5

Direct Methods for Linear Systems Ax=b

A number of problems in engineering and science require simultaneous linear algebraic equations. The problems of obtaining the solution of a chemical plant, the analysis of electronic circuits consisting of frequently invariant elements, analysis of a network under conditions of sinusoidal steady-state excitation, determination of annual cost of education in different colleges or even chemical reactions are some of the examples where the study will lead to the use of simultaneous linear algebraic equations. Hence the need to understand the methods of solution of simultaneous linear algebraic equations.

To yield the exact solution, the direct methods tend to be used for solving a simultaneous linear algebraic equations. The methods given below :

1: *Gauss-Elimination*
2: *Direct-Factorization*
3: *Indirect-Factorization*
4: *Symmetric-Factorization*
5: *Tridiagonal-Factorization*

used for solving a simultaneous linear algebraic equations :

$$\left.\begin{array}{l}a_{11}x_1+a_{12}x_2+\dots\dots\dots+a_{1n}x_n=b_1\\a_{21}x_1+a_{22}x_2+\dots\dots\dots+a_{2n}x_n=b_2\\ \vdots\\ \vdots\\ a_{m1}x_1+a_{m2}x_2+\dots\dots\dots+a_{mn}x_n=b_m\end{array}\right\} \tag{5.1}$$

which in matrix notation can be written as :

$$\begin{bmatrix}a_{11} & a_{12} & \dots\dots & a_{1n}\\ a_{21} & a_{21} & \dots\dots & a_{2n}\\ a_{31} & a_{32} & \dots\dots & a_{3n}\\ \vdots & & & \\ a_{m1} & a_{m2} & \dots\dots & a_{mn}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\ \vdots\\ x_n\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\\ \vdots\\ b_m\end{bmatrix} \Rightarrow Ax=b \tag{5.2}$$

The matrix with known entries a_{ij}, i=1,2,...,m, j=1,2,...,n of the i-th row and j-th column is called the **coefficient matrix** $A=(a_{ij})_{m\times n}$. The matrix with known entire b_i, i= 1, 2,...,m, is called a **column matrix** of the known values $b=b_i$. The matrix with unknown entries $x=x_i$, i=1,2,...,n, called a **column matrix** is to be determined.

If all b_i are zeros, then the system of equations (5.1) is said to be **homogeneous**, otherwise **inhomogeneous** for at least one of b_i not zero.

1 : The homogeneous $n\times n$ system $\mathbf{Ax=0}$ has a trivial solution $x_1=x_2=\dots=x_n=0$, if the coefficient matrix A is nonsingular: $\det A\neq 0$, otherwise nontrivial solution for the singular matrix A:

$\det A = 0$.

2 : The inhomogeneous $n \times n$ system $\mathbf{Ax} = \mathbf{b}$ has a unique solution if the coefficient matrix A is nonsingular; $\det A \neq 0$.

Normally, there are three types of solutions to the system (5.2), that can be found by reducing the **augmented matrix A/b** into **echelon form** :

a : ***Independent system*** : One unique solution if the lines of the system (5.2) are intersecting.

b : ***Inconsistent system*** : No solution if the lines are parallel. A system like this is normally called ill-conditioned system of equations.

c : ***Dependent system*** : Infinitely many solutions if the lines are coinciding.

$x_1 + x_2 = 1$	$x_1 + x_2 = 1$	$x_1 + x_2 = 1$
$x_1 - x_2 = 0$	$x_1 + x_2 = 0$	$2x_1 + 2x_2 = 2$
a	b	c

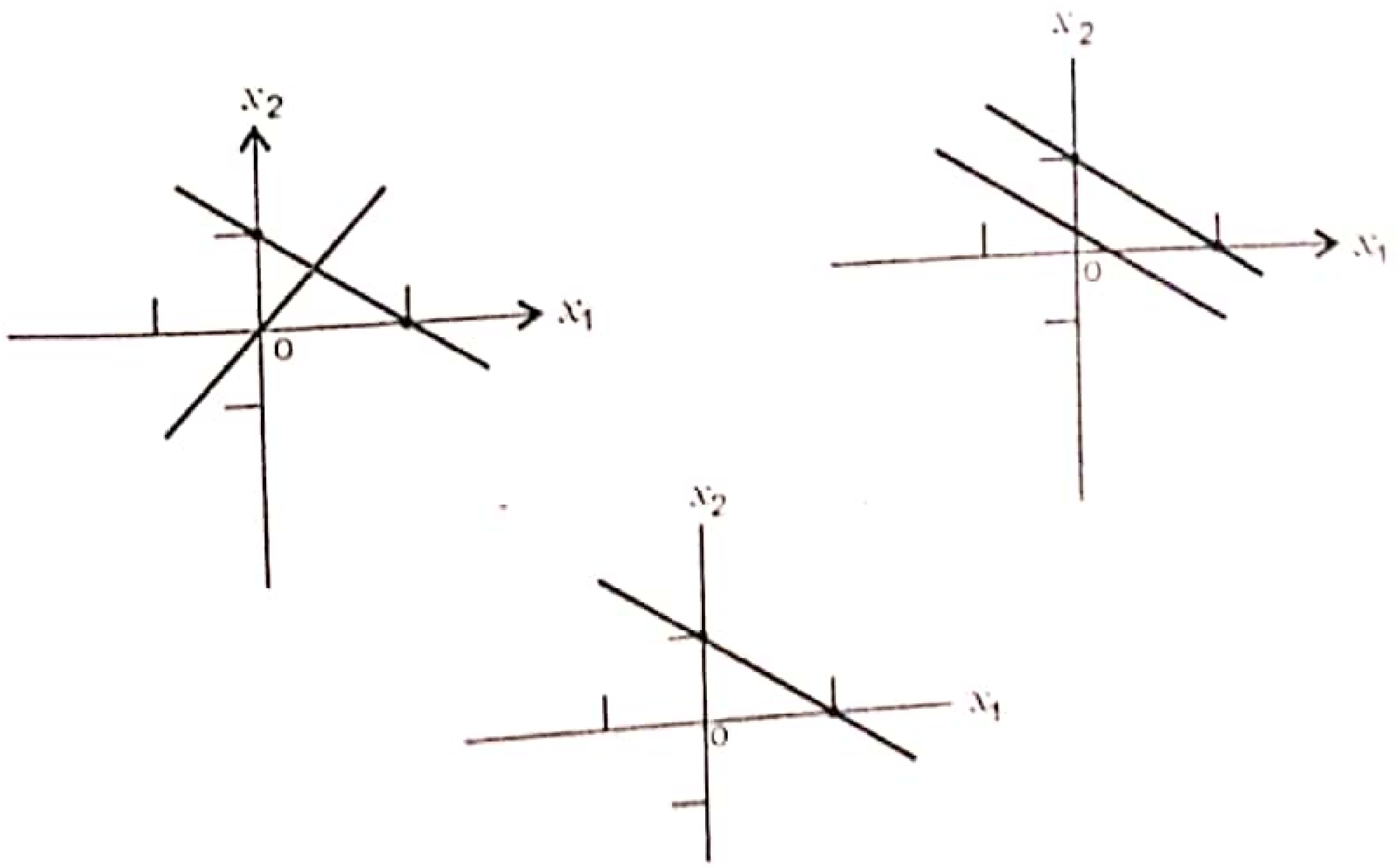

Fig 5.1: Geometrical Interpretation of the System of Linear Equations

The system (5.2) can be solved directly in the following cases :

i : For $\mathbf{A}=\mathbf{D}$, the system of equations (5.2) can be written as :

$$\begin{aligned} a_{11}x_1 \qquad\qquad\qquad &= b_1 \\ a_{22}x_2 \qquad\quad &= b_2 \\ &\vdots \\ a_{nn}x_n &= b_n \end{aligned} \tag{5.3}$$

The solution is given by :

$$x_i=\frac{b_i}{a_{ii}}, \ where\ a_{ii}\neq 0,\ i=1,2,3,\ldots,n.$$

ii : For $\mathbf{A}=\mathbf{L}$, the system of equations (5.2) can be written as :

$$\begin{aligned} a_{11}x_1 \qquad\qquad\qquad\qquad\qquad &= b_1 \\ a_{21}x_1+a_{22}x_2 \qquad\qquad\qquad &= b_2 \\ a_{31}x_1+a_{32}x_2+a_{33}x_3 \qquad\quad &= b_3 \\ \vdots \qquad\qquad\qquad\qquad & \\ a_{n1}x_1+a_{n2}x_2+\ldots \quad +a_{nn}x_n &= b_n \end{aligned} \tag{5.4}$$

The unknowns in the order $x_1,x_2,\ldots,x_{n-1},x_n$:

$$x_1=b_1/a_{11}$$

$$x_2=(b_2-a_{21}x_1)/a_{22}$$

$$x_3=(b_3-a_{31}x_1-a_{32}x_2)/a_{33}$$

$$\vdots$$

$$x_n=(b_n-\sum_{j=1}^{n-1} a_{nj}x_j)/a_{nn},\ where\ a_{ij}\neq 0,\ i=1,2,3,\ldots,n$$

are found by a method is known as **forward substitution method**.

iii : For $\mathbf{A}=\mathbf{U}$, the system of equations (5.2) can be written as :

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n,n}x_n &= b_n
\end{aligned}
\qquad (5.5)
$$

The unknowns in the order $x_n, x_{n-1}, \dots, x_1$:

$$
\begin{aligned}
x_n &= b_n / a_{n,n} \\
x_{n-1} &= (b_{n-1} - a_{n-1,n}\, x_n)/a_{n-1,n-1} \\
&\vdots \\
x_1 &= (b_1 - \sum_{j=2}^{n} a_{1j}x_j)/a_{11}, \quad j = 2,3,\dots,n.
\end{aligned}
$$

are found by a method is known as **backward substitution method.**

Gauss Elimination Method

This method is based on the idea of reducing the coefficient matrix **A** of the linear system (5.2) to an upper triangular matrix **U** for which the 3×3 system can be written as :

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \; : \; E(1) \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \; : \; E(2) \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \; : \; E(3)
\end{aligned}
\qquad (5.6)
$$

Step 1 : To eliminate x_1, multiply the first row by a_{21}/a_{11} and a_{31}/a_{11} and subtract from the second and third rows to obtain :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \quad : E(1) \\ a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 &= b_2^{(2)} \quad : E(2) \\ a_{32}^{(2)}x_2 + a_{33}^{(2)}x_3 &= b_3^{(2)} \quad : E(3) \end{aligned} \tag{5.7}$$

where: $$\left.\begin{aligned} a_{22}^{(2)} &= a_{22} - (a_{21}/a_{11})a_{12}, \quad a_{23}^{(2)} = a_{23} - (a_{21}/a_{11})a_{13}, \\ a_{32}^{(2)} &= a_{32} - (a_{31}/a_{11})a_{12}, \quad a_{33}^{(2)} = a_{33} - (a_{31}/a_{11})a_{13}, \\ b_2^{(2)} &= b_2 - (a_{21}/a_{11})b_1, \quad b_3^{(2)} = b_3 - (a_{31}/a_{11})b_1 \end{aligned}\right\}$$

Step 2 : To eliminate x_2, multiply the second row by $a_{32}^{(2)}/a_{22}^{(2)}$ and subtract from the third row to obtain :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \quad : E(1) \\ a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 &= b_2^{(2)} \quad : E(2) \\ a_{33}^{(3)}x_3 &= b_3^{(3)} \quad : E(3) \end{aligned} \tag{5.8}$$

where: $$\left. a_{33}^{(3)} = a_{33}^{(2)} - (a_{32}^{(2)}/a_{22}^{(2)})a_{23}^{(2)}, \quad b_3^{(3)} = b_3^{(2)} - (a_{32}^{(2)}/a_{22}^{(2)})b_2^{(2)} \right\}$$

The system of equations (5.6), (5.7) and (5.8) are used to develop an **upper triangular system** :

$$\begin{aligned} a_{11}^{(1)}x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 &= b_1^{(1)} \quad : E(1) \\ a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 &= b_2^{(2)} \quad : E(2) \\ a_{33}^{(3)}x_3 &= b_3^{(3)} \quad : E(3) \end{aligned} \tag{5.9}$$

where: $a_{ij}^{(1)} = a_{ij}, \; b_i^{(1)} = b_i, \; i,j = 1,2,3$

This upper triangular system can be solved for the unknowns x_3, x_2, x_1 by back substitution method to obtain :

$$x_3 = b_3^{(3)}/a_{33}^{(3)},\ x_2 = (b_2^{(2)} - a_{23}^{(2)}x_3)/a_{22}^{(2)},\ x_1 = (b_1 - a_{12}x_2 - a_{13}x_3)/a_{11} \tag{5.10}$$

This algorithm for **n=3** is easily extended to one for a general $n \times n$ linear system of equations (5.1) :

$$\begin{aligned}
a_{11}^{(1)}x_1 + a_{12}^{(1)}x_2 + \dots\dots\dots\dots + a_{1n}^{(1)}x_n &= b_1^{(1)} \ : \ E(1)\\
a_{22}^{(2)}x_2 + \dots\dots\dots\dots + a_{2n}^{(2)}x_n &= b_2^{(2)} \ : \ E(2)\\
\ddots \qquad &\\
a_{kk}^{(k)}x_k + \dots\dots + a_{kn}^{(k)}x_n &= b_k^{(k)} \ : \ E(k)\\
\vdots \qquad &\\
a_{ik}^{(k)}x_k + \dots\dots + a_{in}^{(k)}x_n &= b_i^{(k)} \ : \ E(i)\\
\vdots \qquad &\\
a_{nk}^{(k)}x_k + \dots\dots + a_{nn}^{(k)}x_n &= b_n^{(k)} \ : \ E(n)
\end{aligned} \tag{5.11}$$

Assume $a_{kk}^{(k)} \neq 0$ and define the multipliers

$$m_{ik} = \left.\frac{a_{ik}^{(k)}}{a_{kk}^{(k)}}\right\} \ i = k+1, k+2, \dots, n \tag{5.12}$$

Step k : To eliminate x_k, subtract m_{ik} times E(k) from E(i), i= k+1, k+2,....,n. The new coefficients in E(k+1) through E(n) are defined by :

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - \left[\frac{a_{ik}^{(k)}}{a_{kk}^{(k)}}\right] a_{kj}^{(k)} \quad i,j = k+1, k+2, \dots, n$$

$$b_i^{(k+1)} = b_i^{(k)} - \left[\frac{a_{ik}^{(k)}}{a_{kk}^{(k)}}\right] b_k^{(k)} \quad i = k+1, k+2, \dots, n \tag{5.13}$$

The diagonal elements $a_{11}^{(1)}, a_{22}^{(2)}, a_{33}^{(3)}, \ldots\ldots, a_{kk}^{(k)}, \ldots\ldots, a_{nn}^{(n)}$ of the system (5.11) which have been assumed to be nonzero are called the **pivot elements.** In the elimination process, if any one of the pivot elements vanishes or becomes very small compared to other elements in that equation, then the equations have to be rearranged so as to obtain a nonvanishing pivot or to avoid the multiplication by a large number. This strategy is then called the **pivoting.**

Example 01 : ***[pivot vanishes]*** : Solve the system of equations :

$$\left.\begin{aligned} 8x_2+2x_3 &= -7 \\ 3x_1+5x_2+2x_3 &= 8 \\ 6x_1+2x_2+8x_3 &= 26 \end{aligned}\right\} \tag{5.14}$$

Interchange first equation with the second equation to obtain a pivot equation containing x_1 :

$$\begin{aligned} 3x_1+5x_2+2x_3 &= 8 \\ 8x_2+2x_3 &= -7 \\ 6x_1+2x_2+8x_3 &= 26 \end{aligned} \tag{5.15}$$

The system of equations and its augmented matrix form are given by :

$$\begin{array}{ll} \textit{pivot} & \rightarrow 3x_1+5x_2+2x_3 = 8 \\ & \qquad\quad 8x_2+2x_3 = -7 \\ \textit{Eliminate} & \leftarrow 6x_1+2x_2+8x_3 = 26 \end{array} \qquad \begin{pmatrix} 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \\ 6 & 2 & 8 & 26 \end{pmatrix} \tag{5.16}$$

For eliminating x_1 :

$$\begin{aligned} 3x_1+5x_2+2x_3 &= 8 \\ 8x_2+2x_3 &= -7 \\ a_{32}^{(2)}x_2+a_{33}^{(2)}x_3 &= b_3^{(2)} \end{aligned} \qquad \begin{pmatrix} 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & b_3^{(2)} \end{pmatrix}$$

$$
\left.\begin{aligned}
a_{ij}^{(k+1)} &= a_{ij}^{(k)} - \left[a_{ik}^{(k)}/a_{kk}^{(k)}\right]a_{kj}^{(k)} \quad i,j=k+1,k+2,\ldots\ldots,n, \\
b_{i}^{(k+1)} &= b_{i}^{(k)} - \left[a_{ik}^{(k)}/a_{kk}^{(k)}\right]b_{k}^{(k)} \quad i=k+1,k+2,\ldots\ldots,n.
\end{aligned}\right\}
$$

equation (5.13)

is used to obtain $a_{32}^{(2)}=-8,\ a_{33}^{(2)}=4,\ b_3^{(2)}=10$

$$
\begin{aligned}
3x_1+5x_2+2x_3 &= 8 \\
8x_2+2x_3 &= -7 \\
-8x_2+4x_3 &= 10
\end{aligned}
\qquad
\begin{pmatrix} 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \\ 0 & -8 & 4 & 10 \end{pmatrix}
$$

For eliminating x_2 :

$$
\begin{aligned}
3x_1+5x_2+2x_3 &= 8 \\
8x_2+2x_3 &= -7 \\
a_{33}^{(3)}x_3 &= b_3^{(3)}
\end{aligned}
\qquad
\begin{pmatrix} 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \\ 0 & 0 & a_{33}^{(3)} & b_3^{(3)} \end{pmatrix}
$$

equation (5.13) is used to obtain $a_{33}^{(3)}=6,\ b_3^{(3)}=3$

The resulting system is an upper triangular system takes the form :

$$
\begin{aligned}
3x_1+5x_2+2x_3 &= 8 \\
8x_2+2x_3 &= -7 \\
6x_3 &= 3
\end{aligned}
\qquad
\begin{pmatrix} 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \\ 0 & 0 & 6 & 3 \end{pmatrix}
\tag{5.17}
$$

Back substitution method is used to obtain :

$x_3=3/6=1/2$. $x_2=(1/8)(-7-2x_3)=-1$, $x_1=(1/3)(8-5x_2-2x_3)=4$

Example 02 : *[small pivot entries]* : Solve the system of equations by Gauss-elimination method :

$$\left.\begin{aligned} 0.0004x_1 + 1.402x_2 &= 1.406 \\ 0.4003x_1 - 1.502x_2 &= 2.501 \end{aligned}\right\} \text{ exact solution is } x_1 = 10,\ x_2 = 1. \tag{5.18}$$

without Pivoting and with pivoting using four-digit floating point arithmetic.

1 : $\begin{aligned} 0.0004x_1 + 1.402x_2 &= 1.406 \\ 0.4003x_1 - 1.502x_2 &= 2.501 \end{aligned} \qquad \begin{pmatrix} 0.0004 & 1.402 & 1.406 \\ 0.4003 & -1.502 & 2.501 \end{pmatrix}$

For eliminating x_1 :

$$\sim \begin{aligned} 0.0004x_1 + 1.402x_2 &= 1.406 \\ a_{22}^{(2)} x_2 &= b_2^{(2)} \end{aligned} \qquad \begin{pmatrix} 0.0004 & 1.402 & 1.406 \\ 0 & a_{22}^{(2)} & b_2^{(2)} \end{pmatrix}$$

equation (5.13) is used to obtain :

$$\left.\begin{aligned} a_{22}^{(2)} &= a_{22} - (a_{21}/a_{11})a_{12} = -1.502 - 1001(1.402) = -1405.0 \\ b_2^{(2)} &= b_2 - (a_{21}/a_{11})b_1 = 2.501 - (1001)(1.406) = 2.501 - 1407 = -1404 \end{aligned}\right\}$$

The resulting upper triangular system now takes the form :

$$\begin{aligned} 0.0004x_1 + 1.402x_2 &= 1.406 \\ -1405x_2 &= -1404 \end{aligned} \qquad \begin{pmatrix} 0.0004 & 1.402 & 1.406 \\ 0.0 & -1405 & -1404 \end{pmatrix}$$

Back substitution method is used to obtain :

$$x_2 = (1404)/(1405) = 0.9993$$
$$x_1 = (1/0.0004)[1.406 - (1.402)(0.9993)] = 12.5$$

This failure occurs because $|a_{11}|$ is smaller compared to $|a_{12}|$, so that a small round-off error in x_2 will lead to a large error in x_1.

$$0.4003x_1 - 1.502x_2 = 2.501 \qquad \begin{pmatrix} 0.4003 & -1.502 & 2.501 \\ 0.0004 & 1.402 & 1.406 \end{pmatrix}$$

2: $0.0004x_1 + 1.402x_2 = 1.406$

Picked the second equation as the pivot equation.

For **eliminating** x_1 :

$$\begin{aligned} 0.4003x_1 - 1.502x_2 &= 2.501 \\ a_{22}^{(2)} x_2 &= b_2^{(2)} \end{aligned} \qquad \begin{pmatrix} 0.4003 & -1.502 & 2.501 \\ 0.0 & a_{22}^{(2)} & b_2^{(2)} \end{pmatrix}$$

equation (5.13) is used to obtain :

$$\left.\begin{aligned} a_{22}^{(2)} &= a_{22} - (a_{21}/a_{11})a_{12} = 1.402 - (0.0004/0.4003)(-1.502) = 1.404 \\ b_2^{(2)} &= b_2 - (a_{21}/a_{11})b_1 \ = 1.406 - (0.0004/0.4003)(\ 2.501) = 1.404 \end{aligned}\right\}$$

$$\begin{aligned} 0.4003x_1 - 1.502x_2 &= 2.501 \\ 1.404x_2 &= 1.404 \end{aligned} \qquad \begin{pmatrix} 0.4003 & -1.502 & 2.501 \\ 0.0 & 1.404 & 1.404 \end{pmatrix}$$

Back substitution method is used to obtain :

$$x_2 = 1.404/1.404 = 1$$
$$x_1 = (1/0.4003)(2.501 + 1.505) = 10.0.$$

This success occurs because $|a_{21}|$ is not very smaller compared to $|a_{22}|$, so that a small round-off error in x_2 will not lead to a large error in x_1. To overcome these difficulties, the strategies used will be the:

Partial Pivoting : In the first stage of elimination of x_1, the first column $[a_{11}\ a_{21}\ a_{31}]^T$ of the augmented matrix of the given 3x3 system of equations (5.6) is searched for the largest element in magnitude and brought as the first pivot by an interchange of the first row with the row having the largest element in magnitude. In the second stage of elimination of x_2, the second column $[a_{12}^{(1)}\ a_{22}^{(2)}\ a_{32}^{(2)}]^T$ of the augmented matrix of the system of equations (5.7) is searched for the largest element in magnitude among the two elements $a_{22}^{(2)}$ and $a_{32}^{(2)}$ and brought as the second pivot by an interchange of the second row with the row having the largest element in magnitude. This process is then called **partial pivoting.**

Example 03 : ***[partial pivoting]*** : Solve the system of equations :

$$\left.\begin{aligned} 3x_1 - 4x_2 + 5x_3 &= -1 \\ -3x_1 + 2x_2 + x_3 &= 1 \\ 6x_1 + 8x_2 - x_3 &= 35 \end{aligned}\right\}$$

Step 1 : The largest element $|a_{31}|$ in column one is in row three. so interchange row one with row three to put 6 as the first pivot :

$$\begin{aligned} 6x_1 + 8x_2 - x_3 &= 35 \\ -3x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 - 4x_2 + 5x_3 &= -1 \end{aligned} \qquad \begin{pmatrix} 6 & 8 & -1 & 35 \\ -3 & 2 & 1 & 1 \\ 3 & -4 & 5 & -1 \end{pmatrix}$$

For eliminating x_1, equation (5.13) is used to obtain :

$$a_{22}^{(2)} = 6,\ a_{23}^{(2)} = 1/2,\ b_2^{(2)} = 37/2,\ a_{32}^{(2)} = -8,\ a_{33}^{(2)} = 11/2,\ b_3^{(2)} = -37/2$$

$$\begin{aligned} 6x_1 + 8x_2 - x_3 &= 35 \\ 6x_2 + 0.5x_3 &= 37/2 \\ -8x_2 + 5.5x_3 &= -37/2 \end{aligned} \qquad \begin{pmatrix} 6 & 8 & -1 & 35 \\ 0 & 6 & 0.5 & 37/2 \\ 0 & -8 & 5.5 & -37/2 \end{pmatrix}$$

Step 2 : The largest element $|a_{32}|$ in column two is in row three, so interchange row two with row three to put -8 as the second pivot :

$$\begin{aligned} 6x_1 + 8x_2 - x_3 &= 35 \\ -8x_2 + 5.5x_3 &= -37/2 \\ 6x_2 + 0.5x_3 &= 37/2 \end{aligned} \qquad \begin{pmatrix} 6 & 8 & -1 & 35 \\ 0 & -8 & 5.5 & -37/2 \\ 0 & 6 & 0.5 & 37/2 \end{pmatrix}$$

For eliminating x_2, equation (5.13) is used to obtain :

$$a_{33}^{(3)} = 37/8, \ b_3^{(3)} = 37/8$$

$$\begin{aligned} 6x_1 + 8x_2 - x_3 &= 35 \\ -8x_2 + 5.5x_3 &= -37/2 \\ 37/8x_3 &= 37/8 \end{aligned} \qquad \begin{pmatrix} 6 & 8 & -1 & 35 \\ 0 & -8 & 5.5 & -37/2 \\ 0 & 0 & 37/8 & 37/8 \end{pmatrix}$$

Back substitution method is used to obtain :

$$x_3 = 1, x_2 = 3, x_1 = 2.$$

Total Pivoting : At step k :

$$C = \max_{k \le i \le n} |a_{ik}^{(k)}| \tag{5.19}$$

represents the maximum size of the elements of the coefficient matrix **A** of the entire system of equations, beginning at row k and going downward. If the pivot element $|a_{kk}^{(k)}| < C$, then interchange equation $E(k)$ with the equation having $|a_{kk}^{(k)}| = C$. This strategy makes $a_{kk}^{(k)}$ as far away from zero as possible. The element $a_{kk}^{(k)}$ is called the pivot element for step k of the elimination and the process described in this paragraph is called **total pivoting.** If the matrix A is diagonally dominant or real symmetric or positive definite then no pivoting is necessary.

Example 04 : ***[total pivoting]*** : Solve the system of equations

$$\left.\begin{aligned} 3x_1-4x_2+5x_3&=-1\\ -3x_1+2x_2+x_3&=1\\ 6x_1+8x_2-x_3&=35 \end{aligned}\right\} \qquad \begin{pmatrix} 3 & -4 & 5 & -1\\ -3 & 2 & 1 & 1\\ 6 & 8 & -1 & 35 \end{pmatrix}$$

Step 1 : Equation (5.19) is used to search the element of the maximum size of the above system :

$$|a_{11}| = 3 \ and \ \max|a_{1k}| = 5 = C \ \therefore \ |a_{11}| \neq C$$
$$|a_{21}| = 3 \ and \ \max|a_{2k}| = 3 = C \ \therefore \ |a_{21}| = C = 3$$
$$|a_{31}| = 6 \ and \ \max|a_{3k}| = 8 = C \ \therefore \ |a_{31}| \neq C$$

that lies in equation 2, so interchange equation one with equation two :

$$\begin{aligned} -3x_1+2x_2+x_3&=1\\ 3x_1-4x_2+5x_3&=-1\\ 6x_1+8x_2-x_3&=35 \end{aligned} \qquad \begin{pmatrix} -3 & 2 & 1 & 1\\ 3 & -4 & 5 & -1\\ 6 & 8 & -1 & 35 \end{pmatrix}$$

For eliminating x_1 :

$$\begin{aligned} -3x_1+2x_2+x_3&=1\\ a_{22}^{(2)}x_2+a_{23}^{(2)}x_3&=b_2^{(2)}\\ a_{32}^{(2)}x_2+a_{33}^{(2)}x_3&=b_3^{(2)} \end{aligned} \qquad \begin{pmatrix} -3 & 2 & 1 & 1\\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & b_2^{(2)}\\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & b_3^{(2)} \end{pmatrix}$$

equation (5. 3) is used to obtain :

$$a_{22}^{(2)}=-2,\ a_{23}^{(2)}=6,\ b_2^{(2)}=0,\ a_{32}^{(2)}=12,\ a_{33}^{(2)}=1,\ b_3^{(2)}=37$$

$$\begin{aligned} -3x_1+2x_2+x_3&=1\\ -2x_2+6x_3&=0\\ 12x_2+x_3&=37 \end{aligned} \qquad \begin{pmatrix} -3 & 2 & 1 & 1\\ 0 & -2 & 6 & 0\\ 0 & 12 & 1 & 37 \end{pmatrix} \qquad (5.20)$$

Step 2 : Equation (5.19) is used again to search the element of the maximum size of the system (5.20) :

$|a_{22}| = 2$ *and* $\max |a_{2k}| = 6 \therefore |a_{22}| \neq C$

$|a_{32}| = 12$ *and* $\max |a_{3k}| = 12 \therefore |a_{32}| = C = 12$

that lies in equation 3, so interchange equation two with equation three :

$$\begin{aligned} -3x_1 + 2x_2 + x_3 &= 1 \\ 12x_2 + x_3 &= 37 \\ -2x_2 + 6x_3 &= 0 \end{aligned} \qquad \begin{pmatrix} -3 & 2 & 1 & 1 \\ 0 & 12 & 1 & 37 \\ 0 & -2 & 6 & 0 \end{pmatrix}$$

For eliminating x_2 :

$$\begin{aligned} -3x_1 + 2x_2 + x_3 &= 1 \\ 12x_2 + x_3 &= 37 \\ a_{33}^{(3)} x_3 &= b_3^{(3)} \end{aligned} \qquad \begin{pmatrix} -3 & 2 & 1 & 1 \\ 0 & 12 & 1 & 37 \\ 0 & 0 & a_{33}^{(3)} & b_3^{(3)} \end{pmatrix}.$$

equation (5.13) is used to obtain $a_{33}^{(3)} = 37/6,\ b_3^{(3)} = 37/6$.

$$\begin{aligned} -3x_1 + 2x_2 + x_3 &= 1 \\ 12x_2 + x_3 &= 37 \\ (37/6)x_3 &= 37/6 \end{aligned} \qquad \begin{pmatrix} -3 & 2 & 1 & 1 \\ 0 & 12 & 1 & 37 \\ 0 & 0 & \frac{37}{6} & \frac{37}{6} \end{pmatrix}. \qquad (5.21)$$

Back substitution method is used to obtain :

$x_3 = 1, x_2 = 3, x_1 = 2.$

EXAMPLE 04 THROUGH PACKAGE MMFP, COMPUTER PROGRAM 5-01: GAUSSEL
MATRIX OF ORDER: N=3
INPUT ROW WISE ELEMENTS OF A/B:=AUGMENTED :

3.0	-4.0	5.0	-1.0
-3.0	2.0	1.0	1.0
6.0	8.0	-1.0	35.0

```
ROW WISE ELEMENTS OF A AND B :
 3.0000  -4.0000  5.0000  -1.0000
-3.0000   2.0000  1.0000   1.0000
 6.0000   8.0000 -1.0000  35.0000
SOLUTION OF THE ABOVE SYSTEM OF EQUATIONS IS :
0.2000000000D+01   0.3000000000D+01   0.1000000000D+01
```

Ill-Conditioning : There are certain numerical problems in which a small error from any source will cause a large error in the final solution. A problem of this type is called an **ill-conditioned** problem. For example, the system of two linear equations :

$$\left.\begin{aligned} x_1 + 3x_2 &= 4 \\ 3x_1 + 9.00001x_2 &= 12.00001 \end{aligned}\right\} \tag{5.22}$$

has a solution $x_1 = x_2 = 1$, but if the second equation is altered to

$3x_1 + 8.99999x_2 = 12.00002$, then the solution becomes $x_1 = 10$, $x_2 = -2$. A small change in the coefficients has caused a big change in the final solution. Therefore it is essential to use very accurate arithmetic (small round-off errors) when solving an ill-conditioned problem.

Example 05 : ***[ill conditioned problem]*** : Solve the system of two linear equations :

$$\left.\begin{aligned} \frac{3}{8}x_1 + \frac{4}{9}x_2 &= 11 \\ \frac{2}{7}x_1 + \frac{3}{8}x_2 &= 11 \end{aligned}\right\}$$ round the coefficients to 1D and 2D.

by **Gauss-elimination method** for which the exact solution is $x_1 = -56$, $x_2 = 72$.

$$1: \quad \begin{aligned} 0.4x_1 + 0.4x_2 &= 11 \\ 0.3x_1 + 0.4x_2 &= 11 \end{aligned} \qquad \begin{pmatrix} 0.4 & 0.4 & 11 \\ 0.3 & 0.4 & 11 \end{pmatrix}$$

For eliminating x_1, equation (5.13) is used to obtain :

$$\left.\begin{aligned} a_{22}^{(2)} &= a_{22} - (a_{21}/a_{11})a_{12} = (0.4) - (0.3/0.4)(0.4) = 0.1 \\ b_2^{(2)} &= b_2 - (a_{21}/a_{11})b_1 = 11 - (0.3/0.4)(11) = 2.8 \end{aligned}\right\}$$

$$\begin{aligned} 0.4x_1 + 0.4x_2 &= 11 \\ 0.1x_2 &= 2.8 \end{aligned} \qquad \begin{pmatrix} 0.4 & 0.4 & 11 \\ 0.0 & 0.1 & 2.8 \end{pmatrix}$$

Back substitution method is used to obtain :

$x_2 = 28$ and $x_1 = -0.50$.

$$2: \quad \begin{aligned} 0.38x_1 + 0.44x_2 &= 11 \\ 0.29x_1 + 0.38x_2 &= 11 \end{aligned} \qquad \begin{pmatrix} 0.38 & 0.44 & 11 \\ 0.29 & 0.38 & 11 \end{pmatrix}$$

For eliminating x_1, equation (5.13) is used to obtain :

$$\left.\begin{aligned} a_{22}^{(2)} &= a_{22} - (a_{21}/a_{11})a_{12} = 0.38 - (0.29/0.38)(0.44) = 0.0443 \\ b_2^{(2)} &= b_2 - (a_{21}/a_{11})b_1 = 11 - (0.29/0.38)(11) = 2.61 \end{aligned}\right\}$$

$$\begin{aligned} 0.38x_1 + 0.44x_2 &= 11 \\ 0.0443x_2 &= 2.61 \end{aligned} \qquad \begin{pmatrix} 0.38 & 0.44 & 11 \\ 0 & 0.0443 & 2.61 \end{pmatrix}$$

Back substitution method is used to obtain :

$x_2 = 58.9$ and $x_1 = -39.3$.

Calculation (2) is more accurate, since it uses more accurate arithmetic than calculation (1). A problem like this system is called an **ill-conditioned** problem.

Direct Factorization A=LU

Without Pivoting : This method is based on the idea of decomposing the coefficient matrix **A** of the system (5.2) into the product of a lower and **upper** triangular matrices L and U for which the 3×3 system of linear equations (5.6) can be written as :

$$\left.\begin{aligned} Ax &= b \\ LUx &= b \end{aligned}\right\} \tag{5.23}$$

where

$$A=LU$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}, \quad l_{ij}=0,\ j>i \text{ and } u_{ij}=0,\ i>j$$

$$= \begin{pmatrix} l_{11}u_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+l_{22}u_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+l_{33}u_{33} \end{pmatrix} \tag{5.24}$$

To get a unique solution it is convenient to choose either $u_{ii}=1$ or $l_{ii}=1,\ i=1,2,\dots,n$.

Choose $u_{ii}=1,\ i=1,2,\dots,n$, and compare the elements of the resulting matrix **LU** with those of **A** to obtain the elements of the lower triangular matrix **L** and upper triangular matrix **U**.

The comparison of the 1st columns of **A** and **LU** are giving the elements of the 1st column of **L** :

$$\left.\begin{aligned}
&l_{11}u_{11}=a_{11} \Rightarrow l_{11}=a_{11},\ u_{11}=1\\
&l_{21}u_{11}=a_{21} \Rightarrow l_{21}=a_{21}\\
&l_{31}u_{11}=a_{31} \Rightarrow l_{31}=a_{31}\\
&\vdots\\
&l_{i1}=a_{i1},\quad i=1,2,3,\ldots,n.
\end{aligned}\right\}$$

The comparison of the 1st rows of A and LU are giving the elements of the 1st row of **U** :

$$\left.\begin{aligned}
&l_{11}u_{12}=a_{12} \Rightarrow u_{12}=a_{12}/l_{11}\\
&l_{11}u_{13}=a_{13} \Rightarrow u_{13}=a_{13}/l_{11}\\
&\vdots\\
&u_{1j}=a_{1j}/l_{11},\quad j=2,3,\ldots,n.
\end{aligned}\right\}$$

The comparison of the 2nd columns of **A** and **LU** are giving the elements of the 2nd column of **L** :

$$\left.\begin{aligned}
&l_{21}u_{12}+l_{22}u_{22}=a_{22} \Rightarrow l_{22}=a_{22}-l_{21}u_{12},\ u_{22}=1\\
&l_{31}u_{12}+l_{32}u_{22}=a_{32} \Rightarrow l_{32}=a_{32}-l_{31}u_{12}\\
&\vdots\\
&l_{i2}=a_{i2}-l_{i1}u_{12},\quad i=2,3,\ldots,n
\end{aligned}\right\}$$

The comparison of the 2nd rows of **A** and **LU** are giving the elements of the 2nd row of **U** :

$$\left.\begin{array}{l} l_{21}u_{13}+l_{22}u_{23}=a_{23} \quad => \quad u_{23}=(a_{23}-l_{21}u_{13})/l_{22} \\ \vdots \\ u_{2j}=(a_{2j}-l_{21}u_{1j})/l_{22}, \; j=3,4,\ldots,n \end{array}\right\}$$

Similarly, the comparison of the n-th columns of **A** and **LU** is giving the general term for the determination of the elements of the n-th column of **L** :

$$l_{ij}=\left(a_{ij}-\sum_{k=1}^{j-1} l_{ik}u_{kj}\right), \; i=j,\ldots\ldots,n, \; j\geq 2. \tag{5.25}$$

where the summation is taken as zero if the upper limit is less than the lower limit.

Similarly, the comparison of the n-th rows of **A** and **LU** is giving the general term for the determination of the elements of the n-th row of **U**:

$$u_{ij}=\frac{1}{l_{ii}}\left(a_{ij}-\sum_{k=1}^{i-1} l_{ik}u_{kj}\right), \; j=i+1,\ldots\ldots,n, \; i\geq 2 \tag{5.26}$$

Having determined the matrices **L** and **U**, the system (5.23):

$$LUx=b \tag{5.27}$$

now takes the form :

$$\left.\begin{array}{l} Ux=z \\ Lz=b \end{array}\right\} \tag{5.28}$$

The unknowns $z_1,z_2,\ldots\ldots,z_n$ and $x_1,x_2,\ldots\ldots,x_n$ developed in the system (5.28), can be found by forward and backward substitution methods. This method is also known as **triangularisation** or **decomposition** method.

Example 06 : ***[direct factorization]*** : Solve the system of equations :

$$\left.\begin{aligned} x_1+x_2+x_3&=1\\ 4x_1+3x_2-x_3&=6\\ 3x_1+5x_2+3x_3&=4 \end{aligned}\right\} \Rightarrow Ax=b$$

The coefficient matrix A of the given system in terms of L and U :

$A=LU$

$$\begin{pmatrix} 1 & 1 & 1\\ 4 & 3 & -1\\ 3 & 5 & 3 \end{pmatrix}=\begin{pmatrix} l_{11} & 0 & 0\\ l_{21} & l_{22} & 0\\ l_{31} & l_{32} & l_{33} \end{pmatrix}\begin{pmatrix} 1 & u_{12} & u_{13}\\ 0 & 1 & u_{23}\\ 0 & 0 & 1 \end{pmatrix}$$

for which the general terms (5.25) & (5.26) are used for the following assumptions :

$$\left.\begin{aligned} &l_{11}=1,\ l_{21}=4,\ l_{31}=3,\ u_{12}=1,\ u_{13}=1\\ &l_{22}=a_{22}-l_{21}u_{12}=3-(4)(1)=-1\\ &l_{32}=a_{32}-l_{31}u_{12}=5-(3)(1)=2\\ &l_{33}=a_{33}-l_{31}u_{13}-l_{32}u_{23}=[3-(3)(1)-(2)(5)]=-10 \end{aligned}\right\}$$

to obtain the elements of the lower and upper triangular matrices :

$$L=\begin{pmatrix} 1 & 0 & 0\\ 4 & -1 & 0\\ 3 & 2 & -10 \end{pmatrix},\quad U=\begin{pmatrix} 1 & 1 & 1\\ 0 & 1 & 5\\ 0 & 0 & 1 \end{pmatrix}$$

The system (5.28) is used to obtain the unknowns z_1, z_2, z_3 by forward substitution method :

$$Lz=b \Rightarrow \begin{bmatrix} 1 & 0 & 0\\ 4 & -1 & 0\\ 3 & 2 & -10 \end{bmatrix}\begin{bmatrix} z_1\\ z_2\\ z_3 \end{bmatrix}=\begin{bmatrix} 1\\ 6\\ 4 \end{bmatrix}$$

$$z_1=1,\ 4z_1-z_2=6,\ 3z_1+2z_2-10z_3=4,\ \Rightarrow z_1=1,\ z_2=-2,\ z_3=-1/2$$

The system (5.28) is used to obtain the unknowns x_3, x_2, x_1 by backward substitution method :

$$Ux=z \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1/2 \end{bmatrix}$$

$$x_1+x_2+x_3=1,\ x_2+5x_3=-2,\ x_3=-1/2 \Rightarrow x_1=1,\ x_2=1/2,\ x_3=-1/2$$

```
EXAMPLE 06 THROUGH PACKAGE MMFP, COMPUTER PROGRAM 5-02: LUFACT
INPUT ROW WISE ELEMENTS OF A :
1.0   1.0   1.0
4.0   3.0  -1.0
3.0   5.0   3.0
INPUT ELEMENTS OF B :
1.0   6.0   4.0
THE ORDER OF THE MATRIX: N=3
ELEMENTS OF A :
1.0000   1.0000   1.0000
4.0000   3.0000  -1.0000
3.0000   5.0000   3.0000
ELEMENTS OF B :
1.0000   6.0000   4.0000
THE LOWER TRIANGULAR MATRIX L IS :
0.1000000000D+01   0.0000000000D+00   0.0000000000D+00
0.4000000000D+01  -0.1000000000D+01   0.0000000000D+00
0.3000000000D+01   0.2000000000D+01  -0.1000000000D+01
THE UPPER TRIANGULAR MATRIX U IS :
0.1000000000D+01   0.1000000000D+01   0.1000000000D+01
0.0000000000D+00   0.1000000000D+01   0.5000000000D+01
0.0000000000D+00   0.0000000000D+00   0.1000000000D+01
SOLUTION OF THE LINEAR SYSTEM BY LU-FACTORIZATION IS :
0.1000000000D+01   0.5000000000D+00  -0.5000000000D+00
```

Example 07 : Is it possible to decompose the nonsingular matrix

$$A=\begin{pmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{pmatrix}$$ as the product A = LU directly ?

To answer this, let us suppose that A has a direct factorization LU :

$$\begin{pmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}\begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

Equations (5.23), (5.24), (5.25) & (5.26) are used for the following assumptions :

$$l_{11}=1, l_{21}=4, l_{22}=0, l_{31}=-2, l_{32}=7, l_{33}=17, u_{12}=2, u_{13}=6, u_{23}=0$$

that should be substituted in the above equation to obtain :

$$\begin{pmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ -2 & 7 & 17 \end{pmatrix}\begin{pmatrix} 1 & 2 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It is found that A does not have LU-factorization. Sometimes it is not possible to decompose the nonsingular matrix A as the product A=LU directly. To overcome this difficulty, the strategy used will be :

Indirect Factorization PA=LU

Partial Pivoting : This indirect method is based on the idea of introducing a permutation matrix **P** with which the matrix product **PA** can be factored as the product of a **lower** and **upper** triangular matrices **L** and **U** by illustrating a 4×4 nonsingular coefficient matrix **A** :

$$PA=LU \tag{5.29}$$

P is the permutation matrix initially set equal to a unit matrix I. The Lower triangular matrix L can be constructed to have 1's on its diagonal and U will have nonzero diagonal elements :

$$\underset{P}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \underset{A}{\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}} = \underset{L}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}}$$

The unknown elements of the lower and upper triangular matrices can be found by comparing the elements of the product matrix LU with those of PA to obtain :

$$\left.\begin{aligned} u_{ik} &= a_{ik} - \sum_{j=1}^{i-1} l_{ij} u_{jk}, \qquad i=1,2,\ldots\ldots\ldots,k \\ l_{ik} &= \frac{1}{u_{kk}}\left(a_{ik} - \sum_{j=1}^{k-1} l_{ij} u_{jk}\right), \quad i=k+1,k+2,\ldots,n \end{aligned}\right\} \tag{5.30}$$

The solution set of a 4×4 indirect system (5.2) can be found in three stages by illustrating the above 4×4 nonsingular coefficient matrix **A** :

Stage 1 : In the first stage of elimination, the first column of **A** is searched for the largest element in magnitude and brought as the first pivot by an interchange of the first rows of **P** and **A** with the rows having the largest element in magnitude. Form column 1 of **L** and row 1 of **U**.

Stage 2 : In the second stage of elimination, the second column of **U** is searched for the largest element in magnitude among the three elements u_{22}, u_{32}, u_{42} and brought as the second pivot by an interchange of the second rows of **P**, **A**, **L** and **U** with the rows having the largest element in magnitude. Form column 2 of **L** and row 2 of **U**.

Stage 3 : In third stage of elimination, the third column of **U** is searched for the largest element in magnitude among the two elements u_{33}, u_{43} and brought as the third pivot by an interchange of the third rows of **P**, **A**, **L** and **U** with the rows having the largest element in magnitude. Form column 3 of **L** and row 3 of **U** and u_{44}.

Example 08 : *[indirect factorization]* : Solve the system of equations :

$$\left.\begin{aligned} x_1+2x_2+6x_3 &= 1 \\ 4x_1+8x_2-x_3 &= 2 \\ -2x_1+3x_2+5x_3 &= 3 \end{aligned}\right\} \qquad (5.31)$$

The coefficient matrix A of the system (5.31) is factored as :

$$PA=LU \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}.$$

P is the permutation matrix initially set = a unit matrix = I

Stage 1 : The largest element $|a_{21}|=4$ in column 1 of **A** is in row 2 and brought it as the first pivot by an interchange of the first rows of **P** and **A** with the second rows of **P** and **A** :

$$PA=LU \Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 8 & -1 \\ 1 & 2 & 6 \\ -2 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

Equation (5.30) is used to complete the calculations $l_{21}, l_{31}, u_{12}, u_{13}$:

$$\left.\begin{array}{l} u_{11} = a_{11} = 4,\ u_{12} = a_{12} = 8,\ u_{13} = a_{13} = -1, \\ l_{21} = a_{21}/u_{11} = 0.25,\ l_{31} = a_{31}/u_{11} = -0.50 \end{array}\right\}$$

that should be substituted in the above system to obtain :

$$PA = LU \Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 8 & -1 \\ 1 & 2 & 6 \\ -2 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0.25 & * & * \\ -0.50 & * & * \end{pmatrix} \begin{pmatrix} 4 & 8 & -1 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

Stage 2 : Equation (5.30) is used to locate the second pivot u_{22}:

$$\begin{array}{l} u_{22} = a_{22} - l_{21}u_{12} = 2 - (0.25)(8) = 0,\ i = 2, k = 2 \\ l_{32} = (a_{32} - l_{31}u_{12})/u_{22} = (3 + 0.50(8))/0 = \infty,\ i = 3, k = 2 \\ u_{32} = a_{32} - l_{31}u_{12} - l_{32}u_{22} = 3 - (-0.50)(8) - (\infty)(0) = 7,\ i = 3, k = 2 \end{array}$$

which are 0 and 7. The element of the largest magnitude arise from row 3 and brought $u_{22} = 7$ as the second pivot by an interchange of the second rows of **P**, **A**, **L** and **U** with the third rows of **P**, **A**, **L** and **U** :

$$PA = LU \Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 4 & 8 & -1 \\ -2 & 3 & 5 \\ 1 & 2 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -0.50 & 1 & 0 \\ 0.25 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 8 & -1 \\ 0 & 7 & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

Equation (5.30) is used to complete the calculations u_{23}, u_{33}:

$$\left.\begin{array}{l} u_{23} = a_{23} - l_{21}u_{13} = 4.5, \\ u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} = 6.25 \end{array}\right\}$$

that should be substituted in the above system to obtain :

$$PA = LU \Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 4 & 8 & -1 \\ 2 & 3 & 5 \\ 1 & 2 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0.50 & 1 & 0 \\ 0.25 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 8 & -1 \\ 0 & 7 & 4.50 \\ 0 & 0 & 6.25 \end{pmatrix}$$

where $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ is called a **permutation matrix**.

The indirect factorization of the system (5.31) :

$PA = LU$

now takes the form :

$$\left. \begin{array}{c} PAx = Pb \\ or \\ LUx = pb \end{array} \right\} \qquad (5.32)$$

The two stages of the system (5.32) can be written as :

$$\left. \begin{array}{c} Ux = z \\ Lz = pb \end{array} \right\} \qquad (5.33)$$

$$Lz = pb \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -0.50 & 1 & 0 \\ 0.25 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad z_1 = 2, \; z_2 = 4, \; z_3 = 0.5$$

$$Ux = z \Rightarrow \begin{pmatrix} 4 & 8 & -1 \\ 0 & 7 & 4.50 \\ 0 & 0 & 6.25 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0.5 \end{pmatrix}$$

$$\Rightarrow \quad x_3 = 0.08, \; x_2 = 0.52, \; x_1 = -0.52$$

Symmetric Factorization

This method is based on the idea of decomposing the symmetric matrix **A** of th system (5.2) into the product of a **lower** triangular matrix and its transposition matrix (LL^T) or into the product of an **upper** triangular matrix and its transposition matrix (UU^T):

$$A = LL^T$$

or (5.34)

$$A = UU^T \tag{5.35}$$

The decomposition (5.34) for which the system of equations (5.2) :

$$\left.\begin{aligned} Ax &= b \\ LL^T x &= b \end{aligned}\right\} \text{ takes the form } \left.\begin{aligned} L^T x &= z \\ Lz &= b \end{aligned}\right\} \tag{5.36}$$

The unknown values z_i, i=1,2,3,...,n and x_i, i=1,2,3, ...,n of the system (5.37) can be found by forward and backward substitution methods.

1 : In case of decomposition (5.34), the comparison of the elements of **A** and LL^T is giving the general term for the determination of the elements of the lower triangular matrix **L** :

$$\left.\begin{aligned} l_{ii} &= \left[a_{ii} - \sum_{j=1}^{i-1} l_{ij}^2 \right]^{\frac{1}{2}}, \quad i = 1,2,3\dots,n \\ l_{ij} &= \frac{1}{l_{jj}} \left[a_{ij} - \sum_{k=1}^{j-1} l_{jk} l_{ik} \right], \quad l_{ij} = 0, \; i < j \\ i &= j+1, j+2, \dots\dots, n, \; j = 1,2,3,\dots,n \end{aligned}\right\} \tag{5.37}$$

where the summation is taken as zero if the upper limit is less than the lower limit.

2 : In case of decomposition (5.35), the comparison of the elements of **A** and UU^T is giving the general term for the determination of the elements of the upper triangular matrix **U** :

$$\left.\begin{aligned}
u_{nn} &= (a_{nn})^{\frac{1}{2}}, \\
u_{in} &= a_{in}/u_{nn}, \qquad i=1,2,3,\ldots\ldots,(n-1), \\
u_{ij} &= \frac{1}{u_{jj}}\left(a_{ij}-\sum_{k=j+1}^{n} u_{ik}u_{jk}\right), \quad u_{ij}=0, \quad i>j \\
i &= n-2,n-3,\ldots\ldots,1, \quad j=i+1,i+2,\ldots\ldots,n-1, \\
u_{ii} &= \left(a_{ii}-\sum_{k=i+1}^{n} u_{ik}^2\right)^{\frac{1}{2}}, \quad i=n-1,n-2,\ldots\ldots\ldots,1
\end{aligned}\right\} \qquad (5.38)$$

This method is also known as the **square root method** or **Cholesky's method.**

Example 09 : ***[symmetric factorization]*** : Solve the system of equations

$$\begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The coefficient matrix A of the above system with decomposition (5.34):

$$A=LL^T$$
$$\begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix}=\begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix}\begin{bmatrix} l_{11} & l_{21} & l_{31} & l_{41} \\ 0 & l_{22} & l_{32} & l_{42} \\ 0 & 0 & l_{33} & l_{43} \\ 0 & 0 & 0 & l_{44} \end{bmatrix} \qquad (5.39)$$

for which the general term (5.37) is used :

$$\left.\begin{array}{l} l_{11}=2,\ l_{21}=-1/2,\ l_{22}=\sqrt{15/4}\ , \\ l_{31}=0,\ l_{32}=-\sqrt{4/15}\ ,\ l_{33}=\sqrt{56/15}\ , \\ l_{41}=0,\ l_{42}=0,\ l_{43}=-\sqrt{15/56}\ \ and\ l_{44}=\sqrt{209/56} \end{array}\right\}$$

to obtain :

1 : **Lz=b**

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ \frac{-1}{2} & \sqrt{\frac{15}{4}} & 0 & 0 \\ 0 & -\sqrt{\frac{4}{15}} & \sqrt{\frac{56}{15}} & 0 \\ 0 & 0 & -\sqrt{\frac{15}{56}} & \sqrt{\frac{209}{56}} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The forward substitution method is used to obtain the unknowns :

$z_1=1/2,\ z_2=1/\sqrt{60},\ z_3=1/\sqrt{840}$ and $z_4=1/\sqrt{11704}$.

2 : $L^Tx=z$

$$\begin{bmatrix} 2 & \frac{-1}{2} & 0 & 0 \\ 0 & \sqrt{\frac{15}{4}} & -\sqrt{\frac{4}{15}} & 0 \\ 0 & 0 & \sqrt{\frac{56}{15}} & -\sqrt{\frac{15}{56}} \\ 0 & 0 & 0 & \sqrt{\frac{209}{56}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{60}} \\ \frac{1}{\sqrt{840}} \\ \frac{1}{\sqrt{11704}} \end{bmatrix}$$

The backward substitution method is used to obtain the unknowns :

$x_4=1/209$, $x_3=4/209$, $x_2=15/209$ and $x_1=56/209$.

Tridiagonal Factorization

This method is based on the idea of decomposing the **tridiagonal** matrix A of the system (5.2) into the product of a **lower** and **upper** triangular matrices L and U for which the 4×4 tridiagonal system of linear equations (5.2) can be written as :

$$\begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 \\ \gamma_2 & \alpha_2 & \beta_2 & 0 \\ 0 & \gamma_3 & \alpha_3 & \beta_3 \\ 0 & 0 & \gamma_4 & \alpha_4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \qquad (5.40)$$

The LU-decomposition of the coefficient matrix A of the system (5.40) is:

$$A=LU => \begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 \\ \gamma_2 & \alpha_2 & \beta_2 & 0 \\ 0 & \gamma_3 & \alpha_3 & \beta_3 \\ 0 & 0 & \gamma_4 & \alpha_4 \end{bmatrix} = \begin{bmatrix} l_1 & 0 & 0 & 0 \\ \gamma_2 & l_2 & 0 & 0 \\ 0 & \gamma_3 & l_3 & 0 \\ 0 & 0 & \gamma_4 & l_4 \end{bmatrix} \begin{bmatrix} u_1 & \beta_1 & 0 & 0 \\ 0 & u_2 & \beta_2 & 0 \\ 0 & 0 & u_3 & \beta_3 \\ 0 & 0 & 0 & u_4 \end{bmatrix} \qquad (5.41)$$

The system with decomposition (5.41) can be solved by replacing 1's the diagonal elements of the L and U will have to be nonzero diagonal elements :

$$\begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 \\ \gamma_2 & \alpha_2 & \beta_2 & 0 \\ 0 & \gamma_3 & \alpha_3 & \beta_3 \\ 0 & 0 & \gamma_4 & \alpha_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_2 & 1 & 0 & 0 \\ 0 & l_3 & 1 & 0 \\ 0 & 0 & l_4 & 1 \end{bmatrix} \begin{bmatrix} u_1 & \beta_1 & 0 & 0 \\ 0 & u_2 & \beta_2 & 0 \\ 0 & 0 & u_3 & \beta_3 \\ 0 & 0 & 0 & u_4 \end{bmatrix} \qquad (5.42)$$

$$
= \begin{bmatrix} u_1 & \beta_1 & 0 & 0 \\ l_2 u_1 & l_2\beta_1 + u_2 & \beta_2 & 0 \\ 0 & l_3 u_2 & l_3\beta_2 + u_3 & \beta_3 \\ 0 & 0 & l_4 u_3 & l_4\beta_3 + u_4 \end{bmatrix}
$$

In case, if U is replaced to have 1's on its diagonal, then L will have to be nonzero diagonal elements :

$$
\begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 \\ \gamma_2 & \alpha_2 & \beta_2 & 0 \\ 0 & \gamma_3 & \alpha_3 & \beta_3 \\ 0 & 0 & \gamma_4 & \alpha_4 \end{bmatrix} = \begin{bmatrix} l_1 & 0 & 0 & 0 \\ \gamma_2 & l_2 & 0 & 0 \\ 0 & \gamma_3 & l_3 & 0 \\ 0 & 0 & \gamma_4 & l_4 \end{bmatrix} \begin{bmatrix} 1 & u_1 & 0 & 0 \\ 0 & 1 & u_2 & 0 \\ 0 & 0 & 1 & u_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.43}
$$

$$
= \begin{bmatrix} l_1 & l_1 u_1 & 0 & 0 \\ \gamma_2 & \gamma_2 u_1 + l_2 & l_2 u_2 & 0 \\ 0 & \gamma_3 & \gamma_3 u_2 + l_3 & l_3 u_3 \\ 0 & 0 & \gamma_4 & \gamma_4 u_3 + l_4 \end{bmatrix}
$$

For the rearrangements (5.42) and (5.43), compare the elements of the product matrix LU with those of A to obtain the unknown elements of L and U :

$$
\left.\begin{array}{lll}
u_1 & = \alpha_1 & \\
l_2\beta_1 + u_2 = \alpha_2 & & \Rightarrow u_2 = \alpha_2 - l_2\beta_1 \\
l_3\beta_2 + u_3 = \alpha_3 & & \Rightarrow u_3 = \alpha_3 - l_3\beta_2 \\
\vdots & & \\
u_i & = \alpha_i - l_i\beta_{i-1}, & i = 2,3,4,\ldots,n \\
 & & \\
l_2 u_1 & = \gamma_2 & \Rightarrow l_2 = \gamma_2 / u_1 \\
l_3 u_2 & = \gamma_3 & \Rightarrow l_3 = \gamma_3 / u_2 \\
\vdots & & \\
l_i & = \gamma_i / u_{i-1}, & i = 2,3,4,\ldots,n
\end{array}\right\} \text{ for (5.42)} \tag{5.44}
$$

$$\left.\begin{aligned}
&l_1 = \alpha_1 \\
&\gamma_2 u_1 + l_2 = \alpha_2 \quad \Rightarrow l_2 = \alpha_2 - \gamma_2 u_1 \\
&\gamma_3 u_2 + l_3 = \alpha_3 \quad \Rightarrow l_3 = \alpha_3 - \gamma_3 u_2 \\
&\vdots \\
&l_i = \alpha_i - \gamma_i u_{i-1}, \; i = 2,3,4,\ldots,n \\
&l_1 u_1 = \beta_1 \quad \Rightarrow u_1 = \beta_1 / l_1 \\
&l_2 u_2 = \beta_2 \quad \Rightarrow u_2 = \beta_2 / l_2 \\
&\vdots \\
&u_i = \beta_i / l_i, \; i = 1,2,3,\ldots,n
\end{aligned}\right\} \text{ for } (5.43) \tag{5.45}$$

The tridiagonal system (5.40) :

$$\left.\begin{aligned} Ax &= b \\ LUx &= b \end{aligned}\right\} \text{ now takes the form : } \left.\begin{aligned} Ux &= z \\ Lz &= b \end{aligned}\right\} \tag{5.46}$$

The unknowns $z_1, z_2, \ldots, z_n$ of the system (5.46) can be found by forward substitution method :

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ l_2 & 1 & 0 & 0 \\ 0 & l_3 & 1 & 0 \\ 0 & 0 & l_4 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

$$\left.\begin{aligned}
&z_1 = b_1 \\
&l_2 z_1 + z_2 = b_2 \Rightarrow z_2 = b_2 - l_2 z_1 \\
&l_3 z_2 + z_3 = b_3 \Rightarrow z_3 = b_3 - l_3 z_2 \\
&l_4 z_3 + z_4 = b_4 \Rightarrow z_4 = b_4 - l_4 z_3 \\
&\vdots \\
&z_i = b_i - l_i z_{i-1}, \; i = 2,3,4,\ldots,n
\end{aligned}\right\} \tag{5.47}$$

The unknowns $x_1, x_2, \ldots, x_n$ of the system (5.46) can be found by back substitution method :

$$\begin{bmatrix} u_1 & \beta_1 & 0 & 0 \\ 0 & u_2 & \beta_2 & 0 \\ 0 & 0 & u_3 & \beta_3 \\ 0 & 0 & 0 & u_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

$$\left.\begin{aligned} u_4 x_4 \quad &= z_4 \Rightarrow x_4 = z_4/u_4 \\ u_3 x_3 + \beta_3 x_4 &= z_3 \Rightarrow x_3 = (z_3 - \beta_3 x_4)/u_3 \\ u_2 x_2 + \beta_2 x_3 &= z_2 \Rightarrow x_2 = (z_2 - \beta_2 x_3)/u_2 \\ u_1 x_1 + \beta_1 x_2 &= z_1 \Rightarrow x_1 = (z_1 - \beta_1 x_2)/u_1 \\ &\vdots \\ x_1 &= (z_i - \beta_i x_{i+1})/u_i \\ for\ i &= n-1, n-2, \ldots, 1 \\ x_n &= z_n/u_n \end{aligned}\right\} \tag{5.48}$$

Example 10 : ***[tridiagonal system]*** : Use rearrangement (5.42) to solve the system of equations :

$$\begin{pmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -80 \end{pmatrix} \tag{5.49}$$

The coefficient matrix A of the system (5.49) with rearrangement (5.42) is :

$$\begin{pmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_2 & 1 & 0 \\ 0 & l_3 & 1 \end{pmatrix} \begin{pmatrix} u_1 & \beta_1 & 0 \\ 0 & u_2 & \beta_2 \\ 0 & 0 & u_3 \end{pmatrix} \tag{5.50}$$

The system (5.49) is compared with the system (5.40) to obtain the diagonal, superdiagonal, and subdiagonal entries :

$\alpha_i = -4,\ \beta_i = 1,\ \gamma_{i+1} = 1,\ i = 1,2,3$

The general term (5.44) is used for the following assumptions :

$u_1 = \alpha_1 = -4,\ u_2 = -15/4,\ u_3 = -56/15,\ l_2 = -1/4,\ l_3 = -4/15$

that should be substituted in equation (5.50) to obtain :

$$\underbrace{\begin{pmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ 0 & -4/15 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} -4 & 1 & 0 \\ 0 & -15/4 & 1 \\ 0 & 0 & -56/15 \end{pmatrix}}_{U}$$

The system (5.46) is used to obtain the unknowns z_1, z_2, z_3 and x_3, x_2, x_1 :

$Lz = b$

$$\begin{pmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ 0 & -4/15 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -80 \end{pmatrix} \Rightarrow z_1 = 0,\ z_2 = 0,\ z_3 = -80$$

$Ux = z$

$$\begin{pmatrix} -4 & 1 & 0 \\ 0 & -15/4 & 1 \\ 0 & 0 & -56/15 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -80 \end{pmatrix} \Rightarrow x_3 = 150/7,\ x_2 = 40/7,\ x_1 = 10/7$$

```
EXAMPLE 10 THROUGH PACKAGE MMFP. COMPUTER PROGRAM 5-03:TRIDIAG
THE ORDER OF THE MATRIX:  N = 3
INPUT ALPHA,BETA,GAMMA:  -4.0 1.0 1.0
INPUT ELEMENTS OF B:  0.0 0.0 -80.0
ALPHA = -4.0000, BETA = 1.0000, GAMMA = 1.0000
ELEMENTS OF B :
0.0000   0.0000   -80.0000
SOLUTION OF THE TRIDIAGONAL SYSTEM IS :
0.1428571429D+01  0.5714285714D+01  0.2142857143D+02
```

Example 11 : *[tridiagonal system]* : Use rearrangement (5.43) to solve the system of equations of example 10 :

$$\begin{pmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -80 \end{pmatrix} \tag{5.51}$$

Notation and Definitions

A square matrix of order n

A^{-1} inverse of **A**

A^T transpose of **A**

$|A|$ determinant of **A**

$\overline{A}$ conjugate of **A**

M_{ij} minor of a_{ij} in **A**

A_{ij} cofactors of a_{ij} in **A**

O null matrix

I unit matrix of order n

D diagonal matrix of order n

L lower triangular matrix of order n

U upper triangular matrix of order n

P permutation matrix

$\|A\|$ norm of **A**

$\|x\|$ norm of **x**

The matrix A is said to be nonsingular, if $|A| \neq 0$.

Symmetric if $A = A^T$.

Skew-Symmetric if $A = -A^T$.

Orthogonal if $A^{-1} = A^T$.

Hermitian if $A = \overline{A}^T$.

Skew-Hermitian if $-A = \overline{A}^T$.

Unitary if $A^{-1} = \overline{A}^T$.

Null if $a_{ij} = 0, \ i,j = 1(1)n$.

Diagonal matrix D if $a_{ij} = 0, \ i \neq j$.

Unit matrix I if $a_{ij} = 0, \ i \neq j, \ a_{ii} = 1, \ i = 1(1)n$.

Lower triangular matrix L if $a_{ij} = 0, \ j > i$.

Upper triangular matrix U if $a_{ij} = 0, \ i > j$.

Tridiagonal matrix if $a_{ij} = 0, \ for \ |i-j| > 1$.

The minor determinant $M_{ij} \ of \ A = a_{ij}$ is the determinant of the $n-1 \times n-1$ submatrix obtained by deleting the i-th row and j-th column in A.

The Cofactor A_{ij} is defined by $A_{ij}=(-1)^{i+j}M_{ij}$.

The transpose of the matrix of the cofactors of elements of A is called the **adjoint matrix** and is denoted by $Adj(A)$.

The **inverse** of a matrix **A** is defined by $A^{-1}=\frac{1}{|A|}Adj(A)$.

The matrix [A/b] is called the **augmented matrix**. It is formed by appending the column **b** to the **nxn** matrix **A**.

Exercises : Chapter-5

1. Use **echelon form** to determine whether the system is independent, dependent or inconsistent :

$$a:\ \begin{aligned} -x_1+x_2+2x_3&=2\\ 3x_1-x_2+x_3&=6\\ -x_1+3x_2+4x_3&=4 \end{aligned}$$

$$b:\ \begin{aligned} 3x_1+2x_2+x_3&=3\\ 2x_1+x_2+x_3&=0\\ 6x_1+2x_2+4x_3&=6 \end{aligned}$$

$$c:\ \begin{aligned} 3.0x_1+2.0x_2-2.0x_3-5.0x_4&=8.0\\ 0.6x_1+1.5x_2+1.5x_3-5.4x_4&=2.7\\ 1.2x_1-0.3x_2-0.3x_3+2.4x_4&=2.1 \end{aligned}$$

2. Use **Gauss-elimination method** with partial pivoting to solve the following linear system of equations :

$$a:\ \begin{bmatrix} 1 & -1 & 2\\ 4 & 3 & -1\\ 5 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ x_3 \end{bmatrix}=\begin{bmatrix} 3.8\\ -5.7\\ 2.8 \end{bmatrix}$$

$$b:\ \begin{bmatrix} 5 & 10 & -2\\ 2 & -1 & 1\\ 3 & 4 & 0 \end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ x_3 \end{bmatrix}=\begin{bmatrix} -0.30\\ 1.91\\ 1.16 \end{bmatrix}$$

$$c:\begin{bmatrix} 3.1 & 1.4 & 0.91 \\ 5.8 & -1.9 & 3.20 \\ 6.5 & 1.0 & -0.31 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2.9825 \\ 19.4950 \\ 3.6075 \end{bmatrix}$$

$$d:\begin{bmatrix} 3.1 & 1.20 & -0.83 \\ 4.2 & -5.80 & -1.50 \\ 1.4 & 0.32 & 6.50 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2.804 \\ 21.140 \\ 18.760 \end{bmatrix}$$

$$e:\begin{bmatrix} 3.0 & 1 & 2.0 \\ -2.5 & -0.5 & -11.0 \\ 1.8 & 2.8 & 4.0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 16.1 \end{bmatrix}$$

3. Use **Gauss-elimination method** with partial pivoting to solve the following linear system of equations :

$$a:\begin{bmatrix} 4 & 8 & 4 & 0 \\ 1 & 5 & 4 & -3 \\ 1 & 4 & 7 & 2 \\ 1 & 3 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 10 \\ -4 \end{bmatrix}$$

$$b:\begin{bmatrix} 2 & 4 & -4 & 0 \\ 1 & 5 & -5 & -3 \\ 2 & 3 & 1 & 3 \\ 1 & 4 & -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ 18 \\ 8 \\ 8 \end{bmatrix}$$

$$c:\begin{bmatrix} 1 & 1 & 0 & 4 \\ 2 & -1 & 5 & 0 \\ 5 & 2 & 1 & 2 \\ -3 & 0 & 2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \\ -2 \end{bmatrix}$$

4. Use a package MMFP, computer program 5-01: GAUSSEL, based on Gauss-elimination method to check the results of exercises 2 and 3.

5. The equations $\left.\begin{aligned}\frac{5}{6}x_1+\frac{3}{5}x_2&=17\\ \frac{3}{7}x_1+\frac{1}{3}x_2&=8\end{aligned}\right\}$ exact solution is $x_1=42, x_2=-30$

have to be solved by **Gauss elimination**, when :

a : coefficients are rounded to 2D, and use 3 significant figures,

b : coefficients are rounded to 3D, and use 4 significant figures,

6. Solve the linear system $\begin{bmatrix}5 & 3\\ 3 & 2\end{bmatrix}\begin{bmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{bmatrix}=\begin{bmatrix}2 & 5\\ -1 & 0\end{bmatrix}$ by **Gauss-elimination method.**

7. Solve the following linear system of equations by **direct-factorization :**

$$a:\ \begin{bmatrix}3 & 2 & 0\\ 12 & 13 & 6\\ -3 & 8 & 9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}14\\ 40\\ -28\end{bmatrix}$$

$$b:\ \begin{bmatrix}5 & 4 & 1\\ 10 & 9 & 4\\ 10 & 13 & 15\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}3.4\\ 8.8\\ 19.2\end{bmatrix}$$

$$c:\ \begin{bmatrix}1 & -1 & 2.6\\ 0.5 & -3.0 & 3.3\\ -1.5 & -3.5 & -10.4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}-4.94\\ -8.27\\ 10.51\end{bmatrix}$$

$$d:\ \begin{bmatrix}3 & 9 & 6\\ 18 & 48 & 39\\ 9 & -27 & 42\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}23\\ 136\\ 45\end{bmatrix}$$

8. Use a package MMFP, computer program 5-02: LUFACT, based on direct-factorization to check the results of exercise 7.

9. Show that $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has no LU-factorization, but the singular matrix **A+I** does have. Find the permutation matrix **P** such that **PA** has an LU-factorization.

10. Solve the following system of linear equations by **Cholesky's method**, and check the results obtained by using a package MMFP, computer programs 5-01 : GAUSSEL and 5-02 : LUFACT :

$$a : \begin{bmatrix} 9 & 6 & 12 \\ 6 & 13 & 11 \\ 12 & 11 & 26 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 174 \\ 236 \\ 308 \end{bmatrix}$$

$$b : \begin{bmatrix} 0.01 & 0 & 0.03 \\ 0 & 0.16 & 0.08 \\ 0.03 & 0.08 & 0.14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.14 \\ 0.16 \\ 0.54 \end{bmatrix}$$

$$c : \begin{bmatrix} 4 & 6 & 8 \\ 6 & 34 & 52 \\ 8 & 52 & 129 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -160 \\ -452 \end{bmatrix}$$

$$d : \begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix}$$

11. Use rearrangements (5.42) and (5.43) to solve the following **tridiagonal** system of equations, and check the results by using a package MMFP, computer programs 5-01: GAUSSEL, 5-02: LUFACT and 5-03 : TRIDIAG :

$$\mathbf{a} : \left. \begin{aligned} 2x_1 - x_2 \qquad &= 7 \\ -x_1 + 2x_2 - x_3 &= 1 \\ - x_2 + 2x_3 &= 1 \end{aligned} \right\}$$

$$\text{b}: \quad \left.\begin{aligned} 5x_1 - x_2 \qquad &= 9 \\ -x_1 + 5x_2 - x_3 &= 4 \\ - x_2 + 5x_3 &= -6 \end{aligned}\right\}$$

$$\text{c}: \quad \left.\begin{aligned} 4x_1 - x_2 \qquad\qquad\qquad &= 100 \\ -x_1 + 4x_2 - x_3 \qquad\qquad &= 200 \\ - x_2 + 4x_3 - x_4 \qquad &= 200 \\ - x_3 + 4x_4 - x_5 &= 200 \\ - x_4 + 4x_5 &= 100 \end{aligned}\right\}$$

12 . Find the parabola $y = A + Bx + Cx^2$ that passes through the points :

a : (1,4), (2,7) and (3,14)
b : (1,6), (2,5) and (3, 2)
c : (1,2), (2,2) and (4, 8)

Check the results by using a package of computer programs that suits to the linear system obtained.

13 . If $A = \begin{pmatrix} 4 & 1 & 0 \\ 2 & -1 & 2 \\ x & y & -1 \end{pmatrix}$, then :

a : Find values of x and y that make A singular.
b : Find values of x and y that make A nonsingular.
c : Use indirect factorization to solve the linear system of equations for that values of x and y for which A becomes nonsingular :

$$\begin{pmatrix} 4 & 1 & 0 \\ 2 & -1 & 2 \\ x & y & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

14 . Use **indirect factorization** to solve the system of equations :

$$\begin{pmatrix} 0.5 & 0.0 & 1.2 & 2.0 \\ 1.0 & 2.4 & 0.0 & 1.0 \\ 5.0 & -1.0 & 1.6 & 5.0 \\ 2.0 & 2.5 & 3.2 & 1.5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

Use 3 significant figures throughout the computation.

15. A company produces three color television sets: models X, Y and Z. Each model X set requires 2 hours of electronics work, 2 hours of assembly time and 1 hour of finishing time. Each model Y set requires 1, 3 and 1 hours of electronics, assembly and finishing time respectively. Each model z set requires 3, 2 and 2 hours of the same work respectively. There are 100 hours available for electronics, 100 hours available for assembly and 65 hours available for finishing per week. How many of each model should be produced each week if all available time must be used?

16. A furniture manufacturer has 1950 machine hours available each week in the cutting department, 1490 hours in the assembly department and 2160 in the finishing department. Manufacturing a chair requires .2 hours of cutting, .3 hours of assembly and .1 hour of finishing. A cabinet requires .5 hours of cutting, .4 hours of assembly time and .6 hours of finishing. A buffet requires .3 hours of cutting, .1 hours of assembly time and .4 hours of finishing. How many chairs, cabinets and buffets should be produced in order to use all the available production capacity?

17. Three species of bacteria are fed three foods I, II and III. A bacterium of the first species consumes 1.3 units each of foods I and II and 2.3 units of food III each day. A bacterium of the second species consumes 1.1 units of food I, 2.4 units of food II and 3.7 units of food III each day. A bacterium of the third species consumes 8.1 units of I, 2.9 units of II and 5.1 units of III each day. If 16000 units of I, 28000 units of II and 44000 units of III are supplied each day. How many of each species can be maintained in this environment?

18. A company produces three combinations of mixed vegetables which sell in 1 kilogram packages. Italian style combines .3 kilograms of zucchini, .3 of broccoli and .4 of carrots. French style combines .6 kilograms of broccoli and .4 of carrots. Oriental style combines .2 kilograms of zucchini, .5 of broccoli and .3 of carrots. The company has a stock of 16200 kilograms of zucchini, 41400 kilograms of broccoli and 29400 kilograms of carrots. How many packages of each style should they prepare to use up their supplies?