Week 1 Notes

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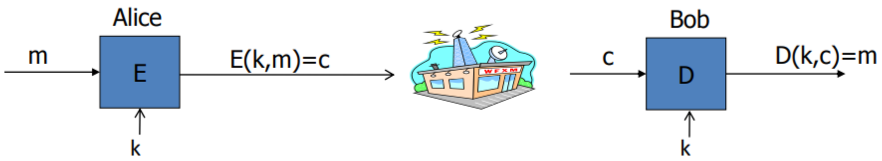
Abstract

Week 1 Notes in the Coursera Course on Cryptography

1 What is Cryptography About?

1.1 Course Overview

- Secure communication: HTTPS, WPA2, GSM, Bluetooth, SSL/TLS
- SSL/TLS has two main parts:
  - Handshake Protocol
  - Record Layer
- Encrypting files on disk: EFS, TrueCrypt
- Symmetric Encryption



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  - Where:
    - E, D: cipher
    - k : secret key (ex: 128 bits)
    - m: plaintext
    - c: ciphertext
  - Encryption Algorithm is **publicly known**
- Use Cases
  - Single use key (one time key)

        * key is used to encrypt a single message
        * new key generated for each email
    - Multi use key (many time key)
        * key encrypts multiple messages
        * same key used to encrypt many files

1.2 What is Cryptography

- Confidentiality and integrity
- Digital signatures, anonymous communication, digital cash
- If something can be done with a trusted authority, it can also be done without one
- Three steps in cryptography
    - Precisely specify threat model
    - Propose a construction
    - Prove that breaking construction under threat mode will solve an underlying hard problem

1.3 History

- Substitution cipher
    - Find most common letter (”E”) via frequencies
    - Use frequency of pairs of letters
- Caesar Cipher
    - Shift letters by three
    - Size of key space: $|\kappa| = 26!$
- Vigenere Cipher
    - Encrypt message m with some cipher k
    - For each letter in m, add it to corresponding letter in k
    - k will ”wrap around” until end of message
    - Take added result and mod 26 to obtain result between 0 and 25
- Rotor Machines
- Data Encryption Standard (DES)
    - keys: 2^{56}, block size: 64 bits
    - today AES is in use

2 Discrete Probability

2.1 Crash Course Part I

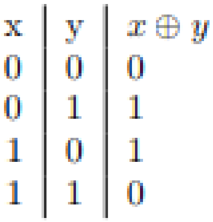
- Let U be some finite set, eg: $U = \{0,1\}^n$
- Probability distribution P over U is a function $P : U \rightarrow [0,1]$, such that $\sum_{X \in U} P(x) = 1$
- EX:
  - Uniform Distribution: for all $x \in U : P(x) = \frac{1}{|U|}$
  - Point Distribution at $x_0 : P(x_0) = 1, \forall x \neq x_0 : P(x) = 0$
- Distribution vector: $(P(000), P(001), P(010), \ldots, P(111))$
- Events
  - For a set $A \subseteq U : Pr[A] = \sum_{x \in U} P(x) \in [0,1]$
  - Set A is an **event**
  - EX: $U = \{0,1\}^8$
  - A = { all x in U such that $lsb_2(x) = 11$ } $\subseteq U$ for the uniform distribution on $0,1^8 : Pr[A] = \frac{1}{4}$
- The Union Bound
  - For events A_1 and A_2: $Pr[A_1 \cup A_2] \leq Pr[A_1] + Pr[A_2]$
  - $A_1 \cap A_2 = \emptyset \rightarrow Pr[A_1 \cup A_2] = Pr[A_1] + Pr[A_2]$
    - $A_1 = \{$all x in $0,1^n$ s.t $lsb_2(x) = 11\}$
    - $Pr[lsb_2(x) = 11$ or $msb_2(x) = 11] = Pr[A1 \cup A_2] \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$
- Random Variables
  - Definition: a random variable X is a function: $X : U \rightarrow V$
  - EX: $X : \{0,1\}^n \rightarrow \{0,1\}; X(y) = lsb(y) \in \{0,1\}$
  - For the uniform distribution on U: $Pr[X = 0] = \frac{1}{2}, Pr[X = 1] = \frac{1}{2}$
  - more generally: random variable X induces a distribution on V: $Pr[X = v] := Pr[X^{-1}(v)]$
- Uniform random variable
  - Let U be some set, eg $U = \{0,1\}^n$
  - We write $r \leftarrow U$ to denote a **uniform random variable** over U for all $a \in U : Pr[r = a] = \dfrac{1}{|U|}$
  - formally r is the identity function: $r(x) = x$for all$x \in U$
  - Let r be a uniform random variable on $\{0,1\}^2$

- Define random variable $X = r_1 + r_2$
- Then $Pr[X = 2] = \frac{1}{4}$
- Hint: $Pr[X = 2] = Pr[r = 11]$

- Randomized Algorithms
  - Deterministic algorithm: $y \leftarrow A(m)$
  - Randomized algorithm: $y \leftarrow A(m; r)$ where $r \leftarrow \{0, 1\}^n$
  - EX: $A(m; k) = E(k, m), y \leftarrow^n A(m)$

2.2 Crash Course Part II

- Independence
  - Events A and B are **independent** if $Pr[A \text{ and } B] = Pr[A] \times Pr[B]$
  - EX: $U = \{0, 1\}^2 = \{00, 01, 10, 11\} and r \leftarrow U$
  - define r.v X and Y as: $X = \text{ lsb}(r), Y = \text{ msb}(r)$
  - $Pr[X = 0 \text{ and } Y = 0] = Pr[r = 00] = \frac{1}{4} = Pr[X = 0] \times Pr[Y = 0]$
- XOR
  - XOR of two strings in $\{0, 1\}^n$ is their bit-wise addition mod 2
  - XOR Chart



x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

    - (Yeah I made that)
  - EX: $0110111 \oplus 1011010 = 1101101$
- Important property of XOR
  - Theorem: Y a random variable over $\{0, 1\}^n$, X is an independent uniform variable on $\{0, 1\}^n$
  - Then $Z := Y \oplus X$ is a uniform variable on $\{0, 1\}^n$
  - Proof for n = 1
    - $Pr[Z = 0] = Pr[(x, y) = (0, 0) \text{ or } (x, y) = (1, 1)]$
    - $Pr[(x, y) = (0, 0)] + Pr[(x, y) = (1, 1)] = \frac{P_0}{2} + \frac{P_1}{2} = \frac{1}{2}$

- The Birthday Paradox
  - Let $r_1, \ldots, r_n \in U$ be independent indentically distributed random variabes
  - When $n = 1.2 \times |U|^{1/2}$ then $Pr[\exists i \neq j : r_i = r_j] \geq 1/2$
  - notation: —U— is the size of U
  - EX: Let $U = \{0,1\}^{128}$
    - * After sampling about 2^{64} random messages from U, some two sampled messages will likely be the same

3 Stream Ciphers

3.1 One Time Pad

- Symmetric Ciphers
  - Def: a cipher defined over (K, M, C) is a pair of "efficient" algorithms (E, D), where:
    - * $E = K \times M \to C$
    - * $D = K \times C \to M$
    - * $\forall m \in M, k \in K : D(k, E(k, m)) = M$
  - E is often randomized, D is always deterministic
- One Time Pad
  - First example of a "secure" cipher
  - $M = C = \{0,1\}^n, K = \{0,1\}^n$
  - key = (random bit string as long as the message)
  - $C := E(k, m) = k \oplus m$
  - $D(k, c) = k \oplus c$
  - $D(k, E(k, m)) = D(k, k \oplus m) = k \oplus (k + m) = (k \oplus k) \oplus m = o \oplus m = m$
  - You are given a message and its OTP encryption (c). Can you compute the OTP key from m and c?
    - * Yes, the key is: $k = m \oplus c$
    - * Very fast encryption/decryption...but long keys (as long as plaintext)
- What is a secure cipher
  - Attacker's abilities: CT only attack (for now)
  - Possible security attempts
    - * 1. Attacker cannot recover secret key. $E(k, m) = m$ would be secure

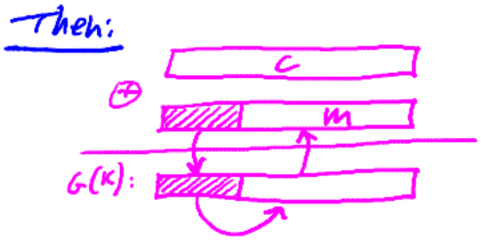
    - 2. Attacker cannot recover all of the plaintext. $E(k, m_0 || m_1) = m_0) || k \oplus m_1$ would be secure
    - Shannon's idea: CT should reveal no "info" about the PT

- Information Theoretic security
  - Definition: A cipher (E, D) over (K, M, C) has **perfect secrecy** if:
    - $\forall m_0, m_1 \in M, (len(m_0) = len(m_1)), \forall c \in C$
    - $Pr[E(K, m_0) = c] = Pr[E(k, m_1) = c]$
    - where k is uniform in K $(k \leftarrow K)$
    - Given ciphertext can't tell if message is m_0 or m_1 (for all m0, m1)
    - most powerful adversary learns nothing about plaintext from ciphertext
    - no ciphertext only attack (other attacks are possible)
  - Lemma: OTP has perfect secrecy. Proof:
    - $\forall m, c : Pr[E(k, m) = c] = \dfrac{\text{number of keys } \kappa \in K \text{ s.l. } E(k, m) = c}{|K|}$
    - If $\forall m, c : \text{number} = \{\kappa \in K : E(k, m) = c\} = \text{const}$
    - Therefore, cipher has perfect secrecy
    - Let $m \in M$ and $c \in C$. How many OTP keys map m to c? **one**
    - However: implies that: $|\kappa| \geq |M| \rightarrow$ hard to use in practice, (key length > message length)

3.2 Pseudorandom Generators

- Stream Ciphers: Making OTP practical
  - idea: replace "random" key by "pseudorandom" key
  - PRG is a function: $G : \{0, 1\}^s \rightarrow \{0, 1\}^n$
  - $n >> s$
  - computable by a <u>deterministic</u> algorithm
  - $C := E(k, m) = m \oplus G(k)$
  - $D(k, c) = C \oplus G(k)$
  - Can a stream cipher have perfect secrecy: No since the key is shorter than the message
  - Stream ciphers cannot have perfect secrecy
    - Need a different definition of security
    - Security will depend on specific PRG
- PRG must be unpredictable
  - Suppose PRG is predictable:

- $\exists i : G(k)|_{1,\dots,i} \rightarrow^{alg} G(k)|_{i+1,\dots,n}$



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- We say that $G : K \rightarrow \{0,1\}^n$ is **unpredictable** if:
    - $\exists$ "eff" alg A and $\exists_0 \leq i \leq n-1 \leq t$
    - $Pr[A(G(k))|_{1,\dots,i} = G(k)|_{i+1}] > \frac{1}{2} + \epsilon$
    - For non-negligible ϵ, eg $\frac{1}{2^{30}}$
    - Definition: PRG is unpredictable if it is not predictable, no "efficient" adv. can predict bit (i+1) for "non-neg" ϵ
    - Suppose $G : K \rightarrow \{0,1\}^n$ is such that for all k: $XOR(G(k)) = 1$. Is G predictable? → Yes given the first (n-1) bits I can predict the n'th bit

- Weak PRGs (do not use for crypto!)
    - glibc random():
    - $r[i] \leftarrow (r[i-3] + r[i-31])\%2^{32}$
    - output $r[i] >> 1$
    - Never use random() for crypto

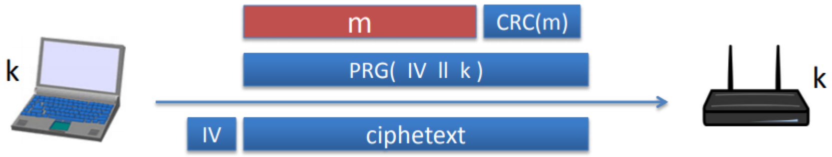
3.3 Negligible vs. non-negligible

- Negligible vs. non-negligible
    - In practice: ϵ is a scalar and
        - ϵ non-neg $\epsilon \geq \frac{1}{2^{30}}$ (likely to happen over 1 GB of data)
        - ϵ negligible $\epsilon \leq \frac{1}{2^{80}}$ (won't happen over life of key)
    - In theory: ϵ is a function: $\epsilon : Z^{\geq 0} \rightarrow R^{\geq 0}$
        - ϵ non-neg $\exists d : \epsilon(\lambda) \geq \frac{1}{\lambda^d}(\epsilon \geq \frac{1}{poly}$, for many $\lambda)$
        - ϵ negligible $\forall d, \lambda \geq \lambda_d : \epsilon(\lambda) \leq \frac{1}{\lambda^d}(\epsilon \leq \frac{1}{poly}$, for large $\lambda)$
    - Few Examples
        - $\epsilon(\lambda) = \frac{1}{2^\lambda}$: negligible
        - $\epsilon(\lambda) = \frac{1}{\lambda^{1000}}$: non-negligible
        - $$\begin{cases} \frac{1}{2^\lambda} & \text{for odd } \lambda \\ \frac{1}{\lambda^{1000}} & \text{for even } \lambda \end{cases}$$

- PRGs: the rigorous theory view
  - PRGs are parametrized by a security parameter: λ
    - PRG becomes "more secure" as λ increases
  - Seed lengths and output lengths grow with λ
  - For every $\lambda = 1, 2, 3, ...$ there is a different PRG: G_λ
  - $G_\lambda : K_\lambda \to \{0,1\}^{n(\lambda)}$
  - We say that the previous equation is **predictable** at position i if:
    - there exists a polynomial time (in λ) algorithm A s.t.
    - $Pr_{k \leftarrow K_\lambda}[A(\lambda, G_\lambda(k))|_{1,\ldots,i} = G_\lambda(k)|_{i+1}] > \frac{1}{2} + \epsilon(\lambda)$
    - For some non-negligible function $\epsilon(\lambda)$

3.4 Attacks on OTP and Stream Ciphers

- Attack 1: Two time pad is insecure
  - Never use stream cipher key more than once
  - $C_1 \leftarrow m_1 \oplus PRG(k)$
  - $C_2 \leftarrow m_2 \oplus PRG(k)$
  - Eavesdropper does: $C_1 + C_2 \to m_1 \oplus m_2$
  - Enough redundancy in English and ASCII encoding that: $m_1 \oplus m_2 \to m_1, m_2$
  - Real World Examples
    - Project Venona
    - MS-PPTP
    - 802.11b WEP



      ·
      · Length of IV: 24 bits
      · Repeated IV after $2^{24} \approx 16$ M frames
      · On some 802.11 cards: IV resets to 0 after power cycle
      · Avoid related keys