Operations Research

Linear Programming: Iterative Methods

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LP: Iterative Methods

1. Simplex Method
2. The "Big M method" or the Charnes method of penalty
3. Two Phase Simplex Method

BASIC CONCEPTS:

- **Basic Solution (BS):** Each solution to any system of equations is called a Basic Solution (BS).
- **Basic Feasible Solution (BFS):** A Basic Feasible Solution (BFS) satisfies the model constraints and has the same number of variables with nonnegative values as there are constraints.
- **Non-Basic Variables (NBV):** The variables that equal to zero at a basic feasible solution point are called Non-Basic Variables (NBV).
- **Basic Variables (BV):** The variables which have values (other than zero) at a basic feasible solution point are called Basic Variables (BV).
- **Slack & Surplus:** Slack is the leftover of a resource. Surplus is the excess of production.
- **Entering Basic Variables (Incoming variable):** The variables that go from non-basic variables (which increase from Zero) to basic variables are called the Entering Basic Variables.
- **Leaving Basic Variables (Outing Variable):** The variables that go from basic variables (which drop to Zero) to non-basic variables are called the Leaving Basic Variables.
- **Minimum Replacement Ratio Test:** The Minimum replacement Ratio Test is to determine which basic variable drops to zero first as the entering basic variable increased.

LP–Iterative Methods: Simplex Method

EXAMPLE: ABC Furniture Company produces computer tables and chairs on a daily basis. Each computer table produced results in Rs. 160 in profit; each chair results in Rs. 200 in profit. The production of computer tables and chairs is dependent on the availability of limited resources: Labor, Wood, and Storage Space. The resource requirements for the production of tables and chairs and the total resources available are as follows.

RESOURCE REQUIREMENTS			
Resources	Computer Table	Chair	Total Available per day
Labor (hour)	2	4	40
Wood (feet)	18	18	216
Storage (Square feet)	24	12	240

The company wants to know the number of computer tables and chairs to produce per day in order to maximize the profit.

SOLUTION: Let 'X_1' and 'X_2' be the number of computer tables and chairs are produced respectively.
The complete formulation of the LP problem is as follow:

Maximize: $Z = 160X_1 + 200X_2$

Subject to:

$$2X_1 + 4X_2 \leq 40 \quad \text{(Labor Constraint)}$$
$$18X_1 + 18X_2 \leq 216 \quad \text{(Wood Constraint)}$$
$$24X_1 + 12X_2 \leq 240 \quad \text{(Storage Constraint)}$$
$$X_1, X_2 \geq 0$$

LP–Iterative Methods: Simplex Method

<u>SOLUTION</u>: Let 'X_1' and 'X_2' be the number of computer tables and chairs are produced respectively. The complete formulation of the LP problem is as follow:

Maximize: $Z = 160X_1 + 200X_2$

Subject to:

$2X_1 + 4X_2 \leq 40$ (Labor Constraint)

$18X_1 + 18X_2 \leq 216$ (Wood Constraint)

$24X_1 + 12X_2 \leq 240$ (Storage Constraint)

$X_1, X_2 \geq 0$

Standard LP Form:

Maximize: $Z = 160X_1 + 200X_2 + 0S_1 + 0S_2 + 0S_3$

Subject to:

$2X_1 + 4X_2 + S_1 = 40$

$18X_1 + 18X_2 + S_2 = 216$

$24X_1 + 12X_2 + S_3 = 240$

$X_1, X_2, S_1, S_2, S_3 \geq 0$

LP–Iterative Methods: Simplex Method

Standard LP Form:

Maximize: $Z = 160X_1 + 200X_2 + 0S_1 + 0S_2 + 0S_3$

Subject to:

$$2X_1 + 4X_2 + S_1 = 40$$
$$18X_1 + 18X_2 + S_2 = 216$$
$$24X_1 + 12X_2 + S_3 = 240$$
$$X_1, X_2, S_1, S_2, S_3 \geq 0$$

Contribution Per Unit	C_j		160	200	0	0	0	
C_{Bi}	Basic Variables (B)	Quantity (Qty)	X_1	X_2	S_1	S_2	S_3	Ratio = [Qty / PE > 0]
0	S_1	40	2	4*	1	0	0	40/4 = 10 ←
0	S_2	216	18	18	0	1	0	216/18 = 12
0	S_3	240	24	12	0	0	1	240/12 = 20
Total Profit (Z_j)		0	0	0	0	0	0	
Net Contribution ($C_j – Z_j$)			160	200 ↑	0	0	0	

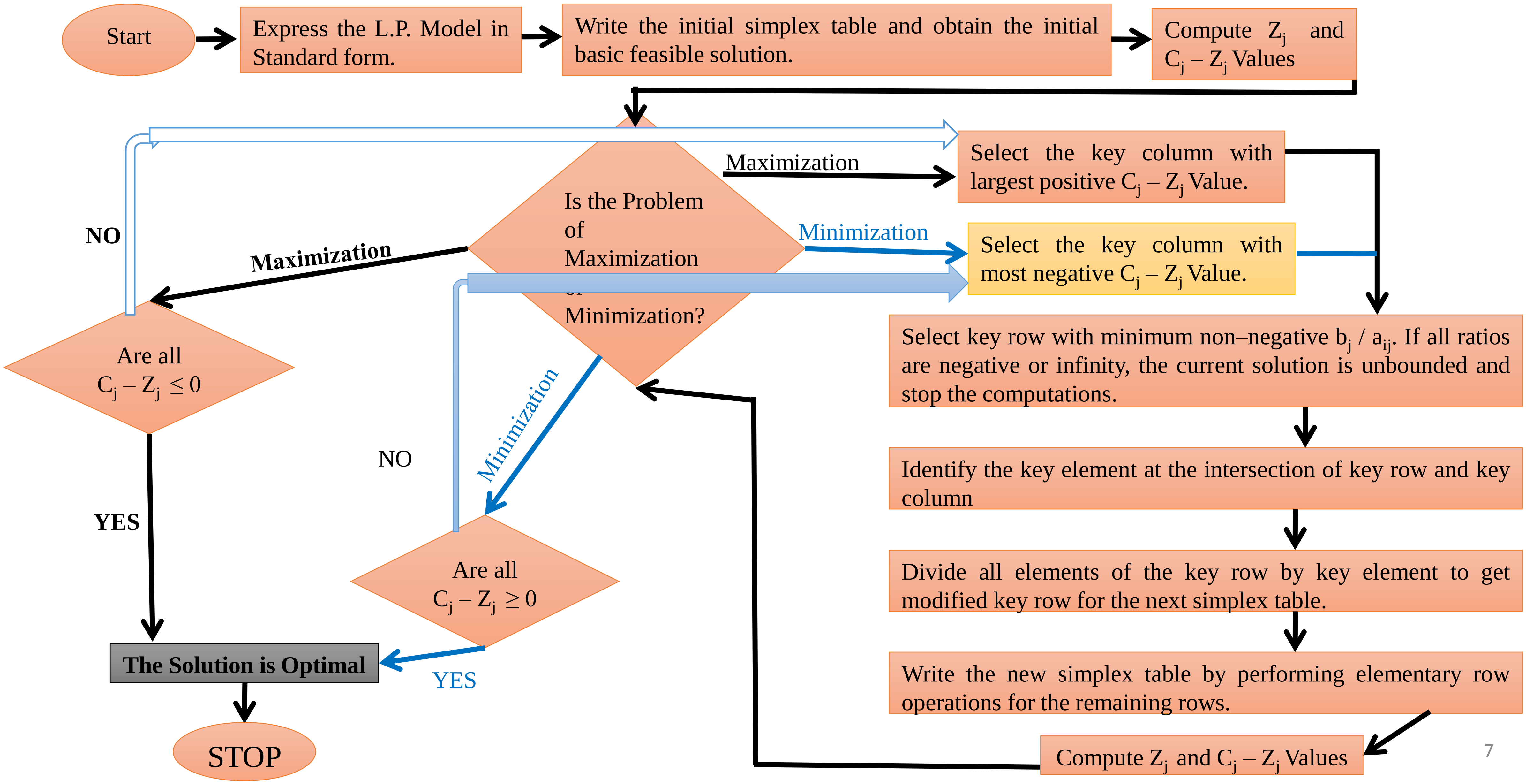
LP–Iterative Methods: Simplex Method

Contribution Per Unit C_j			160	200	0	0	0	
C_{Bi}	Basic Variables (B)	Quantity (Qty)	X_1	X_2	S_1	S_2	S_3	Ratio = [Qty / PE > 0]
200	X_2	10	1/2	1	1/4	0	0	10/(1/2) = 20
0	S_2	36	9*	0	–9/2	1	0	36/9 = 4 ←
0	S_3	120	18	0	–3	0	1	120/18 = 20/3
Total Profit (Z_j)		2000	100	200	50	0	0	
Net Contribution ($C_j – Z_j$)			60 ↑	0	–50	0	0	

Contribution Per Unit C_j			160	200	0	0	0
C_{Bi}	Basic Variables (B)	Quantity (Qty)	X_1	X_2	S_1	S_2	S_3
200	X_2	8	0	1	1/2	–1/18	0
160	X_1	4	1	0	–1/2	1/9	0
0	S_3	48	0	0	6	–2	1
Total Profit (Z_j)		2240	160	200	20	20/3	0
Net Contribution ($C_j – Z_j$)			0	0	–20	–20/3	0

Since all the values of ($C_j – Z_j$) ≤ 0; so, the solution is optimal. Optimal solution is 2240 for Max. 'Z'; at X_1= 4, X_2= 8.

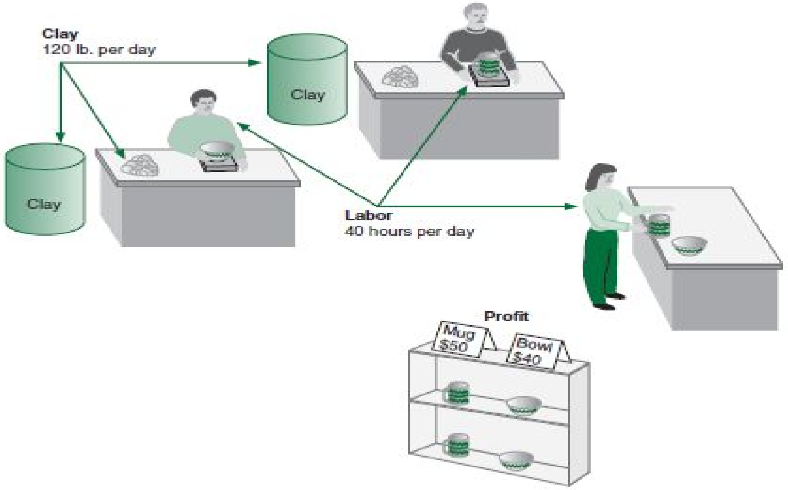
LP–Iterative Methods: Simplex Method



LP–Iterative Methods: Simplex Method

Product Mix

	Resource Requirements		
Product	Labor (Hr./Unit)	Clay (Lb./Unit)	Profit ($/Unit)
Bowl	1	4	40
Mug	2	3	50



- Product mix problem - Beaver Creek Pottery
- How many bowls and mugs should be produced to maximize profits given labor and materials constraints?
- **Resource Availability:** 40 hrs of labor per day (labor constraint)
  120 lbs of clay (material constraint)

Decision Variables: → Let X_1 & X_2 be the <u>number of bowls and mugs produced</u>, respectively.

Objective Function: → Maximize: $Z = 40X_1 + 50X_2$

Constraints: →

$X_1 + 2X_2 \leq 40$ (Labor Constraint)
$4X_1 + 3X_2 \leq 120$ (Clay Constraint)
$X_1, X_2 \geq 0$ (Non–Negativity)

LP MODEL:→ Maximize: $Z = 40X_1 + 50X_2$
(Profit Function)

SUBJECT TO:

$X_1 + 2X_2 \leq 40$ (Labor Constraint)
$4X_1 + 3X_2 \leq 120$ (Clay Constraint)
$X_1, X_2 \geq 0$ (Non–Negativity)

LP–Iterative Methods: Big M Method

Big 'M' Method: Question

Use Big 'M' method to solve the LP problem.

Minimize: $Z = 8X_1 + 4X_2$

Subject to:

$$3X_1 + X_2 \geq 27$$

$$X_1 + X_2 = 21$$

$$X_1 + 2X_2 \leq 40$$

$$X_1 , X_2 \geq 0$$

LP–Iterative Methods: Big M Method

Maximize: $Z = -8X_1 - 4X_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

Subject to:

$$3X_1 + X_2 - S_1 + A_1 = 27$$
$$X_1 + X_2 + A_2 = 21$$
$$X_1 + 2X_2 + S_2 = 40$$
$$X_1, X_2, S_1, S_2, A_1, A_2 \geq 0$$

Contribution Per Unit		C_j	–8	–4	0	0	–M	–M	
C_{Bi}	Basic Variables (B)	Quantity (Qty)	X_1	X_2	S_1	S_2	A_1	A_2	Ratio= [Qty/PE>0]
–M	A_1	27	3*	1	–1	0	1	0	27/3 = 9 ←
–M	A_2	21	1	1	0	0	0	1	21/1 = 21
0	S_2	40	1	2	0	1	0	0	40/1 = 40
Total Profit (Z_j)		–48M	–4M	–2M	M	0	–M	–M	
Net Contribution ($C_j - Z_j$)			–8+4M ↑	–4+2M	–M	0	0	0	

Contribution Per Unit		C_j	–8	–4	0	0	–M	–M	
C_{Bi}	Basic Variables (B)	Quantity (Qty)	X_1	X_2	S_1	S_2	A_1	A_2	Ratio
–8	X_1	9	1	1/3	–1/3	0	1/3	0	27
–M	A_2	12	0	(2/3)*	1/3	0	–1/3	1	18 ←
0	S_2	31	0	5/3	1/3	1	–1/3	0	18.6
Total Profit (Z_j)		–72–12M	–8	–8/3–(2/3)M	8/3–M/3	0	–8/3+M/3	–M	
Net Contribution ($C_j - Z_j$)			0	–4/3+(2/3)M ↑	–8/3+M/3	0	8/3– (2/3)M	0	

LP–Iterative Methods: Big M Method

Contribution Per Unit		C_j	–8	–4	0	0	–M	–M
C_{Bi}	Basic Variables (B)	Quantity (Qty)	X_1	X_2	S_1	S_2	A_1	A_2
–8	X_1	3	1	0	–1/2	0	1/2	–1/2
–4	X_2	18	0	1	1/2	0	–1/2	3/2
0	S_2	1	0	0	–1/2	1	1/2	–5/2
Total Profit (Z_j)		–96	–8	–4	2	0	–2	–2
Net Contribution $(C_j – Z_j)$			0	0	–2	0	–M+2	–M+2

Optimal solution is –96 for Max. 'Z'; at $X_1 = 3$, $X_2 = 18$.
Because Min (Z) = –Max (–Z) = 96; at $X_1 = 3$, $X_2 = 18$.

Use Big 'M' method to solve the LP problem.

Minimize: $Z = -2X_1 - X_2$

Subject to:

$$X_1 + X_2 \geq 2$$
$$X_1 + X_2 \leq 4$$
$$X_1, X_2 \geq 0$$

LP–Iterative Methods: Two Phase Simplex Method

PHASE–1

- ***STEP–1:***
  - *The objective function of the given LP problem must be in the form of Maximized. If it is to be minimized then we convert it into a problem of Maximization by Max Z = – Min (–Z).*
  - *Check whether all the values on the right hand side of the constraints must be positive. If any one of them is negative then multiply that constraint with (–1).*

- **STEP–2:**
  - Express the problem in standard form by introducing slack, surplus and artificial variables. And Assign a cost '–1' to each artificial variable while a cost '0' to all other variables and formulate a new objective function 'Z*'.
  - Write down the auxiliary LP problem in which new objective function is to be maximized subject to the given constraints.

- **STEP–3:** Solve the auxiliary LP problem by simplex method until and obtain the optimum basic feasible solution. If (C_j – Z_j*) row indicates optimal solution as it contains all zero or negative elements and if:
  - the artificial variable appears as a basic variable then the given LP problem has infeasible solution so we terminate our solution in Phase–1.
  - the artificial variable does not appear as a basic variable then the given LP problem has a feasible solution and we proceed to Phase–II.

LP–Iterative Methods: Two Phase Simplex Method

PAHSE–II

- Use the optimum basic feasible solution of Phase–I as a starting solution for the original LP problem.
- Assign the actual costs to the variables in the objective function and a zero cost to every artificial variable in the basis at zero level.
- Delete the artificial variable columns from the table which is eliminated from the basis in Phase–I.
- Apply simplex algorithm to the modified simplex table obtained at the end of Phase–I till an optimum basic feasible is obtained or till there is an indication of unbounded solution.

LP–Iterative Methods: Two Phase Simplex Method

EXAMPLE:

Use Two–Phase Simplex method to solve the LP problem.

Maximize: $Z = 50X_1 + 30X_2$

Subject to:

$$2X_1 + X_2 \geq 18$$
$$X_1 + X_2 \geq 12$$
$$3X_1 + 2X_2 \leq 34$$
$$X_1, X_2 \geq 0$$

STANDARD FORM:

Max. $Z^* = 0X_1 + 0X_2 + 0S_1 + 0S_2 + 0S_3 - A_1 - A_2$

Subject to:

$$2X_1 + X_2 - S_1 + A_1 = 18$$
$$X_1 + X_2 - S_2 + A_2 = 12$$
$$3X_1 + 2X_2 + S_3 = 34$$

LP–Iterative Methods: Two Phase Simplex Method

PHASE - I

Contribution Per Unit		C_j	0	0	0	0	0	–1	–1	
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	X_1	X_2	S_1	S_2	S_3	A_1	A_2	Ratio
–1	A_1	18	2*	1	–1	0	0	1	0	18/2 = 9 ←
–1	A_2	12	1	1	0	–1	0	0	1	12/1 = 12
0	S_3	34	3	2	0	0	1	0	0	34/3 = 11.33
Total Profit (Z_j*)		–30	–3	–2	1	1	0	–1	–1	
Net Contribution ($C_j – Z_j$*)			3 ↑	2	–1	–1	0	0	0	

Contribution Per Unit		C_j	0	0	0	0	0	–1	
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	X_1	X_2	S_1	S_2	S_3	A_2	Ratio
0	X_1	9	1	1/2	–1/2	0	0	0	18
–1	A_2	3	0	1/2 *	1/2	–1	0	1	6 ←
0	S_3	7	0	1/2	3/2	0	1	0	14
Total Profit (Z_j*)		–3	0	–1/2	–1/2	1	0	–1	
Net Contribution ($C_j – Z_j$*)			0	1/2 ↑	1/2	–1	0	0	

LP–Iterative Methods: Two Phase Simplex Method

PHASE–I

Contribution Per Unit		C_j	0	0	0	0	0
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	X_1	X_2	S_1	S_2	S_3
0	X_1	**6**	**1**	**0**	**–1**	**1**	**0**
0	X_2	**6**	**0**	**1**	**1**	**–2**	**0**
0	S_3	**4**	**0**	**0**	**1**	**1**	**1**
Total Profit (Z_j*)		0	0	0	0	0	0
Net Contribution (C_j – Z_j*)			0	0	0	0	0

Since all the values of (C_j – Z_j*) are zero. Thus, we enter to the Phase–II.

PHASE–II

Contribution Per Unit		C_j	50	30	0	0	0	
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	X_1	X_2	S_1	S_2	S_3	Ratio
50	X_1	**6**	**1**	**0**	**–1**	**1**	**0**	---
30	X_2	**6**	**0**	**1**	**1**	**–2**	**0**	6/1 = 6
0	S_3	**4**	**0**	**0**	**1***	**1**	**1**	4/1 = 4 ⟵
Total Profit (Z_j)		480	50	30	–20	–10	0	
Net Contribution (C_j – Z_j)			0	0	20 ↑	10	0	

LP–Iterative Methods: Two Phase Simplex Method

PHASE–II

Contribution Per Unit		C_j	50	30	0	0	0
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	**X_1**	**X_2**	**S_1**	**S_2**	**S_3**
50	X_1	10	1	0	0	2	1
30	X_2	2	0	1	0	–3	–1
0	S_1	4	0	0	1	1	1
Total Profit (Z_j)		560	50	30	0	10	20
Net Contribution ($C_j – Z_j$)			0	0	0	–10	–20

Since all the values of $(C_j – Z_j) \leq 0$; So, we are having optimal solution.

Thus, $Z_j = 560$ at $X_1=10$ and $X_2=2$.

Use Two–Phase Simplex method to solve the LP problem.

Minimize: $Z = –2X_1 – X_2$

Subject to:

$X_1 + X_2 \geq 2$

$X_1 + X_2 \leq 4$

$X_1, X_2 \geq 0$

LP–Iterative Methods: Two Phase Simplex Method

Max. $Z^* = 0X_1+0X_2+0S_1+0S_2-A_1$

Subject to:

$$X_1+X_2-S_1+A_1 = 2$$
$$X_1+X_2+S_2 = 4$$

PHASE–I

Contribution Per Unit		C_j	0	0	0	0	–1	
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	X_1	X_2	S_1	S_2	A_1	Ratio
–1	A_1	2	1*	1	–1	0	1	2/1 = 2 ←
0	S_2	4	1	1	0	1	0	4/1 = 4
Total Profit (Z_j*)		–2	–1	–1	1	0	–1	
Net Contribution (C_j – Z_j*)			1 ↑	1	–1	0	0	

Contribution Per Unit		C_j	0	0	0	0
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	X_1	X_2	S_1	S_2
0	X_1	2	1	1	–1	0
0	S_2	2	0	0	1	1
Total Profit (Z_j*)		0	0	0	0	0
Net Contribution (C_j – Z_j*)			0	0	0	0

LP–Iterative Methods: Two Phase Simplex Method

PHASE–II

Contribution Per Unit		C_j	2	1	0	0	
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	**X_1**	**X_2**	**S_1**	**S_2**	Ratio
2	**X_1**	**2**	**1**	**1**	**–1**	**0**	---
0	**S_2**	**2**	**0**	**0**	**1***	**1**	2/1 = 2 ⟵
Total Profit (Z_j)		4	2	2	–2	0	
Net Contribution ($C_j – Z_j$)			0	–1	2 ↑	0	

Contribution Per Unit		C_j	2	1	0	0
C_{Bi}	**Basic Variables (B)**	**Quantity (Qty)**	**X_1**	**X_2**	**S_1**	**S_2**
2	X_1	4	1	1	0	1
0	S_1	2	0	0	1	1
Total Profit (Z_j)		8	2	2	0	2
Net Contribution ($C_j – Z_j$)			0	–1	0	–2

Since all the values of $(C_j – Z_j) \leq 0$; so, we are having optimal solution. Thus, optimal solution is ‘8’ for Max. ‘Z’; at X_1=4, X_2=0. Because Min (Z) = –Max (–Z) = –8; at X_1=4, X_2=0.

Q & A