OPERATIONS RESEARCH

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LP: SPECIAL CASES Using ITERATIVE METHODS

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LINEAR PROGRAMMING:
SPECIAL CASES USING ITERATIVE METHODS

The following variants are being considered:

1. Tie for the key (Pivot) column
2. Degeneracy (Tie for the Key (Pivot) row)
3. Unbounded solution
4. Multiple solutions
5. Non-existing feasible solution / Infeasible solution

DEGENERACY:
(Tie for the Key (Pivot) row)

- Graphically, degeneracy occurs when three or more constraints intersect in the solution of a problem with two variables

- Degeneracy will arise at the initial stage of the simplex method if at least one basic variable should be zero in the initial basic feasible solution

- Degeneracy will arise at any iteration of the simplex method where more than one variable is eligible to leave the basis (i.e. Tie for the key (pivot) row)

METHOD TO RESOLVE DEGENERACY

- **Step–1:** First find out the rows for which the minimum non–negative ratio is the same (Tie).

- **Step–2:** Now rearrange the columns of the simplex table so that the columns forming the identity (Unit) matrix come first.

- **Step–3:** Now find the minimum ratios of the simplex table columns from left to right one by one which shows the identity matrix, only for the tied rows until tie had not been broken. Whenever tie has been broken, choose that particular row which has minimum positive ratio as a Key row. Use the formula for the minimum ratios is:

  =(Elements of the particular identity matrix column)**/**(Corresponding elements of the Key column)

- After resolve the degeneracy, Simplex method is applied to obtain the optimum solution.

DEGENERACY

Maximize: $Z = 1000X_1 + 4000X_2 + 5000X_3$

Subject to:

$$3X_1 + 3X_3 \leq 22$$
$$X_1 + 2X_2 + 3X_3 \leq 14$$
$$3X_1 + 2X_2 \leq 14$$
$$X_1, X_2, X_3 \geq 0$$

Contribution Per Unit		C_j	1000	4000	5000	0	0	0	Ratio
C_{Bi}	Basic Variables (B)	Quantity (Qty)	X_1	X_2	X_3	S_1	S_2	S_3	
0	S_1	22	3	0	3	1	0	0	22/3
0	S_2	14	1	2	3*	0	1	0	14/3 ←
0	S_3	14	3	2	0	0	0	1	---
Total Profit (Z_j)		0	0	0	0	0	0	0	
Net Contribution ($C_j - Z_j$)			1000	4000	5000 ↑	0	0	0	

DEGENERACY (Cont…)

Contribution Per Unit		C_j	1000	4000	5000	0	0	0	Ratio	
C_{Bi}	Basic Variables	Quantity (Qty)	X_1	X_2	X_3	S_1	S_2	S_3		
0	S_1	8	2	–2	0	1	–1	0	---	
5000	X_3	14/3	1/3	(2/3)	1	0	1/3	0	(14/3) / (2/3) = 7	TIE
0	S_3	14	3	2	0	0	0	1	14 / 2 = 7	
Total Profit (Z_j)		70,000/3	5000/3	10,000/3	5000	0	5000/3	0		
Net Contribution ($C_j – Z_j$)			–2000/3	2000/3 ↑	0	0	–5000/3	0		

Contribution Per Unit		C_j	0	5000	0	1000	4000	0	Ratio	
C_{Bi}	Basic Variables	Quantity (Qty)	S_1	X_3	S_3	X_1	X_2	S_2	(S_1/X_2)	(X_3/X_2)
0	S_1	8	1	0	0	2	–2	–1	---	---
5000	X_3	14/3	0	1	0	1/3	2/3	1/3	0/(2/3)=0	1/(2/3) =3/2
0	S_3	14	0	0	1	3	2*	0	0/2 = 0	0/2 = 0 ←
Total Profit (Z_j)		70,000/3	0	5000	0	5000/3	10,000/3	5000/3		
Net Contribution ($C_j – Z_j$)			0	0	0	–2000/3	2000/3 ↑	–5000/3		

DEGENERACY (Cont...)

Contribution Per Unit		C_j	0	5000	0	1000	4000	0
C_{Bi}	Basic Variables (B)	Quantity (Qty)	S_1	X_3	S_3	X_1	X_2	S_2
0	S_1	22	1	0	1	5	0	–1
5000	X_3	0	0	1	–1/3	–2/3	0	1/3
4000	X_2	7	0	0	1/2	3/2	1	0
Total Profit (Z_j)		28000	0	5000	1000/3	8000/3	4000	5000/3
Net Contribution ($C_j – Z_j$)			0	0	–1000/3	–5000/3	0	–5000/3

Optimal solution is 28000 for Max. 'Z'; at X_1=0, X_2=7, X_3=0.

UNBOUNDED SOLUTION

- *When no variable qualifies to be the outgoing (leaving) variable then a linear programming problem would be unbounded* such situation arise if the incoming (entering) variable could be increased indefinitely without giving negative values to any of the current basic variables. In tabular form, this means that every coefficient in the key column (excluding rows Z_j and $(C_j - Z_j)$) is either negative or zero.

UNBOUNDED SOLUTION

- Solve the given LP problem:

  Maximize: $Z = 10X_1 + 20X_2$

  Subject to:

  $2X_1 + 4X_2 \geq 16$

  $X_1 + 5X_2 \geq 15$

  $X_1, X_2 \geq 0$

Contribution Per Unit C_j			10	20	0	0	–M	–M	Ratio
C_{Bi}	Basic Variables (B)	Quantity (Qty)	X_1	X_2	S_1	S_2	A_1	A_2	
–M	A_1	16	2	4	–1	0	1	0	16/4 = 4
–M	A_2	15	1	5*	0	–1	0	1	15/5 = 3 ←
Total Profit (Z_j)		–31M	–3M	–9M	M	M	–M	–M	
Net Contribution ($C_j – Z_j$)			10+3M	20+9M ↑	–M	–M	0	0	

Contribution Per Unit C_j			10	20	0	0	–M	–M	Ratio
C_{Bi}	Basic Variables (B)	Quantity (Qty)	X_1	X_2	S_1	S_2	A_1	A_2	
–M	A_1	4	(6/5)*	0	–1	4/5	1	–4/5	4/(6/5)=20/6 ←
20	X_2	3	1/5	1	0	–1/5	0	1/5	3/(1/5)=15
Total Profit (Z_j)		60–4M	4 – (6/5)M	20	M	–4–(4/5)M	–M	4M/5+4	
Net Contribution ($C_j – Z_j$)			6+(6/5)M ↑	0	–M	4+(4/5)M	0	–9M/5–4	

UNBOUNDED SOLUTION (Cont…)

Contribution Per Unit		C_j	10	20	0	0	–M	–M	Ratio
C_{Bi}	Basic Variables (B)	Quantity (Qty)	X_1	X_2	S_1	S_2	A_1	A_2	
10	X_1	10/3	1	0	–(5/6)	2/3	5/6	–2/3	---
20	X_2	7/3	0	1	(1/6)*	–1/3	–1/6	1/3	(7/3)/(1/6) = 14 ←
Total Profit (Z_j)		80	10	20	–5	0	5	0	
Net Contribution ($C_j – Z_j$)			0	0	5 ↑	0	–M–5	–M	

Contribution Per Unit		C_j	10	20	0	0	–M	–M	Ratio
C_{Bi}	Basic Variables (B)	Quantity (Qty)	X_1	X_2	S_1	S_2	A_1	A_2	
10	X_1	15	1	5	0	–1	0	1	(Negative Value)
0	S_2	14	0	6	1	–2	–1	2	(Negative Value)
Total Profit (Z_j)		150	10	50	0	–10	0	10	
Net Contribution ($C_j – Z_j$)			0	–30	0	10 ↑	–M	–M–10	

MULTIPLE OPTIMAL SOLUTIONS

After the simplex method finds one optimal basic feasible solution, you can detect other optimal basic feasible solutions if there are any and find them as follows:

Whenever a problem has more than one optimal basic feasible solution, at least one of the non–basic variables has a coefficient of zero in the $(C_j - Z_j)$ row Zero (0), so increasing any such variable will not change the value of the objective function. Therefore, these other optimal basic feasible solutions can be identified (if desired) by performing additional iterations of the simplex method, each time choosing a non–basic variable with Zero (0) coefficient as the entering (incoming) variable.

MULTIPLE OPTIMAL SOLUTIONS

Maximize: $Z = 2000X_1 + 3000X_2$
Subject to:

$6X_1 + 9X_2 \leq 100$
$2X_1 + X_2 \leq 20$
$X_1, X_2 \geq 0$

Max. $Z = 2000X_1 + 3000X_2 + 0S_1 + 0S_2$
Subject to:
$2X_1 + 4X_2 + S_1 = 16$
$X_1 + 5X_2 + S_2 = 15$

Contribution Per Unit		C_j	2000	3000	0	0	Ratio
C_{Bi}	Basic Variables (B)	Quantity (Qty)	X_1	X_2	S_1	S_2	
0	S_1	100	6	9*	1	0	100/9 ←
0	S_2	20	2	1	0	1	20/1
Total Profit (Z_j)		0	0	0	0	0	
Net Contribution ($C_j - Z_j$)			2000	3000 ↑	0	0	

Contribution Per Unit		C_j	2000	3000	0	0	Ratio
C_{Bi}	Basic Variables (B)	Quantity (Qty)	X_1	X_2	S_1	S_2	
3000	X_2	100/9	2/3	1	1/9	0	(100/9)/(2/3)=50/3
0	S_2	80/9	(4/3)*	0	−1/9	1	(80/9)/(4/3)=20/3 ←
Total Profit (Z_j)		100000/3	2000	3000	1000/3	0	
Net Contribution ($C_j - Z_j$)			0 ↑	0	−1000/3	0	

Since all the values in the above table of $(C_j - Z_j) \leq 0$; so, we having optimal solution. ($X_1 = 0$, $X_2 = 100/9$, $Z_j = 100000/3$).

MULTIPLE OPTIMAL SOLUTIONS

Contribution Per Unit		C_j	2000	3000	0	0
C_{Bi}	Basic Variables (B)	Quantity (Qty)	X_1	X_2	S_1	S_2
3000	X_2	20/3	0	1	1/6	1/2
2000	X_1	20/3	1	0	1/12	3/4
Total Profit (Z_j)		100000/3	2000	3000	1000/3	0
Net Contribution ($C_j - Z_j$)			0	0	–1000/3	0

So, the value of the objective function (Z_j) = 100000/3; at X_1=20/3, X_2=20/3

NON–EXISTING FEASIBLE / INFEASIBLE SOLUTIONS

- The infeasibility (mean’s no feasible solutions exist) condition occurs when the LP problem has incompatible constraints. Science final simplex table as shown optimal solution as all (C_j – Z_j) row elements are negative or zero. However, observing the solution basis, if we find any artificial variable is present as a basic variable then the LP problem has infeasible solution. Because the artificial variables have no meaning therefore the values of the artificial variables are totally meaningless.

NON–EXISTING FEASIBLE / INFEASIBLE SOLUTIONS

Solve the given LP problem using Two–Phase Simplex Method:

Maximize: $Z = 5X_1 + 3X_2$

Subject to:

$2X_1 + X_2 \leq 1$

$X_1 + 4X_2 \geq 6$

$X_1, X_2 \geq 0$

Max. $Z^* = 0X_1+0X_2+0S_1+0S_2-A_1$

Subject to:

$2X_1+X_2+S_1 = 1$

$X_1+4X_2-S_2+A_1 = 6$

$X_1, X_2, S_1, S_2, A_1 \geq 0$

Contribution Per Unit	C_j		0	0	0	0	–1	
C_{Bi}	Basic Variables (B)	Quantity (Qty)	X_1	X_2	S_1	S_2	A_1	Ratio
0	S_1	1	2	1*	1	0	0	1/1 = 1 ←
–1	A_1	6	1	4	0	–1	1	6/4 = 3/2
Total Profit (Z_j*)		–6	–1	–4	0	1	–1	
Net Contribution (C_j – Z_j*)			1	4 ↑	0	–1	0	

Contribution Per Unit	C_j		0	0	0	0	–1
C_{Bi}	Basic Variables (B)	Quantity (Qty)	X_1	X_2	S_1	S_2	A_1
0	X_2	1	2	1	1	0	0
–1	A_1	2	–7	0	–4	–1	1
Total Profit (Z_j*)		–2	7	0	4	1	–1
Net Contribution (C_j – Z_j*)			–7	0	–4	–1	0

Since all the elements in the (C_j – Z_j) row are negative or zero (≤ 0), so we are with optimal solution but since 'A_1' artificial variable appear as a basic variable, thus the above LP problem does not have any feasible solution. In other words, the given LP problem has infeasible solution.

QUESTIONS