Analysis Comprehensive Examination

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12h00 – 18h00

Instructions:

This examination consists of 3 parts, A,B and C:

- Part A covers the Core Material, it consists of 8 questions worth 8 marks each for a total of 64 marks. All questions must be answered for full marks.
- Part B covers the Advanced Material on Abstract Measure and Integration Theory. This section consists of 3 questions of which you must attempt 2. Each question is worth 16 marks for a total of 32 marks.
- Part C covers the Advanced Material on Functional Analysis. This section consists of 3 questions of which you must attempt 2. Each question is worth 16 marks for a total of 32 marks.

Please take note of the following:

1. In Parts B and C, if you attempt more than 2 questions, you must clearly indicate which two questions you would like marked. Otherwise only the first 2 answers in the order they appear in your solutions will be graded.
2. The examination is worth a total of 128 marks. A total grade of 75% or 96 marks is required to pass the exam.
3. The examination length is 6 hours. No texts, reference books, calculators, cell phones, or other aids are permitted in the examination.

A. Core material

1. (a) Suppose f_n, $n \geq 1$, is a sequence of real-valued functions of bounded variation on $[0,1]$ which converges pointwise to a function f on $[0,1]$. Assume, in addition, that there exists $M > 0$ such that the variation of f_n on $[0,1]$ does not exceed M for any n, with M independent of n. Show that f has bounded variation on $[0,1]$.

   (b) Construct a sequence f_n, $n \geq 1$, of real-valued functions of bounded variation on $[0,1]$ which converges pointwise to a function f on $[0,1]$ such that f does not have bounded variation on $[0,1]$.

   (c) Suppose f_n, $n \geq 1$, is a sequence of real-valued continuous functions of bounded variation on $[0,1]$ which converges uniformly to a function f on $[0,1]$. Does it follow that f has bounded variation on $[0,1]$?

2. Let $\alpha(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$ Suppose f is of bounded variation on $[0,1]$ and $f \in R(\alpha)$ on $[0,1]$. Prove that f is a constant function.

3. Use residues to compute the value:

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\, dx.$$

4. Prove **Dini's theorem**: If (f_n) is a sequence of real valued continuous functions converging pointwise to a continuous limit function f on a compact set $S \subset \mathbb{R}$, and if $f_n(x) \geq f_{n+1}(x)$, for each $x \in S$ and every $n \in N$, then $f_n \to f$ uniformly on S.

5. Find $f \in L^2([0,2\pi])$ such that

$$f(x) \sim \sum_{n=1}^{\infty} \frac{\cos nx + \sin nx}{\sqrt{n}},$$

   or prove that such function f does not exist.

6. Evaluate the following line integral:

$$\oint_C \left(zy\sin(xy) + (x+y)^2\right) dx + \left((x+y)^2 + zx\sin(xy)\right) dy + \left(yz^3 - \cos(xy)\right) dz,$$

where C is the curve of intersection of surfaces $z = \sqrt{x^2+y^2}$ and $(x-1)^2 + y^2 = 1$, directed counterclockwise when viewed from above.

7. Calculate
$$\sum_{n=0}^{\infty} \int_{-\frac{\pi}{2}}^{\pi} (-1)^n \frac{x^{2n+1}}{(2n)!} dx.$$
Make sure to justify each step.

8. (a) State Cauchy's formula for derivatives.

   (b) Suppose that h is an entire function which obeys $|h(z)| \leq \pi |z|^{11.71}$ for all $|z| \geq 3$. Prove that h is a polynomial of degree at most 11.

B. Measure and Integration Theory

1. Let $(X, \mathcal{M})$ be a measurable space and V be a collection of $\mathcal{M}$-measurable functions. Define $g(x) := \sup\{f(x) : f \in V\}$, $x \in X$.

   (a) Prove (by constructing a counterexample) that g is not necessarily $\mathcal{M}$-measurable.

   (b) If $X = \mathbb{R}$ and each function in V is real-valued and continuous on $\mathbb{R}$, show that g is a Borel measurable function.

2. Let ν be a signed measure on a measurable space $(X, \mathcal{M})$.

   (a) State the Jordan decomposition theorem for ν and define the total variation $|\nu|(X)$.

   (b) Prove that

   $$|\nu|(X) = \sup \sum_{k=1}^{n} |\nu(E_k)|,$$

   where the supremum is taken over all finite disjoint collections $\{E_k\}_{k=1}^{n}$ of measurable subsets of X.

3. Suppose f is a nonnegative integrable function on a σ-finite measure space $(X, \mathfrak{B}, \nu)$. Define

   $$\phi(t) := \nu(\{x : f(x) < t\}), \quad t \geq 0.$$

   (a) State Tonelli's theorem.

   (b) Use Tonelli's theorem to prove that

   $$\int_0^\infty \phi(t)\, dt = \int f\, d\nu.$$

C. Functional Analysis

1. (a) Let X, Y be complex Banach spaces and suppose that $T : X \to Y$ is linear. Prove that the following are equivalent: (i) T is continuous, (ii) T is continuous at $0 \in X$ and (iii) T is bounded, *i.e.* the operator norm, $\|T\|$, of T, is bounded.

   (b) Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator on a complex, separable Hilbert space, $\mathcal{H}$. Show that the range of T is closed if and only if T is bounded below on $\operatorname{Ker}(T)^\perp$, *i.e.* there is a $\delta > 0$ so that $\|Tx\| \geq \delta \|x\|$ for all $x \in \operatorname{Ker}(T)^\perp$, where $\operatorname{Ker}(T)^\perp$ denotes the orthogonal complement of the kernel of T.

   (c) Let X be a Banach space and suppose that $\varphi : X \to \mathbb{C}$ is a linear functional. Prove that φ is bounded if and only if $\varphi^{-1}(\{0\})$ is norm-closed in X.

2. Let $AC[0,1]$ denote the absolutely continuous complex-valued functions on $[0,1]$. Define the linear space

$$\operatorname{Dom}(D) := \left\{ h \in AC[0,1] \,\middle|\, h' \in L^2[0,1] \text{ and } h(0) = 0 \right\} \subset L^2[0,1],$$

   and define the linear map $D : \operatorname{Dom}(D) \to L^2[0,1]$ by $Dh = h'$ for $h \in \operatorname{Dom}(D)$.

   (a) Prove that D is a closed operator, *i.e.* prove that its graph, $G(D)$, is a closed subspace of $L^2[0,1] \oplus L^2[0,1]$.

   (b) Does the closed graph theorem imply that D is bounded? Why or why not?

   (c) Prove whether or not D is bounded by considering the sequence

$$g_n(t) := \left(\frac{1}{1 + e^{-nt}} \right)^{\frac{1}{2}}, \quad \text{and} \quad f_n(t) := g_n(t) - g_n(0) e^{-nt}$$

   Show that f_n converges to the constant function 1 in $L^2[0,1]$, and calculate $\lim_n \|Df_n\|^2$.

3. Let X be a complex Banach space and let $\mathscr{L}(X)$ denote the bounded linear operators on X.

   (a) Show that if $K \in \mathscr{L}(X)$ is compact and $\lambda \in \mathbb{C}$, $\lambda \neq 0$, then $\operatorname{Ran}(K - \lambda I)$ is closed in X.

   (b) Show, with an example, that $\operatorname{Ran}(K)$ need not be closed.

   (c) Consider the linear operator $K : X \to X$ defined on the continuous functions, $X := \mathscr{C}[0,1]$ by

$$(Kf)(t) := \int_0^t \sin(\pi s) f(s) ds.$$

   Prove that K is compact.