newton divided diff : $f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$ ; newton backward : $\Delta f(x_0) = f(x_1) - f(x_0)$

Date

# • Interpolation with equally spaced data :~

-> Newton forward difference $\nabla$ $\quad \nabla f(x_0) = f(x_1) - f(x_0)$

-> Newton backward difference $\Delta$

-> Centered difference

- -> Sitirling
- -> Gauss forward
- -> Gauss backward
- -> Laplace
- -> Bessal

**Q.** The population of a city is taken once in 10 years is given below. Estimate the population in 1955 using Newton formula

Note : When newton comes in Question then 3 things :

① newton divided diff ② newton forward ③ newton backward

↳ unequally spaced data

* in this Question, we have to find 1955 & it is placed in the begining of data so use forward

* If the point is placed towards the end of data then we backward

* If in the middle then use Centered diff formula

Date

| $x$ (Years) | $y$ (population) | | | |
|---|---|---|---|---|
| 1951 | 35 | | | |
| | | 7 | | |
| 1961 | 42 | | 9 | |
| | | 16 | | 1 |
| 1971 | 58 | | 10 | |
| | | 26 | | |
| 1981 | 84 | | | |

diff in intervals — interpolating point — initial point

$h=10, \; x=1955, \; x_0=1951$

$\therefore u = \dfrac{x-x_0}{h} = 0.4 \quad \left(\dfrac{1955-1951}{10}\right)$

$$P(x) = f(x_0) + \frac{u}{1!}\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) \ldots \frac{u(u-1)\ldots(u-(n-1))}{n!}\Delta^n f(x_0)$$

$$P(1955) = 35 + 0.4\times 7 + \frac{(0.4)(0.4-1)\times 9}{2!} + \frac{0.4(0.4-1)(0.4-2)\times 1}{3!}$$

$= 36.784$

## Newton Backward formula :-

$$P(x) = f(x_n) + \frac{v}{1!}\nabla f(x_n) + \frac{v(v+1)}{2!}\nabla^2 f(x_n) + \frac{v(v+1)(v+2)}{3!}\nabla^3 f(x_n) + \ldots \frac{v(v+1)\ldots(v+(n-1))}{n!}\nabla^n f(x_n)$$

$$= 84 + (-0.6)(26) + \frac{(-0.6)(-0.6+1)(10)}{2!} + \frac{(-0.6)(-0.6+1)(-0.6+2)\,1}{3!}$$

$= 67.144$

Date

| $x$ | P | $\nabla f$ | $\nabla^2 f$ | $\nabla^3 f$ |
| --- | --- | --- | --- | --- |
| 1951 | 35 | | | |
| | | 7 | | |
| 1961 | 42 | | 9 | |
| | | 16 | | 1 |
| 1971 | 58 | | 10 | $\nabla^3 f(x_n)$ |
| | | 26 | $\nabla^2 f(x_n)$ | |
| 1981 | 84 | $\nabla f(x_n)$ | | |
| | $f(x_n)$ | | | |

$x = 1975$

$v = \dfrac{1975-1981}{10} = -0.$

$v = \dfrac{x - x_n}{h}$

# Centered Difference :-

## 1.) Gauss forward :-

Use Gauss forward formula to find $y$ when $x = 2.7$

| | $x$ | $y$ | | | | | |
| --- | --- | --- | --- | --- | --- | --- | --- |
| $x_{-2}$ | 1.5 | 37.9 | | | | | |
| | | | 208.3 | | | | |
| $x_{-1}$ | 2 | 246.2 | | -54.2 | | | |
| | | | 163.1 | | 19 | | |
| $x_0$ | 2.5 | 409.3 | | -35.2 | | -12.6 | |
| | | | 127.9 | | 6.4 | | 15.5 |
| $x_1$ | 3 | 537.2 | | -28.8 | | 2.9 | |
| | | | 99.1 | | 9.3 | | |
| $x_2$ | 3.5 | 636.3 | | -19.5 | | | |
| | | | 79.6 | | | | |
| $x_3$ | 4 | 715.9 | | | | | |

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$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u+1)(u-2)}{4!}\Delta^4 y_{-2} + \dots$$

$$u = \frac{x - x_0}{h} = \frac{2.7-2.5}{0.5} = \frac{0.2}{0.5} = 0.4$$

$$y = 409.3 + (0.4)(127.9) + \frac{(0.4)(0.4-1)(-35.2)}{2!} + \frac{(0.4)(0.4-1)(0.4+1)(6.4)}{3!}$$

$$+ \frac{(0.4)(0.4-1)(0.4+1)(0.4-2)(-3.6)}{4!} + \frac{(0.4)(0.4-1)(0.4+1)(0.4-2)(0.4+2)(6.5)}{5!}$$

$= 464.309$

# Gauss Backward :~

$$y = y_0 + u\Delta y_0 + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-1} + \frac{u(u+1)(u-1)(u+2)}{4!}\Delta^4 y_{-2}$$

$$u = \frac{x - x_0}{h} = \frac{2.7-3}{0.5} = \frac{-0.3}{0.5} = -0.6$$

| | $x$ | $y$ | $\Delta f$ | $\Delta^2 f$ | $\Delta^3 f$ | $\Delta^4 f$ | $\Delta^5 f$ |
|---|---|---|---|---|---|---|---|
| $x_{-3}$ | 1.5 | 37.9 | | | | | |
| | | | 208.3 | | | | |
| $x_{-2}$ | 2 | 246.2 | | -45.2 | | | |
| | | | 163.1 | | 10 | | |
| $x_{-1}$ | 2.5 | 409.3 | | -35.2 | | 3.6 | |
| | | | 127.9 | | 6.4 | | 6.5 |
| $x_0$ | 3 | 537.2 | | -28.8 | | 2.9 | |
| | | | 99.1 | | 9.3 | | |
| $x_1$ | 3.5 | 636.3 | | -19.5 | | | |
| | | | 79.6 | | | | |
| $x_2$ | 4 | 715.9 | | | | | |

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Date 9thSep2023 **Lecture 6 :-** Saturday

# Bassel's Formula

$$y = \frac{y_0 + y_1}{2} + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_{-2}}{2}\right] + \frac{u(u-1)(u-1/2)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{u(u+1)(u-1)(u-2)}{4!}\left(\frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2}\right)$$

| | $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ | |
|---|---|---|---|---|---|---|---|
| $x_{-2}$ | 80 | 5026 | | | | | |
| | | | 648 | | | | |
| $x_{-1}$ | 85 | 5074 | | 40 | | | |
| | | | 688 | | -2 | | |
| $x_0$ | 90 | 6362 | | 38 | | 4 | other value is zero |
| | | | 726 | | 2 | | |
| $x_1$ | 95 | 7088 | | 40 | | | |
| | | | 766 | | | | |
| $x_2$ | 100 | 7654 | | | | | |

$$u = \frac{x - x_0}{h} = \frac{91-90}{5} = \frac{1}{5} = 0.2$$

find $y$ when $x = 91$

Date

# Central Difference Interpolation :-

Q. find $u_9$

| $x$ | $f(x)$ | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ | $\Delta^4 f(x)$ |
|---|---|---|---|---|---|
| 0 | $14_{y_{-2}}$ | | | | |
| | | $10_{y_{-2}}$ | | | |
| 4 | $24_{y_{-1}}$ | | $-2_{y_{-2}}$ | | |
| | | $8_{y_{-1}}$ | | $-3_{y_{-2}}$ | |
| 8 | $32_{y_0}$ | | $-5_{y_{-1}}$ | | $10_{y_{-2}}$ |
| | | $3_{y_0}$ | | $7_{y_{-1}}$ | |
| 12 | $35_{y_1}$ | | $2_{y_0}$ | | |
| | | $5_{y_1}$ | | | |
| 16 | $40_{y_2}$ | | | | |

## Gauss forward :-

$u = \frac{x - x_0}{h}$ ← interval diff $= \frac{9-8}{4} = \frac{1}{4} = 0.25$

$$f(a+hu) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-1} + \frac{u(u+1)(u-1)(u-2)}{4!}\Delta^4 y_{-2}$$

$$35 + \frac{0.25}{1}(3) + \frac{(0.25)(0.25-1)(-5)}{2} + \frac{(0.25)(0.25+1)(0.25-1)(7)}{6} + \frac{(0.25)(0.25+1)(0.25-1)(0.25-2)(10)}{24}$$

$= 33.1162$

Date

## Gauss Backward :-

$$f(a+hu) = y_0 + \frac{u}{1!}\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-2} + \frac{u(u-1)(u+1)(u+2)}{4!}\Delta^4 y_{-2}$$

$$= 32 + \frac{(0.25)}{1}(8) + \frac{(0.25)(0.25+1)}{2}(-5) + \frac{(0.25)(0.25-1)(0.25+1)}{6}(-3) + \frac{(0.25)(0.25-1)(0.25+1)(0.25+2)}{24}(10)$$

$$= 33.162$$

## Stirling formula :-

$$f(a+hu) = y_0 + \frac{u}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left(\frac{\Delta^2 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2}$$

$$= 32 + \frac{0.25}{1}\left(\frac{8+3}{2}\right) + \frac{(0.25)^2}{2}(-5) + \frac{(0.25)(0.25^2-1)}{3!}\left(\frac{-5-3}{2}\right) + \frac{(0.25)^2(0.25^2-1)}{4!}(10)$$

$$= 33.162$$

## Bessel's formula :-

$$f(a+hu) = \frac{1}{2}(y_0+y_1) + \frac{(u-1/2)}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right) + \frac{(u-1/2)u(u-1)}{3!}\Delta^3 y_{-1} +$$

$$\frac{(u+1)(u)(u-1)(u-2)}{4!}\left(\frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2}\right)$$

$$= \frac{1}{2}(32+35) + \frac{(0.25-1/2)}{1}(3) + \frac{(0.25)(0.25-1)}{2}\left(\frac{-5+2}{2}\right) + \frac{(0.25-1/2)(0.25)(0.25-1)}{6}(7)$$

$$+ \frac{(0.25+1)(0.25)(0.25-1)(0.25-2)}{24}\left(\frac{10+0}{2}\right) = 33.1162$$

Date

# Numerical Differentiation by Central Diff

1.) Sitirling :-

$$y = y_0 + u\frac{(\Delta y_0 + \Delta y_{-1})}{2} + \frac{u^2\Delta^2 y_{-1}}{2!} + \frac{u(u^2-1)}{3!}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2} + \ldots$$

$$\frac{dy}{du} = \frac{\Delta y_0 + \Delta y_{-1}}{2} + \frac{2u}{2}\Delta^2 y_{-1} + \frac{3u^2-1}{6}\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \ldots$$

$$u = \frac{x - x_0}{h}$$

$$\frac{du}{dx} = \frac{1}{h} \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} + u\Delta^2 y_{-1} + \frac{3u^2-1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)\right]$$

$$= \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} + u\Delta^2 y_{-1} + \frac{3u^2-1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) \ldots\right]$$

**Q.** find $dy/dx$ at $x = 5$

| | $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|---|
| $x_{-2}$ | 0 | 0 | | | | |
| | | | 1 | | | |
| $x_{-1}$ | 2 | 1 | | 0 | | |
| | | | 1 | | 2 | |
| $x_0$ | 4 | 2 | | 2 | | 16 |
| | | | 3 | | 18 | |
| $x_1$ | 6 | 5 | | 20 | | |
| | | | 23 | | | |
| $x_2$ | 8 | 28 | | | | |

Date

Ans: 1.125
or 1.79

$$\frac{dy}{dx} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} + u\Delta^2 y_{-1} + \frac{3u^2-1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{4u^3-2u}{4!}\Delta^4 y_{-2}\right.$$

$$\frac{d^2 u}{dx^2} = \frac{1}{h^2}\left[0 + \Delta^2 y_{-1} + u\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2} + \frac{12u^2-2}{4!}\Delta^4 y_{-2}\right] \times \frac{du}{dx} = \frac{1}{h}$$ so that is why it becomes $1/h^2$

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Date :

# Bessal's Formula :-

$u = \frac{x - x_0}{h}$

$$y = \frac{y_0 + y_1}{2} + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{(u)(u-1)}{2!}\left[\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right] + \frac{(u - 1/2)u(u-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{u(u+1)(u-1)(u-2)}{4!}\left[\frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2}\right] + \ldots$$

$$= 0 + \Delta y_0 + \frac{2u-1}{2!}\left[\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right] + \frac{(2u^3 - 3u^2 + u)}{2 \times 3!}\Delta^3 y_{-1} + \frac{u^4 - 2u^3 - u^2 + 2u}{4!}\left[\frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2}\right] +$$

$$= \left[\Delta y_0 + \frac{2u-1}{2}\left[\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right] + \left[\frac{6u^2 - 6u + 1}{12}\right]\Delta^3 y_{-1} + \frac{4u^3 - 6u^2 - 2u + 2}{48}(\Delta^4 y_{-1} + \Delta^4 y_{-2})\right]\frac{1}{h}$$

$$\frac{d^2 y}{dx^2} = \frac{1}{h^2}\left[\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2} + \frac{12u - 6}{12}\Delta^3 y_{-1} + \frac{12u^2 - 12u - 2}{48}(\Delta^4 y_{-1} + \Delta^4 y_{-2})\right.$$

# Laplace Everts formula :-

$\therefore p = 1 - u$

$\therefore u = \frac{x - x_0}{h}$

$$y_n = \left[p y_0 + \frac{p(p^2-1)}{3!}\Delta^2 y_{-1} + \frac{p(p^2-1)(p^2-2^2)}{5!}\Delta^4 y_{-2} + \ldots\right] +$$

$$\left[u y_1 + \frac{u(u^2-1)}{3!}\Delta^2 y_0 + \frac{u(u^2-1)(u^2-2^2)}{5!}\Delta^4 y_{-1} + \ldots\right]$$

Date

# Assignment 1:

① Derivation of Simpson Rule & Trapezoidal by Newton forward Rule

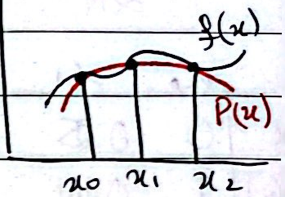
② Derivation of Error in trapezoidal rule & simpson rule

Date

# Simpson Rule :-

more accurate than trapezoidal cuz involves more points

$$\int_a^b f\,dx = \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right)$$




$$\int_a^b f\,dx = \int_{x_0}^{x_n} f\,dx = \int_{x_0}^{x_2} f\,dx + \int_{x_2}^{x_4} f\,dx + \int_{x_4}^{x_6} f\,dx + \ldots + \int_{x_{n-2}}^{x_n} f\,dx$$

$$= \frac{h}{3}\left(f(x_0)+4f(x_1)+f(x_2)\right) + \frac{h}{3}\left(f(x_2)+4f(x_3)+f(x_4)\right) + \frac{h}{3}\left(f(x_4)+4f(x_5)+f(x_6)\right)$$

$$+ \ldots + \frac{h}{3}\left(f(x_{n-2})+4f(x_{n-1})+f(x_n)\right)$$

$$= \frac{h}{3}\left[f(x_0)+f(x_n)+4f(x_1)+4f(x_3)+4f(x_5)\ldots+4f(x_{n-1})+2f(x_2)+2f(x_4)+\ldots + 2f(x_{n-2})\right]$$

$$= \frac{h}{3}\left\{f(x_0)+f(x_n)+4\left[f(x_1)+f(x_3)+f(x_5)+\ldots f(x_{n-1})\right]+2\left[f(x_2)+f(x_4)+\ldots f(x_{n-2})\right]\right.$$

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**Q.** $\int_0^1 \frac{1}{1+x^2}\,dx$ , $N=2$

$h = \frac{b-a}{2N} = \frac{1-0}{2(2)} = \frac{1}{4} = 0.25$

| | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|---|
| $x$ | 0 | 0.25 | 0.5 | 0.75 | 1 |
| $f(x_0)$ | 1 | 0.941 | 0.8 | 0.64 | 0.5 |

$$\Rightarrow \int_0^1 \frac{1}{1+x^2}\,dx = \frac{h}{3}\left[f(x_0) + f(x_4) + 4\left[f(x_1) + f(x_3)\right] + 2f(x_2)\right]$$

## Error Bounds :- (Trapezoidal)

$$E = -\frac{(b-a)h^2}{12} f''(c)$$

$$|E| = \frac{(b-a)h^2}{12} f''(c) \qquad c \in [0,1]$$

**Q.** $I = \int_0^1 e^x\,dx$ $\quad h = 1, 0.5, 0.25$

$f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$$0 \le c \le 1$$

$$f''(0) \le f''(c) \le f''(1)$$

$$e^0 \le f''(c) \le e^1 \qquad \times \text{ by } \frac{b-a}{12}h^2$$

$$\frac{(b-a)h^2}{12} \le \frac{(b-a)h^2}{12} f''(c) \le \frac{(b-a)h^2}{12} e^1$$

$$\frac{1}{12}h^2 \le |E| \le \frac{1}{12}h^2 e^1$$

$h = 1$

$$\frac{1}{12} \le |E| \le \frac{1}{12}e$$

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