx^3}[x^6 - 3x^4 + 3x^2 - 1] = 0$$

$$\Rightarrow \quad \frac{d^2}{dx^2}[6x^5 - 12x^3 + 6x] = 0$$

$$\Rightarrow \quad \frac{d}{dx}[30x^4 - 36x^2 + 6] = 0$$

$$\Rightarrow \quad 30.4x^3 - 36.2x = 0$$

$$\Rightarrow \quad 5x^3 - 3x = 0$$

$$x(5x^2 - 3) = 0$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = \pm\sqrt{\frac{3}{5}}$$

Let

$$x_1 = -\sqrt{\frac{3}{5}}, \quad x_2 = 0, \quad x_3 = \sqrt{\frac{3}{5}}$$

To find $\lambda_1, \lambda_2, \lambda_3$, we assume $f(x)$ is a polynomial of degree 5, since $2\times 3-1=5$

Let $f(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5$

$$\therefore \int_{-1}^{1}(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5)dx$$

$$=\lambda_1\left[a_0+a_1\left(-\sqrt{\frac{3}{5}}\right)+a_2\left(-\sqrt{\frac{3}{5}}\right)^2+a_3\left(-\sqrt{\frac{3}{5}}\right)^3+a_4\left(-\sqrt{\frac{3}{5}}\right)^4+a_5\left(-\sqrt{\frac{3}{5}}\right)^5\right]+\lambda_2a_0$$

$$+\lambda_3\left[a_0-a_1\left(\sqrt{\frac{3}{5}}\right)-a_2\left(\sqrt{\frac{3}{5}}\right)^2+a_3\left(\sqrt{\frac{3}{5}}\right)^3+a_4\left(\sqrt{\frac{3}{5}}\right)^4+a_5\left(\sqrt{\frac{3}{5}}\right)^5\right]$$

$$\Rightarrow \left[a_0x+a_1\frac{x^2}{2}+a_2\frac{x^3}{3}+a_3\frac{x^4}{4}+a_4\frac{x^5}{5}+a_5\frac{x^6}{6}\right]_{-1}^{1}=a_0[\lambda_1+\lambda_2+\lambda_3]$$

$$+a_1\left[-\sqrt{\frac{3}{5}}\lambda_1+\sqrt{\frac{3}{5}}\lambda_3\right]+a_2\left[\frac{3}{5}\lambda_1+\frac{3}{5}\lambda_3\right]+a_3\left[-\left(\sqrt{\frac{3}{5}}\right)^3\lambda_1+\left(\sqrt{\frac{3}{5}}\right)^3\lambda_3\right]$$

$$+a_4\left[\left(\frac{3}{5}\right)^2\lambda_1+\left(\frac{3}{5}\right)^2\lambda_3\right]+a_5\left[-\left(\sqrt{\frac{3}{5}}\right)^5\lambda_1+\left(\sqrt{\frac{3}{5}}\right)^5\lambda_3\right]$$

$$\Rightarrow 2a_1+\frac{2}{3}a_2+\frac{2}{5}a_4=a_0[\lambda_1+\lambda_2+\lambda_3]-\sqrt{\frac{3}{5}}a_1(\lambda_1-\lambda_2)$$

$$+\frac{3}{5}a_2(\lambda_1+\lambda_3)-a_3\left(\sqrt{\frac{3}{5}}\right)^3(\lambda_1-\lambda_3)\frac{3}{5}a_2(\lambda_1+\lambda_2)-a_2\left(\sqrt{\frac{3}{5}}(\lambda_2-\lambda_3)\right)$$

$$+\left(\frac{3}{5}\right)^2a_4[\lambda_1+\lambda_2]-\left(\sqrt{\frac{3}{5}}\right)^5a_5(\lambda_1-\lambda_3)$$

Equating like coefficients, we get

$$\lambda_1+\lambda_2+\lambda_3=2 \tag{2}$$

$$\lambda_1-\lambda_3=0 \quad \Rightarrow \lambda_1=\lambda_3 \tag{3}$$

$$\frac{3}{5}(\lambda_1+\lambda_3)=\frac{2}{3}$$

$$\Rightarrow \qquad \lambda_1+\lambda_3=\frac{10}{9}$$

$$\therefore \qquad 2\lambda_1=\frac{10}{9} \quad \Rightarrow \lambda_1=\frac{5}{9} \quad \text{and } \lambda_3=\frac{5}{9}$$

Substitute in (2), we get

$$\lambda_2 + \frac{10}{9} = 2 \quad \Rightarrow \lambda_2 = 2 - \frac{10}{9} = \frac{8}{9}$$

$$\therefore \quad \int_{-1}^{1} f(x)dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

$$= \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9} f(0)$$

$$= \frac{5}{9}\left[f\left(-\sqrt{0.6}\right) + f\left(\sqrt{0.6}\right)\right] + \frac{8}{9} f(0)$$

WORKED EXAMPLES

Example 1

Apply Gauss two-point formula to evaluate $\int_0^1 \frac{dx}{1+x^2}$.

Solution

Let $$I = \int_0^1 \frac{dx}{1+x^2}$$

We know $\int_{-1}^{1} \frac{dx}{1+x^2} = 2\int_0^1 \frac{dx}{1+x^2}$, since $\frac{1}{1+x^2}$ is even function of x

$\therefore \quad \int_0^1 \frac{dx}{1+x^2} = \frac{1}{2}\int_{-1}^{1} \frac{dx}{1+x^2}$. Here $f(x) = \frac{1}{1+x^2}$

Gauss two-point formula is

$$\int_{-1}^{1} \frac{dx}{1+x^2} = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{\left(1+\frac{1}{3}\right)} + \frac{1}{\left(1+\frac{1}{3}\right)} = \frac{3}{4} + \frac{3}{4} = \frac{3}{2}$$

$$\therefore \quad \int_0^1 \frac{dx}{1+x^2} = \frac{1}{2} \times \frac{3}{2} = \frac{3}{4} = 0.75$$

■

Example 2

Evaluate $\int_0^{\pi/2} \log_e(1+x)dx$ by 2-point Gaussian formula.

Solution

Let $$I = \int_0^{\pi/2} \log_e (1+x)\,dx$$

First transform the interval $[0, \pi/2]$ into the interval $[-1, 1]$

For, this, put $$x = \frac{1}{2}[(b-a)t + (b+a)] \qquad \text{Here } a = 0, \quad b = \frac{\pi}{2}$$

$$x = \frac{1}{2}\left[\frac{\pi}{2}t + \frac{\pi}{2}\right] = \frac{\pi}{4}(t+1) \Rightarrow t = \frac{4}{\pi}x - 1 \qquad \therefore\ dx = \frac{\pi}{4}dt$$

When $x = 0$, $t = -1$ and when $x = \frac{\pi}{2}$, $t = 1$

$$\therefore \qquad I = \int_{-1}^{1} \log\left(1 + \frac{\pi}{4}(t+1)\right)\frac{\pi}{4}dt$$

$$= \frac{\pi}{4}\int_{-1}^{1} \log\left(\frac{\pi}{4}t + 1 + \frac{\pi}{4}\right)dt$$

$$\therefore \qquad I = \frac{\pi}{4}\int_{-1}^{1} f(t)dt, \qquad \text{where } f(t) = \log\left(\frac{\pi}{4}t + 1 + \frac{\pi}{4}\right)$$

By 2-point Gaussian formula,

$$\int_{-1}^{1} f(t)dt = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

But $$f\left(-\frac{1}{\sqrt{3}}\right) = \log_e\left(-\frac{\pi}{4\sqrt{3}} + 1 + \frac{\pi}{4}\right)$$

$$= \log(-0.4534 + 1.7854) = \log_e 1.332 = 0.2867$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \log_e\left(\frac{\pi}{4\sqrt{3}} + 1 + \frac{\pi}{4}\right)$$

$$= \log_e(0.4534 + 1.7854) = \log_e(2.2388) = 0.8059$$

$$\therefore \qquad I = \frac{\pi}{4}[0.2867 + 0.8059] = \frac{\pi}{4}[1.0926] = \mathbf{0.8584}$$ ■

Example 3

Using three point Gaussian formula, evaluate $\int_{0.2}^{1.5} e^{-x^2}dx$.

Solution

Let $$I = \int_{0.2}^{1.5} e^{-x^2} dx$$

We have to transform the interval [0.2, 1.5] into the interval [−1, 1]

For this, Put $x = \frac{1}{2}[(b-a)t + (b+a)]$. Here $a = 0.2, b = 1.5$

$$\therefore \quad x = \frac{1}{2}[1.3t + 1.7] \quad \Rightarrow \quad t = \frac{2x - 1.7}{1.3} \quad \therefore dx = \frac{1.3}{2} dt$$

When $x = 0.2$, $t = \frac{0.4 - 1.7}{1.3} = -1$ and when $x = 1.5$, $t = \frac{3 - 1.7}{1.3} = 1$

$$\therefore \quad I = \int_{-1}^{1} e^{-\frac{1}{4}(1.3t+1.7)^2} \frac{1.3}{2} dt = \frac{1.3}{2} \int_{-1}^{1} e^{\frac{-1}{4}(1.3t+1.7)^2} dt$$

Here $$f(t) = e^{-\frac{1}{4}(1.3t+1.7)^2}$$

Three point quadrature formula is

$$\int_{-1}^{1} f(t) dt = \frac{5}{9}\left[f\left(-\sqrt{0.6}\right) + f\left(\sqrt{0.6}\right)\right] + \frac{8}{9} f(0)$$

But $$f\left(-\sqrt{0.6}\right) = e^{-\frac{1}{4}\left(-1.3\times\sqrt{0.6}+1.7\right)^2} = e^{-0.12} = 0.8869$$

$$f\left(\sqrt{0.6}\right) = e^{-\frac{1}{4}(1.3\times\sqrt{0.6}+1.7)^2} = e^{-1.8319} = 0.1601$$

$$f(0) = e^{-\frac{1}{4}(1.7)^2} = e^{-0.7225} = 0.4855$$

$$\therefore \quad \int_{-1}^{1} f(t) dt = \frac{5}{9}[0.8869 + 0.1601] + \frac{8}{9}(0.4855)$$

$$= 0.5817 + 0.4316 = 1.0133$$

$$\therefore \quad 1 = \frac{1.3}{2} \times 1.0133 = \mathbf{0.6586}$$ ■

Example 4

Evaluate $\int_{1}^{2} \frac{2x dx}{1 + x^4}$, using 3-point Gauss quadrature formula and compare with the actual value.

Solution

Given $$I = \int_1^2 \frac{2xdx}{1+x^4}\,dx.$$

We have to transform the interval [1, 2] into [−1, 1]

For this, Put $x = \frac{1}{2}[(b-a)t + (b+a)]$. Here $a = 1, b = 2$

$$\therefore \quad x = \frac{1}{2}[t+3] \quad \Rightarrow \quad t = 2x-3 \quad \therefore \; dx = \frac{1}{2}dt$$

When $x = 1$, $t = -1$ and when $x = 2$, $t = 1$

$$\therefore \quad I = \int_{-1}^{1} \frac{(t+3)\frac{1}{2}}{1+\frac{1}{16}(t+3)^4}\,dt = \frac{16}{2}\int_{-1}^{1} \frac{(t+3)}{16+(t+3)^4}\,dt$$

$$= 8\int_{-1}^{1} \frac{(t+3)}{16+(t+3)^4}\,dt$$

$$\therefore \quad I = 8\int_{-1}^{1} f(t)dt, \quad \text{where } f(t) = \frac{t+3}{16+(t+3)^4}$$

We know 3-point Gaussian formula is

$$\int_{-1}^{1} f(t)dt = \frac{5}{9}\left[f\left(-\sqrt{0.6}\right) + f\left(\sqrt{0.6}\right)\right] + \frac{8}{9}f(0)$$

But $$f\left(-\sqrt{0.6}\right) = \frac{-\sqrt{0.6}+3}{16+\left(-\sqrt{0.6}+3\right)^4} = \frac{2.2254}{16+24.5263} = \frac{2.2254}{40.5263} = 0.0549$$

$$f\left(\sqrt{0.6}\right) = \frac{\sqrt{0.6}+3}{16+\left(\sqrt{0.6}+3\right)^4} = \frac{3.7746}{218.9943} = 0.0172$$

$$f(0) = \frac{3}{16+3^4} = \frac{3}{97} = 0.0309$$

$$\therefore \quad \int_{-1}^{1} f(t)dt = \frac{5}{9}[0.0549 + 0.0172] + \frac{8}{9}[0.0309]$$

$$= 0.0401 + 0.0275 = 0.0676$$

$$\therefore \quad I = 8(0.0676) = \mathbf{0.5408}$$

We shall now find the exact value by integration.

$$I = \int_1^2 \frac{2x}{1+x^4}\,dx$$

Put $t = x^2 \quad \therefore dt = 2x\,dx$

When $x = 1, \quad t = 1$ and when $x = 2, \quad t = 4$

$$\therefore \quad I = \int_1^4 \frac{dt}{1+t^2} = \left[\tan^{-1} t\right]_1^4 = \tan^{-1} 4 - \tan^{-1} 1$$

$$= 1.3258 - 0.7854 = 0.5404$$

The error is $\quad 0.5408 - 0.5404 = 0.0004$

Example 5

Evaluate $\int_0^2 \frac{x^2+2x+1}{1+(x+1)^4}dx$ by three point Gaussian formula.

Solution

Let
$$I = \int_0^2 \frac{x^2+2x+1}{1+(x+1)^4}dx$$

First transform the interval [0, 2] into the interval [−1, 1]

For this, put
$$x = \frac{1}{2}[(b-a)t + (b+a)], \quad \text{where } a = 0, b = 2$$

$$\therefore \quad x = \frac{1}{2}[2t+2] = t+1 \qquad \Rightarrow t = x-1 \qquad \therefore dx = dt$$

$$\therefore \quad I = \int_{-1}^{1} \frac{(t+2)^2}{1+(t+2)^4}dt$$

$$\therefore \quad I = \int_{-1}^{1} f(t)dt, \qquad \text{where } f(t) = \frac{(t+2)^2}{1+(t+2)^4}$$

By three point Gaussian formula,

$$\therefore \quad I = \int_{-1}^{1} f(t)dt = \frac{5}{9}\left[f\left(-\sqrt{0.6}\right) + f\left(\sqrt{0.6}\right)\right] + \frac{8}{9}f(0)$$

But
$$f\left(-\sqrt{0.6}\right) = \frac{\left(-\sqrt{0.6}+2\right)^2}{1+\left(-\sqrt{0.6}+2\right)^4} = \frac{1.5016}{3.2548} = 0.4613$$

$$f\left(\sqrt{0.6}\right) = \frac{\left(\sqrt{0.6}+2\right)^2}{1+\left(\sqrt{0.6}+2\right)^4} = \frac{7.6984}{60.2654} = 0.1277$$

$$f(0) = \frac{4}{17} = 0.2353$$

$$\therefore \quad I = \frac{5}{9}[0.4613 + 0.1277] + \frac{8}{9}[0.2353]$$

$$= 0.3272 + 0.2092 = \mathbf{0.5364}$$

Example 6

Evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ using Gaussian (i) two points formula (ii) three point formula.

Solution

Given $$I = \int_{-1}^{1} \frac{dx}{1+x^2}.$$

Here $$f(x) = \frac{1}{1+x^2}$$

(i) Gaussian two point formula is

$$\int_{-1}^{1} f(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{1+\left(-\frac{1}{\sqrt{3}}\right)^2} + \frac{1}{1+\left(\frac{1}{\sqrt{3}}\right)^2}$$

$$= \frac{1}{1+\frac{1}{3}} + \frac{1}{1+\frac{1}{3}} = \frac{2}{1+\frac{1}{3}} = \frac{6}{4} = \frac{3}{2} = 1.5$$

(ii) Gaussian three point formula is

$$\int_{-1}^{1} f(x)dx = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9} f(0)$$

$$= \frac{5}{9}\left[\frac{1}{1+\left(-\sqrt{\frac{3}{5}}\right)^2} + \frac{1}{1+\left(\sqrt{\frac{3}{5}}\right)^2}\right] + \frac{8}{9}\left[\frac{1}{1+0}\right]$$

$$= \frac{5}{9}\left[\frac{1}{\left(1+\frac{3}{5}\right)} + \frac{1}{\left(1+\frac{3}{5}\right)}\right] + \frac{8}{9}$$

$$= \frac{5}{9}\left[\frac{5}{8} + \frac{5}{8}\right] + \frac{8}{9}$$

$$= \frac{5}{9} \times \frac{5}{4} + \frac{8}{9} = \frac{25+32}{36} = \frac{57}{36} = 1.5833$$

Note: We shall now find the actual value by integration.

$$\int_{-1}^{1} \frac{1}{1+x^2} dx = 2\int_{0}^{1} \frac{1}{1+x^2} dx \qquad (\because f(x) \text{ is even})$$

$$= 2\left[\tan^{-1} x\right]_0^1$$

$$= 2\left(\tan^{-1} 1 - \tan^{-1} 0\right) = 2\frac{\pi}{4} = \frac{\pi}{2} = 1.57079 = 1.5708$$

$\therefore$ Error in 2 point formula = $|1.5708 - 1.5| = 0.0708$
Error in 3 point formula = $|1.5708 - 1.5833| = 0.0125$
So the error in 3-point formula is less than the error in 2-point formula. ■

Example 7

Obtain $\int_2^3 \frac{\cos 2x}{1+\sin x}dx$ by using Gauss two point and three point rules.

Solution

Let $$I = \int_2^3 \frac{\cos 2x}{1+\sin x}dx$$

Transform the interval [2, 3] into [−1, 1] by the transformation

$$x = \frac{1}{2}[(b-a)t + (b+a)]. \qquad \text{Here } a = 2,\ b = 3$$

$$\therefore \quad x = \frac{1}{2}[t+5] \quad \Rightarrow \quad t = 2x - 5 \qquad \therefore\ dx = \frac{1}{2}dt$$

When $x = 2$, $t = -1$ and when $x = 3$, $t = 1$

$$\therefore \quad I = \int_{-1}^{1} \frac{\cos(t+5)}{1+\sin\left(\frac{t+5}{2}\right)} \frac{1}{2}dt = \int_{-1}^{1} \frac{\frac{1}{2}\cos(t+5)}{1+\sin\left(\frac{t+5}{2}\right)}dt$$

$$\therefore \quad I = \int_{-1}^{1} f(t)dt, \qquad \text{where } f(t) = \frac{\frac{1}{2}\cos(t+5)}{1+\sin\left(\frac{t+5}{2}\right)}$$

(1) Two point Gaussian formula is

$$\int_{-1}^{1} f(t)dt = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Now

$$f\left(-\frac{1}{\sqrt{3}}\right) = \frac{\frac{1}{2}\cos\left(-\frac{1}{\sqrt{3}}+5\right)}{1+\sin\left(\frac{-\frac{1}{\sqrt{3}}+5}{2}\right)}$$

$$= \frac{\frac{1}{2}\cos(4.4226)}{1+\sin(2.2113)} = -\frac{0.1429}{1.8019} = -0.0793$$

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$$f\left(\frac{1}{\sqrt{3}}\right)=\frac{\frac{1}{2}\cos\left(\frac{1}{\sqrt{3}}+5\right)}{1+\sin\left(\frac{\frac{1}{\sqrt{3}}+5}{2}\right)}$$

$$=\frac{\frac{1}{2}\cos(5.5774)}{1+\sin(2.7887)}=\frac{0.3806}{1.3456}=0.2828$$

$\therefore$ $$I=-0.0793+0.2828=\mathbf{0.2035}$$

(2) The three point Gaussian formula is

$$I=\frac{5}{9}\left[f\left(-\sqrt{0.6}\right)+f\left(\sqrt{0.6}\right)\right]+\frac{8}{9}f(0)$$

$$f\left(-\sqrt{0.6}\right)=\frac{\frac{1}{2}\cos\left(-\sqrt{0.6}+5\right)}{1+\sin\left(\frac{-\sqrt{0.6}+5}{2}\right)}$$

$$=\frac{\frac{1}{2}\cos(4.2254)}{1+\sin(2.1127)}=\frac{-0.2340}{1.8567}=-0.1260$$

$$f\left(\sqrt{0.6}\right)=\frac{\frac{1}{2}\cos 5.7746}{1+\sin 2.8873}=\frac{0.4367}{1.2516}=0.3489$$

$$f(0)=\frac{\frac{1}{2}\cos 5}{1+\sin 2.5}=\frac{0.1418}{1.5918}=0.0887$$

$\therefore$ $$I=\frac{5}{9}[-0.1260+0.3489]+\frac{8}{9}(0.0887)$$

$$=0.1238+0.0788=\mathbf{0.2026}$$

■

Example 8

Using Gaussian two and three point formulae evaluate $\int_2^3 \frac{dt}{1+t}$.

Solution

Given $$I=\int_2^3\frac{dt}{1+t}$$

First we transform the interval [2, 3] into the interval [–1, 1]

For this, put $t = \frac{1}{2}[(b-a)x + (b+a)]$. Here $a = 2, b = 3$

$$\therefore t = \frac{1}{2}[x+5] \quad \Rightarrow x = 2t-5 \quad \therefore dt = \frac{1}{2}dx$$

When $t = 2$, $x = -1$ and when $t = 3$, $x = 1$

$$\therefore \quad I = \int_{-1}^{1} \frac{\frac{1}{2}dx}{1+\frac{1}{2}(x+5)} = \int_{-1}^{1} \frac{dx}{x+7}$$

$$\therefore \quad I = \int_{-1}^{1} f(x)dx, \qquad \text{where } f(x) = \frac{1}{x+7}$$

(i) Two point Gaussian formula is

$$I = \int_{-1}^{1} f(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{-\frac{1}{\sqrt{3}}+7} + \frac{1}{\frac{1}{\sqrt{3}}+7}$$

$$= \frac{1}{-0.5774+7} + \frac{1}{0.5774+7}$$

$$= \frac{1}{6.4226} + \frac{1}{7.5774} = 0.1557 + 0.13197 = \mathbf{0.2877}$$

(ii) Three point Gaussian formula is

$$I = \int_{-1}^{1} f(x)dx = \frac{5}{9}\left[f\left(-\sqrt{0.6}\right) + f\left(\sqrt{0.6}\right)\right] + \frac{8}{9}f(0)$$

Now

$$f\left(-\sqrt{0.6}\right) = \frac{1}{-\sqrt{.6}+7} = 0.1606$$

$$f\left(\sqrt{.6}\right) = \frac{1}{\sqrt{.6}+7} = \frac{1}{7.7746} = 0.1286$$

$$f(0) = \frac{1}{7} = 0.1429$$

$$\therefore \quad I = \frac{5}{9}[0.1606 + 0.1286] + \frac{8}{9}(0.1429)$$

$$= 0.1607 + 0.1270 = \mathbf{0.2877}$$

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Exercises 7.2

(1) Evaluate $\int_0^{0.5} e^{-x^2} dx$ by Romberg's method with $h = 0.1$ and $h = 0.2$ and using Trapezoidalrule.

(2) Evaluate $\int_0^2 \frac{dx}{x^2+4}$ by using Romberg's method taking $h = 1, \frac{1}{2}, \frac{1}{4}$.

(3) Calculate $\int_0^{1/2} \frac{x}{\sin x} dx$ using trapezoidal rule with $h = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ and then Romberg's formula.

(4) From the following data evaluate $\int_{1.8}^{3.4} f(x)\, dx$ using Romberg's method with h=0.8, 0.4

x	1.8	2	2.2	2.4	2.6	2.8	3	3.2	3.4
$f(x)$	6.050	7.389	9.025	11.023	13.464	16.445	20.086	24.533	29.964

(5) Evaluate $\int_{0.2}^{1.5} e^{-x^2} dx$ with $h = 0.65$, 0.325 by Romberg's method.

(6) Evaluate $\int_{-2}^{2} e^{-x/2} dx$ by two point Gaussian formula.

(7) Evaluate $\int_0^{\frac{\pi}{2}} \sin x\, dx$ using two point Gaussian formula.

(8) Evaluate $\int_{-1}^{1} (3x^2 + 5x^4) dx$ by using three point Gaussian formula. Compare with actual value.

(9) Evaluate $\int_{-1}^{1} e^{-x^2} \cos x\, dx$ using 3-point Gaussian formula.

(10) Evaluate $\int_0^1 e^x\, dx$ using three point Gaussian formula.

(11) Evaluate $\int_{-1}^{1} (1-x^2)^{1/2} \cos x dx$ using two point and three point Gaussian formula.

Answers 7.2

(1) 1.8521
(2) 0.3927
(3) 0.507070
(4) 23.9181
(5) 0.6586
(6) 4.6854
(7) 1.5432, 1.3911
(8) 4
(9) 1.3247
(10) 1.7128
(11) 0.9985

EULER-MACLAURIN FORMULA FOR NUMERICAL INTEGRATION

Euler-Maclaurin's formula for integration is

$$\frac{1}{h}\int_{a=x_0}^{b=x_0+nh} f(x)dx = f(x_0)+f(x_1)+f(x_2)+\cdots+f(x_n)$$

$$-\frac{1}{2}[f(b)+f(a)]-\frac{h}{12}[f'(b)-f'(a)]+\frac{h^3}{720}[f'''(b)-f'''(a)]$$

$$-\frac{h^5}{30240}\left[f^{(r)}(b)-f^{(r)}(a)\right]+\cdots$$

Where $x_0=a,\ x_n=x_0+nh=b,$ and $h=\frac{b-a}{n}.$

Derivation: Let $y=f(x)$

Divide the interval $[a,b]$ into n equal intervals of width $h=\frac{b-a}{n}$.

$x_0=a,\qquad x_1=x_0+h_1,\qquad x_2=x_0+2h,\ldots,\qquad b=x_n=x_0+nh$

Let $\qquad \Delta F(x)=f(x)$

$\Rightarrow \qquad F(x+h)-F(x)=f(x_0)$

$$\begin{aligned}
\Rightarrow\quad F(x_0+h)-F(x_0)&=f(x_0) &&\Rightarrow F(x_1)-F(x_0)=f(x_0)\\
F(x_1+h)-F(x_1)&=f(x_1) &&\Rightarrow F(x_2)-F(x_1)=f(x_1)\\
F(x_2+h)-F(x_2)&=f(x_2) &&\Rightarrow F(x_3)-F(x_2)=f(x_2)\\
\vdots\qquad & \quad\vdots && \quad\vdots\\
F(x_{n-1}+h)-F(x_{n-1})&=f(x_{n-1}) &&\Rightarrow F(x_n)-F(x_{n-1})=f(x_{n-1})
\end{aligned}$$

Adding all these equations, we get (1)

$$F(x_n)-F(x_0)=f(x_0)+f(x_1)+f(x_2)+\ldots+f(x_{n-1})$$

Since $\Delta F(x)=f(x)$, we define the inverse operator Δ^{-1} as

$$\begin{aligned}
F(x)&=\Delta^{-1}f(x)\\
&=(E-1)^{-1}f(x)\\
&=(e^{hD}-1)^{-1}f(x) \qquad \left[\because E=e^{hD} \text{ where } D=\frac{d}{dx}\right]\\
&=\left[\left(1+\frac{hD}{1!}+\frac{h^2D^2}{2!}+\frac{h^3D^3}{3!}+\ldots\right)-1\right]^{-1} f(x)
\end{aligned}$$

$$=\left[\left(hD+\frac{h^2D^2}{3!}+\frac{h^3D^3}{3!}+\ldots\right)\right]^{-1} f(x)$$

$$=(hD)^{-1}\left[1+\left(\frac{hD}{2!}+\frac{h^2D^2}{3!}+\frac{h^3D^3}{4!}+\frac{h^4D^4}{5!}=\ldots\right)\right]^{-1} f(x)$$

$$=\frac{1}{h}D^{-1}\left[1-\left(\frac{hD}{2!}+\frac{h^2D^2}{3!}+\frac{h^3D^3}{4!}+\frac{h^4D^4}{5!}+\cdots\right)+\left(\frac{hD}{2!}+\frac{h^2D^2}{3!}+\frac{h^3D^3}{4!}+\frac{h^4D^4}{5!}+\ldots\right)^2\right.$$
$$\left.-\left(\frac{hD}{2!}+\frac{h^2D^2}{3!}+\frac{h^3D^3}{4!}+\frac{h^4D^4}{5!}+\ldots\right)^3+\left(\frac{hD}{2!}+\frac{h^2D^2}{3!}\right)^4+\ldots\right]f(x)$$

$$=\frac{1}{h}D^{-1}\left[1-\frac{hD}{2!}+\frac{h^2D^2}{3!}-\frac{h^3D^3}{4!}-\frac{h^4D^4}{5!}\right.$$
$$+\left(\frac{h^2D^2}{4}+\frac{h^3D^3}{6}+\frac{h^4D^4}{36}+\frac{h^4D^4}{24}+\frac{h^5D^5}{72}+\ldots\right)$$
$$\left.-\left(\frac{h^3D^3}{8}+3.\frac{h^2D^2}{4}.\frac{h^2D^2}{6}+\ldots\right)+\left(\frac{h^4D^4}{16}+\ldots\right)\right]f(x)$$

$$=\frac{1}{h}D^{-1}\left[1-\frac{hD}{2!}-h^2\left[\frac{1}{6}-\frac{1}{4}\right]D^2+h^3\left(-\frac{1}{24}+\frac{1}{6}-\frac{1}{8}\right)D^3\right.$$
$$\left.+h^4\left(-\frac{1}{120}+\frac{1}{36}+\frac{1}{24}-\frac{1}{8}+\frac{1}{16}\right)D^4+\ldots\right]f(x)$$

$$=\frac{1}{h}D^{-1}\left[1-\frac{hD}{2}+\frac{h^2D^2}{12}+h^3\left(\frac{-1+4-3}{24}\right)D^3\right.$$
$$\left.+h^4\left(\frac{-6+20+30-90+45}{720}\right)D^4+\ldots\right]f(x)$$

$$=\frac{1}{h}D^{-1}\left[1-\frac{hD}{2}+\frac{h^2D^2}{12}-\frac{h^4D^4}{720}+\ldots\right]f(x)$$

$$=\left[\frac{1}{h}D^{-1}-\frac{1}{2}+\frac{hD}{12}-\frac{h^3D^3}{720}+\ldots\right]f(x)$$

$$\Rightarrow\quad F(x)=\frac{1}{h}D^{-1}f(x)-\frac{1}{2}f(x)+\frac{h}{12}f'(x)-\frac{h^3}{720}f'''(x)+\ldots \tag{2}$$

Putting $x=x_n$ and $x=x_0$ in (2), we get

$$F(x_n)=\frac{1}{h}D^{-1}f(x_n)-\frac{1}{2}f(x_n)+\frac{h}{12}f'(x_n)-\frac{h^3}{720}f'''(x_n)+\ldots \tag{3}$$

$$F(x_0)=\frac{1}{h}D^{-1}f(x_0)-\frac{1}{2}f(x_0)+\frac{h}{12}f'(x_0)-\frac{h^3}{720}f'''(x_0)+\ldots \qquad (4)$$

$(3)-(4)\rightarrow$

$$F(x_n)-F(x_0)=\frac{1}{h}\int_{x_0}^{x_n}f(x)dx-\frac{1}{2}[f(x_n)-f(x_0)]+\frac{h}{2}[f'(x_n)-f'(x_0)]$$

$$-\frac{h^3}{720}[f'''(x_n)-f'''(x_0)]$$

$$\left[\because D^{-1}f(x_n)-D^{-1}f(x_0)=D^{-1}[f(x_n)-f(x_0)]=\int_{x_0}^{x_n}f(x)dx\right]$$

Substituting for L.H.S. from (1), we get

$$f(x_0)+f(x_1)+f(x_2)+\ldots+f(x_{n-1})=\frac{1}{h}\int_{x_0}^{x_n}f(x)dx-\frac{1}{2}[f(x_n)-f(x_0)]$$

$$+\frac{h}{12}[f'(x_n)-f'(x_0)]-\frac{h^3}{720}[f'''(x_n)-f'''(x_0)]+\ldots$$

$$\frac{1}{h}\int_{x_0}^{x_n+h}f(x)dx=[f(x_0)+f(x_1)+\ldots+f(x_{n-1})+f(x_n)]-\frac{1}{2}[f(x_n)+f(x_0)]$$

$$-\frac{h}{12}[f'(x_n)-f'(x_0)]+\frac{h^3}{720}[f'''(x_n)-f'''(x_0)]+\ldots \quad \text{[adding } f(x_0) \text{ and adjusting]}$$

$$\Rightarrow \quad \frac{1}{h}\int_{b}^{a}f(x)dx=\sum_{i=0}^{n}f(x_i)-\frac{1}{2}[f(b)+f(a)]-\frac{h}{12}[f'(b)-f'(a)]$$

$$+\frac{h^3}{720}[f'''(b)-f'''(a)]+\ldots$$

This is called Euler-Maclaurin's formula for integration.

Incase higher derivatives are involved, then the formula is

$$\Rightarrow \quad \frac{1}{h}\int_{a}^{b}f(x)dx=\sum_{i=0}^{n}f(x_i)-\frac{1}{2}[f(b)+f(a)]-\frac{h}{12}[f'(b)-f'(a)]$$

$$+\frac{h^3}{720}[f'''(b)-f'''(a)]-\frac{h^5}{30240}[f^{(v)}(b)-f^{(v)}(a)]+\ldots$$

Note: This formula can be rewritten as

$$\sum_{i=0}^{n}f(x_i)=\frac{1}{h}\int_{a}^{b}f(x)dx+\frac{1}{2}[f(b)+f(a)]+\frac{h}{12}[f'(b)-f'(a)]$$

$$-\frac{h^3}{720}[f'''(b)-f'''(a)]+\frac{h^5}{30240}[f^{(v)}(b)-f^{(v)}(a)]+\ldots$$

This is called Euler-Maclaurin's formula for summation.

WORKED EXAMPLES

Example 1

Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by Euler-Maclaurin formula for integration. Hence find the value of π and compare with the correct value upto 5 decimals.

Solution

Let $$I = \int_0^1 \frac{dx}{1+x^2}$$

Here $$f(x) = \frac{1}{1+x^2}, \quad a = 0, b = 1$$

Euler-Maclaurin formula for integration is

$$\Rightarrow \quad \frac{1}{h}\int_a^b f(x)dx = \sum_{i=0}^{n} f(x_i) - \frac{1}{2}[f(b) + f(a)] - \frac{h}{12}[f'(b) - f'(a)]$$
$$+ \frac{h^3}{720}[f'''(b) - f'''(a)] + \dots \qquad (1)$$

where n = the number of sub-intervals and $h = \frac{b-a}{n}$

Let $n = 6 \quad \therefore h = \frac{1-0}{6} = \frac{1}{6}$

The values of x are $x_0 = 0$, $x_1 = \frac{1}{6}$, $x_2 = \frac{2}{6} = \frac{1}{3}$, $x_3 = \frac{3}{6} = \frac{1}{2}$, $x_4 = \frac{4}{6} = \frac{2}{3}$,
$x_5 = \frac{5}{6}$, $x_6 = 1$

The values of $f(x)$ are given by the table below.

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
x	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$f(x) = \frac{1}{1+x^2}$	1 $f(x_0)$	0.97297 $f(x_1)$	0.9 $f(x_2)$	0.8 $f(x_3)$	0.69231 $f(x_4)$	0.59016 $f(x_5)$	0.5 $f(x_6)$

We have $$f(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$$

$\therefore$ $$f'(x) = (-1)(1+x^2)^{-2}\,2x = -\frac{2x}{(1+x^2)^2}$$

$$f''(x) = -2\frac{\left[(1+x^2)^2 1 - 2x(1+x^2)(2x)\right]}{(1+x^2)^4}$$

$$= -\frac{2(1+x^2)[1+x^2-4x^2]}{(1+x^2)^4} = -\frac{2(1-3x^2)}{(1+x^2)^3}$$

$$f'''(x) = -2\frac{(1+x^2)^3\left[(-6x)-(1-3x^2)3(1+x^2)^2 2x\right]}{(1+x^2)^6}$$

$$= -2\frac{(1+x^2)^2\left[(1+x^2)(-6x)-6(1-3x^2)x\right]}{(1+x^2)^6}$$

$$= \frac{12x\left[1+x^2+(1-3x^2)\right]}{(1+x^2)^4} = 12x\frac{(2-2x^2)}{(1+x^2)^4} = \frac{24x(1-x^2)}{(1+x^2)^4}$$

$\therefore$ $f(a) = f(0) = \frac{1}{1+0} = 1$, $\quad f(b) = f(1) = \frac{1}{1+1} = \frac{1}{2} = 0.5$

$f'(a) = f'(0) = 0$, $\quad f'(b) = f'(1) = \frac{(-2).1}{(1+1)^2} = -\frac{1}{2}$

$f'''(a) = f'''(0) = 0$, $\quad f'''(1) = 24.1\frac{(1-1)}{(1+1)^4} = 0$

$\therefore$
$$\frac{1}{\frac{1}{6}}\int_0^1 \frac{dx}{1+x^2} = f(x_0)+f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)+f(x_6)-\frac{1}{2}[f(1)+f(0)]$$

$$-\frac{1}{12}\left(\frac{1}{6}\right)[f'(1)-f'(0)]+\frac{1}{720}\left(\frac{1}{6}\right)^3[f'''(1)-f'''(0)]$$

$\Rightarrow$
$$6\int_0^1 \frac{dx}{(1+x^2)} = 1+0.97297+0.9+0.8+0.69231+0.59016+0.5$$

$$-\frac{1}{2}[0.5+1]-\frac{1}{72}\left[-\frac{1}{2}-0\right]+\frac{1}{720}\cdot\frac{1}{216}(0)$$

$$= 5.45544-0.75+0.006944 = 4.712384$$

$\therefore$
$$\int_0^1 \frac{dx}{1+x^2} = \frac{4.712384}{6} = \mathbf{0.785397}$$

But
$$\int_0^1 \frac{dx}{(1+x^2)} = [\tan^{-1}x]_0^1 = \tan^{-1}1-\tan^{-1}0 = \frac{\pi}{4}$$

Using Calculator $\quad \frac{\pi}{4} = 0.785398163$

$= 0.78540$ upto 4 places

$\therefore$ Euler-Maclaurin method gives correct answer upto 5 places of decimal. ■

Example 2

Find by Euler's quadrature formula, the value of the integral $\int_0^1 \cos x^2\, dx$.

Solution

Let $$I = \int_0^1 \cos x^2\, dx$$

Here $f(x) = \cos x^2$, $a = 0$, and $b = 1$

Euler-Maclaurin's formula for integration is

$$\frac{1}{h}\int_a^b f(x)dx = \sum_{i=0}^{n} f(x_i) - \frac{1}{2}[f(b) + f(a)] - \frac{h}{12}[f'(b) - f'(a)]$$
$$+\frac{h^3}{720}[f'''(b) - f'''(a)] + \dots$$

where n = the number of sub-intervals and $h = \dfrac{b-a}{n}$.

Let $n = 5$ $\quad \therefore h = \dfrac{1-0}{5} = \dfrac{1}{5}$

The values of x are $x_0 = 0$, $\quad x_1 = \dfrac{1}{5}$, $\quad x_2 = \dfrac{2}{5}$, $\quad x_3 = \dfrac{3}{5}$, $\quad x_4 = \dfrac{4}{5}$, $\quad x_5 = 1$

The values of $f(x)$ are given by table below

x	x_0	x_1	x_2	x_3	x_4	x_5
	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$f(x) = \cos x^2$	1	0.99920	0.98723	0.93590	0.8021	0.54030
	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$	$f(x_5)$

We have

$$f(x) = \cos x^2$$
$$f'(x) = -\sin x^2 .2x = -2x \sin x^2$$
$$f''(x) = -2[\sin x^2 .1 + x\cos x^2 .2x] = -2[\sin x^2 + 2x^2 \cos x^2]$$
$$f'''(x) = -2[\cos x^2 .2x + 2\{x^2(-\sin x^2)2x + \cos x^2 .2x\}]$$
$$= -4x[\cos x^2 - 2x^2 \sin x^2 + 2.\cos x^2] = -4x[3\cos x^2 - 2x^2 \sin x^2]$$

$\therefore \quad f(a) = f(0) = \cos 0 = 1, \qquad f(b) = f(1) = \cos 1 = 0.54030$

$\therefore \quad f'(a) = f'(0) = 0, \qquad f'(b) = f'(1) = -2.1 \sin 1 = -1.68294$

$$f'''(a) = f'''(0) = 0$$

$$\begin{aligned} f'''(b) = f'''(1) &= -4[3\cos 1 - 2\sin 1] \\ &= -4[3(0.54030) - 2(0.84147)] \\ &= -4[1.6209 - 1.68294] = 0.24816 \end{aligned}$$

$$\therefore \quad \frac{1}{\frac{1}{5}}\int_0^1 \cos x^2\, dx = f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) - \frac{1}{2}[f(1) + f(0)]$$

$$- \frac{1}{12}\left(\frac{1}{5}\right)[f'(1) - f'(0)] + \frac{1}{720}\left(\frac{1}{5}\right)^3 [f'''(1) - f'''(0)]$$

$$\Rightarrow \quad 5\int_0^1 \cos x^2\, dx = 1 + 0.99920 + 0.98723 + 0.93590 + 0.8021 + 0.54030$$

$$\begin{aligned} &- \frac{1}{2}[0.54030 + 1] - \frac{1}{60}[-1.68294] + \frac{1}{720} \times \frac{1}{125}[0.24816] \\ &= 5.26473 - 0.77015 + 0.028049 + 0.000002757 \\ &= 4.522632 \end{aligned}$$

$$\therefore \quad \int_0^1 \cos x^2\, dx = \frac{4.522632}{5} = \mathbf{0.904526}$$

■

Example 3

Applying Euler-Maclaurin summation formula, evaluate

$$\frac{1}{51^2} + \frac{1}{53^2} + \frac{1}{55^2} + \ldots + \frac{1}{99^2}$$

Solution

Required the sum of $\frac{1}{51^2} + \frac{1}{53^2} + \frac{1}{55^2} + \ldots + \frac{1}{99^2}$

Here $f(x) = \frac{1}{x^2}$, $\quad h = 2, \quad a = 51 \quad$ and $\quad b = 99$

By Euler – Maclaurin formula for summation

$$\sum_{i=0}^{n} f(x_i) = \frac{1}{h}\int_a^b f(x)dx + \frac{1}{2}[f(b) + f(a)] + \frac{h}{12}[f'(b) - f'(a)]$$

$$- \frac{h^3}{720}[f'''(b) - f'''(a)] + \ldots$$

We have

$$f(x) = \frac{1}{x^2} = x^{-2}$$

$\therefore$ $$f'(x) = -2x^{-3} = -\frac{2}{x^3}, \qquad f''(x) = (-2)(-3)\frac{1}{x^4} = \frac{6}{x^4}$$

and $$f'''(x) = (-2)(-3)(-4)x^{-5} = \frac{-24}{x^5}$$

$$f(a) = f(51) = \frac{1}{51^2} = 0.00038447, \qquad f(b) = f(99) = \frac{1}{99^2} = 0.00010203$$

$$f'(a) = f'(51) = -\frac{2}{51^3} = -0.000015077, \qquad f'(b) = f'(99) = -\frac{2}{99^3} = -0.00000206$$

$$f'''(a) = f'''(51) = -\frac{24}{99^5} = \text{negligible}, \qquad f'''(b) = f'''(99) = -\frac{24}{99^5} = \text{negligible}$$

$\therefore$ $$\frac{1}{51^2} + \frac{1}{53^2} + \frac{1}{55^2} + \dots + \frac{1}{99^2} = \frac{1}{2}\int_{51}^{99} \frac{dx}{x^2} + \frac{1}{2}\left[\frac{1}{99^2} + \frac{1}{51^2}\right] + \frac{2}{12}\left[-\frac{2}{99^3} + \frac{2}{51^3}\right] + \dots$$

$$= \frac{1}{2}\left[\frac{x^{-1}}{-1}\right]_{51}^{99} + \frac{1}{2}[0.00010203 + 0.00038447]$$

$$+ \frac{1}{6}[-0.00000206 + 0.000015077]$$

$$= -\frac{1}{2}\left[\frac{1}{99} - \frac{1}{51}\right] + \frac{1}{2}[0.0004865] + \frac{1}{6}[0.000013017]$$

$$= -\frac{1}{2}[0.01010101 - 0.01960784] + 0.00024325$$

$$+ 0.0000021695$$

$$= 0.00475342 + 0.00024325 + 0.0000021695$$

$$= \mathbf{0.0049988}$$

■

Example 4

By Euler's formula prove that the sum of the cubes of the first n natural numbers is $\dfrac{n^2(n+1)^2}{4}$.

Solution

We have to prove $1^3 + 2^3 + 3^3 + \dots n^3 = \dfrac{n^2(n+1)^2}{4}$

$$\Rightarrow \qquad \sum_{x=1}^{n} x^3 = \frac{n^2(n+1)^2}{2}$$

Here $f(x) = x^3$, $h = 1$, $a = 1$, $b = n$

By Euler's formula for summation

$$\sum_{i=0}^{n} f(x_i) = \frac{1}{h}\int_a^b f(x)dx + \frac{1}{2}[f(b) + f(a)] + \frac{h}{12}[f'(b) - f'(a)] - \frac{h^3}{720}[f'''(b) - f'''(a)]$$

We have

$$f(x) = x^3$$

$$f'(x) = 3x^2, \quad f''(x) = 6x, \quad f'''(x) = 6$$

$\therefore$

$$f(1) = 1, \quad f(n) = n^3$$

$$f'(1) = 3, \quad f'(n) = 3n^2$$

$$f'''(1) = 6, \quad f'''(n) = 6$$

$\therefore$

$$\sum_{x=1}^{n} x^3 = \int_1^n x^3 dx + \frac{1}{2}[n^3 + 1^3] + \frac{1}{12}[3n^2 - 3.1^2] - \frac{1}{720}[6 - 6] + \ldots$$

$$= \left[\frac{x^4}{4}\right]_1^n + \frac{1}{2}[n^3 + 1] + \frac{1}{12}[3n^2 - 3]$$

$$= \frac{1}{4}[n^4 - 1] + \frac{1}{2}[n^3 + 1] + \frac{1}{4}[n^2 - 1]$$

$$= \frac{1}{4}[n^4 - 1 + 2n^3 + 2 + n^2 - 1]$$

$$= \frac{1}{4}[n^4 + 2n^3 + n^2] = \frac{1}{4}n^2[n^2 + 2n + 1] = \frac{n^2(n+1)^2}{4}$$

■

Application of Euler-Maclaurin Formula

Stirling's approximation for factorial

If n is a positive integer, then we know, $n! = 1.2.3.4\ldots(n-1)n$

$$\therefore \quad \log_e n! = \log_e 1 + \log_e 2 + \log_e 3 + \ldots + \log_e (n-1) + \log_e n$$

$$\Rightarrow \quad \log_e n! = \sum_{x=1}^{n} \log_e x \qquad (1)$$

We shall find the sum by Maclaurin's formula.

Here $\quad f(x) = \log_e x, \quad a = 1, \quad b = n, \quad h = 1$

$$\therefore \quad f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}, \quad f'''(x) = \frac{2}{x^3}$$

Maclaurin's formula for summation is

$$\sum_{x=1}^{n} f(x) = \int_1^n f(x)dx + \frac{1}{2}[f(n)+f(x)] + \frac{1}{12}[f'(n)-f'(1)] - \frac{1}{720}[f'''(n)-f'''(1)] + \cdots$$

$$\sum_{x=1}^{n} \log_e x = \int_1^n \log_e x dx + \frac{1}{12}[\log_e n - \log_e 1] + \frac{1}{12}\left[\frac{1}{n}-1\right] - \frac{1}{720}\left[\frac{2}{n^3}-\frac{2}{1}\right] + \cdots$$

$$= [x\log x - x]_1^n + \frac{1}{2}\log_e n + \frac{1}{12}\times\frac{1}{n} - \frac{1}{12} - \frac{1}{360}\times\frac{1}{n^3} + \frac{1}{360} + \ldots$$

$$= n\log n - n - (1\log 1 - 1) \; + \frac{1}{2}\log_e n + \frac{1}{12n} - \frac{1}{12} - \frac{1}{360}\frac{1}{n^3} + \frac{1}{360} + \cdots$$

$$= [n+\tfrac{1}{2}]\log_e n - n + \frac{1}{12n} + \frac{1}{360n^3} + \ldots \; + C, \text{ where C is a constant} \qquad (2)$$

To find C, we use Weddle's formula, namely

$$\lim_{n\to\infty} \frac{2^{4n}(n!)^4}{((2n)!)^2(2n+1)} = \frac{\pi}{2}$$

Taking logarithm, we get

$$\lim_{n\to\infty}[4n\log_e 2 + 4\log_e n! - 2\log_e(2n)! - \log_e(2n+1)] = \log_e \frac{\pi}{2}$$

As $n \to \infty$, $\quad \frac{1}{12n}, \frac{1}{360n^3}, \ldots \to 0$

$\therefore \quad \log_e n! = (n+\tfrac{1}{2})\log_e n - n + C$ approximately (from (1) and (2))

$$\therefore \quad \log\frac{\pi}{2} = \lim_{n\to\infty}\left[4n\log_e 2 + 4\{(n+\tfrac{1}{2})\log_e n - n + C\} - 2\left\{\left(2n+\frac{1}{2}\right)\log_e 2n - 2n + C\right\} - \log_e(2n+1)\right]$$

$$= \lim_{n\to\infty}[4n\{\log_e 2 + \log_e n - \log_e 2n\} + 2\log_e n - \log_e 2n - \log_e(2n+1) + 2C]$$

$$= \lim_{n\to\infty}\left[4n(\log_e 2n - \log_e 2n) + \log_e \frac{n^2}{2n(2n+1)} + 2C\right]$$

$$= \lim_{n\to\infty}\left[2C + \log_e\left\{\frac{n^2}{2n^2\left(2+\frac{1}{n}\right)}\right\}\right]$$

$$= \lim_{n\to\infty}\left[2C + \log_e\left\{\frac{1}{2\left(2+\frac{1}{n}\right)}\right\}\right] \qquad \left[\because \frac{1}{n} \to 0 \text{ as } n \to \infty\right]$$

$$= 2C + \log_e \frac{1}{4} = 2C - \log_e 4$$

$$\Rightarrow \quad 2C = \log_e \frac{\pi}{2} + \log_e 4 = \log_e 2\pi$$

$$\therefore \quad C = \frac{1}{2}\log_e 2\pi = \log_e \sqrt{2\pi}$$

$$\therefore \quad \log_e n! = \left(n+\frac{1}{2}\right)\log_e n - n + \frac{1}{12n} - \frac{1}{360n^3} + \ldots + \log_e \sqrt{2\pi}$$

$$= \log_e n^{(n+1/2)} + \log_e \sqrt{2\pi} - n + \frac{1}{12n} - \frac{1}{360n^3} + \ldots$$

$$\Rightarrow \log_e n! = \log_e \sqrt{2\pi}\,(n)^{n+\frac{1}{2}} - n + \frac{1}{12n} - \frac{1}{360n^3} + \ldots$$

$$\therefore \quad = \log_e \sqrt{2\pi}\,(n)^{n+\frac{1}{2}} + \left(-n + \frac{1}{12n} - \frac{1}{360n^3} + \ldots\right)$$

$$= \log_e \sqrt{2\pi}(n^{n+\frac{1}{2}}) \log_e e^{-n+\frac{1}{12n}-\frac{1}{360n^3}+\ldots}$$

$$\Rightarrow \log_e n! = \log_e \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n+\frac{1}{12n}-\frac{1}{360n^3}+\ldots}$$

$$\therefore \quad n! = \sqrt{2\pi}(n)^{n+\frac{1}{2}}\, e^{-n+\frac{1}{12n}-\frac{1}{360n^3}+\ldots}$$

Corollary: When n is large $\frac{1}{12n}, \frac{1}{360n^3}, \ldots$ are very small and so can be neglected

$$\therefore \quad n! = \sqrt{2\pi}\cdot n^{n+\frac{1}{2}} \cdot e^{-n}$$

WORKED EXAMPLES

Example 1

Calculate $\log_{10} 81!$ correct to six places of decimals and hence find the number of digits in 81!

Solution

Stirling's formula for $n!$ is $n! = \sqrt{2\pi}(n)^{n+\frac{1}{2}} e^{-n+\frac{1}{12n}-\frac{1}{360n^3}+\ldots}$

Taking log to the base 10,

$$\log_{10} n! = \log_{10}\sqrt{2\pi} + \log_{10}(n)^{n+\frac{1}{2}} + \log_{10} e^{-n+\frac{1}{12n}-\frac{1}{360n^3}+\ldots}$$

$$= \frac{1}{2}\log_{10} 2\pi + \left(n+\frac{1}{2}\right)\log_{10} n + \left(-n + \frac{1}{12n} - \frac{1}{360n^3} + \ldots\right)\log_{10} e$$

Put $n = 81$, we get

$$\log_{10} 81! = \frac{1}{2}\log_{10} 2\pi + \left(81+\frac{1}{2}\right)\log_{10} 81 + \left(-81+\frac{1}{12\times 81}-\frac{1}{360\times 81^3}+\ldots\right)$$
$$= 0.3990899 + 155.541529 + (-81 + 0.0010288)(0.4342945)$$
$$= 0.3990899 + 155.541529 - 35.1774077 = 120.7632112$$

$$\Rightarrow \quad \log_{10} 81! = 120.7632112$$

Since the integer part is 120, the number of digits in 81! is 121 ■

Example 2

Using Stirling's approximation for $n!$, find the number of digits in the value of $100C_{50}$.

Solution

Let $x = 100C_{50} = \dfrac{100!}{50!\,50!} = \dfrac{100!}{(50!)^2}$

Taking log to the base 10, we get

$$\therefore \quad \log_{10} x = \log_{10} 100! - \log_{10}(50!)^2$$
$$= \log_{10} 100! - 2\log_{10} 50!$$

Stirlings formula for $n!$ is

$$n! = \sqrt{2\pi}(n)^{n+\frac{1}{2}} e^{-n+\frac{1}{12n}-\frac{1}{360n^3}+\ldots}$$

$$\therefore \quad \log_{10} n! = \log_{10}\sqrt{2\pi} + (n+\tfrac{1}{2})\log_{10} n + \left(-n+\frac{1}{12n}-\frac{1}{360n^3}+\ldots\right)\log_{10} e$$
$$= 0.3990899 + (n+\tfrac{1}{2})\log_{10} n + \left(-n+\frac{1}{12n}-\frac{1}{360n^3}+\ldots\right)\log_{10} e \qquad (1)$$

Put $n = 100$ in (1), then

$$\log_{10} 100! = 0.3990899 + 100.5\log_{10} 100$$
$$+\left(-100+\frac{1}{12\times 100}-\frac{1}{360}\times\frac{1}{100^3}+\ldots\right)(0.4342945)$$
$$= 0.3990899 + 100.5\times 2 + (-100 + 0.00083333)(0.4342945)$$
$$= \mathbf{157.9700018}$$

$\therefore$

Now put $n = 50$ in (1), then

$$\begin{aligned}\log_{10} 50! &= 0.3990899 + 50.5\log_{10} 50 \\ &\quad + \left(-50 + \frac{1}{12\times 50} - \frac{1}{360}\times\frac{1}{50^3} + \ldots\right)(0.4342945) \\ &= 0.3990899 + 85.79798522 \\ &\quad + (-50 + 0.0016667 - 0.000000022 + \ldots)(0.4342945) \\ &= 0.3990899 + 85.79798522 - 21.71400117 = 64.48307395\end{aligned}$$

$\therefore$ $$\begin{aligned}\log_{10} x &= 157.9700018 - 2(64.48307395) \\ &= 157.9700018 - 128.9661479 = 29.0039\end{aligned}$$

$\therefore$ $$x = \text{anti}\log(29.0039)$$

Since the integral part is 29, the number of digits in $100C_{50}$ is 30 ■

Exercises 7.3

(I) Evaluate the following integrals using Euler-Maclaurin formula.

(1) $\int_0^{10} \frac{dx}{1+x}$ correct to fine decimal places.

(2) $\int_{100}^{105} \frac{dx}{x}$ correct to seven decimal places.

(3) $\int_0^1 \frac{dx}{1+x}$ correct to five decimal places.

(4) $\int_0^1 \frac{dx}{1+x}$ and hence find the values of π.

(II) Evaluate the following sums.

(1) $\frac{1}{100} + \frac{1}{101} + \frac{1}{102} + \ldots + \frac{1}{199}$ correct to 6 decimals.

(2) $\frac{1}{(101)^2} + \frac{1}{(103)^2} + \frac{1}{(105)^2} + \ldots$

(3) $\sum_{m=0}^{\infty} \frac{1}{(8+m)^2}$ correct to 6 places of decimals.

(4) $\sum_{n=25}^{49} \frac{1}{(2n+1)^2}$

(5) $\sum_{100}^{200} \frac{1}{x^{4/3}}$

(6) $\frac{1}{400}+\frac{1}{402}+\frac{1}{404}+\ldots+\frac{1}{498}+\frac{1}{500}$

(7) $\frac{1}{201^2}+\frac{1}{203^2}+\ldots+\frac{1}{299^2}$

(8) Find the sum of the fourth powers of first n natural numbers.

(III) Using Stirling approximation for factorials, compute to six places of decimals

(1) $\log_{10} 69!$

(2) $\log_{10} 300!$

(3) $\log_{10} 100C_{40}$

(4) $\log 1000!$

Answers 7.3

(I)

) 2.40011 (2) 0.0487902

) 0.69315 (4) 0.7854, $\pi = 3.1416$

(II)

(1) 0.6956534 (2) 0.0049998

(3) 0.133137 (4) 0.0049988

(5) 0.144539 (6) 0.11382

(7) 0.0008333

(III)

(1) 98.233911 (2) 615.5999

(3) 28.138179 (4) 2567.59914

DOUBLE INTEGRATION

We shall now consider the numerical evaluation of a definite double integral of a function of two independent variables from a set of numerical values of the integrand. **The process of computation of double integral of a function of two independent variables is called mechanical cubature.**

We have seen in calculus that a double integral $\int_c^d \int_a^b f(x,y)\,dx\,dy$ with constant limits can be evaluated by integrating w.r.to x first treating y as constant and then integrating w.r.to y. So we can evaluate a double integral by applying trapezoidal rule or Simpson's rule repeatedly.

Trapezoidal Rule for Double Integral

Let $I = \int_c^d \int_a^b f(x,y)dx\,dy$. The region of integration is the rectangle bounded by the lines $x = a$, $x = b$, $y = c$, $y = d$ in the xy plane.

Divide the interval $[a, b]$ into n equal subintervals of width h at the points

$$a = x_0, x_1, x_2, \ldots, x_n = b$$

and the interval $[c, d]$ into m equal subintervals of width k at the points

$$c = y_0, y_1, \; y_2, \ldots, y_n = d$$

$\therefore \quad x_i = x_0 + ih, \qquad i = 1, 2, 3, \ldots, n$

$\therefore \quad y_j = y_0 + jk, \qquad j = 1, 2, 3, \ldots, m$

Let

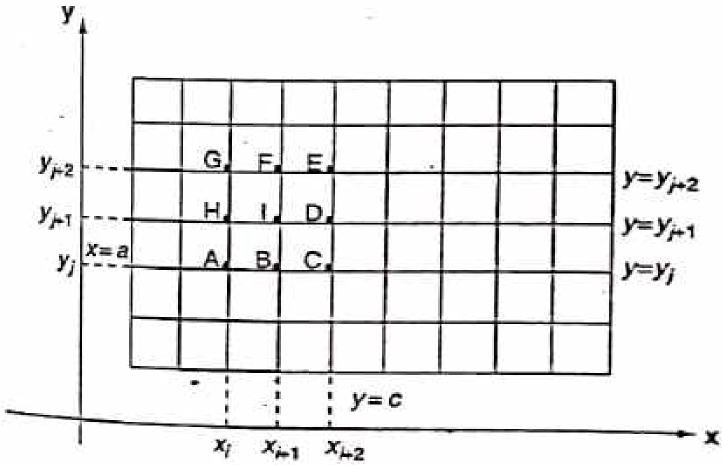
A, B, C be (x_r, y_j), $\qquad r = i, i+1, i+2$

H, I, D be (x_r, y_{j+1}), $\qquad r = i, i+1, i+2$

G, F, E be (x_r, y_{j+2}), $\qquad r = i, i+1, i+2$

Consider the typical rectangle ABIH over which we shall evaluate the integral

$$I_i = \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_i+1} f(x,y)\,dx\,dy$$



By trapezoidal rule for the inner integral with two values, we get

$$I_i = \int_{y_j}^{y_j+1} \frac{h}{2}[f(x_i, y) + f(x_{i+1}, y)]\,dy$$

Again applying trapezoidal rule

$$I_1 = \frac{h}{2}\left[\int_{y_j}^{y_{j+1}} f(x_i, y)dy + \int_{y_j}^{y_{j+1}} f(x_{i+1}, y)dy\right]$$

$$= \frac{h}{2}\left[\frac{k}{2}[f(x_i, y_j) + f(x_i, y_{j+1})] + \frac{k}{2}[f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1})]\right]$$

$$= \frac{hk}{4}[f(x_i, y_j) + f(x_i, y_{j+1}) + f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1})] \qquad (1)$$

To evaluate $\int_{y_j}^{y_{j+2}}\int_{x_i}^{x_{i+2}} f(x,y)\,dx\,dy$, we have to evaluate over the rectangle ACEG, which is the sum of four rectangles.

$$\therefore \int_{y_j}^{y_{j+2}}\int_{x_i}^{x_{i+2}} f(x,y)\,dx\,dy = \int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}} f(x,y)\,dx\,dy + \int_{y_j}^{y_{j+1}}\int_{x_{i+1}}^{x_{i+2}} f(x,y)\,dx\,dy$$

$$+ \int_{y_{j+1}}^{y_{j+2}}\int_{x_i}^{x_{i+1}} f(x,y)\,dx\,dy + \int_{y_{j+1}}^{y_{j+2}}\int_{x_{i+1}}^{x_{i+2}} f(x,y)\,dx\,dy$$

Using (1) to each of these integrals, we get

$$\int_{y_j}^{y_{j+2}}\int_{x_i}^{x_{i+2}} f(x,y)\,dx\,dy = \frac{hk}{4}[f(x_i, y_j) + f(x_i, y_{j+1}) + f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1})]$$

$$+ \frac{hk}{4}[f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1}) + f(x_{i+2}, y_j) + f(x_{i+2}, y_{j+1})]$$

$$+ \frac{hk}{4}[f(x_i, y_{j+1}) + f(x_i, y_{j+2}) + f(x_{i+1}, y_{j+1}) + f(x_{i+1}, y_{j+2})]$$

$$+ \frac{hk}{4}[f(x_{i+1}, y_{j+1}) + f(x_{i+1}, y_{j+2}) + f(x_{i+2}, y_{j+1}) + f(x_{i+2}, y_{j+2})]$$

$$= \frac{hk}{4}\Big[\{f(x_i, y_j) + f(x_{i+2}, y_j) + f(x_{i+2}, y_{j+2}) + f(x_i, y_{j+2})\}$$
$$+ 2\{f(x_{i-1}, y_j) + f(x_i, y_{j+1}) + f(x_{i+2}, y_{j+1}) + f(x_{i+1}, y_{j+2})\}$$
$$+ 4 f(x_{i-1}, y_{j-1})\Big]$$

$= \frac{hk}{4}$[sum of the values of $f(x,y)$ at the corners A, C, E, G

+ 2 (sum of the values at the remaining edges B, H, D

+ 4 (values of $f(x, y)$ at the centre point]

More generally, the value of the double integral is

$I = \dfrac{hk}{4}$[sum of the values of $f(x, y)$ at the corners

+ 2 (sum of the values of $f(x, y)$ at the remaining position of the boundary)

+ 4 (sum of the values of (x, y) at the interior points)]

WORKED EXAMPLES

Example 1

Evaluate $\int_0^2\int_0^2 f(x,y)\,dx\,dy$ **by Trapezoidal rule for the following data:**

y \ x	0	0.5	1	1.5	2
0	2	3	4	5	5
1	3	4	6	9	11
2	4	6	8	11	14

Solution

Let $$I = \int_0^2\int_0^2 f(x,y)\,dx\,dy$$

The given values of x are equally spaced with $h = 0.5$

The given values of y are equally spaced with $k = 1$

y \ x	0	0.5	1	1.5	2
0	$\boxed{2}$	3	4	5	$\boxed{5}$
1	3	4	6	9	11
3	$\boxed{4}$	6	8	11	$\boxed{14}$

∴ the Trapezoidal rule for double integral is

$I = \dfrac{hk}{4}$[Sum of the corner values

+ 2 (sum of the other values on the boundary)

+ 4 (sum of the values at the interior points)]

The corner values are squared.
Other boundary values are indicated by arrows.
The interior values are inside the dotted curve.

$$\therefore \quad I = \frac{0.5 \times 1}{4}[(2+4+14+5)+2(6+8+11+11+5+4+3+3)+4(4+6+9)]$$

$$= \frac{1}{8}[25+102+76] = \mathbf{25.375}$$

Example 2

Use trapezoidal rule to evaluate $\int_1^2 \int_1^2 \frac{dx\,dy}{y+y}$ taking 4 subintervals.

Solution

Let
$$I = \int_1^2 \int_1^2 \frac{1}{x+y}\,dx\,dy$$

Here
$$f(x,y) = \frac{1}{x+y}$$

Number of subintervals is 4. $\quad \therefore h = \frac{2-1}{4} = 0.25$ and $k = 0.25$

$\therefore$ the values of x are 1, 1.25, 1.5, 1.75, 2 and the values of y are 1, 1.25, 1.5, 1.75, 2

$$f(1,1) = \frac{1}{1+1} = \frac{1}{2} = 0.5, \qquad f(1,1.25) = \frac{1}{1+1.25} = \frac{1}{2.25} = 0.4444$$

$$f(1,1.5) = \frac{1}{1+1.5} = \frac{1}{2.5} = 0.4, \qquad f(1,1.75) = \frac{1}{1+1.75} = \frac{1}{2.75} = 0.3636$$

$$f(1,2) = \frac{1}{1+2} = \frac{1}{3} = 0.3333$$

Similarly

$f(1.25,1) = 0.4444,$
$f(1.25,1.25) = 0.4,$
$f(1.25,1.5) = 0.3636,$
$f(1.25,1.75) = 0.3333,$
$f(1.25,2) = 0.3077,$
$f(1.75,1) = 0.3636,$
$f(1.75,1.25) = 0.3333,$
$f(1.75,1.5) = 0.3077,$
$f(1.75,1.75) = 0.2857,$
$f(1.75,2) = 0.2667,$

$f(1.5,1) = 0.4$
$f(1.5,1.25) = 0.3636$
$f(1.5,1.5) = 0.3333$
$f(1.5,1.75) = 0.3077$
$f(1.5,2) = 0.2857$
$f(2,1) = 0.3333$
$f(2,1.25) = 0.3077$
$f(2,1.5) = 0.2857$
$f(2,1.75) = 0.2667$
$f(2,2) = 0.25$

Now we tabulate the values of $f(x,y) = \frac{1}{x+y}$:

$y \backslash x$	1	1.25	1.5	1.75	2
1	$\boxed{0.5}$	0.4444	0.4	0.3636	$\boxed{0.3333}$
1.25	0.444	0.4	0.3636	0.3333	0.3077
1.5	0.4	0.3636	0.3333	0.3077	0.2857
1.75	0.3636	0.3333	0.3077	0.2857	0.2667
2	$\boxed{0.3333}$	0.3077	0.2857	0.2667	$\boxed{0.25}$

$\therefore$ the trapezoidal rule for double integral is

$$I = \frac{hk}{4}\Big[\text{sum of corner values}$$
$$+\ 2\ (\text{sum of other values of } f(x, y) \text{ on the boundary})$$
$$+\ 4\ (\text{sum of the interior values of } f(x, y))\Big]$$

The corner values are squared. Boundary values are indicated by arrows. Interior values are inside the dotted curve.

$$\begin{aligned}
\therefore \qquad I = {} & \frac{0.25\times0.25}{4}[(0.5+0.3333+0.25+0.3333) \\
& +2(0.3077+0.2857+0.2667+0.2667+0.2857+0.3077+0.3636 \\
& +0.4+0.4444+0.4444+0.4+0.3636) \\
& +4(0.4+0.3636+0.3333 \\
& +0.3636+0.3333+0.3077+0.3333+0.3077+0.2857)] \\
= {} & 0.156(1.4166+8.2724+12.1128)=0.3401
\end{aligned}$$

■

Example 3

Evaluate $\int_0^1\int_0^1 \frac{1}{1+x+y}\,dx\,dy$ by Trapezoidal rule.

Solution

Let $\qquad I = \int_0^1\int_0^1 \frac{1}{1+x+y}\,dx\,dy$

Here $f(x,y) = \dfrac{1}{1+x+y}$

We shall take $h = 0.5$, $k = 0.25$

$\therefore$ the values of x are 0, 0.5, 1 and the values of y are 0, 0.25, 0.5, 0.75, 1.

$$\therefore \quad f(0,0) = \frac{1}{1+0+0} = 1, \qquad f(0,0.25) = \frac{1}{1+0+0.25} = \frac{1}{1.25} = 0.8$$

$$f(0,0.5) = \frac{1}{1+0+0.5} = \frac{1}{1.5} = 0.6667, \qquad f(0.75) = \frac{1}{1+0+0.75} = \frac{1}{1.75} = 0.5714$$

$$f(0,1) = \frac{1}{1+0+1} = \frac{1}{2} = 0.5$$

Similarly

$$f(0.5,0) = 0.6667, \qquad f(1,0) = 0.5$$
$$f(0.5,0.25) = 0.5714, \qquad f(1,0.25) = 0.4444$$
$$f(0.5,0.5) = 0.5, \qquad f(1,0.5) = 0.4$$
$$f(0.5,0.75) = 0.4444, \qquad f(1,0.75) = 0.3636$$
$$f(0.5,1) = 0.4, \qquad f(1,1) = 0.3333$$

Now we shall tabulate the values of $f(x, y)$

$y \backslash x$	0	0.5	1
0	$\boxed{1}$	0.6667	$\boxed{0.5}$
0.25	0.8	0.5714	0.4444
0.5	0.6667	0.5	0.4
0.75	0.5714	0.4444	0.3636
1	$\boxed{0.5}$	0.4	$\boxed{0.3333}$

$\therefore$ the Trapezoidal rule for double integral is

$$I = \frac{hk}{4}\,[\text{sum of the corner values} + 2\,(\text{sum of other values of } f(x,y) \text{ on the boundary}) + 4\,(\text{sum of the interior values of } f(x,y))]$$

The corner values are squared.
Other boundary values are indicated by arrows.
The interior values are inside the dotted curve.

$$\therefore \quad I = \frac{0.5\times0.25}{4}[(1+0.5+0.3333+0.5)$$
$$+2(0.4+0.3636+0.4+0.4444+0.6667+0.8+0.6667+0.5714)$$
$$+4(0.5714+0.5+0.4444)]$$
$$=0.03125[2.3333+2(3.9128)+4(1.5158)]$$
$$=0.03125[2.3333+8.6256+6.0632]$$
$$=0.03125(17.0221) = \mathbf{0.5319}$$

Note: The exact value of this double integral is $\log_e \frac{27}{16} = 0.5232$.

So, the error is 0.0087

Example 4

Evaluate $\int_0^1 \int_1^2 \frac{2xy}{(1+x^2)(1+y^2)} dx\, dy$ by Trapezoidal rule with $h = k = 0.25$.

Solution

Let
$$I = \int_0^1 \int_1^2 \frac{2xy}{(1+x^2)(1+y^2)} dx\, dy$$

Here
$$f(x,y) = \frac{2xy}{(1+x^2)(1+y^2)}; \quad h = k = 0.25$$

$\therefore$ the values of x are 1, 1.25, 1.5, 1.75, 2
and the values of y are 0, 0.25, 0.5, 0.75, 1

$$f(1,0) = 0$$
$$f(1,0.25) = \frac{0.5}{2(1.0625)} = 0.2353, \quad f(1,0.5) = \frac{1}{2(1.25)} = 0.4$$
$$f(1,0.75) = \frac{1.5}{2(0.5625)} = 0.48, \quad f(1,1) = \frac{1}{2} = 0.5$$

Similarly

$$f(1.25,0) = 0,$$
$$f(1.25,0.25) = 0.2296,$$
$$f(1.25,0.5) = 0.3902,$$
$$f(1.25,0.75) = 0.4683,$$
$$f(1.25,1) = 0.4848,$$
$$f(1.75,0) = 0,$$
$$f(1.75,0.25) = 0.2027,$$
$$f(1.75,0.5) = 0.3446,$$
$$f(1.75,0.75) = 0.4135,$$
$$f(1.75,1) = 0.4308,$$

$$f(1.5,0) = 0$$
$$f(1.5,0.25) = 0.2172$$
$$f(1.5,0.5) = 0.3672$$
$$f(1.5,0.75) = 0.4431$$
$$f(1.5,1) = 0.4615$$
$$f(2,0) = 0$$
$$f(2,0.25) = 0.1882$$
$$f(2,0.5) = 0.32$$
$$f(2,0.75) = 0.384$$
$$f(2,1) = 0.4$$

Now we shall tabulate the value of $f(x, y)$

$y \backslash x$	1	1.25	1.5	1.75	2
0	$\boxed{0}$	0	0	0	$\boxed{0}$
0.25	0.2353	0.2296	0.2172	0.2027	0.1882
0.5	0.4	0.3902	0.3692	0.3446	0.32
0.75	0.48	0.4683	0.4431	0.4135	0.384
1	$\boxed{0.5}$	0.4878	0.4615	0.4308	$\boxed{0.4}$

$\therefore$ the Trapezoidal rule for double integration is

$$I = \frac{hk}{4}\text{ [sum of the corner values}$$
$$+\ 2\text{ (sum of the other values of } f(x,y) \text{ on the boundary)}$$
$$+\ 4\text{ (sum of the interior values of } f(x,y)\text{)]}$$

The corner values are squared.

Other values of $f(x, y)$ on the boundary are indicated by arrows.

The interior values are inside the dotted curve.

$\therefore$

$$I = \frac{0.25 \times 0.25}{4}[(0 + 0.5 + 0.4 + 0)$$
$$+ 2(0.2353 + 0.4 + 0.48 + 0.4878 + 0.4615$$
$$+ 0.4308 + 0.384 + 0.32 + 0.1882 + 0 + 0 + 0)$$
$$+ 4(0.2296 + 0.3902 + 0.4683 + 0.2172 + 0.3692$$
$$+ 0.4431 + 0.2027 + 0.3446 + 0.4135)]$$
$$= \frac{0.0625}{4}[0.9 + 2(3.3876) + 4(3.0784)] = 0.3123$$

Example 5

Evaluate $\int_1^2 \int_1^2 \frac{1}{x+y} dx\, dy$ using Trapezoidal rule with $h = k = 0.5$ and $h = k = 0.25$. Improve the estimate by Romberg's formula.

Solution

Let
$$I = \int_1^2 \int_1^2 \frac{1}{x+y} dx\, dy$$

Here
$$f(x,y) = \frac{1}{x+y}$$

Case (i) $h = k = 0.5$

The values of x are 1, 1.5, 2 and the values of y are 1, 1.5, 2

$$f(1,1) = \frac{1}{1+1} = \frac{1}{2} = 0.5, \qquad f(1,1.5) = \frac{1}{2.5} = 0.4$$
$$f(1,2) = \frac{1}{3} = 0.3333$$

Similarly

$$f(1.5,1) = 0.4, \qquad f(2,1) = 0.3333$$
$$f(1.5,1.5) = 0.3333, \qquad f(2,1.5) = 0.2857$$
$$f(1.5,2) = 0.2857, \qquad f(2,2) = 0.25$$

We shall tabulate the values of $f(x,y)$

$y \backslash x$	1	1.5	2
1	$\boxed{0.5}$	0.4	$\boxed{0.3333}$
1.5	0.4	0.3333	0.2857
2	$\boxed{0.3333}$	0.2857	$\boxed{0.25}$

Trapezoidal rule for double integration is

$I = \frac{hk}{4}$[sum of the corner values

+ 2 (sum of the other values on the boundary) + 4 (sum of the interior values)]

$$= \frac{0.5 \times 0.5}{4}[(0.5+0.3333+0.25+0.3333)+2(0.4+0.4+0.2857+0.2857)+4(0.3333)]$$

$$= \frac{0.25}{4}[1.4166+2.7428+1.3332] = 0.3433$$

Case (ii) $h = k = 0.25$

By worked example 2 page... $I = 0.3401$

Taking the estimates as $I_1 = 0.3433$ and $I_2 = 0.3401$

By Romberg's formula $I = I_2 + \frac{I_2 - I_1}{3}$

$$= 0.3401 + \frac{0.3401 - 0.3433}{3}$$

$$= 0.3401 - 0.00107 = 0.3390$$

Simpson's Rule for Double Integral

Let $I = \int_c^d \int_a^b f(x,y)\,dx\,dy$

We divide $[a, b]$ into an even number of intervals $2n$ and $[c, d]$ is divided into an even number of intervals $2m$.

Applying Simpson's rule taking 2 intervals in the x-direction and 2 intervals in the y-direction.

Let $h = \frac{b-a}{2n}$, $k = \frac{d-c}{2m}$

The points of division are

$$a = x_0, x_1, \ldots, x_{2n} = b$$
$$c = y_0, y_1, \ldots, y_{2m} = d.$$
$$\therefore \quad x_i = x_0 + ih, \qquad y_j = y_0 + jk$$

The typical sub-rectangles are taken with 2 intervals in x-direction and 2 intervals in y-direction.

Consider $\int_{y_j}^{y_{j+2}} \int_{x_i}^{x_{i+2}} f(x,y)\,dx\,dy$

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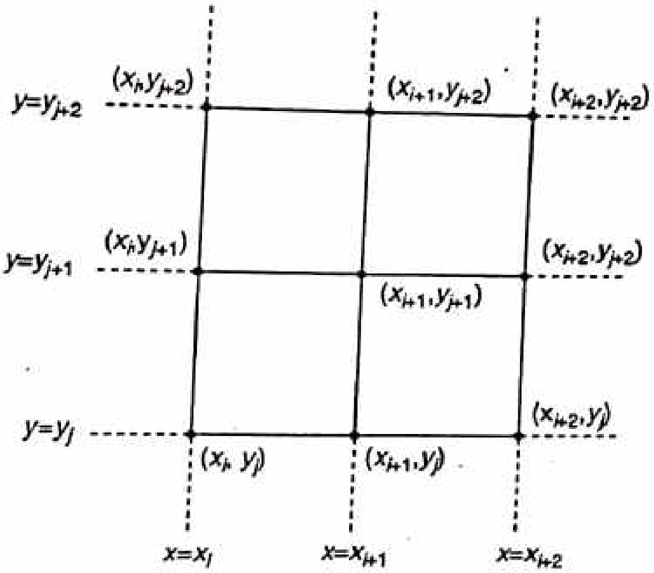


By applying Simpson's rule in the x-direction, for 2 intervals keeping y-constant to the inner integral we get

$$\int_{y_j}^{y_{j+2}}\int_{x_i}^{x_{i+2}} f(x,y)dx\,dy = \int_{y_j}^{y_{j+2}} \frac{h}{3}[f(x_i,y)+4f(x_{i+1},y)+f(x_{i+2},y)]dy$$

$$=\frac{h}{3}\left[\int_{y_j}^{y_{j+2}} f(x_i,y)\,dy+4\int_{y_j}^{y_{j+2}} f(x_{i+1},y)\,dy+\int_{y_j}^{y_{j+2}} f(x_{i+2},y)\,dy\right]$$

Again apply Simpson's rule to each integral in the y-direction.

$$\therefore \quad \int_{y_j}^{y_{j+2}}\int_{x_i}^{x_{i+2}} f(x,y)dx\,dy = \frac{h}{3}\Big[\frac{k}{3}\{f(x_i,y_j)+4f(x_i,y_{j+1})+f(x_i,y_{j+2})\}$$

$$+4.\frac{k}{3}\{f(x_{i+1},y_j)+4f(x_{i+1},y_{j+1})+f(x_{i+1},y_{j+2})\}$$

$$+\frac{k}{3}\{f(x_{i+2},y_j)+4f(x_{i+2},y_{j+1})+f(x_{i+2},y_{j+2})\}\Big]$$

$$=\frac{hk}{9}[[f(x_i,y_j)+f(x_i,y_{j+2})+f(x_{i+2},y_j)+f(x_{i+2},y_{j+2})]$$

$$+4[f(x_i,y_{j+1})+f(x_{i+1},y_{j+2})+f(x_{i+2},y_{j+1})+f(x_{i+1},y_j)]$$

$$+16f(x_{i+1},y_{j+1})]$$

$$I=\frac{hk}{9}\Big[\text{sum of the values at the corners}$$

$$+\ 4\ (\text{sum of values at the other points on the boundary})$$

$$+\ 16\ (\text{value at the interior point})\Big]$$

This can be extended if the number of intervals is 4 or 6 in each direction applying for every pair of intervals.

WORKED EXAMPLES

Example 6

Evaluate $\int_0^1 \int_0^1 e^{x+y} dx\, dy$ using Trapezoidal rule and Simpson's rule.

Solution

Let
$$I = \int_0^1 \int_0^1 e^{x+y} dx\, dy$$

Here
$$f(x, y) = e^{x+y}$$

We shall take $h = 0.5$, $k = 0.5$ and thus dividing into 2 intervals each.

$\therefore$ the value of x are 0, 0.5, 1 and the values of y are 0, 0.5, 1

$\therefore$
$$f(0,0) = 1, \quad f(0.5,0) = 1.6487, \quad f(1,0) = 2.7183$$
$$f(0,0.5) = 1.6487, \quad f(0.5,0.5) = 2.7183, \quad f(1,0.5) = 4.4817$$
$$f(0,1) = 2.7183, \quad f(0.5,1) = 4.4817, \quad f(1,1) = 7.3891$$

We shall tabulate the values

x \ y	0	0.5	1
0	$\boxed{1}$	1.6487	$\boxed{2.7183}$
0.5	1.6487	2.7183	4.4817
1	$\boxed{2.7183}$	4.4817	$\boxed{7.3891}$

(i) Trapezoidal rule for double integral is

$$I = \frac{hk}{4}\Big[\text{sum of the corner values}$$
$$+ 2\,(\text{sum of the other values on the boundary})$$
$$+ 4\,(\text{sum of the interior values})\Big]$$

$$= \frac{0.5\times 0.5}{4}[(1+2.7183+7.3891+2.7183)$$
$$+2(4.4817+4.4817+1.6487+1.6487)$$
$$+4(2.7183)]$$
$$= \frac{0.25}{4}[13.8257+24.5216+10.8732]$$
$$= \mathbf{3.0763}$$

(ii) Simpson's rule for double integral is

$$I = \frac{hk}{9}[\text{sum of the corner values}$$
$$+ 4 \text{ (sum of the other values on the boundary)}$$
$$+ 16 \text{ (sum of the interior values)}]$$

$$= \frac{0.5\times 0.5}{9}[(1+2.7183+7.3891+2.7183)$$
$$+4(4.4817+4.4817+1.6487+1.6487)$$
$$+16(2.7183)]$$
$$= \frac{0.25}{9}[13.8257+49.0432+43.4928]$$
$$= 0.02788[106.3617] = \mathbf{2.9569}$$

Note: Integrating we get the actual value 2.9525 correct to 4 decimals So we notice that the value of Simpson's rule is very close to the actual value. ■

Example 7

Using Simpson's $\frac{1}{3}$ rule evaluate $\int_0^1 \int_0^1 \frac{1}{1+x+y}\,dx\,dy$ by taking $h = k = 0.5$.

Solution

Let $I = \int_0^1 \int_0^1 \frac{1}{1+x+y}\,dx\,dy$

Here $f(x,y) = \frac{1}{1+x+y}$

Given $h = k = 0.5$

$\therefore$ the values of x are 0, 0.5, 1 and the values of y are 0, 0.5, 1.

So the interval is divided into 2 parts.

$f(0,0) = 1$, $f(0,0.5) = 0.6667$, $f(0,1) = \frac{1}{2} = 0.5$

$f(0.5,0) = 0.667$, $f(1,0) = 0.5$

$f(0.5,0.5) = 0.5$, $f(1,0.5) = 0.4$

$f(0.5,1) = 0.4$, $f(1,1) = 0.3333$

We shall tabulate the value

$x \backslash y$	0	0.5	1
0	$\boxed{1}$	0.6667	$\boxed{0.5}$
0.5	0.6667	0.5	0.4
1	$\boxed{0.5}$	0.4	$\boxed{0.3333}$

Simpson's rule for double integral is

$$I = \frac{hk}{9}\Big[\text{sum of the corner values}$$

$$+ 4\,(\text{sum of the other values on the boundary})$$

$$+ 16\,(\text{centre value})\Big]$$

$$= \frac{0.5 \times 0.5}{9}[(1 + 0.5 + 0.3333 + 0.5)$$

$$+ 4(0.4 + 0.4 + 0.6667 + 0.6667) + 16(0.5)]$$

$$= \frac{0.25}{9}[2.3333 + 8.5336 + 8] = \mathbf{0.5241}$$

Note: By integration the actual value is 0.5232 upto 4 places.

Example 8

Evaluate $\int_0^{1/2}\int_0^{1/2} \frac{\sin xy}{1+xy}\,dx\,dy$ by Simpson's rule with $h = k = 0.25$.

Solution

Let $$I = \int_0^{1/2}\int_0^{1/2} \frac{\sin xy}{1+xy}\,dx\,dy$$

Here $$f(x,y) = \frac{\sin xy}{1+xy} \text{ and } h = k = 0.25$$

$\therefore$ the values of x are 0, 0.25, 0.5 and the values of y are 0, 0.25, 0.5

So the interval is divided into 2 parts.

$$f(0,0) = \frac{\sin 0}{1+0} = 0, \quad f(0,0.25) = \frac{\sin 0}{1+0} = 0$$

$$f(0,0.5) = \frac{\sin 0}{1+0} = 0$$

Similarly,

$f(0.25, 0) = 0,$ $\quad f(0.5, 0) = 0$

$f(0.25, 0.25) = 0.0588,$ $\quad f(0.5, 0.25) = 0.1108$

$f(0.25, 0.5) = 0.1108,$ $\quad f(0.5, 0.5) = 0.1979$

We shall tabulate the values of $f(x, y)$

x \ y	0	0.25	0.5
0	$\boxed{0}$	0	$\boxed{0}$
0.25	0	0.0588	0.1108
0.5	$\boxed{0}$	0.1108	$\boxed{0.1979}$

Simpson's rule of double integral is

$$I = \frac{hk}{9}\Big[\text{sum of the corner values} + 4\,(\text{sum of the other values on the boundary}) + 16\,(\text{centre values})\Big]$$

$$= \frac{0.25 \times 0.25}{9}[(0 + 0 + 0.1979 + 0) + 4(0.1108 + 0.1108 + 0 + 0) + 16(0.0588)]$$

$$= \frac{0.0625}{9}[0.1979 + 0.8864 + 0.9408] = 0.01406$$

■

Example 9

Evaluate $\int_2^{2.4}\int_4^{4.4} xy\,dx\,dy$ using Simpson rule, given $h = k = 0.1$.

Solution

$$I = \int_2^{2.4}\int_4^{4.4} xy\,dx\,dy$$

Here $\quad f(x, y) = xy$

Given $\quad h = 0.1, \quad k = 0.1$

∴ the values of x are 4, 4.1, 4.2, 4.3, 4.4 and the values of y are 2, 2.1, 2.2, 2,3, 2.4

The interval is divided into 4 sub intervals.

$$f(4,2)=4\times 2=8,\quad f(4,2.1)=4\times 2.1=8.4$$
$$f(4,2.2)=4\times 2.2=8.8,\quad f(4,2.3)=4\times 2.3=9.2$$
$$f(4,2.4)=4\times 2.4=9.6$$

Similarly,

$f(4.1,2)=8.2$, $f(4.1,2.1)=8.61$, $f(4.1,2.2)=9.02$,
$f(4.1,2.3)=9.43$, $f(4.1,2.4)=9.84$,
$f(4.2,2)=8.4$, $f(4.2,2.1)=8.82$, $f(4.2,2.2)=9.24$,
$f(4.2,2.3)=9.66$, $f(4.2,2.4)=10.08$,
$f(4.3,2)=8.6$, $f(4.3,2.1)=9.02$, $f(4.3,2.2)=9.46$,
$f(4.3,2.3)=9.89$, $f(4.3,2.4)=10.12$,
$f(4.4,2)=8.8$, $f(4.4,2.1)=9.24$, $f(4.4,2.2)=9.68$,
$f(4.4,2.3)=10.12$, $f(4.4,2.4)=10.56$

We shall formulate the table value of $f(x,y)$

x \ y	2	2.1	2.2	2.3	2.4
4	8	8.4	8.8	9.2	9.6
4.1	8.2	8.61	9.02	9.43	9.84
4.2	8.4	8.82	9.24	9.66	10.08
4.3	8.6	9.03	9.46	9.89	10.32
4.4	8.8	9.24	9.68	10.12	10.56

We shall rewrite the integral as sum of 4 integrals taking 2 intervals at a time and apply Simpson's rule

$$I = \int_2^{2.4} \int_4^{4.4} xy\,dx\,dy$$

$$\therefore \quad = \int_2^{2.2} \int_4^{4.2} xy\,dx\,dy + \int_{2.2}^{2.4} \int_4^{4.2} xy\,dx\,dy + \int_2^{2.2} \int_{4.2}^{4.4} xy\,dx\,dy + \int_{2.2}^{2.4} \int_{4.2}^{4.4} xy\,dx\,dy$$

$$= I_1 + I_2 + I_3 + I_4$$

where $I_1 = \int_2^{2.2} \int_4^{4.2} xy\,dx\,dy$

Using Simpson's rule for 2 intervals

$$I_1 = \frac{hk}{9}[\text{sum of the corner values}$$
$$+ 4 \text{ (sum of the other values on the boundary)}$$
$$+ 16 \text{ (centre value)}]$$

$$= \frac{0.1 \times 0.1}{9}[(8 + 8.4 + 9.24 + 8.8) + 4(8.4 + 8.2 + 8.82 + 9.02) + 16(8.61)]$$

$$= \frac{0.01}{9}[34.44 + 137.76 + 137.76]$$

$\Rightarrow$ $\boldsymbol{I_1 = 0.3444}$

$$I_2 = \int_{2.2}^{2.4} \int_4^{4.2} xy\,dx\,dy$$

$$= \frac{0.1 \times 0.1}{9}[(8.8 + 9.24 + 10.08 + 9.6) + 4(9.2 + 9.02 + 9.66 + 9.84) + 16(9.43)]$$

$$= \frac{0.01}{9}[37.72 + 150.88 + 150.88]$$

$\Rightarrow$ $\boldsymbol{I_2 = 0.3772}$

$$I_3 = \int_2^{2.2} \int_{4.2}^{4.4} xy\,dx\,dy$$

$$= \frac{0.1 \times 0.1}{9}[(8.4 + 8.8 + 9.68 + 9.24) + 4(8.82 + 8.6 + 9.24 + 9.46) + 16(9.03)]$$

$$= \frac{0.01}{9}[36.12 + 144.48 + 144.48]$$

$\Rightarrow$ $\boldsymbol{I_3 = 0.3612}$

and $I_4 = \int_{2.2}^{2.4}\int_{4.2}^{4.4} xy\,dx\,dy$

$$= \frac{0.1\times 0.1}{9}[(9.24+9.68+10.56+10.08)+4(9.66+9.46+10.12+10.32)+16(9.89)]$$

$$= \frac{0.01}{9}[39.56+158.24+158.24]$$

$\Rightarrow \quad I_4 = \mathbf{0.3956}$

$\therefore \quad I = 0.3444+0.3772+0.3612+0.3956 = \mathbf{1.4784}$

Note: Directly integrating we get the exact value as 1.4784. So in this problem Simpson's formula gives the exact value. ■

Example 10

Evaluate $\int_{4}^{5.2}\int_{2}^{3.2} \frac{dy\,dx}{xy}$ by Simpson's rule for double integration.

Solution

Let $$I = \int_{4}^{5.2}\int_{2}^{3.2} \frac{dy\,dx}{xy}$$

Here $$f(x,y) = \frac{1}{xy}$$

Let $h = 0.3$ and $k = 0.3$

The order of integration is first w.r.to y and then w.r.to x.
The values of x are 4, 4.3, 4.6, 4.9, 5.2 and the value of y are 2, 2.3, 2.6, 2.9, 3.2
The interval is divided into 4 sub-intervals.

$f(4, 2) = \frac{1}{4\times 2} = \frac{1}{8} = 0.125,$ $\quad f(4.3, 2) = \frac{1}{4.3\times 2} = 0.116279$

$f(4, 2.3) = \frac{1}{4\times 2.3} = 0.108696,$ $\quad f(4.3, 2.2) = \frac{1}{4.3\times 2.3} = 0.101112$

$f(4, 2.6) = \frac{1}{4\times 2.6} = 0.096154,$ $\quad f(4.3, 2.6) = \frac{1}{4.3\times 2.6} = 0.089445$

$f(4, 2.9) = \frac{1}{4\times 2.9} = 0.086207,$ $\quad f(4.3, 2.9) = \frac{1}{4.3\times 2.9} = 0.080192$

$f(4, 3.2) = \frac{1}{4\times 3.2} = 0.078125,$ $\quad f(4.3, 3.2) = \frac{1}{4.3\times 3.2} = 0.072674$

$f(4.6, 2) = \frac{1}{4.6\times 2} = 0.108696,$ $\quad f(4.9, 2) = \frac{1}{4.9\times 2} = 0.102041$

$$f(4.6, 2.3) = \frac{1}{4.6 \times 2.3} = 0.094518$$

$$f(4.6, 2.6) = \frac{1}{4.6 \times 2.6} = 0.083612,$$

$$f(4.6, 2.9) = \frac{1}{4.6 \times 2.9} = 0.074963,$$

$$f(4.6, 3.2) = \frac{1}{4.6 \times 3.2} = 0.067935,$$

$$f(5.2, 2) = \frac{1}{5.2 \times 2} = 0.096154,$$

$$f(5.2, 2.6) = \frac{1}{5.2 \times 2.6} = 0.073964,$$

$$f(5.2, 3.2) = \frac{1}{5.2 \times 3.2} = 0.060096$$

$$f(4.9, 2.3) = \frac{1}{4.9 \times 2.3} = 0.088731$$

$$f(4.9, 2.6) = \frac{1}{4.9 \times 2.6} = 0.078493$$

$$f(4.9, 2.9) = \frac{1}{4.9 \times 2.9} = 0.070373$$

$$f(4.9, 3.2) = \frac{1}{4.9 \times 3.2} = 0.063776$$

$$f(5.2, 2.3) = \frac{1}{5.2 \times 2.3} = 0.083612$$

$$f(5.2, 2.9) = \frac{1}{5.2 \times 2.9} = 0.066313$$

We shall tabulate the values of $f(x, 4)$

x \ y	2	2.3	2.6	2.9	3.2
4	0.125	0.108696	0.096154	0.086207	0.078125
4.3	0.0116279	0.101112	0.089445	0.080192	0.072674
4.6	0.108696	0.094578	0.083612	0.074963	0.067935
4.9	0.102041	0.088731	0.078493	0.070373	0.063776
5.2	0.096154	0.083612	0.073964	0.066313	0.060096

We shall rewrite the given integral as the sum of four integrals taking two intervals at a time and applying Simpson's rule

$$I=\int_{4}^{5.2}\int_{2}^{3.2}\frac{1}{xy}\,dy\,dx$$

$$=\int_{4}^{4.6}\int_{2}^{2.6}\frac{1}{xy}\,dy\,dx+\int_{4}^{4.6}\int_{2.6}^{3.2}\frac{1}{xy}\,dy\,dx+\int_{4.6}^{5.2}\int_{2}^{2.6}\frac{1}{xy}\,dy\,dx+\int_{4.6}^{5.2}\int_{2.6}^{3.2}\frac{1}{xy}\,dy\,dx$$

$$=I_1+I_2+I_3+I_4$$

Now

$$I_1=\int_{4}^{4.6}\int_{2}^{2.6}\frac{1}{xy}\,dy\,dx$$

$$=\frac{hk}{9}[\text{Sum of the corner values} + 4(\text{sum of the other values on the boundary}) + 16(\text{centre value})]$$

$$=\frac{(0.3)(0.3)}{9}[(0.125+0.108696+0.083612+0.096154)+4(0.116279+0.094518+0.089445+0.108696)+16(0.101112)]$$

$$=0.01[0.413462+1.635752+1.617792]$$

$$\Rightarrow \quad I_1=\mathbf{0.036670}$$

$$I_2=\int_{4}^{4.6}\int_{2.6}^{3.2}\frac{1}{xy}\,dy\,dx$$

$$=\frac{(0.3)(0.3)}{9}[(0.096154+0.083612+0.067935+0.078125)+4(0.089445+0.074963+0.072674+0.086207)+16(0.080192)]$$

$$=0.01[0.325826+1.293156+1.283072]$$

$$\Rightarrow \quad I_2=\mathbf{0.029021}$$

$$I_3=\int_{4.6}^{5.2}\int_{2}^{2.6}\frac{1}{xy}\,dy\,dx$$

$$=\frac{(0.3)(0.3)}{9}[(0.108696+0.096154+0.073964+0.083612)+4(0.102041+0.083612+0.078493+0.094518)+16(0.088731)]$$

$$=0.01[0.362426+1.434656+1.419696]$$

$$\Rightarrow \quad I_3=\mathbf{0.032168}$$

and

$$I_4=\int_{4.6}^{5.2}\int_{2.6}^{3.2}\frac{1}{xy}\,dy\,dx$$

$$=\frac{(0.3)(0.3)}{9}[(0.083612+0.073964+0.060096+0.067935)+4(0.078493+0.066313+0.063776+0.074963)+16(0.070373)]$$

$$=0.01[0.285607+1.13418+1.125968]$$

$$\Rightarrow \quad I_4=\mathbf{0.025458}$$

$\therefore \quad I = 0.036670 + 0.029021 + 0.032168 + 0.025458$

$\boldsymbol{I = 0.123317}$

Aliter: We shall apply Simpson's rule to the rows, treating the values 2, 2.3, 2.6, 2.9 and 3.2 as x values and the function values of I row as y_0, y_1, y_2, y_3, y_4

Applying for I row, we get the following table:

x	2	2.3	2.6	2.9	3.2
y	0.125 y_0	0.108696 y_1	0.096154 y_2	0.086207 y_3	0.078125 y_4

$$R_0 = \frac{h}{3}[y_0 + y_4 + 4(y_1 + y_3) + 2(y_2)]$$

$$= \frac{0.3}{3}[(0.125 + 0.078125) + 4(0.108696 + 0.086207) + 2(0.096154)]$$

$$= 0.1[0.203125 + 0.779612 + 0.192308]$$

$$\Rightarrow \quad R_0 = 0.1175045$$

Applying for II row, we get the following table.

x	2	2.3	2.6	2.9	3.2
y	0.116279 y_0	0.101112 y_1	0.089445 y_2	0.080192 y_3	0.072674 y_4

$$R_1 = \frac{h}{3}[y_0 + y_4 + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{0.3}{3}[0.116279 + 0.072674 + 4(0.101112 + 0.080192) + 2(0.089445)]$$

$$= 0.1[0.188953 + 0.725216 + 0.17889]$$

$$= 0.1093059$$

Similarly applying Simpson's rule for III and IV and V rows, we get

$$R_2 = \frac{0.3}{3}[(0.108696 + 0.067935) + 4(0.094518 + 0.074963) + 2(0.083612)]$$

$$= 0.1[0.176631 + 0.677924 + 0.167224]$$

$$= 0.1021779$$

$$R_3 = \frac{0.3}{3}[(0.102041 + 0.063776) + 4(0.088731 + 0.070373) + 2(0.078493)]$$

$$= 0.1[0.165817 + 0.636416 + 0.156986]$$

$$= 0.0959219$$

$$R_4 = \frac{0.3}{3}[(0.096154 + 0.060096) + 4(0.083612 + 0.066313) + 2(0.073964)]$$

$$= 0.1[0.15625 + 0.5997 + 0.147928]$$

$$= 0.0903878$$

Now treating 4, 4.3, 4.6, 4.9, 5.2 as x values and R_0, R_1, R_2, R_3, R_4 as y values and applying Simpson's rule we get

$$I = \frac{0.3}{3}[(R_0 + R_4) + 4(R_1 + R_3) + 2R_2]$$
$$= 0.1[0.1175045 + 0.0903878 + 4(0.1093059 + 0.0959219) + 2(0.1021779)]$$
$$= 0.1[0.2078923 + 0.8209112 + 0.2043558]$$
$$= 0.12331593 = 0.123316$$

Note: By direct integration the actual value of the integral is

$$\int_4^{5.2}\int_2^{3.2} \frac{1}{xy}\,dy\,dx = \int_4^{5.2} \frac{dx}{x} \int_2^{3.2} \frac{1}{y}\,dxy$$
$$= [\log x_e]_4^{5.2} [\log y_e]_2^{3.2}$$
$$= (\log_e 5.2 - \log_e 4)(\log_e 3.2 - \log_e 2)$$
$$= \left[\log_e \frac{5.2}{4}\right]\left[\log_e \frac{3.2}{2}\right]$$
$$= \log_e 1.3 \times \log_e 1.6$$
$$= 0.262364264 \times 0.470003629$$
$$= 0.123312156$$
$$= \mathbf{0.123312}$$

Error $= 0.123312 - 0.123316 = 0.000004$, which is negligible.

Exercises 7.4

(1) Evaluate $\int_2^{2.2}\int_1^{2.6} \frac{1}{x^2+y^2}\,dx\,dy$ using trapezoidal rule taking $h = 0.1$ and $k = 0.8$.

(2) Evaluate $\int_1^5\int_1^5 \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy$ using trapezoidal rule with 4 subintervals.

(3) Evaluate $\int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\,dx\,dy$ using trapezoidal rule with 4 subintervals.

(4) Apply Simpson's rule to evaluate $\int_2^{2.6}\int_4^{4.4} \frac{1}{xy}\,dx\,dy$ by taking $h = 0.2$, $k = 03$.

(5) Evaluate $\int_1^2\int_3^4 \frac{1}{(x+y)^2}\,dx\,dy$ taking $h = k = 0.5$ by Simpson's rule.

(6) Evaluate $\int_0^{\pi/2}\int_0^{\pi/2} \sin(x+y)\,dx\,dy$ using Simpson's rule by taking $h = k = \frac{\pi}{4}$. Also find by trapezoidal rule.

(7) Evaluate $\int_0^{\pi/2}\int_0^{\pi/2} \sqrt{\sin(x+y)}\,dx\,dy$ using Simpson's rule with $h = k = \frac{\pi}{4}$.

(8) Evaluate $\int_0^{\pi/2}\int_{\pi/2}^{\pi} \cos(x+y)\,dx\,dy$ using

(i) Trapezoidal rule and (ii) Simpson's rule with $h = k = \frac{\pi}{4}$.

Answer 7.4

(1) 0.0429 (2) 3.9975 (3) 0.0614
(4) 0.025 (5) 0.0408 (6) 2.0095, 1.7976
(7) 2.1546 (8) (i) −1.7976 (ii) −2.0091

SHORT ANSWER QUESTIONS

1. Evaluate $\int_{1/2}^{1} \frac{1}{x}dx$ by Trapezoidal rule, dividing the range into 4 equal parts.
2. When does Simpson's rule give exact result?
3. Write down trapezoidal rule to evaluate $\int_1^6 f(x)\,dx$ with $h = 0.5$, function $f(x)$ is unknown.
4. In order to evaluate $\int_a^b f(x)dx$ by trapezoidal rule and by Simpson's rule, what is the restriction on the number of intervals?
5. What are the errors in trapezoidal and Simpson's rules of numerical integration?
6. What is the order of error in Simpson's $\frac{1}{3}$ rule?
7. State the local error term in Simpson's $\frac{1}{3}$ rule.
8. State Newton's formula to find $f'(x)$ using the forward differences.
9. What is the order of the error in trapezoidal rule?
10. Why is trapezoidal rule so called?
11. Write down Simpson's $\frac{3}{8}$ rule, assuming 3n intervals.
12. What are the truncation errors in Trapezoidal rule and Simspon's $\frac{1}{3}$ rule?
13. Evaluate $\int_0^1 \frac{dx}{1+x}$, using trapezoidal rule by taking $\Delta x = 0.25$.
14. Using two point Gaussian quadrature formula, evaluate $\int_{-1}^{1} \frac{1}{1+x^2}dx$.
15. Evaluate $\int_{-1}^{1} \frac{1}{1+x^4}dx$ using Gaussian quadrative with 2 points.
16. State Romberg's integration formula to find the value of $I = \int_b^b f(x)dx$, using $h, \frac{h}{2}$.

CHAPTER

Curve Fitting

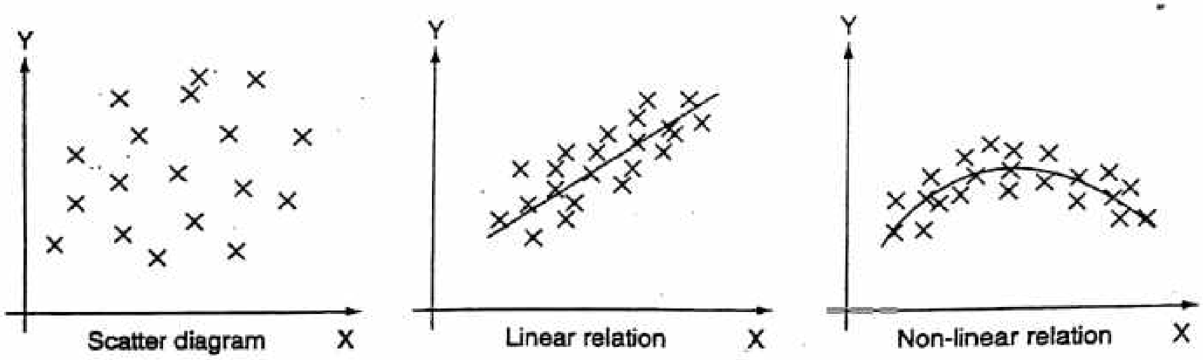
INTRODUCTION

Quite often in engineering and science, we conduct experiments involving two quantities, say x and y. Suppose $x_1, x_2, \ldots, x_n$ be the values of x corresponding to which the values of y are $y_1, y_2 \ldots, y_n$. We would like to know the functional relation between x and y. As a first step we plot the points $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$ with respect to rectangular axes. The resulting set of points plotted is called a **scatter diagram**. From the scatter diagram it is possible to visualize a smooth curve which passes as closely as possible to these points and that approximates the data. Such a curve is called an **approximating curve**.

The equation of this curve between x and y is called an **empirical relation**.

For example, in the scatter diagram if the points cluster around a straight line, then we say that a **linear relationship** exists between x and y. If the points cluster around a curve, then we say a **non-linear relationship** exists between x and y.

The general problem of finding the equation of the approximating curve that fits the given set of data is called **curve fitting.**



If the relation is linear, then we assume the empirical relation of the approximating curve as $y = ax + b$.

The best values of the constants occurring in the equation can be found in different ways. We shall fit a curve by the following methods:

(1) method of least squares
(2) method of averages
(3) method of the sum of exponentials
(4) method of moments

METHOD OF LEAST SQUARES

Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be a set of observed data and let $y = f(x)$ be the relation suggested by the scatter diagram.

Let P_i be the point (x_i, y_i), $i = 1, 2, 3, \ldots, n$

When $x = x_i$, the observed value is y_i and the expected value is $f(x_i)$.

Let $d_i = y_i - f(x_i)$ be the difference between the observed and expected values for each i.

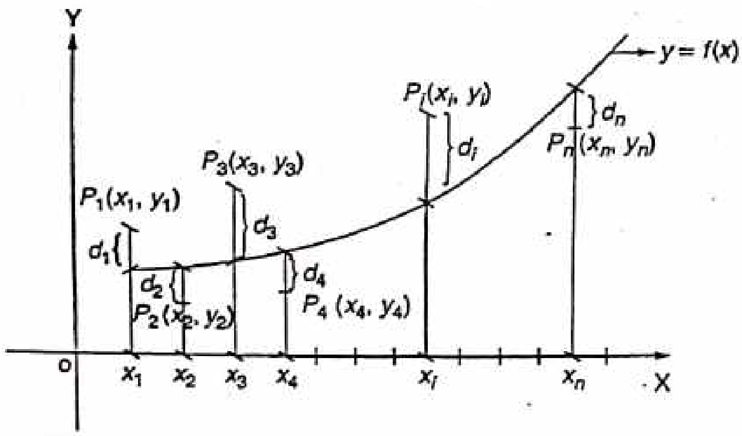
The differences $d_1, d_2, \ldots, d_n$ are called the **residuals**, which may be positive or negative

$$\text{Let } E = \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} [y_i - f(x_i)]^2$$

The principle of least squares is that the sum of the squares of the residuals is minimum.

It is also referred as **Gauss's least square principle.**

The curve fitted by this principle is called the **least square curve.**



The curve for which $E = \sum_{i=1}^{n} d_i^2$ is minimum is called a **best fitting curve** besides all the curves approximating the given set of points.

A line fitted by this principle is known as **least-square line** and a parabola fitted by this principle is known as **least square parabola.**

Note: The method of least squares gives **unique values** for the constants and so the curve fitted is **unique.**

Fit a Straight Line by the Method of Least Squares

Let $(x_1, y_1), (x_2, y_2) \dots (x_n, y_n)$ be a set of observed values of the variables x and y.

Let the relation between x and y be $y = ax + b$ (1)

Then, the sum of squares of residuals is

$$E = \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} [y_i - (ax_i + b)]^2$$

By principle of least squares E is minimum.

Since x_i and y_i are known, E is a function of a and b.

The conditions for E to be minimum are $\dfrac{\partial E}{\partial a} = 0$ and $\dfrac{\partial E}{\partial b} = 0$

$$\therefore \quad \frac{\partial E}{\partial a} = 0 \Rightarrow \sum_{i=1}^{n} 2[y_i - (ax_i + b)](-x_i) = 0$$

$$\Rightarrow \quad -2\sum_{i=1}^{n} [x_i y_i - (ax_i^2 + bx_i)] = 0$$

$$\Rightarrow \quad \sum x_i y_i - a\sum x_i^2 - b\sum x_i = 0$$

$$\Rightarrow \quad \sum x_i y_i = a\sum x_i^2 + b\sum x_i \qquad (2)$$

and

$$\frac{\partial E}{\partial b} = 0 \Rightarrow \sum_{i=1}^{n} 2[y_i - (ax_i + b)](-1) = 0$$

$$\Rightarrow \quad -2\sum_{i=1}^{n} [y_i - ax_i - b] = 0$$

$$\Rightarrow \quad \sum y_i - a\sum x_i - nb = 0$$

$$\Rightarrow \quad \sum y_i = a\sum x_i + nb \qquad (3)$$

Solving (2) and (3), we get the values of a and b.

For these values of a and b, the line $y = ax + b$ is the best fitted line to the data.

The equations (2) and (3) are called the **normal equations** for the least–squares line

$$y = ax + b$$

Note:

(1) The formulae can be remembered by omitting the suffixes:

The line is $y = ax + b$ (1)

The normal equations are $\sum y = a\sum x + nb$

and $\sum xy = a\sum x^2 + b\sum x$

(2) If the values of x are equally spaced with interval h, we can simplify the computation by changing the origin and scale by the transformation

$$X = \frac{x - a}{h} \text{ so that } \sum X = 0.$$

WORKED EXAMPLES

Example 1

Use the method of least squares to fit a straight line to the following data:

x	0	5	10	15	20
y	7	11	16	20	26

Estimate the value of y when $x = 25$.

Solution

We fit a straight line to the given data by the method of least squares.

Since the values of x are equally spaced, let $X = \frac{x-10}{5}$

Then the equation of the line of best fit is

$$y = aX + b \tag{1}$$

Normal equations are

$$\Sigma y = a\Sigma X + nb \tag{2}$$

and

$$\Sigma Xy = a\Sigma X^2 + b\Sigma X \tag{3}$$

To find ΣX, ΣX^2, ΣXy, we form the table

x	y	$X = \frac{x-10}{5}$	Xy	X^2
0	7	−2	−14	4
5	11	−1	−11	1
10	16	0	0	0
15	20	1	20	1
20	26	2	52	4
	80	0	47	10

From the table $n = 5$, $\Sigma X = 0$, $\Sigma y = 80$, $\Sigma Xy = 47$, $\Sigma X^2 = 10$

Substituting in (2), we get $\quad 80 = a.0 + 5b \quad \Rightarrow b = \frac{80}{5} = 16$

Substituting in (3), we get $\quad 47 = a.10 + b.0 \quad \Rightarrow a = \frac{47}{10} = 4.7$

$\therefore$ the equation of the line of best fit is

$$\begin{aligned} y &= 4.7X + 16 \\ &= 4.7\left(\frac{x-10}{5}\right) + 16 \\ &= \frac{4.7}{5}(x-10) + 16 \end{aligned}$$

$$= 0.94x - 0.94(10) + 16$$
$$= 0.94x - 9.4 + 16$$
$$y = 0.94x + 6.6$$

When $x = 25, y = 0.94(25) + 6.6 = \mathbf{30.1}$ ∎

Example 2

In a tensile test of a metal bar the following observation were made where x represents load in tons and y elongation in ten thousands of an inch.

x	1	2	3	4	5	6
y	14	27	41	56	68	75

Using the principle of least squares, find a law of the form $y = ax + b$.

Solution

We fit a straight line by the principle of least squares.

Here the values of x are equally spaced with $h = 1$, but $n = 6$.

So, we choose the origin as the average of the two middle values 3 and 4 and it is $= \frac{3+4}{2} = 3.5$

Let $X = x - 3.5$

Then, the equation of the line of best fit is $y = aX + b$

The normal equations are $\quad \sum y = a\sum X + nb \quad$ (2)

and $\quad \sum Xy = a\sum X^2 + b\sum X \quad$ (3)

x	y	$X = x - 3.5$	Xy	X^2
1	14	−2.5	−35	6.25
2	27	−1.5	−40.5	2.25
3	41	−0.5	−20.5	0.25
4	56	0.5	28	0.25
5	68	1.5	102	2.25
6	75	2.5	187.5	6.25
	281	0	2215	17.5

$n = 6, \sum X = 0, \sum y = 281, \sum Xy = 221.5, \sum X^2 = 17.5$

Substituting in (2), we get, $\quad 281 = a.0 + 6b \quad \Rightarrow b = \frac{281}{6} = 46.8333$

Substituting in (3), we get, $\quad 221.5 = a(17.5) + b.0$

$$\Rightarrow \qquad a = \frac{221.5}{17.5} = 12.6571$$

$\therefore$ the equations of the line of best fit is

$$\begin{aligned} y &= 12.657X + 46.833 \\ &= 12.657(x - 3.5) + 46.833 \\ &= 12.657x - 12.657(3.5) + 46.833 \end{aligned}$$

$$\Rightarrow \qquad \mathbf{y = 12.657x + 2.5335.}$$

■

Example 3

Using 1964 as the origin obtain a straight line trend equation by the method of least squares.

Year	1960	1962	1963	1964	1965	1966	1969
Value	140	144	160	152	168	176	180

Find the trend value of the missing year 1961.

Solution

Let the year be denoted by x, and let the value be denoted by y

Let $$X = x - 1964$$

Then, the equation of the straight line of best fit is

$$y = aX + b \qquad (1)$$

$\therefore$ the normal equations are $$\Sigma y = a\Sigma X + nb \qquad (2)$$

and $$\Sigma Xy = a\Sigma X^2 + b\Sigma X \qquad (3)$$

x	y	$X = x - 1964$	X^2	Xy
1960	140	–4	16	–560
1962	144	–2	4	–288
1963	160	–1	1	–160
1964	152	0	0	0
1965	168	1	1	168
1966	176	2	4	352
1969	180	5	25	900
	1120	1	51	412

$n = 7, \quad \Sigma X = 1, \quad \Sigma y = 1120, \quad \Sigma X^2 = 51, \quad \Sigma Xy = 412$

$\Rightarrow$ The normal equations are

$$1120 = a.1 + 7b$$

$\Rightarrow$ $$1120 = a + 7b \quad (4)$$

and $$412 = a \times 51 + b.1$$

$\Rightarrow$ $$412 = 51a + b \quad (5)$$

$(4) - 7 \times (5) \Rightarrow$ $$1120 - 2884 = a\,(1 - 357)$$

$\Rightarrow$ $$356\,a = 1764$$

$\Rightarrow$ $$a = \frac{1764}{356} = 4.955 = 4.96$$

Substituting in (4), we get

$$7b + 4.96 = 1120$$

$\Rightarrow$ $$7b = 1120 - 4.96 = 1115.04$$

$\Rightarrow$ $$b = \frac{1115.04}{7} = 159.29$$

The straight line trend is $$y = 4.96X + 159.29$$

$\Rightarrow$ $$y = 4.96(x - 1964) + 159.24$$

When $x = 1961$,

$$y = 4.96\,(1961 - 1964) + 159.29$$

$\Rightarrow$ $$y = 4.96(-3) + 159.29$$

$\Rightarrow$ $$y = -14.88 + 159.29 = 144.41$$

$\therefore$ the trend value for the year 1961 is **144.41.** ∎

(a) Fitting Other Type of Equations Reducible to the Form $y = ax + b$

Certain types of equations can be reduced to the linear form by transformation of variables.

This process is usually called **rectification.**

Some of such equations are given below.

1. $y = \dfrac{a}{x} + b$, put $X = \dfrac{1}{x}$, then $y = aX + b$
2. $y = ax^2 + bx$, then $\dfrac{y}{x} = ax + b$,

   Put $Y = \dfrac{y}{x}$, then $Y = ax + b$
3. $y = a + bxy$, put $X = xy$, then $y = a + bX$
4. $y = ax + b.10^x$, then $\dfrac{y}{x} = a + b.\dfrac{10^x}{x}$

   Put $Y = \dfrac{y}{x}$, $X = \dfrac{10^x}{x}$, then $Y = a + bX$

5. $y = ax + b\log_{10} x$, then $\dfrac{y}{x} = a + b.\dfrac{\log_{10} x}{x}$

   Put $Y = \dfrac{y}{x}$, $X = \dfrac{\log_{10} x}{x}$, then $Y = a + bX$

6. $y = ae^{bx}$, then $\log_{10} y = \log_{10} a + bx\log_{10} e$.

   Put $Y = \log_{10} y$, $A = \log_{10} a$ and $B = b\log_{10} e$

   Then, the equation is $Y = A + BX$

7. $y = ax^b$, then $\log_{10} y = \log_{10} a + b\log_{10} x$.

   Put $Y = \log_{10} y$, $A = \log_{10} a$, $X = \log_{10} x$

   Then, the equation is $Y = A + bX$

WORKED EXAMPLES

Example 4

An experiment on the life of a cutting tool at different cutting speeds gave the values given below.

Speed v ft/min	**350**	**400**	**500**	**600**
Life T in min.	**61**	**26**	**7**	**2.6**

It is known that v and T satisfy the relationship $v = a\,T^b$. Using the method of least squares find the best values of a and b.

Solution

Given $v = a\,T^b$

Taking log to the base 10

$$\log_{10} v = \log_{10} a + b\log_{10} T$$

Put $Y = \log_{10} v$, $A = \log_{10} a$, $X = \log_{10} T$

Then, the equation is $Y = A + bX$ (1)

which is linear in x and y.

The normal equations are

$$\Sigma Y = nA + b\Sigma X \quad (2)$$

$$\Sigma XY = A\Sigma X + b\Sigma X^2 \quad (3)$$

We form the table

v	T	$X=\log_{10}T$	$Y=\log_{10}v$	XY	X^2
350	61	1.7853	2.5441	4.542	3.1873
400	26	1.4150	2.6021	3.682	2.0022
500	7	0.8451	2.6990	2.281	0.7142
600	2.6	0.4150	2.7782	1.153	0.1722
		4.4604	10.6234	11.658	6.0759

Substituting in (2) and (3), we get

$$4.4604b + 4A = 10.6234 \quad (4)$$

and

$$6.0759b + 4.4604A = 11.658 \quad (5)$$

$\frac{(4)}{4} \Rightarrow \quad 1.1151b + A = 2.6559 \quad (6)$

$\frac{(5)}{4.4604} \Rightarrow \quad 1.3622b + A = 2.6137 \quad (7)$

$(6)-(7) \Rightarrow \quad -0.2471b = 0.0422$

$\therefore \quad b = \frac{-0.0422}{0.2471} = -0.1708$

$\therefore \quad A = 2.6137 - 1.3622\,(-0.1708) = 2.8464$

$\Rightarrow \quad \log_{10} a = 2.8464 \quad \Rightarrow a = 10^{2.8464} = 702.1$ ■

$\therefore$ the best values of $a =$ **702.1** and $b =$ **−0.1708**

Example 5

The voltage v across a capacitor at time t seconds is given by the following table. Use the principle of least squares to fit a curve of the form $v = ae^{kt}$ to the data.

t	0	2	4	6	8
v	150	63	28	12	5.6

Solution

Given $\quad v = ae^{kt}$.

Taking log to base 10, we get

$$\log_{10}v = \log_{10} a + kt \log_{10} e$$

Put $\quad Y = \log_{10} v, \quad A = \log_{10} a, \quad B = k\log_{10}e, \quad X = \frac{t-4}{2}$

$\therefore$ the equation is $\quad Y = A + BX \quad$ (1)

By method of least squares, the normal equations are

$$\Sigma Y = nA + B\Sigma X \quad (2$$

and $$\Sigma XY = A\Sigma X + B\Sigma X^2 \quad (3)$$

We form the table

t	v	$X = \frac{t-4}{2}$	$Y = \log_{10} v$	XY	X^2
0	150	–2	2.1761	–4.3522	4
2	63	–1	1.7993	–1.7993	1
4	28	0	1.4471	0	0
6	12	1	1.0792	1.0792	1
8	5.6	2	0.7482	1.4964	4
		0	7.2499	–3.5759	10

$n = 5, \quad \Sigma X = 0, \quad \Sigma Y = 7.2499, \quad \Sigma XY = -3.5759, \quad \Sigma X^2 = 10$

Substituting in (1), we get $\quad 7.2499 = 5A + B.0$

$$\Rightarrow \quad A = \frac{7.2499}{5} = 1.45$$

Substituting in (3), we get

$$-3.5759 = A.0 + B.10$$

$$\Rightarrow \quad B = \frac{-3.5759}{10} = -0.3576$$

$$A = 1.45 \qquad \Rightarrow \log_{10} a = 1.45 \Rightarrow a = 10^{1.45} = 28.184$$

$$B = -0.3576$$

$$\Rightarrow \quad k \log_{10} e = -0.3576$$

$$\Rightarrow \quad k = \frac{-0.3576}{\log_{10} e}$$

$$= -0.3576 \log_e 10$$

$$= -0.3576(2.30258) = -0.8234$$

$$\therefore \quad \mathbf{v = 28.184\, e^{-0.8234t}}.$$

(b) Fit a Parabola $y = ax^2 + bx + c$ by the Method of Least Squares

Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be n observed values of x and y. (1)

We have to fit the parabola $y = ax^2 + bx + c$

to this data

The sum of the squares of the residuals is

$$E = \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} [y_i - (ax_i^2 + bx_i + c)]^2$$

Since x_i and y_i. are known, E is a function of a, b, c.
By principle of least squares E is minimum.

$\therefore$ the conditions are $\frac{\partial E}{\partial a} = 0, \frac{\partial E}{\partial b} = 0, \frac{\partial E}{\partial c} = 0$

$$\frac{\partial E}{\partial a} = 0 \quad \Rightarrow \sum_{i=1}^{n} 2[y_i - (ax_i^2 + bx_i + 1)](-x_i^2) = 0$$

$$\Rightarrow \quad -2\Sigma[x_i^2 y_i - ax_i^4 - bx_i^3 - cx_i^2] = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} x_i^2 y_i - \sum_{i=1}^{n} ax_i^4 - b\sum_{i=1}^{n} x_i^3 - c\sum_{i=1}^{n} x_i^2 = 0$$

$$\Rightarrow \quad \Sigma x_i^2 y_i = a\Sigma x_i^4 + b\Sigma x_i^3 + c\Sigma x_i^2 \qquad (2)$$

Similarly, $\frac{\partial E}{\partial b} = 0 \Rightarrow \quad \Sigma x_i y_i = a\Sigma x_i^3 + b\Sigma x_i^2 + c\Sigma x_i$ (3)

and $\frac{\partial E}{\partial c} = 0 \Rightarrow \quad \Sigma y_i = a\Sigma x_i^2 + b\Sigma x_i + nc$ (4)

Solving (2), (3) and (4) we get the values of a, b, c for which the curve $y = ax^2 + bx + c$ is the best fitted curve to the data.

Note: Dropping the suffixes, for simplicity, the normal equations of the least square curve

$$y = ax^2 + bx + c \qquad (1)$$

are

$$\Sigma y = a\Sigma x^2 + b\Sigma x + nc \qquad (2)$$

$$\Sigma xy = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x \qquad (3)$$

$$\Sigma x^2 y = a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2 \qquad (4)$$

They can be remembered as below

Taking Σ of (1) we get (2)

Multiply (1) by x and then taking Σ, we get (3)

Multiplying (1) by x^2 and then taking Σ, we get (4).

WORKED EXAMPLES

Example 1

Fit a parabola of the form $y = ax^2 + bx + c$ to the following data:

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

Solution

We fit a parabola to the given data by the method of least square.

The values of x are equally spaced with $h = 1$

For simplicity in computation we take origin for x.

Put $$X = x - 2.$$

Then the equation of the curve of best fit is

$$y = aX^2 + bX + c \tag{1}$$

The normal equations are

$$\Sigma y = a\Sigma X^2 + b\Sigma X + nc \tag{2}$$

$$\Sigma Xy = a\Sigma X^3 + b\Sigma X^2 + c\Sigma X \tag{3}$$

$$\Sigma X^2 y = a\Sigma X^4 + b\Sigma X^3 + c\Sigma X^2 \tag{4}$$

We form the table

x	y	$X = x - 2$	X^2	X^3	X^4	Xy	X^2y
0	1.0	–2	4	–8	16	–2	4
1	1.8	–1	1	–1	1	–1.8	1.8
2	13	0	0	0	0	0	0
3	2.5	1	1	1	1	2.5	2.5
4	63	2	4	8	16	12.6	25.2
	12.9	0	10	0	34	113	33.5

$$n = 5 \quad \Sigma X = 0, \quad \Sigma y = 12.9, \quad \Sigma X^2 = 10, \quad \Sigma X^3 = 0,$$

$$\Sigma X^4 = 34, \quad \Sigma Xy = 11.3, \quad \Sigma X^2 y = 33.5$$

Substituting in (2), we get $12.9 = 10a + b.0 + 5c$

$\Rightarrow$ $$10a + 5c = 12.9$$

$\Rightarrow$ $$2a + c = 2.58 \tag{5}$$

Substituting in (3), $$11.3 = a.0 + b(10) + c.0$$

$\Rightarrow$ $$10\,b = 11.3 \quad \Rightarrow b = 1.13$$

Substituting in (4), $$33.5 = a.34 + b.0 + c.10$$

$$34a + 10c = 33.5$$

iding by 10, $$3.4a + c = 3.35 \tag{6}$$

– (5) $\Rightarrow$ $$1.4a = 0.77$$

$$a = \frac{0.74}{1.4} = 0.55$$

ubstituting in (5), we get $c = 2.58 - 2(0.55) = 2.58 - 1.10 = 1.48$

he curve of best fit is $y = 0.55X^2 + 1.13X + 1.48$, where $X = x - 2$ ■

Example 2

The following table gives the levels of prices in certain years. Fit a second degree parabola to the data:

Year	1975	1976	1977	1978	1979	1980	1981	1982	1983	1984	1985
Price	88	87	81	78	74	79	85	84	90	92	100

Solution

We fit a parabola, to the given data by the method of least squares.

Let x denote year and y denote price.

For year, take the origin as 1980. Let $X = x - 1980$.

Then the equation of the best fitted curve is

$$y = aX^2 + bX + c \tag{1}$$

The normal equations are

$$\Sigma y = a\Sigma X^2 + b\Sigma X + nc \tag{2}$$

$$\Sigma Xy = a\Sigma X^3 + b\Sigma X^2 + c\Sigma X \tag{3}$$

$$\Sigma X^2 y = a\Sigma X^4 + b\Sigma X^3 + c\Sigma X^2 \tag{4}$$

we form the table

x	y	$X = x - 1980$	X^2	X^3	X^4	Xy	X^2y
1975	88	−5	25	−125	625	−440	2200
1976	87	−4	16	−64	256	−348	1392
1977	81	−3	9	−27	81	−243	729
1978	78	−2	4	−8	16	−156	312
1979	74	−1	1	−1	1	−74	74
1980	79	0	0	0	0	0	0
1981	85	1	1	1	1	85	85
1982	84	2	4	8	16	168	336
1983	90	3	9	27	81	270	810
1984	92	4	16	64	256	368	1472
1985	100	5	25	125	625	500	2500
	938	0	110	0	1958	130	9910

$n = 11, \Sigma y = 938, \Sigma X = 0, \Sigma X^2 = 110, \Sigma X^3 = 0, \Sigma X^4 = 1958, \Sigma Xy = 130, \Sigma X^2 y = 9910$

Substituting in (3) we get, $130 = a.0 + b.110 + c.0 \quad \Rightarrow b = \frac{130}{110} = 1.1818$

Substituting in (2), we get $938 = a \times 110 + b.0 + 11 \times c$

$\Rightarrow \quad 110a + 11c = 938$

$\Rightarrow \quad 10a + c = 85.2727 \quad \ldots(5)$

Substituting in (4), $\quad 9910 = 1958a + 0.b + 110c$

$\Rightarrow \quad 1958a + 110c = 9910$

Dividing by 110, $\quad 17.8a + c = 90.0909 \quad \ldots(6)$

$(6) - (5) \Rightarrow \quad 7.8a = 4.8182$

$\therefore \quad a = \frac{4.8182}{7.8} = 0.61771$

$\therefore \quad c = 85.2727 - 10(0.61771) = 85.2727 - 6.1771 = 79.0956$

Thus $a = 0.62$, $b = 1.18$, $c = 79.10$

The curve of best fit is $y = 0.6X^2 + 1.18X + 79.10$, where $X = x - 1880$ ■

Example 3

Fit a second degree parabola of the form $y = ax^2 + bx + c$ to the following data taking x as the independent variable.

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	11	10	9

and introducing the new variable u and v by the equations $u = x - 5$, $v = y - 7$.

Solution

We fit a parabola to the given data by the method of least squares.

Since the given new variables are u and v and $u = x - 5, v = y - 7$, taking u as the independent variable, and v as the dependent variable the equation of the parabola in u and v is

$$v = au^2 + bu + c \quad \ldots(1)$$

The normal equations are

$$\Sigma v = a\Sigma u^2 + b\Sigma u + nc \quad \ldots(2)$$

$$\Sigma uv = a\Sigma u^3 + b\Sigma u^2 + c\Sigma u \quad \ldots(3)$$

$$\Sigma u^2 v = a\Sigma u^4 + b\Sigma u^3 + c\Sigma u^2 \quad \ldots(4)$$

x	y	$u = x - 5$	$v = y - 7$	u^2	u^3	u^4	uv	u^2v
1	2	–4	–5	16	–64	256	20	–80
2	6	–3	–1	9	–27	81	3	–9
3	7	–2	0	4	–8	16	0	0
4	8	–1	1	1	–1	1	–1	1
5	10	0	3	0	0	0	0	0
6	11	1	4	1	1	1	4	4
7	11	2	4	4	8	16	8	16
8	10	3	3	9	27	81	9	27
9	9	4	2	16	64	256	8	32
		0	11	60	0	708	51	–9

$n = 9, \quad \sum u = 0, \quad \sum u^2 = 60, \quad \sum u^3 = 0,$

$\sum u^4 = 708, \quad \sum v = 11, \quad \sum uv = 51, \quad \sum u^2 v = -9$

∴ Substituting in (2), (3) and (4), we get

$$11 = a \times 60 + b \times 0 + 9c$$

$\Rightarrow$ $$60a + 9c = 11$$

$\Rightarrow$ $$\frac{60}{9}a + c = \frac{11}{9}$$

$\Rightarrow$ $$6.6667a + c = 1.2222 \qquad (5)$$

$$51 = a.0 + b.60 + c.0$$

$\Rightarrow$ $$60b = 51 \Rightarrow b = \frac{51}{60} = 0.85 \qquad (6)$$

and $$-9 = a \times 708 + b.0 + c \times 60$$

$\Rightarrow$ $$708a + 60c = -9$$

$\Rightarrow$ $$\frac{708}{60}a + c = \frac{-9}{60}$$

$\Rightarrow$ $$11.8a + c = -0.15 \qquad (7)$$

(7) – (5) $\Rightarrow$ $$5.1333a = -1.3722 \Rightarrow a = \frac{-1.3722}{5.1333} = -0.2673$$

(5) $\Rightarrow$ $$c = 1.2222 - 6.6667 \times (-0.2673)$$
$$= 1.222 + 1.7820 = 3.0042$$

∴ $$a = -0.2673, \quad b = 0.85, \quad c = 3.0042$$

∴ $$v = -0.2673\, u^2 + 0.85u + 3.0042$$

$\Rightarrow$ $$y - 7 = -0.2673\, (x - 5)^2 + 0.85\, (x - 5) + 3.0042$$

$\Rightarrow$ $$y = 7 - 0.2673\, (x^2 - 10x + 25) + 0.85x - 4.25) + 3.0042$$

$\Rightarrow$ $$y = -0.2673x^2 + 3.523x - 0.9283$$ ■

Example 4

Fit a parabola of the form $y = ax^2 + bx + c$ to the following data by the method of least squares and predict the value of y when $x = 70$.

X	71	68	73	69	67	65	66	67
Y	69	72	70	70	68	67	68	64

Solution

We have to fit a parabola of the form $y = ax^2 + bx + c$ by the method of least squares.

Let $\quad X = x - 67,\ Y = y - 70$

$\therefore$ the equation is $\quad Y = aX^2 + bX + c \qquad (1)$

The normal equations are

$$\Sigma Y = a\Sigma X^2 + b\Sigma X + nc \qquad (2)$$

$$\Sigma XY = a\Sigma X^3 + b\Sigma X^2 + c\Sigma X \qquad (3)$$

$$\Sigma X^2Y = a\Sigma X^4 + b\Sigma X^3 + c\Sigma X^2 \qquad (4)$$

we form the table

x	y	$X = x - 67$	$Y = y - 70$	X^2	X^3	X^4	XY	X^2Y
71	69	+4	−1	16	64	256	−4	−16
68	72	+1	2	1	1	1	2	2
73	70	+6	0	36	216	1296	0	0
69	70	+2	0	4	8	16	0	0
67	68	0	−2	0	0	0	0	0
65	67	−2	−3	4	−8	16	6	−12
66	68	−1	−2	1	−1	1	2	−2
67	64	0	−6	0	0	0	0	0
		10	−12	62	280	1586	6	−28

$n = 10,\ \Sigma X = 10,\ \Sigma Y = -12,\ \Sigma X^2 = 62,\ \Sigma X^3 = 280,$

$\Sigma X^4 = 1586,\ \Sigma XY = 6,\ \Sigma X^2Y = -28$

Substituting in (2), (3) and (4), we get

$$-12 = a \times b2 + b \times 10 + 8c$$

$$7.75a + 1.25b + c = -1.5 \quad \text{(5) [Dividing by 8]}$$

$\Rightarrow$

$$6 = a \times 280 + b \times 62 + c \times 10$$

$$28a + 6.2b + c = 0.6 \quad \text{(6) [Dividing by 10]}$$

$\Rightarrow$

and

$$-28 = a \times 1586 + b \times 280 + c \times 62$$

$\Rightarrow$

$$25.58a + 4.52b + c = -0.45 \quad \text{(7) [Dividing by 62]}$$

$(6) - (5) \Rightarrow \quad 20.25a + 4.5b = 2.1$

$\Rightarrow \quad 4.09\,a + b = 0.4242 \quad (8)$

$(6) - (7) \Rightarrow \quad 2.42a + 1.68b = 0.15$

$\Rightarrow \quad 1.44\,a + b = 0.089 \quad (9)$

$(8) - (9) \Rightarrow \quad 2.65a = 0.3352 \quad \Rightarrow a = \dfrac{0.3352}{2.65} = 0.1265$

$(9) \Rightarrow$

$$b = 0.089 - 1.44 \times 0.1265$$
$$= 0.089 - 0.182 = -0.093$$
$$c = 0.6 - 28 \times 0.1265 - 6.2\,(-0.093)$$
$$= 0.6 - 3.542 + 0.5766 = -2.3654$$

$\therefore$ The equation of the parabola is

$$Y = 0.1265\,X^2 - 0.093X - 2.3654$$

$\Rightarrow \quad y - 70 = 0.1265\,(x - 67\,)^2 - 0.093(x - 67\,) - 2.3654$

$\Rightarrow \quad y = 0.1265x^2 - (0.093 + 16.95)x$

$$+ 567.8585 + 6.231 - 2.3654 + 70$$

$\Rightarrow \quad y = 0.1265x^2 - 17.043\,x + 641.724$

When $x = 70$, $\quad y = 0.1265 \times 70^2 - 17.043 \times 70 + 641.724$

$\Rightarrow \quad y = 619.85 - 1193.01 + 641.724 = 68.564 = \mathbf{69}$ ■

Exercises 8.1

By method of least squares fit the indicated curve to the given data:

(1) Fit a straight line to the data:

x	0	1	2	3	4
y	1	1.8	3.3	4.5	6.3

(2) Fit a straight line to the data:

x	75	80	93	65	87	71	98	68	84	77
y	82	78	86	72	91	80	95	72	89	74

(3) Find the least square line for the data below:

x	1	2	4	6	8	9	11	14
y	1	2	4	4	5	7	8	9

smoothen the data in the above by using the least square line.

(4) The weights of calf taken at weekly intervals are supplied below. Fit a straight line and calculate the average rate of growth per week.

Age x	1	2	3	4	5	6	7	8	9	10
Weight y	52.5	58.7	65.0	70.2	75.4	81.1	87.2	95.5	102.2	106.4

(5) Fit a curve of the form $y = ax + bx^2$ to the data:

x	1	2	3	4	5
y	1.8	5.1	8.9	14.1	19.8

Calculate y when $x = 2$.

$\left[\textbf{Hint: } \frac{y}{x} = a + bx, \text{ Put } Y = \frac{y}{x}, \text{ then } y = a + bx\right]$

(6) Given

x	2	4	6	8
y	5.5	14.5	26.2	41.8

Fit a law of the form $y = ax + bx^2$ by computing the least square line.

$\left[\textbf{Hint: } \text{Since } \frac{y}{x} = a + bx, \text{ put } Y = \frac{y}{x}, \text{then } Y = a + bx\right]$

(7) For the law of the $y = \frac{a}{x} + bx$, find the best values of a and b from the following data

x	1	2	3	4	5	6
y	5.4	6.3	8.2	10.2	12.6	15

(8) Fit an equation of the form $y = ae^{bx}$ for the data

x	1	2	3	4
y	1.65	2.70	4.50	7.35

(9) The pressure and volume of a gas are known to be related by the equation $pv = c$, a constant. In an experiment, the following volumes of quantity of gas were observed for the pressures specified. Fit the same equation taking P as independent variable.

P (kg/sq.cm)	0.5	1.0	1.5	2.0	2.5	3.0
v (litres)	1.62	1.00	0.75	0.62	0.52	0.46

(10) Growth of bacteria (y) in a culture after x hours is given in the following table. Fit a curve of the form $y = ab^x$ by method of least squares.

House x	0	1	2	3	4	5	6
No. of bacteria y	32	47	65	92	132	190	275

Estimate y when $x = 7$.

(11) Find the least squares parabola for the data:

x	0	2	4	6	8	10
y	1	3	13	31	57	91

(12) Find the least square parabola to the data:

x	0	1	2	3	4
y	1	5	10	22	38

Answers 8.1

(1) $y = 1.33x + 0.72$,

(2) $y = 0.66x + 29.13$

(3) The least square line is $y = 0.62x + 0.75$
The estimated values of y obtained from this line corresponding to $x = 1, 2, \ldots, 9, 11, 14$ are the smoothening values. The smoothened values of y are 1.37, 1.99, 3.23, 4.47, 5.71, 633, 7.57, 9.43

(4) $y = 6.16x + 45.74$, rate of growth = 6.16

(5) When $x = 2$, $y = 486$

(6) $Y = 1.95 + 0.41x$ and $y = 1.95x + 0.41x^2$

(7) $a = 2.7907$, $b = 2.413$

(8) $y = e^{0.4994x}$

(9) $Pv^{0.422} = 0.9972$

(10) $y = 32.15\,(1.427)^x$; when $x = 7$, $y = 387$

(11) $y = x^2 - x + 1$

(12) $y = 2.2x^2 + 0.3x + 1.4$

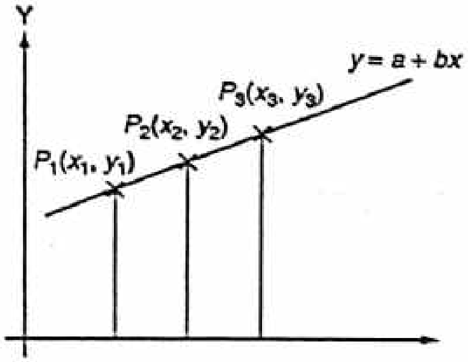
8.2 METHOD OF GROUP AVERAGES

Let (x_1, y_1). (x_2, y_2), (x_3, y_3), ..., (x_n, y_n) be the set of observed values of the related variables x and y.

Let $y = a + bx$ (1)

be the assumed linear relationship between the two variables.

To determine the constants a and b, we need two equations in a and b.



When $x = x_1$, the observed value of y is y_1 and the estimated values of y from (1) is $a + bx_1$
Their difference is called **residual** or **deviation.**

Thus $d_1 = y_1 - (a + bx_1)$ is the residual for the point (x_1, y_1).

Similarly for the other points we find the residuals

$d_2 = y_2 - (a + bx_2)$, $d_3 = y_3 - (a + bx_3)$, ..., $d_n = y_n - (a + bx_n)$

Some of the residuals may be positive, some of them may be negative or zero. The method of group averages is based on the assumption that the sum of the residuals is zero.

ie. $\sum_{i=1}^{n} d_i = 0$.

Since we require two equations in a and b to find out the values of a and b, we divide the given data into two groups such that the sum of residuals is zero for each group.

Let the first group contain r observations, say, (x_1, y_1), (x_2, y_2), (x_3, y_3), ..., (x_r, y_r) and the second group contain remaining $(n - r)$ observations.

$$\therefore \quad \sum_{i=1}^{r} d_i = 0$$

$$\Rightarrow \sum_{i=1}^{r} [y_i - (a + bx_i)] = 0$$

$$\Rightarrow \quad \sum_{i=1}^{r} y_i - \sum_{i=1}^{r} a - \sum_{i=1}^{r} bx_i = 0$$

$$\Rightarrow \quad \sum_{i=1}^{r} y_i - ra - b_i \sum_{i=1}^{r} x_i = 0$$

$$\Rightarrow \quad \sum_{i=1}^{r} y_i = ra + b \sum_{i=1}^{r} x_i$$

Dividing by r, we get

$$a + b\frac{\sum_{i=1}^{r} x_i}{r} = \frac{\sum_{i=1}^{r} y_i}{r}$$

$$\Rightarrow \quad a + b\bar{x}_1 = \bar{y}_1 \tag{2}$$

where

$$\bar{x}_1 = \frac{\sum_{i=1}^{r} x_i}{r}, \qquad \bar{y}_1 = \frac{\sum_{i=1}^{r} y_i}{r}$$

Similarly $$\sum_{i=r+1}^{n} d_i = 0$$

$\Rightarrow$ $$\sum_{i=r+1}^{n} [y_i - (a + bx_i)] = 0$$

$\Rightarrow$ $$\sum_{i=r+1}^{n} y_i - \sum_{i=r+1}^{n} a - \sum_{i=r+1}^{n} bx_i = 0$$

$\Rightarrow$ $$\sum_{i=r+1}^{n} y_i = \sum_{i=r+1}^{n} a + b\sum_{i=r+1}^{n} x_i$$

$\Rightarrow$ $$\sum_{i=r+1}^{n} y_i = (n-r)a + b\sum_{i=r+1}^{n} x_i$$

Dividing by $(n-r)$, $$a + b\frac{\sum_{i=r+1}^{n} x_i}{(n-r)} = \frac{\sum_{i=r+1}^{n} y_i}{(n-r)}$$

$\Rightarrow$ $$a + b\bar{x}_2 = \bar{y}_2 \qquad (3)$$

where $$\bar{x}_2 = \frac{\sum_{i=r+1}^{n} x_i}{n-r}, \qquad \bar{y}_2 = \frac{\sum_{i=r+1}^{n} y_i}{n-r}$$

Solving (2) and (3), we get a and b.

Substituting the values of a and b in (1), we get the line fitted to the data.

Note:

(1) Since $\bar{x}_1$ and $\bar{y}_1$ are the average values of x and y in the first group and $\bar{x}_2$, $\bar{y}_2$ are the average values of x and y in the second group, **the method is known as method of averages.**

(2) If we divide the data into two groups in a different way such that the sum of the residuals is zero in each group, then the values of a and b will be different, and so the equation will be different. This is the serious draw back of this method.

(3) Since $y = a + bx$ is satisfied by $(\bar{x}_1, \bar{y}_1)$ and $(\bar{x}_2, \bar{y}_2)$, this line is the line joining the points $(\bar{x}_1, \bar{y}_1)$ and $(\bar{x}_2, \bar{y}_2)$. So the equation of the line fitted to the data is $\dfrac{y-\bar{y}_1}{\bar{y}_2-\bar{y}_1} = \dfrac{x-\bar{x}_1}{\bar{x}_2-\bar{x}_1}$

(4) In practice, we divide the data into two groups which contain the same number of points.

WORKED EXAMPLES

Example 1

Fit a straight line of the form $y = a + bx$ by the method of group averages for the following data:

x	0	5	10	15	20	25
y	12	15	17	22	24	30

Solution

Given 6 sets of values, we divide it into two groups each containing 3 sets of values.

Group I

X	Y
0	12
5	15
10	17
$\Sigma x = 15$	$\Sigma y = 44$

Group II

X	Y
15	22
20	24
25	30
$\Sigma x = 60$	$\Sigma y = 76$

$$\therefore \quad \bar{x}_1 = \frac{15}{3} = 5, \quad \bar{y}_1 = \frac{44}{3} = 14.6666$$

$$\bar{x}_2 = \frac{60}{3} = 20, \quad \bar{y}_2 = \frac{76}{3} = 25.3333$$

$\therefore$ the equation of the line fitted to the data is

$$\frac{y-\bar{y}_1}{\bar{y}_2-\bar{y}_1} = \frac{x-\bar{x}_1}{x_2-\bar{x}_1}$$

$$\Rightarrow \quad \frac{y-14.6666}{25.3333-14.6666} = \frac{x-5}{20-5}$$

$$\Rightarrow \quad \frac{y-14.6666}{10.6667} = \frac{x-5}{15}$$

$$\Rightarrow \quad y-14.6666 = \frac{10.6667}{15}(x-5)$$

$$= 0.7111(x-5)$$

$$\Rightarrow \quad y = 0.7111x - 0.71111 \times 5 + 14.6666$$

$$\Rightarrow \quad \mathbf{y = 0.7111x + 11.1111}$$

Example 2

The weights of a calf taken at weekly intervals are given below. Fit a straight line by method of averages.

Age in weeks	1	2	3	4	5	6	7	8	9	10
Weight	52.5	58.7	65	70.2	75.4	81.1	87.2	95.5	102.2	108.4

Calculate also the average rate of growth per week.

Solution

Let x denote the age in weeks and y denote the corresponding weight.
We divide the given data of 10 values into two groups of 5 values each.

Group I

x	y
1	52.5
2	58.7
3	65.0
4	70.2
5	75.4
$\Sigma x = 15$	$\Sigma y = 321.8$

Group II

x	y
6	81.1
7	87.2
8	95.5
9	102.2
10	108.4
$\Sigma x = 40$	$\Sigma y = 474.4$

$$\bar{x}_1 = \frac{15}{5} = 3, \quad \bar{y}_1 = \frac{321.8}{5} = 64.36$$

$$\bar{x}_2 = \frac{40}{5} = 8, \quad \bar{y}_2 = \frac{474.4}{5} = 94.88$$

Equation of the straight line fitted to data is

$$\frac{y - \bar{y}_1}{\bar{y}_2 - \bar{y}_1} = \frac{x - \bar{x}_1}{\bar{x}_2 - \bar{x}_1}$$

$$\Rightarrow \quad \frac{y - 64.36}{94.88 - 64.36} = \frac{x-3}{8-3}$$

$$\Rightarrow \quad \frac{y - 64.36}{30.52} = \frac{x-3}{5}$$

$$\Rightarrow \quad y - 64.36 = \frac{30.52}{5}(x-3)$$

$$= 6.104(x-3)$$

$$= 6.104x - 18.312$$

$$\therefore \quad y = 6.104x - 18.312 + 64.36$$

$$\Rightarrow \quad \mathbf{y = 6.104x + 46.048}$$

The rate of growth is $\frac{dy}{dx} = \mathbf{6.104}$ ■

Example 3

Experimental values of two connected quantities x, y are given below:

x	1	2	3	4	5	6
y	2.6	5.4	8.7	12.1	16	20.2

If the relation between x and y is $y = ax + bx^2$ where a and b are constants, find the best values of a and b.

Solution

The given equation is $y = ax + bx^2$

$\Rightarrow \quad y = x(a + bx)$

$\Rightarrow \quad \dfrac{y}{x} = a + bx$

Put $Y = \dfrac{y}{x}$ and $X = x$. Then the equation is $Y = a + bX$, which is linear in X and Y.

Divide the given set of 6 values into two groups of 3 values each.

Group I

$X = x$	y	$Y = \dfrac{y}{x}$
1	2.6	2.6
2	5.4	$\dfrac{5.4}{2} = 2.7$
3	8.7	$\dfrac{8.7}{3} = 2.9$
$\Sigma X = 6$		$\Sigma Y = 8.2$

Group II

$X = x$	y	$Y = \dfrac{y}{x}$
4	12.1	$\dfrac{12.1}{4} = 3.025$
5	16	$\dfrac{16}{5} = 3.2$
6	20.2	$\dfrac{20.2}{6} = 3.3667$
$\Sigma X = 15$		$\Sigma Y = 9.5917$

$$\therefore \quad \bar{X}_1 = \frac{6}{3} = 2, \quad \bar{Y}_1 = \frac{8.2}{3} = 2.7333$$

$$\bar{X}_2 = \frac{15}{3} = 5, \quad \bar{Y}_2 = \frac{9.5917}{3} = 3.1972$$

$\therefore$ the linear equation in X and Y is

$$\frac{Y - \bar{Y}_1}{\bar{Y}_2 - \bar{Y}_1} = \frac{X - \bar{X}_1}{\bar{X}_2 - \bar{X}_1}$$

$$\Rightarrow \quad \frac{Y - 2.7333}{3.1972 - 2.7333} = \frac{X - 2}{5 - 2}$$

$$Y - 2.7333 = \frac{0.4639}{3}(X - 2)$$

$$= 0.1546X - 2 \times 0.1546$$

$$\Rightarrow \quad Y = 0.1546X - 0.3093 + 2.7333$$

$$= 0.1546X + 2.424$$

$$\Rightarrow \quad \frac{y}{x} = 0.1546x + 2.424$$

$$\Rightarrow \quad y = 0.1546x^2 + 2.424x$$

$$\therefore \quad \boldsymbol{a = 2.424, \quad b = 0.1546}$$

■

Example 4

The data in the following table fit a formula of the type $y = ax^n$. Find the values of a and n and the formula by the method of group averages.

x	10	20	30	40	50	60	70	80
y	1.06	1.33	1.52	1.68	1.81	1.91	2.01	2.11

Solution

Given $$y = ax^n$$
Taking log to the base 10,
$$\log_{10} y = \log_{10} a + n \log_{10} x$$
Put $Y = \log_{10} y$, $X = \log_{10} x$ and $A = \log_{10} a$

$\therefore$ the equation is $Y = A + nX$, which is linear in X and Y.

Divide the given data into two groups of 4 values each.

Group I

x	$X = \log_{10} x$	y	$Y = \log_{10} y$
10	1	1.06	0.0253
20	1.3010	1.33	0.1239
30	1.4771	1.52	0.1818
40	1.6021	1.68	0.2253
	5.3802		0.5563

Group II

x	$X = \log_{10} x$	y	$Y = \log_{10} y$
50	1.6989	1.81	0.2577
60	1.7782	1.91	0.2810
70	1.8451	2.01	0.3032
80	1.9031	2.11	0.3243
	7.2253		1.1662

$$\overline{X}_1 = \frac{5.3802}{4} = 1.3451, \qquad \overline{Y}_1 = \frac{0.5563}{4} = 0.1391$$
$$\overline{X}_2 = \frac{7.2253}{4} = 1.8063, \qquad \overline{Y}_2 = \frac{1.1662}{4} = 0.2916$$

$\therefore$ the equation in X and Y is
$$\frac{Y - \overline{Y}_1}{\overline{Y}_2 - \overline{Y}_1} = \frac{X - \overline{X}_1}{\overline{X}_2 - \overline{X}_1}$$
$$\Rightarrow \quad \frac{Y - 0.1391}{0.2916 - 0.1391} = \frac{X - 1.3451}{1.8063 - 1.3451}$$
$$\Rightarrow \quad \frac{Y - 0.1391}{0.1525} = \frac{X - 1.3451}{0.4612}$$
$$\Rightarrow \quad Y - 0.1391 = \frac{0.1525}{0.4612}(X - 1.3451) = 0.3307(X - 1.3451)$$
$$\Rightarrow \quad Y = 0.3307X - 0.3307(1.3451) + 0.1391$$

$$\Rightarrow \quad y = 0.3307X - 0.4448 + 0.1391$$
$$= 0.3307X - 0.3057$$
$$\therefore \quad A = -0.3057, \; n = 0.3307$$
$$\Rightarrow \quad \log_{10} a = -0.3057 \Rightarrow a = 10^{-0.3057} = 0.4947$$
$\therefore$ the equation is $\mathbf{y = 0.4947x^{0.3307}}$ ■

Example 5

Fit a curve of the form $y = \dfrac{x}{a+bx}$ to the following data by the method of averages.

x	8	10	15	20	30	40
y	13	14	15.4	16.3	17.2	17.8

Hence find the values of a and b.

Solution

Given $\quad y = \dfrac{x}{a+bx} \quad \Rightarrow \dfrac{1}{y} = \dfrac{a+bx}{x}$

$$\Rightarrow \quad \frac{1}{y} = \frac{a}{x} + b$$

Put $\quad \dfrac{1}{y} = Y$ and $\dfrac{1}{x} = X$

$\therefore$ the equation is $Y = aX + b$, which is linear in x and y.
Divide the given data into two groups of 3 values each.

Group I

x	$X = \frac{1}{x}$	y	$Y = \frac{1}{y}$
8	$\frac{1}{8} = 0.125$	13	$\frac{1}{13} = 0.0769$
10	$\frac{1}{10} = 0.1$	$\frac{1}{14}$	$\frac{1}{14} = 0.0714$
15	$\frac{1}{15} = 0.0667$	15.4	$\frac{1}{15.4} = 0.0649$
	0.29167		0.2132

Group II

x	$X = \frac{1}{x}$	y	$Y = \frac{1}{y}$
20	$\frac{1}{20} = 0.05$	16.3	$\frac{1}{16.3} = 0.0613$
30	$\frac{1}{10} = 0.0333$	17.2	$\frac{1}{17.2} = 0.0581$
40	$\frac{1}{40} = 0.025$	17.8	$\frac{1}{17.8} = 0.05620$
	0.1083		0.1756

$$\overline{X}_1 = \frac{0.29167}{3} = 0.0972, \quad \overline{Y}_1 = \frac{0.2132}{3} = 0.0711$$
$$\overline{X}_2 = \frac{0.1083}{3} = 0.0361, \quad \overline{Y}_2 = \frac{0.1756}{3} = 0.0585$$

$\therefore$ the equation in X and Y is $\dfrac{Y-\overline{Y}_1}{\overline{Y}_2-\overline{Y}_1}=\dfrac{X-\overline{X}_1}{\overline{X}_2-\overline{X}_1}$

$$\Rightarrow \quad \frac{Y-0.0711}{0.0585-0.0711}=\frac{X-0.0972}{0.0361-0.0972}$$

$$\Rightarrow \quad \frac{Y-0.0711}{-0.0126}=\frac{X-0.0972}{-0.0611}$$

$$\Rightarrow \quad Y-0.0711=\frac{0.0126}{0.0611}(X-0.0972)$$

$$=0.2062(X-0.0972)$$

$$=0.2062X-0.2062\times 0.0972$$

$$=0.2062X-0.0200$$

$$\Rightarrow \quad Y=0.2062X+0.0711-0.0200$$

$$\Rightarrow \quad Y=0.2062X+0.0511$$

$$\therefore \quad \mathbf{a=0.2062,\ b=0.0511}$$

Type 2. Equations of the form $y=a+bx+cx^2$

We reduce this equation to the above type and solve.

Let (x_1, y_1) be a particular point on $y=a+bx+cx^2$ satisfying the equation.

Then, $$y_1=a+bx_1+cx_1^2$$

$$\therefore \quad y-y_1=b(x-x_1)+c(x^2-x_1^2)$$

$$=(x-x_1)[b+c(x+x_1)]$$

$$\Rightarrow \quad \frac{y-y_1}{x-x_1}=b+c(x+x_1)$$

$$\text{Put } Y=\frac{y-y_1}{x-x_1},\ X=x+x_1$$

Then, we get $Y=b+cX$, which is linear in X and Y.

We use group average method to find b and c. ■

Example 6

The data given below will fit a formula of the type $y=a+bx+cx^2$. Find the formula.

x	87.5	84.0	77.8	63.7	46.7	36.9
y	292	283	270	235	197	181

Solution

Given $y=a+bx+cx^2$ (1)

Taking (87.5, 292) as a particular point on (1), we get

$$292=a+b(87.5)+c(87.5)^2 \quad (2)$$

(1) – (2) $\Rightarrow$ $$y-292=b[x-87.5]+c[x^2-(87.5)^2]$$

$$=(x-87.5)\,[b+c(x+87.5)]$$

$$\Rightarrow \qquad \frac{y-292}{x-87.5} = b + c(x+87.5)$$

Put $\qquad Y = \dfrac{y-292}{x-87.5}, \qquad X = x+87.5$

$\therefore \qquad Y = b + cX$

To find b and c, we use the group average method.
Divide the given data into two groups of 3 values each.

Group I

x	y	$x-87.5$	$y-292$	$Y=\dfrac{y-292}{x-87.5}$	$X=x+87.5$
87.5	292	0	0	–	–
84.0	283	–3.5	–9	2.5714	171.5
77.8	270	–9.7	–22	2.2680	165.3
				4.8394	336.8

$$\overline{X}_1 = \frac{336.8}{2} = 168.4; \quad \overline{Y}_1 = \frac{4.8394}{2} = 2.4197$$

Group II

x	y	$x-87.5$	$y-292$	$Y=\dfrac{y-292}{x-87.5}$	$X=x+87.5$
63.7	2.35	–23.8	–57	2.3950	151.2
46.7	197	–40.8	–95	2.3284	134.2
36.9	181	–50.6	–111	2.1937	124.4
				6.9171	409.8

$$\overline{X}_2 = \frac{409.8}{3} = 136.6, \quad \overline{Y}_2 = \frac{6.9171}{3} = 2.3057$$

$\therefore$ the equation in X and Y is $\qquad \dfrac{Y-\overline{Y}_1}{\overline{Y}_2-\overline{Y}_1} = \dfrac{X-\overline{X}_1}{\overline{X}_2-\overline{X}_1}$

$$\Rightarrow \qquad \frac{Y-2.4197}{2.3057-2.4197} = \frac{X-168.4}{136.6-168.4}$$

$$\Rightarrow \qquad \frac{Y-2.4197}{-0.114} = \frac{X-168.4}{-31.8}$$

$$\Rightarrow \qquad Y-2.4197 = \frac{0.114}{31.8}(X-168.4)$$

$$= 0.00358(X-168.4)$$

$$= 0.00358X-0.6029$$

$$\Rightarrow \quad Y = 0.00358X+2.4197-0.6029$$

$$\Rightarrow \quad Y = 0.00358X+1.8168$$

$$\frac{y-292}{x-87.5} = 0.00358(x+87.5)+1.8168$$

$$\frac{y-292}{x-87.5} = 0.00358x+2.13005$$

$$y-292 = 0.00358x(x-87.5)+2.13005(x-87.5)$$

$$\mathbf{y = 0.00358x^2 + 1.8168x + 105.62}$$

Exercises 8.2

By the method of group averages fit the indicated curve.

(1) Fit $y = ax + b$ to the data

x	50	70	100	120
y	12	15	21	25

(2) The following numbers relate to the flow of water over a triangular notch:

H	1.2	1.4	1.6	1.8	2.0	2.4
Q	4.2	6.1	8.5	11.5	14.9	23.5

H denotes the head of water (in feet), Q the quantity (in cubic feet) of water flowing per second. If the law is $Q = CH^n$, find the best values of C and n.

[**Hint:** $\log_{10}Q = \log_{10}C + n\log_{10}H$; put $Y = \log_{10}Q$, $A = \log_{10}C$, $X = \log_{10}H$

$\therefore \quad Y = A + nX$]

(3) Fit a curve of the form $y = kx^n$ given:

x	25.9	259	2590	25900
y	0.308	0.209	0.148	0.098

(4) Fit a curve of the form $y = a + bx + cx^2$ for the data:

x	2	2.5	3	3.5	4	4.5	5	5.5	6
y	18	17.8	17.5	17	15.8	14.8	13.3	11.7	9

Answers 8.2

(1) $y = 0.19x + 2.1$

(2) $C = 2.6$, $n = 2.5$

(3) $y = 0.2673x^{-0.0118}$

(4) $y = 15.8 + 2.1x - 0.5x^2$

8.3 METHOD OF THE SUM OF EXPONENTIALS

We shall now discuss another type of curve fitting known as exponential curve fitting.

Let $(x_1,y_1),(x_2,y_2),\ldots(x_n,y_n)$ be the observed data for the variables x and y.

Let
$$y=f(x)=A_1e^{\lambda_1 x}+A_2e^{\lambda_2 x}+\ldots A_ne^{\lambda_n x} \tag{1}$$

be the curve fitted to the data, where $A_1,A_2,\ldots A_n;\lambda_1,\lambda_2,\ldots,\lambda_n$ are constants to be determined. We shall assume n is fixed and is known.

We know that $y=f(x)$ is the solution of the differential equation of the form

$$\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{dy^{n-2}}{dx^{n-2}}+\ldots+a_ny=0. \tag{2}$$

where $a_1,a_2,\ldots,a_n$ are unknown constants and $\lambda_1,\lambda_2,\ldots,\lambda_n$ are the roots of the auxiliary equation

$$\lambda^n+a_1\lambda^{n-1}+\ldots+a_n=0 \tag{3}$$

By Froberg's method (1965), we can compute the derivatives numerically at the n points and substitute in (2) which will result in a system of n linear equations in the n unknowns $a_1,a_2,\ldots,a_n$, which can be solved. Then $\lambda_1,\lambda_2,\ldots,\lambda_n$ can be obtained and finally $A_1,A_2,\ldots,A_n$ can be obtained from (1) by the method of least squares or the method of averages.

Thus $y=A_1e^{\lambda_1 x}+A_2e^{\lambda_2 x}+\ldots+A_ne^{\lambda_n x}$ is fitted to the given data.

Note: The drawback of this method is the unreliability of the results, because of the accuracy of the derivatives decreases as the order n increases.

We shall now discuss a method given by Moore in 1974 which gives more reliable results. For simplicity, we shall consider the case when $n=2$.

Let the curve to be fitted to the data be

$$y=A_1e^{\lambda_1 x}+A_2e^{\lambda_2 x} \tag{4}$$

This function is the solution of the second order differential equation

$$\frac{d^2y}{dx^2}=a_1\frac{dy}{dx}+a_2y \tag{5}$$

where a_1,a_2 are constants to be determined and λ_1,λ_2 are the roots of the auxiliary equation

$$\lambda^2=a_1\lambda+a_2.$$

Let 'a' be the initial value of x.

Integrating (5) w.r.to x, we get

$$\int_a^x \frac{d^2y}{dx^2}dx = a_1\int_a^x \frac{dy}{dx}dx + a_2\int_a^x y dx$$

$$\Rightarrow \quad y'(x) - y'(a) = a_1[y(x) - y(a)] + a_2\int_a^x y dx$$

where $y'(x) = \frac{dy}{dx}$.

Again integrating w.r.to x, we get

$$\int_a^x y'(x)dx - \int_a^x y'(a)dx = a_1\int_a^x y(x)dx - a_1\int_a^x y(a)dx + a_2\int_a^x\int_a^x y\, dxdx$$

$$\Rightarrow \quad y(x) - y(a) - y'(a)(x-a) = a_1\int_a^x y(x)dx - a_1 y(a)(x-a) + a_2\int_a^x\int_a^x y dxdx$$

We know the formula

$$\int_a^x \cdots \int_a^x f(x)\underbrace{dxdx\ldots dx}_{n \text{ times}} = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)dt$$

$$\int_a^x\int_a^x y(x)dxdx = \int_a^x (x-t)y(t)dt$$

$$\therefore \quad y(x) - y(a) - (x-a)y'(a) = a_1\int_a^x y(x)dx - a_1(x-a)y(a) + a_2\int_a^x (x-t)y(t)dt \qquad (6)$$

Let (x_1, y_1) and (x_2, y_2) be two data points such that $a - x_1 = x_2 - a$, then

$$(6) \Rightarrow y(x_1) - y(a) - (x_1 - a)y'(a) = a_1\int_a^{x_1} y(x)dx - a_1(x_1 - a)y(a) + a_2\int_a^{x_1}(x_1 - t)y(t)dt \qquad (7)$$

$$\text{and} \quad y(x_2) - y(a) - (x_2 - a)y'(a) = a_1\int_a^{x_2} y(x)dx - a_1(x_2 - a)y(a) + a_2\int_a^{x_2}(x_2 - t)y(t)dt \qquad (8)$$

$$(7)+(8) \Rightarrow \quad y(x_1) + y(x_2) - 2y(a) = a_1\left[\int_a^{x_1} y(x)dx + \int_a^{x_2} y(x)dx\right]$$

$$+a_2\left[\int_a^{x_1}(x_1 - t)y(t)dt + \int_a^{x_2}(x_2 - t)y(t)dt\right]$$

$[\because a - x_1 = x_2 - a, \text{ other terms cancel out}]$

Taking two different a, x_1, x_2 and evaluating these integrals by numerical method, say Simpson's $\frac{1}{3}$ rule, we get two linear equations in a_1 and a_2.

Solving these equations, we get a_1 and a_2.

Solving $\lambda^2 - a_1\lambda - a_2 = 0$, we get the values of λ_1 and λ_2.

Finally, we find the values of A_1 and A_2 by the method of least squares or averaging method.

Thus $y = A_1e^{\lambda_1 x} + A_2e^{\lambda_2 x}$ is fitted to the data.

WORKED EXAMPLES

Example 1

Fit a function of the form $y = A_1 e^{\lambda_1 x} + A_2 e^{\lambda_2 x}$ to the following data.

x	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
y	1.175	1.336	1.510	1.698	1.904	2.129	2.376	2.646	2.942

Solution

The curve $y = A_1 e^{\lambda_1 x} + A_2 e^{\lambda_2 x}$ fitted to the given data is the solution of the differential equation

$$\frac{d^2 y}{dx^2} = a_1 \frac{dy}{dx} + a_2 y$$

The auxiliary equation is

$$\lambda^2 = a_1 \lambda + a_2 \tag{1}$$

Let λ_1, λ_2 be the roots of the auxiliary equation.

The coefficients a_1, a_2 are given by

$$y(x_1) + y(x_2) - 2y(a) = a_1 \left[\int_a^{x_1} y(x)\,dx + \int_a^{x_2} y(x)\,dx \right] + a_2 \left[\int_a^{x_1} (x_1 - t) y(t)\,dt + \int_a^{x_2} (x_2 - t) y(t)\,dt \right] \tag{2}$$

where a is the initial value of x and x_1 and x_2 are two data points such that

$$a - x_1 = x_2 - a \quad \Rightarrow a = \frac{x_1 + x_2}{2}$$

To evaluate the integrals, we use Simpson's $\frac{1}{3}$ rule.

So, we choose x_1, x_2 such that there are even number of intervals.

$\therefore$ choose $x_1 = 1$ and $x_2 = 1.4$

$$\therefore \qquad a = \frac{1 + 1.4}{2} = \frac{2.4}{2} = 1.2$$

Substituting in (2), we get

$$y(1)+y(1.4)-2y(1.2)=a_1\left[\int_{1.2}^{1}y(x)dx+\int_{1.2}^{1.4}y(x)dx\right]$$
$$+a_2\left[\int_{1.2}^{1}(1-t)y(t)dt+\int_{1.2}^{1.4}(1.4-t)y(t)dt\right]$$

The given table is

x	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
y	1.175	1.336	1.510	1.698	1.904	2.129	2.376	2.646	2.942

$$\therefore \quad 1.175+1.904-2(1.510)=a_1\left[-\int_{1}^{1.2}y(x)dx+\int_{1.2}^{1.4}y(x)dx\right]$$
$$+a_2\left[-\int_{1}^{1.2}(1-t)y(t)dt+\int_{1.2}^{1.4}(1.4-t)y(t)dt\right]$$

$$\Rightarrow \quad 0.059=a_1\left[-\int_{1}^{1.2}y(x)dx+\int_{1.2}^{1.4}y(x)dx\right]$$
$$+a_2\left[-\int_{1}^{1.2}(1-t)y(t)dt+\int_{1.2}^{1.4}(1.4-t)y(t)dt\right]$$

We shall evaluate these integrals by Simpson's $\frac{1}{3}$ rule

$$\therefore \quad \int_{1}^{1.2}y(x)dx=\frac{0.1}{3}[1.175+1.510+4(1.336)]$$
$$=0.2676$$

x	1	1.1	1.2
y(x)	1.175	1.336	1.510

$$\int_{1.2}^{1.4}y(x)dx=\frac{0.1}{3}[1.510+1.904+4(1.698)]$$
$$=0.3402$$

x	1.2	1.3	1.4
y	1.510	1.698	1.904

$$\int_{1}^{1.2}(1-t)y(t)dt=\frac{0.1}{3}[0-0.302+4(-0.1336)]$$
$$=-0.0279$$

t	1	1.1	1.2
(1 − t)y(t)	0	−0.1336	−0.302

$$\therefore \quad \int_{1.2}^{1.4}(1.4-t)y(t)dt=\frac{0.1}{3}[0.302+0+4(0.1698)]$$
$$=0.0327$$

t	1.2	1.3	1.4
(1.4 − t)y(t)	0.302	0.1698	0

$$\therefore \quad 0.059=a_1[-0.2676+0.3402]+a_2[0.0279+0.0327]$$
$$\Rightarrow \quad 0.0726a_1+0.0606a_2=0.059$$
$$\Rightarrow \quad 726a_1+606a_2=590$$
$$\therefore \quad 1.1980a_1+a_2=0.9736 \quad (3)$$

[Dividing by 606]

Now take $x_1 = 1.4$ and $x_2 = 1.8$. We get $a = \frac{1.4+1.8}{2} = 1.6$

$$y(1.4)+y(1.8)-2y(1.6) = a_1\left[\int_{1.6}^{1.4} y(x)\,dx + \int_{1.4}^{1.8} y(x)\,dx\right]$$
$$+a_2\left[\int_{1.6}^{1.4}(1.4-t)y(t)dt + \int_{1.4}^{1.8}(1.8-t)y(t)dt\right]$$

$$\Rightarrow \quad 1.904+2.942-2(2.376) = a_1\left[-\int_{1.4}^{1.6} y(x)dx + \int_{1.4}^{1.8} y(x)dx\right]$$
$$+a_2\left[-\int_{1.4}^{1.6}(1.4-t)y(t)dt + \int_{1.4}^{1.8}(1.8-t)y(t)dt\right]$$

$$\Rightarrow \quad 0.094 = a_1\left[-\int_{1.4}^{1.6} y(x)dx + \int_{1.4}^{1.8} y(x)dx\right] + a_2\left[-\int_{1.4}^{1.6}(1.4-t)y(t)dt + \int_{1.4}^{1.8}(1.8-t)y(t)dt\right]$$

We shall evaluate these integrals by Simpson's $\frac{1}{3}$ rule.

$$\therefore \quad \int_{1.4}^{1.6} y(x)dx = \frac{0.1}{3}[1.904+2.376+4(2.129)]$$
$$= \frac{0.1}{3}[12.796] = 0.4265$$

x	1.4	1.5	1.6
$y(x)$	1.904	2.129	2.376

$$\int_{1.6}^{1.8} y(x)dx = \frac{0.1}{3}[2.376+2.942+4(2.646)]$$
$$= \frac{0.1}{3}[15.902]$$
$$= 0.5301$$

x	1.6	1.7	1.8
$y(x)$	2.376	2.646	2.942

$$\int_{1.4}^{1.6}(1.4-t)y(t)\,dt = \frac{0.1}{3}[0-0.4752+4(-0.2129)]$$
$$= \frac{0.1}{3}[-1.3268]$$
$$= -0.0442$$

t	1.4	1.5	1.6
$(1.4-t)y(t)$	0	-0.2129	-0.4752

$$\int_{1.6}^{1.8}(1.8-t)y(t)\,dt = \frac{0.1}{3}[0.4752+0+4(0.2646)]$$
$$= \frac{0.1}{3}[1.5336]$$
$$= 0.0511$$

t	1.6	1.7	1.8
$(1.8-t)y(t)$	0.4752	0.2646	0

$$\therefore \quad 0.094 = a_1[-0.4265+0.5301] + a_2[0.0442+0.0511]$$
$$= 0.1036a_1 + 0.0953a_2$$

$$\Rightarrow 0.1036a_1 + 0.0953a_2 = 0.094$$

$$\Rightarrow \quad 103.6a_1 + 95.3a_2 = 94$$

$$\Rightarrow \quad 1.0871a_1 + a_2 = 0.9864 \qquad (4)$$

[Dividing by 95.3]

$(3)-(4) \Rightarrow \quad 0.1109a_1 = -0.0128$

$$\Rightarrow \quad a_1 = \frac{-0.0128}{0.1109} = -0.1154$$

$$\therefore \quad a_2 = 0.9736 - 1.1980(-0.1154) = 1.1118$$

Substituting in (1), we get

$$\lambda^2 = -0.1154\lambda + 1.1118$$

$$\Rightarrow \quad \lambda^2 + 0.1154\lambda - 1.1118 = 0$$

$$\Rightarrow \quad \lambda = \frac{-0.1154 \pm \sqrt{(0.1154)^2 - 4(-1.1118)}}{2}$$

$$= \frac{-0.1154 \pm 2.1120}{2}$$

$$= 0.9983, -1.1137$$

We take $\lambda_1 = 1$ and $\lambda_2 = -1.1$

$$\therefore \quad y = A_1 e^{x} + A_2 e^{-1.1x}$$

To find A_1 and A_2, we use the method of least squares

The sum of the squares of the residuals is

$$E = \sum_i \left[y_i - \left(A_1 e^{x_i} + A_2 e^{-1.1x_i} \right) \right]^2$$

According to the method of least squares E is minimum if $\frac{\partial E}{\partial A_1} = 0$ and $\frac{\partial E}{\partial A_2} = 0$

$$\therefore \quad \frac{\partial E}{\partial A_1} = 0$$

$$\Rightarrow \quad \sum_i 2\left[y_i - \left(A_1 e^{x_i} + A_2 e^{-1.1x_i} \right) \right](-e^{x_i}) = 0$$

$$\Rightarrow \quad \sum_i \left[y_i - \left(A_1 e^{x_i} + A_2 e^{-1.1x_i} \right) \right] e^{x_i} = 0$$

$$\Rightarrow \quad \sum_i \left[y_i e^{x_i} - \left(A_1 e^{2x_i} + A_2 e^{-0.1x_i} \right) \right] = 0$$

$$\Rightarrow \quad \sum_i y_i e^{x_i} - A_1 \sum_i e^{2x_i} - A_2 \sum_i e^{-0.1x_i} = 0$$

$$\Rightarrow \quad \sum_i y_i e^{x_i} = A_1 \sum_i e^{2x_i} + A_2 \sum_i e^{-0.1x_i} \qquad (5)$$

and
$$\frac{\partial E}{\partial A_2} = 0$$

$\Rightarrow$
$$\sum_i 2\left[y_i - \left(A_1 e^{x_i} + A_2 e^{-1.1x_i}\right)\right]\left[-e^{-1.1x_i}\right] = 0$$

$\Rightarrow$
$$\sum_i \left[y_i - \left(A_1 e^{x_i} + A_2 e^{-1.1x_i}\right)\right] e^{-1.1x_i} = 0$$

$\Rightarrow$
$$\sum_i \left[y_i e^{-1.1x_i} - A_1 e^{(-0.1)x_i} - A_2 e^{(-2.2)x_i}\right] = 0$$

$\Rightarrow$
$$\sum_i y_i e^{-1.1x_i} - A_1 \sum_i e^{(-0.1)x_i} - A_2 \sum_i e^{(-2.2)x_i} = 0$$

$\Rightarrow$
$$\sum_i y_i e^{-1.1x_i} = A_1 \sum_i e^{(-0.1)x_i} + A_2 \sum_i e^{(-2.2)x_i} \tag{6}$$

The equations (5) and (6) are called the normal equations.
Now we shall form the table for computation

x_i	y_i	e^{x_i}	$e^{(-1.1)x_i}$	$y_i e^{x_i}$	$y_i e^{(-1.1)x_i}$	$e^{(-0.1)x_i}$	e^{2x_i}	$e^{(-2.2)x_i}$
1	1.175	2.7183	0.3329	3.1940	0.3912	0.9048	7.3892	0.1108
1.1	1.336	3.0042	0.2982	4.0136	0.3984	0.8958	9.0250	0.0889
1.2	1.510	3.3201	0.2671	5.0134	0.4033	0.8869	11.0232	0.0714
1.3	1.698	3.6693	0.2393	6.2305	0.4063	0.8780	13.4637	0.0573
1.4	1.904	4.0552	0.2144	7.7211	0.4082	0.8694	16.4446	0.0460
1.5	2.129	4.4817	0.1920	9.5415	0.4088	0.8607	20.0855	0.0369
1.6	2.376	4.9530	0.1720	11.7683	0.4087	0.8521	24.5325	0.0296
1.7	2.646	5.4739	0.1541	14.4839	0.4077	0.8437	29.9641	0.0238
1.8	2.942	6.0496	0.1381	17.7979	0.4063	0.8353	36.5982	0.0191
				79.7642	3.6389	7.8267	168.526	0.4838

Substituting in (5) and (6), we get

$$79.7642 = A_1 \times 168.526 + A_2 \times 7.8267$$

Dividing by 7.8267, we get

$$21.5322 A_1 + A_2 = 10.1913 \tag{7}$$

and
$$3.6389 = A_1(7.8267) + A_2(0.4838)$$

Dividing by 0.4838, we get,

$$16.1776 A_1 \div A_2 = 7.5215 \tag{8}$$

$(7)-(8) \Rightarrow \qquad 5.3556A_1 = 2.6698$

$\Rightarrow \qquad A_1 = 0.4985$

Substituting in (7), we get

$$A_2 = 10.1913 - 21.5321(0.4985)$$
$$= 10.1913 - 10.7338$$

$\Rightarrow \qquad A_2 = -0.5425$

$\therefore \qquad y = 0.4985e^{x} - 0.5425e^{-1.1x}$

which is the equation of the curve fitted to the given data. ■

Example 2

Fit a curve of the form whose equation is $y = A_1e^{\lambda_1 x} + A_2e^{\lambda_2 x}$, using the following data.

x	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
y	1.54	1.67	1.81	1.97	2.15	2.35	2.58	2.83	3.11

Solution

The curve given by $y = A_1e^{\lambda_1 x} + A_2e^{\lambda_2 x}$ fitted to the data is the solution of the differential equation

$$\frac{d^2y}{dx^2} = a_1\frac{dy}{dx} + a_2 y$$

The auxiliary equation is

$$\lambda^2 = a_1\lambda + a_2 \tag{1}$$

Let λ_1, λ_2 be the root of the auxiliary equation
The co-efficients a_1, a_2 are given by

$$y(x_1) + y(x_2) - 2y(a) = a_1\left[\int_a^{x_1} y(x)\,dx + \int_a^{x_2} y(x)\,dx\right]$$
$$+ a_2\left[\int_a^{x_1}(x_1 - t)\,y(t)\,dt + \int_a^{x_2}(x_2 - t)\,y(t)\,dt\right] \tag{2}$$

Where a is the value of x and x_1, x_2 are two data points such that

$$a - x_1 = x_2 - a$$

$\Rightarrow \qquad a = \dfrac{x_1 + x_2}{2}$

To evaluate the integrals, we use Simpson's $\frac{1}{3}$ rule.

So, we choose x_1, x_2 in such way that there are even number of intervals between x_1 and x_2 and the last value of x

$\therefore$ choose $x_1 = 1$ and $x_2 = 1.4$

$$\therefore \qquad a = \frac{1+1.4}{2} = \frac{2.4}{2} = 1.2$$

Substituting in (2), we get

$$y(1) + y(1.4) - 2y(1.2) = a_1\left[\int_{1.2}^{1} y(x)\,dx + \int_{1.2}^{1.4} y(x)\,dx\right]$$
$$+ a_2\left[\int_{1.2}^{1} (1-t)y(t)\,dt + \int_{1.2}^{1.4} (1.4-t)y(t)\,dt\right]$$

The given data is

x	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
y	1.54	1.67	1.81	1.97	2.15	2.35	2.58	2.83	3.11

$$\therefore \quad 1.54 + 2.15 - 2 \times 1.81 = a_1\left[-\int_1^{1.2} y(x)\,dx + \int_{1.2}^{1.4} y(x)\,dx\right]$$
$$+ a_2\left[-\int_1^{1.2}(1-t)y(t)\,dt + \int_{1.2}^{1.4}(1.4-t)y(t)\,dt\right]$$

$$\Rightarrow \quad 0.07 = a_1\left[-\int_1^{1.2} y(x) + \int_{1.2}^{1.4} y(x)\,dx\right]$$
$$+ a_2\left[-\int_1^{1.2}(1-t)y(t)\,dt + \int_{1.2}^{1.4}(1.4-t)y(t)\,dt\right]$$

We shall evaluate these integrals by Simpson's $\frac{1}{3}$ rule.

$$\therefore \quad \int_1^{1.2} y(x)\,dx = \frac{0.1}{3}[1.54 + 1.81 + 4 \times 1.67]$$
$$= \frac{0.1}{3}[10.03] = 0.3343$$

x	1	1.1	1.2
y(x)	1.54	1.67	1.81

$$\int_{1.2}^{1.4} y(x)\,dx = \frac{0.1}{3}[1.81 + 2.15 + 4 \times 1.97]$$
$$= \frac{0.1}{3}[11.84] = 0.3947$$

x	1.2	1.3	1.4
y(x)	1.81	1.97	2.15

$$\int_1^{1.2}(1-t)y(t)\,dt = \frac{0.1}{3}[0 - 0.362 + 4(-0.167)]$$
$$= \frac{0.1}{3}[-1.03] = -0.0343$$

t	1	1.1	1.2
(1 − t)y(t)	0	–0.167	–0.362

$$\int_{1.2}^{1.4}(1.4-t)y(t)\,dt = \frac{0.1}{3}[0.362 + 0 + 4(0.197)]$$
$$= \frac{0.1}{3}[1.15] = 0.0383$$

t	1.2	1.3	1.4
(1.4 − t)y(t)	0.362	0.197	0

$$\therefore \quad 0.07 = a_1[-0.3343+0.3947] + a_2[0.0343+0.0383]$$

$$\Rightarrow \quad 0.0604\, a_1 + 0.0726\, a_2 = 0.07$$

Dividing by 0.0604, we get

$$a_1 + 1.2020\, a_2 = 1.1589 \quad (3)$$

Now take $x_1 = 1.4$ and $x_2 = 1.8$

$$\therefore \quad a = \frac{1.4+1.8}{2} = 1.6$$

Substituting in (2), we get

$$y(1.4)+y(1.8)-2y(1.6) = a_1\left[\int_{1.6}^{1.4} y(x)dx + \int_{1.6}^{1.8} y(x)dx\right] + a_2\left[\int_{1.6}^{1.4}(1.4-t)y(t)dt + \int_{1.6}^{1.8}(1.8-t)y(t)dt\right]$$

$$\Rightarrow \quad 2.15+3.11-2\times 2.58 = a_1\left[-\int_{1.4}^{1.6} y(x)dx + \int_{1.6}^{1.8} y(x)dx\right] + a_2\left[-\int_{1.4}^{1.6}(1.4-t)y(t)dt + \int_{1.6}^{1.8}(1.8-t)y(t)dt\right]$$

$$0.1 = a_1\left[-\int_{1.4}^{1.6} y(x)dx + \int_{1.6}^{1.8} y(x)dx\right] + a_2\left[-\int_{1.4}^{1.6}(1.4-t)y(t)dt + \int_{1.6}^{1.8}(1.8-t)y(t)dt\right]$$

We shall evaluate these integrals by Simpson's $\frac{1}{3}$ rule

$$\therefore \quad \int_{1.4}^{1.6} y(x)dx = \frac{0.1}{3}[2.15+2.58+4\times 2.35] = \frac{0.1}{3}[14.13] = 0.471$$

x	1.4	1.5	1.6
$y(x)$	2.15	2.35	2.58

$$\int_{1.6}^{1.8} y(x)dx = \frac{0.1}{3}[2.58+3.11+4\times 2.83] = \frac{0.1}{3}[17.01] = 0.567$$

x	1.6	1.7	1.8
$y(x)$	2.58	2.83	3.11

$$\int_{1.4}^{1.6}(1.4-t)y(t)dt = \frac{0.1}{3}[0-0.516+4(-0.235)] = \frac{0.1}{3}[-1.456] = -0.0485$$

t	1.4	1.5	1.6
$(1.4-t)y(t)$	0	-0.235	-0.516

$$\int_{1.6}^{1.8}(1.8-t)y(t)dt = \frac{0.1}{3}[0.516+0+4(0.283)]$$

t	1.6	1.7	1.8
$(1.8-t)y(t)$	0.516	0.283	0

$$= \frac{0.1}{3}[1.648] = 0.0549$$

$\therefore$ $$0.1 = a_1[-0.471+0.567] + a_2[-(-0.0485)+0.0549]$$

$\Rightarrow 0.096a_1 + 0.1034a_2 = 0.1$

Dividing by 0.096, we get

$$a_1 + 1.0771a_2 = 1.0417 \qquad (4)$$

$(3)-(4) \Rightarrow$ $$0.1249a_2 = 0.1172$$

$\Rightarrow$ $$a_2 = \frac{0.1172}{0.1249} = 0.9384$$

Substituting in (4), we get

$\Rightarrow$ $$a_1 = 1.0417 - 1.0771 \times 0.9384 = 0.0309$$

Substituting in (1), we get

$$\lambda^2 = 0.0309\lambda + 0.9384$$

$\Rightarrow$ $$\lambda^2 - 0.0309\lambda - 0.9384 = 0$$

$\Rightarrow$ $$\lambda = \frac{0.0309 \pm \sqrt{(0.0309)^2 - 4(-0.9384)}}{2}$$

$$= \frac{0.0309 \pm 1.9377}{2}$$

$$= 0.9843, -0.9534$$

We take $\lambda_1 = 0.98$ and $\lambda_2 = -0.95$

$\therefore$ $$y = A_1 e^{0.98x} + A_2 e^{-0.95x}$$

We shall now find A_1, A_2 by the method of least squares.
The sum of the squares of the residuals is

$$E = \sum_i \left[y_i - \left(A_1 e^{0.98x_i} + A_2 e^{-0.95x_i}\right)\right]^2$$

According to the method of least squares E is minimum for the best fitted curve.

The normal equations are given by $\frac{\partial E}{\partial A_1} = 0$ and $\frac{\partial E}{\partial A_2} = 0$.

$$\frac{\partial E}{\partial A_1} = 0$$

$$\Rightarrow \quad \sum_i 2\left[y_i - \left(A_1 e^{0.98x_i} + A_2 e^{-0.95x_i}\right)\right]\left[-e^{0.98x_i}\right] = 0$$

$$\Rightarrow \quad \sum_i \left[y_i - \left(A_1 e^{0.98x_i} + A_2 e^{-0.95x_i}\right)\right] e^{0.98x_i} = 0 \qquad \text{[Dividing by } -2]$$

$$\Rightarrow \quad \sum_i \left[y_i\, e^{0.98x_i} - A_1 e^{e^{1.96x_i}} - A_2 e^{0.03x_i}\right] = 0$$

$$\Rightarrow \quad \sum_i y_i\, e^{0.98x_i} - A_1 \sum_i e^{1.96x_i} - A_2 \sum_i e^{0.03x_i} = 0$$

$$\Rightarrow \quad \sum_i y_i\, e^{0.98x_i} = A_1 \sum_i e^{1.96x_i} + A_2 \sum_i e^{0.03x_i} \qquad (5)$$

and

$$\frac{\partial E}{\partial A_2} = 0$$

$$\Rightarrow \quad \sum_i \left\{2\left[y_i - \left(A_1 e^{0.98x_i} + A_2 e^{-0.95x_i}\right)\right]\left(-e^{0.95x_i}\right) = 0\right\}$$

$$\Rightarrow \quad \sum_i \left[y_i - \left(A_1 e^{0.98x_i} + A_2 e^{-0.95x_i}\right)\right] e^{0.95x_i} = 0$$

$$\Rightarrow \quad \sum_i \left[y_i\, e^{-0.95x_i} - A_1 e^{e^{0.03x_i}} - A_2 e^{-1.9x_i}\right] = 0$$

$$\Rightarrow \quad \sum_i y_i\, e^{-0.95x_i} - A_1 \sum_i e^{0.03x_i} - A_2 \sum_i e^{-1.9x_i} = 0$$

$$\Rightarrow \quad \sum_i y_i\, e^{-0.95x_i} = A_1 \sum_i e^{0.03x_i} + A_2 \sum_i e^{-1.9x_i} \qquad (6)$$

x_i	y_i	$e^{0.98x_i}$	$e^{-0.95x_i}$	$y_i e^{0.98x_i}$	$y_i\, e^{-0.95x_i}$	$e^{1.96x_i}$	$e^{0.03x_i}$	$e^{-1.9x_i}$
1	1.54	2.6645	0.3867	4.1033	0.5955	7.0993	1.0305	0.1496
1.1	1.67	2.9388	0.3517	4.9078	0.5873	8.6365	1.0336	0.1237
1.2	1.81	3.2414	0.3198	5.8669	0.5788	10.5066	1.0367	0.1023
1.3	1.97	3.5751	0.2908	7.0429	0.5729	12.7815	1.0398	0.0846
1.4	2.15	3.9432	0.2645	8.4779	0.5687	15.5491	1.0429	0.0699
1.5	2.35	4.3492	0.2405	10.2206	0.5652	18.9158	1.0460	0.0578
1.6	2.58	4.7970	0.2187	12.3763	0.5642	23.0116	1.0492	0.0478
1.7	2.83	5.2910	0.1989	14.9735	0.5629	27.9943	1.0523	0.0396
1.8	3.11	5.8357	0.1809	18.1490	0.5626	34.0558	1.0555	0.0327
Total				86.1182	5.1581	158.5505	9.3865	0.708

Substituting in the normal equations (5) and (6), we get

$$86.1182 = A_1(158.5505) + A_2(9.3865)$$

Dividing by 9.3685, we get

$$16.8913\,A_1 + A_2 = 9.1747 \qquad (7)$$

and

$$5.1581 = A_1(9.3865) + A_2(0.708)$$

Dividing by 0.708, we get

$$13.2578\,A_1 + A_2 = 7.2855 \qquad (8)$$

$(7)-(8) \Rightarrow \quad 3.6335 A_1 = 1.8892$

$\Rightarrow \quad A_1 = \frac{1.8892}{3.6335} = 0.5199$

Substituting in (8), we get $A_2 = 7.2855 - 13.2578(0.5199)$

$= 7.2855 - 6.8927 = 0.3928$

Take $A_1 = 0.52$, and $A_2 = 0.39$

$\therefore \quad \mathbf{y = 0.52e^{0.98x} + 0.39e^{-0.95x}}$

Which is the equation of the curve fitted to the given data. ■

Exercise 8.3

(1) Fit a curve of the form $y = A_1e^{\lambda_1 x} + A_2e^{\lambda_2 x}$ for the following data

x	2	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6
y	3.63	4.46	5.47	6.70	8.19	10.02	12.25	14.97	18.29

Answer 8.3

(1) $y = -0.51e^{-x} + 0.50e^{x}$

METHOD OF MOMENTS

Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be a set of n observations of the related variables x and y.

Let the values of x be equally spaced with interval $h = \Delta x$, say

ie. $x_{i+1} - x_i = \Delta x$ for all $i = 1, 2, 3, \ldots, n-1$.

For such a set of points, we define

the first moment $\quad \mu_1 = \Sigma y\,\Delta x = \Delta x \Sigma y$

the second moment $\quad \mu_2 = \Sigma\,xy\,\Delta x = \Delta x \Sigma xy$

the third moment $\quad \mu_3 = \Sigma x^2 y\,\Delta x = \Delta x \Sigma x^2 y$ and so on.

These moments are called the moments of the observed values of y.

Let $y = f(x)$be the curve to be fitted the data.

Then, the first moment $\gamma_1 = \int y\, dx = \int f(x)\, dx$

the second moment $\gamma_2 = \int xy\, dx = \int x f(x)\, dx$

and the third moment $\gamma_3 = \int x^2 y\, dx = \int x^2 f(x)\, dx$ and so on.

These moments are called the moments of the computed values of y or expected values of y.

The method of moments is based on the assumption that the moments of the observed values of y is equal to the moments of the expected values of y.

i. e. $\mu_i = r_i\ \forall i$

It can be proved that

$$\Delta x \Sigma y = \int_{x_1 - \frac{\Delta x}{2}}^{x_n + \frac{\Delta x}{2}} f(x) dx$$

$$\Delta x \Sigma xy = \int_{x_1 - \frac{\Delta x}{2}}^{x_n + \frac{\Delta x}{2}} x f(x) dx$$

$$\Delta x^2 \Sigma xy = \int_{x_1 - \frac{\Delta x}{2}}^{x_n + \frac{\Delta x}{2}} x^2 f(x) dx \text{ and so on.}$$

These equations are known as **observation equations.**

If the form of $f(x)$ is known, then using the observation equations the constants can be determined.

If $f(x)$ is linear ie. $f(x) = ax + b$, then the equation of the curve fitted is $y = ax + b$

If $\quad a = x_1 - \dfrac{\Delta x}{2}$ and $b = x_n + \dfrac{\Delta x}{2}$, then $\Delta x \Sigma y = \int_{\alpha}^{\beta} (ax + b) dx$

and $\quad \Delta x\ \Sigma xy = \int_{\alpha}^{\beta} x(ax + b) dx$

then we get, $\quad \Delta x\ \Sigma y = \dfrac{a}{2}(\beta^2 - \alpha^2) + b(\beta - \alpha)$

Solving these two equations we find a and b and hence the equation $y = ax+b$

WORKED EXAMPLES

Example 1

By the method of moments, obtain a straight line to fit the data.

x	1	2	3	4
y	0.30	0.64	1.32	5.40

Solution

We fit a straight line to the given data by the method of moments.
Given values of x are equally spaced with $\Delta x = 1$.
Here $x_1 = 1$, $x_n = 4$

$$\therefore \quad \alpha = x_1 - \frac{\Delta x}{2} = 1 - \frac{1}{2} = \frac{1}{2} \text{ and } \beta = x_n + \frac{\Delta x}{2} = 4 + \frac{1}{2} = \frac{9}{2}$$

Let the line fitted be $\quad y = ax + b \quad$ (1)

Then, the observation equations are

$$\frac{a}{2}(\beta^2 - \alpha^2) + b(\beta - \alpha) = \Delta x.\Sigma y \quad (2)$$

$$\frac{a}{3}(\beta^3 - \alpha^3) + \frac{b}{2}(\beta^2 - \alpha^2) = \Delta x.\Sigma xy \quad (3)$$

Now $\quad \Sigma y = 0.30 + 0.64 + 1.32 + 5.40 = 7.66$

and $\quad \Sigma xy = 1(.30) + 2(0.64) + 3(1.32) + 4(5.40) = 27.14$

Substituting in (2) and (3) we get

$$\frac{a}{2}\left[\frac{81}{4} - \frac{1}{4}\right] + b\left[\frac{9}{2} - \frac{1}{2}\right] = 7.66$$

$$\Rightarrow \quad 10a + 4b = 7.66 \quad (4)$$

$$\Rightarrow \quad 2.5a + b = 1.915$$

and $$\frac{a}{3}\left[\frac{729}{8} - \frac{1}{8}\right] + \frac{b}{2}\left[\frac{81}{4} - \frac{1}{4}\right] = 27.14$$

$$30.333a + 10b = 27.14 \quad (5)$$

Dividing by 10, $\quad 3.0333a + b = 2.714$

(5) – (4) $\Rightarrow \quad a = \frac{0.799}{0.533} = 1.499$

$$\therefore \quad b = 1.915 - 2.5(1.499) = -1.8325$$

$\therefore$ the equation of the straight line is $\mathbf{y = 1.499x - 1.8325}$

Example 2

By method of moments fit a straight line to the data:

x	1	2	3	4
y	0.17	0.18	0.23	0.32

Solution

We fit a straight line to the given data by the method of moments.
Given values of x are equally spaced with $\Delta x = 1$. Here $x_1 = 1$, $x_n = 4$

$$\therefore \quad \alpha = 1 - \frac{1}{2} = \frac{1}{2} \text{ and } \beta = 4 + \frac{1}{2} = \frac{9}{2}$$

Let the line fitted be $y = ax + b$ (1)

Then, the observation equations are (2)

$$\frac{a}{2}\left(\beta^2 - \alpha^2\right) + b(\beta - \alpha) = \Delta x \ \Sigma y \quad (3)$$

$$\frac{a}{3}\left(\beta^3 - \alpha^3\right) + \frac{b}{2}\left(\beta^2 - \alpha^2\right) = \Delta x \Sigma xy$$

$$\Sigma y = 0.17 + 0.18 + 0.23 + 0.32 = 0.9$$

Now $\Sigma xy = 0.17 + 2(0.18) + 3(0.23) + 4(0.32) = 2.5$

Substituting in (2),

$$\frac{a}{2}\left[\frac{81}{4} - \frac{1}{4}\right] + b\left[\frac{9}{2} - \frac{1}{2}\right] = 0.9$$

$$10a + 4b = 0.9 \quad (4)$$

$\Rightarrow$ $2.5a + b = 0.225$

$\Rightarrow$

Substituting in (3),

$$\frac{a}{3}\left[\frac{729}{8} - \frac{1}{8}\right] + \frac{b}{2}\left[\frac{81}{4} - \frac{1}{4}\right] = 2.5$$

$$30.3333a + 10b = 2.5 \quad (5)$$

$\Rightarrow$ $3.0333a + b = 0.25$

$\Rightarrow$

(5) − (4) $\Rightarrow$ $0.5333a = 0.025$

$\Rightarrow$ $a = \dfrac{0.025}{0.5333} = 0.047$

$b = 0.225 - 2.5(0.047) = 0.1075$

$\therefore$

$\therefore$ the equation of the straight line is $\mathbf{y = 0.047x + 0.1075}$

Exercises 8.4

(1) Fit a straight line by the method of moments to the following data:

x	1	2	3	4
y	16	19	23	26

(2) Fit a straight line by the method of moments to the following data:

x	1	3	5	7	9
y	1.5	2.8	4.0	4.7	6.0

(3) Fit a straight line by the method of moments to the following data:

x	1	2	3	4
y	1.7	1.8	2.3	3.2

CHAPTER

9

Initial Value Problems for Ordinary Differential Equations

9.1 INTRODUCTION

10-05-19

Many problems in science and engineering can be reduced to the problem of solving differential equations satisfying certain conditions. For example, spring-mass systems, resistor-capacitor-inductance circuits, bending of beams, chemical reactions, pendulums, simple harmonic motions, the motion of a rotating mass and so on. Using analytical methods we can solve only standard types of differential equations. But the differential equations appearing in practical problems of engineering and science are not of the standard types and are complex. So, it is rather impossible to find closed form solutions. Such equations are solved by numerical methods where the solution is given as a table of values of the function at various values of the independent variable. This is called point wise solution. Such methods are called pointwise methods or single step methods because they use y_n at $x = x_n$ to find y_{n+1} at $x = x_{n+1}$. In contrast we have methods like Milne's and Adam's which use $y_n, y_{n-1}, \ldots$ to find y_{n+1}. So they are called multistep methods.

We shall consider some of the methods commonly used to solve differential equations.

9.2 TAYLOR'S SERIES METHOD

Consider the first order differential equation $\boxed{\dfrac{dy}{dx} = f(x, y)}$ with initial condition $\boxed{y(x_0) = y_0}$.

We can expand $y(x)$ as a power series in $(x - x_0)$ in the neighbourhood of x_0 by Taylor's series.

$\rightarrow y(x = x_0 + x_1) =$

$\therefore$ $$y(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \frac{y'''(x_0)}{3!}(x - x_0)^3 + \ldots$$

Put $$h = x - x_0 \quad \therefore x = x_0 + h$$

$\therefore$ $$y(x) = y(x_0) + hy'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \ldots$$

$$y(x) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots$$

If $x = x_1 = x_0 + h$, then

$$y(x_1) = y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots$$

Now y can be expanded as Taylor's series about $x = x_1$ and we have

$$y(x_1 + h) = y(x_2) = y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \ldots$$

Continuing this way, we find

$$y_{r+1} = y_r + \frac{h}{1!}y_r' + \frac{h^2}{2!}y_r'' + \frac{h^3}{3!}y_r''' + \ldots$$

where $y_{r+1} = y(x_{r+1})$ $y_{r+1} = y(x_{r+1})$, $r = 1, 2, 3, \ldots$

The solution $y(x)$ is given as a sequence $y_0, y_1, y_2, \ldots$

Note:

(1) If we calculate the value of y_2 by omitting h^3 and higher powers of h, the truncation error will be kh^3 where k is a constant and the corresponding Taylor's series is said to be of **second order**.

If h is small and terms after n terms are neglected the error is $\frac{h^n}{n!}f^{(n)}(\theta)$, where $x_0 < \theta < x_1$ if $x_1 - x_0 = h$

Drawback:

(2) It is a single step method. If $f(x, y)$ is complicated function, higher derivatives cannot be found and so the method can not be used. This is the draw back of the method. End

WORKED EXAMPLES

Example 1 14-05-19

Solve $y' = x + y$, $y(0) = 1$ by Taylor's series method. Find the values of y at $x = 0.1$ and $x = 0.2$.

$h = x_2 - x_1$

Solution

Given $$y' = x + y, \qquad y(0) = 1$$

$\therefore$ $$x_0 = 0, \qquad y_0 = 1, \qquad h = 0.1$$

Taylor's series is $y_{r+1} = y_r + hy'_r + \frac{h^2}{2!}y''_r + \frac{h^3}{3!}y'''_r + \dots$ (1)

We have $y' = x + y$

Differentiating we get, $y'' = 1 + y'$

$$y''' = y''$$

At $(x_0, y_0) = (0,1)$, $y'_0 = 0 + 1 = 1$

$$y''_0 = 1 + y'_0 = 1 + 1 = 2$$

$$y'''_0 = y''_0 = 2$$

Substituting $r = 0$ in (1), we get $y_1 = y_0 + hy'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \dots$

$$y_1 = 1 + (0.1)1 + \frac{(0.1)^2}{2!}2 + \frac{(0.1)^3}{3!}2$$

$$y_1 = 1 + 0.1 + 0.01 + \frac{0.001}{3} = 1.1103$$

Now $x_1 = 0.1$, $y_1 = 1.1103$, $h = 0.1$

Put $r = 1$ in (1), we get $y_2 = y_1 + \frac{h}{1!}y'_1 + \frac{h^2}{2!}y''_1 + \dots$

At $(x_1, y_1) = (0.1, 1.1103)$, $y'_1 = x_1 + y_1 = 0.1 + 1.1103 = 1.2103$

$$y''_1 = 1 + y'_1 = 1 + 1.2103 = 2.2103$$

$$y'''_1 = y''_1 = 2.2103$$

$\therefore$ $y_2 = 1.1103 + (0.1)(1.2103) + \frac{(0.1)^2}{2!}(2.2103) + \frac{(0.1)^3}{3!}(2.2103)$

$$= 1.1103 + 0.12103 + (0.01)(1.10515) + (0.001)(0.368383)$$

$$= 1.1103 + 0.12103 + 0.0110515 + \cancel{0.00368383}$$

0.000368383

$$y_2 = 1.2427$$

$\therefore$ the solution is $y_1 = y(0.1) = \mathbf{1.1103}$

$$y_2 = y(0.2) = \mathbf{1.2338}$$

Example 2

Using Taylor's series method find y at $x = 0$, if $\frac{dy}{dx} = x^2y - 1$, $y(0) = 1$. , h = 0·1

Solution

Given $y' = x^2y - 1$, $y(0) = 1$

$\therefore$ $x_0 = 0$, $y_0 = 1$, $h = 0.1$

Taylor's series is $y_{r+1} = y_r + hy'_r + \frac{h^2}{2!}y''_r + \frac{h^3}{3!}y'''_r + \dots$, $r = 0,1,2,\dots$ (1)

we have $y' = x^2y - 1$

$\therefore$ $y'' = x^2y' + 2xy$

and $\qquad y''' = x^2y'' + 2xy' + 2(xy' + y.1)$

At $(x_0, y_0) = (0,1)$, $\qquad y'_0 = 0 - 1 = -1$

$\because \qquad y''_0 = 0 + 0 = 0$

$$y'''_0 = 0 + 0 + 0 + 2.1 = 2$$

Putting $r = 0$ in (1), we get

$$y_1 = y_0 + hy'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \ldots$$

$$\therefore \quad y_1 = y(0.1) = 1 + (0.1)(-1) + 0 + \frac{(0.1)^3}{3!}.2 + \ldots$$

$$= 1 - 0.1 + \frac{0.001}{3} = 0.9 + 0.0003333 = 0.9003$$

$$\therefore \quad \mathbf{y_1 = y(0.1) = 0.9003}$$

■

Example 3

Using Taylor's series method find $y(1.1)$ and $y(1.2)$ correct to four places given $y' = xy^{\frac{1}{3}}$ and $y(1) = 1$.

Solution

Given $\qquad y' = xy^{\frac{1}{3}}, \qquad y(1) = 1$

$\therefore \qquad x_0 = 1, \qquad y_0 = 1$

Required $\qquad y(1.1),\ y(1.2) \qquad \therefore h = 0.1$

Taylor's series is $\qquad y_{r+1} = y_r + hy'_r + \frac{h^2}{2!}y''_r + \frac{h^3}{3!}y'''_r + \ldots, \qquad r = 0,1,2,\ldots \qquad (1)$

we have $\qquad y' = xy^{\frac{1}{3}}$

$$\therefore \quad y'' = x.\frac{1}{3}y^{-2/3}y' + y^{1/3}.1 = \frac{1}{3}x.y^{-2/3}.y' + y^{1/3}$$

and

$$y''' = \frac{1}{3}\left[x\left(y^{-2/3}.y'' + y'\left(\frac{-2}{3}\right)y^{-5/3}.y'\right) + y^{-2/3}y'\right] + \frac{1}{3}y^{-2/3}.y'.1$$

$$= \frac{x}{3}\left(y^{-2/3}y'' - \frac{2}{3}y^{-5/3}(y')^2\right) + \frac{2}{3}y^{-2/3}y'$$

At $(x_0, y_0) = (1,1)$, $\qquad y'_0 = 1.1 = 1$

$$y''_0 = \frac{1}{3}.1 + 1 = \frac{4}{3}$$

$$y'''_0 = \frac{1}{3}\left[1.\frac{4}{3} - \frac{2}{3}.1.1^2\right] + \frac{2}{3}.1.1 = \frac{1}{3}.\frac{2}{3} + \frac{2}{3} = \frac{8}{9}$$

Putting $r=0$ in (1), we get

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \dots$$

$$\therefore \quad y(1.1) = y_1 = 1 + (0.1).1 + \frac{(0.1)^2}{2}.\frac{4}{3} + \frac{(0.1)^3}{3!}\frac{8}{9} + \dots$$

$$= 1 + 0.1 + 0.006666 + 0.000148$$

$$\mathbf{y_1 = 1.1068}$$

Now $x_1 = 1.1, \quad y_1 = 1.1068, \quad h = 0.1$

Putting $r=1$,we get

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \dots$$

At $(x_1, y_1) = (1.1, 1.1068)$

$$y_1' = (1.1)(1.1068)^{\frac{1}{3}} = 1.1378$$

$$y_1'' = \frac{1}{3}(1.1)(1.1068)^{-2/3}(1.1378) + (1.1068)^{1/3}$$

$$= 0.3899 + 1.0344 = 1.4243$$

$$y_1''' = \frac{x_1}{3}\left[y_1^{-2/3} y_1'' - \frac{2}{3}y_1^{-5/3}(y_1')^2\right] + \frac{2}{3}y_1^{-2/3}.y_1'$$

$$y_1''' = \frac{1.1}{3}\left[(1.1068)^{-2/3}(1.4243) - \frac{2}{3}(1.1068)^{-5/3}(1.1378)^2\right] + \frac{2}{3}(1.1068)^{-2/3}(1.1378)$$

$$= \frac{1.1}{3}\left[(0.93459)(1.4243) - \frac{2}{3}(0.84441)(1.29459)\right] + \frac{2}{3}(0.93459)(1.1378)$$

$$= \frac{1.1}{3}[1.33114 - 0.72878] + 0.7089$$

$$= \frac{11}{3}(0.60236) + 0.7089 = 0.2209 + 0.7089 = 0.9298$$

0.2265
+0.3549

$$\therefore \quad y_2 = y(1.2) = 1.1068 + (0.1)1.1378 + \frac{(0.1)^2}{2}(1.4243) + \frac{(0.1)^3}{6}(0.9298)$$

$$= 1.1068 + 0.11378 + 0.0071215 + 0.000155$$

$$= 1.2279$$

Thus $\quad \mathbf{y(1.1) = 1.1068, \quad y(1.2) = 1.2279}$ ■

Note:

Since this equation $\frac{dy}{dx} = xy^{1/3}$ is a simple variable separable equation we can solve and find the exact solution. We shall compare the computed value with the exact value.

$$\frac{dy}{y^{1/3}} = x dx \Rightarrow \frac{3}{2}y^{2/3} = \frac{x^2}{2} + C$$

where $x = 1, y = 1, \Rightarrow C = 1$

So, solution is $3y^{2/3} = x^2 + 2 \Rightarrow y^{2/3} = \frac{x^2+2}{3} \quad \therefore y = \left(\frac{x^2+2}{3}\right)^{\frac{3}{2}}$

When $x = 1.1$, $\quad y = \left[\frac{(1.1)^2+2}{3}\right]^{3/2} = (1.07)^{3/2} = \mathbf{1.1068}$

When $x = 1.2$, $\quad y = \left[\frac{(1.2)^2+2}{3}\right]^{3/2} = (1.146667)^{3/2} = \mathbf{1.2278}$

	y	
x	**Taylor series value**	**Actual value**
1.1	1.1068	1.1068
1.2	1.2279	1.2278

Example 4

Using Taylor's series method find y at $x = 1.1$ and 1.2 by solving $\frac{dy}{dx} = x^2 + y^2$, given $y(1) = 2.3$.

Solution

Given $\quad \frac{dy}{dx} = x^2 + y^2, \qquad y(1) = 2.3$

$\Rightarrow \quad y' = x^2 + y^2, \qquad x_0 = 1, \qquad y_0 = 2.3$

Required y at $x = 1.1$, $\qquad x = 1.2$, $\qquad \therefore h = 0.1$

Taylor's series is

$$y_{r+1} = y_r + hy'_r + \frac{h^2}{2!}y''_r + \frac{h^3}{3!}y'''_r + \ldots, \qquad r = 0,1,2,\ldots \qquad (1)$$

we have $\quad y' = x^2 + y^2$

$\therefore \quad y'' = 2x + 2yy'$

and $\quad y''' = 2 + 2\{yy'' + y'y'\} = 2\{1 + yy'' + (y')^2\}$

At $(x_0, y_0) = (1, 2, 3)$ x

$$y'_0 = x_0^2 + y_0^2 = 1 + (2.3)^2 = 6.29$$

$$\therefore y''_0 = 2x_0 + 2y_0y'_0 = 2 + 2(2.3)(6.29) = 30.934$$

$$y_0''' = 2\{1 + y_0 y_0'' + (y_0')^2\}$$
$$= 2\{1 + (2.3)(30.934) + (6.29)^2\} = 223.4246$$

Putting $r = 0$ in (1), we get,

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots$$

$\therefore$
$$y_1 = 2.3 + (0.1)(6.29) + \frac{(0.1)^2}{2}(30.934) + \frac{(0.1)^3}{6}(223.4246)$$
$$= 2.3 + 0.629 + \frac{(0.01)}{2}(30.934) + \frac{(0.001)}{6}(223.4246) = 3.1209$$

$\therefore$ $\quad$ $\mathbf{y(1.1) = 3.1209}$

Now $x_1 = 1.1, \quad y_1 = 3.1209$

Putting $r = 1$ in (1), we get,

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \ldots$$

At $(x_1, y_1) = (1.1, 3.1209)$

$$y_1' = x_1^2 + y_1^2 = (1.1)^2 + (3.1209)^2 = 10.95$$
$$y_1'' = 2x_1 + 2y_1y_1' = 2(1.1) + 2(3.1209)(10.95) = \underline{70.5477}$$
$$y_1''' = 2\{1 + y_1y_1'' + (y_1')^2\} = 2[1 + 3.1209\,(70.5477) + (10.95)^2] = 682.1496$$

$\therefore$
$$y_2 = 3.1209 + (0.1)(10.95) + \frac{(0.1)^2}{2}(70.5477) + \frac{(0.1)^3}{6}(682.1496) = 4.6823$$
$$x_2 = x_1 + h = 1.1 + 0.1 = 1.2$$

$\therefore$
$$y_2 = y(1.2) = 4.6823$$

Hence $\quad$ $\mathbf{y(1.1) = 3.1209,\ y(1.2) = 4.6823}$ ■

Example 5

Use Taylor series method to find $y(0.1)$, $y(0.2)$, given that $\frac{dy}{dx} = 3e^x + 2y$, $y(0) = 0$ correct to 4 decimal accuracy.

Solution

Given equation is $\frac{dy}{dx} = 3e^x + 2y$ and $y(0) = 0$

$\therefore$ $\quad x = 0, \quad y = 0$

Required $y(0.1)$, $y(0.2)$, $\therefore h = 0.1$

Taylor's series is $$y_{r+1} = y_r + hy'_r + \frac{h^2}{2!}y''_r + \frac{h^3}{3!}y'''_r + \ldots, \qquad r = 0,1,2,\ldots \qquad (1)$$

We have $y' = 3e^x + 2y$

$\therefore$ $y'' = 3e^x + 2y'$

and $y''' = 3e^x + 2y''$

Putting $r = 0$ in (1), we get,

$$y_1 = y_0 + hy'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \ldots$$

At $(x_0, y_0) = (0, 0)$

$$y'_0 = 3e^0 + 2\times 0 = 3$$

$$y''_0 = 3e^0 + 2y'_0 = 3 + 2\times 3 = 9$$

$$y'''_0 = 3e^0 + 2y''_0 = 3 + 2\times 9 = 21$$

$\therefore$ $$y_1 = 0 + (0.1)3 + \frac{(0.1)^2}{2!}\times 9 + \frac{(0.1)^3}{3!}\times 21 + \ldots$$

$$= 0.3 + 0.045 + 0.0035 + \ldots = 0.3485$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$\therefore$ $\mathbf{y(0.1) = 0.3485}$

Now $x_1 = 0.1, y_1 = 0.3485$

Putting $r = 1$ in (1), we get $y_2 = y_1 + hy'_1 + \frac{h^2}{2!}y''_1 + \frac{h^3}{3!}y'''_1 + \ldots$

At $(x_1, y_1) = (0.1, 0.3485)$

$$y'_1 = 3e^{0.1} + 2(0.3485) = 3(1.1052) + 0.697 = \underline{4.0126}$$

$$y''_1 = 3(1.1052) + 2(4.0126) = \underline{11.3408}$$

$$y'''_1 = 3(1.1052) + 2(11.3408) = 25.9972$$

$\therefore$ $$y_2 = 0.3485 + (0.1)\times(4.0126) + \frac{(0.1)^2}{2!}(11.3408) + \frac{(0.1)^3}{3!}(25.9972)$$

$$= 0.3485 + 0.4013 + 0.0567 + 0.00433 = 0.8108$$

$$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$$

$\therefore$ $\mathbf{y(0.2) = 0.8108}$

$\therefore$ $\mathbf{y(0.1) = 0.3485, y(0.2) = 0.8108}$ ■

EULER'S METHOD AND MODIFIED EULER'S METHOD

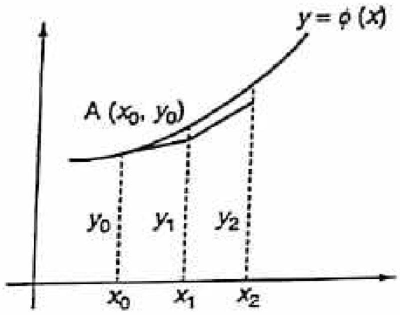
17-05-19

Euler's method is one of the simplest and oldest methods of finding numerical solution of differential equation. It is a step-by-step method because the values of y are computed by short steps ahead for equal intervals of the independent variable. This crude method is rarely used in practice but it explains the principle of methods based on Taylor's series.

A. Euler's method or simple Euler's method

Consider the differential equation

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$



Let $x_0, x_1, x_2, \ldots$ be equidistant values of x where

$x_1 - x_0 = h = x_2 - x_1 = \ldots$ so that $x_1 = x_0 + h, \quad x_2 = x_0 + 2h, \ldots$

In Euler's method we approximate a curve in a small interval by a straight line.

Let $y = \phi(x)$ be the curve representing the actual solution.

Equation of the tangent at (x_0, y_0) is

$$y - y_0 = \left(\frac{dy}{dx}\right)_{(x_0, y_0)} (x - x_0)$$

$$= f(x_0, y_0)(x - x_0)$$

$\Rightarrow$
$$y = y_0 + f(x_0, y_0)(x - x_0)$$

Since the curve in the interval $(x_0, x_0 + h)$ is approximated by this straight line, the value of y at $x = x_1$ is approximately

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$
$$= y_0 + hf(x_0, y_0)$$

Similarly the curve in the interval (x_1, x_2) is approximated by the line through (x_1, y_1) and having slope $f(x_1, y_1)$

$\therefore$
$$y_2 = y_1 + hf(x_1, y_1)$$

Proceeding in this way we get the general formula

$$y_{n+1} = y_n + hf(x_n, y_n), \quad n = 0,1,2,3,\ldots$$

This is called Euler's algorithm:

Note:

(1) Since $x_n = x_0 + nh$ and $y_n = y(x_n)$, the above formula can be simply written as $y(x+h) = y(x) + hf(x,y)$

(2) The draw back of this method is that it is either too slow (in case h is small) or too inaccurate (in case h is not small).

(3) The computed y's will deviate more and more from the true y's as we proceed further along X-axis, due to cumulative rounding errors.

These drawbacks have led to a modification of Euler's method aiming at more accurate results.

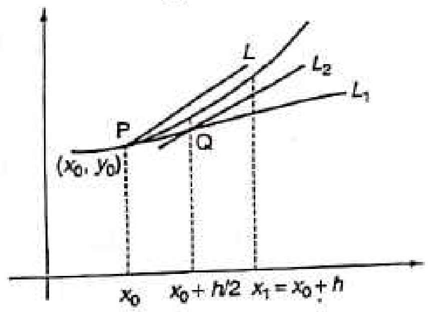
B. Modified Euler's method

We start with the tangent L_1 at (x_0, y_0).

Let the ordinate at $x_0 + \dfrac{h}{2}$ intersect the tangent L_1 at Q, where $y = y_0 + \dfrac{h}{2} f(x_0, y_0)$.

We find the slope at Q.

ie $\left(\dfrac{dy}{dx}\right)_Q = f\left[x_0 + \dfrac{h}{2}, y_0 + \dfrac{h}{2} f(x_0, y_0)\right]$



We draw the line through Q with this as slope and let this line be L_2.

We then draw the line L through (x_0, y_0) parallel to L_2.

This line L is taken as an approximation to the curve in the interval (x_0, x_1).

Equation of L is $\quad y - y_0 = f\left(x_0 + \dfrac{h}{2}, y_0 + \dfrac{h}{2} f(x_0, y_0)\right)(x - x_0)$

When $x = x_1 = x_0 + h$, $y = y_0 + f\left(x_0 + \dfrac{h}{2}, y_0 + \dfrac{h}{2} f(x_0, y_0)\right)h$.

ie.
$$y_1 = y_0 + hf\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right)$$

Proceeding this way, in general we have

$$y_{n+1} = y_n + hf\left[x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right], \quad n = 0,1,2,\ldots$$

where $x_n = x_0 + nh$, $y_n = y(x_n)$

Note:

he above formula is one version of the modified formula. It gives a better approximation. this method we took average of points.

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(2) Another version of Euler modified formula is given in terms of the average of the slopes at the end points of an interval.

That is $y_{n+1} = y_n + h\dfrac{(y'_n + y'_{n+1})}{2}$

$$= y_n + \frac{h}{2}[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))]$$

This formula is also referred as improved Euler formula by some authors.

(3) The improved and modified Euler's methods give a better approximation to y_{n+1} than Euler's formula.

(4) Local error in improved and modified methods are $\boxed{O(h^3)}$ MCQ's

$h = \frac{b-a}{N}$

WORKED EXAMPLES

Example 1 X

Using Euler's method find y for $x = 0.1$ given $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$, $y(0) = 1$. $h = 0.5$

Solution

Given equation is $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$, $y(0) = 1$

$$f(x, y) = \frac{y-x}{y+x}$$

We shall divide $(0, 0.1)$ into 2 parts with $h = 0.05$

$$y(0) = 1 \Rightarrow x_0 = 0, \quad y_0 = 1$$

Euler's algorithm is $y_{n+1} = y_n + hf(x_n, y_n), n = 0, 1, 2, \ldots$

$$x_n = x_0 + nh, \quad y_n = y(x_n)$$

When $n = 0$, $\quad y_1 = y_0 + hf(x_0, y_0)$

$$= 1 + 0.05\left[\frac{y_0 - x_0}{y_0 + x_0}\right] = 1 + 0.05(1) = 1.05$$

When $n = 1$, $\quad y_2 = y_1 + hf(x_1, y_1),\ x_1 = x_0 + h = 0.05$

$$= 1.05 + (0.05)\left(\frac{y_1 - x_1}{y_1 + x_1}\right)$$

$$= 1.05 + (0.05)\left(\frac{1.05 - 0.05}{1.05 + 0.05}\right)$$

$$= 1.05 + 0.05\left(\frac{1}{1.1}\right) = 1.0955$$

$\therefore \quad$ **$y(0.1) = 1.0955$**

Example 2

Apply modified Euler's method to find $y(0.2)$ and $y(0.4)$ given $y' = x^2 + y^2$, $y(0) = 1$ by taking $h = 0.2$.

Solution

Given equation is $y' = x^2 + y^2$, $y(0) = 1$

$\therefore$ $f(x, y) = x^2 + y^2$, and $x_0 = 0$, $y_0 = 1$, $h = 0.2$

Modified Euler's formula is

$$y_{n+1} = y_n + hf\left[x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right], \quad n = 0,1,2, \ldots$$

Putting $n = 0$, we get, $y_1 = y_0 + hf\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right]$

$$= 1 + 0.2\, f\left[0 + \frac{0.2}{2}, 1 + \frac{0.2}{2} f(0,1)\right]$$ $= 1+0.2f[0.1, 1+0.1(1)$

But $f(0,1) = 0 + 1 = 1$ $= 1+0.2f[0.1, 1.1]$

$\therefore$ $y_1 = 1 + 0.2f[0.1, 1.1]$ $= 1.244$

$$= 1 + 0.2[(0.1)^2 + (1.1)^2] = 1.244$$

$\therefore$ $y(0.2) = 1.244$

Now $x_1 = 0.2$, $y_1 = 1.244$

Putting $n = 1$, we get,

$$y_2 = y_1 + hf\left[x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x_1, y_1)\right]$$

$$= 1.244 + (0.2)\, f\left[0.2 + \frac{0.2}{2}, 1.244 + \frac{0.2}{2} f(0.2, 1.244)\right]$$

$$= 1.244 + (0.2) f[0.3, 1.244 + 0.1(.2^2 + 1.244^2)]$$

$$= 1.244 + (0.2) f[0.3, 1.403]$$

$$= 1.244 + (0.2)\left[(0.3)^2 + (1.403)^2\right] = 1.6557$$

$\therefore$ $y(0.4) = 1.6557$

Thus $y(0.2) = 1.244$, $y(0.4) = 1.6557$ ■

Example 3

Solve $\frac{dy}{dx} = \log_{10}(x + y), y(0) = 2$ by Euler's modified method and find the values of $y(0.2)$, $y(0.4)$ and $y(0.6)$ by taking $h = 0.2$.

Solution

Given $$\frac{dy}{dx} = \log_{10}(x+y), \quad y(0) = 2$$

$\therefore$ $$f(x,y) = \log_{10}(x+y)$$

and $$x_0 = 0, \quad y_0 = 2, \quad h = 0.2$$

Modified Euler formula is

$$y_{n+1} = y_n + hf[x_n + \frac{h}{2}, y_n + \frac{h}{2}f(x_n, y_n)], n = 0,1,2, \ldots$$

Putting $n = 0$, we get

$$y_1 = y_0 + hf\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2}f(x_0, y_0)\right]$$

But $f(x_0, y_0) = \log_{10}(x_0 + y_0) = \log_{10}(2) = 0.3010$

$\therefore$ $$y_1 = 2 + (0.2)f\left[0 + \frac{.2}{2}, 2 + \frac{0.2}{2}(0.3010)\right]$$

$$= 2 + (0.2)f[0.1, 2.0301]$$

$$= 2 + (0.2)\log(0.1 + 2.0301)$$

$$= 2 + (0.2)\log(2.1301) = 2.0657$$

$\therefore$ $$\mathbf{y(0.2) = 2.0657}$$

Now $x_1 = 0.2, \quad y_1 = 2.0657$

Putting $n = 1$, we get

$$y_2 = y_1 + hf\left[x_1 + \frac{h}{2}, y_1 + \frac{h}{2}f(x_1, y_1)\right]$$

$$= 2.0657 + (0.2)f\left[0.2 + \frac{0.2}{2}, 2.0657 + \frac{0.2}{2}f(0.2, 2.0657)\right]$$

$$= 2.0657 + (0.2)f[0.3, 2.0657 + 0.1\log(2.2657)]$$

$$= 2.0657 + (0.2)f[0.3, 2.1012]$$

$$= 2.0657 + (0.2)\log[0.3 + 2.1012]$$

$$= 2.0657 + 0.07608 = 2.14178 = 2.1418$$

$$\mathbf{y(0.4) = 2.1418}$$

Now $x_2 = 0.4, y_2 = 2.1418$

Putting $n = 2$, we get

$$y_3 = y_2 + hf\left[x_2 + \frac{h}{2}, y_2 + \frac{h}{2}f(x_2, y_2)\right]$$

$$= 2.1418 + 0.2f\left[0.4 + \frac{0.2}{2}, 2.1418 + \frac{0.2}{2}\log(2.5418)\right]$$

$$= 2.1418 + 0.2f[0.5, 2.1823]$$

$$= 2.1418 + 0.2\log 2.6823 = 2.2275$$

$$y(0.4) = \mathbf{2.2275}$$

Thus $\quad y(0.2) = \mathbf{2.0657}, y(0.4) = \mathbf{2.1418}, y(0.6) = \mathbf{2.2275}$ ■

Example 4

Using modified Euler's method solve given that $y' = y - x^2 + 1, y(0) = 0.5$ find $y(0.2)$. with $h = 0.1$

Solution

Given equation is $\quad y' = y - x^2 + 1, \quad x_0 = 0, \quad y_0 = 0.5$

$\therefore \quad f(x,y) = y - x^2 + 1 \quad$ Take $h = 0.1$

We shall find $y(0.2)$ in two steps.
Modified Euler's formula is

$$y_{n+1} = y_n + hf\left[x_n + \frac{h}{2}, y_n + \frac{h}{2}f(x_n, y_n)\right], \quad n = 0,1,2,\ldots$$

Putting $n = 0$, we get,

$$y_1 = y_0 + hf\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2}f(x_0, y_0)\right]$$

$$= 0.5 + (0.1)f\left[0 + \frac{0.1}{2}, 0.5 + \frac{0.1}{2}f(0, 0.5)\right]$$

$$= 0.5 + (0.1)f[0.05, 0.5 + 0.05(0.5 + 1)]$$

$$= 0.5 + (0.1)f[0.05, 0.575]$$

$$= 0.5 + (0.1)\left[0.575 - (0.05)^2 + 1\right] = 0.65725$$

Now $x_1 = 0.1, y_1 = 0.65725$ ie. $y(0.1) = \mathbf{0.65725}$

Putting $n = 1$, we get,

$$y_2 = y_1 + hf\left[x_1 + \frac{h}{2}, y_1 + \frac{h}{2}f(x_1, y_1)\right]$$

$$= 0.65725 + 0.1f[0.1 + 0.05, 0.65725 + 0.05(1.64725)]$$

$$= 0.65725 + 0.1f[0.15, .7396]$$

$$= 0.65725 + 0.1[0.7396 - (0.15)^2 + 1]$$

$$= 0.65725 + 0.17171 = 0.82896$$

$\therefore \quad y(0.2) = \mathbf{0.82896}$ ■

Exercises 9.1

(1) Use Taylor's series method solution to solve $\frac{dy}{dx} = x^2 - y$, $y(0) = 1$. Find $y(0.1)$, $y(0.2)$, $y(0.3)$, $y(0.4)$ 2.

(2) Use Taylor's series method to find y at $x = 0.1, 0.2$ given $\frac{dy}{dx} - 2y = 3e^x$, $y(0) = 0$.

(3) Use Taylor's series method to find y at $x = 0.1$ if $\frac{dy}{dx} = x^2y - 1$, $y(0) = 0$.

(4) Use Taylor's series method to solve $\frac{dy}{dx} = 1 + xy$ with $y(0) = 2$ and find $y(0.1)$, $y(0.2)$, $y(0.3)$.

(5) By Taylor's method find the solution with 3 terms for the initial value problem $\frac{dy}{dx} = x^3 + y$, $y(1) = 1$, and obtain $y(1.1)$, $y(1.2)$.

(6) Using Taylor's series method find $y(0.1)$ and $y(0.2)$ given $\frac{dy}{dx} = 3x + \frac{y}{2}$, $y(0) = 1$.

(7) Use Euler's method to find $y(0.2)$, $y(0.4)$ from $\frac{dy}{dx} = x + y$, $y(0) = 1$ with $h = 0.2$.

(8) Given $\frac{dy}{dx} = \frac{y - x}{y + x}$ in $0 \le x \le 0.1$ taking $h = 0.02$ and $y(0) = 1$, find y when $x=0.1$.

(9) Using Euler's modified method find $y(0.2)$, $y(0.4)$, $y(0.6)$ given $\frac{dy}{dx} = y - x^2$, $y(0) = 1$.

(10) Using Euler's modified method compute $y(0.1)$ with $h = 0.1$ from $y' = y - \frac{2x}{y}, y(0) = 1$.

(11) Consider the initial value problem $\frac{dy}{dx} = y - x^2 + 1, y(0) = 0.5$ using the modified Euler's method, find $y(0.2)$.

(12) Using modified Euler's method solve, given that $y' = 1 - y$, $y(0) = 0$ find $y(0.1)$, $y(0.2)$ and $y(0.3)$.

(13) By modified Euler's method find $y(1.2)$, given $\frac{dy}{dx} = (y - x^2)^3$, $y(1) = 0$, $h = 0.2$.

(14) Solve $\frac{dy}{dx} = x + y$ given $y(1) = 0$, and find $y(1.1)$ and $y(1.2)$ by Taylor's series method.

(15) Using Modified Euler's method find $y(0.1)$ and $y(0.2)$ given $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$.

Answers 9.1

(1) 0.9052, 0.8213, 0.7492, 0.6897

(2) 0.3487, 0.8113

(3) 0.9003

(4) 2.1103, 2.2430, 2.4011

(5) $y(1.1) = 1.225$; $y(1.2) = 1.512$

(6) 1.0665, 1.1672

(7) 1.2, 1.48

(8) 1.0928

(9) 1.218, 1.467, 1.737

(10) $y(0.1) = 1.0955$

(11) $y(0.2) = 0.828$, taking $h = 0.2$

(12) 0.095, 0.18098, 0.25878

(13) $y(1.2) = -0.2735$

(14) $y(1.1) = 0.1104$, $y(12) = 0.2424$

(15) $y(0.1) = 1.1105$, $y(0.2) = 1.2503$

9.3 RUNGE-KUTTA METHOD (R-K METHOD)

21-05-19 :

Runge-Kutta methods are more accurate than the earlier methods we have seen. Two German mathematicians, Runge and Kutta developed algorithms to solve a differential equation efficiently. The advantage of this method is that it requires only values of the function at some specified points. These methods agree with Taylor series expansion upto the terms of h^r, where r is the order of the Runge-Kutta method and it differs from method to method.

In these methods two or more estimates of Δy, the increment in y, are computed and a linear combination of these estimates are used to determine Δy and hence the next value of y, is $y(x+h) = y(x) + \Delta y$. Since the derivations are complicated we shall state here only the algorithms to solve $\dfrac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$

1. First-order Runge-Kutta method

Euler's method is $y(x+h) = y(x) + hf(x, y)$

$$\therefore \quad y_1 = y_0 + hf(x_0, y_0), \quad y_2 = y_1 + hf(x_1, y_1), \quad \text{etc.}$$

In general, $$y_{n+1} = y_n + hf(x_n, y_n), \quad n = 0, 1, 2, \ldots$$

2. Second order Runge-Kutta method

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$\Delta y = \frac{1}{2}[k_1 + k_2]$$

$$y_{n+1} = y_n + \Delta y, \quad n = 0, 1, 2, 3, \ldots$$

Note that Δy is the mean of k_1 and k_2. Some times Δy is taken as k_2, in which case second order Runge-Kutta method is modified Euler method.

3. Third-order Runge-Kutta method

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf(x_n + h, y_n + 2k_2 - k_1)$$

$$\Delta y = \frac{1}{6}(k_1 + 4k_2 + k_3)$$

Then $$y_{n+1} = y_n + \Delta y, \; n = 0, 1, 2, \ldots$$

Note that Δy is the weighted mean of k_1, k_2, k_3

4. Fourth-order Runge-Kutta method

The fourth order Runge-Kutta method is most widely used and is popular and so it is referred to as the Runge-Kutta method. In problems we use fourth-order method unless otherwise specified.

Fourth-order algorithm is

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

and

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_{n+1} = y_n + \Delta y, \; n = 0, 1, 2, \ldots$$

Note that Δy is the weighted mean of k_1, k_2, k_3, k_4.

WORKED EXAMPLES

Example 1

Using Runge-Kutta method of Fourth order solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2, 0.4$.

Solution

Given $\quad \frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}, \quad y(0) = 1$

Then $\quad f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}, \quad x_0 = 0, \quad y_0 = 1, \quad h = 0.2$

Required the values of y when $x = 0.2$ and $x = 0.4$

The fourth-order Runge-Kutta method is

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3),$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$y_{n+1} = y_n + \Delta y, \qquad n = 0, 1, 2, \ldots$$

Put $n = 0$ to find y_1

Then, $\quad k_1 = hf(x_0, y_0) = h\left(\frac{y_0^2 - x_0^2}{y_0^2 + x_0^2}\right) = 0.2\left(\frac{1-0}{1+0}\right) = \mathbf{0.2}$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2f\,[0.1,\ 1 + 0.1]$$

$$= 0.2f\,[0.1,\ 1.1]$$

$$= (0.2)\left(\frac{(1.1)^2 - (0.1)^2}{(1.1)^2 + (0.1)^2}\right) = (0.2)\left(\frac{1.20}{1.22}\right) = \mathbf{0.1967}$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.2)f\left(0.1, 1 + \frac{0.1967}{2}\right)$$

$$= (0.2)f[0.1, 1.09835]$$

$$= (0.2)\left(\frac{(1.09835)^2 - (0.1)^2}{(1.09835)^2 + (0.1)^2}\right)$$

$$= (0.2)\left(\frac{1.2064 - 0.01}{1.2064 + 0.01}\right) = (0.2)\left(\frac{1.1964}{1.2164}\right) = \mathbf{0.1967}$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = hf[0.2, 1 + 0.1967]$$

$$= 0.2f[0.2,\ 1.1967]$$

$$= (0.2)\left(\frac{(1.1967)^2 - (0.2)^2}{(1.1967)^2 + (0.2)^2}\right)$$

$$= (0.2)\left(\frac{1.4321 - 0.04}{1.4321 + 0.04}\right) = (0.2)\left(\frac{1.3921}{1.4721}\right) = \mathbf{0.1891}$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.2 + 2\times 0.1967 + 2\times 0.1967 + 0.1891) = \mathbf{0.19598}$$

$$\therefore \quad y_1 = y_0 + \Delta y = 1 + 0.19598 = 1.19598 = \mathbf{1.196}$$

Now $x_1 = 0.2, y_1 = 1.196, h = 0.2$

Put $n = 1$ to find y_2

Then

$$k_1 = hf(x_1, y_1) = 0.2f(0.2,\ 1.196)$$

$$= (0.2)\left(\frac{(1.196)^2 - (0.2)^2}{(1.196)^2 + (0.2)^2}\right)$$

$$= (0.2)\left(\frac{1.4304 - 0.04}{1.4304 + 0.04}\right) = (0.2)\left(\frac{1.3904}{1.4704}\right) = 0.1891$$

Similarly $k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.2f(0.3,\ 1.2906) = \mathbf{0.1795}$

$$\therefore \quad k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.2f(0.3,\ 1.2858) = \mathbf{0.1793}$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2f(0.4, 1.3753) \quad = \mathbf{0.1688}$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.1891 + 2\times 0.1795 + 2\times 0.1793 + 0.1688] = \mathbf{0.1792}$$

$\therefore \qquad y_2 = y_1 + \Delta y = 1.196 + 0.1792 = 1.3752$

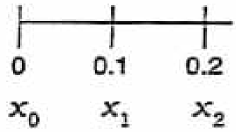
Thus $\qquad \mathbf{y(0.2) = 1.196, \quad y(0.4) = 1.3752}$ ■

Example 2

Apply Runge-Kutta method to find approximate value of y for $x = 0.2$ in steps of 0.1 if $\frac{dy}{dx} = x + y^2$ given that $y = 1$ when $x = 0$.

Solution

Given $\quad \frac{dy}{dx} = x + y^2, \quad y = 1$, $x = 0$,



Then $f(x, y) = x + y^2, \quad x_0 = 0,\ y_0 = 1$, and $h = 0.1$

Required $y(0.2)$. Since we have to compute with $h = 0.1$, first we find $y(0.1)$ and then $y(0.2)$. Fourth order Runge-Kutta formula is

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

$$y_{n+1} = y_n + \Delta y, \qquad n = 0, 1, 2 \ldots$$ $y_1 = y_0 + \Delta y$

Put $n = 0$ to find y_1

Then $\qquad k_1 = hf(x_0, y_0) = 0.1[x_0 + y_0^2] = 0.1(0 + 1) = 0.1$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1f\left[0 \div \frac{0.1}{2}, 1 + \frac{0.1}{2}\right] = 0.1f[0.05, 1.05]$$

$$= 0.1[0.05 + (1.05)^2] = \mathbf{0.1153}$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1f[0.05, 1 + .0576] = 0.1f[0.05, 1.0576]$$

$$= 0.1[0.05 + (1.0576)^2] = \mathbf{0.1169}$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f[0.1, 1 + 0.1169]$$
$$= 0.1[0.1 + (1.1169)^2] = \mathbf{0.1347}$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}(0.1 + 2\times 0.1153 + 2\times 0.1169 + 0.1347)$$
$$= \frac{1}{6}(0.1 + 0.2306 + 0.2338 + 0.1347) = \mathbf{0.1165}$$

$$y_1 = y_0 + \Delta y = \mathbf{1.1165}$$

$\therefore$ $\quad x_1 = x_0 + h = 0.1, \quad y_1 = 1.1165, \quad h = 0.1$

Now

Put $n = 1$ to find y_2

Then

$$k_1 = hf(x_1, y_1)$$
$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$\therefore$

$$k_1 = 0.1f(0.1, 1.1165) = 0.1[0.1 + (1.1165)^2] = \mathbf{0.1347}$$
$$k_2 = 0.1f\left[0.1 + \frac{0.1}{2}, 1.1165 + \frac{0.1347}{2}\right]$$
$$= 0.1f[0.15, 1.1839] = 0.1[0.15 + (1.1839)^2] = \mathbf{0.1552}$$
$$k_3 = 0.1f\left[0.15, 1.1165 + \frac{0.1552}{2}\right]$$
$$= 0.1f[0.15, 1.1941] = 0.1[0.15 + (1.1941)^2] = \mathbf{0.1576}$$
$$k_4 = 0.1f[0.1 + 0.1, 1.1165 + 0.1576]$$
$$= 0.1f(0.2, 1.2741) = 0.1[0.2 + (1.2741)^2] = \mathbf{0.1823}$$
$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}(0.1347 + 2\times 0.1552 + 2\times 0.1576 + 0.1823) = \mathbf{0.1571}$$

$\therefore$ $\quad y_2 = y_1 + \Delta y = 1.1165 + 0.1571 = 1.2736$

Thus $\quad \mathbf{y(0.1) = 1.1165}$ **and** $\mathbf{y(0.2) = 1.2736}$

Example 3

Using Runge-Kutta method of $\boxed{\text{order 4,}}$ find $y(0.2)$ for the **equation** $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$, $y(0) = 1$. Take $h = 0.2$.

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Solution

Given equation is $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}, \quad y(0) = 1, \quad h = 0.2$

Then $f(x,y) = \dfrac{y-x}{y+x}, \quad x_0 = 0, \quad y_0 = 1,$

Required $y(0.2)$

Fourth order Runge-Kutta formula is

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left[x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right]$$

$$k_3 = hf\left[x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right]$$

$$k_4 = hf[x_n + h, y_n + k_3]$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

$$y_{n+1} = y_n + \Delta y_n, \qquad n = 0, 1, 2, \ldots$$

Put $n = 0$ to find $y_1 = y(0.2)$

$\therefore$ $$k_1 = hf(x_0, y_0) = 0.2\left(\frac{1-0}{1+0}\right) = \mathbf{0.2}$$

$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right] = 0.2f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right)$$

$$= 0.2f(0.1, 1.1) = 0.2\left(\frac{1.1-0.1}{1.1+0.1}\right) = \mathbf{0.1667}$$

$$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right] = 0.2f\left(0.1, 1 + \frac{0.1667}{2}\right)$$

$$= 0.2f(0.1, 1.0833) = 0.2\left(\frac{1.0833-0.1}{1.0833+0.1}\right) = \mathbf{0.1662}$$

$$k_4 = hf[x_0 + h, y_0 + k_3] = 0.2f(0.2, 1.1662) = 0.2\left(\frac{1.1662-0.2}{1.1662+0.2}\right) = \mathbf{0.1414}$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.2 + 2(0.1667) + 2(0.1662) + 0.1414] = \mathbf{0.1679}$$

$\therefore$ $$y_1 = y_0 + \Delta y = 1 + 0.1679 = \mathbf{1.1679}$$

Example 4

Apply Runge-Kutta method to $\dfrac{dy}{dx} = -2x - y$, $y(0) = -1$ and find $y(0.2)$ with $h = 0.1$.

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Solution

Given $\dfrac{dy}{dx} = -2x - y, \quad y(0) = -1, \quad h = 0.1$

Then, $f(x,y) = -2x - y, \quad x_0 = 0, \quad y_0 = -1, \quad h = 0.1$

Fourth order Runge-Kutta formula is

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left[x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right]$$

$$k_3 = hf\left[x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right]$$

$$k_4 = hf[x_n + h, y_n + k_3]$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

$$y_{n+1} = y_n + \Delta y, \quad n = 0, 1, 2, \ldots$$

Put $n = 0$ to find y_1

Then $k_1 = hf(x_0, y_0) = 0.1f(0,-1) = 0.1[(-2)\times 0 - (-1)] = 0.1$

$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right] = 0.1f\left(\frac{0.1}{2}, -1 + \frac{0.1}{2}\right)$$

$$= 0.1f(0.05, -0.95)$$

$$= 0.1[(-2)(0.05) + (0.95)] = 0.0850$$

$$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right] = 0.1f\left(0.05, -1 + \frac{0.085}{2}\right)$$

$$= 0.1f(0.05, -0.9575)$$

$$= 0.1[(-2)(0.05) + 0.9595] = 0.0858$$

$$k_4 = hf[x_0 + h, y_0 + k_3] = 0.1f[0.1, -1 + 0.0858]$$

$$= 0.1f[0.1, -0.9142]$$

$$= 0.1[(-2)(0.1) + 0.9142] = 0.0714$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.1 + 2(0.085) + 2(0.0858) + 0.0714]$$

$$= \frac{1}{6}[0.513] = 0.0855$$

$\therefore$ $$y_1 = y_0 + \Delta y = -1 + 0.0855 = -0.9145$$

Now $$x_1 = x_0 + h = 0.1, \quad y_1 = -0.9145$$

Put $n = 1$ to find y_2

Then $k_1 = hf(x_1, y_1) = 0.1f(0.1, -0.9145)$

$$= 0.1[(-2)\times.1 + 0.9145] = 0.1(0.7145) = 0.07145$$

$$k_2 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right] = 0.1f\left(0.1 + \frac{.1}{2}, -0.9145 + \frac{0.0715}{2}\right)$$

$$= 0.1f(0.15, -0.8788)$$

$$= 0.1(-2\times0.15 + 0.8788) = 0.0579$$

$$k_3 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right] = 0.1f\left(0.1 + \frac{0.1}{2}, -0.9145 + \frac{0.0579}{2}\right)$$

$$= 0.1f(0.15, -0.8856) = 0.1[(-2)\times0.15 + 0.8856]$$

$$= (0.1)(0.5856) = 0.0586$$

$$k_4 = hf[x_1 + h, y_1 + k_3] = 0.1f(0.1 + 0.1, -0.9145 + 0.0586)$$

$$= 0.1f(0.2, -0.8559)$$

$$= 0.1[(-2)\times0.2 + 0.8559] = 0.1(0.4559) = 0.0456$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.07145 + 2(0.0579) + 2(0.0586) + 0.0456]$$

$$= \frac{1}{6}[0.3501] = 0.0584$$

$\therefore$ $$y_2 = y_1 + \Delta y = -0.9145 + 0.0584 = -0.8562$$

$\therefore$ $$y(0.2) = -0.8562$$

9.7 RUNGE-KUTTA METHOD FOR THE SOLUTION OF SIMULTANEOUS EQUATIONS AND SECOND ORDER EQUATIONS

We have seen Runge-Kutta method for solving first order differential equations. We shall now describe the solution of second order differential equations by Runge-Kutta method. A second order differential equation can be reduced to a system of simultaneous first order equations. These first order equations can be solved by Runge-Kutta method.

Runge-Kutta Method for Simultaneous Equations

Consider the system of equations

$$\frac{dy}{dx} = f(x, y, z), \qquad y(x_0) = y_0$$

$$\frac{dz}{dx} = g(x, y, z), \qquad z(x_0) = z_0$$

where x is independent variable and y, z are dependent variables.

Starting at (x_0, y_0, z_0) the increments Δy and Δz in y and z for the increment h in x are computed by means of the formulae.

$$k_1 = hf(x_0, y_0, z_0) \qquad l_1 = hg(x_0, y_0, z_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) \qquad l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) \qquad l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) \qquad l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad \Delta z = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

Then $\quad y_1 = y_0 + \Delta y \qquad z_1 = z_0 + \Delta z$

Thus we get (x_1, y_1, z_1), where $\qquad x_1 = x_0 + h$

To find (x_2, y_2, z_2), we repeat the above algorithm replacing (x_0, y_0, z_0) by (x_1, y_1, z_1) ie. we start with (x_1, y_1, z_1)

WORKED EXAMPLES

Example 1

Solve the system of differential equations $\frac{dy}{dx} = xz + 1$, $\frac{dz}{dx} = -xy$ for $x = 0.3$, using fourth order Runge-Kutta method with the values $x = 0$, $y = 0$, $z = 1$.

Solution

Given system of equations is

$$\frac{dy}{dx} = xz + 1$$

$$\frac{dz}{dx} = -xy$$

and $\qquad x_0 = 0,\ y_0 = 0,\ z_0 = 1$ and $h = 0.3$

Here $f(x, y, z) = xz + 1$ and $g(x, y, z) = -xy$

Required the values of y and z when $h = 0.3$

$$k_1 = hf(x_0, y_0, z_0),$$
$$= h[x_0 z_0 + 1],$$
$$= (0.3)(0+1) = \mathbf{0.3}$$

$$l_1 = hg(x_0, y_0, z_0)$$
$$= h(-x_0 y_0)$$
$$= (0.3)(0) = \mathbf{0}$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$
$$= h\left[\left(x_0 + \frac{h}{2}\right)\left(z_0 + \frac{l_1}{2}\right) + 1\right]$$
$$= (0.3)\left[\left(0 + \frac{0.3}{2}\right)(1+0) + 1\right]$$
$$= (0.3)\left[\left(0.\frac{3}{2}\right) + 1\right]$$
$$= (0.3)(1.15) = \mathbf{0.345}$$

$$l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$
$$= h\left[-\left(x_0 + \frac{h}{2}\right)\left(y_0 + \frac{k_1}{2}\right)\right]$$
$$= (0.3)\left[-\left(0 + \frac{0.3}{2}\right)\left(0 + \frac{0.3}{2}\right)\right]$$
$$= -(0.3)\left(\frac{0.09}{4}\right) = \mathbf{-0.00675}$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$
$$= h\left[\left(x_0 + \frac{h}{2}\right)\left(z_0 + \frac{l_2}{2}\right) + 1\right]$$
$$= (0.3)\left[\left(0 + \frac{0.3}{2}\right)\left(1 + \frac{(-0.00675)}{2}\right) + 1\right]$$
$$= (0.3)\left[\left(\frac{0.3}{2}\right)(1 - 0.003375) + 1\right]$$
$$= (0.3)\left[\left(0.\frac{3}{2}\right)(0.996625) + 1\right]$$
$$= (0.3)(1.14949) = \mathbf{0.34485}$$

$$l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$
$$= h\left(-\left(x_0 + \frac{h}{2}\right)\left(y_0 + \frac{k_2}{2}\right)\right)$$
$$= (0.3)\left(-\left(0 + \frac{0.3}{2}\right)\left(0 + \frac{0.345}{2}\right)\right)$$
$$= -(0.3)(0.15)(0.1725)$$
$$= \mathbf{-0.00776}$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= h[(x_0 + h)(z_0 + l_3) + 1]$$
$$= (0.3)[(0 + 0.3)(1 - (0.00776)) + 1]$$
$$= (0.3)[(0.3)(0.09224) + 1]$$
$$= (0.3)[1.297672] = \mathbf{0.389302}$$

$$l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= h[-(x_0 + h)(y_0 + k_3)]$$
$$= -(0.3)[(0 + 0.3)(0 + 0.34485)]$$
$$= \mathbf{-0.0310365}$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$
$$= \frac{1}{6}[0.3 + 2(0.345) + 2(0.34485) + (0.389302)]$$
$$= \frac{1}{6}[0.3 + 0.69 + 0.6897 + 0.389302]$$
$$= \frac{1}{6}[2.069002] = \mathbf{0.344834}$$

$$\Delta z = \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$$
$$= \frac{1}{6}[0 + 2(-0.00675) + 2(-0.00776) + (-0.0310365)]$$
$$= -\frac{1}{6}[0.01350 + 0.01552 + 0.0310365]$$
$$= -\frac{1}{6}[0.0600565] = \mathbf{-0.01001}$$

Now the increments,

$$y_1 = y_0 + \Delta y = 0 + 0.344834 = 0.344834$$

and

$$z_1 = z_0 + \Delta z = 1 - 0.01001 = 0.98999$$

$\therefore$ **$x = 0.3$, $y = 0.344834$ and $z = 0.98999$** ■

Example 2

Using Runge-Kutta method of fourth order, find the approximate values of x and y at $t = 0.2$ for the following system, $\frac{dx}{dt} = 2x + y$, $\frac{dy}{dt} = x - 3y$, $t = 0$, $x = 0$ $y = 0.5$.

Solution

Given system of equations are

$$\frac{dx}{dt} = 2x + y$$
$$\frac{dy}{dt} = x - 3y$$

and $t = 0$, $x = 0$, $y = 0.5$.

Here t is the independent variable and x, y are dependent variables

Here $f(t,x,y) = 2x + y$ and $g(t,x,y) = x - 3y$, $t_0 = 0$, $x_0 = 0$, $y_0 = 0.5$.

Required the values of x and y when $t = 0.2$ $\therefore h = 0.2$.

$$\therefore \quad k_1 = hf(t_0, x_0, y_0), = h[2x_0 + y_0] = 0.2[2 \times 0 + 0.5] = \mathbf{0.1}$$

$$l_1 = g(t_0, x_0, y_0) = h[x_0 - 3y_0] = 0.2[0 - 3(0.5)] = \mathbf{-0.3}$$

$$k_2 = hf\left(t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}, y_0 + \frac{l_1}{2}\right) = h\left[2\left(x_0 + \frac{k_1}{2}\right) + y_0 + \frac{l_1}{2}\right]$$
$$= (0.2)\left[2\left(0 + \frac{0.1}{2}\right) + \left(0.5 - \frac{0.3}{2}\right)\right]$$
$$= (0.2)[0.1 + 0.35] = (0.2)(0.45) = \mathbf{0.09}$$

$$l_2 = hg\left(t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}, y_0 + \frac{l_1}{2}\right) = h\left[x_0 + \frac{k_1}{2} - 3\left(y_0 + \frac{l_1}{2}\right)\right]$$
$$= (0.2)\left[0 + \frac{0.1}{2} - 3\left(0.5 - \frac{0.3}{2}\right)\right]$$
$$= (0.2)[0.05 - 1.05] = \mathbf{-0.2}$$

$$k_3 = hf\left(t_0+\frac{h}{2}, x_0+\frac{k_2}{2}, y_0+\frac{l_2}{2}\right)$$
$$= h\left[2\left(x_0+\frac{k_2}{2}\right)+y_0+\frac{l_2}{2}\right]$$
$$= (0.2)\left[2\left(0+\frac{0.09}{2}\right)+0.5-\frac{0.2}{2}\right]$$
$$= (0.2)[0.09+0.4]$$
$$= (0.2)(0.49) = \mathbf{0.098}$$

$$l_3 = hg\left(t_0+\frac{h}{2}, x_0+\frac{k_2}{2}, y_0+\frac{l_2}{2}\right)$$
$$= h\left[x_0+\frac{k_2}{2}-3\left(y_0+\frac{l_2}{2}\right)\right]$$
$$= (0.2)\left[0+\frac{0.09}{2}-3\left(0.5-\frac{0.2}{2}\right)\right]$$
$$= (0.2)[0.045-3(0.4)]$$
$$= (0.2)[-1.155] = \mathbf{-0.231}$$

$$k_4 = hf(t_0+h, x_0+k_3, y_0+l_3),$$
$$= h[2(x_0+k_3)+y_0+l_3]$$
$$= (0.2)[2(0+0.098)+0.5-0.231]$$
$$= (0.2)[0.196+0.269]$$
$$= (0.2)(0.465) = \mathbf{0.093}$$

$$l_4 = hg(t_0+h, x_0+k_3, y_0+l_3)$$
$$= h[x_0+k_3-3(y_0+l_3)]$$
$$= (0.2)[0+0.098-3(0.5-0.231)]$$
$$= (0.2)[0.098-0.807]$$
$$= (0.2)(-0.709) = \mathbf{-0.1418}$$

Now the increments,

$$\Delta x = \frac{1}{6}[k_1+2k_2+2k_3+k_4]$$
$$= \frac{1}{6}[0.1+2(0.09)+2(0.098)+(0.093)]$$
$$= \frac{1}{6}[0.569] = \mathbf{0.09483}$$

$$\Delta y = \frac{1}{6}[l_1+2l_2+2l_3+l_4]$$
$$= \frac{1}{6}[-0.3+2(-0.2)+2(-0.231)-0.1418]$$
$$= -\frac{1}{6}[1.3038] = \mathbf{-0.2173}$$

$\therefore$
$$x_1 = x_0+\Delta x = 0+0.09483 = 0.09483$$
$$y_1 = y_0+\Delta y = 0.5-0.2173 = 0.2827$$

$\therefore$
$$\mathbf{t = 0.2,\quad x = 0.09483,\quad y = 0.2827}$$ ■

Runge-Kutta Method for Second Order Equations

Consider the second order differential equation $y'' = g(x, y, y')$, given $y(x_0) = y_0$, $y'(x_0) = y_0'$

Put $y' = z$. Hence $y'' = \dfrac{dz}{dx} = z'$

So, the second order differential equation is reduced to two simultaneous first order equations

$$y' = z \text{ and } z' = g(x, y, z)$$

$$\Rightarrow \quad \frac{dy}{dx} = z \text{ and } \frac{dz}{dx} = g(x, y, z)$$

Here $f(x, y, z) = z$, x is independent variable, y and z are dependent variables.
These equations can be solved by using the algorithm in 9.4.1.

WORKED EXAMPLES

Example 3

Given $y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$ find the value of $y(0.1)$ by R-K method of fourth order.

Solution

Given equation is $\quad y'' = -(xy' + y), \qquad y(0) = 1, y'(0) = 0$

Put $\quad y' = z \qquad \therefore \; y'' = \dfrac{dz}{dx}$

$\therefore$ the equation is reduced to a system of first order simultaneous equations.

$$\frac{dy}{dx} = z \qquad \text{and} \qquad \frac{dz}{dx} = -(xz + y)$$

Here $\quad f(x, y, z) = z, \qquad$ and $\quad g(x, y, z) = -(xz + y)$

To find $y(0.1)$

we have $\quad x_0 = 0, \quad y_0 = 1, \quad z_0 = y_0' = 0, \quad h = 0.1$

By Runge-Kutta method,

$$k_1 = hf(x_0, y_0, z_0) = (0.1)f(0,1,0) = \mathbf{0},$$

$$l_1 = hg\,(x_0, y_0, z_0) = (0,1)\;g\;(0,1,0) = (0.1)\,[-(0+1)] = \mathbf{-0.1}$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$= 0.1f\left[0 + \frac{0.1}{2}, 1 + 0, 0 + \frac{(-0.1)}{2}\right]$$

$$= 0.1f[0.05, 1, -0.05] = 0.1(-0.05) = \mathbf{-0.005}$$

$$l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$= 0.1g[0.05, 1, -0.05] = (-0.1)[(0.05)(-0.05) + 1] = \mathbf{-0.0998}$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= 0.1f\left[0.05, 1 + \frac{(-.005)}{2}, 0 + \left(\frac{-0.0998}{2}\right)\right]$$

$$= 0.1f[0.05, 0.9975, -0.0499] = 0.1(-0.0499) = \mathbf{-0.00499}$$

$$l_3 = hg\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right]$$

$$= 0.1g[0.05, 0.9975, -0.0499]$$

$$= (-0.1)[(0.05)(-0.0499) + 0.9975] = \mathbf{-0.0995}$$

$$k_4 = hf[x_0 + h, y_0 + k_3, z_0 + l_3]$$
$$= 0.1f[0.1, 1 + (-0.00499), 0 + (-0.0995)]$$
$$= 0.1f[0.1, 0.995, -0.0995] = 0.1(-0.0995) = \mathbf{-0.00995}$$

Now $$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[0 + 2(-0.005) + 2(-0.00499) + (-0.00995)] = \mathbf{-0.00498}$$

$\therefore$ $$y_1 = y_0 + \Delta y = 1 - 0.00498 = 0.9950$$

$\therefore$ $\mathbf{y(0.1) = 0.9950}$ ■

Example 4

Consider the second order initial value problem $y'' - 2y' + 2y = e^{2t}\sin t$ with $y(0) = -0.4$ and $y'(0) = -0.6$. Using fourth order Runge-Kutta method, find $y(0.2)$.

Solution

Given equation is $y'' - 2y' + 2y = e^{2t}\sin t$, $y(0) = -0.4$ and $y'(0) = -0.6$

$y'(0) = -0.6$ Assume $t = x$, x is independent variable, y and z are dependent variables

Put $y' = z$, then $y'' = z'$

$\therefore$ the equation is $z' - 2z + 2y = e^{2x}\sin x$ $\Rightarrow z' = 2z - 2y + e^{2x}\sin x$

That is the equation is reduced to a system of first order simultaneous equations.

$\therefore$ $$\frac{dy}{dx} = z \quad \text{and} \quad \frac{dz}{dx} = 2z - 2y + e^{2x}\sin x$$

Here $f(x, y, z) = z$ and $g(x, y, z) = 2z - 2y + e^{2x}\sin x$

To find $y(0.2)$

we have $x_0 = 0$, $y_0 = -0.4$ $z_0 = y'(0) = -0.6$ and $h = 0.2$

By Runge-Kutta method,

$$k_1 = hf(x_0, y_0, z_0)$$
$$= 0.2f(0, -0.4, -0.6) = 0.2(-0.6) = \mathbf{-0.12}$$

$$l_1 = hg(x_0, y_0, z_0)$$
$$= 0.2g(0, -0.4, -0.6)$$
$$= 0.2[2(-0.6) - 2(-0.4) + e^0\sin 0] = \mathbf{0.136}$$

$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right]$$
$$= 0.2f[0 + 0.1, -0.4 + (-0.06), -0.6 + 0.068]$$
$$= 0.2f[0.1, -0.46, -0.532] = 0.2(-0.532) = \mathbf{-0.1064}$$
$$l_2 = 0.2g[0.1, -0.46, -0.532]$$
$$= 0.2[2(-0.532) - 2(-0.46) + e^{0.2}\sin(0.1)]$$
$$= 0.2[-1.064 + 0.92 + 0.1219] = \mathbf{-0.0044}$$
$$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right]$$
$$= 0.2f\left[0.1, -0.4 + \left(\frac{-0.1064}{2}\right), -0.6 + \left(\frac{-0.0044}{2}\right)\right]$$
$$= 0.2f[0.1, -0.4532, -0.6022] = 0.2(-0.6022) = \mathbf{-0.1204}$$
$$l_3 = hg\left[x_0 + \frac{h}{2}, y + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right]$$
$$= 0.2g[0.1, -0.4532, -0.6022]$$
$$= 0.2[2(-0.6022) + 2(0.4532) + e^{0.2}\sin(0.1)]$$
$$= 0.2[-1.2044 + 0.9064 + 0.1219] = \mathbf{-0.0352}$$
$$k_4 = hf[x_0 + h, y_0 + k_3, z_0 + l_3]$$
$$= 0.2f[0.2, -0.4 + (-0.1204), -0.6 - 0.0352]$$
$$= 0.2f[0.2, -0.5204, -0.6352] = 0.2(-0.6352) = \mathbf{-0.127}$$

Now $$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$
$$= \frac{1}{6}[-0.12 + 2(-0.1064) + 2(-0.1204) + (-0.127)]$$
$$= \frac{1}{6}[-0.7006] = -0.1168$$

$\therefore$ $$y_1 = y_0 + \Delta y = -0.4 - 0.1168 = \mathbf{-0.5168}$$

$\Rightarrow$ $$\mathbf{y(0.2) = -0.5168}$$ ■

Example 5

Given $\dfrac{d^2y}{dx^2} - y^3 = 0$, $y(0) = 10$, $y'(0) = 5$. Evaluate $y(0.1)$ using Runge-Kutta method.

Solution

Given $$y'' - y^3 = 0$$

$\Rightarrow$ $$y'' = y^3$$

Put $y' = z \quad \therefore y'' = z'$

That is the given equation is reduced to a system of first order simultaneous equations

$$\frac{dy}{dx} = z \quad \text{and} \quad \frac{dz}{dx} = y^3$$

Here $f(x,y,z) = z$ and $g(x,y,z) = y^3$

To find y(0.1)

we have $x_0 = 0,\ y_0 = 10,\ z_0 = y'(0) = 5$ and $h = 0.1$

By Runge-Kutta method,

$$k_1 = hf(x_0, y_0, z_0) = hz_0 = 0.1(5) = \mathbf{0.5}$$

$$l_1 = hg(x_0, y_0, z_0) = 0.1y_0^3 = 0.1(10)^3 = \mathbf{100}$$

$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right]$$
$$= hf\left[0 + \frac{(0.1)}{2}, 10 + \frac{0.5}{2}, 5 + \frac{100}{2}\right]$$
$$= 0.1f[0.05, 10.25, 55] = 0.1(55) = \mathbf{5.5}$$

$$l_2 = hg\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right]$$
$$= hg[0.05, 10.25, 55] = 0.1(10.25)^3 = \mathbf{107.689}$$

$$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right]$$
$$= hf\left[0.05, 10 + \frac{55}{2}, 5 + \frac{107.689}{2}\right]$$
$$= (0.1)f[0.05, 12.75, 58.8445] = (0.1)(58.8445) = \mathbf{5.8845}$$

$$l_3 = hg\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right]$$
$$= (0.1)g[0.05, 12.75, 58.8445] = (0.1)(12.75)^3 = \mathbf{207.267}$$

$$k_4 = hf[x_0 + h, y_0 + k_3, z_0 + l_3]$$
$$= 0.1f[0.1, 10 + 5.8845, 5 + 207.267]$$
$$= 0.1f[0.1, 15.8845, 212.267] = 0.1(212.267) = \mathbf{21.2267}$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$
$$= \frac{1}{6}[0.5 + 2(5.5) + 2(5.8845) + 21.2267] = \mathbf{7.416}$$

$$\therefore \quad y_1 = y_0 + \Delta y = 10 + 7.416 = \mathbf{17.416}$$

$$\therefore \quad \mathbf{y(0.1) = 17.416}$$

■

Exercises 9.2

(1) Solve, using fourth order Runge-Kutta method $\frac{dy}{dx}+\frac{y}{x}=\frac{1}{x^2}$, $y(1)=1$. Evaluate the value of y when $x = 1.1$ [Take $h = 0.5$].

(2) Using Runge-Kutta method of fourth order, solve for y at $x = 1.2, 1.4$ from $\frac{dy}{dx}=\frac{2xy+e^x}{x^2+xe^x}$ with $y(1)=0$.

(3) Find $y(0.8)$ given that $y'=y-x^2, y(0.6)=1.739$ by using Runge-Kutta method of fourth order. Take $h = 0.1$.

(4) Find $y(0.3)$ given that $\frac{dy}{dx}+y+xy^2=0, y(0)=1$ by taking $h = 0.1$, using Runge-Kutta method.

(5) Apply Runge-Kutta method of order 4 to find $y(0.2)$ given that $\frac{dy}{dx}=3xy+\frac{y}{2}$ $y(0)=1$ taking $h = 0.1$.

(6) Compute $y(0.1)$ and $y(0.2)$ by Runge-Kutta method of 4th order for the differential equation $\frac{dy}{dx}=xy+y^2, y(0)=1$.

(7) Given $\frac{dy}{dx}+\frac{y^2-2x}{y^2+x}$, $y(0)=1$, find y for $x = 0.1, 0.2, 0.3$ by R-K Method.

(8) Given the equation $\frac{dy}{dx}=y-\frac{2x}{y}$, $y(0)=1$, by R.K method find y at $x = 0.1, 0.4$.

(9) Given $\frac{dy}{dx}=x^3+\frac{y}{2}$, $y(1)=2$ find $y(1.1)$ and $y(1.2)$ using R-K method.

(10) Use the Runga-Kutta method to determine the approximate value of y at $x = 0.1$ if y satisfies the differential equation $\frac{d^2y}{dx^2}-x^2\frac{dy}{dx}-2xy=1$ with $y(0)=1$, $y'(0)=0$.

(11) Using Runge-Kutta method solve $y''-xy'^2+y^2=0$, $y(0)=1, y'(0)=0$, and find $y(0.2)$, $y'(0.2)$ [Take $h = 0.2$].

(12) Solve using Rung-Kutta method $y''=xy'-4y, y(0)=3$, $y'(0)=0$ and find $y(0.1)$.

(13) Find $y(0.1)$ by Runge-Kutta method $y''+2xy'-4y=0, y(0)=0.2$, $y'(0)=0.5$.

(14) Given that $\frac{dy}{dx}=x^3+y, y(0)=2$. Compute $y(0.2)$, $y(0.4)$, $y(0.6)$ by Runge-Kutta method of fourth order.

(15) Solve the system of differential equations $\frac{dy}{dx}=xz+1, \frac{dz}{dx}=-xy$ for $x=0.6$, using 4th order Runge-Kutta method with initial values $x=0.3$, $y=0.3448$, $z=0.99$.

(16) Solve the system $\frac{dy}{dx}=z, \frac{dz}{dx}=-x\,z^2-y^2$ for $x=0.2$, Correct to 4 decimal places, given $x=0$, $y=1$, $z=0$.

Answers 9.2

(1) 0.9958,
(2) 0.1402, 0.2705
(3) 1.8763; 2.0142
(4) $y(0.1)=0.9006, y(0.2)=0.8046, y(0.3)=0.7144$
(5) 2.5005
(6) 1.1169, 1.2774
(7) 1.0911, 1.1677, 1.2352
(8) 1.1832, 1.3416
(9) 2.2213, 2.4914
(10) $y(0.1) = 1.0053$
(11) $y(0.2) = 0.9801, y'(0.2) = -0.178$
(12) $y(0.1) = 2.9399$
(13) $y(0.1) = 0.2542$
(14) 2.4432; 2.9903; 3.6805
(15) $x = 0.6, y = 0.7738, z = 0.9121$
(16) $x = 0.2, y = 0.9801, z = -0.1970$

9.5 MILNE'S PREDICTOR–CORRECTOR METHOD

Consider the differential equation $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.

Divide the range of x into a number of subintervals of equal width h.
If x_i and x_{i+1} are consecutive points then $x_{i+1} = x_i + h$.

By Euler's formula we get

$$y_{i+1} = y_i + hf(x_i, y_i),\ i = 1, 2, 3, \ldots \qquad (1)$$

By one form of Euler's modified formula, we get

$$y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}, y_{i+1})],\ i = 1, 2, \ldots \qquad (2)$$

The value y_{i+1} is first estimated by (1) and this value is substituted in the R.H.S of (2).
Then we get a better approximation of y_{i+1} from (2). This value is again substituted in (2) to find a still better value of y_{i+1}.

This process is repeated until we get two consecutive approximations of y_{i-1} are almost same. This method of refining an initially rough estimate by means of a more accurate formula is called a **predictor-corrector method**.

The formula (1) is called the **predictor formula** and formula (2) is called the **corrector formula**.

Milne's method

Consider $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.

We know Newton's forward formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

where $$u = \frac{x - x_0}{h} \Rightarrow x = x_0 + hu$$

Replace y by $y' = \left(\frac{dy}{dx}\right)$, then

$$y' = y_0' + u\Delta y_0' + \frac{u(u-1)}{2!}\Delta^2 y_0' + \frac{u(u-1)(u-2)}{3!}\Delta^3 y'_0 + \dots \quad (1)$$

Integrating w.r.to x in the interval $[x_0, x_0 + 4h]$, we get

$$\int_{x_0}^{x_0+4h} y'dx = \int_{x_0}^{x_0+4h}\left[y_0' + u\Delta y_0' + \left(\frac{u(u-1)}{2!}\right)\Delta^2 y_0' + \dots\right]dx$$

$$\Rightarrow \quad [y(x)]_{x_0}^{x_0+4h} = h\int_0^4 \left[y_0' + \Delta y_0' + \left(\frac{u^2-u}{2!}\right)\Delta^2 y_0' + \dots\right]du$$

$[\because$ when $x = x_0$, $u = 0$, $x = x_0 + 4h$; $u = 4]$

$$\Rightarrow \quad y(x_0+4h) - y(x_0) = h\left[y_0'u + \frac{u^2}{2}\Delta y_0' + \frac{1}{2}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)\Delta^2 y_0' + \dots\right]_0^4$$

$$\Rightarrow \quad y_4 - y_0 = h\left[4y_0' + 8\Delta y_0' + \frac{20}{3}\Delta^2 y_0' + \dots\right]$$

$$= h\left[4y_0' + 8(y_1' - y_0') + \frac{20}{3}(y_2' - 2y_1' + y_0') + \frac{8}{3}(y_3' - 3y_2' + 3y_1' - y_0')\right]$$

[neglecting higher order differences]

$$= \frac{4h}{3}[2y_1' - y_2' + 2y_3']$$

$$\therefore \quad y_4 = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$$

Since x_0, x_1, x_2, x_3, x_4 are any 5 consecutive values of x, the above equation can be written generally as

$$\boxed{y_{n+1} = y_{n-3} + \frac{4h}{3}[2y_{n-2}' - y_{n-1}' + 2y_n']} \quad (2)$$

This is called Milne's Predictor formula.

Note that this formula can be used to predict the value of y_4 when the values of y_0, y_1, y_2, y_3 are known.

To obtain a corrector formula, integrating (1) w.r.to x in the interval $[x_0, x_0, + 2h]$, we get

$$\int_{x_0}^{x_0-2h} y'dx = \int_{x_0}^{x_0+2h}\left[y_0' + u\Delta y_0' + \frac{u(u-1)}{2!}\Delta^2 y_0' + \dots\right]dx$$

$$[y(x)]_{x_0}^{x_0+2h} = h\int_0^2\left[y_0' + u\Delta y_0' + \frac{u^2-u}{2!}\Delta^2 y_0{}' \frac{(u^3-3u^2+2u)}{3!}\Delta^3 y_0' + \dots\right]du$$

$$[y(x)]_{x_0}^{x_0-2h} = h\int_0^2\left[y_0' + u\Delta y_0' + \frac{u^2-u}{2!}\Delta^2 y_0' + \frac{(u^3-3u^2+2u)}{3!}\Delta^3 y_0' + \dots\right]du$$

$$\Rightarrow \quad y(x_0+2h)-y(x_0)=h\left[y_0'\,u+\frac{u^2}{2}\Delta y_0'+\frac{1}{2!}\left(\frac{u^3}{3}-\frac{u^2}{2}\right)\Delta^2 y_0'+\frac{1}{3!}\left(\frac{u^4}{4}-u^3+u^2\right)\Delta^3 y_0'+\ldots\right]_0^2$$

$$\Rightarrow \quad y_2-y_0=h\left[2y_0'+2\Delta y_0'+\frac{1}{3}\Delta^2 y_0'+\ldots\right]$$

$$\Rightarrow \quad y_2-y_0=h\left[2y_0'+2(y_1'-y_0')+\frac{1}{3}(y_2'-2y_1'+y_0')\right]$$

$$=h\left[2y_0'+2(y_1'-y_0')+\frac{1}{3}(y_2'-2y_1'+y_0')\right]$$

[neglecting higher powers]

$$\Rightarrow \quad y_2=y_0+\frac{h}{3}[y_0'+4y_1'+y_2']$$

Since x_0, x_1, x_2 are consecutive values of x the above relation can be written generally as

$$\boxed{y_{n+1}=y_{n-1}+\frac{h}{3}[y_{n-1}'+4y_n'+y_{n+1}']} \tag{3}$$

This is known as Milne's Corrector formula.

Note:

(1) In problems if the first 4 values of y are not given we have to determine them by using Taylor's series or Euler's or Runge-Kutta method. Usually, these values will be given.

(2) To apply Milne's Predictor-Corrector method, we need four starting values of y. Hence this method is a **multi-step method**.

WORKED EXAMPLES

Example 1

Solve $y'=\frac{1}{2}(1+x^2)y^2$, $y(0)=1$, $y(0.1)=1.06$, $y(0.2)=1.12$, $y(0.3)=1.21$. Compute $y(0.4)$, using Milne's Predictor Corrector formula.

Solution

Given equation is $\quad y'=\frac{1}{2}(1+x^2)y^2$

Here $\quad f(x,y)=\frac{1}{2}(1+x^2)y^2$ and $h=0.1$

Also given $x_0=0,\quad x_1=0.1,\quad x_2=0.2,\quad x_3=0.3,$

and $\quad y_0=1,\quad y_1=1.06,\quad y_2=1.12,\quad y_3=1.21$

Required $y(0.4)=y_4$ at $x_4=0.4$

Milne's Predictor formula is

$$y_{n+1}=y_{n-3}+\frac{4h}{3}[2y'_{n-2}-y'_{n-1}+2y'_n] \tag{1}$$

Putting $n = 3$, we get

$$y_4 = y_0 + \frac{4(.1)}{3}[2y'_1 - y'_2 + 2y'_3]$$

we have

$$y' = \frac{1}{2}(1 + x^2)y^2$$

$$\therefore \quad y_1' = \left[\frac{1}{2}(1 + x^2)y^2\right]_{(x_1, y_1)}$$

$$= \frac{1}{2}\left[(1 + x_1^2)y_1^2\right] = \frac{1}{2}\left[1 + (0.1)^2\right](1.06)^2 = 0.5674$$

$$y_2' = \left[\frac{1}{2}(1 + x^2)y^2\right]_{(x_2, y_2)}$$

$$= \frac{1}{2}\left[(1 + x_2^2)y_2^2\right] = \frac{1}{2}[1 + (0.2)^2](1.12)^2 = 0.6522$$

$$y_3' = \left[\frac{1}{2}(1 + x^2)y^2\right]_{(x_3, y_3)}$$

$$= \frac{1}{2}\left[(1 + x_3^2)y_3^2\right] = \frac{1}{2}[1 + (0.3)^2](1.21)^2 = 0.7979$$

Substituting in (2), we get

$$y_4 = 1 + \frac{0.4}{3}\left[2(0.5674) - 0.6522 + 2(0.7979)\right] = 1.2771$$

This is the Predictor value.

The accuracy of the value may be increased using corrector formula.

Milne's corrector formula is

$$y_{n+1} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}] \quad (3)$$

Put $n = 3$ in (3), then

$$y_4 = y_2 + \frac{0.1}{3}[y'_2 - 4y'_3 + y'_4] \quad (4)$$

Since y'_4 is not known, we find y'_4.

Now

$$y'_4 = \left[\frac{1}{2}(1 + x^2)y^2\right]_{(x_4, y_4)}$$

$$= \frac{1}{2}\left[(1 + x_4^2)y_4^2\right] = \frac{1}{2}[1 + (0.4)^2](1.2771)^2 = 0.9460$$

Substituting in (4) we get

$$y_4 = 1.12 + \frac{0.1}{3}[0.6522 + 4(0.7979) + 0.9460]$$

$$= 1.12 + 0.1597 = 1.2797$$

$\therefore$ the corrector value is $y_4 = y(0.4) = 1.2797$

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Example 2

Given $5xy' + y^2 = 2$, $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$, $y(4.3) = 1.0143$ **compute** $y(4.4)$ **using Milne's method.**

Solution

Given equation is $\quad 5xy' + y^2 = 2 \quad \Rightarrow 5xy' = y^2 - 2$

$$\Rightarrow y' = \frac{2 - y^2}{5x}$$

Here $\quad f(x, y) = \frac{y^2 - 2}{5x}$ and $h = 0.1$

Also given $x_0 = 4$, $x_1 = 4.1$, $x_2 = 4.2$, $x_3 = 4.3$,
and $y_0 = 1$, $y_1 = 1.0049$, $y_2 = 1.0097$, $y_3 = 1.0143$

Required $y(4.4) = y_4$ **at** $x_4 = 4.4$

Milen's Predictor formula is

$$y_{n+1} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n] \qquad (1)$$

Putting $n = 3$ **in (1) we get,** $y_4 = y_0 + \frac{4h}{3}[2y'_1 - y'_2 + 2y'_3]$

we have $\quad y' = \frac{2 - y^2}{5x}$

$$\therefore \quad y'_1 = \left(\frac{2 - y^2}{5x}\right)_{(x_1, y_1)} = \frac{2 - y_1^2}{5x_1} = \frac{2 - (1.0049)^2}{5(4.1)} = \mathbf{0.0483}$$

$$y'_2 = \left(\frac{2 - y^2}{5x}\right)_{(x_2, y_2)} = \frac{2 - y_2^2}{5x_2} = \frac{2 - (1.0097)^2}{5(4.2)} = \mathbf{0.0467}$$

and

$$y'_3 = \left(\frac{2 - y^2}{5x}\right)_{(x_3, y_3)} = \frac{2 - y_3^2}{5x_3} = \frac{2 - (1.0143)^2}{5(4.3)} = \mathbf{0.0452}$$

Substituting in (2), we get

$$\Rightarrow \quad y_4 = 1 + \frac{4(0.1)}{3}[2(0.0483) - 0.0467 + 2(0.0452)]$$

$$\mathbf{y_4 = 1.0187}$$

This is the predictor value of y_4.
An improved value of y_4 may be obtained by the corrector formula.

$$y_{n+1} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}] \qquad (3)$$

Putting $n = 3$ **in (3) we get,** $\quad y_4 = y_2 + \frac{h}{3}[y'_2 + 4y'_3 + y'_4] \qquad (4)$

But y_4' is not known, so we find y'_4.

Now $$y'_4 = \left(\frac{2-y^2}{5x}\right)_{(x_4,y_4)} = \frac{2-y_4^2}{5x_4} = \frac{2-(1.0187)^2}{5(4.4)} = \mathbf{0.0437}$$

$\therefore$ $$y_4 = 1.0097 + \frac{(0.1)}{3}[0.0467 + 4(0.0452) + 0.0437] = \mathbf{1.0187}$$

$\therefore$ the corrector value is $\mathbf{y_4 = y(4.4) = 1.0187}$ ■

Example 3

Given $y' = xy + y^2$, $y(0) = 1$, find $y(0.1)$ by Taylor's method, $y(0.2)$ by Euler's method, $y(0.3)$ by Runge-Kutta method and $y(0.4)$ by Milne's method.

Solution

Given equation is $y' = xy + y^2$

Here $f(x,y) = xy + y^2$

Also given $x_0 = 0$, $y_0 = 1$, and $h = 0.1$

when $x_1 = 0.1, y_1 = y(0.1)$ is required by **Taylor's series method.**

By Taylor's formula,

$$y_1 = y_0 + \frac{h}{1!}y'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \dots \quad (1)$$

we have $$y' = xy + y^2$$

$\therefore$ $$y'' = x.y' + y + 2yy'$$
$$y''' = x.y'' + y' + y' + 2(y.y'' + y'.y')$$
$$= xy'' + 2y' + 2yy'' + 2(y')^2$$

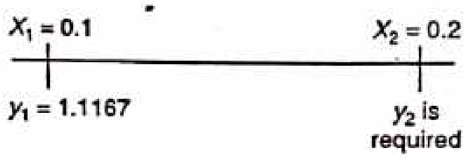
At $(x_0, y_0) = (0, 1)$, $$y'_0 = 0\times1 + 1 = 1$$
$$y''_0 = 0\times1 + 1 + 2\times1\times1 = 3$$
$$y'''_0 = 0 + 2\times1 + 2\times1\times3 + 2\times1^2 = 10$$

$\therefore$ $$y_1 = 1 + 0.1(1) + \frac{(0.1)^2}{2}3 + \frac{(0.1)^3}{6}\times10 = 1.1167$$

$\therefore$ $$\mathbf{y_1 = y(0.1) = 1.1167}$$

Next to find $y_2 = y(0.2)$, we use **Euler's method**

By Euler's formla,



$$y_2 = y_1 + hf(x_1, y_1)$$
$$= 1.1167 + 0.1f(0.1, 1.1167)$$
$$= 1.1167 + 0.1\left[0.1(1.1167) + (1.1167)^2\right] \quad [\because y' = f(x,y)]$$
$$= 1.2526$$
$$\mathbf{y_2 = y(0.2) = 1.2526}$$

$X_3 =$

y_3 is required

$X_2 = 0.2$

$y_2 = 1.2626$

Next to find $y_3 = y(0.3)$. We use **Runge-Kutta method**

we have $\quad y' = f(x,y) = xy + y^2$

By R-K formula, $\quad y_3 = y_2 + \Delta y$

where $\quad \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Now $\quad k_1 = hf(x_2, y_2), \quad k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right)$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right), \quad k_4 = hf(x_2 + h, y_2 + k_3)$$

$\therefore$
$$k_1 = 0.1f(0.2, 1.2526) = 0.1[0.2(1.2526) + (1.2526)^2] = \mathbf{0.1820}$$

$$k_2 = 0.1f\left(0.2 + \frac{0.1}{2}, 1.2526 + \frac{0.1820}{2}\right)$$
$$= 0.1f(0.25, 1.3436)$$
$$= 0.1[0.25(1.3436) + (1.3436)^2] = \mathbf{0.2141}$$

$$k_3 = 0.1f\left(0.25, 1.2526 + \frac{0.2141}{2}\right)$$
$$= 0.1f(0.25, 1.3597)$$
$$= 0.1[0.25(1.3597) + (1.3597)^2] = \mathbf{0.2189}$$

$$k_4 = 0.1f(0.2 + 0.1, 1.2526 + 0.2189)$$
$$= 0.1f(0.3, 1.4715)$$
$$= 0.1[0.3(1.4715) + (1.4715)^2] = \mathbf{0.2607}$$

$\therefore$
$$\Delta y = \frac{1}{6}[0.1820 + 2(0.2141) + 2(0.2189) + 0.2607]$$
$$= \frac{1}{6}(1.3087) = \mathbf{0.2181}$$

$\therefore$
$$y_3 = 1.2526 + 0.2181 = \mathbf{1.4707}$$

$\therefore$
$$y_3 = y(0.3) = \mathbf{1.4707}$$

Now we know 4 values

$X_0 = 0$	$X_1 = 0.1$	$X_2 = 0.2$	$X_3 = 0.3$	$X_4 = 0.4$
$y_0 = 1$	$y_1 = 1.1167$	$y_2 = 1.2526$	$y_3 = 1.4707$	y_4 is required

We have to find y_4 by Milne's method.

Milne's Predictor formula is

$$y_{n+1} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n] \quad (1)$$

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Putting $n = 3$ in (1), we get, $y_4 = y_0 + \frac{4h}{3}[2y'_1 - y'_2 + 2y'_3]$

$$y' = xy + y^2$$

we have

$$y'_1 = (xy + y^2)_{(x_1, y_1)} = [0.1\,(1.1167) + (1.1167)^2] = \mathbf{1.3587}$$

$\therefore$

$$y'_2 = (xy + y^2)_{(x_2, y_2)} = [0.2(1.2526) + (1.2526)^2] = \mathbf{1.820}$$

$$y'_3 = (xy + y^2)_{(x_3, y_3)} = [0.3(1.4707) + (1.4707)^2] = \mathbf{2.6042}$$

$$y_4 = 1 + \frac{4(0.1)}{3}[2(1.3587) - 1.820 + 2(2.6042)]$$

$\therefore$

$$y_4 = \mathbf{1.8141}$$

$\Rightarrow$

This is the predictor value of y_4.

To find the corrector value we use Milne's corrector formula.

$$y_{n+1} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}] \qquad (2)$$

Putting $n = 3$ in (2), we get, $y_4 = y_2 + \frac{h}{3}[y'_2 + 4y'_3 + y'_4]$

But y'_4 is not known and so we find y'_4

Now

$$y'_4 = (y')_{(x_4, y_4)} = (x_4 y_4 + y_4^2) = 0.4(1.8141) + (1.8141)^2 = \mathbf{4.0166}$$

$\therefore$

$$y_4 = 1.2526 + \frac{0.1}{3}[1.820 + 4(2.6042) + 4.0166] = \mathbf{1.7944}$$

Since the Predictor and corrector values of y_4 are not close to each other, we improve the values of y_4 repeating the corrector process taking $y_4 = 1.7944$ as predictor value.

$\therefore$ Corrector value is $\quad y_4 = y_2 + \frac{h}{3}[y'_2 + 4y'_3 + y'_4]$

But

$$y'_4 = (y')_{(x_4, y_4)} = x_4 y_4 + y_4^2$$

$$= 0.4(1.7944) + (1.7944)^2 = \mathbf{3.9376}$$

$\therefore$ the corrector value is $y_4 = 1.2526 + \frac{0.1}{3}[1.820 + 4\,(2.6042) + 3.9376] = \mathbf{1.7917}$

$\therefore$ the value for two decimal places is $y_4 = \mathbf{1.79}$ ■

Example 4

Given $\frac{dy}{dx} = x^3 + y, y(0) = 2, y(0.2) = 2.073, y(0.4) = 2.452, y(0.6) = 3.023$. Compute $y(0.8)$ by Milne's predictor-corrector method taking $h = 0.2$.

Solution

Given equation is $y' = x^3 + y$

Here $f(x,y) = x^3 + y$ and $h = 0.2$

Also given $x_0 = 0,\quad x_1 = 0.2,\quad x_2 = 0.4,\quad x_3 = 0.6,\quad x_4 = 0.8$

$y_0 = 2,\quad y_1 = 2.073\quad y_2 = 2.452,\quad y_3 = 3.023,$

y_4 is required by Milne's method.

Milne's Predictor formula is $y_{n+1} = y_{n-3} + \dfrac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n]$ (1)

Putting $n = 3$ in (1), we get $y_4 = y_0 + \dfrac{4h}{3}[2y'_1 - y'_2 + 2y'_3]$

we have $y' = x^3 + y$

$$\therefore \quad y'_1 = (y')_{(x_1,y_1)} = x_1^3 + y_1 = (0.2)^3 + 2.073 = \mathbf{2.081}$$

$$y'_2 = (y')_{(x_2,y_2)} = x_2^3 + y_2 = (0.4)^3 + 2.452 = \mathbf{2.516}$$

$$y'_3 = (y')_{(x_3,y_3)} = x_3^3 + y_3 = (0.6)^3 + 3.023 = \mathbf{3.239}$$

$$\therefore \quad y_4 = 2 + \frac{4(0.2)}{3}[2(2.081) - 2.516 + 2(3.239)]$$

$$\Rightarrow \quad y_4 = \mathbf{4.1664}$$

This is the Predictor value.
We shall now find the corrector value.

Milne's Corrector formula is (2)

$$y_{n+1} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}]$$

Putting n = 3, we get $y_4 = y_2 + \dfrac{h}{3}[y'_2 + 4y'_3 + y'_4]$

But y'_4 is not known and so we find y'_4.

Now $y'_4 = (y')_{(x_4,y_4)} = x_4^3 + y_4 = (0.8)^3 + 4.1664 \doteq 4.6784$

$$\therefore \quad y_4 = 2.452 + \frac{0.2}{3}[2.516 + 4(3.239) + 4.6784] = \mathbf{3.7954}$$

Since the Predictor and corrector values differ much, we repeat the process taking $y_4 = 3.7954$ as the Predictor value.

Corrector value is $y_4 = y_2 + \dfrac{h}{3}[y'_2 + 4y'_3 + y'_4]$

But $y'_4 = (y')_{(x_4,y_4)} = x_4^3 + y_4 = (0.8)^3 + 3.7954 = 4.3074$

$$\therefore \quad y_4 = 2.452 + \frac{0.2}{3}[2.516 + 4(3.239) + 4.3074]$$

$$= 2.452 + \frac{0.2}{3}(19.7794) = 3.7706$$

Again $\quad y_4 = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$

where $\quad y_4' = (y')_{(x_4, y_4)} = x_4^3 + y_4 = (0.8)^3 + 3.7706 = 4.2826$

$\therefore \quad y_4 = 2.452 + \frac{0.2}{3}[2.516 + 4(3.239) + 4.2826]$

$\mathbf{y_4 = 3.7690}$ ■

Example 5

Use Taylor series method to solve $\frac{dy}{dx} = x^2 + y^2 - 2$, $y(0) = 1$, at $x = \pm 0.1, 0.2$, continue the solution of the problem at $x = 0.3$ using Milne's method.

Solution

Given equation is $y' = x^2 + y^2 - 2$

Here $\quad x_0 = 0, \quad y_0 = 1 \quad$ and $\quad h = 0.1$

We have to find the values of y at $x = -0.1$, $\quad x = 0.1$, and $\quad x = 0.2$

In the usual notation the corresponding y's are denoted by y_{-1}, y_1 and y_2

Taylor series algorithm is

$$y_{n+1} = y_n + \frac{h}{1!}y_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n''' + \ldots \tag{1}$$

where y_n^r is the r^{th} derivative of y at (x_n, y_n), $\quad n = 0, 1, 2, \ldots$

$$\therefore \quad y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots \tag{2}$$

We have $\quad y' = x^2 + y^2 - 2$

$\therefore \quad y'' = 2x + 2yy'$

and $\quad y''' = 2 + 2\{yy'' + (y')^2\}$ $\quad$ (3)

At $(x_0, y_0) = (0, 1)$ $\quad y_0' = 0 + 1 - 2 = -1$

$y_0'' = 2.0 + 2.1(-1) = -2$

$y_0''' = 2 + 2\{1.(-2) + (-1)^2\} \;= 2 + (-2) = 0$

$\therefore \quad y_1 = 1 + (0.1)(-1) + \frac{(.1)^2}{2}(-2) + 0 = \mathbf{0.89}$

Putting $h = -0.1$, we get $x_0 + h = -0.1$

$\therefore$ $$y_{-1} = 1 + (-0.1) + \frac{(-0.1)^2}{2}(-2) + 0 = 1.09$$

$\therefore$ $$\mathbf{y(-0.1) = 1.09}$$

Put $n = 1$ in (1), we get $$y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h3}{3!}y_1''' + \ldots \quad (4)$$

At $(x_1, y_1) = (0.1, 0.89)$ $$y_1' = (0.1)^2 + (0.89)^2 - 2 = -1.1979$$

$$y_1'' = 2(0.1) + 2(0.89) + (-1.1979) = -1.932$$

$$y_1''' = 2 + 2\{(0.89)(-1.932) + (-1.1979)^2\} = 1.43$$

Substituting in (4), we get, $$y_2 = 0.89 + (0.1)(-1.1979) + \frac{(0.1)^2}{2}(-1.932) + \frac{(0.1)^3}{6}(1.43)$$

$$= 0.89 - 0.11979 - 0.00996 + 0.00024 = \mathbf{0.7605}$$

Thus we have 4 values $\quad x_{-1} = -0.1, \quad x_0 = 0, \quad x_1 = 0.1, \quad x_2 = 0.2$

$$y_{-1} = 1.09, \quad y_0 = 1, \quad y_1 = 0.89, \quad y_2 = 0.7605$$

Now, using Milne's Predictor corrector formula we have to find $y(0.3) = y_3$

Milne's Predictor formula is $$y_{n+1} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n] \quad (5)$$

Putting $n = 2$ in (5), we get $y_3 = y_{-1} + \frac{4h}{3}[2y_0' - y_1' + 2y_2']$

We have $y_0' = -1$, $\quad y_1' = -1.1979$ and y_2' is not known. We shall find it.

Now $$y_2' = (y')_{(x_2, y_2)} = x_2^2 + y_2^2 - 2$$

$$= (0.2)^2 + (0.7605)^2 - 2 = \mathbf{-1.3816}$$

$\therefore$ $$y_3 = 1.09 + \frac{4(0.1)}{3}[2(-1) + 1.1979 + 2(-1.3816)] = \mathbf{0.6146}$$

$$\mathbf{y_3 = 0.6146}$$

This is the Predictor value.
We shall now find the corrector value of y_3.

Milne's Corrector formula is $$y_{n+1} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}] \quad (6)$$

Putting $n = 2$ in (6), we get $y_3 = y_1 + \frac{h}{3}[y_1' + 4y_2' + y_3']$

Now y_3' is not known and we find y_3'

Now $$y_3' = (y')_{(x_3, y_3)} = x_3^2 + y_3^2 - 2 = (0.3)^2 + (0.6146)^2 - 2 = -1.5322$$

$\therefore$ $$y_3 = 0.89 + \frac{0.1}{3}[-1.1979 + 4(-1.3816) + (-1.5322)] = 0.6148$$

Hence $\quad \mathbf{y(0.3) = 0.6148}$

ADAM'S PREDICTOR AND CORRECTOR METHOD

We have seen Milne's method is derived by using Newton's forward difference formula. But Adam's method is derived using Newton's backward difference formula.

We shall state the formulae without proof.

To solve $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$ the predictor formula is

$$y_{n+1} = y_n + \frac{h}{24}[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}], \quad n = 3,4,5,\ldots$$

Corrector formula is

$$y_{n+1} = y_n + \frac{h}{24}[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}], \quad n = 2,3,\ldots$$

Note: To apply the predictor formula, we need four starting values of y which will be usually given, otherwise y can be calculated by any of the methods we have seen. Fourth order Runge-Kutta method is the most suitable one. It is found that Adam's method is more stable. **It is also known as Adam-Bashforth method.**

WORKED EXAMPLES

Example 1

Evaulate $y(1.4)$ given $y' = \frac{1}{x^2} - \frac{y}{x}$, $y(1) = 1$, $y(1.1) = 0.996$, $y(1.2) = 0.986$, $y(1.3) = 0.972$ by Adam's-Bash forth formula.

Solution

Given equation is $f(x,y) = \frac{1}{x^2}$

Here $\quad f(x,y) = \frac{1}{x^2} - \frac{y}{x} \quad$ and $\quad h = 0.1$

Also given $\quad x_0 = 1, \quad x_1 = 1.1, \quad x_2 = 1.2, \quad x_3 = 1.3$

and $\quad y_0 = 1, \quad y_1 = 0.996, \quad y_2 = 0.986, \quad y_3 = 0.972$

Required $y(1.4) = y_4$ at $x_4 = 1.4$

Adam's Predictor formula is

$$y_{n+1} = y_n + \frac{h}{24}[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}], \quad n = 3,4,5,\ldots$$

Putting $n = 3$, we get

$$y_4 = y_3 + \frac{h}{24}[55y_3' - 59y_2' + 37y_1' - 9y_0'] \quad (1)$$

we have $$y' = \frac{1}{x^2} - \frac{y}{x}$$

$\therefore$ $$y_0' = \left[\frac{1}{x^2} - \frac{y}{x}\right]_{(x_0, y_0)} = \frac{1}{x_0^2} - \frac{y_0}{x_0} = \frac{1}{1} - \frac{1}{1} = 0$$

$$y_1' = \left[\frac{1}{x^2} - \frac{y}{x}\right]_{(x_1, y_1)} = \frac{1}{x_1^2} - \frac{y_1}{x_1} = \frac{1}{(1.1)^2} - \frac{0.996}{1.1}$$

$$= 0.8264 - 0.9055 = -0.0791$$

$$y_2' = \left[\frac{1}{x^2} - \frac{y}{x}\right]_{(x_2, y_2)} = \frac{1}{x_2^2} - \frac{y_2}{x_2} = \frac{1}{(1.2)^2} - \frac{0.986}{(1.2)}$$

$$= 0.6944 - 0.8217 = -0.1273$$

$$y_3' = \left[\frac{1}{x^2} - \frac{y}{x}\right]_{(x_3, y_3)} = \left[\frac{1}{x_3^2} - \frac{y_3}{x_3}\right] = \frac{1}{(1.3)^2} - \frac{0.972}{1.3} = 0.5917 - 0.7477 = -0.156$$

Substituting in (1), the predictor value is

$$y_4 = 0.972 + \frac{0.1}{24}[55(-0.156) - 59(-0.1273) + 37(-0.0791) - 9(0)]$$

$$= 0.972 + 0.004167[-3.996] = 0.972 - 0.01665 = 0.9554$$

Thus the Predictor value is $\mathbf{y(1.4) = 0.9954}$

We improve this by using Adam's corrector formula

$$y_{n+1} = y_n + \frac{h}{24}[9y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}'], \quad n = 2, 3, \ldots$$

Putting $n = 3$, we get

$$y_4 = y_3 + \frac{0.1}{24}[9y_4' + 19y_3' - 5y_2' + y_1']$$

Here y_4' is not known and we shall find it

Now $$y_4' = (y')_{(x_4, y_4)} = \left[\frac{1}{x_4^2} - \frac{y_4}{x_4}\right]_{(x_4, y_4)} = \frac{1}{(1.4)^2} - \frac{0.9554}{1.4}$$

$$= 0.5102 - 0.6824 = -0.1722$$

Now substituting in the corrector formula (2) we get

$$y_4 = 0.972 + 0.004167[9(-0.1722) + 19(-0.156) - 5(-0.1273) - 0.0791]$$
$$= 0.972 + 0.004167[-3.9564] = 0.972 - 0.01649 = 0.9555$$

$\therefore$ the corrector value is $\mathbf{y_4 = y(1.4) = 0.9555}$ ■

Example 2

Given $\frac{dy}{dx} = x^2(1+y), y(1) = 1, y(1.1) = 1.233, y(1.2) = 1.548, y(1.3) = 1979,$ evaluate $y(1.4)$ by Adam-Bashforth method.

Solution

Given equation is $y' = x^2(1+y)$.

Here $\quad f(x,y) = x^2(1+y)$ and $h = 0.1$

Also given $\quad x_0 = 1, \quad x_1 = 1.1, \quad x_2 = 1.2, \quad x_3 = 1.3$

and $\quad y_0 = 1, \quad y_1 = 1.233, \quad y_2 = 1.548, \quad y_3 = 1.979$

Required $y_4 = y(1.4)$, at $x_4 = 1.4$

Adam's Predictor formula is

$$y_{n+1} = y_n + \frac{h}{24}[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}], \quad n = 3,4,5,\ldots$$

Putting $n = 3$, we get

$$y_4 = y_3 + \frac{h}{24}[55y'_3 - 59y'_2 + 37y'_1 - 9y'_0] \qquad (1)$$

$$y' = x^2(1+y)$$

we have

$$y'_0 = [x_0^2(1+y)]_{(x_0,y_0)} = x_0^2(1+y_0) = 1^2(1+1) = \mathbf{2}$$

$\therefore$

$$y'_1 = [x^2(1+y)]_{(x_1,y_1)} = x_1^2(1+y_1) = (1.1)^2(1+1.233) = \mathbf{2.7019}$$
$$y'_2 = [x^2(1+y)]_{(x_2,y_2)} = x_2^2(1+y_2) = (1.2)^2(1+1.548) = \mathbf{3.6691}$$
$$y'_3 = [x^2(1+y)]_{(x_3,y_3)} = x_3^2(1+y_3) = (1.3)^2(1+1.979) = \mathbf{5.0345}$$

Substituting in (1), we get the Predictor value

$$y_4 = 1.979 + \frac{0.1}{24}[55(5.0345) - 59(3.6691) + 37(2.7019) - 9(2)]$$
$$= 1.979 + 0.004167[142.3909]$$
$$= 1.979 + 0.5933 = \mathbf{2.5723}$$

$\therefore$ the Predictor value is $\mathbf{y_4 = 2.5723}$

We improve this by the Adam's corrector formula

$$y_{n+1} = y_n + \frac{h}{24}[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}], \quad n = 2,3,\ldots \qquad (2)$$

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Putting $n=3$, we get $\quad y_4 = y_3 + \dfrac{h}{24}[9y_4' + 19y_3' - 5y_2' + y_1']$

Here y_4' is not known so we shall find it

$$y_4' = [x^2(1+y)]_{(x_4,y_4)} = x_4^2(1+y_4) = (1.4)^2(1+2.5723) = 7.0017$$

$\therefore$ the corrector value is

$$\begin{aligned} y_4 &= 1.979 + \frac{0.1}{24}[9(7.0017) + 19(5.0345) - 5(3.6691) + 2.7019] \\ &= 1.979 + 0.004167[143.0272] \\ &= 1.979 + 0.59599 = 2.57499 \end{aligned}$$

$\therefore$ the corrector value is $\quad \mathbf{y_4 = y(1.4) = 2.575}$

■

Example 3

Find $y(0.1)$, $y(0.2)$, $y(0.3)$ from $y' = x - y^2$ by using Runge-Kutta method of order 4 using step value $h = 0.1$ and then find $y(0.4)$ by Adam's method with $y(0) = 1$.

Solution

The given equation is $y' = x - y^2$

Here $\quad f(x,y) = x - y^2 \quad$ and $\quad h = 0.1$

Also given $x_0 = 0$, $y_0 = 1$

We shall find $y(0.1)$, $y(0.2)$, $y(0.3)$ by 4th order Runge-Kutta method.

By Runge–Kutta method

$$y_{n+1} = y_n + \Delta y, \qquad n = 0,1,2,\ldots$$

where
$$\Delta y = \frac{1}{2}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = hf(x_n, y_n), \qquad k_2 = hf\left[x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right]$$

$$k_3 = hf\left[x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right], \qquad k_4 = hf\left[x_n + h, y_n + k_3\right]$$

Starting with y_0, we shall find y_1

Put $n = 0$, $\quad y_1 = y_0 + \Delta y \quad$ and $\quad x_1 = x_0 + h = 0.1$

$$k_1 = hf(x_0, y_0) = h[x_0 - y_0^2] = 0.1(0 - 1^2) = -0.1$$

$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right] = hf\left[0 + \frac{0.1}{2}, 1 + \frac{(-0.1)}{2}\right]$$

$$= hf[0.05, 0.95] = 0.1[(0.05) - (0.95)^2] = 0.8525$$

$$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right] = hf\left[0.05, 1 + \frac{(-0.8525)}{2}\right]$$

$$= 0.1f[0.05, 0.5738] = 0.1(0.05 - (0.5738)^2) = \mathbf{-0.0279}$$

$$k_4 = hf[x_0 + h, y_0 + k_3] = hf[0.1, 1 - 0.0279] = (0.1)f[0.1, 0.9721]$$

$$= 0.1(0.1 - (0.9721)^2) = \mathbf{-0.0845}$$

$$\therefore \quad \Delta y = \frac{1}{6}[-0.1 + 2(-0.8525) + 2(-0.0279) + (-0.0845)] = \mathbf{-0.3242}$$

$$\therefore \quad y_1 = 1 + (-0.3242) = \mathbf{0.6758}$$

$$\therefore \quad \mathbf{y(0.1) = 0.6758}$$

Put $n = 1$ to find y_2 ie. to find y_2 when $x_2 = 0.2$.

$$\therefore \quad k_1 = hf(x_1, y_1) = 0.1f(0.1, 0.6758) = 0.1[0.1 - (0.6758)^2] = \mathbf{-0.0357}$$

$$k_2 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right] = hf\left[0.1 + \frac{0.1}{2}, 0.6758 + \frac{(-0.0357)}{2}\right]$$

$$= 0.1f[0.15, 0.6758 - 0.0178]$$

$$= 0.1f[0.15, 0.658] = 0.1[0.15 - (0.658)^2] = -0.0283$$

$$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right] = hf[0.15, 0.6758 - 0.0141]$$

$$= 0.1f[0.15, 0.6617] = 0.1[0.15 - (0.6617)^2] = -0.0288$$

$$k_4 = hf[x_1 + h_3, y_1 + k_3] = 0.1f[0.1 + 0.1, 0.6758 - 0.0288]$$

$$= 0.1f[0.2, 0.647] = 0.1[0.2 - (0.647)^2] = -0.0219$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[-0.0357 + 2(-0.0283) + 2(-0.0288) - 0.0219] = -0.0286$$

$$\therefore \quad y_2 = y_1 + \Delta y = 0.6758 - 0.0286 = \mathbf{0.6472}$$

Put $n = 2$ to find y_3

$$\therefore \quad k_1 = hf(x_2, y_2) = (0.1)\, f(0.2, 0.6472)$$

$$= (0.1)[0.2 - (0.6472)^2] = 0.1(-0.2189) = -0.0219$$

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$$k_2 = hf\left[x_2+\frac{h}{2}, y_2+\frac{k_1}{2}\right] = (0.1)f\left(0.2+\frac{0.1}{2}\left(0.6472-\frac{0.0219}{2}\right)\right)$$
$$= (0.1)f(0.25, 0.6363)$$
$$= (0.1)[0.25-(0.6363)^2]$$
$$= (0.1)[0.25-0.4049] = -0.01549 = -0.0155$$

$$k_3 = hf\left[x_2+\frac{h}{2}, y_2+\frac{k_2}{2}\right] = (0.1)f\left(0.2+\frac{0.1}{2}, \left(0.06472-\frac{0.0155}{2}\right)\right)$$
$$= (0.1)f(0.25, 0.6395)$$
$$= (0.1)[0.25-(0.6395)^2] = (0.1)(0.25-0.4090) = -0.0159$$

$$k_4 = hf[x_2+h, y_2+k_3] = (0.1)f(0.2+0.1, 0.6472-0.0159)$$
$$= (0.1)f(0.3, 0.6313) = (0.1)[0.3-(0.6313)^2] = -0.0099$$

$\therefore$
$$\Delta y = \frac{1}{6}[-0.0219+2(-0.0155)+2(-0.0159)+(-0.0099)]$$
$$= -\frac{1}{6}(0.0946) = -0.01573 = -0.0157$$

$\therefore$
$$y_3 = 0.6472-0.0157 = 0.6315$$

We shall now find $y_4 = y(0.4)$ by Adam's Predictor-Corector formula

Adam's Predictor formula is

$$y_{n+1} = y_n + \frac{h}{24}[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}], \qquad n = 3, 4, 5, \ldots$$

Putting $n = 3$, we get

$$y_4 = y_3 + \frac{h}{24}[55y'_3 - 59y'_2 + 37y'_1 - 9y'_0] \tag{1}$$

Now
$$y'_0 = (y')_{(x_0, y_0)} = x_0 - y^2{}_0 = 0 - 1^2 = -1$$
$$y'_1 = x_1 - y^2{}_1 = 0.1-(0.6758)^2 = -0.3567$$
$$y'_2 = x_2 - y^2{}_2 = 0.2-(0.6472)^2 = 0.2-0.4189 = -0.2189$$
$$y'_3 = x_3 - y^2{}_3 = 0.3-(0.6315)^2 = 0.3-0.3988 = -0.0988$$

$\therefore$
$$y_4 = 0.6315+\frac{0.1}{24}[55(-0.0988)-59(-0.2189)+37(-0.3567)-9(-1)]$$
$$= 0.6315+\frac{0.1}{24}[-5.434+12.9151-13.1979+9]$$
$$= 0.6315+\frac{0.1}{24}(-.03282) = 0.6315-0.0137 = \mathbf{0.6178}$$

Adam's Corrector formula is

$$y_{n+1} = y_n + \frac{h}{24}[9y'_{n+1} - 19y'_n + 5y'_{n-1} - y'_{n-2}],\ n = 2,3,...$$

Putting $n = 3$, we get

$$y_4 = y_3 + \frac{h}{24}[9y'_4 - 19y'_3 + 5y'_2 - y'_1], \qquad (2)$$

But $\quad y'_4 = x_4 - y_4^2 = 0.4 - (0.5494)^2 = 0.4 - 0.3018 = 0.0982$

Substituting in (2), we get the corrector value

$$y_4 = 0.6315 + \frac{0.1}{24}[9(0.0982) + 19(-0.0988) - 5(-0.2189) - (0.3567)]$$

$$= 0.6315 + \frac{0.1}{24}(-0.2556) = 0.6315 - 0.001065$$

$\therefore$ the corrector value of $\mathbf{y(0.4) = 0.6326}$ ∎

Example 4

Consider $\frac{dy}{dx} = y - x^2 + 1, y(0) = 0.5$

(i) Using the modified Euler method find $y(0.2)$.
(ii) Using R-K fourth order method find $y(0.4)$ and $y(0.6)$.
(iii) Using Adam-Bashforth Predictor-Corrector method find $y(0.8)$.

Solution

Given equation is $y' = y - x^2 + 1$

Here $\quad f(x, y) = y - x^2 + 1 \quad$ and $\quad h = 0.2$

Also given $\quad x_0 = 0, \quad y_0 = 0.5$

(i) By modified Euler method, we shall find $y(0.2)$

Modified Euler formula is

$$y_{n+1} = y_n + hf\left[x_n + \frac{h}{2}, y_n + \frac{h}{2}f(x_n, y_n)\right], n = 0,1,2\ ...$$

Putting $n = 0$, we get $\quad y_1 = y_0 + hf\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2}f(x_0, y_0)\right]$

$$= 0.5 + 0.2f\left[0 + \frac{0.2}{2}, 0.5 + \frac{0.2}{2}f(0, 0.5)\right]$$

$$= 0.5 + 0.2f[0.1, 0.5 + 0.15]$$

$$= 0.5 + 0.2f[0.1, 0.5 + 0.15]$$

$$y_1 = 0.5 + 0.2[0.65 - (0.1)^2 + 1] = 0.5 + 0.2[1.64] = 0.828$$

$\therefore$ $\boldsymbol{y(0.2) = 0.828}$

When $x_1 = 0.2, y_1 = 0.828$

(ii) Now we shall find $y(0.4)$ and $y(0.6)$ by 4th order R-K method

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = hf(x_n, y_n)$, $\quad k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right), \quad k_4 = hf(x_n + h, y_n + k_3)$$

$$y_{n+1} = y_n + \Delta y, \qquad n = 0, 1, 2, \ldots$$

Since y_1 is known we shall find y_2

$\therefore$ **Put $n = 1$,** Then $y_2 = y_1 + \Delta y$

$$k_1 = hf(x_1, y_1) = h[y_1 - x^2_1 + 1] = 0.2[0.828 - (0.2)^2 + 1] = 0.3576$$

$$k_2 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right] = 0.2f\left[0.2 + 0.1, 0.828 + \frac{0.3576}{2}\right]$$

$$= 0.2f[0.3, 1.0068]$$

$$= 0.2[1.0068 - (0.3)^2 + 1] = \mathbf{0.3834}$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.2f\left[0.2 + 0.1, 0.828 + \frac{0.3834}{2}\right]$$

$$= 0.2f[0.3, 1.0197]$$

$$= 0.2[1.0197 - (0.3)^2 + 1] = 0.3859$$

$$k_4 = hf[x_1 + h, y_1 + k_3] = 0.2f(0.2 + 0.2, 0.828 + 0.3859)$$

$$= 0.2f(0.4 + 1.2139) = 0.2[1.2139 - (0.4)^2 + 1] = \mathbf{0.4108}$$

$\therefore$ $$\Delta y = \frac{1}{6}[0.3576 + 2(0.3834) + 2(0.3859) + 0.4108] = \mathbf{0.3845}$$

$\therefore$ $$y_2 = y_1 + \Delta y = 0.828 + 0.3845 = 1.2125$$

$\Rightarrow$ $\boldsymbol{y(0.4) = 1.2125}$

$\therefore$ $$y_2 = 1.2125, \qquad x_2 = 0.4$$

Now $y_3 = y_2 + \Delta y$, where $\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$k_1 = hf(x_2, y_2), \qquad k_2 = hf\left[x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right]$$

$$k_3 = hf\left[x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right], \quad k_4 = hf[x_2 + h, y_2 + k_3]$$

Now $k_1 = 0.2f[0.4, 1.2125] = 0.2(1.2125 - (0.4)^2 + 1) = 0.4105$

$$k_2 = 0.2f\left[0.4 + 0.1, 1.2125 + \frac{0.4105}{2}\right]$$

$$= 0.2f[0.5, 1.4178] = 0.2(1.4178 - (0.5)^2 + 1) \quad = \mathbf{0.4336}$$

$$k_3 = 0.2f\left[0.4 + 0.1, 1.2125 + \frac{0.4336}{2}\right]$$

$$= 0.2f[0.5, 1.4293] = 0.2(1.4293 - (0.5)^2 + 1) \quad = \mathbf{0.4359}$$

$$k_4 = 0.2f[0.4 + 0.2, 1.2125 + 0.4359]$$

$$= 0.2f[0.6, 1.6484] = 0.2(1.6484 - (0.6)^2 + 1) \quad = \mathbf{0.4577}$$

$$\Delta y = \frac{1}{6}[0.4105 + 2(0.4336) + 2(0.4359) + 0.4577)] = \mathbf{0.4345}$$

$$y_3 = 1.2125 + 0.4345 = \mathbf{1.6470}$$

ie. when $x_3 = 0.6$, $y_3 = 1.6470$

(iii) Given $x_0 = 0$, $\quad x_1 = 0.2$, $\quad x_2 = 0.4$, $\quad x_3 = 0.6$

$y_0 = 0.5$, $\quad y_1 = 0.828$, $\quad y_2 = 1.2125$, $\quad y_3 = 1.6470$

We have to find y_4 when $x_4 = 0.8$, by Adam's Predictor-Corrector method.
Adam's Predictor formula is

$$y_{n+1} = y_n + \frac{h}{24}[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}], \qquad n = 3, 4, 5, \ldots$$

Putting $n = 3$, we get

$$y_4 = y_3 + \frac{h}{24}[55y'_3 - 59y'_2 + 37y'_1 - 9y'_0], \tag{1}$$

Since $y' = y - x^2 + 1$

$$y'_0 = [y']_{(x_0, y_0)} = y_0 - x_0^2 + 1 = 0.5 - 0 + 1 \qquad = 1.5$$

$$y'_1 = [y']_{(x_1, y_1)} = y_1 - x_1^2 + 1 = 0.828 - (0.2)^2 + 1 \quad = 1.788$$

$$y'_2 = [y']_{(x_2, y_2)} = y_2 - x_2^2 + 1 = 1.2125 - (0.4)^2 + 1 = 2.0525$$

$$y'_3 = [y']_{(x_3, y_3)} = y_3 - x_3^2 + 1 = 1.6470 - (0.6)^2 + 1 = 2.287$$

Substituting in (1), we get the Predictor value is

$$y_4 = 1.6470 + \frac{0.2}{24}[55(2.287) - 59(2.0525) + 37(1.788) - 9(1.5)]$$

$$y_4 = 1.6470 + 0.4779 = \mathbf{2.1249}$$

Adam's Corrector formula is

$$y_{n+1} = y_n + \frac{h}{24}[9y'_{n+1} - 19y'_n + 5y'_{n-1} + y'_{n-2}], \qquad n = 2, 3, \ldots$$

Putting $n = 3$, we get

$$y_4 = y_3 + \frac{h}{24}[9y_4' + 19y_3' + 5y_2' - y_1'], \quad (2)$$

Since y_4' is not known, we shall find it.

Now $y_4' = (y')_{(x_4, y_4)} = y_4 - x_4^2 + 1 = 2.1249 - (0.8)^2 + 1 = 2.4849$

Substituting in (2), we get the Corrector value

$$y_4 = 1.6470 + \frac{0.2}{24}[9(2.4849) + 19(2.287) - 5(2.0525) + (1.788)]$$
$$= 1.6470 + 0.4778 = 2.1249$$

Thus the corrector value is $\mathbf{y(0.8) = 2.1249}$ ∎

Exercises 9.3

(1) Solve numerically the differential equation $y' = \frac{1}{x+y}$, $y(0) = 2$, taking the starting values $y(0.2) = 2.0933$, $y(0.4) = 2.1755$, $y(0.6) = 2.2493$. Find the values of $y(0.8)$, using Milne's Predictor-corrector method.

(2) Solve $y' = y - \frac{2x}{y}$, $y(0) = 1$, $y(0.1) = 1.0954$, $y(0.2) = 1.1832$, $y(0.3) = 1.2649$, compute $y(0.4)$ and $y(0.5)$ by Milne's method.

(3) Apply Milne's method to solve $y' = 2 - xy^2$, $y(0) = 10$, $h = 0.2$ and find $y(1)$.

(4) Using Milne's method find $y(2)$ if $y(x)$ is a solution of $\frac{dy}{dx} = \frac{x+y}{2}$, given $y(0) = 2$, $y(0.5) = 2.636$, $y(1) = 3.595$ and $y(1.5) = 4.968$.

(5) Using Milne's method find $y(0.4)$ given $\frac{dy}{dx} = 1 + xy$, $y(0) = 2$.

Use Taylor's series to find the starting values.

(6) Compute the first 3 steps of the initial value problem $\frac{dy}{dx} = \frac{x-y}{2}$, $y(0) = 1$ by Taylor's series method and next step by Milne's method with step length $h = 0.1$.

(7) Given $\frac{dy}{dx} = x(x^2 + y^2)e^{-x}$, $y(0) = 1$, find $y(0.1)$, $y(0.2)$, $y(0.3)$ by Taylor's method and find $y(0.4)$ by Milne's predictor corrector method.

(8) Determine the value of $y(0.4)$ using Milne's method given $y' = xy + y^2$, $y(0) = 1$; use Taylor series to get the values of $y(0.1)$, $y(0.2)$ and $y(0.3)$.

(9) Given $\frac{dy}{dx} = \frac{xy}{2}$, $y(0) = 1$, $y(0.1) = 1.01$, $y(0.2) = 1.022$, $y(0.3) = 1.023$. Using Adam's method find $y(0.4)$.

(10) Using Adam-Bashforth method find $y(4.4)$ given $5xy' + y^2 = 2$, $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$ and $y(4.3) = 1.0143$.

(11) Find y at $x = 0.4$ if $\dfrac{dy}{dx} = 2e^x - y$ with $y(0) = 2$, $y(0.1) = 2.01$, $y(0.2) = 2.04$, $y(0.3) = 2.09$ by Adam-Bashforth method.

Answers 9.3

(1) $y_p(0.8) = 2.3162$, $y_c(0.8) = 2.3164$

(2) $y_p(0.4) = 1.3415$, $y_c(0.4) = 1.3416$, $y_p(0.5) = 1.4141$, $y_c(0.5) = 1.4142$

(3) $y_c(1) = 1.6505$,

(4) $y_p(2) = 6.8710$, $y_c(2) = 6.8732$

(5) $y_p(0.4) = 2.5885$, $y_c(0.4) = 2.5885$

(6) $x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, y_1 = 0.954, y_2 = 0.915$, $y_3 = 0.883$, $y_{4,c} = 0.856$, $y_{4,c} = 0.857$

(7) 1.0047, 1.01813, 1.03975, $y(0.4) = 1.0709$

(8) $y(0.1) = 1.1167$, $y(0.2) = 1.2767$, $y(0.3) = 1.5023$, $y(0.4) = 1.8370$

(9) $y_p = 1.0408$, $y_c = 1.0410$,

(10) $y_p = 1.0186$, $y_c = 1.0187$

(11) $y_p = 2.1616$, $y_c = 2.1615$

9.7 PICARD'S METHOD

9.7.1 Picard's Method of Successive Approximations

We have seen Taylor's series method gives the solution of an ordinary differential equation as a series. Picard method is also a method which gives the solution as a series.
Consider the first order differential equation.

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

We have

$$dy = f(x,y)dx.$$

Integrating from x_0 to x, the corresponding y values are y_0 and y.

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\, dx$$

$$\Rightarrow \quad [y]_{y_0}^{y} = \int_{x_0}^{x} f(x,y)\, dx$$

$$\Rightarrow \quad y - y_0 = \int_{x_0}^{x} f(x,y)\, dx$$

$$\Rightarrow \quad y = y_0 + \int_{x_0}^{x} f(x,y)\, dx \qquad (1)$$

This equation is complicated because the unknown function y appears inside the integral as well as outside.

This type of equation is called **an integral equation** and it is solved by successive approximation or iteration.

The first approximation of y is obtained by putting y_0 for y in the integrand.

$$\therefore \qquad y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

Now the second approximation is obtained by putting $y^{(1)}$ in the integrand of (1)

$$\therefore \qquad y^{(2)} = y_0 + \int_{x_0}^{x} f\left(x, y^{(1)}\right) dx$$

The process is repeated and we get the n[th] approximation $y^{(n)} = y_0 + \int_{x_0}^{x} f\left(x, y^{(n-1)}\right) dx$.

This method is known as **picard's method.**

Note:

(1) The sequence of approximations $y^{(1)}, y^{(2)}, \ldots$ converges to the solution y_s of $\frac{dy}{dx} = f(x,y),\ y(x_0) = y_0$, if the function $f(x,y)$ is bounded in the neighbourhood of (x_0, y_0) and satisfies

Lip schitz condition

i.e. $|f(x,y) - f(x,y_s)| = k|x - y_s|$, where k is a constant.

(2) In practice this method is not convenient if the integrand is complicated.

WORKED EXAMPLES

Example 1

Given $\frac{dy}{dx} = x + y,\ y(0) = 1$. Find the value of y when $x = 0.1,\ x = 0.2$ by picard's method. Check the result with the exact value.

Solution

Given equation is $\qquad \frac{dy}{dx} = x + y$

Here $\qquad f(x,y) = x + y \qquad$ and $\qquad x_0 = 0,\ y_0 = 1$

By Picard's method,

$$y = y_0 + \int_{x_0}^{x} f(x,y)\, dx \tag{1}$$

First approximation is $\qquad y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$

$$= y_0 + \int_0^x (x+1)dx = 1 + \left[\frac{x^2}{2} + x\right]_0^x = 1 + \frac{x^2}{2} + x$$

$$\Rightarrow \qquad y^{(1)} = 1 + x + \frac{x^2}{2}$$

Second approximation is

$$y^{(2)} = 1 + \int_0^x f(x, y^{(1)})dx$$

$$= 1 + \int_0^x (x + y^{(1)})dx \qquad [\because f(x+y) = x+y]$$

$$= 1 + \int_0^x \left(x + 1 + x + \frac{x^2}{2}\right)dx$$

$$= 1 + \int_0^x \left(1 + 2x + \frac{x^2}{2}\right)dx$$

$$= 1 + \left[x + 2\frac{x^2}{2} + \frac{1}{2}\frac{x^3}{3}\right]_0^x$$

$$\Rightarrow \qquad y^{(2)} = 1 + x + x^2 + \frac{x^3}{6}$$

Third approximation is

$$y^{(3)} = 1 + \int_0^x f(x, y^{(2)})dx$$

$$= 1 + \int_0^x (x + y^{(2)})dx$$

$$= 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{6}\right)dx$$

$$= 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{6}\right)dx$$

$$= 1 + \left[x + 2\frac{x^2}{2} + \frac{x^3}{3} + \frac{1}{6}\frac{x^4}{4}\right]_0^x$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

Fourth approximation is

$$y^{(4)} = 1 + \int_0^x f(x, y^{(3)})dx$$

$$= 1 + \int_0^x (x + y^{(3)})dx$$

$$=1+\int_0^x\left(x+1+x+x^2+\frac{x^3}{3}+\frac{x^4}{24}\right)dx$$

$$=1+\int_0^x\left(1+2x+x^2+\frac{x^3}{3}+\frac{x^4}{24}\right)dx$$

$$=1+\left[x+2\frac{x^2}{2}+\frac{x^3}{3}+\frac{1}{3}\frac{x^4}{4}+\frac{x^5}{24.5}\right]_0^x$$

$$\Rightarrow \qquad y^{(4)}=1+x+x^2+\frac{x^3}{3}+\frac{x^4}{12}+\frac{x^5}{120}$$

The fifth approximation is

$$y^{(5)}=1+\int_0^x f(x,y^{(4)})\,dx$$

$$=1+\int_0^x (x+y^{(4)})\,dx$$

$$=1+\int_0^x\left(x+1+x+x^2+\frac{x^3}{3}+\frac{x^4}{12}+\frac{x^5}{120}\right)dx$$

$$=1+\int_0^x\left(1+2x+x^2+\frac{x^3}{3}+\frac{x^4}{12}+\frac{x^5}{120}\right)dx$$

$$=1+\left[x+2\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{3.4}+\frac{1}{12}.\frac{x^5}{5}+\frac{1}{120}.\frac{x^6}{6}\right]_0^x$$

$$=1+x+x^2+\frac{x^3}{3}+\frac{x^4}{12}+\frac{x^5}{60}+\frac{x^6}{720}$$

When $x=0.1$,

$$y^{(5)}=1+0.1+(0.1)^2+\frac{(0.1)^3}{3}$$

$$y^{(5)}=1+0.1+(0.1)^2+\frac{(0.1)^3}{3}+\frac{(0.1)^4}{12}+\frac{(0.1)^5}{60}+\frac{(0.1)^6}{720}$$

$$=1+0.1+0.01+\frac{0.001}{3}+\frac{0.0001}{12}+\frac{0.0001}{60}+\frac{0.000001}{720}$$

$$=1.110341835$$

When $x=0.2$,

$$y^{(5)}=1+0.2+(0.2)^2+\frac{(0.2)^3}{3}+\frac{(0.2)^4}{12}+\frac{(0.2)^5}{60}+\frac{(0.2)^6}{720}$$

$$=1+0.2+0.04+\frac{0.008}{3}+\frac{0.0016}{12}+\frac{0.00032}{60}+\frac{0.000064}{720}$$

$$=1.242805$$

When $x=0.1$, $\quad$ $\boldsymbol{y=1.1103418}$

When $x=0.2$, $\quad$ $\boldsymbol{y=1.242805}$

The actual solution of the first order linear equation $\frac{dy}{dx}-y=x$ is $y=2e^x-x-1$

When $x = 0.1$, $y = 2.2103418 - 0.1 - 1 = 1.110341836$

When $x = 0.2$, $y = 1.242805516$.

Comparing with the actual value, we find that the computed value of Picard's method is correct upto last decimal. ■

Example 2

Use picards method to approximate the value of y when $x = 0.1$, given that $y = 1$, when $x = 0$ and $\frac{dy}{dx} = 3x + y^2$.

Solution

Given equation is $\frac{dy}{dx} = 3x + y^2$

Here $f(x,y) = 3x + y^2$ and $x_0 = 0$, $y_0 = 1$

Picards formula is

$$y = y_0 + \int_{x_0}^{x} f(x,y)dx$$

$$= 1 + \int_{0}^{x} f(x,y)dx$$

The first approximation is

$$y^{(1)} = 1 + \int_0^x f(x, y_0)dx$$

$$= 1 + \int_0^x (3x + y_0^2)dx$$

$$= 1 + \int_0^x (3x + 1)dx$$

$$= 1 + \left[3\frac{x^2}{2} + x\right]_0^x$$

$$= 1 + 3\frac{x^2}{2} + x = 1 + x + \frac{3}{2}x^2$$

The Second approximation is

$$y^{(2)} = 1 + \int_0^x f(x, y^{(1)})dx$$

$$= 1 + \int_0^x \left[3x + (y^{(1)})^2\right]dx$$

$$= 1 + \int_0^x \left[3x + \left(1 + x + \frac{3}{2}x^2\right)^2\right]dx$$

$$= 1 + \int_0^x \left[3x + 1 + x^2 + \frac{9}{4}x^4 + 2x + 3x^2 + 3x^3\right] dx$$

$$= 1 + \int_0^x \left[1 + 5x + 4x^2 + 3x^3 + \frac{9}{4}x^4\right] dx$$

$$= 1 + \left[x + 5\frac{x^2}{2} + 4\frac{x^3}{3} + 3\frac{x^4}{4} + \frac{9}{4}\frac{x^5}{5}\right]_0^x$$

$$\Rightarrow \quad y^{(2)} = 1 + x + 5\frac{x^2}{2} + 4\frac{x^3}{3} + 3\frac{x^4}{4} + 9\frac{x^5}{20}$$

The third approximation involves squares of $y^{(2)}$. which is a big expression.

So we stop with $y^{(2)}$.

When $x = 0.1$ $\quad y^{(2)} = 1 + 0.1 + 5\frac{(0.1)^2}{2} + 4\frac{(0.1)^3}{3} + 3\frac{(0.1)^4}{4} + 9\frac{(0.1)^5}{20}$

$$= 1.1 + 0.025 + 0.0013333 + 0.000075 + 0.0000045 = 1.1264$$

When $\quad \mathbf{x = 0.1,\ y = 1.1264}$ ■

Example 3

Use Picard's method to find the value of y when $x = 0.1$ given $\frac{dy}{dx} = \frac{y - x}{y + x}$, $y(0) = 1$.

Solution

Given $\quad \frac{dy}{dx} = \frac{y - x}{y + x}$ and $y = 1$ when $x = 0$

Here $\quad f(x, y) = \frac{y - x}{y + x}$, $x_0 = 0$, $y_0 = 1$

Picards formula is $\quad y = y_0 + \int_{x_0}^{x} f(x, y)\,dx = 1 + \int_0^x f(x, y)\,dx$

The first approximation is $y^{(1)} = 1 + \int_0^x f(x, y_0)\,dx = 1 + \int_0^x \frac{y_0 - x}{y_0 + x}\,dx$

$$= 1 + \int_0^x \frac{1 - x}{1 + x}\,dx$$

$$= 1 + \int_0^x \frac{2 - (1 + x)}{1 + x}\,dx$$

$$= 1 + \int_0^x \left(\frac{2}{1 + x} - 1\right) dx$$

$$= 1 + \left[2\log_e(1 + x) - x\right]_0^x$$

$$= 1 + 2\log_e(1 + x) - x - (2\log_e(1 + 0) - 0)$$

$$= 1 + 2\log_e(1 + x) - x$$

The second approximation is

$$y^{(2)} = 1 + \int_0^x f(x, y^{(1)})\,dx$$

$$= 1 + \int_0^x \frac{y^{(1)} - x}{y^{(1)} + x}\,dx$$

$$= 1 + \int_0^x \frac{1 + 2\log_e(1+x) - x - x}{1 + 2\log(1+x) - x + x}\,dx$$

$$= 1 + \int_0^x \frac{1 + 2\log_e(1+x) - 2x}{1 + 2\log_e(1+x)}\,dx$$

$$= 1 + \int_0^x \left[1 - \frac{2x}{1 + 2\log_e(1+x)}\right]dx$$

$$= 1 + \int_0^x dx - 2\int_0^x \frac{x}{1 + 2\log_e(1+x)}\,dx$$

$$= 1 + x - 2\int_0^x \frac{x}{1 + 2\log_e(1+x)}\,dx$$

This integral cannot be evaluated in closed form. Hence we cannot proceed. So, we take $y^{(1)}$ as the approximate solution.

When $x = 0.1,\ y^{(1)} = 1 + 2\log_e(1+0.1) - 0.1$

$= 0.9 + 0.19062 = 1.09062$

$\therefore$ when $\mathbf{x = 0.1,\ y = 1.09062}$ ■

Exercises 9.4

(1) Using picards method solve $\dfrac{dy}{dx} = -xy$ with $x_0 = 0, y_0 = 1$ upto third approximation. Hence evaluate y when $x = 0.2$.

(2) Find the solution of $\dfrac{dy}{dx} = 1 + xy$ which passes through the point (0, 1). Find y correct to three decimal places when $x = 0.2$ and $x = 0.4$.

(3) Find the solution with $x_0 = 1, y = 3$ by picards methods.

$$\left[y = \frac{x^4}{12} - \frac{5}{6}x^3 + \frac{7}{2}x^2 - \frac{35}{0}x + \frac{73}{12}\right]$$

Answers 9.4

(1) $y = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{8} - \dfrac{x^6}{48}$, $y = 0.9802$ (2) $y = 1.203, 1.505$

(3) $y = \dfrac{x^4}{12} - \dfrac{5}{6}x^3 + \dfrac{7}{2}x^2 - \dfrac{35}{6}x + \dfrac{73}{12}$

SHORT ANSWER QUESTIONS

1. State Taylor series algorithm for the first order differential equation.
2. Write the merits and demerits of the Taylor's method of Solution.
3. Write down the Euler algorithm to solve the differential equation $\frac{dy}{dx} = f(x,y)$.
4. Given $y' = x + y$, $y(0) = 1$ find $y(0.1)$ by Euler's method.
5. State modified Euler algorithm to solve $y' = f(x, y)$, $y(x_0) = y_0$.
6. State the modified Euler's formula to solve $y' = f(x, y)$, $y(x_0) = y_0$ at $x = x_0 + h$.
7. Write the Runge-Kutta algorithm of 4th order to solve $\frac{dy}{dx} = f(x,y)$. with $y(x_0) = y_0$.
8. State the special advantage of Runge-Kutta method over Taylor series method.
9. State the difference between single step and multistep methods in solving ordinary differential equations numerically?
10. What is a predictor-corrector method of solving a differential equation $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$?
11. State Milne's Predictor corrector formula.
12. How many pairs of prior values are required to predict the next value in Milne's method?
13. State Milne's Predictor-corrector formula to solve $\frac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$, $y(x_0 + h) = y_1$, $y_1(x_0 + 2h) = y_2$, $y(x_0 + 3h) = y_3$ at $x = x_0 + 4h$.
14. State the Taylor series formula to find $y(x_1)$ for solving $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$.
15. Write down the modified Euler's formula for ordinary differential equation.
16. By Taylor series method find $y(1.1)$ given that $y' = x + y$ and $y(1) = 0$.
17. Find the Taylor series expansion upto x^3 term satisfying $2y' + y = x + 1, y(0) = 1$.
18. Using modified Euler's method, evaluate $y(1.1)$ if $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}, y(1) = 1$.
19. Write the Adam's predictor-corrector formula.
20. Using Euler's modified method, find y at $x = 0.1$ if $\frac{dy}{dx} = 1 + xy, y(0) = 2$.
21. Using Euler's method find y at $x = 0.1$, if $\frac{dy}{dx} = 1 + xy, y(0) = 2$.
22. Using Taylor's series method find $y(1.1)$ given that $y' = x + y$, $y(1) = 0$.
23. Find $y(0.2)$ for the equation $y' = y + e^x$, given that $y(0) = 0$ by using Euler's method.

Boundary Value Problems in Ordinary and Partial Differential Equation

INTRODUCTION

The areas of applications of partial differential equations is too large compared to ordinary differential equations. Partial differential equations occur especially in problems involving wave phenomena and heat conduction. In practical problems we seek to obtain unique solution of ordinary and partial differential equations subject to certain specific conditions which are called boundary value conditions. The differential equation with boundary value conditions is called a boundary-value problem.

For a differential equation when conditions are prescribed at the same point, we call them as **initial conditions.** When conditions are prescribed at different points, we call them as **boundary conditions.** The initial conditions and boundary conditions together are known as **boundary-value conditions.**

For example: For the one-dimensional wave-equation $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$, the conditions $u(0,t) = 0$ and $u(l,t) = 0$ which are prescribed at different points are called boundary conditions.

But the conditions $u(x, 0) = x^2$, $\frac{\partial u}{\partial t}(x, 0) = 0$, which are prescribed at the same point are initial conditions.

All the four conditions $u(0,t) = 0$, $u(l,t) = 0$, $u(x, 0) = x^2$, $\frac{\partial u}{\partial t}(x, 0) = 0$ together are called boundary-value conditions.

The partial differential equation with these four conditions is called a boundary-value problem.

Certain types of boundary value problems can be solved by replacing the derivatives or partial derivations by approximate differences and thus reducing them to a difference equation. Thereby the given boundary value problem is converted to a system of linear equations which are solved by iteration methods.

FINITE DIFFERENCE METHODS FOR SOLUTION OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

Consider the boundary value problem $\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=Q(x)$ with the boundary conditions $y(x_0)=\alpha$, $y(x_n)=\beta$. We divide the interval $[x_0, x_n]$ into n equal subintervals of width h, by the points $x_0, x_1, x_2, \ldots, x_n$ where $x_r = x_0 + rh, r = 1, 2, \ldots, n$.

Let $y_0, y_1, \ldots, y_n$ be the corresponding values of y. In this method, the derivatives in the differential equations are replaced by their finite difference approximations, namely,

$$\left(\frac{dy}{dx}\right)_{(x_i,\, y_i)} = y_i' = \frac{y_{i+1}-y_i}{h} \tag{1}$$

$$\left[\because \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \frac{f(x+h)-f(x)}{h}\right]$$

Also $$y_i' = \frac{y_{i+1}-y_{i-1}}{h} \tag{2}$$

$$\left(\frac{d^2y}{dx^2}\right)_{(x_i,\, y_i)} = y_i'' = \frac{y_{i+1}-2y_i+y_{i-1}}{h^2} \tag{3}$$

Substituting these values in the given differential equation we get a difference equation, which can be solved using the boundary conditions.

Note: (1) is called forward difference approximation, where as (2) is a central difference approximation for y_i' and (3) is called central difference approximation for y_i''.

WORKED EXAMPLES

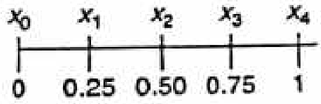
Example 1

Using the finite difference method, find $y(0.25)$, $y(0.5)$ and $y(0.75)$ satisfying the differential equation $\frac{d^2y}{dx^2}+y=x$ subject to the boundary conditions $y(0)=0$, $y(1)=2$.

Solution

Given $$y'' + y = x,\ y(0) = 0,\ y(1) = 2$$

Here $$h = 0.25 = \frac{1}{4}; \qquad x_0 = 0, \qquad x_n = 1 \text{ and } n = 4$$



We know $y_i'' = \dfrac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$, $i = 1, 2, 3$

$\therefore$ the equation becomes $$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + y_i = x_i, \qquad i = 1, 2, 3$$

$$\Rightarrow \quad \frac{y_{i+1} - 2y_i + y_{i-1}}{\frac{1}{16}} + y_i = x_i \qquad (1)$$

$$\Rightarrow \quad 16y_{i+1} - 31y_i + 16y_{i-1} = x_i, \qquad i = 1, 2, 3$$

Given $y_0 = 0, \quad x_0 = 0, \quad x_4 = 1, \quad y_4 = 2, \quad x_1 = 0.25, \quad x_2 = 0.5, \quad x_3 = 0.75$

If $i = 1$, then (1)

$$\Rightarrow \quad 16y_2 - 31y_1 + 16y_0 = x_1$$
$$\Rightarrow \quad 16y_2 - 31y_1 = 0.25$$
$$\Rightarrow \quad y_2 - 1.9375y_1 = 0.0156 \qquad (2)$$

If $i = 2$, then (1)

$$\Rightarrow \quad 16y_3 - 31y_2 + 16y_1 = x_2$$
$$\Rightarrow \quad 16y_3 - 31y_2 + 16y_1 = 0.5$$
$$\Rightarrow \quad y_3 - 1.9375y_2 + y_1 = 0.0313 \qquad (3)$$

If $i = 3$, then (1)

$$\Rightarrow \quad 16y_4 - 31y_3 + 16y_2 = x_3$$
$$\Rightarrow \quad 16y_4 - 31y_3 + 16y_2 = 0.75$$
$$\Rightarrow \quad 16 \times 2 - 31y_3 + 16y_2 = 0.75$$
$$\Rightarrow \quad -31y_3 + 16y_2 = -31.25$$
$$\Rightarrow \quad -1.9375y_3 + y_2 = -1.9531 \qquad (4)$$

Solve (2), (3), (4) to find y_1, y_2, y_3

$$(3) \times 1.9375 \Rightarrow \quad 1.9375y_3 - 3.7539y_2 + 1.9375y_1 = 0.0606 \qquad (5)$$
$$(4) + (5) \Rightarrow \quad y_2 - 3.7539y_2 + 1.9375y_1 = -1.9531 + 0.0606$$
$$\Rightarrow \quad -2.7539y_2 + 1.9375y_1 = -1.8925$$

Dividing by 2.7539, $\quad -y_2 + 0.703y_1 = -0.6872$

(2) is Adding, $\quad \dfrac{y_2 - 1.9375y_1 = 0.0156}{-1.234y_1 = -.6716}$

$\Rightarrow \qquad y_1 = 0.5442$

$\therefore \qquad y_2 = 0.0156 + 1.9375 \times 0.5442 = 1.06999 \approx 1.07$

Substituting in (3), $\qquad y_3 = 0.0313 + 1.9375 \times 1.07 - 0.5442 = 1.5602$

Thus, $\quad y_1 = y(0.25) = \mathbf{0.5442}, \quad y_2 = y(0.5) = \mathbf{1.07}, \quad y_3 = y(0.75) = \mathbf{1.5602}$

Note: The equations (2), (3), (4) could be solved by Gauss elimination method. ■

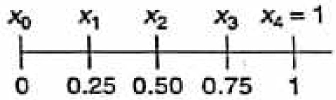
Example 2

Solve the equation $y'' = x + y$ with the boundary conditions $y(0) = y(1) = 0$.

Solution

Given $y'' = x + y, \qquad y(0) = y(1) = 0$

$\therefore \qquad x_0 = 0, \quad x_n = 1, \quad y_0 = 0, \quad y_n = 0$



We shall divide the interval (0, 1) into 4 equal parts

$\therefore \qquad n = 4, \quad h = \dfrac{1-0}{4} = \dfrac{1}{4} = 0.25$

$\therefore \qquad x_0 = 0, \quad x_1 = x_0 + h = 0.25, \quad x_2 = 0.5, \quad x_3 = 0.75, \quad x_4 = 1$

We know $\qquad y_i'' = \dfrac{y_{i+1} - 2y_i + y_{i-1}}{h^2}, \quad i = 1, 2, 3$

$\therefore$ the equation becomes $\qquad \dfrac{y_{i+1} - 2y_i + y_{i-1}}{16} = x_i + y_i, \qquad i = 1, 2, 3$

$\Rightarrow \qquad 16y_{i+1} - 32y_i + 16y_{i-1} = x_i + y_i, \qquad i = 1, 2, 3$

$\Rightarrow \qquad 16y_{i+1} - 33y_i + 16y_{i-1} = x_i, \qquad i = 1, 2, 3$

Put $i = 1$, then $\qquad 16y_2 - 33y_1 + 16y_0 = x_1$

$\Rightarrow \qquad 16y_2 - 33y_1 + 16 \times 0 = 0.25$

$\Rightarrow \qquad 16y_2 - 33y_1 = 0.25$

Put $i = 2$, then $\quad 16y_3 - 33y_2 + 16y_1 = x_2$

$\Rightarrow \quad 16y_3 - 33y_2 + 16y_1 = 0.5$

Put $i = 3$, then $\quad 16y_4 - 33y_3 + 16y_2 = x_3$

$\Rightarrow \quad 16 \times 0 - 33y_3 + 16y_2 = 0.75$

$\Rightarrow \quad -33y_3 + 16y_2 = 0.75$

The equations are

$$-33y_1 + 16y_2 = 0.25 \tag{1}$$

$$16y_1 - 33y_2 + 16y_3 = 0.5 \tag{2}$$

$$16y_2 - 33y_3 = 0.75 \tag{3}$$

We use Gauss elimination method to find y_1, y_2 and y_3

$$\frac{(2)}{16} \Rightarrow \quad y_1 - 2.0625y_2 + y_3 = 0.03125 \tag{4}$$

$$\frac{(3)}{33} \Rightarrow \quad 0.4848y_2 - y_3 = 0.0227 \tag{5}$$

Adding,
$$y_1 - 1.5777y_2 = 0.05395 \tag{6}$$

$$\frac{(1)}{33} \Rightarrow \quad -y_1 + 0.4848y_2 = 0.00758$$

Adding,
$$-1.0929y_2 = 0.06153$$

$$\Rightarrow \quad y_2 = -\frac{0.0615}{1.0929} = -0.0563$$

Substituting in (6), $\quad y_1 = 0.05395 + 1.5777(-0.0563) = -0.0349$

Substituting in (5), we get $\quad y_3 = 0.4848(-0.0563) - 0.0227$

$$= -0.04999 = -0.05$$

$$\therefore \quad \boldsymbol{y_1 = y(0.25) = -0.0349, \quad y_2 = y(0.5) = -0.0563, \quad y_3 = y(0.75) = -0.05} \quad \blacksquare$$

Example 3

Solve $xy'' + y = 0$, $y(1) = 1$, $y(2) = 2$ with $h = 0.25$ by using finite difference method.

Solution

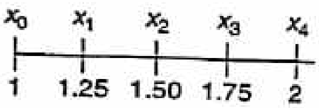
Given $\quad xy'' + y = 0,\ y(1) = 1,\ y(2) = 2,\ h = 0.25$

$\therefore \quad x_0 = 1$ and $x_n = 2$

Since $\quad h = 0.25, \quad n = 4$

$x_1 = 1.25, \quad x_2 = 1.5, \quad x_3 = 1.75$

$y_0 = 1, \quad y_4 = 2$



We know $$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}, \quad i = 1, 2, 3$$

$$= 16(y_{i+1} - 2y_i + y_{i-1}), \quad i = 1, 2, 3$$

$\therefore$ the equation becomes

$$16x_i(y_{i+1} - 2y_i + y_{i-1}) + y_i = 0, \quad i = 1, 2, 3$$

Put $i = 1$, then $$16x_1(y_2 - 2y_1 + y_0) + y_1 = 0$$

$$20(y_2 - 2y_1 + 1) + y_1 = 0 \qquad \left[\because x_1 = 1.25 = \frac{5}{4}\right]$$

$\Rightarrow$ $$20y_2 - 39y_1 + 20 = 0$$

Put $i = 2$, then $$16x_2(y_3 - 2y_2 + y_1) + y_2 = 0$$

$$24(y_3 - 2y_2 + y_1) + y_2 = 0 \qquad \left[\because x_2 = 1.5 = \frac{3}{2}\right]$$

$\Rightarrow$ $$24y_3 - 47y_2 + 24y_1 = 0$$

Put $i = 3$, then $$16x_3(y_4 - 2y_3 + y_2) + y_3 = 0$$

$$28(y_4 - 2y_3 + y_2) + y_3 = 0 \qquad \left[\because x_3 = 1.75 = \frac{7}{4}\right]$$

$\Rightarrow$ $$28 \times 2 - 55y_3 + 28y_2 = 0$$

$\Rightarrow$ $$28y_2 - 55y_3 + 56 = 0$$

$\therefore$ the equations are

$$39y_1 - 20y_2 = 20 \qquad (1)$$

$$24y_1 - 47y_2 + 24y_3 = 0 \qquad (2)$$

$$28y_2 - 55y_3 = -56 \qquad (3)$$

$\frac{(2)}{24} \Rightarrow$ $$y_1 - 1.9583y_2 + y_3 = 0 \qquad (4)$$

$\frac{(3)}{55} \Rightarrow$ $$0.5091y_2 - y_3 = -1.0182 \qquad (5)$$

Adding, $$y_1 - 1.4492y_2 = -1.0182$$

$\frac{(1)}{39} \Rightarrow$ $$-y_1 - 0.5128y_2 = 0.5128 \qquad (6)$$

Subtracting, $$-0.9364y_2 = -1.5310 \qquad \Rightarrow y_2 = 1.635$$

From(5), $$y_3 = 0.5091 \times 1.635 + 1.0182 = 1.851$$

From(6), $$y_1 = 0.5128 \times 1.635 + 0.5128 = 1.315$$

Thus

$$\boldsymbol{y_1 = y(1.25) = 1.351, \; y_2 = y(1.50) = 1.635, \; y_3 = y(1.75) = 1.851} \quad \blacksquare$$

Exercises 10.1

(1) Solve the boundary value problem for $x = 0.5$ by finite difference method $\frac{d^2y}{dx^2} + y + 1 = 0$, $y(0) = y(1) = 0$.

(2) Solve by finite difference method:

$y'' + xy' + y = 3x^2 + 2$; $y(0) = 0$, $y(1) = 1$ by taking $h = 0.25$.

(3) Using finite difference method, solve the differential equation $y'' + xy' - 2y = 2(x+1)$, $y(0) = 0$, $y'(1) = 0$ with $h = \frac{1}{3}$.

(4) Solve the equation $y'' - 64y + 10 = 0$, with $y(0) = y(1) = 0$ taking $h = 0.5$ using finite difference method.

(5) Solve $y' - xy = 0$, given $y(0) = -1$, $y(1) = 2$ taking $n = 2$, using finite difference method.

(6) Solve the equation $xy'' + xy' - y = 1$, $y(0) = 0$, $y(2) = 1$ and $h = 0.2$ using finite difference method.

Answers 10.1

(1) Take $n = 4$, $y(0.5) = 0.1403$ (2) 0.062, 0.250, 0.562 (3) –0.5556, –0.8889, –1
(4) $y_1 = 0.1389$ (5) $y(0,5) = 0.4706$ (6) 0.2, 0.4, 0.6, 0.8

10.2 NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

We have seen numerical solution of second order ordinary differential equations by replacing the derivatives by the approximate differences. Similarly partial differential equations are also solved numerically by finite difference method. The partial derivatives appearing in the equation and the boundary conditions are replaced by their finite difference approximations. Thereby the given equation is converted into a system of a linear equations which are solved by iteration methods.

10.2.1 Classifications of Second Order Partial Differential Equations

Let u be a function of two independent variables x and y. Consider the partial differential equation of u of the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \tag{1}$$

where A, B, C are functions of x and y or constants.

The second degree terms are linear whereas the part $F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$ may be non-linear. So the equation (1) is called a second order **quasi-linear equation**. Incase $F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$ is linear then (1) is called a second order linear equation.

At a point (x, y) in the xy-plane, the equation (1) is

(a) elliptic if $B^2 - 4AC < 0$

(b) Parabolic if $B^2 - 4AC = 0$

(c) Hyperbolic if $B^2 - 4AC > 0$

Note:

(1) If at all points of a region R if $B^2 - 4AC \lesseqgtr 0$, then the equation is elliptic or parabolic or hyperbolic in R.

(2) It is quite possible an equation elliptic in one region may be parabolic in another region and hyperbolic in yet another region.

WORKED EXAMPLES

Example 1

Test the nature of (i) $xu_{xx} + u_{yy} = 0$

(ii) $xu_{xx} + yu_{yy} = 0$

(iii) $(x+1)u_{xx} - 2(x+2)u_{xy} + (x+3)u_{yy} = 0$

(iv) $x^2 f_{xx} + (1-y^2) f_{yy} = 0$

Solution

(i) **Given $xu_{xx} + u_{yy} = 0$**

Here $A = x$, $B = 0$, $C = 1$

$$\therefore \quad B^2 - 4AC = 0 - 4x = -4x \begin{cases} < 0 & \text{if } x > 0 \\ > 0 & \text{if } x < 0 \\ = 0 & \text{if } x = 0 \end{cases}$$

So, the equation is elliptic if $x > 0$ ie. at all points on the right side of y-axis satisfying the equation. The equation is hyperbolic if $x < 0$ ie. at all points on the left side of y-axis satisfying the equation and parabolic if $x = 0$ ie at all points of y-axis satisfying the equation.

(ii) **Given $xu_{xx} + yu_{yy} = 0$**

Here $A = x$, $B = 0$, $C = y$

$\therefore \quad B^2 - 4AC = 0 - 4xy$

$$= -4xy > 0 \text{ if } xy < 0 \quad \text{ie. } x \text{ and } y \text{ have opposite signs}$$
$$< 0 \text{ if } xy > 0 \quad \text{ie. } x \text{ and } y \text{ have same sign}$$
$$= 0 \text{ if } xy = 0 \quad \text{ie. } x = 0 \text{ or } y = 0.$$

So, the equation is hyperbolic in the 2nd and 4th quadrants, elliptic in the 1st and 3rd quadrants and parabolic at all points in the x-axis and y-axis which satisfy the equation.

(iii) **Given** $(x+1)u_{xx} - 2(x+2)u_{xy} + (x+3)u_{yy} = 0$

Here $A = x+1, \quad B = -2(x+2), \quad C = x+3$

$$\therefore \quad B^2 - 4AC = 4(x+2)^2 - 4\cdot(x+1)(x+3)$$
$$= 4\{x^2 + 4x + 4 - (x^2 + 4x + 3)\}$$
$$= 4 > 0, \text{ for any } x, y$$

$\therefore$ the equation is hyperbolic for all (x, y) satisfying the equation

(iv) **Given** $x^2 f_{xx} + (1 - y^2) f_{yy} = 0$

Here $A = x^2, \qquad B = 0, \qquad C = 1 - y^2$

$$B^2 - 4AC = 0 - 4x^2(1 - y^2) = 4x^2(y^2 - 1)$$

If $x^2(y^2 - 1) > 0$, then $y^2 - 1 > 0$ (as $x^2 > 0$ for all $x \neq 0$)

$\Rightarrow \qquad y < -1$ or $y > 1$ for all $x \neq 0$

$\therefore$ the equation is hyperbolic if $y < -1$ or $y > 1$ and for all $x \neq 0$

If $\qquad x^2(y^2 - 1) < 0 \Rightarrow y^2 - 1 < 0 \quad (\text{as } x^2 > 0 \ \forall\, x \neq 0)$

$\Rightarrow \qquad -1 < y < 1; \quad x \neq 0$

$\therefore$ the equation is elliptic if $-1 < y < 1$ and $x \neq 0$

If $x^2(y^2 - 1) = 0$, then $x = 0$ or $y = \pm 1$

$\therefore$ the equation is parabolic if $x = 0$

ie. on the y-axis or on the line $y = -1$ or $y = 1$ ■

Depending on the nature of the equations, methods of solving will differ.

Some well known equations, we consider for numerical solution are the following:

(1) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ is Laplace's equation

i.e. two dimensional heat equation in steady-state. It is elliptic type.

(2) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y)$ –Poisson's equation.

It is elliptic type

(3) $\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$ – one dimensional wave equation.

It is hyperbolic type

(4) $\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}$ – one dimensional heat equation.

It is parabolic type.

Finite Difference Approximations to Partial Derivatives

Consider a rectangular region in the xy-plane with sides parallel to the axes.

Divide this region into a network of rectangles of sides h and k units by drawing lines $x = ih$, $y = jk$, $i, j = 0, 1, 2, 3, \ldots$ The points of intersection of these family of lines are called **mesh points or lattice points** or **grid points**.

The point of intersection of $x = ih$, $y = jk$ is called the grid point (i, j)

If u is a function of two independent variables x and y then the value of $u(x,y)$ at the grid point (i,j) is denoted by $u_{i,j}$. Thus $u(x,y)=u(ih,jk)=u_{i,j}$

Consider $u(x,y_0)$, where x is a variable and y_0 is fixed then $u(x,y_0)$ is single variable function.

The Taylor's series expansion of it in the neighbourhood of x_0 is

$$u(x,y_0)=u(x_0,y_0)+(x-x_0)u_x(x_0,y_0)+\frac{(x-x_0)^2}{2!}u_{xx}(x_0,y_0)+\ldots$$

Put $\qquad x = x_0 + h$, then

$$u(x_0+h,y_0)=u(x_0,y_0)+hu_x(x_0,y_0)+\frac{h^2}{2!}u_{xx}(x_0,y_0)+\ldots$$

If h is small, we approximate as

$$u(x_0+h,y_0)=u(x_0,y_0)+hu_x(x_0,y_0) \tag{1}$$

Replacing h by $-h$ in (1), we get

$$u(x_0-h,y_0)=u(x_0,y_0)-hu_x(x_0,y_0) \tag{2}$$

(1) $\Rightarrow$ $$hu_x(x_0,y_0)=u(x_0+h,y_0)-u(x_0,y_0)$$

$\Rightarrow$ $$u_x(x_0,y_0)=\frac{u(x_0+h,y_0)-u(x_0,y_0)}{h}$$

This is called the forward difference approximation for $u_x(x_0,y_0)$

(2) $\Rightarrow$ $$u_x(x_0,y_0)=\frac{u(x_0,y_0)-u(x_0-h,y_0)}{h}$$

This is called the backward difference approximation for $u_x(x_0,y_0)$

(1) – (2) $\Rightarrow$ $u(x_0+h,y_0)-u(x_0-h,y_0)=2h\, u_x(x_0,y_0)$

$\Rightarrow$ $$u_x(x_0,y_0)=\frac{u(x_0+h,y_0)-u(x_0-h,y_0)}{2h}$$

This is called the central difference approximation to $u_x(x_0,y_0)$.

The central difference approximation is better than the forward and backward difference approximations.

In the same way the approximation for 2nd derivative is

$$u_{xx}(x_0,y_0)=\frac{u_x(x_0+h,y_0)-u_x(x_0,y_0)}{h}$$

Using backward difference approximations for $u_x(x_0+h,y_0)$ and $u_x(x_0,y_0)$ we get

$$u_x(x_0+h,y_0)=\frac{u(x_0+h,y_0)-u(x_0,y_0)}{h}$$

and $$u_x(x_0,y_0)=\frac{u(x_0,y_0)-u(x_0-h,y_0)}{h}$$

$\therefore$ $$u_{xx}(x_0,y_0)=\frac{1}{h}\left[\frac{u(x_0+h,y_0)-u(x_0,y_0)}{h}-\frac{u(x_0,y_0)-u(x_0-h,y_0)}{h}\right]$$

$$u_{xx}(x_0,y_0)=\frac{u(x_0+h,y_0)-2u(x_0,y_0)+u(x_0-h,y_0)}{h^2}$$

In the same way, the first and second partial derivatives w.r.to y can be obtained.
The forward difference approximation is

$$u_y(x_0,y_0)=\frac{u(x_0,y_0+k)-u(x_0,y_0)}{k}$$

The backward difference approximation is

$$u_y(x_0,y_0)=\frac{u(x_0,y_0)-u(x_0,y_0-k)}{k}$$

The central difference approximation is

$$u_y(x_0,y_0)=\frac{u(x_0,y_0+k)-u(x_0,y_0-k)}{2k}$$

and
$$u_{yy}(x_0,y_0)=\frac{u(x_0,y_0+k)-2u(x_0,y_0)+u(x_0,y_0-k)}{k^2}$$

Since $u(x,y)=u(ih,jk)=u_{i,j}$, the above formulae can be written as

$$u_x=\frac{u_{i+1,j}-u_{i,j}}{h} \quad \text{– forward difference}$$

$$u_x=\frac{u_{i,j}-u_{i-1,j}}{h} \quad \text{– backward difference}$$

$$u_x=\frac{u_{i+1,j}-u_{i-1,j}}{2h} \quad \text{– central difference}$$

$$u_y=\frac{u_{i,j+1}-u_{i,j}}{k} \quad \text{– forward difference}$$

$$u_y=\frac{u_{i,j}-u_{i,j-1}}{k} \quad \text{– backward difference}$$

$$u_y=\frac{u_{i,j+1}-u_{i,j-1}}{2k} \quad \text{– central difference}$$

$$u_{xx}=\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}$$

$$u_{yy}=\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{k^2}$$

Solution of Laplace Equation $u_{xx}+u_{yy}=0$

Given the equation $u_{xx}+u_{yy}=0$, the solution $u(x,y)$ is satisfied by every point in a region subject to certain boundary conditions. Consider a rectangular region R for which $u(x,y)$ is known at the boundary. For simplicity, take R to be a square region and divide it into small square regions of side h.

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Replacing u_{xx} and u_{yy} by differences

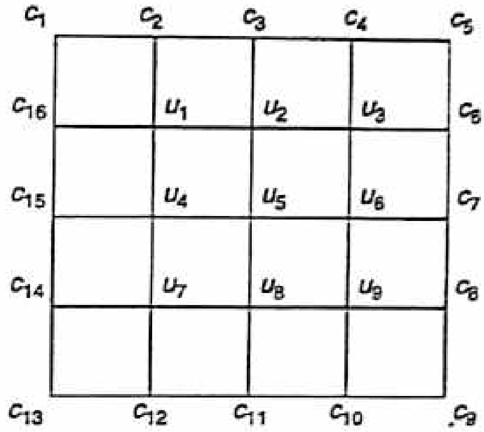


Figure 10.1

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$u_{yy} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}$$

the equation becomes

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} = 0$$

$$\Rightarrow \quad u_{i+1,j} - 4u_{i,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} = 0$$

$$\Rightarrow \quad u_{i,j} = \frac{1}{4}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right] \tag{1}$$

This shows that the value of u at any interior mesh point is the average value of the four adjacent mesh points on the lines through $u_{i,j}$.

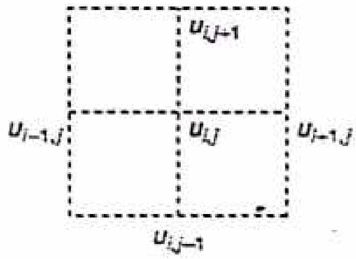


Figure 10.2

ie. the average of the mesh point values above, below, left and right of $u_{i,j}$

The formula (1) is called the **standard five point formula (SFPF)**

We know that Laplace equation remains invariant when the coordinate axes are rotated through 45°. So we can also use the following formula.

$$u_{i,j} = \frac{1}{4}\left[u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j-1} + u_{i+1,j-1}\right] \tag{2}$$

This shows that the value of $u_{i,j}$ is the average of the values of the four mesh points through the diagonals of $u_{i,j}$.

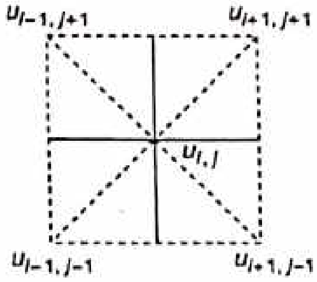


Figure 10.3

The formula (2) is called the **diagonal five point formula (DFPF) as in Fig 10.3.**

Now to find the initial values of u at the interior mesh points, we first find u_5 at the centre of the square by taking the mean of the 4 boundary values at the ends of the horizontal and vertical through u_5.

$$\therefore \quad u_5 = \frac{1}{4}\left[c_{15} + c_7 + c_3 + c_{11}\right]$$

Next we compute the initial values of the centers of the large squares with u_5 as a corner. ie. we find u_1, u_3, u_7, u_9 by DFPF

$$u_1 = \frac{1}{4}(u_5 + c_1 + c_3 + c_{15})$$

$$u_3 = \frac{1}{4}(u_5 + c_5 + c_3 + c_7)$$

$$u_7 = \frac{1}{4}(u_5 + c_{13} + c_{11} + c_{15})$$

$$u_9 = \frac{1}{4}(u_5 + c_9 + c_{11} + c_7)$$

The values at the remaining four mesh points u_2, u_4, u_6, u_8 are got by using SFPF Thus, we have

$$u_2 = \frac{1}{4}(u_1 + u_3 + c_3 + u_5)$$

$$u_4 = \frac{1}{4}(u_5 + c_{15} + u_1 + u_7)$$

$$u_6 = \frac{1}{4}(u_5 + c_7 + u_3 + u_9)$$

$$u_8 = \frac{1}{4}(u_7 + u_9 + u_5 + u_{11})$$

Having got all the values of $u_{i,j}$, the accuracy of these values are improved by any one of the following iteration methods.

1. Gauss Jacobi method:

If $u_{i,j}^{(n)}$ be the n^{th} iterative value of $u_{i,j}$, then an iterative procedure to SFPF is given by

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i+1,j}^{(n)} + u_{i-1,j}^{(n)} + u_{i,j+1}^{(n)} + u_{i,j-1}^{(n)}\right] \text{ for the interior mesh points.}$$

2. Gauss-Seidel method

The iterative formula for SFPF is given by

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i+1,j}^{(n)} + u_{i-1,j}^{(n+1)} + u_{i,j+1}^{(n)} + u_{i,j-1}^{(n+1)}\right], \qquad n = 0, 1, 2, 3, \ldots$$

Here we use the latest available iterative values and hence the rate of convergence will be faster by this method. Infact, it is nearly twice faster than Jacobi method.

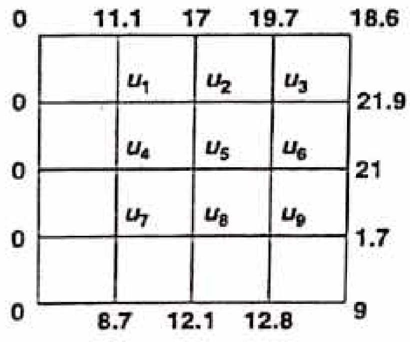
The application of Gauss-Seidel iteration to solve boundary value problems was due to Liebmann. So this method is known as **Liebmann's method.**

Note: It can be proved that the error in the diagonal formula is four times than in the standard formula. So, as far as possible we use the standard five point formula.

WORKED EXAMPLES

Example 1

Determine by iteration method the values at the interior lattice points of a square region of the harmonic function u whose boundary values are given as shown in the figure below



Solution

Since u is harmonic, it satisfies Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Let $u_1, u_2, u_3, \ldots, u_9$ be the values of u at the interior points.

We find the initial values of $u_1, u_2, \ldots, u_9$.

Then by iteration improve these values.

Now
$$u_5 = \frac{1}{4}[0+21+17+12.1] \quad = 12.5 \quad \text{[SFPF]}$$
$$u_1 = \frac{1}{4}[0+12.5+0+17] \quad = 7.4 \quad \text{[DFPF]}$$
$$u_3 = \frac{1}{4}[12.5+18.6+17+21] \quad = 17.3 \quad \text{[DFPF]}$$
$$u_7 = \frac{1}{4}[0+12.5+0+12.1] \quad = 6.2 \quad \text{[DFPF]}$$
$$u_9 = \frac{1}{4}[12.1+21+12.5+9] \quad = 13.7 \quad \text{[DFPF]}$$
$$u_2 = \frac{1}{4}[7.4+17+17.3+12.5] \quad = 13.6 \quad \text{[SFPF]}$$
$$u_4 = \frac{1}{4}[0+7.4+12.5+6.2] \quad = 6.5 \quad \text{[SFPF]}$$
$$u_6 = \frac{1}{4}[12.5+17.3+21+13.7] = 16.1 \quad \text{[SFPF]}$$
$$u_8 = \frac{1}{4}[6.2+12.5+13.7+12.1] = 11.1 \quad \text{[SFPF]}$$

We now start the iteration to improve these values. We start with u_1 and proceed along horizontal.
Liebmann's iteration formula is

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j+1}^{(n)} + u_{i,j-1}^{(n+1)}\right], \qquad n = 0,1,2,\ldots$$

I Iteration: SFPF

$$u_1^{(1)} = \frac{1}{4}[0+u_2+u_4+11.1] \quad = \frac{1}{4}[13.6+6.5+11.1] \quad = 7.8$$
$$u_2^{(1)} = \frac{1}{4}[u_1^{(1)}+u_3+17+u_5] \quad = \frac{1}{4}[7.8+17.3+12.5+17] \quad = 13.7$$
$$u_3^{(1)} = \frac{1}{4}[u_2+21.9+19.7+u_6] = \frac{1}{4}[13.7+21.9+19.7+16.1] = 17.9$$
$$u_4^{(1)} = \frac{1}{4}[0+u_5+u_1^{(1)}+u_7] \quad = \frac{1}{4}[12.5+7.8+6.2] \quad = 6.6$$
$$u_5^{(1)} = \frac{1}{4}[u_4^{(1)}+u_6+u_2^{(1)}+u_8] \quad = \frac{1}{4}[6.6+16.1+13.7+11.1] \quad = 11.9$$
$$u_6^{(1)} = \frac{1}{4}\left[u_5^{(1)}+21+u_3^{(1)}+u_9\right] \quad = \frac{1}{4}\left[11.9+21+17.9+13.7\right] = 16.1$$

$$u_7^{(1)} = \frac{1}{4}\left[0 + u_8 + 8.7 + u_4^{(1)}\right] \quad = \frac{1}{4}[11.1 + 8.7 + 6.6] \quad = 6.6$$

$$u_8^{(1)} = \frac{1}{4}\left[u_7^{(1)} + u_9 + u_5^{(1)} + 12.1\right] = \frac{1}{4}[6.6 + 13.7 + 11.9 + 12.1] = 11.1$$

$$u_9^{(1)} = \frac{1}{4}\left[u_8^{(1)} + 17 + 12.8 + u_6^{(1)}\right] = \frac{1}{4}[11.1 + 17 + 12.8 + 16.1] = 14.3$$

II Iteration:

$$u_1^{(2)} = \frac{1}{4}\left[0 + u_2^{(1)} + 11.1 + u_4^{(1)}\right] \quad = \frac{1}{4}[13.7 + 11.1 + 6.6] \quad = 7.9$$

$$u_2^{(2)} = \frac{1}{4}\left[17 + u_5^{(1)} + u_1^{(1)} + u_3^{(1)}\right] \quad = \frac{1}{4}[17 + 11.9 + 7.9 + 17.9] \quad = 13.7$$

$$u_3^{(2)} = \frac{1}{4}\left[u_2^{(1)} + 21.9 + 19.7 + u_6^{(1)}\right] = \frac{1}{4}[13.7 + 21.9 + 19.7 + 16.1] = 17.9$$

$$u_4^{(2)} = \frac{1}{4}\left[u_1^{(1)} + u_7^{(1)} + 0 + u_5^{(1)}\right] \quad = \frac{1}{4}[7.9 + 6.6 + 11.9] \quad = 6.6$$

$$u_5^{(2)} = \frac{1}{4}\left[u_2^{(2)} + u_8^{(1)} + u_4^{(2)} + u_6^{(1)}\right] \quad = \frac{1}{4}[13.7 + 11.1 + 6.6 + 16.1] \quad = 11.9$$

$$u_6^{(2)} = \frac{1}{4}\left[u_3^{(2)} + u_9^{(1)} + u_5^{(2)} + 21\right] \quad = \frac{1}{4}[17.9 + 14.3 + 11.9 + 21] \quad = 16.3$$

$$u_7^{(2)} = \frac{1}{4}\left[u_4^{(2)} + 8.7 + 0 + u_8^{(1)}\right] \quad = \frac{1}{4}[6.6 + 8.7 + 11.1] \quad = 6.6$$

$$u_8^{(2)} = \frac{1}{4}\left[u_5^{(2)} + 12.1 + u_7^{(2)} + u_9^{(1)}\right] \quad = \frac{1}{4}[11.9 + 12.1 + 6.6 + 14.3] \quad = 11.2$$

$$u_9^{(2)} = \frac{1}{4}\left[17 + u_8^{(2)} + 12.8 + u_6^{(2)}\right] \quad = \frac{1}{4}[17 + 11.2 + 12.8 + 16.3] \quad = 14.3$$

III Iteration:

$$u_1^{(3)} = \frac{1}{4}[0 + u_2^{(2)} + u_4^{(2)} + 11.1] \quad = \frac{1}{4}[13.7 + 6.6 + 11.1] \quad = 7.9$$

$$u_2^{(3)} = \frac{1}{4}[17 + u_5^{(2)} + u_1^{(3)} + u_3^{(2)}] \quad = \frac{1}{4}[17 + 11.9 + 7.9 + 17.9] \quad = 13.7$$

$$u_3^{(3)} = \frac{1}{4}[19.7 + u_6^{(2)} + u_2^{(3)} + 21.9] = \frac{1}{4}[19.7 + 16.3 + 13.7 + 21.9] = 17.9$$

$$u_4^{(3)} = \frac{1}{4}[0 + u_5^{(2)} + u_1^{(3)} + u_7^{(2)}] \quad = \frac{1}{4}[11.9 + 7.9 + 6.6] \quad = 6.6$$

$$u_5^{(3)} = \frac{1}{4}[u_2^{(3)} + u_8^{(2)} + u_4^{(3)} + u_6^{(2)}] \quad = \frac{1}{4}[13.7 + 11.2 + 6.6 + 16.3] \quad = 11.9$$

$$u_6^{(3)} = \frac{1}{4}[u_3^{(3)} + u_9^{(2)} + u_5^{(3)} + 21] \quad = \frac{1}{4}[17.9 + 14.3 + 11.9 + 21] \quad = 16.3$$

$$u_7^{(3)} = \frac{1}{4}[0 + u_8^{(2)} + u_4^{(3)} + 8.7] \quad = \frac{1}{4}[11.2 + 6.6 + 8.7] \qquad = 6.6$$

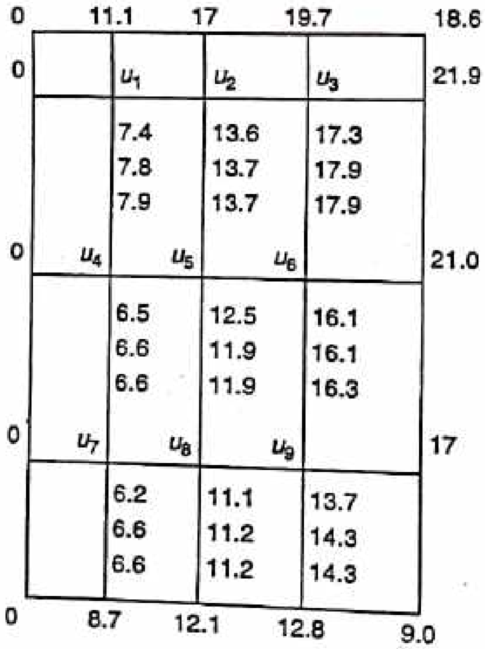
$$u_8^{(3)} = \frac{1}{4}[u_5^{(3)} + 12.1 + u_7^{(3)} + u_9^{(2)}] = \frac{1}{4}[11.9 + 12.1 + 6.6 + 14.3] = 11.2$$

$$u_9^{(3)} = \frac{1}{4}[17 + u_8^{(3)} + 12.8 + u_6^{(3)}] \quad = \frac{1}{4}[17 + 11.2 + 12.8 + 16.3] = 14.3$$

We notice that II and III iteration values are almost same

$\therefore$ $\quad u_1 = 7.9, \quad u_2 = 13.7, \quad u_3 = 17.9, \quad u_4 = 6.6, \quad u_5 = 11.9,$
$\quad u_6 = 16.3, \quad u_7 = 6.6, \quad u_8 = 11.2, \quad u_9 = 14.3$

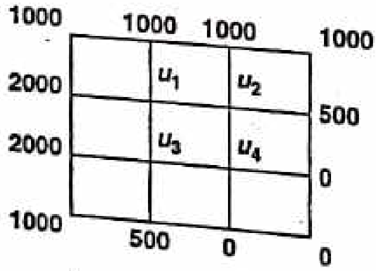
Remark: We have worked out elaborately. We can find the values of u's at each grid point indicating the values at the point itself as in the table schematically.
Using the latest available values.



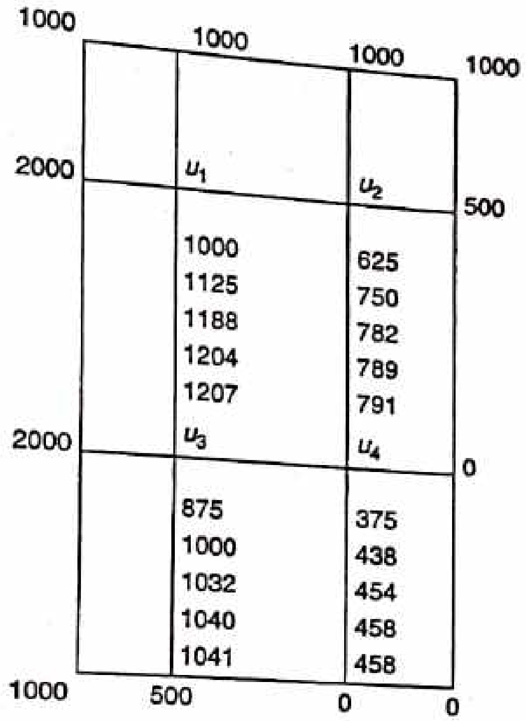
■

Example 2

Given the values of $u(x, y)$ on the boundary of the square in the figure, evaluate the function $u(x, y)$ satisfying the Laplace equation $\nabla^2 u = 0$ at the pivotal points of the figure by Gauss-Seidel method.



Solution



To get the initial values of u_1, u_1, u_3, u_4 we can not use either SFPF or DFPF to find any values. So we shall assume $u_4 = 0$ as the initial value.

Then

$$u_1 = \frac{1}{4}[1000+0+2000+1000] = 1000 \qquad [DFPF]$$

$$u_2 = \frac{1}{4}[1000+0+1000+500] = 625 \qquad [SFPF]$$

$$u_3 = \frac{1}{4}[1000+500+0+2000] = 875 \qquad [SFPF]$$

$$u_4 = \frac{1}{4}[625+0+875+0] = 375 \qquad [SFPF]$$

I Iteration:

$$u_1^{(1)} = \frac{1}{4}[1000+875+2000+625] = 1125$$

$$u_2^{(1)} = \frac{1}{4}[1000+375+1125+500] = 750$$

$$u_3^{(1)} = \frac{1}{4}[1125+500+2000+375] = 1000$$

$$u_4^{(1)} = \frac{1}{4}[0+750+0+1000] = 438$$

II Iteration:

$$u_1^{(2)} = \frac{1}{4}[1000+1000+2000+750] = 1188$$
$$u_2^{(2)} = \frac{1}{4}[1188+500+100+438] = 782$$
$$u_3^{(2)} = \frac{1}{4}[2000+438+1188+500] = 1032$$
$$u_4^{(2)} = \frac{1}{4}[1032+0+782+0] = 454$$

III Iteration:

$$u_1^{(3)} = \frac{1}{4}[2000+782+1000+1032] = 1204$$
$$u_2^{(3)} = \frac{1}{4}[1204+500+454+1000] = 789$$
$$u_3^{(3)} = \frac{1}{4}[1204+500+2000+454] = 1040$$
$$u_4^{(3)} = \frac{1}{4}[1040+0+0+789] = 458$$

IV Iteration:

$$u_1^{(4)} = \frac{1}{4}[2000+789+1000+1040] = 1207$$
$$u_2^{(4)} = \frac{1}{4}[1207+500+1000+458] = 791$$
$$u_3^{(4)} = \frac{1}{4}[1207+500+2000+458] = 1041$$
$$u_4^{(4)} = \frac{1}{4}[1041+0+0+791] = 458$$

V Iteration:

$$u_1^{(5)} = 1208, \quad u_2^{(5)} = 792, \quad u_3^{(5)} = 1042, \quad u_4^{(5)} = 458$$

We take $\boldsymbol{u_1 = 1208, \ u_2 = 792, \ u_3 = 1042, \ u_4 = 458}$ ■

Example 3

By iteration method, solve the elliptic equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ over a square region of side 4, satisfying the boundary conditions

(i) $u(0,y) = 0$ for $0 \le y \le 4$ (ii) $u(4,y) = 12 + y$ for $0 \le y \le 4$
(iii) $u(x,0) = 3x$ for $0 \le y \le 4$ (iv) $u(x,4) = x^2$ for $0 \le y \le 4$

By dividing the square into 16 square mashes of side 1 and always correcting the computed values to two places of decimals, obtain the values of *u* at 9 interior pivotal points.

Solution

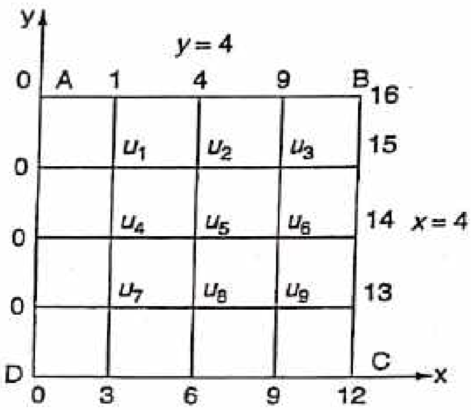
Given equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

The boundary condition $u(0, y) = 0,\ 0 \le y \le 4$ gives all boundary values zero on y-axis.

$u(x, 0) = 3x,\quad 0 \le x \le 4$ means on x-axis the values are 0, 3, 6, 9, 12

$u(4, y) = 12 + y,\quad 0 \le x \le 4$ gives the values on $x = 4$ as 12, 13, 14, 15, 16.

$u(x, 4) = x^2,\quad 0 \le x \le 4$ means on $y = 4$ the boundary values are 0, 1, 4, 9, 16.



Let us take the 9 interior grid points as $u_1, u_2, u_3, \ldots, u_9$
First we have to find the initial values.

$$u_5 = \frac{1}{4}(0 + 14 + 4 + 6) = 6 \qquad \text{[SFPF]}$$

$$u_1 = \frac{1}{4}(0 + u_5 + 0 + 4) = \frac{1}{4}(6 + 4) = 2.5 \qquad \text{[DFPF]}$$

$$u_3 = \frac{1}{4}(u_5 + 16 + 14 + 4) = \frac{1}{4}(6 + 34) = 10 \qquad \text{[DFPF]}$$

$$u_7 = \frac{1}{4}(u_5 + 0 + 6 + 0) = \frac{1}{4}(6 + 6) = 3 \qquad \text{[DFPF]}$$

$$u_9 = \frac{1}{4}(u_5 + 12 + 6 + 14) = \frac{1}{4}(6 + 32) = 9.5 \qquad \text{[DFPF]}$$

The other four values u_2, u_4, u_6, u_8 can be found by SFPF

$$u_2 = \frac{1}{4}(4+6+2.5+10) = 5.63$$

$$u_4 = \frac{1}{4}(0+6+2.5+3) = 2.88$$

$$u_6 = \frac{1}{4}(10+9.5+6+14) = 9.88$$

$$u_8 = \frac{1}{4}(6+6+3+9.5) = 6.13$$

I Iteration:

$$u_1^{(1)} = \frac{1}{4}(1+2.88+0+5.63) = 2.38$$

$$u_2^{(1)} = \frac{1}{4}(4+6+2.38+10) = 5.60$$

$$u_3^{(1)} = \frac{1}{4}(9+9.88+5.6+15) = 9.87$$

$$u_4^{(1)} = \frac{1}{4}(2.38+3+0+6) = 2.85$$

$$u_5^{(1)} = \frac{1}{4}(5.6+6.13+2.85+9.88) = 6.12$$

$$u_6^{(1)} = \frac{1}{4}(9.87+9.5+6.12+14) = 9.87$$

$$u_7^{(1)} = \frac{1}{4}(2.85+3+0+6.13) = 3.00$$

$$u_8^{(1)} = \frac{1}{4}(6.12+6+3+9.5) = 6.16$$

$$u_9^{(1)} = \frac{1}{4}(9.87+9+6.16+13) = 9.51$$

II Iteration:

$$u_1^{(2)} = \frac{1}{4}[1+u_4^{(1)}+0+u_2^{(1)}] = \frac{1}{4}[1+2.85+0+5.60] = 2.36$$

$$u_2^{(2)} = \frac{1}{4}[u_1^{(2)}+u_3^{(1)}+4+u_5^{(1)}] = \frac{1}{4}[2.36+9.87+6.12+4] = 5.59$$

$$u_3^{(2)} = \frac{1}{4}[u_2^{(2)}+15+9+u_6^{(1)}] = \frac{1}{4}[5.59+15+9+9.87] = 9.87$$

$$u_4^{(2)} = \frac{1}{4}[u_1^{(2)}+u_7^{(1)}+0+u_5^{(1)}] = \frac{1}{4}[2.36+3+6.12] = 2.87$$

$$u_5^{(2)} = \frac{1}{4}[u_2^{(2)}+u_8^{(1)}+u_4^{(2)}+u_6^{(1)}] = \frac{1}{4}[5.59+6.16+2.87+9.87] = 6.12$$

$$u_6^{(2)} = \frac{1}{4}[u_3^{(2)} + u_9^{(1)} + u_5^{(2)} + 14] \quad = \frac{1}{4}[9.87 + 9.51 + 6.12 + 14] \quad = 9.88$$

$$u_7^{(2)} = \frac{1}{4}[u_4^{(2)} + 3 + 0 + u_8^{(1)}] \quad = \frac{1}{4}[2.87 + 3 + 6.16] \quad = 3.01$$

$$u_8^{(2)} = \frac{1}{4}[u_5^{(2)} + 6 + u_7^{(2)} + u_9^{(1)}] \quad = \frac{1}{4}[6.12 + 6 + 3.01 + 9.51] \quad = 6.16$$

$$u_9^{(2)} = \frac{1}{4}[u_6^{(2)} + 9 + u_8^{(2)} + 13] \quad = \frac{1}{4}[9.88 + 22 + 6.16] \quad = 9.51$$

III Iteration:

$$u_1^{(3)} = \frac{1}{4}[1 + u_4^{(2)} + 0 + u_2^{(2)}] \quad = \frac{1}{4}[1 + 2.87 + 5.59] \quad = 2.37$$

$$u_2^{(3)} = \frac{1}{4}[u_1^{(3)} + u_3^{(2)} + 4 + u_5^{(2)}] \quad = \frac{1}{4}[2.37 + 9.87 + 4 + 6.12] \quad = 5.59$$

$$u_3^{(3)} = \frac{1}{4}[u_2^{(3)} + 15 + 9 + u_6^{(2)}] \quad = \frac{1}{4}[5.59 + 24 + 9.88] \quad = 9.87$$

$$u_4^{(3)} = \frac{1}{4}[u_1^{(3)} + u_7^{(2)} + 0 + u_5^{(2)}] \quad = \frac{1}{2}[2.37 + 3.01 + 6.12] \quad = 2.88$$

$$u_5^{(3)} = \frac{1}{4}[u_2^{(3)} + u_8^{(2)} + u_4^{(3)} + u_6^{(2)}] = \frac{1}{4}[5.59 + 6.16 + 2.88 + 9.88] = 6.13$$

$$u_6^{(3)} = \frac{1}{4}[u_3^{(3)} + u_9^{(2)} + u_5^{(3)} + 14] \quad = \frac{1}{4}[9.87 + 9.51 + 6.13 + 14] \quad = 9.88$$

$$u_7^{(3)} = \frac{1}{4}[u_4^{(3)} + 3 + 0 + u_8^{(2)}] \quad = \frac{1}{4}[2.88 + 3 + 6.16] \quad = 3.01$$

$$u_8^{(3)} = \frac{1}{4}[u_5^{(3)} + 6 + u_7^{(3)} + u_9^{(2)}] \quad = \frac{1}{4}[6.13 + 6 + 3.01 + 9.51] \quad = 6.16$$

$$u_9^{(3)} = \frac{1}{4}[u_6^{(3)} + 9 + u_8^{(3)} + 13] \quad = \frac{1}{4}[9.88 + 22 + 6.16] \quad = 9.51$$

From the II and III iterations we can take the values of u correct to 2 places as

$$u_1 = 2.37, \quad u_2 = 5.59, \quad u_3 = 9.87, \quad u_4 = 2.88, \quad u_5 = 6.13,$$
$$u_6 = 9.88, \quad u_7 = 3.01, \quad u_8 = 6.16, \quad u_9 = 9.51$$ ■

10.2.4 Poisson Equation

An equation of the form $u_{xx} + u_{yy} = f(x, y)$ is called Poisson's equation. (1)
Proceeding as in Laplace equations we get the standard five point formula for poisson equation as

$$u_{i-1,j}+u_{i+1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j}=h^2 f(ih, jh)$$

$$\Rightarrow \quad u_{i,j}=\frac{1}{4}[u_{i-1,j}+u_{i+1,j}+u_{i,j+1}+u_{i,j-1}-h^2 f(ih, jh)] \qquad (2)$$

By applying (2) at each interior mesh point, we obtain a system of linear equations in the Pivotal values i, j.

We solve by Gauss-Siedel method

method $u_{i,j}=\frac{1}{4}[u_{i-1,j}+u_{i+1,j}+u_{i,j+1}+u_{i,j-1}-f(i,j)]$ if $h=1$.

WORKED EXAMPLES

Example 4

Solve the Poisson equation

$$\nabla^2 u=-10(x^2+y^2+10), \quad 0\le x\le 3, \quad 0\le y\le 3, \quad u=0$$

on the boundary.

Solution

Given $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-10(x^2+y^2+10), \quad 0\le x\le 3, \quad 0\le y\le 3$$

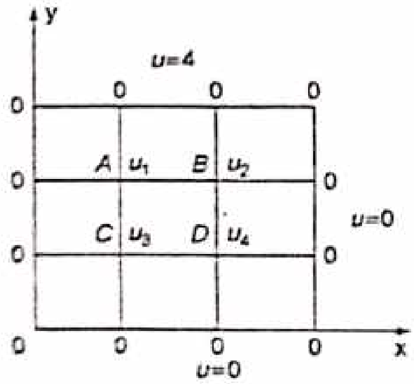
The region is a square bounded by $x=0$, $x=3$, $y=0$, $y=3$

Divide it into a square mesh with side $h=1$.

Let u_1, u_2, u_3, u_4 be the values of u at the interior mesh points A, B, C, D

A is (1,2), B is (2,2), $C(1,1)$, $D(2,1)$.

$\Rightarrow$



The difference equation is $u_{i-1,j}+u_{i+1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j}=-10(i^2+j^2+10)$

$$\Rightarrow \quad u_{i,j}=\frac{1}{4}[u_{i-1,j}+u_{i+1,j}+u_{i,j+1}+u_{i,j-1}+10(i^2+j^2+10)] \qquad (1)$$

Applying formula (1) at A (1,2) u is $u_1=\frac{1}{4}[0+u_2+0+u_3+10(1+4+10)]$

$$\Rightarrow \quad u_1=\frac{1}{4}[u_2+u_3+150] \qquad (2)$$

At B, u is $u_2 = \frac{1}{4}[u_1 + 0 + 0 + u_4 + 10(2^2 + 2^2 + 10)]$

$\Rightarrow$ $u_2 = \frac{1}{4}[u_1 + u_4 + 180]$ (3)

At C, u is $u_3 = \frac{1}{4}[0 + u_1 + 0 + u_4 + 10(1 + 1 + 10)]$

$\Rightarrow$ $u_3 = \frac{1}{4}[u_1 + u_4 + 120]$ (4)

At D, u is $u_4 = \frac{1}{4}[0 + u_2 + u_3 + 0 + 10(4 + 1 + 10)]$

$\Rightarrow$ $u_4 = \frac{1}{4}[u_2 + u_3 + 150]$ (5)

(2) is $4u_1 = u_2 + u_3 + 150 \Rightarrow 4u_1 - u_2 - u_3 = 150$ (2)′

(3) is $4u_2 = u_1 + u_4 + 180 \Rightarrow -u_1 + 4u_2 - u_4 = 180$ (3)′

(4) is $4u_3 = u_1 + u_4 + 120 \Rightarrow -u_1 + 4u_3 - u_4 = 120$ (4)′

(5) is $4u_4 = u_2 + u_3 + 150 \Rightarrow -u_2 - u_3 + 4u_4 = 150$ (5)′

Eliminate u_1; (3)′ − (4)′ $\Rightarrow 4u_2 - 4u_3 = 60$

$\Rightarrow$ $u_2 - u_3 = 15$ (6)

$4 \times (3)' \Rightarrow$ $-4u_1 + 16u_2 - 4u_4 = 720$

(2)′ $\Rightarrow$ $4u_1 - u_2 - u_3 = 150$

Adding, $15u_2 - u_3 - 4u_4 = 870$

(5)′ is $-u_2 - u_3 + 4u_4 = 150$

Adding, $14u_2 - 2u_3 = 1020$

(6) × 2 $\Rightarrow$ $2u_2 - 2u_3 = 30$

Subtracting, $12u_2 = 990 \Rightarrow u_2 = \frac{990}{12} = 82.5$

$\therefore$ $u_3 = u_2 - 15 = 82.5 - 15\ u_3 = 67.5$

(2)′ $\Rightarrow$ $4u_1 = 150 + u_2 + u_3 = 150 + 82.5 + 67.5 = 300$

$\therefore$ $u_1 = \frac{300}{4} = 75$

(5) $\Rightarrow$ $u_4 = \frac{1}{4}[82.5 + 67.5 + 150] = 75$

Thus $\mathbf{u_1 = u_4 = 75,\ u_2 = 82.5,\ u_3 = 67.5}$

Note: In the above problem $\nabla^2 u = -10(x^2 + y^2 + 10)$ which is symmetric in x and y. That is when x and y are interchanged the equation is not changed. So the function is symmetric about the line $y = x$. So the solution is symmetric about $y = x$.

The boundary conditions are also symmetric

$\therefore$ $u_1 = u_4$

This is the reason we got $u_1 = u_4$

This observation will be helpful in solving, when the number of mesh points are more in a square.

Example 5

Solve $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 8x^2y^2$ in the square mesh given $u = 0$ on the four boundaries dividing the square into 16 subsquares of length 1 unit.

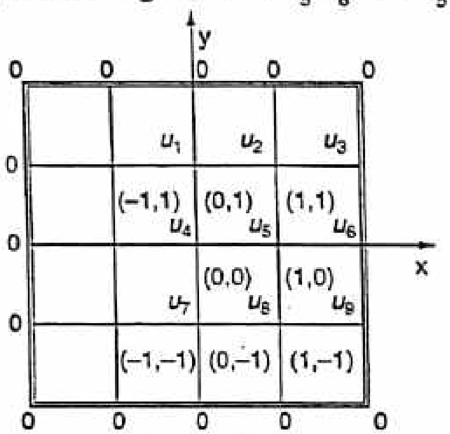
Solution

Given $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 8x^2y^2$

Here $h = 1$

Let $u_1, u_2, \ldots, u_9$ be the values of u at the interior grid points.

Choose u_5 as origin and the x, y axes along the lines $u_5\,u_6$ and $u_5\,u_2$



Since the equation is symmetric in x and y and the boundary values are same when x and y are interchanged.

The values of u at the mesh points symmetric about $y = x$ and about the axes will be the same

$$\therefore \quad u_1 = u_3 = u_7 = u_9$$

$$u_2 = u_8, \text{ and } u_4 = u_6, \text{ but } u_2 = u_6 \text{ (about } y = x)$$

$$\therefore \quad u_2 = u_4 = u_6 = u_8$$

Hence we have to find only u_1, u_2, u_5

The difference equation given by the SFPF for poisons equation is

$$u_{i,j} = \frac{1}{4}[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j-1} - 8i^2 j^2] \quad (1)$$

$$[\because f(x,y) = 8x^2y^2]$$

$$u_5 = \frac{1}{4}[u_4 + u_6 + u_2 + u_8] \qquad [\because i = 0, j = 0]$$

$$= \frac{1}{4}[4u_2] = u_2 \qquad [\because u_2 = u_4 = u_6 = u_8]$$

$$\therefore \quad u_2 = u_5 \quad (2)$$

$$u_2 = \frac{1}{4}[u_1 + u_3 + u_5 + 0 - 8] \qquad [\because i = 0, j = 1]$$

$$= \frac{1}{4}[u_1 + u_3 + u_5] \qquad [\because u_1 = u_3]$$

$$= \frac{1}{4}[2u_1 + u_5] = \frac{1}{4}[2u_1 + u_2] \qquad [\because u_2 = u_5] \quad (3)$$

$\Rightarrow$ $$4u_2 = 2u_1 + u_2 \quad \Rightarrow \quad 3u_2 = 2u_1$$

Now $$u_1 = \frac{1}{4}[0 + u_2 + 0 + u_4 - 8(-1)^2.1^2] \qquad [\because i = -1, j = 1] \quad (4)$$

$$= \frac{1}{4}[2u_2 - 8] \qquad [\because u_2 = u_4]$$

$\Rightarrow$ $$u_1 = \frac{1}{2}(u_2 - 4) \quad \Rightarrow \quad 2u_1 = u_2 - 4$$

Substitute in (3), $$3u_2 = u_2 - 4 \quad \Rightarrow \quad 2u_2 = -4 \quad \Rightarrow \quad u_2 = -2$$

$$u_1 = -3, \quad u_5 = -2$$

Hence $$\boldsymbol{u_1 = u_3 = u_7 = u_9 = -3}$$

and $$\boldsymbol{u_2 = u_4 = u_6 = u_8 = -2, \quad u_5 = -2}$$

Example 6

Solve the poisson equation $u_{xx} + u_{yy} = -81xy$, $0 < x < 1$, $0 < y < 1$. Given that $u(0, y) = 0$, $u(x, 0) = 0$, $u(1, y) = 100$, $u(x, 1) = 100$ and $h = \frac{1}{3}$.

Solution

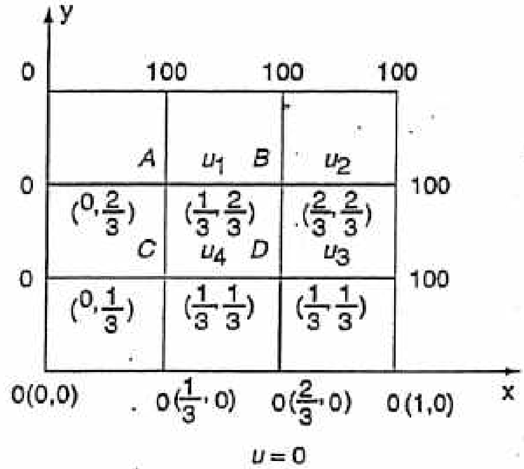
Given $u_{xx} + u_{yy} = -81xy$, $0 < x < 1, 0 < y < 1$. $h = \frac{1}{3}$.

The equation is symmetric about x and y.

When x and y are interchanged it is unaffected and so symmetric about $y = x$.

The boundary conditions are also symmetric about $y = x$.

Let u_1, u_2, u_3, u_4 be the values of u at the mesh points.



$$u_1 = u_3$$

$\therefore$

The difference equation by SFPF is

$$u_{i,j} = \frac{1}{4}[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^2 f(ih, jh)] \tag{1}$$

$$\left[\because h = \frac{1}{3}, f(ih, jh) = -81h^2 ij\right]$$

$$u_{i,j} = \frac{1}{4}[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^4 81ij]$$

$$= \frac{1}{4}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} + ij\right] \qquad \left[\because h^4 = \frac{1}{81}\right]$$

Now

$$u_1 = \frac{1}{4}\left[0 + u_2 + 100 + u_4 + \frac{1}{3}\cdot\frac{2}{3}\right] \qquad \left[\because i = \frac{1}{3}, j = \frac{2}{3}\right]$$

$$\Rightarrow \quad 4u_1 = \left[u_2 + u_4 + \frac{902}{9}\right] \Rightarrow 4u_1 - u_2 - u_4 = \frac{902}{9} \tag{1}$$

$$u_2 = \frac{1}{4}\left[u_1 + 100 + 100 + u_3 + \frac{2}{3}\cdot\frac{2}{3}\right] \qquad \left[\because i = \frac{2}{3}, j = \frac{2}{3}\right]$$

$$\Rightarrow \quad 4u_2 = 2u_1 + 200 + \frac{4}{9} \Rightarrow 2u_2 = u_1 + 100 + \frac{2}{9} \qquad [\because u_1 = u_3]$$

$$\Rightarrow \quad 2u_2 - u_1 = \frac{902}{9} \tag{2}$$

$$u_4 = \frac{1}{4}\left[0 + u_1 + 0 + u_3 + \frac{1}{3}\cdot\frac{1}{3}\right]$$

$$= \frac{1}{4}\left[2u_1 + \frac{1}{9}\right] \tag{3}$$

Substitute for u_4 in (1) we get

$$4u_1 - u_2 - \frac{1}{4}\left(2u_1 + \frac{1}{9}\right) = \frac{902}{9}$$

$$\Rightarrow \quad \frac{7}{2}u_1 - u_2 = \frac{1}{36} + \frac{902}{9}$$

$$\Rightarrow \quad 7u_1 - 2u_2 = \frac{3609}{18} \tag{4}$$

$$(2)\times 7 \Rightarrow \quad -7u_1 + 14u_2 = \frac{6314}{9} \tag{5}$$

$$(4)+(5) \Rightarrow \quad 12u_2 = \frac{3609}{18} + \frac{6314}{9} = \frac{16237}{18}$$

$$\therefore \quad u_2 = \frac{16237}{12\times 18} = 75.2$$

Substituting u_2 in (2) $\quad u_1 = 2 \times 75.2 - \dfrac{902}{9} = 50.18$

Substituting u_1 in (3) $\quad u_4 = \dfrac{1}{4}\left[2 \times 50.18 + \dfrac{1}{9}\right] = 25.12$

$\therefore \qquad \mathbf{u_1 = 50.18 = u_3,\ u_2 = 75.2,\ u_4 = 25.12}$

ONE DIMENSIONAL HEAT EQUATION

The one dimensional heat equation is $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2},\ \alpha^2 = \dfrac{k}{c\rho}$.

(where c is the specific heat of the material, ρ is the density and k is the thermal conductivity).
It is a parabolic equation.

Put $\alpha^2 = \dfrac{1}{a}$ then the equation can be written as

$$\frac{\partial^2 u}{\partial x^2} - a.\frac{\partial u}{\partial t} = 0 \quad \Rightarrow \quad u_{xx} - au_t = 0 \tag{1}$$

We shall see two methods to solve for $u(x, t)$ subject to the boundary conditions.

$$u(x, 0) = T_0, \tag{2}$$

$$u(l, t) = T_1 \tag{3}$$

the initial conditions $\qquad u(x, 0) = f(x), \qquad 0 < x < l \qquad (4)$

Schmidt's Method [Explicit Method]

Consider a rectangular mesh in the x-t plane with side lengths h in the x-direction and k in the t-direction. Let us denote the mesh point $(x, y) = (ih, jk)$ as simply i, j.
Then we have the approximate relations of differences

$$u_t = \frac{u_{i,j+1} - u_{i,j}}{k} \qquad i, j = 0, 1, 2, \cdots$$

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

Substituting in the equations (1), we get

$$\frac{u_{i+1,j} - 1u_{i,j} + u_{i-1,j}}{h^2} - a\left(\frac{u_{i,j+1} - u_{i,j}}{k}\right) = 1$$

$$\Rightarrow \qquad u_{i,j+1} - u_{i,j} = \frac{k}{ah^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right]$$

$$= \lambda\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right]$$

where $\lambda = \dfrac{k}{ah^2} = \dfrac{k\alpha^2}{h^2}$ is the mesh ratio parameter.

$$\therefore \qquad u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j} \tag{5}$$

The boundary conditions can be written as

$$u_{0,j} = T_0 \quad \text{and} \quad u_{l,j} = T_1, \quad \text{where} \quad nh = l, \quad j = 1,2,3,...$$

and the initial condition can be written as

$$u_{x,0} = f(ih),\ i = 1,2,3,...$$

We choose the time interval k in such a way that the coefficient of $u_{i,j}$ in (5) is zero.

$$\therefore \qquad 1-2\lambda = 0 \quad \Rightarrow \lambda = \frac{1}{2} \quad \Rightarrow \frac{k\alpha^2}{h^2} = \frac{1}{2} \quad \Rightarrow k = \frac{h^2}{2\alpha^2}$$

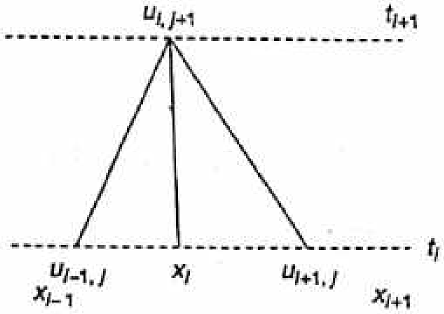
Then the equation becomes $\qquad u_{i,j+1} = \lambda\left[u_{i+1,j} + u_{i-1,j}\right]$

$$\Rightarrow \qquad u_{i,j+1} = \frac{1}{2}\left[u_{i+1,j} + u_{i-1,j}\right] \tag{6}$$

This equation is called the Bender-Schmidt recurrence equation.

Note:

(1) The equation (6) gives the value of u at the $(i, j+1)^{th}$ interior mesh point as the average of the values of u at the surrounding points x_{i+1} and x_{i-1} at previous time t_j



(2) Schmidt method is called the **explicit method** because each subsequent computation u at $t = t_{j+1}$ is explicitly given from the previous u-value at $t = t_j$

(3) Bender-Schmidt method is valid only for $0 \le \lambda \le \frac{1}{2} \Rightarrow k \le \frac{h^2}{2\alpha^2}$. To obtain more accurate results h should be small and hence k should be very small. This makes the computation pretty long because many time-intervals would be required to cover the prescribed region. The solution will become unstable if λ exceeds $\frac{1}{2}$. The reason for this in the explicit method is the difference in orders of the finite-difference approximations for the spatial derivative $\frac{\partial^2 u}{\partial x^2}$ and the time derivative $\frac{\partial u}{\partial t}$.

The next method, proposed by Crank and Nicolson in 1947 is a technique that makes these finite difference approximations of the same order. This method does not restrict λ and also reduces the amount of computations.

10.3.2 Crank-Nicolson Method [Implicit Method]

One dimensional heat equations is $u_{xx} = au_t$ (1)

where $a = \frac{1}{\alpha^2} = \frac{c\rho}{k}$ with boundary conditions $u(0,t) = T_0$, $u(l,t) = T_1$ and the initial condition $u(x,0) = f(x);\ 0 < x < l$

In this method $\frac{\partial^2 u}{\partial x^2}$ is replaced by the average of its central-difference approximations on the j^{th} and $(j+1)^{th}$ time rows.

$$\therefore \qquad u_{xx} = \frac{1}{2}\left[\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}\right]$$

and
$$u_t = \frac{u_{i,j+1} - u_{i,j}}{k}$$

$\therefore$ the equation (1) becomes

$$\frac{1}{2h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\right] = a\left(\frac{u_{i,j+1} - u_{i,j}}{k}\right)$$

$$\Rightarrow \qquad \frac{k}{ah^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\right] = 2\left[u_{i,j+1} - u_{i,j}\right]$$

Put $\lambda = \frac{k}{ah^2} = \frac{\alpha^2 k}{h^2}$

$$\therefore \qquad \lambda\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\right] = 2\left(u_{i,j+1} - u_{i,j}\right)$$

$$\lambda u_{i-1,j+1} - 2(1+\lambda)u_{i,j+1} + \lambda u_{i+1,j+1} = -\lambda u_{i-1,j} + 2(\lambda - 1)u_{i,j} - \lambda\, u_{i+1,j} \qquad (2)$$

This is called the Crank-Nicolson difference formula

Note:

(1) The left hand side of the formula contains three unknown values of u at the $(j+1)^{th}$ time level while all the three values on the right hand side are known values of u at the j^{th} level. Hence Crank-Nicolson formula is called a 2-level **implicit relation** because it does not give the value of u at $t = t_{j+1}$ directly in terms of u at $t = t_j$

(2) To simplify the scheme, we choose k such that $\lambda = 1 \Rightarrow \frac{k\alpha^2}{h^2} = 1 \Rightarrow k = \frac{h^2}{\alpha^2}$,
The Crank-Nicolson formula becomes

$$u_{i-1,j+1} - 4u_{i,j+1} + u_{i+1,j+1} = -u_{i-1,j} - u_{i+1,j}$$

$$\therefore \qquad u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\right] \qquad (3)$$

If there are n internal mesh points on each row then the formula (2) gives a system of n simultaneous equations for the n unknown values in terms of the known boundary values. These equations can be soled by Gauss Seidel method. Similarly the values at the other internal mesh points on each row can be found.

WORKED EXAMPLES

Example 1

Solve $\frac{\partial^2 u}{\partial x^2} = 2\frac{\partial u}{\partial t}$, $u(0, t) = 0$, $u(4, t) = 0$ and $u(x, 0) = x(4 - x)$, choosing $h = k = 1$ and using Bender-Schmidt formula find the values upto $t = 5$.

Solution

Given
$$\frac{\partial^2 u}{\partial x^2} = 2\frac{\partial u}{\partial t}$$

$\therefore$
$$a = 2 \Rightarrow \frac{1}{\alpha^2} = 2 \qquad \Rightarrow \alpha^2 = \frac{1}{2}$$

We have to choose
$$h = k = 1 \qquad \therefore \lambda = \frac{k\alpha^2}{h^2} = \frac{1}{2}$$

Bender-Schmidt recurrence formula for $\lambda = \frac{1}{2}$ is

$$u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right] \tag{1}$$

Now
$$x = ih = 1 \qquad \Rightarrow x_i = i \qquad [\because h = 1]$$
and
$$t = jk = j \qquad \Rightarrow t_j = j \qquad [\because k = 1]$$

x varies from 0 to 4 $\Rightarrow i = 0, 1, 2, 3, 4, j \geq 0$
The boundary value conditions become

$$u_{0,j} = 0 \text{ and } u_{4,j} = 0 \quad \forall\, j = 0, 1, 2, \ldots$$
$$u_{i,0} = i(4 - i), \qquad i = 0, 1, 2, 3, 4.$$
$$u_{0,0} = 0, \quad u_{1,0} = 3, \quad u_{2,0} = 4, \quad u_{3,0} = 3, \quad u_{4,0} = 0$$

We shall tabulate the meshpoint values of u

If $j = 0$, then $u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right]$

If $i = 1$, then $u_{1,1} = \frac{1}{2}(0 + 4) = 2$

If $i = 2$, then $u_{2,1} = \frac{1}{2}\left[u_{1,0} + u_{3,0}\right] = \frac{1}{2}[3 + 3] = 3$

If $i = 3$, then $u_{3,1} = \frac{1}{2}\left[u_{2,0} + u_{4,0}\right] = \frac{1}{2}[4 + 0] = 2$

Similarly for $j = 1, 2, 3, 4, 5$. The values can be found and they are tabulated. ∎

Now we shall tabulate the mesh point value of u

$j \backslash i$	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0	0.5	0.75	0.5	0

Example 2

Find the values of the function $u(x, t)$ satisfying the differential equation $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ and the boundary conditions

$u(0, t) = 0 = u(8, t)$ and $u(x, 0) = 4x - \frac{1}{2}x^2$ at the points $x = i$, $i = 0, 1, 2, 3, 4, \ldots 8$ and $t = \frac{1}{8}j$, $j = 0, 1, 2, 3, 4, 5$.

Solution

Given $$\frac{\partial u}{\partial t} = 4\frac{\partial u}{\partial x^2} \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{1}{4}\frac{\partial u}{\partial t}$$

$\therefore$ $$a = \frac{1}{4} \Rightarrow \frac{1}{\alpha^2} = \frac{1}{4} \Rightarrow \alpha^2 = 4.$$

Given $i = 0, 1, 2, \ldots, 8$ and $j = 0, 1, 2, 3, 4, 5$

$\therefore h = 1, k = \frac{1}{8}$, So, $\lambda = \frac{k\alpha^2}{h^2} = \frac{1}{8}.4 = \frac{1}{2}$

Bender-Schmidt recurrence formula for $\lambda = \frac{1}{2}$ is

$$u_{i,j+1} = \frac{1}{2}\left[u_{i+1,j} + u_{i-1,j}\right]$$

The boundary conditions are $u(0, t) = 0 \Rightarrow u_{0,j} = 0 \ \forall j$

$$u(8, t) = 0 \Rightarrow u_{8,j} = 0 \ \forall j$$

and $$u_{i,0} = 4i - \frac{1}{2}i^2, \quad i = 1, 2, 3, \ldots, 8.$$

For $i = 0, 1, 2, \ldots, 8$, we get $u_{0,0} = 0$, $u_{1,0} = 3.5$, $u_{2,0} = 6$, $u_{3,0} = 7.5$, $u_{4,0} = 8$, $u_{5,0} = 7.5$, $u_{6,0} = 6$, $u_{7,0} = 3.5$, $u_{8,0} = 0$

These are the entries in the first row. Similarly we can find the other rows mesh points.

Now we shall tabulate the mesh point values of u

j \ i	0	1	2	3	4	5	6	7	8
0	0	3.5	6	7.5	8	7.5	6	3.5	0
1	0	3	5.5	7	7.5	7	5.5	3	0
2	0	2.75	5	6.5	7	6.5	5	2.75	0
3	0	2.5	4.625	6	6.5	6	4.625	2.5	0
4	0	2.3125	4.25	5.5525	6	5.5625	4.25	2.125	0
5	0	2.125	3.9875	5.125	2.5625	5.125	3.9875	2.125	0

■

Example 3

Solve $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$, $u(0, t) = 0$, $u(4, t) = 0$ and $u(x, 0) = 4(4 - x)$ choosing $h = k = 1$ using Bender-schmidt formula.

Solution

Given $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \Rightarrow u_{xx} = u_t,$$

Here $a = 1 \Rightarrow \alpha^2 = 1$ $h = 1, \quad k = 1$ $\therefore \lambda = \dfrac{\alpha^2 k}{h^2} = 1$

The Bender-Schmidt formula for $\left(\lambda \neq \dfrac{1}{2}\right)$ is

$$u_{i,j+1} = \lambda\left(u_{i+1,j} + u_{i-1,j}\right) + (1 - 2\lambda)u_{i,j}$$

Since $\lambda = 1$, the equation becomes

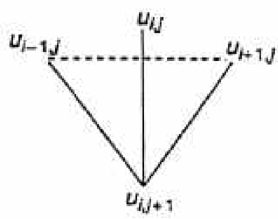
$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j} \tag{1}$$

Now $x = ih = i$, $y = jk = j$, $u(x, y) = u_{i,j}$ $i = 0, 1, 2, 3, 4.$

The boundary conditions $u(0,t) = 0 \Rightarrow u_{0,j} = 0 \ \forall \ j$

$u(4,t) = 0 \Rightarrow u_{4,j} = 0 \ \forall \ j$

and $u(x,0) = x(4-x) \Rightarrow u_{i,j} = i(4-i)$



For $i = 0, 1, 2, 3, 4$, we get $u_{0,0} = 0$, $u_{1,0} = 3$, $u_{2,0} = 4$

$u_{3,0} = 3$, $u_{4,0} = 0$

which are the first row mesh points. Similarly the other rows mesh points are computed.

Now we shall tabulate the mesh point values of u

j \ i	0	1	2	3	4
0	0	3	4	3	0
1	0	1	2	1	0
2	0	1	0	1	0
3	0	–1	2	–1	0
4	0	3	–4	3	0

Example 4

Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 \le x \le 1$, $t \ge 0$ with $u(x, 0) = x(1 - x)$, $0 < x < 1$, $u(0, t) = u(1, t) = 0$ $\forall$ $t > 0$. Using explicit method with $\Delta x = 0.2$ for 3 time steps.

Solution

Given $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \Rightarrow u_{xx} = u_t.$$

Here $$a = 1 \quad \Rightarrow \frac{1}{\alpha^2} = 1 \quad \Rightarrow \alpha^2 = 1 \quad \text{and} \quad \Delta x = 0.2 \quad \Rightarrow h = 0.2$$

Here k is not given. We shall choose k such that $\lambda = \frac{1}{2}$

But $$\lambda = \frac{k\alpha^2}{h^2} \quad \Rightarrow \frac{1}{2} = \frac{k.1}{0.2^2} \quad \Rightarrow k = 0.02$$

$$x = ih = 0.2i, \qquad y = kj = 0.02j$$

Then Bender-Schmidt formula is

$$u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]$$

The boundary condition are $u(0,t) = 0 \quad \Rightarrow u_{0,j} = 0 \ \forall j$

$$u(1, t) = 0 \ \forall, t > 0 \quad \Rightarrow \quad u_{5,j} = 0 \ \forall j$$

$$u(x, 0) = x(1-x) \quad \Rightarrow u_{i,0} = 0.2i(1-0.2i), \ i = 0, 1, 2, 3, 4, 5$$

$\therefore$
$$u_{0,0} = 0, \qquad u_{1,0} = 0.2(1-0.2) = 0.16,$$
$$u_{2,0} = 0.4(1-0.4) = 0.24, \qquad u_{3,0} = 0.6(1-0.6) = 0.24$$
$$u_{4,0} = 0.8(1-0.8) = 0.16, \qquad u_{5,0} = 1(1-1) = 0$$

These values are the first row mesh points.

For 3 time steps the mesh point values are given in the table

$j \backslash i$	0	1	2	3	4	5
0	0	0.16	0.24	0.24	0.16	0
1	0	0.12	0.20	0.2	0.12	0
2	0	0.10	0.16	0.16	0.10	0
3	0	0.08	0.13	0.13	0.08	0

■

Example 5

Solve the equation $\frac{\partial^2 u}{\partial x^2} = 16\frac{\partial u}{\partial t}$, $0 \le x \le 4$, $t > 0$ with the conditions $u(0, t) = 0$, $u(4, t) = 8$ and $u(x, 0) = \frac{1}{2}x(8-x)$, taking $\Delta x = \frac{1}{2}$ and $\Delta t = 1$ upto 5 time units.

Solution

Given $\qquad \frac{\partial^2 u}{\partial x^2} = 16\frac{\partial u}{\partial t} \quad \Rightarrow u_{xx} = 16\, u_t$

Now $\qquad \Delta x = \frac{1}{2} \quad \Rightarrow h = \frac{1}{2}, \ \Delta t = 1 \quad \Rightarrow k = 1,$

and $\qquad a = 16 \quad \Rightarrow \frac{1}{\alpha^2} = 16 \quad \Rightarrow \alpha^2 = \frac{1}{16}$

$\therefore \qquad \lambda = \frac{k\alpha^2}{h^2} \quad = \frac{1}{16} \cdot \frac{1}{\frac{1}{4}} = \frac{1}{4}$

Since $\lambda \ne \frac{1}{2}$, we have to use the general Schmidt's formula

$$u_{i,j+1} = \lambda u_{i-1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i+1,j}$$

$$\lambda = \frac{1}{4} \ \Rightarrow u_{i,j+1} = \frac{1}{4}\ u_{i-1,j} + \left(1 - \frac{1}{2}\right)u_{i,j} + \frac{1}{4}u_{i+1,j}$$

$$= \frac{1}{4}[u_{i-1,j} + 2u_{i,j} + u_{i+1,j}]$$

$$x = ih = \frac{1}{2}i, \qquad t = jk = j$$

$$u(x, y) = u(ih, j) = u_{i,j}$$

The boundary value conditions are $u(0, t) = 0 \ \Rightarrow u_{0j} = 0 \ \forall\ j = 0, 1, 2, 3, 4, 5$

When $x = 4$, $t = 8$, $\qquad u(4, t) = 8 \ \Rightarrow u(4, j) = 8 \ \Rightarrow u_{8,j} = 8 \ \forall\ j$

$$u(x, 0) = \frac{x}{2}(8 - x) \qquad \Rightarrow u\left(\frac{1}{2}i, 0\right) = \frac{1}{4}i\left(8 - \frac{i}{2}\right)$$

$$\Rightarrow \quad u_{i,0} = \frac{1}{8}i(16 - i)$$

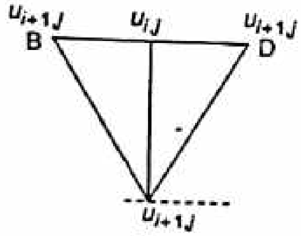
$$u_{1,0} = \frac{1}{8}(16-1) = \frac{15}{8} = 1.875, \quad u_{2,0} = \frac{1}{4}(16-2) = 3.5$$

$$u_{3,0} = \frac{3}{8}(16-3) = \frac{39}{8} = 4.875,$$

$$u_{4,0} = 6, \qquad u_{5,0} = 6.875$$

$$u_{6,0} = 7.5, \qquad u_{9,0} = 7.875$$

$$u_{1,1} = \frac{1}{4}[0 + 2 \times 1.875 + 3.5] = 1.8125 \text{ and so on}$$



$A = \frac{1}{4}[B + 2C + D]$

Now we shall tabulate the mesh point value of u

$j \backslash i$	0	1	2	3	4	5	6	7	8
0	0	1.875	3.5	4.875	6	6.875	7.5	7.875	8
1	0	1.8125	3.4375	4.8125	5.9375	6.8125	7.4375	7.8125	8
2	0	1.7656	3.3750	4.7500	5.8750	6.75	7.3750	7.7656	8
3	0	1.7266	3.3164	4.6875	5.8125	6.6875	7.3164	7.7266	8
4	0	1.6924	3.2617	4.6260	5.7500	6.6260	7.2617	7.6924	8
5	0	1.6616	3.2105	4.5660	5.6880	6.5660	7.2105	7.6616	8

■

Example 6

Using Crank-Nicolson's implicit scheme, solve 16 $u_t = u_{xx}$, $0 < x < 1$, $t > 0$, given that $u(x, 0) = 0$, $u(0, t) = 0$, $u(1, t) = 100t$, compute u for one-time step.

Solution

Given $\qquad u_{xx} = 16\, u_t$

Here $\qquad a = 16 \quad \Rightarrow \frac{1}{\alpha^2} = 16 \quad \Rightarrow \alpha^2 = \frac{1}{16}$

Here (0, 1) is divided, choose $h = \frac{1}{4}$.

Choose k such that $\lambda = \frac{k\alpha^2}{h^2} = 1 \quad \Rightarrow \frac{k}{16} \cdot \frac{1}{(1/4)^2} = 1 \quad \Rightarrow k = 1$

Since $\lambda = 1$, we use the simplified form of Crank-Nicolson formula

$$u_{i,j+1} = \frac{1}{4}[u_{i-1,j} + u_{i+1,j} + u_{i-1,j+1} + u_{i+1,j+1}]$$

$$x = ih = \frac{1}{4}i \text{ and } t = jk = j$$

$$u(x,y) = u\left(\frac{i}{4}, j\right) = u_{i,j} \text{ –Mesh points}$$

Boundary value conditions are

$$u(0,t) = 0 \qquad \Rightarrow u_{0,j} = 0 \quad \forall j$$

$$u(x,0) = 0 \qquad \Rightarrow u_{i,0} = 0 \quad \forall i$$

When $x = 1,\ t = 4,\ u(1,t) = 100t, \Rightarrow\ u_{i,j} = 100j, \quad j = 0,1,2,\ldots$

We have to compute the values of u for only one time step.
Put $j = 0$, $i = 0, 1, 2, 3, 4$.

$$u_1 = \frac{1}{4}[0+0+0+u_2] = \frac{1}{4}u_2 \qquad (1)$$

$$u_2 = \frac{1}{4}[0+0+u_1+u_3]$$

$\Rightarrow$
$$u_2 = \frac{1}{4}[u_1+u_3] \qquad (2)$$

$$u_3 = \frac{1}{4}[0+0+u_2+100]$$

$\Rightarrow$
$$u_3 = \frac{1}{4}[u_2+100] \qquad (3)$$

$j \backslash i$	0	1	2	3	4
0	0	0	0	0	0
-1	0	u_1	u_2	u_3	100

Substituting for u_1 and u_3 in (2), we get

$$u_2 = \frac{1}{4}\left[\frac{1}{4}u_2 + \frac{1}{4}(u_2+100)\right]$$

$$= \frac{1}{16}[2u_2+100] = \frac{1}{8}(u_2+50)$$

$\Rightarrow$
$$7u_2 = 50 \qquad \Rightarrow u_2 = \frac{50}{7} = 7.1429$$

$$u_1 = \frac{1}{4}(7.1429) = 1.7857 \text{ and } u_3 = \frac{1}{4}[7.1429+100] = 26.7857$$

$\therefore$ $\quad \mathbf{u_1 = 1.7857, \quad u_2 = 7.1429, \quad u_3 = 26.7857}$ ■

$\therefore$

Example 7

Solve $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, $0 < x < 5, t > 0$ given that $u(x, 0) = 20$, $u(0, t) = 0$, $u(5, t) = 100$. Compute u for one time step with h=1 by Crank-Nicolson method.

Solution

Given $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t} \Rightarrow u_{xx} = u_t$

Here $a = 1 \Rightarrow \dfrac{1}{\alpha^2} = 1 \Rightarrow \alpha^2 = 1 \text{ and } h = 1$

k is not given, so choose k such that $\lambda = 1$ $\qquad \Rightarrow \dfrac{k\alpha^2}{h^2} = 1 \Rightarrow k = 1$

Since $\lambda = 1$, the simplified Crank-Nicolson formula is used to compute only one time step.

$$u_{i,j+1} = \frac{1}{4}[u_{i-1,j+1} + u_{i-1,j} + u_{i+1,j+1} + u_{i+1,j}] \tag{1}$$

where $x = i\,h = i$, $t = k\,j = j$

$\therefore$

$$u(x,t) = u(i,j) = u_{i,j}$$
$$u(x,0) = u_{i,0} = 20, \quad i = 1, 2, 3, 4$$
$$u(0,t) = u_{0,j} = 0$$

On the line $x = 0$, $u = 0$ for all t. That is for all j

j \ i	0	1	2	3	4	5
0	0	20	20	20	20	100
1	0	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	$u_{4,1}$	100

$\therefore$

That is $\quad u_{0,0} = 0, \quad u_{0,1} = 0 \quad$ and $\quad u(5, t) = 100.$

That is on the line $\quad x = 5, \quad u = 100$

$i = 5, \quad u = 100 \quad$ for all t.

Using formula (1) $\quad u_{1,0} = 20, \quad u_{2,0} = u_{3,0} = 20, \quad u_{4,0} = 20$

$$u_{1,1} = \frac{1}{4}[0 + u_{2,1} + 0 + 20] = \frac{1}{4}[u_{2,1} + 20] \tag{2}$$

$$u_{2,1} = \frac{1}{4}[u_{1,1} + u_{3,1} + 20 + 20] = \frac{1}{4}[u_{1,1} + u_{3,1} + 40] \qquad (3)$$

$$u_{3,1} = \frac{1}{4}[u_{2,1} + u_{4,1} + 20 + 20] = \frac{1}{4}[u_{2,1} + u_{4,1} + 40] \qquad (4)$$

$$u_{4,1} = \frac{1}{4}[u_{3,1} + 100 + 20 + 100] = \frac{1}{4}[u_{3,1} + 220] \qquad (5)$$

Substituting (5) in (4),

$$u_{3,1} = \frac{1}{4}[u_{2,1} + \frac{1}{4}(u_{3,1} + 220) + 40]$$

$$\Rightarrow \qquad 16u_{3,1} = 4u_{2,1} + u_{3,1} + 220 + 160$$

$$\Rightarrow \qquad 15u_{3,1} - 4u_{2,1} = 380 \qquad (6)$$

Substituting (2) in (3),

$$u_{2,1} = \frac{1}{4}\left[\frac{1}{4}(u_{2,1} + 20) + u_{3,1} + 40\right]$$

$$= \frac{1}{16}[u_{2,1} + 20 + 4u_{3,1} + 160]$$

$$\Rightarrow \qquad 15u_{2,1} = 4u_{3,1} + 180$$

$$\Rightarrow \qquad -4u_{3,1} + 15u_{2,1} = 180 \qquad (7)$$

$$(6)\times 4 \Rightarrow \qquad 60u_{3,1} - 16u_{2,1} = 1520$$

$$(7)\times 15 \Rightarrow \qquad -60u_{3,1} + 225u_{2,1} = 2700$$

Adding, $\qquad 209u_{2,1} = 4220 \Rightarrow u_{2,1} = \frac{4220}{209} = 20.19$

$$\therefore \qquad u_{1,1} = \frac{1}{4}[20.19 + 20] = 10.05$$

$$(6) \Rightarrow \qquad 15u_{3,1} = 4 \times 20.19 + 380 = 460.76$$

$$\Rightarrow \qquad u_{3,1} = 30.72$$

$$u_{4,1} = \frac{1}{4}[30.72 + 220] = 62.68$$

$$\therefore \qquad u_{1,1} = 10.05, \quad u_{2,1} = 20.19, \quad u_{3,1} = 30.72, \quad u_{4,1} = 62.68$$ ■

Example 8

Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ subject to the conditions $u(x, 0) = \sin \pi x$, $0 \le x \le 1$ $u(0, t) = u(1, t) = 0$, using Crank-Nicolson method taking $h = \frac{1}{3}$, $k = \frac{1}{36}$.

Solution

Given $u_{xx} = u_t$

Here $a = 1 \Rightarrow \frac{1}{a^2} = 1 \Rightarrow a^2 = 1, \quad h = \frac{1}{3}, \quad k = \frac{1}{36}$

$\therefore$ $\lambda = \frac{ka^2}{h^2} = \frac{1}{36} \cdot \frac{1}{\frac{1}{9}} = \frac{1}{4}$

Crank-Nicolson formula is

$$\lambda u_{i+1,j+1} - 2(1+\lambda)u_{i,j+1} + \lambda u_{i-1,j+1} = -\lambda u_{i+1,j} + 2(\lambda - 1)u_{i,j} - \lambda u_{i-1,j}$$

Since $\lambda = \frac{1}{4}$, we get

$$\frac{1}{4}u_{i+1,j+1} - 2.\frac{5}{4}u_{i,j+1} + \frac{1}{4}u_{i-1,j+1} = -\frac{1}{4}u_{i+1,j} + 2\left(-\frac{3}{4}\right)u_{i,j} - \frac{1}{4}u_{i-1,j}$$

Multiply by 4, (1)

$$u_{i+1,j+1} - 10u_{i,j+1} + u_{i-1,j+1} = -u_{i+1,j} - 6u_{i,j} - u_{i-1,j}$$

$$x = ih = \frac{1}{3}i, \quad t = kj = \frac{1}{36}j \quad \text{and} \quad 0 \le x \le 1 \quad \therefore i = 0,1,2,3.$$

The boundary conditions are $u(0,t) = 0 \quad \Rightarrow u_{0,j} = 0 \quad \forall j$

$$u(1,t) = 0 \quad \Rightarrow u\left(3, \frac{1}{36}j\right) = 0 \quad \Rightarrow u_{3,j} = 0 \quad \forall j$$

$$u(x,0) = \sin \pi x \quad \Rightarrow u\left(\frac{i}{3}, 0\right) = \sin\frac{\pi}{3}i \Rightarrow u_{i,0} = \sin\frac{\pi}{3}i, \quad i = 0,1,2,3$$

j \ i	0	1	2	3
0	0	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0
1	0	$u_{1,1}$	$u_{2,1}$	0

$$u_{0,0} = 0, \quad u_{0,1} = 0, \quad u_{3,0} = 0, \quad u_{3,1} = 0$$

$$u_{1,0} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \quad \text{and} \quad u_{2,0} = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

We shall find one time step values.

Putting $j = 0$, $i = 1$ in (1), we get

$$u_{2,1} - 10u_{1,1} + u_{0,1} = -u_{2,0} - 6u_{1,0} - u_{0,0}$$

$$\Rightarrow \quad u_{2,1} - 10u_{1,1} + 0 = -\frac{\sqrt{3}}{2} - 6.\frac{\sqrt{3}}{2} - 0$$

$$\Rightarrow \quad u_{2,1} - 10u_{1,1} = \frac{-7\sqrt{3}}{2} \qquad (1)$$

Putting $j = 0$, $i = 2$, in (1) we get,

$$u_{3,1} - 10u_{2,1} + u_{1,1} = -u_{3,0} - 6u_{2,0} - u_{1,0}$$

$$\Rightarrow \quad 0 - 10u_{2,1} + u_{1,1} = -0 - 6.\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad -10u_{2,1} + u_{1,1} = \frac{-7\sqrt{3}}{2}$$

(1)×10 $\quad \Rightarrow \quad 10u_{2,1} - 100u_{1,1} = -10\left(\frac{7\sqrt{3}}{2}\right)$

(2) is $\quad -10u_{2,1} + u_{1,1} = \frac{-7\sqrt{3}}{2} \qquad (2)$

Adding, $\quad -99u_{1,1} = -11\left(\frac{7\sqrt{3}}{2}\right)$

$$\Rightarrow \quad u_{1,1} = \frac{77\sqrt{3}}{198} = 0.67$$

Substituting in (1), we get $\quad u_{2,1} = \frac{-7\sqrt{3}}{2} + 10 \times 0.67 = 0.64$

Thus we have computed the value of u for one time step.

$$\mathbf{u_{1,1} = 0.67, \quad u_{2,1} = 0.64}$$ ■

Example 9

Using Crank-Nicolson scheme Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ given $u(x, 0) = u(0, t) = 0$, $u(1, t) = t$, choosing $h = 0.5$, $k = \frac{1}{8}$.

Solution

Given $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \Rightarrow u_{xx} = u_t$

Here

$$a = 1 \Rightarrow \frac{1}{\alpha^2} = 1 \Rightarrow \alpha^2 = 1; \ h = 0.5, \ k = \frac{1}{8}$$

$$\therefore \quad \lambda = \frac{k\alpha^2}{h^2} = \frac{1}{8} \cdot \frac{1}{0.5^2} = \frac{1}{2} (\neq 1)$$

So we have to use Crank-Nicolson general formula

$$\lambda u_{i+1,j+1} - 2(1+\lambda)u_{i,j+1} + \lambda u_{i-1,j+1} = -\lambda u_{i+1,j} + 2(\lambda - 1)u_{i,j} - \lambda u_{i-1,j}$$

Putting $\lambda = \frac{1}{2}$, $\frac{1}{2}u_{i+1,j+1} - 2.\frac{3}{2}u_{i,j+1} + \frac{1}{2}u_{i-1,j+1} = -\frac{1}{2}u_{i+1,j} - 2.\frac{1}{2}u_{i,j} - \frac{1}{2}u_{i-1,j}$

Multiply by 2,

$$u_{i+1,j+1} - 6u_{i,j+1} + u_{i-1,j+1} = -u_{i+1,j} - 2u_{i,j} - u_{i-1,j} \tag{1}$$

$$x = ih = \frac{i}{2}, \quad t = kj = \frac{j}{8}, \quad 0 \le x \le 1 \quad \Rightarrow i = 0,1,2$$

The boundary value conditions are

$$u(x,0) = 0 \quad \Rightarrow u\left(\frac{i}{2}, 0\right) = 0 \quad \Rightarrow u_{i,0} = 0, \quad i = 0,1,2$$

$$u(0,t) = 0 \quad \Rightarrow u\left(0, \frac{j}{8}\right) = 0 \quad \Rightarrow u_{0,j} = 0 \quad \forall\, j \ge 0$$

$$\therefore \quad u(1,t) = t \quad \Rightarrow u\left(2, \frac{j}{8}\right) = \frac{j}{8} \quad \Rightarrow u_{2,j} = \frac{j}{8}, \quad j \ge 0$$

We shall find one-step values.

$u_{0,0} = 0$, $u_{1,0} = 0$, $u_{2,0} = 0$, $u_{0,1} = 0$

$j \backslash i$	0	1	2
0	0	0	0
1	0	$u_{1,1}$	$\frac{1}{8}$

Putting $j = 0$, $i = 1$ in (1), we get

$$u_{2,1} - 6u_{1,1} + u_{0,1} = -u_{2,0} - 2u_{1,0} - u_{0,0}$$

$$\Rightarrow \quad \frac{1}{8} - 6u_{1,1} + 0 = 0 \quad \Rightarrow 6u_{1,1} = \frac{1}{8} \quad \Rightarrow u_{1,1} = 0.0208$$

$$\therefore \quad \boldsymbol{u_{1,1} = 0.0208}$$

■

Example 10

Solve by Crank-Nicolson's method $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ for $0 < x < 1$, $t > 0$ given that $u(0, t) = 0$, $u(1, t) = 0$ and $u(x, 0) = 100x(1 - x)$. Compute u for one time step with $h = \frac{1}{4}$.

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Solution

Given $u_{xx} = u_t$

Here $a = 1 \Rightarrow \frac{1}{\alpha^2} = 1 \Rightarrow \alpha^2 = 1, \quad h = \frac{1}{4}$ and k is not given

So, we choose k such that $\lambda = 1 \Rightarrow \frac{k\alpha^2}{h^2} = 1 \Rightarrow k = \frac{h^2}{\alpha^2} = \frac{1}{16}$

Since $\lambda = 1$, we use Crank-Nicolson simplified formula (1)

$$u_{i,j+1} = \frac{1}{4}[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}]$$

$$x = ih = \frac{1}{4}i, \quad t = jk = \frac{1}{16}j, \quad 0 \le x \le 1 \text{ and so } i = 0,1,2,3,4$$

The boundary value conditions are

$$u(0,t) = 0 \Rightarrow u\left(0, \frac{j}{16}\right) = 0 \Rightarrow u_{0,j} = 0 \quad \forall\, j \ge 0$$

$$u(1,t) = 0 \Rightarrow u\left(1, \frac{j}{16}\right) = 0 \Rightarrow u_{4,j} = 0 \quad \forall\, j \ge 0 \qquad [\because x = 1 \Rightarrow i = 4]$$

$$u(x,0) = 100x(1-x) \Rightarrow u\left(\frac{i}{4}, 0\right) = 100\frac{i}{4}\left(1 - \frac{i}{4}\right)$$

$$\Rightarrow \quad u_{i,0} = \frac{25}{4}i(4-i), \quad i = 0,1,2,3,4$$

We have to compute on time step values of u

$$u_{0,0} = 0, \quad u_{0,1} = 0, \quad u_{4,0} = 0, \quad u_{4,1} = 0$$

$$u_{1,0} = 18.75, \quad u_{2,0} = 25, \quad u_{3,0} = \frac{35}{4} \times 3 = 18.75$$

$j \backslash i$	0	1	2	3	4
0	0	18.75	25	18.75	0
1	0	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	0

Putting $i = 1$, $j = 0$ in (1), we get

$$u_{1,1} = \frac{1}{4}[u_{0,1} + u_{2,1} + u_{0,0} + u_{2,0}]$$

$$\Rightarrow \quad u_{1,1} = \frac{1}{4}[0 + u_{2,1} + 0 + 25] = \frac{1}{4}[u_{2,1} + 25] \qquad (2)$$

Putting $i = 2, j = 0$ in (1), we get

$$u_{2,1} = \frac{1}{4}[u_{1,1} + u_{3,1} + u_{1,0} + u_{3,0}]$$

$$= \frac{1}{4}[u_{1,1} + u_{3,1} + 18.75 + 18.75]$$

$\Rightarrow$ $$u_{2,1} = \frac{1}{4}[u_{1,1} + u_{3,1} + 37.5] \tag{3}$$

Putting $i = 3, j = 0$ in (1), we get

$$u_{3,1} = \frac{1}{4}[u_{2,1} + u_{4,1} + u_{2,0} + u_{4,0}] = \frac{1}{4}[u_{2,1} + 25] \tag{4}$$

From (2) and (4), we get $u_{1,1} = u_{3,1}$

$\therefore$ (3) $\Rightarrow$ $$u_{2,1} = \frac{1}{4}[u_{1,1} + u_{1,1} + 37.5]$$

$\Rightarrow$ $$4u_{2,1} = 2u_{1,1} + 37.5$$

$\Rightarrow$ $$4u_{2,1} = 2.\frac{1}{4}[u_{2,1} + 25] + 37.5 \quad \text{[Using (2)]}$$

$$= \frac{1}{2}[u_{2,1} + 25] + 37.5$$

$\Rightarrow$ $$8u_{2,1} = u_{2,1} + 25 + 75$$

$\Rightarrow$ $$7u_{2,1} = 100 \quad \Rightarrow \quad u_{2,1} = \frac{100}{7} = 14.29$$

$\therefore$ $$u_{1,1} = \frac{1}{4}[14.29 + 25] = 9.82$$

and $$u_{3,1} = 9.82$$

Thus $$\boldsymbol{u_{1,1} = 9.82 = u_{3,1}, \qquad u_{2,1} = 14.29}$$ ■

Example 11

Obtain the Crank-Nicholson finite difference method by taking $\lambda = \frac{kc^2}{h^2} = 1$. Hence, find $u(x, t)$ in the rod for two time steps for the heat equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ given $u(x, 0) = \sin \pi x$, $u(0, t) = 0$, $u(1, t) = 0$. Take $h = 0.2$.

Solution

Given $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \Rightarrow u_{xx} = u_t$

For derivation of Crank-Nicholson formula for one dimensional heat equation $\frac{\partial^2 u}{\partial x^2} = a\frac{\partial u}{\partial t}$ where $a = \frac{1}{\alpha^2}$ refer page [In this question c is α]

Here $\quad a = 1 \Rightarrow \frac{1}{\alpha^2} = 1 \Rightarrow \alpha^2 = 1$ and $h = 0.2$

Now $\quad \lambda = 1 \Rightarrow \frac{k\alpha^2}{h^2} = 1 \Rightarrow \frac{k}{(0.2)^2} = 1 \Rightarrow k = 0.04$

Since $\lambda = 1$, we use the simplified Crank-Nicolson formula

$$u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i-1,j} + u_{i+1,j+1} + u_{i+1,j}\right] \tag{1}$$

where $x = i\,h = 0.2\,i$, $\quad t = k\,j = 0.04\,j$ and $0 \le x \le 1$

When $x = 0$, $i = 0$ and when $x = 1$, $i = \frac{1}{0.2} = 5$ $\qquad \therefore i = 0, 1, 2, 3, 4, 5$

The boundary value conditions are $u(0,t) = 0 \quad \Rightarrow u_{0,j} = 0 \forall j$

$$u(1,t) = 0 \quad \Rightarrow u_{5,j} = 0 \forall j$$

and $\quad u(x, 0) = \sin \pi x \quad \Rightarrow u(0.2i, 0) = \sin\pi(0.2i) = \sin\frac{\pi}{5}i$

$$\Rightarrow \quad u_{i,0} = \sin\frac{\pi i}{5}, \quad i = 0, 1, 2, 3, 4, 5$$

On the line $x = 0$, $\; u = 0 \forall t$ ie. when i = 0, $\; u = 0 \;\forall j$
and on the line $x = 1$, $\; u = 0 \forall j$ ie. when i = 5, $\; u = 0 \;\forall j$
For two time steps we have to compute for $j = 0, 1, 2$

$$\therefore \quad u_{0,0} = 0, \quad u_{0,1} = 0, \quad u_{0,2} = 0, \quad u_{5,0} = 0, \quad u_{5,1} = 0, \quad u_{5,2} = 0$$

$$\therefore \quad u_{1,0} = \sin\frac{\pi}{5} = 0.5878 \quad , \quad u_{2,0} = \sin\frac{2\pi}{5} = 0.9510$$

$$u_{3,0} = \sin\frac{3\pi}{5} = 0.9511 \quad , \quad u_{4,0} = \sin\frac{4\pi}{5} = 0.5878$$

$$u_{5,0} = \sin\pi = 0.$$

We shall tabulate the values

j \ i	0	1	2	3	4	5
0	0	0.5878	0.9510	0.9511	0.5878	0
1	0	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	$u_{4,1}$	0
2	0	$u_{1,2}$	$u_{2,2}$	$u_{3,2}$	$u_{4,2}$	0

We shall now compute the unknown u's.

Putting $j = 0$ in (1) we get the second row values of u's

$$u_{i,1} = \frac{1}{4}[u_{i-1,1} + u_{i-1,0} + u_{i+1,1} + u_{i+1,0}],\ i = 1,2,3,4,5$$

Put $i = 1$, then

$$u_{1,1} = \frac{1}{4}[u_{0,1} + u_{0,0} + u_{2,1} + u_{2,0}]$$

$$= \frac{1}{4}[0 + 0 + u_{2,1} + 0.9510]$$

$$= \frac{1}{4}[u_{2,1} + 0.9510] \qquad (2)$$

Put $i = 2$, then

$$u_{2,1} = \frac{1}{4}[u_{1,1} + u_{1,0} + u_{3,1} + u_{3,0}]$$

$$= \frac{1}{4}[u_{1,1} + 0.5878 + u_{3,1} + 0.9511]$$

$$u_{2,1} = \frac{1}{4}[u_{1,1} + u_{3,1} + 1.5389] \qquad (3)$$

Put $i = 3$, then

$$u_{3,1} = \frac{1}{4}[u_{2,1} + u_{2,0} + u_{4,1} + u_{4,0}]$$

$$= \frac{1}{4}[u_{2,1} + 0.9510 + u_{4,1} + 0.5878]$$

$$u_{3,1} = \frac{1}{4}[u_{2,1} + u_{4,1} + 1.5388] \qquad (4)$$

Put $i = 4$, then

$$u_{4,1} = \frac{1}{4}\left[u_{3,1} + u_{3,0} + u_{5,1} + u_{5,0}\right]$$

$$= \frac{1}{4}[u_{3,1} + 0.9511 + 0 + 0]$$

$$u_{4,1} = \frac{1}{4}[u_{3,1} + 0.9511] \qquad (5)$$

Substituting (5) in (4) we get

$$u_{3,1} = \frac{1}{4}\left[u_{2,1} + \frac{1}{4}u_{3,1} + \frac{1}{4}(0.9511) + 1.5388\right]$$

$$= \frac{1}{16}[4u_{2,1} + u_{3,1} + 7.1063]$$

$$\Rightarrow \quad 16u_{3,1} = 4u_{2,1} + u_{3,1} + 7.1063$$

$$\Rightarrow \quad 15u_{3,1} - 4u_{2,1} = 7.1063 \qquad (6)$$

Now substituting (2) in (3), we get

$$u_{2,1} = \frac{1}{4}\left[\frac{1}{4}u_{2,1} + \frac{1}{4}(0.9510) + u_{3,1} + 1.5389\right]$$

$$= \frac{1}{16}[u_{2,1} + 4u_{3,1} + 7.1066]$$

$\Rightarrow$ $$16u_{2,1} = u_{2,1} + 4u_{3,1} + 7.1066$$

$\Rightarrow$ $$4u_{3,1} - 15u_{2,1} = -7.1066 \qquad (7)$$

$(6) \times 15 \Rightarrow$ $$225u_{3,1} - 60u_{2,1} = 106.5945$$

$(7) \times 4 \Rightarrow$ $$16u_{3,1} - 60u_{2,1} = -28.4264$$

Subtracting we get, $209u_{3,1} = 135.0209 \quad \Rightarrow \quad u_{3,1} = 0.6460$

Substituting in (6), we get

$$15(0.6460) - 4u_{2,1} = 7.1063$$

$\Rightarrow$ $$4u_{2,1} = 15(0.6460) - 7.1063 = 2.5837$$

$\Rightarrow$ $$u_{2,1} = \frac{2.5837}{4} = 0.6459$$

$\therefore$ $$u_{1,1} = \frac{1}{4}[0.6459 + 0.9510] = 0.3992$$

$$u_{4,1} = \frac{1}{4}[0.6460 + 0.9511] = 0.3993$$

To find the third row values of u, put $j = 1$ in (1)

$\therefore$ $$u_{i,2} = \frac{1}{4}[u_{i-1,2} + u_{i-1,1} + u_{i+1,2} + u_{i+1,1}]$$

Asad Ali

Put $i = 1$, then $$u_{1,2} = \frac{1}{4}[u_{0,2} + u_{0,1} + u_{2,2} + u_{2,1}]$$

$$= \frac{1}{4}[0 + 0 + u_{2,2} + 0.6459]$$

$\Rightarrow$ $$u_{1,2} = \frac{1}{4}[u_{2,2} + 0.6459] \qquad (8)$$

Put $i = 2$, then $$u_{2,2} = \frac{1}{4}[u_{1,2} + u_{1,1} + u_{3,2} + u_{3,1}]$$

$$= \frac{1}{4}[u_{1,2} + 0.3992 + u_{3,2} + 0.6460]$$

$$u_{2,2} = \frac{1}{4}[u_{1,2} + u_{3,2} + 1.0452] \qquad (9)$$

Put $i = 3$, then $$u_{3,2} = \frac{1}{4}[u_{2,2} + u_{2,1} + u_{4,2} + u_{4,1}]$$

$$= \frac{1}{4}[u_{2,2} + 0.6459 + u_{4,2} + 0.3993]$$

$$u_{3,2} = \frac{1}{4}[u_{2,2} + u_{4,2} + 1.0452] \qquad (10)$$

Put $i = 4$, then

$$u_{4,2} = \frac{1}{4}[u_{3,2} + u_{3,1} + u_{5,2} + u_{5,1}]$$

$$= \frac{1}{4}[u_{3,2} + 0.6460 + 0 + 0]$$

$$u_{4,2} = \frac{1}{4}[u_{3,2} + 0.6460] \qquad (11)$$

Substituting (11) in (10) we get,

$$u_{3,2} = \frac{1}{4}\left[u_{2,2} + \frac{1}{4}u_{3,2} + \frac{1}{4}(0.6460) + 1.0452\right]$$

$$= \frac{1}{16}[4u_{2,2} + u_{3,2} + 4.8268]$$

$$\Rightarrow \qquad 15u_{3,2} - 4u_{2,2} = 4.8268 \qquad (12)$$

Substituting (8) in (9), we get,

$$u_{2,2} = \frac{1}{4}\left[\frac{1}{4}u_{2,2} + \frac{1}{4}(0.6459) + u_{3,2} + 1.0452\right]$$

$$= \frac{1}{16}[u_{2,2} + 4u_{3,2} + 4.8267]$$

$$\Rightarrow \qquad 15u_{2,2} = 4u_{3,2} + 4.8267$$

$$\Rightarrow \qquad 4u_{3,2} - 15u_{2,2} = -4.8267 \qquad (13)$$

$(12) \times 15 \Rightarrow \qquad 225u_{3,2} - 60u_{2,2} = 72.402$

$(13) \times 4 \quad \Rightarrow \qquad 16u_{3,2} - 60u_{2,2} = -19.3068$

Subtracting we get, $\qquad 209\, u_{3,2} = 91.7088 \qquad \Rightarrow \quad u_{3,2} = 0.4388$

$\therefore \qquad 15u_{2,2} = 4(0.4388) + 4.8267 = 6.5819 \qquad \Rightarrow \quad u_{2,2} = 0.4388$

$$\therefore \qquad u_{4,2} = \frac{1}{4}[0.4388 + 0.6460] = 0.2712$$

$$\text{and} \qquad u_{1,2} = \frac{1}{4}[0.4388 + 0.6459] = 0.2712$$

Thus the two time step values are given in the table

j \ i	0	1	2	3	4	5
0	0	0.5878	0.9510	0.9511	0.5878	0
I	0	0.3992	0.6459	0.6460	0.3993	0
2	0	0.2712	0.4388	0.4388	0.2712	0

■

ONE-DIMENSIONAL WAVE EQUATION

One dimensional wave equation (of vibrating string) is

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow a^2 u_{xx} - u_{tt} = 0 \tag{1}$$

This is hyperbolic type.

We shall solve this for $u(x, t)$ using finite difference method subject to the boundary conditions.

$u(0, t) = 0$, $u(l, t) = 0$ (2)

and the initial conditions $u(x, 0) = f(x)$, $u_t(x, 0) = 0$ (3)

Assume equal interval h for x variable and equal interval k for t variable and $x = ih$, $t = jk$ and $u(x, t) = u(ih, jk) = u_{i,j}$

We know the partial derivatives in terms of differences

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$u_{tt} = \left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}\right)$$

$\therefore$ the equation (1) becomes

$$a^2\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right) - \left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}\right) = 0$$

$$\Rightarrow \quad \frac{k^2 a^2}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) = u_{i,j+1} - 2u_{i,j} + u_{i,j-1}$$

Let $\lambda = \frac{ak}{h}$, then the equation becomes

$$\lambda^2 (u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) = u_{i,j+1} - 2u_{i,j} + u_{i,j-1}$$

$$\therefore \quad u_{i,j+1} = \lambda^2 (u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + 2u_{i,j} - u_{i,j-1}$$

$$\Rightarrow \quad u_{i,j+1} = 2(1-\lambda^2)u_{i,j} + \lambda^2 (u_{i+1,j} + u_{i-1,j}) - u_{i,j-1} \tag{4}$$

This equation is an **explicit formula for the solution of the wave equation** because it gives the value of u at $t = t_{j+1}$, explicitly in terms of the values of u at $t = t_j$ and $t = t_{j-1}$

The simplest form of the equation (4) is obtained by putting

$$1 - \lambda^2 = 0 \quad \Rightarrow \lambda^2 = 1 \quad \Rightarrow \lambda = 1 \quad \Rightarrow \frac{k}{h} = \frac{1}{a} \quad \Rightarrow k = \frac{h}{a}$$

$\therefore$ the equation (4) becomes,

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1} \tag{5}$$

The boundary conditions are

$$u(0,t) = 0 \quad \Rightarrow u(0, jk) = 0 \quad \Rightarrow u_{0,j} = 0$$

and

$$u(l,t) = 0 \Rightarrow u_{n,j} = 0 \quad \text{if } l = nh$$

The initial conditions are $u(x, 0) = f(x) \Rightarrow u(ih, 0) = f(ih)$

$\Rightarrow$ $u_{i,0} = f(ih),\ i = 0, 1, 2, 3, \ldots$

We shall use central difference approximation for

$$u_t = \frac{u_{i,j+1} - u_{i,j-1}}{2k}$$

When $t = 0$, $u_t(x, 0) = 0 \Rightarrow \dfrac{u_{i,j+1} - u_{i,j-1}}{2k} = 0 \quad \Rightarrow \dfrac{u_{i,j+1} - u_{i,j-1}}{2k} = 0$

When $j = 0$,

$$u_{i,1} - u_{i,-1} = 0$$

$$\Rightarrow \quad u_{i,1} = u_{i,-1} \text{ for all } i$$

Putting $j = 0$ in (5), we get

$$u_{i,1} = u_{i+1,0} + u_{i-1,0} - u_{i,-1}$$

$$= u_{i+1,0} + u_{i-1,0} - u_{i,1}$$

$$\Rightarrow \quad 2u_{i,1} = u_{i+1,0} + u_{i-1,0}$$

$$\Rightarrow \quad u_{i,1} = \frac{1}{2}[u_{i+1,0} + u_{i-1,0}]$$

For $i = 1, 2, 3, \ldots$, we get the values of u in the second row.

Note: The period of vibration of the string of length l given by $a^2u_{xx} - u_{tt} = 0$ is $\dfrac{2l}{a}$, we have to find the values of u at the mesh points for one period, unless otherwise specified.

WORKED EXAMPLES

Example 1

Solve $16u_{xx} = u_{tt}$, $u(0, t) = u(5, t) = 0$, $u(x, 0) = x^2(5 - x)$, $u_t(x, 0) = 0$ taking $h = 1$ and upto one half of the period of vibration.

Solution

Given $$16u_{xx} - u_{tt} = 0$$

Here $$a^2 = 16 \quad \Rightarrow a = 4.$$

Now $$\lambda = \frac{ak}{h} = \frac{4k}{h}$$

Choose $\lambda = 1$ and $h = 1$, then $k = \dfrac{1}{4}$

Since $\lambda = 1$, the simplest difference equation is

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1} \tag{1}$$

Now $x = ih = i$, $\quad t = jk = \dfrac{j}{4}$, $\quad u(x, y) = u\left(i, \dfrac{j}{4}\right) = u_{i,j}$

From the boundary conditions $u(0, t) = u(5, t) = 0$, it is obvious $0 \le x \le 5$ and length of the string $l = 5$

$\therefore$ period of vibration $= \frac{2l}{a} = 2\left(\frac{5}{4}\right) = 2.5$ secs.

Half the period = 1.25 secs.

So, we have to find values of t upto $t = 1.25 \quad \Rightarrow \frac{j}{4} = 1.25 \quad \Rightarrow j = 5$

$$u(0,t) = 0 \quad \Rightarrow u\left(0, \frac{j}{4}\right) = 0 \quad \Rightarrow u_{0,j} = 0 \quad \forall j = 0,1,2,3,4,5 \tag{2}$$

$$u(5,t) = 0 \quad \Rightarrow u\left(5, \frac{j}{4}\right) = 0 \quad \Rightarrow u_{5,j} = 0 \quad \forall j = 0,1,2,3,4,5 \tag{3}$$

$$u(x,0) = x^2(5-x) \Rightarrow u(i,0) = i^2(5-i)$$

$$\Rightarrow \qquad u_{i,0} = i^2(5-i), \qquad i = 0, 1, 2, 3, 4, 5. \tag{4}$$

When $t = 0$, $\quad u_t(x,0) = 0 \quad \Rightarrow \quad \frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0,$

Since $\quad t = \frac{j}{4}, \; t = 0 \quad \Rightarrow j = 0$

When $j = 0$, $\quad u_{i,1} - u_{i,-1} = 0 \quad \Rightarrow u_{i,1} = u_{i,-1} \quad \forall i = 1,2,3,4,5 \qquad (5)$

(2) $\Rightarrow \quad u_{0,0} = 0, \quad u_{0,1} = 0, \quad u_{0,2} = 0, \quad \ldots, \quad u_{0,5} = 0$

(3) $\Rightarrow \quad u_{5,0} = 0, \quad u_{5,1} = 0, \quad \ldots, \quad u_{5,5} = 0$

(4) $\Rightarrow \quad u_{1,0} = 4, \quad u_{2,0} = 12, \quad u_{3,0} = 18, \quad u_{4,0} = 16$

Putting $j = 0$ in (1), we get $u_{i,1} = u_{i+1,0} + u_{i-1,0} - u_{i,-1}$

$$= u_{i+1,0} + u_{i-1,0} - u_{i,1} \qquad \text{[Using (5)]}$$

$$\Rightarrow \qquad 2u_{i,1} = u_{i+1,0} + u_{i-1,0}$$

$$\Rightarrow \qquad u_{i,1} = \frac{1}{2}[u_{i+1,0} + u_{i-1,0}] \tag{6}$$

$j \backslash i$	0	1	2	3	4	5
0	0	4	12	18	16	0
1	0	6	11	14	9	0
2	0	7	8	2	–2	0
3	0	2	–2	–8	–7	0
4	0	–9	–14	–11	–6	0
5	0	–16	–18	–12	–4	0

Put $i = 1$,

$$u_{1,1} = \frac{1}{2}(u_{2,0} + u_{0,0}) = \frac{1}{2}(12 + 0) = 6$$

$$u_{2,1} = \frac{1}{2}(u_{3,0} + u_{1,0}) = \frac{1}{2}(18 + 4) = 11$$

$$u_{3,1} = \frac{1}{2}(u_{4,0} + u_{2,0}) = \frac{1}{2}(16 + 12) = 14$$

$$u_{4,1} = \frac{1}{2}(u_{5,0} + u_{3,0}) = \frac{1}{2}(0 + 18) = 9$$

To fill up the 3rd row put $j = 1$ in (1)

$$\therefore \quad u_{i,2} = u_{i+1,1} + u_{i-1,1} - u_{i,0}$$

Putting $i = 1, 2, 3, 4$, we get

$$u_{1,2} = u_{2,1} + u_{0,1} - u_{1,0} = 11 + 0 - 4 = 7$$
$$u_{2,2} = u_{3,1} + u_{1,1} - u_{2,0} = 14 + 6 - 12 = 8$$
$$u_{3,2} = u_{4,1} + u_{2,1} - u_{3,0} = 9 + 11 - 18 = 2$$
$$u_{4,2} = u_{5,1} + u_{3,1} - u_{4,0} = 0 + 14 - 16 = -2$$

To fill up the 4th row put $j = 2$ in (1)

$$\therefore \quad u_{i,3} = u_{i+1,2} + u_{i-1,2} - u_{i,1}$$

Putting $i = 1, 2, 3, 4$ we get

$$u_{1,3} = u_{2,2} + u_{0,2} - u_{1,1} = 8 + 0 - 6 = 2$$
$$u_{2,3} = u_{3,2} + u_{1,2} - u_{2,1} = 2 + 7 - 11 = -2$$
$$u_{3,3} = u_{4,2} + u_{2,2} - u_{3,1} = -2 + 8 - 14 = -8$$
$$u_{4,3} = u_{5,2} + u_{3,2} - u_{4,1} = 0 + 2 - 9 = -7$$

To fill up the 5th row put $j = 3$ in (1)

$$u_{i,4} = u_{i+1,3} + u_{i-1,3} - u_{i,2}$$

Putting $i = 1, 2, 3, 4$, we get

$$u_{1,4} = u_{2,3} + u_{0,3} - u_{1,2} = -2 + 0 - 7 = -9$$
$$u_{2,4} = u_{3,3} + u_{1,3} - u_{2,2} = -8 + 2 - 8 = -14$$
$$u_{3,4} = u_{4,3} + u_{2,3} - u_{3,2} = -7 - 2 - 2 = -11$$
$$u_{4,4} = u_{5,3} + u_{3,3} - u_{4,2} = 0 - 8 + 2 = -6$$

To fill up the last row put $j = 4$ in (1)

$$u_{i,5} = u_{i+1,4} + u_{i-1,4} - u_{i,3}$$

Putting $i = 1, 2, 3, 4$, we get $u_{1,5} = u_{2,4} + u_{0,4} - u_{1,3} = -14 + 0 + 2 = -16$

$$u_{2,5} = u_{3,4} + u_{1,4} - u_{2,3} = -11 - 9 + 2 = -18$$

$$u_{3,5} = u_{4,4} + u_{2,4} - u_{3,3} = -6 - 14 + 8 = -12$$

$$u_{4,5} = u_{5,4} + u_{3,4} - u_{4,3} = -0 - 11 + 7 = -4$$

■

Example 2

Solve the wave equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, 0 < x < 1, t > 0,$

$$u(0,t) = u(1,t) = 0, t > 0, u(x,0) = \begin{cases} 1, & 0 \le x \le \frac{1}{2} \\ -1, & \frac{1}{2} < x \le 1 \end{cases}$$

and $\dfrac{\partial u}{\partial t}(x,0) = 0$, using $h = k = 0.1$ for 3 time steps.

Solution

Given $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2} \quad \Rightarrow u_{xx} - u_{tt} = 0$

Here $a^2 = 1 \quad \Rightarrow a = 1. \quad h = k = 0.1$

$\therefore \quad \lambda = \dfrac{ak}{h} = 1\dfrac{(0.1)}{0.1} = 1$

Since $\lambda = 1$, the simplest equation is $u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$ (1)

$x = ih = (0.1)i, \quad t = jk = (0.1)j, \quad u(x, y) = u((0.1)i, (0.1)j) = u_{i,j}$

Since $0 \le x \le 1, \quad x = 1 \quad \Rightarrow 0.1\, i = 1 \quad \Rightarrow i = 10.$ So $i = 0, 1, 2, \ldots, 10$

Since t takes 3 steps, $j = 0, 1, 2, 3$

Given $u(x,0) = \begin{cases} 1, & 0 \le x \le \frac{1}{2} \\ -1, & \frac{1}{2} < x \le 1 \end{cases}$

$\therefore \quad x = \dfrac{1}{2} \quad \Rightarrow 0.1i = \dfrac{1}{2} \quad \Rightarrow i = 5$

$\therefore \quad u_{i,0} = \begin{cases} 1 & \text{if } i = 0,1,2,3,4,5 \\ -1 & \text{if } i = 6,7,8,9,10 \end{cases}$

$$u_{0,0} = u_{1,0} = u_{2,0} = u_{3,0} = u_{4,0} = u_{5,0} = 1$$

$$u_{6,0} = u_{7,0} = u_{8,0} = u_{9,0} = u_{10,0} = -1$$

$$u(0,t) = 0 \quad \Rightarrow u_{0,j} = 0 \quad \forall j = 1,2,3$$

$j \backslash i$	0	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	−1	−1	−1	−1	−1
1	0	1	1	1	1	0	0	−1	−1	−1	0
2	0	0	1	1	0	0	0	0	−1	0	0
3	0	0	1	0	0	0	0	0	1	0	0

$u(0,t) = 0$ for $t > 0 \quad \Rightarrow u_{0,j} = 0$ for $j = 1,2,3$

$u(1,t) = 0$ for $t > 0 \quad \Rightarrow u_{10,j} = 0$ for $j = 1,2,3.$

When $t = 0,\ u(x, 0) = 0 \qquad \Rightarrow \dfrac{u_{i,j+1} - u_{i,j-1}}{2k} = 0$

$\Rightarrow$

$$u_{i,j+1} - u_{i,j-1} = 0$$

Now $t=0 \Rightarrow j=0$

When $j = 0$, $\qquad u_{i,1} - u_{i,-1} = 0 \qquad \Rightarrow u_{i,1} = u_{i,-1}$

(1) $\Rightarrow$

$$u_{i,1} = u_{i+1,0} + u_{i-1,0} - u_{i,-1}$$

$$= u_{i+1,0} + u_{i-1,0} - u_{i,1}$$

$\Rightarrow$

$$2u_{i,1} = u_{i+1,0} + u_{i-1,0}$$

$\Rightarrow$

$$u_{i,1} = \frac{1}{2}(u_{i+1,0} + u_{i-1,0})$$

Putting $i = 1, 2, \ldots, 9$, we get

$$u_{1,1} = \frac{1}{2}(u_{2,0} + u_{0,0}) = \frac{1}{2}(1+1) = 1$$

$$u_{2,1} = \frac{1}{2}(u_{3,0} + u_{1,0}) = \frac{1}{2}(1+1) = 1$$

$$u_{3,1} = \frac{1}{2}(u_{4,0} + u_{2,0}) = \frac{1}{2}(1+1) = 1$$

$$u_{4,1} = \frac{1}{2}(u_{5,0} + u_{3,0}) = \frac{1}{2}(1+1) = 1$$

$$u_{5,1} = \frac{1}{2}(u_{6,0} + u_{4,0}) = \frac{1}{2}(-1+1) = 0$$

$$u_{6,1} = \frac{1}{2}(u_{7,0} + u_{5,0}) = \frac{1}{2}(-1+1) = 0$$

$$u_{7,1} = \frac{1}{2}(u_{8,0} + u_{6,0}) = \frac{1}{2}(-1-1) = -1$$

$$u_{8,1} = \frac{1}{2}(u_{9,0} + u_{7,0}) = \frac{1}{2}(-1-1) = -1$$

$$u_{9,1} = \frac{1}{2}(u_{10,0} + u_{8,0}) = \frac{1}{2}(-1-1) = -1$$

To find the values of u in the third row put $j = 1$ in (1)

$$u_{i,2} = u_{i+1,1} + u_{i-1,1} - u_{i,0}$$

Putting $i = 1, 2, 3, \ldots, 9$, we get

$$u_{1,2} = (u_{2,1} + u_{0,1} - u_{1,0}) = 1+0-1 \quad = 0$$
$$u_{2,2} = (u_{3,1} + u_{1,1} - u_{2,0}) = 1+1-1 \quad = 1$$
$$u_{3,2} = (u_{4,1} + u_{2,1} - u_{3,0}) = 1+1-1 \quad = 1$$
$$u_{4,2} = (u_{5,1} + u_{3,1} - u_{4,0}) = 0+1-1 \quad = 0$$
$$u_{5,2} = (u_{6,1} + u_{4,1} - u_{5,0}) = 0+1-1 \quad = 0$$
$$u_{6,2} = (u_{7,1} + u_{5,1} - u_{6,0}) = -1+0+1 \quad = 0$$
$$u_{7,2} = (u_{8,1} + u_{6,1} - u_{7,0}) = -1+0+1 \quad = 0$$
$$u_{8,2} = (u_{9,1} + u_{7,1} - u_{8,0}) = -1+(-1)+1 \quad = -1$$
$$u_{9,2} = (u_{10,1} + u_{8,1} - u_{9,0}) = 0+(-1)-(-1) = 0$$

To find the values of u in the 4th row put $j = 2$ in (1)

$$u_{i,3} = u_{i+1,2} + u_{i-1,2} - u_{i,1}$$

Putting $i = 1, 2, \ldots, 9$, we get

$$u_{1,3} = u_{2,2} + u_{0,2} - u_{1,1} = 1+0-1 \quad = 0$$
$$u_{2,3} = u_{3,2} + u_{1,2} - u_{2,1} = 1+1-1 \quad = 1$$
$$u_{3,3} = u_{4,2} + u_{2,2} - u_{3,1} = 0+1-1 \quad = 0$$
$$u_{4,3} = u_{5,2} + u_{3,2} - u_{4,1} = 0+1-1 \quad = 0$$
$$u_{5,3} = u_{6,2} + u_{4,2} - u_{5,1} = 0+0-0 \quad = 0$$
$$u_{6,3} = u_{7,2} + u_{5,2} - u_{6,1} = 0+0-0 \quad = 0$$
$$u_{7,3} = u_{8,2} + u_{6,2} - u_{7,1} = -1+0-(-1) \quad = 0$$
$$u_{8,3} = u_{9,2} + u_{7,2} - u_{8,1} = 0+0-(-1) \quad = 1$$
$$u_{9,3} = u_{10,2} + u_{8,2} - u_{9,1} = 0+(-1)-(-1) = 0$$

■

Example 3

Approximate the solution to the wave equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$, $0 < x < 1$, $t > 0$, $u(0, t) = 0$, $u(1, t) = 0$, $t > 0$, $u(x, 0) = \sin 2\pi x$, $0 \le x \le 1$ and $\dfrac{\partial u}{\partial t}(x, 0) = 0$, $0 \le x \le 1$ with $\Delta x = 0.25$ and $\Delta t = 0.25$ for 3 time steps.

Solution

Given $u_{xx} - u_{tt} = 0$

Here $a^2 = 1 \Rightarrow a = 1$; $\Delta x = h = 0.25$, $\Delta t = k = 0.25$

$$\therefore \quad \lambda = \frac{ak}{h} = 1\frac{(.25)}{0.25} = 1$$

Since $\lambda = 1$, the difference equation is

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1} \quad (1)$$

$$x = ih = 0.25i = \frac{1}{4}i \quad \text{and} \quad y = jk = 0.25j = \frac{1}{4}j$$

$$u(x,y) = u\left(\frac{i}{4}, \frac{j}{4}\right) = u_{i,j}$$

Since $0 \le x \le 1$, i varies from 0 to 1 and $\frac{i}{4} = 1 \Rightarrow i = 4$

We have to compute for 3 time steps. $\therefore j = 0, 1, 2, 3$

$$\Rightarrow \quad u(1,t) = 0 \text{ for } t > 0 \quad \Rightarrow u\left(4, \frac{j}{4}\right) = 0$$

$$u_{4,j} = 0 \text{ for } j = 1, 2, 3 \quad (2)$$

$$u(0,t) = 0 \text{ for } t > 0 \quad \Rightarrow u_{0,j} = 0,\ j = 1, 2, 3 \quad (3)$$

$$\Rightarrow \quad u(x,0) = \sin 2\pi x \quad \Rightarrow u\left(\frac{i}{4}, 0\right) = \sin\frac{2\pi}{4}i$$

$$\therefore \quad u_{i,0} = \sin\frac{\pi}{2}i, \qquad i = 0, 1, 2, 3, 4 \quad (4)$$

$$u_{0,1} = 0, \quad u_{0,2} = 0, \quad u_{0,3} = 0 \quad \text{(from 2)}$$

$$u_{4,1} = 0, \quad u_{4,2} = 0, \quad u_{4,3} = 0 \quad \text{(from 3)}$$

$$u_{0,0} = \sin 0 = 0, \quad u_{1,0} = \sin\frac{\pi}{2} = 1, \quad u_{2,0} = \sin\pi = 0$$

$$u_{3,0} = \sin\frac{3\pi}{2} = -1, \quad u_{4,0} = \sin 2\pi = 0$$

So we have the first row values of u.

$j \backslash i$	0	1	2	3	4
0	0	1	0	−1	0
1	0	0	0	0	0
2	0	−1	0	1	0
3	0	0	0	0	0

To find second row mesh point values of u.

When $t = 0$ $u_t(x,0) = 0 \Rightarrow \dfrac{u_{i,j+1} - u_{i,j-1}}{2k} = 0 \Rightarrow u_{i,j+1} - u_{i,j-1} = 0$

$[\therefore t = 0 \Rightarrow j = 0]$

When $j = 0$, $\quad u_{i,1} - u_{i,-1} = 0 \Rightarrow u_{i,1} = u_{i,-1}$

When $j = 0$, (1) $\Rightarrow$

$$u_{i,1} = u_{i+1,0} + u_{i-1,0} - u_{i,-1}$$

$$= u_{i+1,0} + u_{i-1,0} - u_{i,1}$$

$\Rightarrow$

$$2u_{i,1} = u_{i+1,0} + u_{i-1,0}$$

$\Rightarrow$

$$u_{i,1} = \frac{1}{2}(u_{i+1,0} + u_{i-1,0}) \tag{5}$$

Putting $i = 1, 2, 3$, we get

$$u_{1,1} = \frac{1}{2}(u_{2,0} + u_{0,0}) = 0$$

$$u_{2,1} = \frac{1}{2}(u_{3,0} + u_{1,0}) = \frac{1}{2}(-1+1) = 0$$

$$u_{3,1} = \frac{1}{2}(u_{4,0} + u_{2,0}) = \frac{1}{2}(0+0) = 0$$

To find the third row values of u, put $j = 1$ in (1)

$\therefore$

$$u_{i,2} = u_{i+1,1} + u_{i-1,1} - u_{i,0}$$

Putting $i = 1, 2, 3$, we get

$$u_{1,2} = u_{2,1} + u_{0,1} - u_{1,0} = 0 + 0 - 1 = -1$$

$$u_{2,2} = u_{3,1} + u_{1,1} - u_{2,0} = 0 + 0 - 0 = 0$$

$$u_{3,2} = u_{4,1} + u_{2,1} - u_{3,0} = 0 + 0 - (-1) = 1$$

To find the fourth row values of u, put $j = 2$ in (1)

$\therefore$

$$u_{i,3} = u_{i+1,2} + u_{i-1,2} - u_{i,1}$$

Putting $i = 1, 2, 3$, we get,

$$u_{1,3} = u_{2,2} + u_{0,2} - u_{1,1} = 0 + 0 - 0 = 0$$

$$u_{2,3} = u_{3,2} + u_{1,2} - u_{2,1} = 1 + (-1) - 0 = 0$$

$$u_{3,3} = u_{4,2} + u_{2,2} - u_{3,1} = 0 + 0 - 0 = 0$$

■

Example 4

Solve numerically $4u_{xx} = u_{tt}$ with the boundary conditions $u(0, t) = 0$, $u(4, t) = 0$, $t > 0$ and the initial conditions $u_t(x, 0) = 0$, $u(x, 0) = x(4 - x)$, taking $h = 1$ for four time steps.

Solution

Given $4u_{xx} = u_{tt}$

Here $a^2 = 4 \Rightarrow a = 2;\ h = 1$

We shall choose k such that $\lambda = 1 \Rightarrow \frac{ak}{h} = 1 \Rightarrow 2k = 1 \Rightarrow k = \frac{1}{2}$

Since $\lambda = 1$, the difference equation is

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1} \tag{1}$$

$$x = ih = i, \qquad t = jk = \frac{1}{2}j$$

From boundary condition, $0 \le x \le 4$ and so $i = 0, 1, 2, 3, 4$

$$u(x, y) = u\left(i, \frac{j}{2}\right) = u_{i,j}$$

Since we have to compute for 4 time steps $j = 0, 1, 2, 3, 4$

$$u(0, t) = 0 \Rightarrow u\left(0, \frac{1}{2}j\right) = 0 \Rightarrow u_{0,j} = 0 \qquad j = 1, 2, 3, 4 \tag{2}$$

$$u(4, t) = 0 \Rightarrow u\left(4, \frac{1}{2}j\right) = 0 \Rightarrow u_{4,j} = 0 \qquad j = 1, 2, 3, 4 \tag{3}$$

$\Rightarrow$ $u(x, 0) = x(4 - x) \Rightarrow u(i, 0) = i(4 - i) \Rightarrow u_{i,0} = i(4 - i),\ i = 0, 1, 2, 3, 4$ (4)

$\therefore$ $u_{0,1} = u_{0,2} = u_{0,3} = u_{0,4} = 0$ [From (2)]

and $u_{4,1} = u_{4,2} = u_{4,3} = u_{4,4} = 0$ [From (3)]

From (4), $u_{0,0} = 0,\ u_{1,0} = 3,\ u_{2,0} = 4,\ u_{3,0} = 3,\ u_{4,0} = 0$

So we have the first row values of u

j \ i	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	0	0	0	0
3	0	−2	−3	−2	0
4	0	−3	−4	−3	0

Now to find other rows mesh point values of u

When $t = 0$, $u_t(x, 0) = 0 \Rightarrow \frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0 \Rightarrow u_{i,j+1} - u_{i,j-1} = 0$

When $j = 0$ $u_{i,1} - u_{i,-1} = 0$ $[\because\ t = 0 \Rightarrow j = 0]$

$$\Rightarrow \qquad u_{i,1} = u_{i,-1} \tag{5}$$

When $j = 0$, (1) $\Rightarrow$ $$u_{i,1} = u_{i+1,0} + u_{i-1,0} - u_{i,-1}$$

$$= u_{i+1,0} + u_{i-1,0} - u_{i,1}$$ [Using (5)]

$\Rightarrow$ $$2u_{i,1} = u_{i+1,0} + u_{i-1,0}$$

$\Rightarrow$ $$u_{i,1} = \frac{1}{2}[u_{i+1,0} + u_{i-1,0}]$$

When $i = 1$, $$u_{1,1} = \frac{1}{2}[u_{2,0} + u_{0,0}] = \frac{1}{2}[4+0] = 2$$

When $i = 2$, $$u_{2,1} = \frac{1}{2}[u_{3,0} + u_{1,0}] = \frac{1}{2}[3+3] = 3$$

When $i = 3$, $$u_{3,1} = \frac{1}{2}[u_{4,0} + u_{2,0}] = \frac{1}{2}[0+4] = 2$$

To find the values of u in the third row put $j = 1$ in (1)

$\therefore$ $$u_{i,2} = u_{i+1,1} + u_{i-1,1} - u_{i,0}$$

Putting $i = 1, 2, 3$ we get

$$u_{1,2} = u_{2,1} + u_{0,1} - u_{1,0} = 3+0-3 = 0$$
$$u_{2,2} = u_{3,1} + u_{1,1} - u_{2,0} = 2+2-4 = 0$$
$$u_{3,2} = u_{4,1} + u_{2,1} - u_{3,0} = 0+3-3 = 0$$

To find the values of u in the 4th row, put $j = 2$ in (1)

$\therefore$ $$u_{i,3} = u_{i+1,2} + u_{i-1,2} - u_{i,1}$$

Putting $i = 1, 2, 3$, we get

$$u_{1,3} = u_{2,2} + u_{0,2} - u_{1,1} = 0+0-2 = -2$$
$$u_{2,3} = u_{3,2} + u_{1,2} - u_{2,1} = 0+0-3 = -3$$
$$u_{3,3} = u_{4,2} + u_{2,2} - u_{3,1} = 0+0-2 = -2$$

To find the values of u in the 5th row, put $j = 3$ in (1)

$\therefore$ $$u_{i,4} = u_{i+1,3} + u_{i-1,3} - u_{i,2}$$

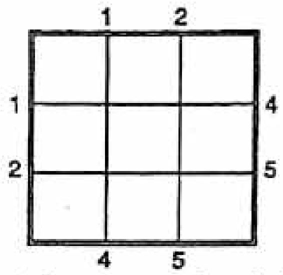
Putting $i = 1, 2, 3$, we get

$$u_{1,4} = u_{2,3} + u_{0,3} - u_{1,2} = -3+0-0 = -3$$
$$u_{2,4} = u_{3,3} + u_{1,3} - u_{2,2} = -2+(-2)-0 = -4$$
$$u_{3,4} = u_{4,3} + u_{2,3} - u_{3,2} = 0+(-3)-0 = -3$$

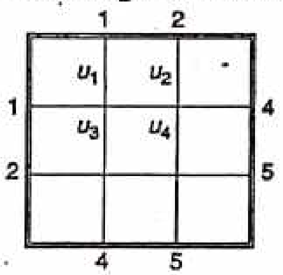
Exercises 10.2

(I) Elliptic equations:

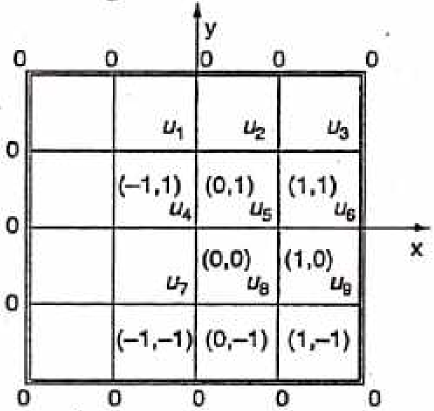
(1) Solve $u_{xx} + u_{yy} = 0$ for the following mesh with boundary values as shown in the figure. Iterate until the maximum difference between the successive values at any point is less than 0.005.



(2) Solve $u_{xx} + u_{yy} = 0$ numerically for the following mesh with boundary conditions as shown below.



(3) Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 8x^2y^2$ in the square mesh given $u = 0$ on the four boundaries dividing the square into 16 subsquares of length 1 unit.



(4) Using central-difference approximation solve $\nabla^2 u = 0$ over the square region of side n, satisfying the boundary condition

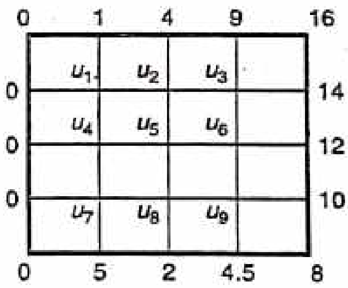
(i) $u(0,y) = 0,\ 0 \le y \le 4$

(ii) $u(4,y) = 12 + y,\ 0 \le y \le 4$

(iii) $u(x,0) = 3x,\ 0 \le y \le 4$

(iv) $u(x,4) = x^2,\ 0 \le y \le 4$.

(5) Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ in $0 \le x \le 4, 0 \le y \le 4$ given that $u(0, y) = 0$, $u(4, y) = 8 + 2y$, $u(x, 0) = \dfrac{x^2}{2}$, $u(x, 4) = x^2$ with $\Delta x = \Delta y = 1$.



(II) Parabolic equations:

(6) Solve $\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2}$ subject to the conditions $u(x, 0) \doteq \sin \pi x$, $0 \le x \le 1$, $u(0, t) = u(1, t) = 0$ Using Schmidt method taking $h = \frac{1}{3}, k = \frac{1}{36}$.

(7) Solve $u_{xx} = 32\ u_t =$ taking $h = 0.25$ for $t > 0$, $0 < x < 1$ and $u(x, 0) = 0$, $u(0, t) = 0$, $u(1, t) = t$ using Schmidt method.

(8) Using Crank-Nicolson method solve $u_{xx} = 16\ u_t$, $0 < x < 1$, $t > 0$ given $u(x, 0) = 0$, $u(0, t) = 0$ and $u(1, t) = 100\ t$, choosing $h = \frac{1}{4}$.

(9) Solve by Crank – Nicoloson method $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, 0 < x < 2$, $t > 0$, $u(0, t) = u(2, t) = 0$, $t > 0$ and $u(x, 0) = \sin \frac{\pi x}{2}, 0 \le x \le 2$ using $h = 0.5$, $k = 0.25$ for two time steps.

(10) Using Crank-Nicolson method solve $u_t = u_{xx}$ in $0 \le x \le 1$, $t > 0$, given $u(0, t) = 0\ u(l, t)$

$$\text{and } u(x, 0) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2(1-x), & \frac{1}{2} < x \le 1 \end{cases}$$

Take $h = 0.1$ and find u for one step in t.

(III) Hyperbolic equations:

(11) Solve $y_{tt} = y_{xx}$ upto $t = 0.5$, with spacing of 0.1 subject to $y(0, t) = 0$, $y(1, t) = 0$, $y_t(x, 0) = 10 + x(1 - x)$.

(12) Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $0 < x < t$, $t > 0$ given $u(x, 0) = 100(x - x^2)$, $\frac{\partial u}{\partial t}(x, 0) = 0$, $u(0, t) = u(1, t) = 0$, $t > 0$ by finite difference method for one time step with $h = 0.25$.

(13) Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, 0 < x < 1, t > 0$, given $u(x, 0) = 0$, $u_t(x, 0) = u(0, t) = 0$ and $u(1, t) = 100 \sin \pi t$. Compute u for 4 time steps with $h = 0.25$.

(14) Solve $25u_{xx} - u_{tt} = 0$ for u at the pivotal points, given $u(0, t) = u(5, t) = 0$, $u_t(x, 0) = 0$ and

$$u(x,0) = \begin{cases} 2x & \text{for } 0 \le x \le 2.5 \\ 2(5-x) & \text{for } 2.5 < x \le 5 \end{cases} \quad \text{for one half period of vibration.}$$

(15) Solve the boundary value problem $u_{tt} = u_{xx} = 0$ with the conditions $u(0, t) = u(1, t) = 0$, $u(x, 0) = \frac{1}{2}x(1-x)$ and $u_t(x, 0) = 0$ taking $h = k = 0.1$ for $0 \le t \le 0.4$. Compare your solution with the exact solution at $x = 0.5$ and $t = 0.3$.

Answers 10.2

(1) 1.999, 2.999, 3.999

(2) $u_1 = u_4 = 1.333$, $u_2 = u_3 = 1.6667$

(3) $u_1 = u_3 = u_7 = u_9 = -3$

$u_2 = u_4 = u_6 = u_8 = -2$, $u_5 = -2$

(4) 2.37, 5.59, 9.87, 2.88, 6.13, 9.88, 3.01, 6.16, 9.51

(5) $u_1 = 2$, $u_2 = 4.9$, $u_3 = 9$, $u_4 = 2.07$, $u_5 = 4.69$, $u_6 = 8.07$, $u_7 = 1.57$, $u_8 = 3.71$, $u_9 = 6.57$]

(6)

$j \backslash i$	0	1	2	3	4	5
0	0	0.5878	0.9511	0.9511	0.5878	0
1	0	0.4756	0.7695	0.7695	0.4756	0
2	0	0.3848	0.6225	0.6225	0.3848	0
3	0	0.3113	0.5036	0.5036	0.3113	0
4	0	0.2518	0.4074	0.4074	0.2518	0
5	0	0.2037	0.3296	0.3296	0.2037	0

(7)

$j \backslash i$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	0	0	.5	2
3	0	0	0.25	1	3
4	0	0.125	0.5	1.625	4

(8) $u_1 = 1.7857$, $u_2 = 7.1429$, $u_3 = 26.7857$

(9) $u_1 = 0.2857$, $u_2 = 0.1429$, $u_3 = 0.2857$, $u_4 = 0.0816$, $u_5 = 0.1837$, $u_6 = 0.0816$

(10)

$j \backslash i$	0	1	2	3	4	5	6	7	8	9	10
0	0	0.2	0.4	0.6	0.8	1	0.8	0.6	0.4	0.2	0
1	0	0.1989	0.3956	0.5834	0.7381	0.7691	0.7381	0.5834	0.3956	0.1989	0

(11)

j \ i	0	1	2	3	4	5	6	7	8	9	10
0	0	10.09	10.16	10.21	10.24	10.25	10.24	10.21	10.16	10.09	10
1	0	5.08	10.15	10.20	10.23	10.24	10.23	10.20	10.15	10.08	0
2	0	0.06	5.12	10.17	10.20	10.21	10.20	10.17	10.12	0.06	0
3	0	0.04	0.08	5.12	10.15	10.16	10.15	10.12	10.08	0.04	0
4	0	0	0.02	0.04	0.06	5.08	10.09	10.08	10.06	10.04	0
5	0	0	0	0	0	0	0	0	0	0	0

(12)

j \ i	0	1	2	3	4
0	0	18.75	25	18.75	0
1	0	12.5	18.75	12.5	0

(13)

j \ i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	70.7106
2	0	0	0	70.7106	100
3	0	0	70.7106	100	70.7106
4	0	70.7106	100	70.7106	0

(14)

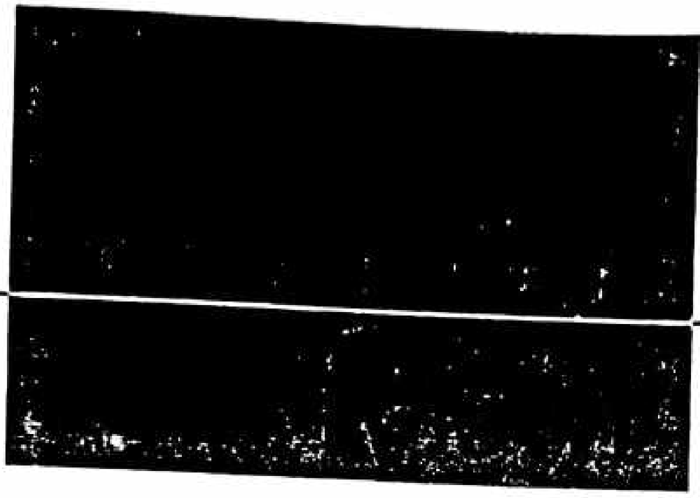
j \ i	0	1	2	3	4	5
0	0	2	4	4	2	0
1	0	2	3	3	2	0
2	0	1	1	1	1	0
3	0	−1	−1	−1	−1	0
4	0	−2	−3	−3	−2	0
5	0	−2	−4	−4	−2	0

(15) Find the exact solution when $t = 0.3$ or $j = 3$

i	1	2	3	4	5
Numerical solution	0.02	0.04	0.06	0.075	0.08
Exact solution	0.02	0.04	0.06	0.075	0.08

SHORT ANSWER QUESTIONS

1. Identify the nature of the equation $u_{xx} + 2u_{xy} + u_{yy} = 0$.
2. Identify the type of the equation $f_{xx} + 2f_{xy} + f_{yy} = 0$.
3. Classify the partial differential equation $u_{xx} + 2u_{xy} + 4u_{yy} = 0, x > 0, y > 0$.
4. For what values of x and y, the equation $xf_{xx} + yf_{yy} = 0, x > 0, y > 0$ is elliptic?
5. Obtain the finite difference approximation for the differential equation $\frac{d^2y}{dx^2} + 2y = 0$.
6. Name at least two numerical methods that are used to solve one dimensional diffusion equation.
7. Write down Laplace equation and its finite difference analogue and the standard five point formula.
8. State the explicit finite difference scheme for one dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$.
9. Find the explicit finite differences scheme for one dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$.
10. Write down the Crank-Nicolson formula to solve $u_t = u_{xx}$.
11. Write down the implicit formula to solve one dimensional heat flow equation $u_{xx} = \frac{1}{c^2} u_t$.
12. Why is Crank - Nicolson's scheme is called an implicit scheme?
13. What type of equations can be solved by using Crank-Nicolson's formula?
14. Write down the five point finite difference formula for solving $\nabla^2 f = 0$.
15. State Leibmann's iteration process formula.
16. Write down the finite difference form of the Poisson equation $\nabla^2 u = f(x, y)$.
17. Write down the Schemidt's explicit formula for solving heat flow equation.
18. Write down an explicit formula to solve numerically the heat equation $u_{xx} - au_t = 0$.
19. Write down the Crank-Nicolson difference scheme to solve $u_{xx} = au_t$ with $u(0, t) = T_0$, $(l, t) = T_1$ and the initial condition as $u(x, 0) = f(x)$.
20. Write down the Forward difference approximation to $u_x(x_0, y_0)$.
21. Write down the backward difference approximation of $u_x(x_0, y_0)$.
22. Write down the finite difference approximation to u_{xx}, where $u = u(x, y)$ taking, h, k as the step sizes at the point (x, y).



Difference Equations

INTRODUCTION

Difference equations may be considered as the discrete analogue of differential equations. Discrete functions occur in many physical and engineering problems. So, difference equations are the natural choice for dealing with discrete situations.

Basic concepts of difference equations are similar to differential equations.

Definition 11.1: Difference equation

A difference equation is an equation involving an independent variable x, dependent variable y_x and the successive differences of y_x or successive values $y_x, y_{x+1}, y_{x+2} \cdots$

Example (1) $\Delta^2 y_x + 5\Delta y_x - y_x = \cos x$

(2) $\Delta^2 y_x + 3\Delta y_x = x^2$

(3) $y_{x+2} - 3y_{x+1} + y_x = 3^x$ are difference equations.

Since $\Delta y_x = y_{x+1} - y_x$, $\Delta^2 y_x = y_{x+2} - 2y_{x+1} + y_x$, every difference equation can be expressed in terms of the successive values $y_x, y_{x+1}, \ldots$

Definition 11.2: Order of the difference equation

The **order of the difference** equation is the difference between the highest and lowest suffixes of y_x involved in it.

Example (1) order of $y_{x+2} - 3y_{x+1} + y_x = 3^x$ is $x+2-x=2$.

(2) order of $\Delta^2 y_x + 5\Delta^2 y_x - y_x = \cos x$ is 2, since the highest difference is of order 2.

(3) order of $\Delta^2 y_x + 3\Delta y_x = x^2$ is 2.

Definition 11.3: Degree of a difference equation

The degree of a difference equation expressed in terms of values is the highest power of the y's. The degree of the difference equations (1), (2), (3) in the example is 1.

But the degree of $y_{x+1}^3 - 2y_x y_{x-1} + y_{x+1}^2$ is 3.

Definition 11.4: Solution of a difference equation

The solution of a difference equation is a function which satisfies it.

Definition 11.5: General solution of a linear difference equation

The general solution of a linear difference equation of order n is a solution which contains n arbitrary constants.

Definition 11.6: Particular solution

A particular solution of a difference equation is a solution obtained from the general solution by giving particular values to the arbitrary constants.

For example: $y_x = A.2^x + B.3^x$ is the general solution of the difference equation.

$y_{x+2} - 5y_{x+1} + 4y_x = 0.$

Putting $A = 1$, $B = -1$, we get the particular solution $y_x = 2^x - 3^x$

LINEAR DIFFERENCE EQUATION

The general form of a linear difference equation of order r with constant coefficients is

$$a_0 y_{x+r} + a_1 y_{x+r-1} + a_2 y_{x+r-2} + \dots + a_{r-1} y_{x+1} + a_r y_x = f(x) \tag{1}$$

where $a_0, a_1, a_2, \dots a_{r-1}, a_r$ are constants.

The equation of the form

$$a_0 y_{x+r} + a_1 y_{x+r-1} + a_1 y_{x+r-2} + \dots + a_{r-1} + y_{x+1} + a_r y_x = 0 \tag{2}$$

is called the corresponding homogenous linear difference equation of order r.

SOLUTION OF A DIFFERENCE EQUATION

As in the case of linear differential equations, we have the following results for linear difference equations.

1. If $y_x = f_1(x)$, $y_x = f_2(x), \dots, y_x = f_x(x)$ are r independent solutions of the linear difference equation (2), then their linear combination $c_1 f_1(x) + c_2 f_2(x) + \dots + c_r f_r(x)$, where $c_1, c_2, \dots, c_r$ are constants, is also a solution of (2).

   This solution containing r arbitrary constants is called the **general solution of (2)**.

   The general solution of (2) is called the **complementary function** of (1).

   If u_x is a particular solution of (1), then $y_x = c_1 f_1(x) + c_2 f_2(x) + \dots + c_r f_r(x) + u_x$ is the general solution of (1).

If the particular solution is denoted by P.I and the complementary function is denoted by C.F, then the solution is $y_x = \text{C.F} + \text{P.I.}$

Note: In the linear difference equation $y_x, y_{x+1},\ldots$ occur in the first degree and there is no product of $y_x, y_{x+1},\ldots$ occur in the equation. **This is a characteristic of linear difference equation.**

FORMATION OF A DIFFERENCE EQUATION

A difference equation is formed by eliminating the arbitrary constants in an ordinary relation between the variables.

WORKED EXAMPLES

Example 1

Form the difference equation corresponding to the family of curves $y = ax + bx^2$.

Solution

Given $\quad y = ax + bx^2$

Here $\quad y_x = y$

$\therefore \quad y_x = ax + bx^2$

Since y_x is a polynomial of degree 2, the second order differences are constants.
So, we consider 3 values and get three equations

$$y_x = ax + bx^2 \tag{1}$$

$$y_{x+1} = a(x+1) + b(x+1)^2 \tag{2}$$

$$y_{x+2} = a(x+2) + b(x+2)^2 \tag{3}$$

From this system of linear equations in a and b, the eliminant of a and b is

$$\begin{vmatrix} y_x & x & x^2 \\ y_{x+1} & x+1 & (x+1)^2 \\ y_{x+2} & x+2 & (x+2)^2 \end{vmatrix} = 0$$

Expanding this determinant by first column, we get

$$y_x[(x+1)(x+2)^2 - (x+2)(x+1)^2] - y_{x+1}[x(x+2)^2 - (x+2)x^2]$$
$$+ y_{x+2}[x(x+1)^2 - (x+1)x^2] = 0$$

$$\Rightarrow \quad y_x[(x+1)(x+2)\{x+2-(x+1)\}] - y_{x+1}[x(x+2)\{x+2-x\}]$$
$$+ y_{x+2}[x(x+1)\{x+1-x\}] = 0$$

$$\Rightarrow \quad y_x(x^2+3x+2) - 2y_{x+1}(x^2+2x) + y_{x+2}(x^2+x) = 0$$

$$\Rightarrow \quad \mathbf{(x^2+x)y_{x+2} - 2(x^2+2x)y_{x+1} + (x^2+3x+2)y_x = 0}$$

which is the required difference equation. ∎

Example 2

Form the difference equation by eliminating a and b using $y_x = ax + b.2^x$.

Solution

Given $$y_x = ax + b.2^x \quad (1)$$

$\therefore$ $$y_{x+1} = a(x+1) + b\,2^{x+1} \quad (2)$$

and $$y_{x+2} = a(x+2) + b\,2^{x+2} \quad (3)$$

From this system of linear equations in a and b, the eliminant of a and b is

$$\begin{vmatrix} y_x & x & 2^x \\ y_{x+1} & x+1 & 2^{x+1} \\ y_{x+2} & x+2 & 2^{x+2} \end{vmatrix} = 0$$

$\Rightarrow$ $$2^x \begin{vmatrix} y_x & x & 1 \\ y_{x+1} & x+1 & 2 \\ y_{x+2} & x+2 & 4 \end{vmatrix} = 0$$

$\Rightarrow$ $$\begin{vmatrix} y_x & x & 1 \\ y_{x+1} & x+1 & 2 \\ y_{x+2} & x+2 & 4 \end{vmatrix} = 0 \qquad [\because 2^x \neq 0]$$

$\Rightarrow$ $$y_x[4(x+1) - 2(x+2)] - y_{x+1}[4x - (x+2)] + y_{x+2}[2x - (x+1)] = 0$$

$\Rightarrow$ $$y_x(2x) - y_{x+1}(3x-2) + y_{x+2}(x-1) = 0$$

$\Rightarrow$ $$\mathbf{(x-1)y_{x+2} - (3x-2)y_{x+1} + 2xy_x = 0}$$

which is the required difference equation. ■

Example 3

Derive the difference equation, given $y_x = (A + Bx)2^x$.

Solution

Given $$y_x = (A + Bx)2^x \quad (1)$$

$\Rightarrow$ $$y_x = A.2^x + Bx.2^x \quad (2)$$

$\therefore$ $$y_{x+1} = A.2^{x+1} + B(x+1).2^{x+1} \quad (3)$$

and $$y_{x+2} = A2^{x+2} + B(x+2)2^{x+2}$$

Treating these equations as linear equations in A and B, the eliminant of A and B is

$$\begin{vmatrix} y_x & 2^x & x2^x \\ y_{x+1} & 2^{x+1} & (x+1)2^{x+1} \\ y_{x+2} & 2^{x+2} & (x+2)2^{x+2} \end{vmatrix} = 0$$

$$\Rightarrow \quad 2^x.2^x \begin{vmatrix} y_x & 1 & x \\ y_{x+1} & 2 & 2(x+1) \\ y_{x+2} & 4 & 4(x+2) \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} y_x & 1 & x \\ y_{x+1} & 2 & 2(x+1) \\ y_{x+2} & 4 & 4(x+2) \end{vmatrix} = 0 \qquad [\because 2^{2x} \neq 0]$$

$$\Rightarrow \quad y_x[8(x+2)-8(x+1)] - y_{x+1}[4(x+2)-4x] + y_{x+2}[2(x+1)-2x] = 0$$

$$\Rightarrow \quad 8y_x - 8y_{x+1} + 2y_{x+2} = 0$$

$$\Rightarrow \quad \mathbf{y_{x+2} - 4y_{x+1} + 4y_x = 0}$$

which is the required difference equation. ■

Example 4

Find the difference equation generated by $y = \dfrac{a}{x} + b$.

Solution

Given $\quad y = \dfrac{a}{x} + b$

Denoting y by y_x, we get $\quad y_x = \dfrac{a}{x} + b$

$\therefore \quad y_{x+1} = \dfrac{a}{x+1} + b$

and $\quad y_{x+2} = \dfrac{a}{x+2} + b$

Treating these equations as simultaneous linear equations in a and b, the eliminant of a and b is

$$\begin{vmatrix} y_x & \frac{1}{x} & 1 \\ y_{x+1} & \frac{1}{x+1} & 1 \\ y_{x+2} & \frac{1}{x+2} & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad y_x\left(\frac{1}{x+1}-\frac{1}{x+2}\right)-y_{x+1}\left(\frac{1}{x}-\frac{1}{x+2}\right)+y_{x+2}\left(\frac{1}{x}-\frac{1}{x+1}\right)=0$$

$$\Rightarrow \quad y_x\left[\frac{x+2-x-1}{(x+1)(x+2)}\right]-y_{x+1}\left[\frac{x+2-x}{x(x+2)}\right]+y_{x+2}\left(\frac{x+1-x}{x(x+1)}\right)=0$$

$$\Rightarrow \quad \frac{y_x}{(x+1)(x+2)}-\frac{2y_{x+1}}{x(x+2)}+\frac{y_{x+2}}{x(x+1)}=0$$

Multiplying by $x(x + 1)(x + 2)$, we get

$$xy_x-2(x+1)y_{x+1}+(x+2)y_{x+2}=0$$

$$\Rightarrow \quad \mathbf{(x+2)y_{x+2}-2(x+1)y_{x+1}+xy_x=0}$$

which is the required difference equation. ■

Example 5

Form the difference equation satisfied by the Fibonacci sequence 1, 1, 2, 3, 5, 8, ...

Solution

Let y_n denote the n^{th} term of the sequence 1, 1, 2, 3, 5, 8, 13, ...

We observe that any term starting with the third term is the sum of the proceeding two numbers

$\therefore$

$$y_3 = y_2 + y_1$$
$$y_4 = y_3 + y_2$$
$$\vdots \quad \vdots$$

$\therefore$ the difference equation is $\quad y_{n+2} = y_{n+1} + y_n, \quad n = 1, 2, 3, \ldots$

and $\quad y_1 = 1, \; y_2 = 1.$

This is the difference equation of the **Fibonacci sequence.**

Note: The general term of any sequence is usually denoted by any of these symbols y_n, $y(n)$, y_k, $y(k)$ y_x, $y(x)$ etc. ■

Example 6

Find the difference equation with initial conditions to find the number of n - bit strings that do not have two consecutive zeros. Hence find the number of 5 bit strings.

Solution

Let y_n denote the number of n – bit strings that do not contain two consecutive zeros.
Then the sequence $\{y_n\}$ gives all terms.
We have to find the difference equation satisfied by this sequence $\{y_n\}$,
If $n = 1$, the string is 1 – bit string, So, it is 0 or 1.
Hence it does not contain two zeros.

$$\therefore \qquad y_1 = 2$$

If $n = 2$, the string is a 2 – bit string which may be 11,10,01,00
So, there are three 2-bit strings not containing two zeros.

$$\therefore \qquad y_2 = 3$$

Now assume $n \geq 3$
we shall consider an arbitrary n-bit string having no two consecutive zeros.
It may end with 0 or 1.

Case (i): Suppose the n-bit string ends with zero, then the $(n-1)^{th}$ bit must be 1 and there is no restriction on the $(n-2)^{th}$ bit.

$\therefore$ the number of n-bit strings ending with zero is $y_{n-2.}$

Case (ii): Suppose the n-bit string ends with 1, then the $(n-1)^{th}$ bit can be 1 or 0.

$\therefore$ the number of such strings is $y_{n-1.}$

$\therefore$ the total number of n-bit strings with no two consecutive zeros is

$$y_n = y_{n-1} + y_{n-2},\ n \geq 3 \text{ with the conditions } y_1 = 2,\ y_2 = 3$$

So, the number of 5-bit strings is $y_5 = y_4 + y_3$

$$\text{But} \qquad y_4 = y_3 + y_2$$
$$y_3 = y_2 + y_1 = 3 + 2 = 5$$
$$\therefore \qquad y_4 = 5 + 3 = 8$$
$$\text{and} \qquad y_5 = 8 + 5 = 13$$

■

Exercises 11.1

Find the difference equation satisfied by

(1) $y_x = A.3^x + B.5^x$ (2) $y_x = 2ax^2 + b$

(3) $y_x = A\,3^x + x\,3^{x-1}$ (4) $y_n = a2^n + b(-2)^n$

Answers 11.1

(1) $y_{x+2} - 8y_{x+1} + 15y_x = 0$ (2) $(2x+1)y_{x+2} - 4(x+1)y_{x+1} + (2x+3)y_x = 0$

(3) $y_{x+1} - 3y_x = 0$ (4) $y_{n+2} - 4y_n = 0$

LINEAR HOMOGENEOUS DIFFERENCE EQUATION WITH CONSTANT COEFFICIENTS

The general form of a linear homogeneous difference equation with constant coefficients is

$$a_0 y_{x+k} + a_1 y_{x+k-1} + a_2 y_{x+k-2} + \dots + a_{k-1} y_{x+1} + a_k y_x = 0 \tag{1}$$

Let $y = m^x, m \neq 0$ be a trial solution of (1)

Substituting in (1), we get

$$a_0 m^{x+k} + a_1 m^{x+k-1} + a_2 m^{x+k-2} + \dots + a_{k-1} m^{x+1} + a_k m^x = 0$$

$$\Rightarrow \quad m^x \left[a_0 m^k + a_1 m^{k-1} + a_2 m^{k-2} + \dots + a_{k-1} m + a_k \right] = 0$$

$$\Rightarrow \quad a_0 m^k + a_1 m^{k-1} + a_2 m^{k-2} + \dots + a_{k-1} m + a_k = 0 \quad [\because m^x \neq 0] \tag{2}$$

The equation (2) of degree k in m is called the auxiliary equation or the characteristic equation of (1).

Let $m_1, m_2, m_3, \dots, m_k$ be the roots of (2).

Depending on the nature of roots we have the solution of (1)

Case (1): The roots are all real and different

If the roots $m_1, m_2, m_3, \dots, m_k$ are real and different, then

$$y_x = c_1 m_1^x + c_2 m_2^x + \dots + c_k m_k^x \text{ is}$$

the general solution of (1), where $c_1, c_2, \dots, c_k$ are arbitrary constants

Case (2): Some roots are real and repeated

If $m_1 = m_2 = m$ and $m_3, m_4, \dots, m_k$ are different, then

$y_x = (c_1 + c_2 x) m^x + c_3 m_3^x + c_4 m_4^x \dots + c_k m_k^x$ is the general solution of (1), where $c_1, c_2, c_3 \dots c_k$ are arbitrary constants.

Suppose $m_1 = m_2 = m_3 = m_4 = m$ and $m_5, m_6, \dots, m_k$ are different, then the general solution is $y_x = (c_1 + c_2 x + c_3 x^2 + c_4 x^3) m^x + c_5 m_5^x + \dots + c_5 m_k^x$.

Case (3): Non-repeated complex roots

Let $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ be the complex roots and $m_3, m_4, \dots, m_k$ are real and different. Then the general solution is

$$y_x = (c_1 \cos x\theta + c_2 \sin x\theta) r^x + c_3 m_3^x + c_4 m_4^x + \dots + c_k m_k^k.$$

where $r = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1} \dfrac{\beta}{\alpha}$

Let $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, $m_3 = \alpha + i\beta$, $m_4 = \alpha - i\beta$ be repeated complex roots and $m_5, m_6, \dots, m_k$ are real and different roots.

Then the general solution is

$$y_x = [(c_1 + c_2 x)\cos x\theta + (c_3 + c_4 x)\sin x\theta]r^x + c_5 m_5^x + c_6 m_6^x + \ldots + c_x m_x^x$$

where $r = \sqrt{\alpha^2 + \beta^2}$, $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$ and $c_1, c_2, c_3, \ldots, c_x$ are arbitrary constants.

Working Rule

(1) The equation (1) can be written as

$$a_0 E^k y_x + a_1 E^{k-1} y_x + a_2 E^{k-2} y_x + \ldots + a_{k-1} E y_x + a_k y_x = 0$$

$$\Rightarrow \quad \left[a_0 E^k + a_1 E^{k-1} + a_2 E^{k-2} + \ldots + a_{k-1} E + a_k\right] y_x = 0$$

(2) Replacing E by m, we get the auxiliary equation

$$a_0 m^k + a_1 m^{k-1} + a_2 m^{k-2} + \ldots + a_{k-1} m + a_k = 0$$

(3) If $m_1, m_2, m_3, \ldots, m_k$ are the roots, then we can write the general solution as discussed in case (1), (2) and (3)

SOME BASIC RESULTS OF DIFFERENCE OPERATOR TO SOLVE DIFFERENCE EQUATIONS

Let us take $f(x)$ be the first difference of the function $F(x)$, then $\Delta F(x) = f(x)$

Now
$$\begin{aligned}
\sum_{x=1}^{n} f(x) &= f(1) + f(2) + f(3) + \ldots + f(n) \\
&= f(1) + Ef(1) + E^2 f(1) + \ldots + E^n f(x) \\
&= \left(1 + E + E^2 + \ldots + E^n\right) f(1) \\
&= \frac{E^n - 1}{E - 1} f(1) \\
&= \frac{E^n - 1}{E - 1} \Delta F(1) && [\because \Delta F(1) = f(1)] \\
&= \frac{E^n - 1}{\Delta} \Delta E(1) && [\because 1 + \Delta = E \Rightarrow E - 1 = \Delta] \\
&= (E^n - 1) F(1) \\
&= E^n F(1) - F(1) \\
&= F(n+1) - F(1)
\end{aligned}$$

$$= [F(x)]_1^{n+1} \text{, in symbols.}$$

$$\sum_{x=1}^{n} f(x) = [\Delta^{-1} f(x)]_1^{n+1}$$

Note: 17-05-19

(1) Thus Δ^{-1} is an operator, when operated on a function $f(x)$ yields $F(x)$ whose first difference is $f(x)$. So we find Σ behaves like Δ^{-1}

(2) The above finite difference summation is sometimes referred to as finite integration, Since it employs the operator Δ^{-1}, similar to D^{-1} in integral calculus.

(3) Since $\Delta[F(x)+c] = \Delta F(x) = f(x)$

$$\Rightarrow \qquad \Delta^{-1} f(x) = F(x) + c$$

We have the following table of finite integrals

(1) $\Delta^{-1} a^x = \dfrac{a^x}{a-1}$ if $a \neq 1$

(2) $\Delta^{-1} [x]^r = \dfrac{[x]^{r+1}}{r+1}$

(3) $\Delta^{-1} \sin ax = -\dfrac{1}{2\sin\frac{a}{2}} \cos a\left(x - \dfrac{1}{2}\right)$

(4) $\Delta^{-1} \cos ax = \dfrac{1}{2\sin\frac{a}{2}} \sin a\left(x - \dfrac{1}{2}\right)$

(5) $\Delta^{-1} [x(x!)] = x!$

(6) $\Delta^{-1}[u_x \Delta v_x] = u_x v_x - \Delta^{-1}[v_{x+1} \Delta u_x]$

This is called the rule for finite integral by parts.

Note: $\Delta C = 0$ $\qquad \therefore \Delta^{-1}(0) = C$

$\Delta(ax+b) = a$ $\qquad \therefore \Delta^{-1}(a) = ax + b$

WORKED EXAMPLES

Example 1

Obtain the solution of the difference equation $y_{x+1} - 5y_x = 0$.

Solution

The given equation is $\qquad y_{x+1} - 5y_x = 0$

$$\Rightarrow \qquad Ey_x - 5y_x = 0$$

$$\Rightarrow \qquad (E-5)y_x = 0$$

Auxiliary equation is $m - 5 = 0, \Rightarrow m = 5.$
The root is real

$\therefore$ the general solution is $\boldsymbol{y_x = C.5^x}$

Example 2

Obtain the solution of the difference equation $3u_{x+1} + 2u_x = 0$.

Solution

The given equation is $$3u_{x+1} + 2u_x = 0$$

$\Rightarrow$ $$3Eu_x + 2u_x = 0$$

$$(3E + 2)u_x = 0$$

Auxiliary equation is $3m + 2 = 0 \Rightarrow m = -\frac{2}{3}$

The root is real.

$\therefore$ the general solution is $\boldsymbol{u_x = C\left(-\frac{2}{3}\right)^x}$,

Example 3

Solve the difference equation $y_{x+2} - 6y_{x+1} + 8y_x = 0$.

Solution

Asad Ali

The given equation is $$y_{x+2} - 6y_{x+1} + 8y_x = 0$$

$$E^2y_x - 6Ey_x + 8y_x = 0$$

$$\left(E^2 - 6E + 8\right)y_x = 0$$

$\therefore$ auxiliary equation is $$m^2 - 6m + 8 = 0$$

$\Rightarrow$ $(m - 4)(m - 2) = 0 \Rightarrow m = 2, 4$

The roots are real and different

$\therefore$ the general solution is $\boldsymbol{y_x = C_1 2^x + C_2 4^x}$

Example 4

Solve $9y_{x+2} - 6y_{x+2} + y_x = 0$.

Solution

The given equation is $$9y_{x+2} - 6y_{x-1} + y_x = 0$$

$\Rightarrow$ $$9E^2y_x - 6Ey_x + y_x = 0$$

$\Rightarrow$ $$\left(9E^2 - 6E + 1\right)y_x = 0$$

Auxiliary equation is $9m^2 - 6m + 1 = 0$

$\Rightarrow$ $(3m-1)^2 = 0 \Rightarrow m = \frac{1}{3}, \frac{1}{3}$

The roots are real and equal.

$\therefore$ the general solution is $y_x = (C_1 + C_2 x)\left(\frac{1}{3}\right)^x$ ∎

Example 5

Solve $y_{x+2} - 4y_{x+1} + 13y_x = 0$.

Solution

The given equation is $y_{x+2} - 4y_{x+1} + 13y_x = 0$

$\Rightarrow$ $E^2 y_x - 4Ey_x + 13y_x = 0$

$\Rightarrow$ $(E^2 - 4E + 13)y_x = 0$

Auxiliary equation is $m^2 - 4m + 13 = 0$

$$\Rightarrow \quad m = \frac{4 \pm \sqrt{4^2 - 4 \times 1.13}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm i6}{2} = 2 \pm i3$$

The roots are complex with $\alpha = 2$ and $\beta = 3$

$$\therefore \quad r = \sqrt{\alpha^2 + \beta^2} = \sqrt{2^2 + 3^2} = \sqrt{13}$$

and $\theta = \tan^{-1}\frac{\beta}{\alpha} = \tan^{-1}\left(\frac{3}{2}\right)$

$\therefore$ the general solution is $y_x = (C_1 \cos x\theta + C_2 \sin x\theta) r^x$

$$= (\mathbf{C_1 \cos x\theta + C_2 \sin x\theta})\left(\sqrt{\mathbf{13}}\right)^x$$ ∎

Example 6

Solve $2y_{n+2} - 5y_{n+1} + 2y_n = 0$ with the condition $y_0 = 0, y_1 = 1$.

Solution

The given equation is $2y_{n+2} - 5y_{n+1} + 2y_n = 0$

$\Rightarrow$ $2E^2 y_n - 5Ey_n + 2y_n = 0$

$\Rightarrow$ $(2E^2 - 5E + 2)y_n = 0$

Auxiliary equation is $2m^2 - 5m + 2 = 0$

$\Rightarrow$ $2m^2 - 4m - m + 2 = 0$

$\Rightarrow$ $2m(m-2) - (m-2) = 0$

$\Rightarrow$ $(2m-1)(m-2) = 0 \Rightarrow m = \frac{1}{2}, 2.$

The roots are real and distinct.

$\therefore$ the general solution is $y_n = C_1\left(\frac{1}{2}\right)^n + C_2 2^n$ (1)

Given $y_0 = 0$ and $y_1 = 1$

Putting $n = 0$ and 1 in (1), we get $y_0 = C_1\left(\frac{1}{2}\right)^0 + C_2 \cdot 2^0$

$\Rightarrow$ $0 = C_1 + C_2 \Rightarrow C_1 = -C_2.$ (2)

and $y_1 = C_1 \frac{1}{2} + C_2 2$

$\Rightarrow$ $1 = \frac{C_1}{2} + 2C_2$

$\Rightarrow$ $C_1 + 4C_2 = 2$

$\Rightarrow$ $-C_2 + 4C_2 = 2 \Rightarrow 3C_2 = 2 \Rightarrow C_2 = \frac{2}{3}$

$\therefore$ $C_1 = -\frac{2}{3}$ [using (2)]

$\therefore$ particular solution is $y_n = -\frac{2}{3}\left(\frac{1}{2}\right)^n + \frac{2}{3} \cdot 2^n = \frac{2}{3}\left[2^n - \frac{1}{2^n}\right]$ ■

Example 7

Solve the difference equation $y_{n+3} - 3y_{n+1} + 2y_n = 0$ given $y_1 = 0, y_2 = 8$ and $y_3 = -2$.

Solution

The given equation is $y_{n+3} - 3y_{n+1} + 2y_n = 0$

$\Rightarrow$ $E^3 y_n - 3Ey_n + 2y_n = 0$

$\Rightarrow$ $(E^3 - 3E + 2)y_n = 0$

Auxiliary equation is $m^3 - 3m + 2 = 0$

Since the sum of the coefficients is zero, $m=1$ is a ...
The other roots are given by

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -3 & 2 \\ & 0 & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & 0 \end{array}$$

$\Rightarrow$ $$m^2+m-2=0$$

$$(m+2)(m-1)=0$$

$\Rightarrow$ $$m=-2,1$$

$\therefore$ the roots are $m = \underline{1}, \underline{1}, \underline{-2}$ with two equal roots

$\therefore$ the general solution is

$$y_n=(C_1+C_2n)1^n+C_3(-2)^n$$

$$y_n=C_1+C_2n+C_3(-2)^n$$

$\Rightarrow$

Given $$y_1=0,\ y_2=8,\ y_3=-2.$$

putting $n = 1, 2, 3$ in (1), we get

$$y_1=C_1+C_2.1+C_3(-2)$$

$$0=C_1+C_2-2C_3$$ (1)

$\Rightarrow$

$\Rightarrow$ $$C_1+C_2-2C_3=0$$

$$y_2=C_1+C_2.2+C_3(-2)^2$$

$$8=C_1+2C_2+4C_3$$

$\Rightarrow$ (2)

$\Rightarrow$ $$C_1+2C_2+4C_3=8$$

and $$y_3=C_1+3C_2+C_3(-2)^3$$

$$-2=C_1+3C_2-8C_3$$

$\Rightarrow$ (3)

$\Rightarrow$ $$C_1+3C_2-8C_3=-2$$

$2\times(2)\Rightarrow$ $$2C_1+2C_2-4C_3=0$$ (4)

(3) is $$C_1+2C_2+4C_3=8$$

Adding $$3C_1+4C_2=8$$

$2\times(3)\Rightarrow$ $$2C_1+4C_2+8C_3=16$$ (5)

(4) is $$C_1+3C_2-8C_3=-2$$

Adding $$3C_1+7C_2=14$$ (6)

$(6)-(5)\Rightarrow$ $$3C_2=6 \Rightarrow C_2=2$$

Substituting in (5), $3C_1+4\cdot 2=8.$ $\Rightarrow$ $3C_1=0$ $\Rightarrow C_1=0$

Substituting in (2), $0+2-2C_3=0 \Rightarrow 2C_3=2 \Rightarrow C_3=1$

$\therefore$ the particular solution is $\mathbf{y_n = 2n + (-2)^n}$ ■

Example 8

If x_k satisfies the difference equation
$x_{k+1} - 2x_k \cos\alpha + x_{k-1} = 0$, $k = 1, 2, \ldots$ and the conditions $x_0 = 0, x_1 = 1$, then show that $x_k = \dfrac{\sin k\alpha}{\sin\alpha}$ is a solution if $\sin\alpha \neq 0$, where α a is constant.

Solution

The given equation is $x_{k+1} - 2x_k \cos\alpha + x_{k-1} = 0, \quad k = 1,2,\ldots$
Rewrite the equation replacing k by $k + 1$

$$\therefore \quad x_{k+2} - 2x_{k+1}\cos\alpha + x_k = 0, \quad k = 0,1,2,3,..$$

$$\Rightarrow \quad E^2 x_k - 2\cos\alpha\, E x_k + x_k = 0$$

$$\Rightarrow \quad (E^2 - 2\cos\alpha\, E + 1)x_k = 0$$

Auxiliary equation is $m^2 - 2\cos\alpha . m + 1 = 0$

$$\Rightarrow \quad m = \frac{2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4}}{2}$$

$$= \frac{2\cos\alpha \pm \sqrt{-4(1-\cos^2\alpha)}}{2}$$

$$= \frac{2\cos\alpha \pm 2i\sin\alpha}{2}$$

$$= \cos\alpha \pm i\sin\alpha$$

The roots are complex with $'\alpha' = \cos\alpha, \quad '\beta' = \sin\alpha$

$$r^2 = \alpha^2 + \beta^2 = \cos^2\alpha + \sin^2\alpha = 1 \quad \Rightarrow r = 1$$

and $\theta = \tan^{-1}\dfrac{\sin\alpha}{\cos\alpha} = \tan^{-1}(\tan\alpha) = \alpha$

$\therefore$ the general solution is $x_k = (C_1\cos k\alpha + C_2\sin k\alpha)1^x$

$$\Rightarrow \quad x_k = C_1\cos k\alpha + C_2\sin k\alpha \qquad (1)$$

Given $x_0 = 0, \quad x_1 = 1$

Putting $k = 0$ and 1 in (1), we get

$$x_0 = C_1 \cos 0 + C_2 \sin 0 \qquad \Rightarrow 0 = C_1$$

and $$x_1 = C_1 \cos\alpha + C_2 \sin\alpha$$

$\Rightarrow$ $$1 = C_2 \sin\alpha$$

$\therefore$ $$C_2 = \frac{1}{\sin\alpha}, \quad \sin\alpha \neq 0$$

Particular solution is $$x_k = 0 \cdot \cos k\alpha + \frac{1}{\sin\alpha}\sin k\alpha$$

$\Rightarrow$ $$x_k = \frac{\sin k\alpha}{\sin\alpha}, \quad \sin\alpha \neq 0$$

Exercises 11.2

Solve the following linear homogenous difference equation.

(1) $y_{x+1} + 15y_x = 0$ (2) $y_{x+1} + ay_x = 0$

(3) $y_{x+1} - y_x = 0$ (4) $y_{x+1} - 2y_x = 0$

(5) $y_{x+2} - 2y_{x+1} + 4y_x = 0$ (6) $y_{x+2} - 2y_{x+1} + 2y_x = 0$

(7) $y_{x+2} - 6y_{x+1} + 6y_x = 0$ (8) $y_{x+3} - y_{x+2} - y_{x+1} - 2y_x = 0$

(9) $y_{k+2} - 6y_{k+1} + 25y_k = 0$ (10) $y_{n+3} + y_{n+2} - 8y_{n+1} - 12y_n = 0$

(11) $u_{x+3} - 3u_{x+1} - 2u_x = 0$ (12) $(E^3 + E^2 - 9E - 9)u_x = 0$

Answers 11.2

(1) $y_x = C(-15)^x$ (2) $y_x = C(-a)^x$

(3) $y_x = C.1^x \equiv C$ (4) $y_x = C.2^x$

(5) $y_x = 2^x\left[C_1 \cos\frac{x\pi}{3} + C_2 \sin\frac{x\pi}{3}\right]$ (6) $y_x = \left(\sqrt{2}\right)^x\left[C_1 \cos\frac{\pi x}{4} + C_2 \sin\frac{\pi x}{4}\right]$

(7) $y_x = (C_1 + C_2 x)3^x$ (8) $y_x = C_1 2^x + C_2 \cos\left(\frac{2\pi x}{3}\right) + C_3 \sin\left(\frac{2\pi x}{3}\right)$

(9) $y_k = 5^k\left[C_1 \cos k\theta + C_2 \sin k\theta\right]$, where $\theta = \tan^{-1}\left(-\frac{4}{3}\right)$

(10) $y_n = C_1 3^n + (C_2 + C_3 x)(-2)^n$ (11) $u_x = C_1 2^x + (C_2 + C_3 x)2^x$

(12) $y_x = C_1(-1)^x + C_2 3^x + C_3(-3)^x$

NON-HOMOGENEOUS LINEAR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

05-04-19

The general form of non-homogeneous linear difference equation with constant coefficients is

$$a_0 y_{x+k} + a_1 y_{x+k-1} + a_2 y_{x+k-2} + \ldots + a_{k-1} y_{x+1} + a_k y_x = f(x) \qquad (1)$$

where $a_0, a_1, a_2 \ldots, a_k$ are constants.

The equation (1) can be written as

$$aE^k y_x + a_1 E^{k-1} y_x + a_2 E^{k-2} y_x + \ldots + a_{k-1} E^k y_x + a_k y_x = f(x)$$

$$\Rightarrow \qquad \left[a_0 E^k + a_1 E^{k-1} + a_2 E^{k-2} + \ldots + a_{k-1} E + a_k\right] y_x = f(x) \qquad (2)$$

To fi nd the complementary function, solve

$$(a_0 E^k + a_1 E^{k-1} + a_2 E^{k-2} + \cdots + a_{k-1} E + a_k) y_x = 0 \qquad (3)$$

The general solution of (3) is called the complementary function of (1).

To find the complementary function

Refer 11.4, page

The particular solution is usually called particular integral and denoted as P.I.

$$\therefore \qquad \text{P.I} = \frac{1}{a_0 E^k + a_1 E^{k-1} + \ldots a_k} f(x)$$

$$= \frac{1}{\phi(E)} f(x)$$

The general solution of (1) is $y_x = \text{C.F} + \text{P.I.}$

Evaluation of Particular Integrals

1. Particular Integral – Type I:

If $f(x) = a^x$, then $\text{P.I} = \frac{1}{\phi(E)} a^x$

$$= \frac{a^x}{\phi(a)} \qquad \text{if } \phi(a) \neq 0$$

If $\phi(a) = 0$, then $(E-a)^r$ is a factor of $\phi(E)$.

$$\therefore \qquad \phi(E) = (E-a)^r g(E) \quad \text{where } g(a) \neq 0$$

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$$P.I = \frac{1}{(E-a)^r g(E)} a^x$$

$$\therefore \quad = \frac{1}{g(a)(E-a)^r} a^x \quad [\because g(a) \neq 0]$$

$$= \frac{1}{g(a)} x(x-1)(x-2)\ldots(x-\overline{r-1})a^{x-r}$$

[Refer 11.5 page,...]

WORKED EXAMPLES

Example 1

Solve $y_{x+1} - 5y_x = 2^x$.

Solution

The given equation is $\quad y_{x+1} - 5y_x = 2^x$

$$Ey_x - 5y_x = 2^x$$

$\Rightarrow$ $\quad (E-5)y_x = 2^x$

$\Rightarrow$ To find the complementary function, solve $(E-5)y_x = 0$

Auxiliary equation is $\quad m - 5 = 0 \quad \Rightarrow m = 5$

$$CF = C.5^x$$

$\therefore$
$$P.I = \frac{1}{E-5}2^x = \frac{2^x}{2-5} = -\frac{1}{3}2^x$$

$\therefore$ the general solution is $\quad y_x = C.F + P.I$

$\Rightarrow$
$$y_x = C.5^x + \left(-\frac{1}{3}\right)2^x$$

Example 2

Solve the difference equation $u_{x+1} - 4u_x = 5^x$.

Solution

The given equation is $\quad u_{x+1} - 4u_x = 5^x$

$\Rightarrow$ $\quad Eu_x - 4u_x = 5^x$

$\Rightarrow$ $\quad (E-4)u_x = 5^x$

To find the complimentary function solve $(E-4)u_x = 0$

Auxiliary equation is $\quad m - 4 = 0 \quad \Rightarrow m = 4$

$\therefore$ $\quad C.F = C.4^x$

$$P.I = \frac{1}{E-4}5^x = \frac{5^x}{5-4} = 5^x$$

$\therefore$ the general equation is $\quad u_x = C.F + P.I$

$\Rightarrow \quad$ $$u_x = C.4^x + 5^x$$

Example 3

Solve the difference equation $u_{x+1} + a u_x = b$, where a and b are constants.

Solution

The given equation is $u_{x+1} + a u_x = b$

$\Rightarrow \quad Eu_x + au_x = b$

$\Rightarrow \quad (E + a)u_x = b$

To find the complementary function, solve $(E + a)u_x = 0$

Auxiliary equation is $\quad m + a = 0 \quad \Rightarrow m = -a$

$\therefore \quad C.F = C\,(-a)^x$

$$P.I = \frac{1}{E+a}b$$

$$= b\frac{1}{E+a}1^x = \frac{b}{1+a}, \quad a \neq -1$$

$\therefore$ the general equation is $\quad u_x = C.F + P.I$

$\Rightarrow \quad$ $$u_x = C\,(-a)^x + \frac{b}{1+a}, \quad a \neq -1$$

Example 4

Solve $y_{x+2} - 2y_{x+1} + y_x = 2^x$, where $y_0 = 2$, $y_1 = 1$.

Solution

The given equation is $\quad y_{x+2} - 2y_{x+1} + y_x = 2^x$

$\Rightarrow \quad E^2y_x - 2Ey_x + y_x = 2^x$

$\Rightarrow \quad (E^2 - 2E + 1)y_x = 2^x$

To find the complementary function, solve $(E^2 - 2E + 1)y_x = 0$

Auxiliary equation is $m^2 - 2m + 1 = 0 \quad \Rightarrow (m-1)^2 = 0 \quad \Rightarrow m = 1, 1$

The roots are real and equal.

$$C.F = (C_1 + C_2x)1^x = C_1 + C_2x$$

$\therefore$

$$P.I = \frac{1}{E^2 - 2E + 1}2^x$$

$$= \frac{1}{(E-1)^2}2^x = \frac{2^x}{(2-1)^2} = 2^x$$

$\therefore$ the general solution is $\quad y_x = C.F + P.I$

$\Rightarrow \quad y_x = C_1 + C_2x + 2^x \quad (1)$

Given $\quad y_0 = 2,\ y_1 = 1$

Putting $x = 0$ and 1 in (1), we get $\quad y_0 = C_1 + C_2.0 + 2^0$

$\Rightarrow \quad 2 = C_1 + 1 \Rightarrow C_1 = 1$

and $\quad y_1 = C_1 + C_2.1 + 2$

$\Rightarrow \quad 1 = C_1 + C_2 + 2 \quad \Rightarrow C_2 = 1 - 2 - 1 = -2$

$\therefore$ the solution is $\quad \mathbf{y_x = 1 - 2x + 2^x}$ ■

Example 5

Solve the difference equation $y_{x+2} - 5y_{x+1} + 6y_x = 4^x$ where $y_0 = 0,\ y_1 = 1$.

Solution

The given equation is $y_{x+2} - 5y_{x+1} + 6y_x = 4^x$

$\Rightarrow \quad E^2y_x - 5Ey_x + 6y_x = 4^x$

$\Rightarrow \quad (E^2 - 5E + 6)y_x = 4^x$

To find the complementary function, solve $(E^2 - 5E + 6)y_x = 0$

Auxiliary equation is $\quad m^2 - 5m + 6 = 0$

$\Rightarrow \quad (m-3)(m-2) = 0 \quad \Rightarrow m = 3,\ 2$

$\therefore \quad C.F = C_1 2^x + C_2 3^x$

$$P.I = \frac{1}{E^2 - 5E + 6}4^x$$

$$= \frac{4^x}{4^2 - 5\times 4 + 6} = \frac{4^x}{16 - 20 + 6} = \frac{4^x}{2}$$

$\therefore$ the general solution is $\quad y_x = C.F + P.I$

$$\Rightarrow \quad y_x = C_1 2^x + C_2 3^x + \frac{4^x}{2} \qquad (1)$$

Given $\quad y_0 = 0,\ y_1 = 1$

Putting $x = 0$ and 1 in (1), we get $y_0 = C_1 2^0 + C_2 3^0 + \frac{4^0}{2}$

$$0 = C_1 + C_2 + \frac{1}{2}$$

$$\Rightarrow \quad 2C_1 + 2C_2 = -1 \qquad (2)$$

and $\quad y_1 = C_1.2 + C_2.3 + \frac{4}{2}$

$$\Rightarrow \quad 1 = 2C_1 + 3C_2 + 2$$

$$\Rightarrow \quad 2C_1 + 3C_2 = -1 \qquad (3)$$

$$(3)-(2) \Rightarrow \quad C_2 = 0$$

$$\therefore \quad 2C_1 = -1 \Rightarrow C_1 = -\frac{1}{2}$$

$\therefore$ the solution is $\quad y_x = -\frac{1}{2}.2^x + \frac{4^x}{2} = \frac{1}{2}\left[4^x - 2^x\right]$

Example 6

Solve the difference equation $y_{x+2} - 8y_{x+1} + 16y_x = 4^x$.

Solution

the given equation is

$$y_{x+2} - 8y_{x+1} + 16y_x = 4^x$$

$$\Rightarrow \quad E^2 y_x - 8Ey_x + 16y_x = 4^x$$

$$\Rightarrow \quad (E^2 - 8E + 16)y_x = 4^x$$

$$\Rightarrow \quad (E-4)^2 y_x = 4^x$$

To find the complementary function, solve $(E-4)^2 y_x = 0$

Auxiliary equation is $\quad (m-4)^2 = 0 \quad \Rightarrow m = 4, 4.$

The roots are real and equal.

$$\frac{1}{(E-4)^2}4^x$$

$$C.F = (C_1 + C_2x)4^x$$

$$\frac{x}{2(E-4)}4^{x-1}$$

$$P.I = \frac{1}{(E-4)^2}4^x$$

$$\frac{x(x-1)}{2}4^{x-2}$$

$$= \frac{x(x-1)}{2!}4^{x-2} \qquad [\because \phi(E) = 0, \text{ Refer 11.7.1 page,...}]$$

$\therefore$ the general solution is

$\Rightarrow$ $$y_x = C.F + P.I$$

$\Rightarrow$ $$y_x = (C_1 + C_2x)4^x + \frac{x(x-1)}{2!}4^{x-2}$$

$\Rightarrow$ $$y_x = (C_1 + C_2x)4^x + \frac{x(x-1)}{2 \cdot 4^2}4^x$$

$$y_x = \left[C_1 + C_2x + \frac{x(x-1)}{32}\right]4^x$$ ■

Example 7

Solve the difference equation $u_{x+2} - 2u_{x+1} + u_x = 7$.

Solution

the given equation is $$u_{x+2} - 2u_{x+1} + u_x = 7.$$

$\Rightarrow$ $$E^2u_x - 2Eu_x + u_x = 7$$

$\Rightarrow$ $$(E^2 - 2E + 1)u_x = 7$$

$\Rightarrow$ $$(E-1)^2u_x = 7$$

To find the complementary function, solve $(E-1)^2u_x = 0$

Auxiliary equation is $(m-1)^2 = 0 \quad \Rightarrow m = 1, 1$

The roots are real and equal

$\therefore$ $$C.F = (C_1 + C_2x)1^x = C_1 + C_2x$$

$$P.E = \frac{1}{(E-1)^2}7$$

$$= 7\frac{1}{(E-1)^2}1^x$$

$$= 7 \cdot \frac{x(x-1)}{2!}1^{x-2} \qquad [\because \phi(1) = 0]$$

$$= \frac{7}{2}x(x-1)$$

$\therefore$ the general solution is $\quad u_x = C.F + P.I$

$$\Rightarrow \quad \boldsymbol{u_x = C_1 + C_2 x + \frac{7}{2}x(x-1)}$$ ■

Example 8

Solve the equation $u_{x+3} - 5u_{x+2} + 8u_{x+1} - 4u_x = 3.2^x$.

Solution

The given equation $\quad u_{x+3} - 5u_{x+2} + 8u_{x+1} - 4u_x = 3 \cdot 2^x$

$$\Rightarrow \quad E^3u_x - 5E^2u_x + 8Eu_x - 4u_x = 3 \cdot 2^x$$

$$\Rightarrow \quad \left(E^3 - 5E^2 + 8E - 4\right)u_x = 3 \cdot 2^x$$

To find the complementary function, solve $\left(E^3 - 5E^2 + 8E - 4\right)u_x = 0$

Auxiliary equation is

$$m^3 - 5m^2 + 8m - 4 = 0$$

Since sum of the coefficients is zero, $m = 1$ is a root.
The other roots are given by

$$m^2 - 4m + 4 = 0$$

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 8 & -4 \\ & 0 & 1 & -4 & 4 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$$\Rightarrow \quad (m-2)^2 = 0$$

$$\Rightarrow \quad m = 2,\ 2$$

$\therefore$ the roots are $m_1 = 1$, $m_2 = 2$, $m_3 = 2$ with two equal roots.

$$\therefore \quad C.F = C_1 1^x + \left(C_2 + C_3 x\right)2^x$$

$$= C_1 + \left(C_2 + C_3 x\right)2^x$$

$$P.I = \frac{1}{E^3 - 5E^2 + 8E - 4}3 \cdot 2^x$$

$$= \frac{1}{(E-1)(E-2)^2}3 \cdot 2^x$$

$$= 3\frac{1}{(2-1)(E-2)^2}2^x$$

$$= 3.\frac{1}{(E-2)^2}2^x \qquad [\because \phi(2) = 0]$$

$$= 3\frac{x(x-1)}{2!}2^{x-2}$$

$$= \frac{3}{8}x(x-1)2^x.$$

$\therefore$ the general solution is $u_x = \text{C.F} + \text{P.I}$

$$= C_1 + (C_2 + C_3 x)2^x + \frac{3}{8}x(x-1)2^x$$

$$\Rightarrow \quad u_x = C_1 + \left[C_2 + C_3 x + \frac{3}{8}x(x-1)\right]2^x$$

2. Particular Integral – Type 2:

If $f(x) = x^r$, where r is a non – negative integer, then

$$\text{P.I} = \frac{1}{\phi(E)} f(x)$$

$$= \frac{1}{\phi(E)} x^r$$

$$= \frac{1}{\phi(1+\Delta)} x^r$$

$$= [\phi(1+\Delta)]^{-1} x^r$$

Expanding as a polynomial in Δ,

$$\text{P.I} = \left(b_0 + b_1\Delta + b_2\Delta^2 + \ldots + b_r\Delta^r\right)x^r$$

WORKED EXAMPLES

Example 9

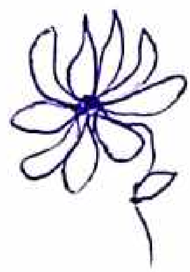
Solve the difference equation $3y_{x+1} - y_x = x$.

Solution

The given equation is $3y_{x+1} - y_x = x$

$$\Rightarrow \quad 3Ey_x - y_x = x$$

$$\Rightarrow \quad (3E-1)y_x = x$$



To find the complementary function, solve $(3E-1)y_x = 0$

Auxiliary equation is $3m - 1 = 0 \quad \Rightarrow m = \frac{1}{3}$

$$\therefore \quad \text{C.F} = C\left(\frac{1}{3}\right)^x$$

$$\text{P.I} = \frac{1}{3E-1}x$$

$$= \frac{1}{3(1+\Delta)-1}x \qquad \because E = 1+\Delta$$

$$= \frac{1}{2+3\Delta}x \qquad (1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots$$

$$= \frac{1}{2\left(1+\frac{3\Delta}{2}\right)}x$$

$$= \frac{1}{2}\left(1+\frac{3\Delta}{2}\right)^{-1}x$$

$$= \frac{1}{2}\left[1-\frac{3\Delta}{2}+\frac{3\Delta^2}{4}+\cdots\right]x$$

$$= \frac{1}{2}\left[1-\frac{3\Delta}{2}\right]x$$

$$= \frac{1}{2}\left[x-\frac{3}{2}\Delta x\right]$$

$$= \frac{1}{2}\left[x-\frac{1}{2}(x+1-x)\right] = \frac{1}{2}\left[x-\frac{1}{2}\right] = \frac{2x-1}{4}$$

$\therefore$ the general solution is $\quad y_x = \text{C.F} + \text{P.I}$

$\Rightarrow \quad \boldsymbol{y_x = C\cdot\left(\frac{1}{3}\right)^x + \frac{1}{4}(2x-1)}$ ■

Example 10

Solve the equation $y_{x+1} - 2y_x = x^2$.

Solution

The given equation is $\quad y_{x+1} - 2y_x = x^2$

$\Rightarrow \quad Ey_x - 2y_x = x^2$

$\Rightarrow \quad (E-2)y_x = x^2$

To find the complementary function, solve $(E-2)y_x = 0$

Auxiliary equation is $\quad m-2=0 \Rightarrow m=2$

$\therefore \quad \text{C.F} = C\cdot 2^x$

$$\text{P.I} = \frac{1}{E-2}x^2$$

$$= \frac{1}{1+\Delta-2}x^2$$

$$= -\frac{1}{(1-\Delta)}x^2$$

$$= -(1-\Delta)^{-1}x^2$$

$$= -\left(1+\Delta+\Delta^2+\cdots\right)x^2$$

$$= -\left(1+\Delta+\Delta^2\right)x^2 \qquad [\because \Delta^3 x^2 = 0]$$

$$= -\left[x^2+\Delta x^2+\Delta^2 x^2\right]$$

Now $\Delta x^2 = (x+1)^2 - x^2 = x^2+2x+1-x^2 = 2x+1$

and $\Delta^2 x^2 = \Delta(\Delta x^2) = \Delta(2x+1) = \Delta(2x)+\Delta(1) = 2\Delta x + 0 = 2[x+1-x] = 2$

$\therefore$ $$\text{P.I} = -\left[x^2+2x+1+2\right] = -\left[x^2+2x+3\right]$$

$\therefore$ the general solution is $y_x = \text{C.F} + \text{P.I}$

$\Rightarrow$ $$\boldsymbol{y_x = C\ 2^x - \left(x^2+2x+3\right).}$$ ■

Example 11

Solve the difference equation $y_{x+2} - 4y_x = 9x^2$.

Solution

The given equation is $$y_{x+2} - 4y_x = 9x^2$$

$\Rightarrow$ $$E^2 y_x - 4y_x = 9x^2$$

$\Rightarrow$ $$\left(E^2-4\right)y_x = 9x^2$$

To find the complementary function, solve $\left(E^2-4\right)y_x = 0$

Auxiliary equation is

$$m^2 - 4 = 0 \Rightarrow m = \pm 2.$$

$\therefore$ $$\text{C.F} = C_1 2^x + C_2(-2)^x.$$

$$\text{P.I} = \frac{1}{E^2-4}9x^2$$

$$= 9\cdot\frac{1}{(1+\Delta)^2-4}x^2$$

$$= 9 \cdot \frac{1}{1 + 2\Delta + \Delta^2 - 4} x^2$$

$$= 9 \cdot \frac{1}{-3 + 2\Delta + \Delta^2} x^2$$

$$= -\frac{9}{3} \frac{1}{\left[1 - \left(\frac{2}{3}\Delta + \frac{1}{3}\Delta^2\right)\right]} x^2$$

$2\left(\frac{2}{3}\Delta\right)\left(\frac{1}{3}\Delta^2\right)$

$\frac{4}{9}$

$$= -3\left[1 - \left(\tfrac{2}{3}\Delta + \tfrac{1}{3}\Delta^2\right)\right]^{-1} x^2$$

$$= -3\left[1 + \left(\tfrac{2}{3}\Delta + \tfrac{1}{3}\Delta^2\right) + \left(\tfrac{2}{3}\Delta + \tfrac{1}{3}\Delta^2\right)^2 + \cdots\right] x^2$$

$$= -3\left[1 + \tfrac{2}{3}\Delta + \tfrac{1}{3}\Delta^2 + \tfrac{4}{9}\Delta^2\right] x^2$$

$$= -3\left[1 + \tfrac{2}{3}\Delta + \tfrac{7}{9}\Delta^2\right] x^2 \qquad \left[\because \Delta^3 x^2 = 0\right]$$

$$= -3\left[x^2 + \tfrac{2}{3}\Delta x^2 + \tfrac{7}{9}\Delta^2 x^2\right]$$

Now $\quad \Delta x^2 = (x+1)^2 - x^2 = x^2 + 2x + 1 - x^2 = 2x + 1$

and $\quad \Delta^2 x^2 = \Delta\left(\Delta x^2\right) \quad = \Delta(2x+1) = \Delta(2x) + \Delta(1) = 2\Delta x + 0 = 2[x + 1 - x] = 2$

$\because$
$$\text{P.I} = -3\left[x^2 + \frac{2}{3}(2x+1) + \frac{7}{9} \times 2\right]$$

$$= -3\left[x^2 + \frac{2}{3}(2x+1) + \frac{14}{9}\right]$$

$$= -\frac{1}{3}\left[9x^2 + 6(2x+1) + 14\right] = -\frac{1}{3}\left[9x^2 + 12x + 20\right]$$

$\therefore$ the general solution is $y_x = \text{C.F} + \text{P.I}$

$\Rightarrow$
$$\mathbf{y_x = C_1 2^x + C_2(-2)^x - \frac{1}{3}[9x^2 + 12x + 20]}$$ ■

Example 12

Solve the equation $y_{x+2} - 3y_{x+1} + 2y_x = x^2 + x$.

Solution

The given equation is $\quad y_{x+2} - 3y_{x+1} + 2y_x = x^2 + x$

$\Rightarrow \quad E^2 y_x - 3E y_x + 2y_x = x^2 + x$

$\Rightarrow \quad \left(E^2 - 3E + 2\right) y_x = x^2 + x$

To find the complementary function solve $(E^2 - 3E + 2)y_x = 0$

Auxiliary equation is $m^2 - 3m + 2 = 0$

$\Rightarrow$ $$(m-2)(m-1) = 0 \quad \Rightarrow m = 1, 2$$

$\therefore$ $$C.F = C_1 \cdot 1^x + C_2 \cdot 2^x = C_1 + C_2 2^x$$

$$P.I = \frac{1}{E^2 - 3E + 2}(x^2 + x)$$

$$= \frac{1}{(E-1)(E-2)}(x^2 + x)$$

$$= \frac{1}{(1+\Delta-1)(1+\Delta-2)}(x^2 + x)$$

$$= -\frac{1}{\Delta(1-\Delta)}(x^2 + x)$$

$$= -\frac{1}{\Delta}(1-\Delta)^{-1}(x^2 + x)$$

$$= -\frac{1}{\Delta}[1 + \Delta + \Delta^2 + \cdots][x^2 + x]$$

$$= -\frac{1}{\Delta}[x^2 + x + \Delta(x^2 + x) + \Delta^2(x^2 + x)] \quad [\because \Delta^3(x^2 + x) = 0]$$

Now $$\Delta(x^2 + x) = \Delta(x^2) + \Delta x = (x+1)^2 - x^2 + (x + 1 - x) = x^2 + 2x + 1 - x^2 + 1 = 2x + 2$$

and $$\Delta^2(x^2 + x) = \Delta(\Delta(x^2 + x)) = \Delta(2x + 2) = 2\Delta x + \Delta(2) = 2(x + 1 - x) + 0 = 2$$

$\therefore$ $$P.I = -\frac{1}{\Delta}[x^2 + x + 2x + 2 + 2] = -\frac{1}{\Delta}[x^2 + 3x + 4]$$

We shall express $x^2 + 3x + 4$ in factorial polynomial form.

$$x^2 + 3x + 4 = x(x-1) + 4x + 4 = [x]^2 + 4[x] + 4[x]^0$$

We know that $$\frac{1}{\Delta}[x]^r = \frac{[x]^{r+1}}{r+1}$$

$\therefore$ $$\frac{1}{\Delta}[x]^2 = \frac{[x]^{2+1}}{2+1} = \frac{[x]^3}{3}$$

$$\frac{1}{\Delta}[x] = \frac{[x]^2}{2} \text{ and } \frac{1}{\Delta}[x]^0 = [x]$$

$\therefore$ $$P.I = -\frac{1}{\Delta}\{[x]^2 + 4[x] + 4[x]^0\}$$

$$= -\left\{\frac{1}{\Delta}[x]^2 + 4\frac{1}{\Delta}[x] + 4\frac{1}{\Delta}[x]^0\right\}$$

$$= -\left[\frac{[x]^3}{3} + 4\frac{[x]^2}{2} + 4[x]\right]$$

$$= -\left[\frac{1}{3}x(x-1)(x-2) + 2x(x-1) + 4x\right]$$

$$= -\frac{1}{3}\left[x^3 - 3x^2 + 2x + 6x^2 - 6x + 12x\right]$$

$$= -\frac{1}{3}\left[x^3 + 3x^2 + 8x\right] = -\frac{x}{3}\left[x^2 + 3x + 8\right]$$

$\therefore$ the general solution is

$$y_x = C.F + P.I$$

$$\Rightarrow \quad \boldsymbol{y_x = C_1 + C_2 2^x - \frac{x}{3}(x^2 + 3x + 8)}$$

■

Example 13

Find the general solution of the difference equation $y_{x+2} - y_{x+1} + y_x = x^2$.

Solution

The given equation is $$y_{x+2} - y_{x+1} + y_x = x^2$$

$$\Rightarrow \quad E^2 y_x - E y_x + y_x = x^2$$

$$\Rightarrow \quad \left(E^2 - E + 1\right) y_x = x^2$$

To find the complimentary function, solve $\left(E^2 - E + 1\right) y_x = 0$

Auxiliary equation is $$m^2 - m + 1 = 0$$

$$\Rightarrow \quad m = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

The roots are complex with $\alpha = \frac{1}{2}, \beta = \frac{\sqrt{3}}{2}$

$$\therefore \quad r = \sqrt{\alpha^2 + \beta^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

and $$\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$$

$$\therefore \quad C.F = \left(C_1 \cos\frac{\pi}{3}x + C_2 \sin\frac{\pi}{3}x\right)1^x$$

$$= \left(C_1 \cos\frac{\pi}{3}x + C_2 \sin\frac{\pi}{3}x\right)$$

$$\text{P.I} = \frac{1}{E^2 - E + 1}x^2$$

$$= \frac{1}{(1+\Delta)^2 - (1+\Delta) + 1}x^2$$

$$= \frac{1}{1 + 2\Delta + \Delta^2 - 1 - \Delta + 1}x^2$$

$$= \frac{1}{1 + \Delta + \Delta^2}x^2$$

$$= \frac{1}{1 + (\Delta + \Delta^2)}x^2$$

$$= \left[1 + (\Delta + \Delta^2)\right]^{-1}x^2$$

$$= \left(1 - (\Delta + \Delta^2) + (\Delta + \Delta^2)^2 + \dots\right)x^2$$

$$= \left(1 - \Delta - \Delta^2 + \Delta^2\right)x^2 \qquad [\because \Delta^3 x^2 = 0]$$

$$= (1 - \Delta)x^2$$

$$= x^2 - \Delta x^2$$

$$= x^2 - \left[(x+1)^2 - x^2\right]$$

$$= x^2 - \left[x^2 + 2x + 1 - x^2\right] = x^2 - 2x - 1$$

$\therefore$ the general solution is $\quad y_x = \text{C.F} + \text{P.I}$

$$\Rightarrow \quad y_x = C_1 \cos\frac{\pi}{3}x + C_2 \sin\frac{\pi}{3}x + x^2 - 2x - 1$$ ■

07-05-19. Tuesday.

3. Particular Integral – Type 3:

If $f(x) = a^x g(x)$, where $g(x)$ is a polynomial in x, then

$$\text{P.I} = \frac{1}{\phi(E)}a^x g(x).$$

$$= a^x \frac{1}{\phi(aE)}g(x)$$

WORKED EXAMPLES

Example 14 ✓

Solve the difference equation $y_{x+1} - ay_x = (2x+1)a^x$.

Solution

The given equation is $\quad y_{x+1} - ay_x = (2x+1)a^x$

$\Rightarrow \quad Ey_x - ay_x = (2x+1)a^x$

$\Rightarrow \quad (E-a)y_x = (2x+1)a^x$

To find the complementary function, solve $(E-a)y_x = 0$

Auxiliary equation is $\quad m - a = 0 \quad \Rightarrow m = a$

$\therefore \quad \text{C.F} = C\ a^x$

$$\text{P.I} = \frac{1}{E-a}(2x+1)a^x$$

$$= a^x \frac{1}{aE-a}(2x+1)$$

$$= \frac{a^x}{a} \cdot \frac{1}{E-1}(2x+1)$$

$$= a^{x-1}\left[\frac{1}{1+\Delta-1}\right](2x+1)$$

$$= a^{x-1}\frac{1}{\Delta}(2[x]+1)$$

by integration $\Rightarrow a^{x-1}[x^2+x]$

$$= a^{x-1}\Delta^{-1}[2[x]+1]$$

$$= a^{x-1}\left[2\Delta^{-1}[x]+\Delta^{-1}(1)\right]$$

$y_x = y_c + y_p$

$$= a^{x-1}\left[2\frac{[x]^2}{2}+x\right]$$

$= Ca^x + a^{x-1}(x^2+x)$

$$= a^{x-1}\left[[x]^2+x\right]$$

$$= a^{x-1}\left[x(x-1)+x\right] = x^2a^{x-1}$$

$\therefore$ the general solution is $\quad y_x = \text{C.F} + \text{P.I}$

$$\boldsymbol{y_x = C.a^x + x^2a^{x-1}}$$

Example 15

Solve the difference equation $y_{x+1} - 2y_x = 4^x(x^2 - 2x)$.

Solution

The given equation is $y_{x+1} - 2y_x = 4^x(x^2 - 2x)$

$$\Rightarrow \quad Ey_x - 2y_x = 4^x(x^2 - 2x)$$

$$\Rightarrow \quad (E-2)y_x = 4^x(x^2 - 2x)$$

To find the complementary function, solve $(E-2)y_x = 0$

Auxiliary equation is $m - 2 = 0 \quad \Rightarrow m = 2$

$$\therefore \quad C.F = C.2^x$$

$$P.I = \frac{1}{E-2}4^x(x^2-2x)$$

$$= 4^x \frac{1}{4E-2}(x^2-2x)$$

$$= \frac{4^x}{2}\frac{1}{(2E-1)}(x^2-2x)$$

$$= \frac{4^x}{2}\frac{1}{[2(1+\Delta)-1]}(x^2-2x)$$

$$= \frac{4^x}{2}\cdot\frac{1}{[1+2\Delta]}(x^2-2x)$$

$$= \frac{4^x}{2}(1+2\Delta)^{-1}(x^2-2x)$$

$$= \frac{4^x}{2}[1-2\Delta+4\Delta^2+\ldots][x^2-2x]$$

$$= \frac{4^x}{2}[1-2\Delta+4\Delta^2][x^2-2x] \qquad [\because \Delta^3x^2 = 0 \ \Delta^3 x = 0]$$

$$= \frac{4^x}{2}[x^2-2x-2\Delta(x^2-2x)+4\Delta^2(x^2-2x)]$$

Now $\Delta(x^2-2x) = \Delta x^2 - 2\Delta x = (x+1)^2 - x^2 - 2(x+1-x) = x^2+2x+1-x^2-2 = 2x-1$

and $\Delta^2(x^2-2x) = \Delta[\Delta(x^2-2x)] = \Delta(2x-1) = 2\Delta x + \Delta(-1) = 2[x+1-x]+0 = 2$

$$\therefore \quad P.I = \frac{4^x}{2}[x^2-2x-2(2x-1)+4\cdot 2] = \frac{4^x}{2}[x^2-6x+10]$$

$\therefore$ the general solution is $\quad y_x = \text{C.F} + \text{P.I}$

$$\Rightarrow \quad y_x = C\,2^x + \frac{4^x}{2}\left[x^2 - 6x + 10\right]$$

■

Example 16 ✓

Solve the difference equation $y_{x+2} + y_{x+1} - 56y_x = 2^x\left(x^2 - 2\right)$.

Solution

The given equation is $\quad y_{x+2} + y_{x+1} - 56y_x = 2^x\left(x^2 - 2\right)$

$$\Rightarrow \quad E^2y_x + Ey_x - 56y_x = 2^x\left(x^2 - 2\right)$$

$$\Rightarrow \quad \left(E^2 + E - 56\right)y_x = 2^x\left(x^2 - 2\right)$$

To find the complementary function, solve $\left(E^2 + E - 56\right)y_x = 0$

Auxiliary equation is $\quad m^2 + m - 56 = 0$

$$\Rightarrow \quad (m+8)(m-7) = 0 \quad \Rightarrow \quad m = -8, 7$$

$$\therefore \quad \text{C.F} = C_1(-8)^x + C_2 \cdot 7^x.$$

$$\text{P.I} = \frac{1}{E^2 + E - 56}2^x\left(x^2 - 2\right)$$

$$= 2^x\frac{1}{(2E)^2 + 2E - 56}\left(x^2 - 2\right)$$

$$= 2^x\frac{1}{4E^2 + 2E - 56}\left(x^2 - 2\right)$$

$$= 2^x\frac{1}{2\left(2E^2 + E - 28\right)}\left(x^2 - 2\right)$$

$$= 2^{x-1}\frac{1}{2(1+\Delta)^2 + 1 + \Delta - 28}\left(x^2 - 2\right)$$

$$= 2^{x-1}\frac{1}{2\left(1 + 2\Delta + \Delta^2\right) + 1 + \Delta - 28}\left(x^2 - 2\right)$$

$$= 2^{x-1}\frac{1}{-25 + 5\Delta + 2\Delta^2}\left(x^2 - 2\right)$$

$$= -\frac{2^{x-1}}{25}\cdot\frac{1}{\left[1-\left(\frac{1}{5}\Delta+\frac{2}{25}\Delta^2\right)\right]}(x^2-2)$$

$$= -\frac{2^{x-1}}{25}\left[1-\left(\frac{1}{5}\Delta+\frac{2}{25}\Delta^2\right)\right]^{-1}(x^2-2)$$

$$= -\frac{2^{x-1}}{25}\left[1+\left(\frac{1}{5}\Delta+\frac{2}{25}\Delta^2\right)+\left(\frac{1}{5}\Delta+\frac{2}{25}\Delta^2\right)^2+\dots\right]\left[x^2-2\right]$$

$$= -\frac{2^{x-1}}{25}\left[1+\frac{1}{5}\Delta+\frac{2}{25}\Delta^2+\frac{1}{25}\Delta^2\right]\left[x^2-2\right] \qquad \left[\because \Delta^3x^2=0\right]$$

$$= -\frac{2^{x-1}}{25}\left[1+\frac{1}{5}\Delta+\frac{3}{25}\Delta^2\right]\left[x^2-2\right]$$

$$= -\frac{2^{x-1}}{25}\left[x^2-2+\frac{1}{5}\Delta(x^2-2)+\frac{3}{25}\Delta^2(x^2-2)\right]$$

Now $\quad \Delta(x^2-2)=\Delta(x^2)-\Delta(2)=(x+1)^2-x^2-0=x^2+2x+1-x^2=2x+1$

and $\quad \Delta^2(x^2-2)=\Delta\left[\Delta(x^2-2)\right]=\Delta[2x+1]=2\Delta(x)+\Delta(1)=2(x+1-x)+0=2.$

$$\therefore \qquad \text{P.I} = -\frac{2^{x-1}}{25}\left[x^2-2+\frac{1}{5}(2x+1)+\frac{3}{25}\times 2\right]$$

$$= -\frac{2^{x-1}}{25}\left[x^2+\frac{2}{5}x-2+\frac{1}{5}+\frac{6}{25}\right]$$

$$= -\frac{2^{x-1}}{25}\left[x^2+\frac{2}{5}x-\frac{39}{25}\right] = -\frac{2^{x-1}}{625}\left[25x^2+10x-39\right]$$

$\therefore$ the general solution is $y_x = \text{C.F} + \text{P.I}$

$$\Rightarrow \qquad \boldsymbol{y_x = C_1(-8)^x + C_2 7^x - \frac{2^{x-1}}{625}\left[25x^2+10x-39\right]} \qquad \blacksquare$$

Example 17

Solve the equation $y_{x+2}-7y_{x+1}-8y_x=2^x.x^2$.

Solution

The given equation is $\qquad y_{x+2}-7y_{x+1}-8y_x=2^x x^2$

$$\Rightarrow \qquad E^2y_x-7Ey_x-8y_x=2^x x^2$$

$$\Rightarrow \qquad (E^2-7E-8)y_x=2^x.x^2$$

To find the complementary function, solve $(E^2-7E-8)y_x=0$

Auxiliary equation is $m^2-7m-8=0$

$\Rightarrow \quad (m-8)(m+1)=0 \quad \Rightarrow m=-1,\ 8$

$\therefore \quad \text{C.F} = C_1(-1)^x + C_2\cdot 8^x$

$$\text{P.I} = \frac{1}{E^2-7E-8}2^x x^2$$

$$= 2^x \frac{1}{(2E)^2-7(2E)-8}x^2$$

$$= 2^x \frac{1}{4E^2-14E-8}x^2$$

$$= \frac{2^x}{2}\cdot\frac{1}{[2E^2-7E-4]}x^2$$

$$= 2^{x-1}\frac{1}{2(1+\Delta)^2-7(1+\Delta)-4}x^2$$

$$= 2^{x-1}\frac{1}{2(1+2\Delta+\Delta^2)-7-7\Delta-4}x^2$$

$$= 2^{x-1}\frac{1}{[-9-3\Delta+2\Delta^2]}x^2$$

$$= -\frac{2^{x-1}}{9}\frac{1}{\left[1+\left(\frac{1}{3}\Delta-\frac{2}{9}\Delta^2\right)\right]}x^2$$

$$= -\frac{2^{x-1}}{9}\left[1+\left(\frac{1}{3}\Delta-\frac{2}{9}\Delta^2\right)\right]^{-1}x^2$$

$$= -\frac{2^{x-1}}{9}\left[1-\left(\frac{1}{3}\Delta-\frac{2}{9}\Delta^2\right)+\left(\frac{1}{3}\Delta-\frac{2}{9}\Delta^2\right)^2+\dots\right]x^2$$

$$= -\frac{2^{x-1}}{9}\left[1-\frac{1}{3}\Delta+\frac{2}{9}\Delta^2+\frac{1}{9}\Delta^2\right]x^2 \qquad [\because \Delta^3x^2=0$$

$$= -\frac{2^{x-1}}{9}\left[1-\frac{1}{3}\Delta+\frac{1}{3}\Delta^2\right]x^2$$

$$= -\frac{2^{x-1}}{9}\left[x^2-\frac{1}{3}\Delta x^2+\frac{1}{3}\Delta^2x^2\right]$$

Now $$\Delta x^2 = (x+1)^2 - x^2 = x^2 + 2x + 1 - x^2 = 2x + 1$$

and $$\Delta^2(x^2) = \Delta\left[\Delta(x^2)\right] = \Delta(2x+1) = 2\Delta x + \Delta(1) = 2(x+1-x) + 0 = 2$$

$\therefore$ $$P.I = -\frac{2^{x-1}}{9}\left[x^2 - \frac{1}{3}(2x+1) + \frac{1}{3}\times 2\right]$$

$$= -\frac{2^{x-1}}{27}\left[3x^2 - 2x - 1 + 2\right] = -\frac{2^{x-1}}{27}(3x^2 - 2x + 1)$$

$\therefore$ the general solution is $y_x = C.F + P.I$

$\Rightarrow$ $$y_x = C_1(-1)^x + C_2 8^x - \frac{2^{x-1}}{27}(3x^2 - 2x + 1)$$ ■

Example 18

Solve the difference equation $y_{x+2} - 5y_{x+1} + 6y_x = 4^x(x^2 - x + 5)$.

Solution

The given equation is $$y_{x+2} - 5y_{x+1} + 6y_x = 4^x\left(x^2 - x + 5\right)$$

$\Rightarrow$ $$E^2 y_x - 5Ey_x + 6y_x = 4^x\left(x^2 - x + 5\right)$$

$\Rightarrow$ $$\left(E^2 - 5E + 6\right)y_x = 4^x\left(x^2 - x + 5\right)$$

To find the complementary function, solve $\left(E^2 - 5E + 6\right)y_x = 0$

Auxiliary equation is $m^2 - 5m + 6 = 0$

$\Rightarrow$ $(m-2)(m-3) = 0$ $\Rightarrow m = 2, 3$

$\therefore$ $$C.F = C_1.2^x + C_2\cdot 3^x$$

$$P.I = \frac{1}{E^2 - 5E + 6}4^x\left(x^2 - x + 5\right)$$

$$= 4^x\frac{1}{(4E)^2 - 5(4E) + 6}\left(x^2 - x + 5\right)$$

$$= 4^x\frac{1}{16E^2 - 20E + 6}\left(x^2 - x + 5\right)$$

$$= \frac{4^x}{2}\cdot\frac{1}{(8E^2 - 10E + 3)}\left(x^2 - x + 5\right)$$

$$= \frac{4^x}{2}\cdot\frac{1}{(8(1+\Delta)^2 - 10(1+\Delta) + 3)}\left(x^2 - x + 5\right)$$

$$= \frac{4^x}{2} \cdot \frac{1}{\left[8\left(1+2\Delta+\Delta^2\right)-10-10\Delta+3\right]}\left(x^2-x+5\right)$$

$$= \frac{4^x}{2} \cdot \frac{1}{\left[1+6\Delta+8\Delta^2\right]}\left(x^2-x+5\right)$$

$$= \frac{4^x}{2}\left[1+(6\Delta+8\Delta^2)\right]^{-1}\left(x^2-x+5\right)$$

$$= \frac{4^x}{2}\left[1-\left(6\Delta+8\Delta^2\right)+\left(6\Delta+8\Delta^2\right)^2-\ldots\right]\left(x^2-x+5\right)$$

$$= \frac{4^x}{2}\left[1-6\Delta-8\Delta^2+36\Delta^2\right]\left(x^2-x+5\right)$$

$$= \frac{4^x}{2}\left[1-6\Delta+28\Delta^2\right]\left(x^2-x+5\right)$$

$$= \frac{4^x}{2}\left[x^2-x+5-6\Delta\left(x^2-x+5\right)+28\Delta^2\left(x^2-x+5\right)\right]$$

Now $\Delta(x^2-x+5) = \Delta x^2 - \Delta x + \Delta(5)$

$$= (x+1)^2 - x^2 - (x+1-x) + 0 = x^2+2x+1-x^2-1 = 2x$$

and $\Delta^2\left(x^2-x+5\right) = \Delta\left[\Delta\left(x^2-x+5\right)\right] = \Delta(2x) = 2[x+1-x] = 2$

$\therefore$ $$\text{P.I} = \frac{4^x}{2}\left[x^2-x+5-12x+28\times 2\right] = \frac{4^x}{2}\left[x^2-13x+61\right]$$

$\therefore$ the general solution is $y_x = \text{C.F} + \text{P.I}$

$$\Rightarrow \quad y_x = C_1 2^x + C_2 3^x + \frac{4^x}{2}(x^2-13x+61)$$

4. Particular Integral – Type 4:

If $f(x) = \cos\alpha x$ or $\sin\alpha x$

then $$\text{P.I} = \frac{1}{\phi(E)} f(x)$$

$$= \frac{1}{\phi(E)}(\cos\alpha x \text{ or } \sin\alpha x)$$

(i) If $f(x) = \cos\alpha x$, then $$\text{P.I} = \frac{1}{\phi(E)}\cos\alpha x$$

$$= \text{R.P of } \frac{1}{\phi(E)} e^{i\alpha x}$$

$$= \text{R.P of } \frac{1}{\phi(E)} a^x, \text{ where } a = e^{i\alpha},$$

which can be obtained by type 1.

If $f(x) = \sin \alpha x$, then
$$\text{P.I} = \frac{1}{\phi(E)} \sin \alpha x$$
$$= \text{I.P of } \frac{1}{\phi(E)} e^{i\alpha x}$$
$$= \text{I.P of } \frac{1}{\phi(E)} a^x, \text{ where } a = e^{i\alpha},$$

which can be obtained by type 1. ■

WORKED EXAMPLES

Example 19

Solve the equation $y_{x+1} - y_x = \sin 2x$.

Solution

The given equation is $y_{x+1} - y_x = \sin 2x$

$\Rightarrow$ $Ey_x - y_x = \sin 2x$

$\Rightarrow$ $(E-1)y_x = \sin 2x$

To find the complementary function, solve $(E-1)y_x = 0$

Auxiliary equation is $m - 1 = 0 \quad \Rightarrow m = 1$

$\therefore$ $\text{C.F} = C_1 1^x = C_1$.

$$\text{P.I} = \frac{1}{E-1} \sin 2x$$
$$= \frac{1}{1+\Delta-1} \sin 2x$$
$$= \frac{1}{\Delta} \sin 2x$$
$$= \Delta^{-1}(\sin 2x)$$
$$= -\frac{1}{2\sin\frac{2}{2}} \cos 2\left(x - \frac{1}{2}\right) \quad \text{[Refer 11.5, page,...]}$$
$$= -\frac{1}{2\sin 1} \cos(2x-1)$$

$\therefore$ the general solution is $\quad y_x = \text{C.F} + \text{P.I}$

$\Rightarrow \quad y_x = C_1 - \dfrac{1}{2\sin 1}\cos(2x-1)$ ■

Example 20

Solve $\Delta y_x + \Delta^2 y_x = \cos x.$

Solution

The given equation is

$$\Delta y_x + \Delta^2 y_x = \cos x$$

$$\Rightarrow \quad (E-1)y_x + (E-1)^2 y_x = \cos x$$

$$\Rightarrow \quad Ey_x - y_x + \left(E^2 - 2E + 1\right)y_x = \cos x$$

$$\Rightarrow \quad Ey_x - y_x + E^2 y_x - 2Ey_x + y_x = \cos x$$

$$\Rightarrow \quad E^2 y_x - Ey_x = \cos x$$

$$\Rightarrow \quad \left(E^2 - E\right)y_x = \cos x$$

To find the complementary function, solve $\left(E^2 - E\right)y_x = 0$

Auxiliary equation is

$$\therefore \quad m^2 - m = 0 \Rightarrow m(m-1) = 0 \Rightarrow m = 0,\ 1$$

$$\text{C.F} = C_1 0^x + C_2 . 1^x = C_2$$

$$\text{P.I} = \frac{1}{E^2 - E}\cos x$$

$$= \text{R.P of } \frac{1}{E^2 - E}e^{ix}$$

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$$= \text{R.P of } \frac{1}{E^2 - E}a^x, \quad \text{where } a = e^i$$

$$= \text{R.P of } \frac{a^x}{a^2 - a}$$

$$= \text{R.P of } \frac{e^{ix}}{e^{i2} - e^i}$$

$$= \text{R.P of } \frac{\cos x + i\sin x}{\cos 2 + i\sin 2 - (\cos 1 + i\sin 1)}$$

$$= \text{R.P of } \frac{\cos x + i\sin x}{(\cos 2 - \cos 1) + i(\sin 2 - \sin 1)}$$

$$= \text{R.P of } \frac{(\cos x + i\sin x)\left[(\cos 2 - \cos 1) - i(\sin 2 - \sin 1)\right]}{(\cos 2 - \cos 1)^2 + (\sin 2 - \sin 1)^2}$$

$$= \text{R.P of } \frac{\left[\cos x(\cos 2-\cos 1)+\sin x(\sin 2-\sin 1)+i\{\sin x(\cos 2-\cos 1)-\cos x(\sin 2-\sin 1)\}\right]}{\cos^2 2+\cos^2 1-2\cos 2\cos 1+\sin^2 2+\sin^2 1-2\sin 2\sin 1}$$

$$= \frac{\cos x(\cos 2-\cos 1)+\sin x(\sin 2-\sin 1)}{\cos^2 2+\sin^2 2+\cos^2 1+\sin^2 1-2(\cos 2\cos 1+\sin 2\sin 1)}$$

$$= \frac{\cos x\cos 2+\sin x\sin 2-(\cos x\cos 1+\sin x\sin 1)}{1+1-2\cos(2-1)}$$

$$= \frac{\cos(x-2)-\cos(x-1)}{2(1-\cos 1)}$$

$$= -\frac{2\sin\left(\frac{x-2+x-1}{2}\right)\sin\left(\frac{x-2-x+1}{2}\right)}{2.2\sin^2\frac{1}{2}}$$

$$= -\frac{\sin\left(x-\frac{3}{2}\right)\sin\left(-\frac{1}{2}\right)}{2\sin^2\frac{1}{2}} = \frac{\sin\left(x-\frac{3}{2}\right)}{2\sin\frac{1}{2}}$$

$\therefore$ the general solution is $y_x = \text{C.F} + \text{P.I}$

$$\Rightarrow \qquad y_x = C_2 + \frac{\sin\left(x-\frac{3}{2}\right)}{2\sin\frac{1}{2}}$$

■

Example 21

Solve the difference equation $y_{x+2} - y_x = \cos\frac{x}{2}$.

Solution

The given equation is $\qquad y_{x+2} - y_x = \cos\frac{x}{2}$

$$\Rightarrow \qquad E^2 y_x - y_x = \cos\frac{x}{2}$$

$$\Rightarrow \qquad (E^2-1)y_x = \cos\frac{x}{2}$$

To find the complementary function, solve $(E^2-1)y_x = 0$

Auxiliary equation is $\qquad m^2 - 1 = 0 \qquad \Rightarrow m = \pm 1$

$$\therefore \qquad \text{C.F} = C_1(-1)^x + C_2.1^x = C_1(-1)^x + C_2$$

$$P.I = \frac{1}{E^2-1}\cos\left(\frac{x}{2}\right)$$

$$= \text{R.P of } \frac{1}{E^2-1}.e^{i\frac{x}{2}}$$

$$= \text{R.P of } \frac{1}{E^2-1}a^x, \quad \text{where } a = e^{\frac{i}{2}}$$

$$\therefore \quad P.I = \text{R.P of } \frac{1}{a^2-1}a^x \qquad [\because a \neq \pm 1]$$

$$= \text{R.P of } \frac{e^{i\frac{x}{2}}}{\left(e^{i/2}\right)^2-1}$$

$$= \text{R.P of } \frac{e^{i\frac{x}{2}}}{e^{i}-1}$$

$$= \text{R.P of } \frac{\cos\frac{x}{2}+i\sin\frac{x}{2}}{\cos 1+i\sin 1-1}$$

$$= \text{R.P of } \frac{\cos\frac{x}{2}+i\sin\frac{x}{2}}{(\cos 1-1)+i\sin 1}$$

$$= \text{R.P of } \frac{[(\cos 1-1)-i\sin 1]\left[(\cos\frac{x}{2}+i\sin\frac{x}{2})\right]}{(\cos 1-1)^2+\sin^2 1}$$

$$= \text{R.P of } \frac{\left[\cos 1\cos\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}\sin 1+i\left\{(\cos 1-1)\sin\frac{x}{2}-\sin 1\cos\frac{x}{2})\right\}\right]}{\cos^2 1-2\cos 1+1+\sin^2 1}$$

$$= \frac{\cos 1\cos\frac{x}{2}+\sin\frac{x}{2}\sin 1-\cos\frac{x}{2}}{\cos^2 1+\sin^2 1-2\cos 1+1}$$

$$= \frac{\cos\left(\frac{x}{2}-1\right)-\cos\frac{x}{2}}{2-2\cos 1}$$

$$= -2\frac{\sin\frac{1}{2}\left(\frac{x}{2}-1+\frac{x}{2}\right)\sin\frac{1}{2}\left(\frac{x}{2}-1-\frac{x}{2}\right)}{2(1-\cos 1)}$$

$$= -\frac{\sin\left(\frac{x-1}{2}\right)\sin\left(-\frac{1}{2}\right)}{2\sin^2\frac{1}{2}} = \frac{\sin\left(\frac{x-1}{2}\right)}{2\sin\frac{1}{2}}$$

$\therefore$ the general solution is $y_x = \text{C.F} + \text{P.I}$

$$\Rightarrow \qquad y_x = C_1(-1)^x + C_2 + \frac{\sin\left(\frac{x-1}{2}\right)}{2\sin\frac{1}{2}}$$

Example 22

Solve the equation $y_{x+3} - y_x = \cos x$.

Solution

The given equation is $y_{x+3} - y_x = \cos x$

$$\Rightarrow \qquad E^3 y_x - y_x = \cos x$$

$$\Rightarrow \qquad (E^3 - 1) y_x = \cos x$$

To find the complementary function, solve $(E^3 - 1) y_x = 0$

Auxiliary equation is $m^3 - 1 = 0$

$$\Rightarrow \qquad (m-1)(m^2 + m + 1) = 0$$

$$\Rightarrow \qquad m - 1 = 0 \text{ or } m^2 + m + 1 = 0$$

$$\Rightarrow \qquad m = 1 \text{ or } m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

The roots are $m_1 = 1,\ m_2 = \frac{-1+i\sqrt{3}}{2},\ m_3 = \frac{-1-i\sqrt{3}}{2}$

Here $\alpha = -\frac{1}{2},\ \beta = \frac{\sqrt{3}}{2}$

$$\therefore \qquad r = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{4}{4}} = 1$$

and $\tan\theta = \dfrac{\frac{\sqrt{3}}{2}}{-1/2} = -\sqrt{3} \quad \Rightarrow \quad \theta = \tan^{-1}\left(-\sqrt{3}\right) = -\dfrac{\pi}{3}$

$$\therefore \qquad \text{C.F} = C_1 1^x + \left[C_2 \cos\left(-\frac{\pi}{3}x\right) + C_3 \sin\left(-\frac{\pi}{3}x\right)\right] 1^x$$

$$= C_1 + C_2 \cos\frac{\pi}{3}x - C_3 \sin\frac{\pi}{3}x$$

$$\text{P.I} = \frac{1}{E^3 - 1} \cos x$$

$$\Rightarrow \qquad \text{P.I} = \text{R.P of } \frac{1}{E^3 - 1} e^{ix}$$

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$$= \text{R.P of } \frac{1}{E^3-1}a^x, \text{ where } a = e^i$$

$$\therefore \quad \text{P.I} = \text{R.P of } \frac{a^x}{a^3-1}$$

$$= \text{R.P of } \frac{e^{ix}}{e^{i3}-1}$$

$$= \text{R.P of } \frac{\cos x + i\sin x}{\cos 3 + i\sin 3 - 1}$$

$$= \text{R.P of } \frac{\cos x + i\sin x}{(\cos 3 - 1) + i\sin 3}$$

$$= \text{R.P of } \frac{(\cos x + i\sin x)[(\cos 3-1) - i\sin 3]}{(\cos 3-1)^2 + \sin^2 3}$$

$$= \frac{\cos x\,(\cos 3 - 1) + \sin x \sin 3}{\cos^2 3 - 2\cos 3 + 1 + \sin^2 3}$$

$$= \frac{\cos x \cos 3 + \sin x \sin 3 - \cos x}{\cos^2 3 + \sin^2 3 + 1 - 2\cos 3}$$

$$= \frac{\cos(x-3) - \cos x}{2 - 2\cos 3}$$

$$= \frac{\cos(x-3) - \cos x}{2(1-\cos 3)}$$

$$= -\frac{2\sin\left(\frac{x-3+x}{2}\right)\sin\left(\frac{x-3-x}{2}\right)}{2\cdot 2\sin^2\frac{3}{2}}$$

$$= -\frac{\sin\left(x-\frac{3}{2}\right)\sin\left(-\frac{3}{2}\right)}{2\sin^2\frac{3}{2}}$$

$$= \frac{\sin\left(x-\frac{3}{2}\right)\sin\frac{3}{2}}{\sin^2\frac{3}{2}} = \frac{\sin\left(x-\frac{3}{2}\right)}{\sin\frac{3}{2}}$$

$\therefore$ the general solution is $\quad y_x = \text{C.F} + \text{P.I}$

$$\Rightarrow \quad \mathbf{y_x = C_1 + C_2\cos\frac{\pi}{3}x - C_3\sin\frac{\pi}{3}x + \frac{\sin\left(x-\frac{3}{2}\right)}{\sin\frac{3}{2}}}$$

5. Particular Integral – Type 5:

If $$f(x) = f_1(x) + f_2(x)$$

where $$f_1(x) = a^x, x^r, \cos\alpha x, \sin\alpha x$$

and $$f_2(x) = a^x, x^r, \cos\alpha x, \sin\alpha x$$

then
$$\text{P.I} = \frac{1}{\phi(E)} f(x)$$
$$= \frac{1}{\phi(E)}[f_1(x) + f_2(x)]$$
$$= \frac{1}{\phi(E)} f_1(x) + \frac{1}{\phi(E)} f_2(x)$$
$$= \text{P.I}_1 + \text{P.I}_2$$

$\therefore$ the general solution is $$\mathbf{y_x = C.F + P.I_1 + P.I_2}$$

WORKED EXAMPLES

Example 23

Solve $y_{x+1} + 3y_x = x + 2^x$.

Solution

The given equation is $$y_{x+1} + 3y_x = x + 2^x$$
$$\Rightarrow \quad Ey_x + 3y_x = x + 2^x$$
$$\Rightarrow \quad (E+3)y_x = x + 2^x$$

To find the complementary function, solve $(E+3)y_x = 0$

Auxiliary equation is $$m + 3 = 0 \quad \Rightarrow m = -3$$

$$\therefore \quad \text{C.F} = C.(-3)^x$$

$$\text{P.I} = \frac{1}{E+3}(2^x + x)$$
$$= \frac{1}{E+3}2^x + \frac{1}{E+3}x$$
$$= \text{P.I}_1 + \text{P.I}_2$$

where
$$\text{P.I}_1 = \frac{1}{E+3}2^x$$
$$= \frac{2^x}{2+3} = \frac{2^x}{5}$$

and

$$\text{P.I}_2 = \frac{1}{E+3}x$$
$$= \frac{1}{1+\Delta+3}x$$
$$= \frac{1}{4+\Delta}x$$
$$= \frac{1}{4\left(1+\frac{\Delta}{4}\right)}x$$
$$= \frac{1}{4}\left(1+\frac{\Delta}{4}\right)^{-1}x$$
$$= \frac{1}{4}\left[1-\frac{\Delta}{4}+\frac{\Delta^2}{16}-\cdots\right]x$$
$$= \frac{1}{4}\left[x-\frac{1}{4}\Delta x\right]$$
$$= \frac{1}{4}\left[x-\frac{1}{4}(x+1-x)\right] = \frac{1}{4}\left[x-\frac{1}{4}\right] = \frac{1}{16}[4x-1]$$

$\therefore$ the general solution is

$$y_x = \text{C.F} + \text{P.I}_1 + \text{P.I}_2$$
$$\boldsymbol{y_x = C(-3)^x + \frac{2^x}{5} + \frac{1}{16}(4x-1)}$$

Example 24

Solve the difference equation $y_{x+1} - 2y_x = x + 3^x$.

Solution

The given equations is $\quad y_{x+1} - 2y_x = x + 3^x$

$\Rightarrow \quad Ey_x - 2y_x = x + 3^x$

$\Rightarrow \quad (E-2)y_x = x + 3^x$

To find the complementary function, solve $(E-2)y_x = 0$

Auxiliary equation is $\quad m - 2 = 0 \quad \Rightarrow m = 2$

$\therefore \quad \text{C.F} = C.2^x$

$$\text{P.I} = \frac{1}{E-2}\left(x+3^x\right)$$
$$= \frac{1}{E-2}x + \frac{1}{E-2}3^x$$
$$= \text{P.I}_1 + \text{P.I}_2$$

where
$$P.I_1 = \frac{1}{E-2}x$$
$$= \frac{1}{1+\Delta-2}x$$
$$= \frac{1}{-1+\Delta}x$$
$$= -\frac{1}{(1-\Delta)}x$$
$$= -(1-\Delta)^{-1}x$$
$$= -(1+\Delta+\Delta^2+\cdots)x$$
$$= -(x+\Delta x) = -(x+x+1-x) = -(x+1)$$

and
$$P.I_2 = \frac{1}{E-2}3^x = \frac{3^x}{3-2} = 3^x$$

$\therefore$ the general solution is
$$y_x = C.F + P.I_1 + P.I_2$$
$\Rightarrow$
$$y_x = C2^x - (x+1) + 3^x$$
$\Rightarrow$
$$\mathbf{y_x = C2^x + 3^x - x - 1}$$

Example 25

Solve the difference equation $y_{x+2} - 6y_{x+1} + 8y_x = 2^x + 6x$.

Solution

The given equation is
$$y_{x+2} - 6y_{x+1} + 8y_x = 2^x + 6x$$
$\Rightarrow$
$$E^2y_x - 6Ey_x + 8y_x = 2^x + 6x$$
$\Rightarrow$
$$(E^2 - 6E + 8)y_x = 2^x + 6x$$

To find the complementary function, solve $(E^2 - 6E + 8)y_x = 0$

Auxiliary equation is $m^2 - 6m + 8 = 0$

$\Rightarrow$
$$(m-4)(m-2) = 0 \quad \Rightarrow m = 4, 2$$
$\therefore$
$$C.F = C_1.2^x + C_2.4^x$$
$$P.I = \frac{1}{E^2-6E+8}(2^x + 6x)$$
$$= \frac{1}{E^2-6E+8}2^x + \frac{1}{E^2-6E+8}6x$$
$$= P.I_1 + P.I_2$$

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where
$$P.I_1 = \frac{1}{E^2-6E+8}2^x$$
$$= \frac{1}{(E-4)(E-2)}2^x$$
$$= \frac{1}{(2-4)(E-2)}2^x$$
$$= -\frac{1}{2}\frac{1}{(E-2)}2^x$$
$$= -\frac{1}{2}x2^{x-1} = -\frac{x}{4}2^x \qquad \left[\because \frac{1}{E-a}a^x = xa^{x-1}\right]$$

and
$$P.I_2 = \frac{1}{E^2-6E+8}6x$$
$$= \frac{1}{(1+\Delta)^2-6(1+\Delta)+8}6x$$
$$= \frac{1}{1+2\Delta+\Delta^2-6-6\Delta+8}6x$$
$$= \frac{1}{3-4\Delta+\Delta^2}6x$$
$$= \frac{1}{3\left(1-\left(\frac{4}{3}\Delta-\frac{1}{3}\Delta^2\right)\right)}6x$$
$$= 2\left[1-\left(\frac{4}{3}\Delta-\frac{1}{3}\Delta^2\right)\right]^{-1}x$$
$$= 2\left[1+\frac{4}{3}\Delta-\frac{1}{3}\Delta^2+\left(\frac{4}{3}\Delta-\frac{1}{3}\Delta^2\right)^2+\cdots\right]x \qquad [\because \Delta^2x=0]$$
$$= 2\left[1+\frac{4}{3}\Delta\right]x$$
$$= 2\left[x+\frac{4}{3}\Delta x\right]$$
$$= 2\left[x+\frac{4}{3}(x+1-x)\right] = 2\left[x+\frac{4}{3}\right] = \frac{2}{3}(3x+4)$$

$\therefore$ the general solution is $y_x = C.F + P.I_1 + P.I_2$
$$\Rightarrow \qquad \boldsymbol{y_x = C_1.2^x + C_2.4^x - \frac{x2^x}{4} + \frac{2}{3}(3x+4)}$$

Example 26

Solve the difference equation $y_x - 7y_{x-1} + 10y_{x-2} = 3^x + x$.

Solution

The given equation is $y_x - 7y_{x-1} + 10y_{x-2} = 3^x + x$

Replacing x by $x + 2$, we get

$$y_{x+2} - 7y_{x+1} + 10y_x = 3^{x+2} + (x+2)$$

$$\Rightarrow \quad E^2 y_x - 7Ey_x + 10y_x = 3^2.3^x + (x+2)$$

$$\Rightarrow \quad (E^2 - 7E + 10)y_x = 9.3^x + (x+2)$$

To find the complementary function, solve $(E^2 - 7E + 10)y_x = 0$

Auxiliary equation is $m^2 - 7m + 10 = 0$

$$\Rightarrow \quad (m-5)(m-2) = 0 \qquad \Rightarrow m = 2, 5$$

$$\therefore \quad C.F = C_1.2^x + C_2.5^x$$

$$P.I = \frac{1}{E^2 - 7E + 10}\left[9.3^x + (x+2)\right]$$

$$= \frac{1}{E^2 - 7E + 10} 9.3^x + \frac{1}{E^2 - 7E + 10}(x+2)$$

$$= P.I_1 + P.I_2$$

where

$$P.I_1 = \frac{1}{E^2 - 7E + 10} 9.3^x$$

$$= 9\frac{3^x}{3^2 - 7\times 3 + 10} = 9\frac{3^x}{9 - 21 + 10} = -\frac{9}{2}3^x$$

and

$$P.I_2 = \frac{1}{E^2 - 7E + 10}(x+2)$$

$$= \frac{1}{(1+\Delta)^2 - 7(1+\Delta) + 10}(x+2)$$

$$= \frac{1}{1 + 2\Delta + \Delta^2 - 7 - 7\Delta + 10}(x+2)$$

$$= \frac{1}{4 - 5\Delta + \Delta^2}(x+2)$$

$$= \frac{1}{4\left[1 - \left(\frac{5}{4}\Delta - \frac{\Delta^2}{4}\right)\right]}(x+2)$$

$$= \frac{1}{4}\left[1 - \left(\frac{5}{4}\Delta - \frac{1}{4}\Delta^2\right)\right]^{-1}(x+2)$$

$$=\frac{1}{4}\left[1+\frac{5}{4}\Delta-\frac{1}{4}\Delta^2+\left(\frac{5}{4}\Delta-\frac{1}{4}\Delta^2\right)^2+\cdots\right](x+2)$$

$$=\frac{1}{4}\left[1+\frac{5}{4}\Delta\right][x+2] \qquad [\because \Delta^2x=0]$$

$$=\frac{1}{4}\left[x+2+\frac{5}{4}\Delta(x+2)\right]$$

$$=\frac{1}{4}\left[(x+2)+\frac{5}{4}(\Delta x+\Delta(2))\right]$$

$$=\frac{1}{4}\left[x+2+\frac{5}{4}\{x+1-x+0\}\right]$$

$$=\frac{1}{4}\left[x+2+\frac{5}{4}\right]=\frac{1}{16}(4x+13)$$

$\therefore$ the general solution is $y_x = C.F + P.I_1 + P.I_2$

$$\Rightarrow \qquad \boldsymbol{y_x = C_1.2^x + C_2.5^x - \frac{9}{2}3^x + \frac{1}{16}(4x+13)}$$ ■

Exercises 11.3 α

Solve the following non-homogeneous differential equations.

(1) $3u_{x+1} - u_x = x$

(2) $y_{x+2} - 5y_{x+1} + 6y_x = 5^x$

(3) $y_{x+2} - 6y_{x+1} + 8y_x = 4^x$

(4) $6y_{x+2} + 5y_{x+1} - 6y_x = 2^x$

(5) $y_{x+2} - 4y_{x+1} + 3y_x = 4^x.x$

(6) $y_{n+2} - 2y_{n+1} + y_n = n^2 2^n$

(7) $y_{x+3} + y_x = x+1$, given $y_0 = y_1 = y_2 = 0$.

(8) $y_{n+2} - 4y_n = n^2 + n - 1$

(9) $y_{x+2} - 16y_x = \cos\left(\frac{x}{2}\right)$

(10) $y_{k+2} + y_k = \sin\left(\frac{k\pi}{2}\right)$

(11) $y_{x+2} + y_{x+1} - 12y_x = 3^x + 108$

(12) $y_{x+3} - y_x = \cos x + (x+3)$

(13) $y_{x+1} = 2y_x - 1,\ y_0 = 5$

(14) $y_{x+1} + y_x = 2$

(15) $y_{x+1} - y_x = x;\ y_0 = \frac{1}{2}$.

Answers 11.3

(1) $y_x = C\left(\frac{1}{3}\right)^x + \frac{1}{4}(2x-3)$

(2) $y_x = C_1 2^x + C_2 3^x + \frac{5^x}{6}$

(3) $y_x = C_1 2^x + C_2 4^x + \frac{1}{2}x.4^{x-1}$

(4) $y_x = C_1\left(\frac{2}{3}\right)^x + C^2\left(-\frac{3}{2}\right)^x + \frac{2^x}{28}$

(5) $y_x = C_1 + C_2 3^x + \frac{4^x}{9}(3x-16)$

(6) $y_x = C_1 + C_2 x + 2^x(x^2-8x+20)$

(7) $y_x = \frac{2}{\sqrt{3}} \sin\frac{\pi x}{3} + \frac{1}{2}x(x-3)$

(8) $y_x = C_1 \cdot 2^n + C_2 \cdot (-2)^n - \frac{(9n^2+21n+17)}{27}$

(9) $y_x = C_1 \cdot 4^x + C_2 \cdot (-4)^x + \dfrac{\cos\left(\frac{x}{2}-1\right) - 16\cos\frac{x}{2}}{257 - 32\cos 1}$

(10) $y_x = A\cos k\frac{\pi}{2} + B\sin k\frac{\pi}{2} + \frac{k}{2}\sin(k-1)\frac{\pi}{2}$

(11) $y_x = C_1(-4)^x + C_2 3^x$

(12) $y_x = C_1 + C_2\cos\frac{2\pi}{3}x + C_3\sin\frac{2\pi}{3}x + \dfrac{\sin\left(x-\frac{3}{2}\right)}{2\sin\frac{3}{2}} + \frac{x(x+3)}{6}$

(13) $y_x = 4.2^x + 1$ (14) $y_x = C\cdot(-1)^x + 1$ (15) $y_x = \frac{1}{2}(x^2 - x + 1)$

FIRST ORDER LINEAR DIFFERENCE EQUATION WITH VARIABLE COEFFICIENTS

First Order Linear Homogeneous Difference Equation with Variable Coefficients

Consider the homogeneous linear equation

$$y_{x+1} - P(x)y_x = 0$$

$$\Rightarrow \qquad y_{x+1} = P(x)y_x$$

Putting $x = 0, 1, 2, \ldots, x-1$, we get $y_1 = P(0)y_0$, where $P(0) \neq 0$

$$y_2 = P(1)y_1 = P(1)P(0)y_0$$
$$y_3 = P(2)y_2 = P(2)P(1)P(0)y_0$$
$$\vdots$$
$$y_x = P(x-1)P(x-2)\ldots P(2)P(1)P(0)y_0$$

If $y_0 = c$, a constant, then $y_x = c\prod_{x=0}^{x-1} P(x)$ is the solution of the given equation

If $P(0) = 0$, then we give the values of x from 1 to $x-1$.

$\therefore$ we get

$$y_x = P(x-1)P(x-2)\ldots P(1)y_1$$
$$= y_1 \prod_{x=1}^{x-1} P(x)$$

$\Rightarrow y_x = c\prod_{x=1}^{x-1} P(x)$, where $y_1 = c$ is a constant, which is the solution of the given equation

WORKED EXAMPLES

Example 1

Solve $y_{x+1} - \frac{1}{x}y_x = 0$.

Solution

The given equation is $$y_{x+1} - \frac{1}{x}y_x = 0$$

$$\Rightarrow \quad y_{x+1} = \frac{1}{x}y_x$$

Put $x = 1, 2, 3, \ldots, x-1$, then

$$y_2 = y_1$$
$$y_3 = \frac{1}{2}y_2 = \frac{1}{2}.1.y_1$$
$$y_4 = \frac{1}{3}y_3 = \frac{1}{3}\cdot\frac{1}{2}y_1$$
$$\vdots$$
$$y_x = \frac{1}{(x-1)}\frac{1}{(x-2)}\cdots\frac{1}{3}\cdot\frac{1}{2}.y_1$$

If $y_1 = c$ is a constant, then the solution is

$$y_x = \frac{c}{(x-1)(x-2)\ldots.3.2.1}$$

$$\Rightarrow \quad y_x = \frac{c}{(x-1)!}$$ ■

Example 2

Solve $(x-1)y_{x+1} - (x+1)y_x = 0$ if $y_2 = 1$.

Solution

The given equation is $$(x-1)y_{x+1} - (x+1)y_x = 0$$

$$\Rightarrow \quad y_{x+1} = \frac{x+1}{x-1}y_x$$

Put $x = 2, 3, 4, \ldots, x-1$, then we get

$$y_3 = \frac{3}{1}y_2 = \frac{3}{1}.1$$

$$y_4 = \frac{4}{2}y_3 = \frac{4}{2}.\frac{3}{1}.$$

$$y_5 = \frac{5}{3}y_4 = \frac{5}{3}.\frac{4}{2}.\frac{3}{1}$$

$$\vdots$$

$$y_x = \frac{x}{x-2}\cdot\frac{x-1}{x-3}\cdot\frac{x-2}{x-4}\cdot\frac{x-3}{x-5}\cdots\frac{6}{4}\cdot\frac{5}{3}\cdot\frac{4}{2}\cdot\frac{3}{1}$$

$\Rightarrow$ $y_x = \frac{1}{2}x(x-1)$ is the solution of the given equation. ■

Example 3

Solve $y_{x+1} - x^2 y_x = 0$.

Solution

The given equation is $y_{x+1} - x^2 y_x = 0$

$\Rightarrow$ $$y_{x+1} = x^2 y_x$$

Put $x = 1, 2, 3, \ldots, x-1$, then we get

$$y_2 = 1^2 y_1$$

$$y_3 = 2^2 y_2 = 2^2.1^2 y_1$$

$$y_4 = 3^2 y_3 = 3^2.2^2.1^2 y_1$$

$$\vdots$$

$$y_x = (x-1)^2(x-2)^2 \ldots 3^2.2^2.1^2.y_1$$

If $y_1 = c$, a constant, then $y_x = (x-1)^2(x-2)^2 \ldots 3^2 2^2 1^2.c$

$\Rightarrow$ $$y_x = c\{(x-1)(x-2)\ldots 3.2.1\}^2$$

$\Rightarrow$ $y_x = c[(x-1)!]^2$ is the solution of the given equation.

First Order Linear Non-homogeneous Difference Equation with Variable Coefficients

Consider the equation

$$y_{x+1} - P(x)y_x = q(x) \tag{1}$$

$\therefore$ the homogeneous equation is $y_{x+1} - P(x)y_x = 0$ (2)

We can find the solution of (2) by 17.7.1, page...

Let v_x be the solution of (2)

$\therefore$ $$v_{x+1} - P(x)v_x = 0 \quad (3)$$

Let $y_x = v_x w_x$ be the solution of (1)

Then $$y_{x+1} = v_{x+1} w_{x+1}$$

Substituting in (1), we get $$v_{x+1}w_{x+1} - P(x)v_x w_x = q(x)$$

$\Rightarrow$ $$v_{x+1}w_{x+1} - v_{x+1}w_x = q(x) \quad \text{[using (3)]}$$

$\Rightarrow$ $$v_{x+1}[w_{x+1} - w_x] = q(x)$$

$\Rightarrow$ $$v_{x+1}\Delta w_x = q(x)$$

$\Rightarrow$ $$\Delta w_x = \frac{q(x)}{v_{x+1}}$$

$\therefore$ $$w_x = \Delta^{-1}\left[\frac{q(x)}{v_{x+1}}\right] + c$$

$\therefore$ $$y_x = v_x\left\{\Delta^{-1}\left[\frac{q(x)}{v_{x+1}}\right] + c\right\}$$

which is the solution of (1).

WORKED EXAMPLES

Example 4

Solve the difference equation $y_{x+1} - (x+1)y_x = 2^x(x+1)!$.

Solution

The given equation is $$y_{x+1} - (x+1)y_x = 2^x(x+1)! \quad (1)$$

Here $$p(x) = x+1,\ q(x) = 2^x(x+1)!$$

Consider $$y_{x+1} - (x+1)y_x = 0 \quad (2)$$

$\Rightarrow$ $$y_{x+1} = (x+1)y_x$$

Put x = 0, 1, 2, 3, …, (x − 1), then we get

$$\begin{aligned}
y_1 &= 1.y_0 \\
y_2 &= 2.y_1 = 2.1.y_0 \\
y_3 &= 3y_2 = 3.2.1.y_0 \\
&\vdots \\
y_x &= x(x-1)(x-2)\ldots 3.2.1.y_0 \\
&= x!\, y_0 \\
&= kx!, \quad \text{where } y_0 = k
\end{aligned}$$

Let $v_x = kx!$ be the solution of (2)

Let $y_x = v_x w_x$ be the solution of (1)

$$\therefore \qquad y_x = v_x\left[c + \Delta^{-1}\left(\frac{q(x)}{v_{x+1}}\right)\right]$$

$$= kx!\left[c + \Delta^{-1}\frac{2^x(x+1)!}{k(x+1)!}\right]$$

$$= kx!\left[c + \Delta^{-1}\left(\frac{2^x}{k}\right)\right]$$

$$= \frac{kx!}{k}\left[ck + \Delta^{-1}(2^x)\right]$$

$$\Rightarrow \qquad \boldsymbol{y_x = x!\left[ck + \frac{2^x}{2-1}\right] = x!\left[c_1 + 2^x\right], \quad c_1 = ck}$$ ■

Example 5

Solve the difference equation $y_{x+1} - e^{2x-1}y_x = 5xe^{x^2}$.

Solution

The given equation is $\qquad y_{x+1} - e^{2x-1}y_x = 5xe^{x^2} \qquad (1)$

Here $\qquad P(x) = e^{2x-1}, \; q(x) = 5xe^{x^2}$

Consider $\qquad y_{x+1} - e^{2x-1}y_x = 0 \qquad (2)$

$$\Rightarrow \qquad y_{x+1} = e^{2x-1}y_x$$

Putting $x = 1, 2, 3, \ldots, (x-1)$, we get

$$y_2 = e^1 y_1$$
$$y_3 = e^3 y_2 = e^3 e^1 y_1$$
$$y_4 = e^5 y_3 = e^5.e^3.e^1 y_1$$
$$\vdots$$
$$y_x = e^{2x-3}e^{2x-5}\ldots e^5 e^3 e^1 y_1$$

$$\Rightarrow \qquad y_x = k.e^1 e^3 e^5 \ldots e^{2x-5}e^{2x-3}$$
$$= k\,e^{1+3+5+\ldots+(2x-3)}$$

$= ke^{(x-1)^2}$ [$\because$ sum of first n odd numbers is n^2. Here $n = x-1$].

Take $\qquad v_x = ke^{(x-1)^2}$

$$v_{x+1} = ke^{(x+1-1)^2} = ke^{x^2}$$

$\therefore$

$\therefore$ the general solution of (1) is $$y_x = v_x\left[c+\Delta^{-1}\left(\frac{q(x)}{v_{x+1}}\right)\right]$$

$$= ke^{(x-1)^2}\left[c+\Delta^{-1}\frac{5xe^{x^2}}{ke^{x^2}}\right]$$

$$= ke^{(x-1)^2}\left[c+\frac{5}{k}\Delta^{-1}x\right]$$

$$= e^{(x-1)^2}\left[ck+5\Delta^{-1}[x]\right]$$

$$= e^{(x-1)^2}\left[ck+5\frac{[x]^2}{2}\right]$$

$\Rightarrow$ $$y_x = e^{(x-1)^2}\left[c_1+\frac{5}{2}x(x-1)\right], \text{ where } c_1 = ck$$

Exercises 11.4

(1) Solve $y_{x+1}-(x+1)y_x=0$

(2) Solve $(x+2)y_{x+1}+xy_x=0$

(3) Solve $xy_{x+1}-(x+1)y_x=1$

(4) Solve $(x+1)y_{x+1}-(2x+1)y_x=0$

(5) Solve $3x^2y_{x+1}=(2x-1)y_x$

Answers 11.4

(1) $y_x = cx!$

(2) $y_x = -\dfrac{2c}{x(x+1)}$

(3) $y_x = cx-1$

(4) $y_x = c.\dfrac{1.3.5...(2x-3)(2x-1)}{x!}$

(5) $y_x = c.\dfrac{1.3.5...(2x-5)(2x-3)}{3^x.1^2.2^2...(x-1)^2}$

SHORT ANSWER QUESTIONS

1. Define a difference equation.
2. What is the solution of the difference equation?
3. Define the general solution of difference equation.
4. Define the particular solution of a difference equation.
5. What is the order of the difference equation?
6. What is the degree of the difference equation?
7. Is $\Delta^2 y_x + 2\Delta y_x + y_x = x^2$ a difference equation?
8. Form the difference equation by eliminating a and b from the relation $y_x = a.2^x+b.3^x$.
9. Form the difference equation by eliminating A and B from $y_x = A.2^x+B.5^x$.
10. Form the difference equation by eliminating a and b from the relation $y_x = a.2^x+b.(-2)^x$.

11. Solve $y_{x+2} - 4y_{x+1} + 4y_x = 0$.
12. Solve the difference equation $y_{x+3} - 3y_{x+1} + 2y_x = 0$.
13. Show that $y_n = 1 - \frac{2}{n}$ is a solution of the difference equation.
14. Find the solution of $y_{x+1} + 3y_x = 0$.
15. Find the solution of $y_{x+3} + y_x = 0$.
16. Find the solution of $y_{n+2} - 16y_n = 0$.
17. Solve $y_{x+2} - 8y_{x+1} + 15y_x = 0$.
18. Solve $y_{x+2} - 4y_x = 0$.
19. Solve $y_{n+2} - 6y_{n+1} + 8y_n = 0$.
20. Find the solution of the difference equation $(E^2 - 5E + 6)y_n = 0$.
21. Find the P.I of $u_{n+2} - 4u_{n+1} + 4u_n = 2^n$.
22. Find the particular Integral of $(E^2 - 7E + 12)y_n = 2^n$.
23. Find the particular integral of $y_{x+2} - 6y_{x+1} + 9y_x = 3^x$.
24. Find the particular integral of $y_{n+2} - 4y_n = n - 1$.

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Index

Asal Ali

128
9/06

Numerical Analysis
Sivaramakrishna Das C Vija