Week 1 Notes

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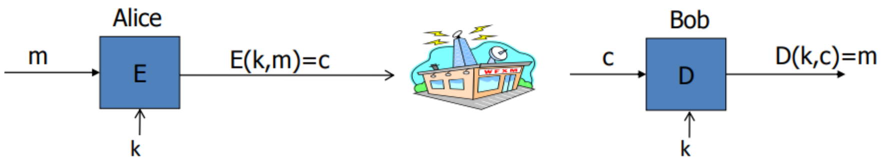
Abstract

Week 1 Notes in the Coursera Course on Cryptography

1 What is Cryptography About?

1.1 Course Overview

- Secure communication: HTTPS, WPA2, GSM, Bluetooth, SSL/TLS
- SSL/TLS has two main parts:
  - Handshake Protocol
  - Record Layer
- Encrypting files on disk: EFS, TrueCrypt
- Symmetric Encryption



  -
  - Where:
    - E, D: cipher
    - k : secret key (ex: 128 bits)
    - m: plaintext
    - c: ciphertext
  - Encryption Algorithm is **publicly known**
- Use Cases
  - Single use key (one time key)

    - key is used to encrypt a single message
    - new key generated for each email
  - Multi use key (many time key)
    - key encrypts multiple messages
    - same key used to encrypt many files

1.2 What is Cryptography

- Confidentiality and integrity
- Digital signatures, anonymous communication, digital cash
- If something can be done with a trusted authority, it can also be done without one
- Three steps in cryptography
  - Precisely specify threat model
  - Propose a construction
  - Prove that breaking construction under threat mode will solve an underlying hard problem

1.3 History

- Substitution cipher
  - Find most common letter (”E”) via frequencies
  - Use frequency of pairs of letters
- Caesar Cipher
  - Shift letters by three
  - Size of key space: $|\kappa| = 26!$
- Vigenere Cipher
  - Encrypt message m with some cipher k
  - For each letter in m, add it to corresponding letter in k
  - k will ”wrap around” until end of message
  - Take added result and mod 26 to obtain result between 0 and 25
- Rotor Machines
- Data Encryption Standard (DES)
  - keys: 2^{56}, block size: 64 bits
  - today AES is in use

2 Discrete Probability

2.1 Crash Course Part I

- Let U be some finite set, eg: $U = \{0,1\}^n$
- Probability distribution P over U is a function $P : U \to [0,1]$, such that $\sum_{X \in U} P(x) = 1$
- EX:
  - Uniform Distribution: for all $x \in U : P(x) = \frac{1}{|U|}$
  - Point Distribution at $x_0 : P(x_0) = 1, \forall x \neq x_0 : P(x) = 0$
- Distribution vector: $(P(000), P(001), P(010), \ldots, P(111))$
- Events
  - For a set $A \subseteq U : Pr[A] = \sum_{x \in U} P(x) \in [0,1]$
  - Set A is an **event**
  - EX: $U = \{0,1\}^8$
  - A = { all x in U such that $lsb_2(x) = 11$ } $\subseteq U$ for the uniform distribution on $0,1^8 : Pr[A] = \frac{1}{4}$
- The Union Bound
  - For events A_1 and A_2: $Pr[A_1 \cup A_2] \leq Pr[A_1] + Pr[A_2]$
  - $A_1 \cap A_2 = \emptyset \to Pr[A_1 \cup A_2] = Pr[A_1] + Pr[A_2]$
    - $A_1 = \{$all x in $0,1^n$ s.t $lsb_2(x) = 11\}$
    - $Pr[lsb_2(x) = 11$ or $msb_2(x) = 11] = Pr[A1 \cup A_2] \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$
- Random Variables
  - Definition: a random variable X is a function: $X : U \to V$
  - EX: $X : \{0,1\}^n \to \{0,1\}; X(y) = lsb(y) \in \{0,1\}$
  - For the uniform distribution on U: $Pr[X = 0] = \frac{1}{2}, Pr[X = 1] = \frac{1}{2}$
  - more generally: random variable X induces a distribution on V: $Pr[X = v] := Pr[X^{-1}(v)]$
- Uniform random variable
  - Let U be some set, eg $U = \{0,1\}^n$
  - We write $r \leftarrow U$ to denote a **uniform random variable** over U for all $a \in U : Pr[r = a] = \dfrac{1}{|U|}$
  - formally r is the identity function: $r(x) = x$for all$x \in U$
  - Let r be a uniform random variable on $\{0,1\}^2$

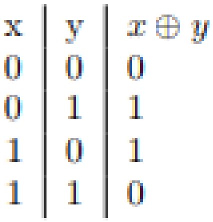
  - Define random variable $X = r_1 + r_2$
  - Then $Pr[X = 2] = \dfrac{1}{4}$
  - Hint: $Pr[X = 2] = Pr[r = 11]$

- Randomized Algorithms
  - Deterministic algorithm: $y \leftarrow A(m)$
  - Randomized algorithm: $y \leftarrow A(m; r)$ where $r \leftarrow \{0,1\}^n$
  - EX: $A(m; k) = E(k, m), y \leftarrow^n A(m)$

2.2 Crash Course Part II

- Independence
  - Events A and B are **independent** if $Pr[A \text{ and } B] = Pr[A] \times Pr[B]$
  - EX: $U = \{0,1\}^2 = \{00, 01, 10, 11\} and r \leftarrow U$
  - define r.v X and Y as: $X = \text{ lsb}(r), Y = \text{ msb}(r)$
  - $Pr[X = 0 \text{ and } Y = 0] = Pr[r = 00] = \frac{1}{4} = Pr[X = 0] \times Pr[Y = 0]$

- XOR
  - XOR of two strings in $\{0,1\}^n$ is their bit-wise addition mod 2
  - XOR Chart



x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

    - (Yeah I made that)
  - EX: $0110111 \oplus 1011010 = 1101101$

- Important property of XOR
  - Theorem: Y a random variable over $\{0,1\}^n$, X is an independent uniform variable on $\{0,1\}^n$
  - Then $Z := Y \oplus X$ is a uniform variable on $\{0,1\}^n$
  - Proof for n = 1
    - $Pr[Z = 0] = Pr[(x, y) = (0, 0) \text{ or } (x, y) = (1, 1)]$
    - $Pr[(x, y) = (0, 0)] + Pr[(x, y) = (1, 1)] = \dfrac{P_0}{2} + \dfrac{P_1}{2} = \dfrac{1}{2}$

- The Birthday Paradox
  - Let $r_1, \ldots, r_n \in U$ be independent indentically distributed random variabes
  - When $n = 1.2 \times |U|^{1/2}$ then $Pr[\exists i \neq j : r_i = r_j] \geq 1/2$
  - notation: —U— is the size of U
  - EX: Let $U = \{0,1\}^{128}$
    - ∗ After sampling about 2^{64} random messages from U, some two sampled messages will likely be the same

3 Stream Ciphers

3.1 One Time Pad

- Symmetric Ciphers
  - Def: a cipher defined over (K, M, C) is a pair of "efficient" algorithms (E, D), where:
    - ∗ $E = K \times M \rightarrow C$
    - ∗ $D = K \times C \rightarrow M$
    - ∗ $\forall m \in M, k \in K : D(k, E(k,m)) = M$
  - E is often randomized, D is always deterministic
- One Time Pad
  - First example of a "secure" cipher
  - $M = C = \{0,1\}^n, K = \{0,1\}^n$
  - key = (random bit string as long as the message)
  - $C := E(k,m) = k \oplus m$
  - $D(k,c) = k \oplus c$
  - $D(k, E(k,m)) = D(k, k \oplus m) = k \oplus (k+m) = (k \oplus k) \oplus m = o \oplus m = m$
  - You are given a message and its OTP encryption (c). Can you compute the OTP key from m and c?
    - ∗ Yes, the key is: $k = m \oplus c$
    - ∗ Very fast encryption/decryption...but long keys (as long as plaintext)
- What is a secure cipher
  - Attacker's abilities: CT only attack (for now)
  - Possible security attempts
    - ∗ 1. Attacker cannot recover secret key. $E(k,m) = m$ would be secure

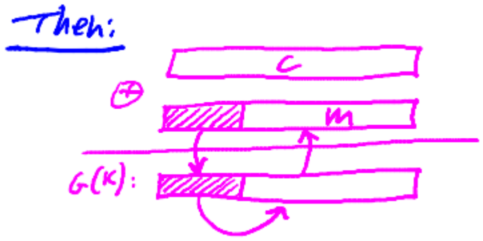
        - 2. Attacker cannot recover all of the plaintext. $E(k, m_0||m_1) = m_0)||k \oplus m_1$ would be secure
        - Shannon's idea: CT should reveal no "info" about the PT

- Information Theoretic security
    - Definition: A cipher (E, D) over (K, M, C) has **perfect secrecy** if:
        - $\forall m_0, m_1 \in M, (len(m_0) = len(m_1)), \forall c \in C$
        - $Pr[E(K, m_0) = c] = Pr[E(k, m_1) = c]$
        - where k is uniform in K ($k \leftarrow K$)
        - Given ciphertext can't tell if message is m_0 or m_1 (for all m0, m1)
        - most powerful adversary learns nothing about plaintext from ciphertext
        - no ciphertext only attack (other attacks are possible)
    - Lemma: OTP has perfect secrecy. Proof:
        - $\forall m, c : Pr[E(k, m) = c] = \dfrac{\text{number of keys } \kappa \in K \text{ s.l. } E(k, m) = c}{|K|}$
        - If $\forall m, c : \text{number} = \{\kappa \in K : E(k, m) = c\} = \text{const}$
        - Therefore, cipher has perfect secrecy
        - Let $m \in M$ and $c \in C$. How many OTP keys map m to c? **one**
        - However: implies that: $|\kappa| \geq |M| \rightarrow$ hard to use in practice, (key length > message length)

3.2 Pseudorandom Generators

- Stream Ciphers: Making OTP practical
    - idea: replace "random" key by "pseudorandom" key
    - PRG is a function: $G : \{0, 1\}^s \rightarrow \{0, 1\}^n$
    - $n >> s$
    - computable by a <u>deterministic</u> algorithm
    - $C := E(k, m) = m \oplus G(k)$
    - $D(k, c) = C \oplus G(k)$
    - Can a stream cipher have perfect secrecy: No since the key is shorter than the message
    - Stream ciphers cannot have perfect secrecy
        - Need a different definition of security
        - Security will depend on specific PRG
- PRG must be unpredictable
    - Suppose PRG is predictable:

  - $\exists i : G(k)|_{1,\ldots,i} \rightarrow^{alg} G(k)|_{i+1,\ldots,n}$



  -
  - We say that $G : K \rightarrow \{0,1\}^n$ is **unpredictable** if:
    - $\exists$ ”eff” alg A and $\exists_0 \leq i \leq n-1 \leq t$
    - $Pr[A(G(k))|_{1,\ldots,i} = G(k)|_{i+1}] > \frac{1}{2} + \epsilon$
    - For non-negligible ϵ, eg $\frac{1}{2^{30}}$
    - Definition: PRG is unpredictable if it is not predictable, no ”efficient” adv. can predict bit (i+1) for ”non-neg” ϵ
    - Suppose $G : K \rightarrow \{0,1\}^n$ is such that for all k: $XOR(G(k)) = 1$. Is G predictable? → Yes given the first (n-1) bits I can predict the n’th bit

- Weak PRGs (do not use for crypto!)
  - glibc random():
  - $r[i] \leftarrow (r[i-3] + r[i-31])\%2^{32}$
  - output $r[i] >> 1$
  - Never use random() for crypto

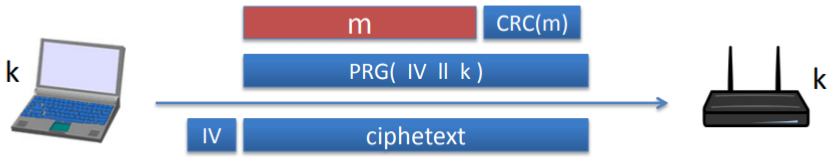
3.3 Negligible vs. non-negligible

- Negligible vs. non-negligible
  - In practice: ϵ is a scalar and
    - ϵ non-neg $\epsilon \geq \frac{1}{2^{30}}$ (likely to happen over 1 GB of data)
    - ϵ negligible $\epsilon \leq \frac{1}{2^{80}}$ (won’t happen over life of key)
  - In theory: ϵ is a function: $\epsilon : Z^{\geq 0} \rightarrow R^{\geq 0}$
    - ϵ non-neg $\exists d : \epsilon(\lambda) \geq \frac{1}{\lambda^d}(\epsilon \geq \frac{1}{poly}$, for many $\lambda)$
    - ϵ negligible $\forall d, \lambda \geq \lambda_d : \epsilon(\lambda) \leq \frac{1}{\lambda^d}(\epsilon \leq \frac{1}{poly}$, for large $\lambda)$
  - Few Examples
    - $\epsilon(\lambda) = \frac{1}{2^\lambda}$: negligible
    - $\epsilon(\lambda) = \frac{1}{\lambda^{1000}}$: non-negligible
    - $$\begin{cases} \frac{1}{2^\lambda} & \text{for odd } \lambda \\ \frac{1}{\lambda^{1000}} & \text{for even } \lambda \end{cases}$$

- PRGs: the rigorous theory view
  - PRGs are parametrized by a security parameter: λ
    - PRG becomes "more secure" as λ increases
  - Seed lengths and output lengths grow with λ
  - For every $\lambda = 1, 2, 3, ...$ there is a different PRG: G_λ
  - $G_\lambda : K_\lambda \rightarrow \{0,1\}^{n(\lambda)}$
  - We say that the previous equation is **predictable** at position i if:
    - there exists a polynomial time (in λ) algorithm A s.t.
    - $Pr_{k \leftarrow K_\lambda}[A(\lambda, G_\lambda(k))|_{1,\ldots,i} = G_\lambda(k)|_{i+1}] > \frac{1}{2} + \epsilon(\lambda)$
    - For some non-negligible function $\epsilon(\lambda)$

3.4 Attacks on OTP and Stream Ciphers

- Attack 1: Two time pad is insecure
  - Never use stream cipher key more than once
  - $C_1 \leftarrow m_1 \oplus PRG(k)$
  - $C_2 \leftarrow m_2 \oplus PRG(k)$
  - Eavesdropper does: $C_1 + C_2 \rightarrow m_1 \oplus m_2$
  - Enough redundancy in English and ASCII encoding that: $m_1 \oplus m_2 \rightarrow m_1, m_2$
  - Real World Examples
    - Project Venona
    - MS-PPTP
      - Have one key for interaction between server and client
      - Another one between client and server
    - 802.11b WEP



      -
      - Length of IV: 24 bits
      - Repeated IV after $2^{24} \approx 16$ M frames
      - On some 802.11 cards: IV resets to 0 after power cycle
      - Avoid related keys
    - Disk encryption (one-time pad does not work)

        - · As you make changes to a file, the encrypted contents can leak as a hacker can figure out where in memory those changes were made

- Attack 2: OTP provides no integrity, is malleable
    - (m ⊕ k) ⊕ p (p is the message used by the hacker)
    - Decrypting the previous expression will yield m ⊕ p
    - OTP is malleable because it is easy to change ciphertext

3.5 Real World Stream Ciphers

- Old Example 1: RC4
    - Input: variable sized seed
    - Expands into 2048 bits, executes very simple loop to generate one byte/round
    - Used in HTTPS and WEP
    - Weaknesses: bias in initial output, probability of (0,0) is $\frac{1}{256^2} + \frac{1}{256^3}$, Related key attacks
- Old Example 2: CSS(Content Scrambling System)
    - Badly broken
    - Linear feedback shift register
        - ∗ a register that contains cells
        - ∗ certain taps are in certain cells that feed into an XOR
        - ∗ at every clock cycle register shifts to the left
    - DVD encryption uses this
    - seed = 5 bytes
- Modern Example: eStream
    - PRG: $\{0,1\}^s * R \rightarrow \{0,1\}^n$
    - Nonce: a non-repeating value for a given key
    - First input is a seed, R is the nonce
    - $E(k, m : r) = m \oplus PRG(k; r)$
    - the pair k, r is never used more than once
    - can reuse the key because (k, r) is unique
    - eStream: Salsa20 (SW + HW)
        - ∗ 128,256 bit seed
        - ∗ 64 bit nonce
        - ∗ $\{0,1\}^{128 \text{ or } 256} * \{0,1\}^{64} \rightarrow \{0,1\}^n$
        - ∗ Salsa20(k;r) := $H(k, (r, 0))||H(k, (r, 1))||$...
        - ∗ Expand the states into 64 bytes long

3.6 PRG Security Defs

- Let $G : K \rightarrow \{0,1\}^n$ be a PRG
- Goal: define what it means that: $[K \leftarrow \kappa,$ output G(k)]
- This distrubtion of pseudo random string is "indistinguishiable" from a truly uniform distribution
- $[r \leftarrow \{0,1\}^n,$output r]
- Statistical Test
  - Statistical test on $\{0,1\}^n$: an algorithm A s.t. A(x) outputs 0 or 1
  - Takes input x and decides whether its random or not
  - Examples:
    - * $A(x) = 1$ iff $|\#0(x) - \#1(x)| \leq 10 * \sqrt{n}$
    - * $A(x) = 1$ iff $|\#00(x) - \frac{n}{4}| \leq 10 * \sqrt{n}$
    - * $A(x) = 1$ iff max_num_of_0(x) $\leq 10 * \sqrt{n}$
- Advantage: $Adv_{PRG}[A,G] := |Pr[A(G(k)) = 1] - Pr[A(r) = 1]| \leftarrow [0,1]$
- If the advantage is close to 1, it means that the statistical test behaved differently when we gave it pseudorandom inputs versus random inputs
- If the advantage is close to 0, it means the pseudorandom inputs and random inputs behaved the same (or similarly)
- Supose $G : K \rightarrow \{0,1\}^n$ satisfies $msb(G(k)) = 1$ for 2/3rds of the keys in K
  - Define statistical test A(x) as: if[msb = 1 output 1 else output 0]
  - Then: $Adv_{PRG} = |Pr[A(G(k)) = 1 - Pr[A(r) = 1]| = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$
  - "A breaks G with advantage 1/6"
- Secure PRGs: Crypto definition
  - Def: We say that $G : K \rightarrow \{0,1\}^n$ is a **secure PRG** if:
  - $\forall$ "effcient" statistical tests $A : adv_{PRG}[A,G]$is negligible (close to 0)
  - Are there provably secure PRGs? Unknown
- A secure PRG is unpredictable
  - We show that if a PRG is predictable it is insecure
  - Suppose A is an efficient algorithm s.t. $pr[A(G(k))|_{1,\ldots,i} = G(k)|_{i+1}] = \frac{1}{2} + \epsilon$
  - for non-negligible ϵ (EX: $\epsilon = \frac{1}{1000}$)
  - Define statistical test B as: $B(x) = [if A(X|_{1,\ldots,i}) = X_{i+1} output 1 else output 0]$

  - Give the statistical test a truly random string R what is the probability that it outputs 1; its $\frac{1}{2}$
  - Give it a pseudorandom sequence, it is likely to output 1 as $\geq \frac{1}{2} + \epsilon$
  - Difference between the two is ϵ

- An unpredictable PRG is secure
  - Let $G : K \rightarrow \{0, 1\}^n$ be PRG
  - Yaos Theorem: if $\forall i \in 0, ..., n - 1$PRG G is unpredictable at position i then G is a secure PRG
  - If next bit predictors cannot distinguish G from random then no statistical test can
  - Suppose a generator where from the last $\frac{n}{2}$ bits of G(k) it is easy to compute the first $\frac{n}{2}$ bits
  - Is G predictable? Yes because G is not secure, by Yaos theorem then G is predictable

- More generally
  - Let P_1 and P_2 be two distributions over $\{0, 1\}^n$
  - By definition P1 and P2 are **computationally indistinguishiable** (denoted P1 $\approx$ P2)
  - If $\forall$ efficient statistical tests A $|Pr_{P1}[A(x) = 1 - Pr_{P2}[A(x) = 1| <$ negligible
  - EX: a PRG is secure if $k \leftarrow K : G(k) \approx uniform(\{0, 1\}^n)$

3.7 Semantic Security

- What is a secure cipher
  - Attacker's abilities: obtain one ciphertext (for now)
  - Possible security requiremnts:
    - Attempt 1: attacker cannot require secret key
      - $E(k, m) = m$ encrypt key k with message m just outputs m
    - Attempt 2: attacker cannot recover all of plaintext
      - $E(k, m_0||m_1) = m_0||E(k, m_1)$ we encrypt two messages together, outputs the first message but concatenates the encryption of the second message
      - Shannon: Ciphertext should reveal no information about the plaintext
    - Shannon's perfect secrecy
      - (E, D) has perfect secrecy if $\forall m_0, m_1 \in M(|m_0 = m_1)$
      - If the attacker sees the ciphertext he doesn't know if it came from the distribution of encryptin m_0 or m_1

    - This is true for all messages of the same length
    - The same distributions don't need to be identical but can be similar

- Semantic Security
  - For b=0,1 define experiments Exp(0) and Exp(1) as
  - Adversary A: $m_0, m_1 \in M : |m_0| = |m_1|$ to Challenger b
  - Challenger b: $c \leftarrow E(k, m_b)$
  - For b=0,1 $W_b :=$ [event that $Exp(b) = 1$]
  - $Adv_{SS}[A, E] := |Pr[W_0] - Pr[W_1]| \in [0, 1]$
  - If in both experiemnts, the adversary output 1 with the same probability, he/she wasn't able to distinguish the two experiemnts
  - However if they were able to output 1 with different probability then he/she was able to distinguish the two experiments
- Definition of Semantic Security
  - E is semantically secure if for all "efficient" A $Adv_{SS}[A, E]$ is negligible for all explicit $m_0, m_1 \in M : E(k, m_0 \approx E(k, m_1)$
  - Example 1
    - Suppose efficient A can always deduce LSB of PT from CT $\rightarrow E = (E, D)$ is not semantically secure
    - $m_0, lsb(m_0) = 0; m_1, lsb(m_1), Adv_{ss}[B, E] = |Pr[Exp(0) = 1] - Pr[Exp(1) = 1]| = |0 - 1| = 1$
    - This is a concept of perfect secrecy
  - Example 2
    - For all A: $Adv_{SS}[A, E] = |Pr[A(k \oplus m_0 = 1)] - Pr[A(k \oplus m_1) = 1]| = 0$
    - Where $|m_0| = |m_1|$ therefore the previous example is 0 because the both values are the same