**Dynamical Systems**

Dynamical system is a mathematical model to represent a function that develops in time. It is defined as a set of {M,T,D} where M is the manifold, T is the one-dimensional directed space, and D is an operator that maps M onto itself. In simpler words, dynamical system at any particular time has a given state represented by a set of real numbers that can be characterised by a point in a phase space. Physically, it can be inferred that a dynamical system is an ensemble of particles whose state variables follow differential equations and vary with time as these equations involve time derivatives. Thus, for a particular time interval from a current state, only one future state is obtained, making the future deterministic.

The manifold M is defined as the state space, the directed space T is time, and D is the development equation or operator, defining how the state traverse through time. A state u(t) consisting of state variables describes the state of the system at time t which is obtained by applying the operator D on the current state. Therefore, we only look at Markovian systems. Over time on applying the operator D, several successive states u(t) with are obtained which form the trajectories in M. Such solutions of the differential equations help us to understand various phenomenon in a more comprehensive manner by shifting the scale from microscopic level to a macroscopic level.

Typical examples of a dynamical system include oscillating pendulum, Lorenz 63.

**2.1 Physical Point Mass:**

A simple dynamical system is a point mass that propagates through a physical space. Let x denote the position of a particle in some domain Ω and v(x,t) define the velocity of the point mass, such as:

(2.1)

Hence, the trajectory of the point mass can be obtained by solving:

,

where . Above equation is an example of development operator as it helps to obtain a future state x(t) from current state x0. In a sophisticated way, this can be represented as

A trajectory can be understood as two different interpretations: (i) the path through which a point particle traverses from the start point defined by set of initial conditions, or (ii) the set of points or states which can be reached from a particular point for t > 0 or set of particles from which the initial state can be achieved when going backwards in time i.e. t < 0.

Usually, considering the complexity that can arise with higher derivatives, only first order derivatives are considered. It has a twofold effect, firstly intuitively the interpretation is easier and secondly, in continuous cases the development equation always has a form such as (2.1). In case, equations with higher derivatives are defining the state space, effort is made to reduce the nth order differential equation in a set of n first-order differential equation. For eg, if the dynamics of a mass point is defined as:

Where, denotes the position and denotes the acceleration, order of differential equation is reduced by increasing the number of equations, which is shown as follows:

Thus, above method of reducing the order of differential equations by increasing the number of equations, emphasises on the equivalence between order and dimension.

**General Aspects of Dynamical System:**

Here, we proceed to define the different elements of dynamical system, namely state space or manifold M, time T and development operator D.

**State Space:**

State space consists of a minimum set of variables which mathematically completely define the dynamical system at a time. These set of variables are called state variables. The values of such a set of variables is known as the *state* of the system. And the set of all possible states is defined as the state space.

We can group dynamical systems in different categories based on their structure of state space. The system can be low dimensional such as undamped pendulum with two state variable , the angle and it’s derivative or of high dimensions such as n dimensional Lotka-Voltera equations.

The system can be discrete, continuous, or hybrid. If the state takes values from a finite-set , then it’s a discrete system. For eg. A light switch makes a discrete dynamical with two values, . A continuous system is defined when the state takes values from a Euclidean space for some , as in the motion of a pendulum. A hybrid system has a part of the state taking value from finite set, while the other part is taking values from a space. Using a computer to control the motion of a pendulum is one such example of a hybrid system.

**Time:**

The order of states in a dynamical system is given by the orderly sequence of the time space. The future states of the system must be completely determined by the state variables at a given time. The evolution of states under time variable can be discrete or continuous. If the time is discrete, then the evolution of system takes place in discrete time steps, which are usually takes as integer values . The state of the system is thus defined at time t as . Many a times, a dynamical system is defined such that it takes as a state at time t and gives the output state at next time. Therefore, in cases where we start at state , and feed the initial conditions to development operator, we obtain and subsequently obtain the sequence of states . Such a system where the states at all times are obtained by the development operator and initial condition defines a dynamical system. On other hand, when the state evolves continuously through time, it’s called as a continuous dynamical system. As time evolves, the state of system evolves simultaneously through state space. The development operator thus defines how the state would evolve through time.

**Lagrangian Coherent Structure:**

LCS acronym was termed by Haller & Yuan(2000) to describe the skeletons of Lagrangian particle dynamics which form the most attracting, repelling and shearing material surfaces. LCS are distinguished separatices in dynamical systems, similar to stable and unstable manifolds of time-independent systems. These invariant manifolds divide dynamically distinct regions in the flow and enforce a major influence on nearby trajectories over time. The kind of this influence may vary, but it invariably creates a coherent surface for which the underlying LCS serves as a skeleton.

Attracting LCS are the separatices which attracts the nearby trajectories over the time with the maximum intensity for the times Therefore, the attracting LCS are responsible for making the centrepiece where the nearby trajectories compile for the forward-evolving trajectory patterns over the time interval . Per se, in unsteady flows these LCS make the theoretical centrepieces of tracer filaments.

Accordingly, repelling LCS are the separatices which repel the nearby trajectories over the with the maximum intersity for times . Such strong repulsive elements cause the nearby trajectories to diverge and propagate to different areas in domain. In particular, repelling LCSs serve as the theoretical centrepieces of major local stretching regions as seen in the unsteady flows. In backward time, repelling LCSs become attracting LCSs and vice versa.

**Haller** suggested that

LCS examples seen in real world includes oil spills, floating debris, chlorophyll patterns in the ocean, spores in the atmosphere.

**FTLE:**

The finite-time Lypunov exponent FTLE, denoted by , is a scalar value which is used to define the stretching for the trajectory of point in the time interval [t, t + T].