Operations Research

Linear Programming : Sensitivity Analysis

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Linear Programming: Sensitivity Analysis

- **Sensitivity analysis (post-optimality analysis)**
  - refers to an analysis of the effect on the optimal solution of changes in the parameters of problem on the current optimal solution.

The following questions arise in connection with performing the sensitivity analysis.

1. How do *changes made to the coefficients of the objective function affect the optimal solution*?

2. How do *changes made to the constants on the right hand side of the constraint affect the optimal solution*?

Example

XYZ manufacturing company has a division that produces two models of grates, model–A and model–B. To produce each model–A grate requires '3' g. of cast iron and '6' minutes of labor. To produce each model–B grate requires '4' g. of cast iron and '3' minutes of labor. The profit for each model–A grate is Rs.2 and the profit for each model–B grate is Rs.1.50. One thousand g. of cast iron and 20 hours of labor are available for grate production each day. Because of an excess inventory of model–A grates, Company's manager has decided to limit the production of model–A grates to no more than 180 grates per day.

Solve the given LP problem and perform sensitivity analysis.

LP MODEL: Let X_1 and X_2 be the number of model–A and model–B grates respectively. The complete LP model is as follow:

Maximum: $Z = 2X_1 + 1.5X_2 \rightarrow 2X_1 + (3/2)X_2$

Subject to:

$$3X_1 + 4X_2 \leq 1000$$
$$6X_1 + 3X_2 \leq 1200$$
$$X_1 \leq 180$$
$$X_1, X_2 \geq 0$$

Example…

Contribution Per Unit	C_j		2	3/2	0	0	0
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	X_1	X_2	S_1	S_2	S_3
0	S_1	1000	3	4	1	0	0
0	S_2	1200	6	3	0	1	0
0	S_3	180	1	0	0	0	1
Total Profit (Z_j)		0	0	0	0	0	0
Net Contribution ($C_j – Z_j$)			**2**	**3/2**	**0**	**0**	**0**
0	S_1	460	0	4	1	0	–3
0	S_2	120	0	3	0	1	–6
2	X_1	180	1	0	0	0	1
Total Profit (Z_j)		360	2	0	0	0	2
Net Contribution ($C_j – Z_j$)			**0**	**3/2**	**0**	**0**	**–2**
0	S	300	0	0	1	–4/3	5
						1/3	–2
						0	1
						1/2	–1
Net Contribution (C			**0**	**0**	**0**	**–1/2**	**1**
0	S	60	0	0	1/5	–4/15	1
3/2	X_2	160	0	1	2/5	–1/5	0
2	X_1	120	1	0	–1/5	4/15	0
Total Profit (Z_j)		480	2	3/2	1/5	7/30	0
Net Contribution ($C_j – Z_j$)			**0**	**0**	**–1/5**	**–7/30**	**0**

‘Z = 480; at ‘X_1=120’ and ‘X_2=160’. Thus, XYZ manufacturing company realizes a maximum profit of Rs.480 per day by producing 120 model–A grates and 160 model–B grates per day.

Since all the values of (C_j – Z_j) ≤ 0; so, the solution is optimal.

LP: Sensitivity Analysis

How do *changes made to the coefficients of the objective function affect the optimal solution?*

Change in the Coefficients of the Objective Function:

Now two questions are to be answered:

1. How do changes made to the *coefficients of non–basic variable affect the optimal solution*?

2. How do changes made to the *coefficients of basic variable affect the optimal solution*?

LP: Sensitivity Analysis…

- How do *changes made to the coefficients of non–basic variable* affect the optimal solution?

  - According to the above optimal simplex table, S_1, & S_2 are the non–basic variables.
    - Now the question arises that how much have the objective function coefficients to be changed before S_1, & S_2 would enter the solution and replaces one of the basic variables (X_1, X_2 & S_3)?
    - According to the simplex algorithm as long as the row $(C_j - Z_j) \leq 0$, there will not be any change in the optimal solution. Thus, $C_j - Z_j \leq 0 \rightarrow C_j \leq Z_j$

Range for a non–basic variable:

The range of 'C_j' values for non–basic variables are: "$-\infty \leq C_j \leq Z_j$".

The range of values for 'S_1' is: "$-\infty \leq C_j \leq 1/5$"

The range of values for 'S_2' is: "$-\infty \leq C_j \leq 7/30$"

LP: Sensitivity Analysis…

- **How do changes made to the coefficients of basic variable affect the optimal solution?**
  - X_1, X_2 & S_3 are the basic variables.

Change in the Coefficient of the Basic Variable 'X_1': The change in 'X_1' is '2+h'.

Contribution Per Unit		C_j	2+h	3/2	0	0	0
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	**X_1**	**X_2**	**S_1**	**S_2**	**S_3**
0	S_3	60	0	0	1/5	–4/15	1
3/2	X_2	160	0	1	2/5	–1/5	0
2+h	X_1	120	1	0	–1/5	4/15	0
Total Profit (Z_j)		480+120h	2+h	3/2	(1–h)/5	(7+8h)/30	0
Net Contribution ($C_j – Z_j$)			0	0	– (1–h)/5	– (7+8h)/30	0

Since all the values in the $(C_j – Z_j) \leq 0$, so the solution is optimal.

From 'S_1' Column: $(C_j – Z_j) \leq 0 \rightarrow –(1–h)/5 \leq 0 \rightarrow h \leq 1$

It means that 'S_1' will not enter the basis unless the profit per unit for 'X_1' will increase to '2+h =2 +1 = 3'; if it goes above, the solution will not remain optimal.

From 'S_2' Column: $(C_j – Z_j) \leq 0 \rightarrow –(7+8h)/30 \leq 0 \rightarrow h \geq –(7/8)$

It means that 'S_2' will not enter the basis unless the profit per unit for 'X_1' will reduce to '2+h =2 – (7/8) = 9/8'; if it goes below, the solution will not remain optimal.

Range of Optimality for 'X_1':

Here, the value of 'C_1' ranges from '(9/8) to 3' **(i.e. $1.125 \leq C_1 \leq 3$).** In this interval of 'C_1', the optimality is unaffected. So, we conclude that the contribution to the profit of a model–A grate can assume values between Rs.1.125 and Rs.3 without changing the optimal solution.

LP: Sensitivity Analysis…

Change in the Coefficient of the Basic Variable 'X_2': The change in 'X_2' is '(3/2)+k'

Contribution Per Unit C_j			2	(3/2)+k	0	0	0
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	**X_1**	**X_2**	**S_1**	**S_2**	**S_3**
0	S_3	60	0	0	1/5	–4/15	1
(3/2)+k	X_2	160	0	1	2/5	–1/5	0
2	X_1	120	1	0	–1/5	4/15	0
Total Profit (Z_j)		480+160k	2	(3/2)+k	(1+2k)/5	(7–6k)/30	0
Net Contribution ($C_j – Z_j$)			0	0	– (1+2k)/5	– (7–6k)/30	0

Since all the values in the ($C_j – Z_j$) ≤ 0, so the solution is optimal.

<u>From 'S_1' Column</u>: ($C_j – Z_j$) ≤ 0 → –(1+2k)/5 ≤ 0 → k ≥ (–1/2)

It means that 'S_1' will not enter the basis unless the profit per unit for 'X_2' will decrease to '(3/2)+k =(3/2)–(1/2)= 1'; if it goes below, the solution will not remain optimal.

<u>From 'S_2' Column</u>: ($C_j – Z_j$) ≤ 0 → – (7–6k)/30 ≤ 0 → k ≤ 7/6

It means that 'S_2' will not enter the basis unless the profit per unit for 'X_2' will increase to '(3/2)+k =(3/2) + (7/6)= 15/6'; if it goes above, the solution will not remain optimal.

Range of Optimality for 'X_2':

Here, the value of 'C_2' ranges from '1 to 15/6' (i.e. $1 \le C_2 \le 2.5$). In this interval of 'C_2', the optimality is unaffected. So, we conclude that the contribution to the profit of a model–B grate can assume values between Rs.1 and Rs.2.5 without changing the optimal solution.

LP: Sensitivity Analysis…

Change in the Coefficient of the Basic Variable 'S_3': The change in 'S_3' is '0+L'.

Contribution Per Unit C_j			2	3/2	0	0	0+L
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	**X_1**	**X_2**	**S_1**	**S_2**	**S_3**
0+L	S_3	60	0	0	1/5	–4/15	1
3/2	X_2	160	0	1	2/5	–1/5	0
2	X_1	120	1	0	–1/5	4/15	0
Total Profit (Z_j)		480+60L	2	3/2	(1+L)/5	(7–8L)/30	0+L
Net Contribution ($C_j – Z_j$)			0	0	– (1+L)/5	– (7–8L)/30	0

Since all the values in the $(C_j – Z_j) \leq 0$, so the solution is optimal.

<u>From 'S_1' Column</u>: $(C_j – Z_j) \leq 0 \rightarrow –(1+L)/5 \leq 0 \rightarrow L \geq –1$

It means that 'S_1' will not enter the basis unless the profit per unit for 'S_3' will decrease to '0+L =0–1= –1'; if it goes below, the solution will not remain optimal.

<u>From 'S_2' Column</u>: $(C_j – Z_j) \leq 0 \rightarrow – (7–8L)/30 \leq 0 \rightarrow L \leq 7/8$

It means that 'S_2' will not enter the basis unless the profit per unit for 'S_3' will increase to '0+L = 0 + (7/8)= 7/8'; if it goes above, the solution will not remain optimal.

Range of Optimality for 'S_3':

Here, the value of 'C_5' ranges from '–1 to 7/8' (i.e. $–1 \leq C_5 \leq 0.875$). In this interval of 'C_5', the optimality is unaffected

LP: Sensitivity Analysis…

- **Change to the Constants on the Right–Hand side of the Constraint Inequalities:**

The LP problem is:

Maximum: $Z = 2X_1 + 1.5X_2 \rightarrow 2X_1 + (3/2)X_2$

Subject to:

$$3X_1 + 4X_2 \leq 1000$$
$$6X_1 + 3X_2 \leq 1200$$
$$X_1 \leq 180$$
$$X_1, X_2 \geq 0$$

Analyzing the 1st Constraint: Now, consider an 'h' increase in the right–hand side of the first constraint then the LP model constraints become:

$$3X_1 + 4X_2 \leq 1000 + 1h$$
$$6X_1 + 3X_2 \leq 1200 + 0h$$
$$X_1 \leq 180 + 0h$$

Contribution Per Unit	C_j		2	3/2	0	0	0
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	**X_1**	**X_2**	**S_1**	**S_2**	**S_3**
0	S_1	1000 +1h	3	4	1	0	0
0	S_2	1200 + 0h	6	3	0	1	0
0	S_3	180 + 0h	1	0	0	0	1
Total Profit (Z_j)		0	0	0	0	0	0
Net Contribution ($C_j – Z_j$)			2	3/2	0	0	0

Contribution Per Unit	C_j		2	3/2	0	0	0
C_{Bi}	Basic Variables (B)	Quantity (Qty)	X_1	X_2	S_1	S_2	S_3
0	S_3	60+(h/5)	0	0	1/5	–4/15	1
3/2	X_2	160+(2h/5)	0	1	2/5	–1/5	0
2	X_1	120–(h/5)	1	0	–1/5	4/15	0
Total Profit (Z_j)		480+(h/5)	2	3/2	1/5	7/30	0
Net Contribution ($C_j – Z_j$)			0	0	– 1/5	– 7/30	0

LP: Sensitivity Analysis…

- **Change to the Constants on the Right–Hand side of the Constraint Inequalities:**
  - Remember that as the requirement of the simplex method is that the values in the quantity column not be negative.
  - If any value in the quantity column becomes negative then the current solution will no longer be feasible and a new variable will enter the solution.

Thus, the inequalities: $60+(h/5) \geq 0$, $160+(2h/5) \geq 0$ & $120-(h/5) \geq 0$; are solved for 'h':

- $60+(h/5) \geq 0 \rightarrow (h/5) \geq (-60) \rightarrow h \geq -300$;
- $160+(2h/5) \geq 0 \rightarrow (2h/5) \geq -160 \rightarrow h \geq -400$;
- $120-(h/5) \geq 0 \rightarrow -(h/5) \geq -120 \rightarrow h \leq 600$.

Since: $b_1 = 1000 + h$ then $h = b_1 - 1000$;

The value '$h = b_1 - 1000$' is put into the inequalities $h \geq -300$, $h \geq -400$ & $h \leq 600$ as follows:

$h \geq -300$	$h \geq -400$	$h \leq 600$
$b_1 - 1000 \geq -300$	$b_1 - 1000 \geq -400$	$b_1 - 1000 \leq 600$
$b_1 \geq -300+1000$	$b_1 \geq -400+1000$	$b_1 \leq 600+1000$
$b_1 \geq 700$	$b_1 \geq 600$	$b_1 \leq 1600$

Thus, the range of 'b_1' is: $600 \leq b_1 \leq 1600$. Under these conditions, XYZ Manufacturing Company should produce '$120-(h/5)$' model–A grates and '$160+(2h/5)$' model–B grates.

LP: Sensitivity Analysis…

- **Change to the Constants on the Right–Hand side of the Constraint Inequalities:**

The LP problem is:
Maximum: $Z = 2X_1 + 1.5X_2 \rightarrow 2X_1 + (3/2)X_2$
Subject to:

$$3X_1 + 4X_2 \leq 1000$$
$$6X_1 + 3X_2 \leq 1200$$
$$X_1 \leq 180$$
$$X_1, X_2 \geq 0$$

Analyzing the 2nd Constraint:

Now, consider a ‘k’ increase in the right–hand side of the first constraint then the LP model constraints become:

$$3X_1 + 4X_2 \leq 1000 + 0k$$
$$6X_1 + 3X_2 \leq 1200 + 1k$$
$$X_1 \leq 180 + 0k$$

Contribution Per Unit		C_j	2	3/2	0	0	0
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	X_1	X_2	S_1	S_2	S_3
0	S_1	1000 +0k	3	4	1	0	0
0	S_2	1200 + 1k	6	3	0	1	0
0	S_3	180 + 0k	1	0	0	0	1
Total Profit (Z_j)		0	0	0	0	0	0
Net Contribution ($C_j – Z_j$)			2	3/2	0	0	0

Contribution Per Unit		C_j	2	3/2	0	0	0
C_{Bi}	Basic Variables (B)	Quantity (Qty)	X_1	X_2	S_1	S_2	S_3
0	S_3	60–(4k/15)	0	0	1/5	–4/15	1
3/2	X_2	160–(k/5)	0	1	2/5	–1/5	0
2	X_1	120+(4k/15)	1	0	–1/5	4/15	0
Total Profit (Z_j)		480+(7k/30)	2	3/2	1/5	7/30	0
Net Contribution ($C_j – Z_j$)			0	0	– 1/5	– 7/30	0

LP: Sensitivity Analysis…

- **Change to the Constants on the Right–Hand side of the Constraint Inequalities:**

Thus, the inequalities: $60-(4k/15) \geq 0$, $160-(k/5) \geq 0$ & $120+(4k/15) \geq 0$; are solved for 'k':

- $60 - (4k/15) \geq 0 \rightarrow -(4k/15) \geq -60 \rightarrow k \leq 225$,
- $160-(k/5) \geq 0 \rightarrow -(k/5) \geq -160 \rightarrow k \leq 800$ &
- $120+(4k/15) \geq 0 \rightarrow (4k/15) \geq -120 \rightarrow k \geq -450$

Since: $b_2 = 1200 + k$ then $k = b_2 - 1200$;

The value '$k = b_2 - 1200$' is put into the inequalities $k \leq 225$, $k \leq 800$ & $k \geq -450$ as follows:

$k \leq 225$ $b_2 - 1200 \leq 225$ $b_2 \leq 225+1200$ $b_2 \leq 1425$	$k \leq 800$ $b_2 - 1200 \leq 800$ $b_2 \leq 800+1200$ $b_2 \leq 2000$	$k \geq -450$ $b_2 - 1200 \geq -450$ $b_2 \geq -450+1200$ $b_2 \geq 750$

Thus, the range of 'b_2' is: $750 \leq b_2 \leq 2000$. Under these conditions, XYZ Company should produce '$120+(4k/15)$' model–A grates and '$160-(k/5)$' model–B grates.

LP: Sensitivity Analysis…

- **Change to the Constants on the Right–Hand side of the Constraint Inequalities:**

The LP problem is:
Maximum: $Z = 2X_1 + 1.5X_2 \rightarrow 2X_1 + (3/2)X_2$
Subject to:

$$3X_1 + 4X_2 \leq 1000$$
$$6X_1 + 3X_2 \leq 1200$$
$$X_1 \leq 180$$
$$X_1, X_2 \geq 0$$

Analyzing the 3rd Constraint:

Now, consider an 'L' increase in the right–hand side of the first constraint then the LP model constraints become:

$$3X_1 + 4X_2 \leq 1000 + 0L$$
$$6X_1 + 3X_2 \leq 1200 + 0L$$
$$X_1 \leq 180 + 1L$$

Contribution Per Unit		C_j	2	3/2	0	0	0
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	**X_1**	**X_2**	**S_1**	**S_2**	**S_3**
0	S_1	1000 + 0L	3	4	1	0	0
0	S_2	1200 + 0L	6	3	0	1	0
0	S_3	180 + 1L	1	0	0	0	1
Total Profit (Z_j)		0	0	0	0	0	0
Net Contribution ($C_j - Z_j$)			2	3/2	0	0	0

Contribution Per Unit	C_j		2	3/2	0	0	0
C_{Bi}	Basic Variables (B)	Quantity (Qty)	X_1	X_2	S_1	S_2	S_3
0	S_3	60+L	0	0	1/5	–4/15	1
3/2	X_2	160	0	1	2/5	–1/5	0
2	X_1	120	1	0	–1/5	4/15	0
Total Profit (Z_j)		480	2	3/2	1/5	7/30	0
Net Contribution ($C_j - Z_j$)			0	0	– 1/5	– 7/30	0

LP: Sensitivity Analysis…

- **Change to the Constants on the Right–Hand side of the Constraint Inequalities:**
  - Remember that as the requirement of the simplex method is that the values in the quantity column not be negative. If any value in the quantity column becomes negative then the current solution will no longer be feasible and a new variable will enter the solution.

Thus, the inequalities:

$$60+L \geq 0,$$
$$160 \geq 0 \ \&$$
$$120 \geq 0;$$

are solved for ‘L’: $60+L \geq 0 \rightarrow L \geq -60$.

We will get: $L \geq -60$.

Since: $b_3 = 180 + L$ then $L = b_3 - 180$;

The value ‘$L = b_3 - 120$’ is put into the inequality $L \geq -60$ as follows:

$L \geq -60 \rightarrow b_3 - 180 \geq -60 \rightarrow b_3 \geq -60+180 \rightarrow b_3 \geq 120$

Thus, the range of ‘b_3’ is: $120 \leq b_3 \leq \infty$.

LP: Sensitivity Analysis…

Shadow Price for 1st Resource: As we showed earlier that if the right hand side of constraint–1 is increased by ‘h’ units then the optimal solution is:
$X_1 = 120–(h/5)$ & $X_2 = 160+(2h/5)$.
Profit → $Z = 2X_1 + 1.5X_2 = 2X_1 + (3/2)\,X_2$
$Z = (2)\,[120–(h/5)] + (3/2)[160+(2h/5)]$
$Z = (480+h/5)$

- Now, let ‘h=1’, we find that $Z = (480+(1/5))= 480.20$. → Shadow price for the 1st resource is 480.20–480 = Rs. 0.20.

Shadow Price for 2nd Resource: the optimal solution is: X_1 = ‘120+(4k/15)’ & X_2 = ‘160–(k/5)’.
Profit → $Z = 2X_1 + 1.5X_2 = 2X_1 + (3/2)\,X_2$
$Z = (2)[\,120+(4k/15)] +(3/2)[\,160–(k/5)]$
$Z = 480+(7k/30)$

- Now, let ‘k=1’, we find that $Z = [480+(7(1)/30]= 480.23$. →Shadow price for the 2nd resource is 480.23–480 = Rs.0.23.

Shadow Price for 3rd Resource: As we showed earlier that if the right hand side of constraint–3 is increased by ‘L’ units then the optimal solution is:
X_1 = ‘120’ & X_2 = ‘160’.
Profit → $Z = 2X_1 + 1.5X_2 = 2X_1 + (3/2)\,X_2$
$Z = (2)[120] +(3/2)[160]$
$Z = 480$

- The shadow price for this resource is ‘Zero’.

- Due to the excess of this resource constraint “$X_1 \leq 180$” is not binding on the optimal solution (X_1=120 & X_2=160).
- While, Constraint–1 and Constraint–2, which hold with equality at the optimal solution (X_1=120 & X_2=160), are said to be binding constraints.
- The objective function cannot be increased without increasing these resources. They have positive shadow prices.

Q & A