Analysis Comprehensive Examination

Examining Committee: R. M. Slevinsky (coordinator), R. Clouâtre, C. Cowan

Name: ____________________, Student I.D.#: ____________________

9:00 a.m. – 3:00 p.m., April 29, 2022

The exam consists of three parts worth 50 marks, 40 marks, and 40 marks, respectively. The passing criterion is at least 97.5 marks overall. Students have 6 hours to complete the exam.

Part A: Core Analysis

This unit consists of five questions worth 10 marks each. Answer all questions.

Part B: Differential Equations

This unit consists of five questions worth 10 marks each. Answer four questions. You may attempt as many questions as you like in this unit; however, if you attempt more than four questions, you must clearly indicate which answers you want us to mark. In the absence of any explicit indication, we will mark the first four questions.

Part C: Numerical Analysis

This unit consists of five questions worth 10 marks each. Answer four questions. You may attempt as many questions as you like in this unit; however, if you attempt more than four questions, you must clearly indicate which answers you want us to mark. In the absence of any explicit indication, we will mark the first four questions.

Part A: Core Analysis

This unit consists of five questions worth 10 marks each. Answer all questions.

1. [10 marks] Let X be a compact metric space and let $f : X \to \mathbb{R}$ be a function. Show that f is continuous if and only if f is uniformly continuous.

2. [10 marks] Let $U \subset \mathbb{R}^n$ be an open convex subset. Let $f : U \to \mathbb{R}$ be a differentiable function. Assume that there is $\gamma > 1$ with the property that

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| \leq |\boldsymbol{x} - \boldsymbol{y}|^\gamma, \quad \boldsymbol{x}, \boldsymbol{y} \in U.$$

   Show that f is constant.

3. [10 marks] Let X be a compact metric space. For each positive integer n, let $f_n : X \to \mathbb{R}$ be a continuous function. Assume that the sequence (f_n) converges pointwise on X to some continuous function $f : X \to \mathbb{R}$. Assume also that for every $x \in X$, we have that

$$f_n(x) \leq f_{n+1}(x), \quad n \geq 1.$$

   Show that (f_n) converges *uniformly* to f on X.

4. [10 marks] Let $f : [0, 1] \to \mathbb{R}$ be a non-negative Lebesgue integrable function. For each positive integer n, define $g_n : [0, 1] \to \mathbb{R}$ as

$$g_n(x) = \begin{cases} f(x) & \text{if } f(x) \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

   Show that

$$\lim_{n\to\infty} \int g_n = \int f.$$

5. [10 marks] For each $R \geq 2$, let $\Gamma_R = \{z \in \mathbb{C} : |z| = R\}$. Find the value of

$$\lim_{R\to\infty} \int_{\Gamma_R} \frac{z^{2022} + 3}{z^{2024} + z^2 + 1} dz.$$

Part B: Differential Equations

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PDE/ODE Cheat sheet.

Theorem 1. (Maximum Principle)

- (Elliptic maximum principle) Suppose $u = u(x)$ is a smooth solution of

$$-\Delta u(x) = f(x) \geq 0 \qquad \text{in } \Omega,$$

  where Ω is a bounded domain in $\mathbb{R}^N$. Then $\min_{\overline{\Omega}} u = \min_{\partial\Omega} u$.

- (Parabolic maximum principle). Let Ω be a bounded domain in $\mathbb{R}^N$ and for $0 < T < \infty$ set $\Omega_T := \Omega \times (0, T]$ and

  $$\Gamma_T := (\Omega \times \{t = 0\}) \cup (\partial\Omega \times (0, T)) = \text{“ bottom and sides of cylinder } \Omega_T\text{”}.$$

  Then if $u = u(x, t)$ is a smooth solution of

  $$u_t - \Delta u = f(x, t) \geq 0 \quad \text{in } \Omega_T,$$

  then $\min_{\overline{\Omega_T}} u = \min_{\Gamma_T} u$.

1. [10 marks] In this question let $| \cdot |$ denote the Euclidean norm on $\mathbb{R}^N$.
   Let $f : \mathbb{R}^N \times [0, \infty) \to \mathbb{R}^N$ smooth with the property for all $R > 0$ there is some C_R such that

   $$\sup_{t \geq 0, |x| \leq R} \{|f(x, t)| + \|D_x f(x, t)\|\} \leq C_R, .$$

   where $\|A\| := \max_{|z|=1} |Az|$ for a matrix $A \in \mathbb{R}^{N \times N}$. Show for all $x_0 \in \mathbb{R}^N$ there is a local solution of the following ODE

   $$\begin{cases} x'(t) = f(x(t), t), & t > 0, \\ x(0) = x_0. \end{cases} \tag{1}$$

   Recall to do this you need to show there is some $T > 0$ and $x \in C([0, T]; \mathbb{R}^N)$ which satisfies

   $$x(t) = x_0 + \int_0^t f(x(\tau), \tau) d\tau, \qquad 0 \leq t \leq T. \tag{2}$$

   (You do **not** need to show the equivalence of these two notions of solution).

2. [10 marks] Consider

   $$\begin{cases} x'(t) = f(x(t), t), & t > 0, \\ x(0) = x_0. \end{cases} \tag{3}$$

   where f is given below. In this question we examine when there is a global solution versus when there is no global solution of (3). You will apply the following theorem

   Theorem 2. (Blow up alternative) Suppose $f : \mathbb{R}^N \times [0, \infty) \to \mathbb{R}^N$ is smooth and suppose there is ‘maximal solution’ of (3) on $(0, T_{max})$ and $T_{max} < \infty$ (ie. so there is no global solution). Then

   $$\liminf_{t \nearrow T_{max}} |x(t)| = \infty.$$

   (a) [5 marks] Suppose f is of the form $f(x, t) = -\alpha(t)|x|^2 x$ where $1 \leq \alpha(t) \leq 2$ is smooth on $[0, \infty)$. Show the local solution of (3) with this f can be extended to a global solution, ie. defined on $[0, \infty)$. You do **not** need to prove the extension exists you just need to prove that if there is a smooth solution $x(t)$ on $(0, T)$ with T finite then $\liminf_{t \nearrow T} |x(t)| < \infty$.

   (b) [5 marks] Suppose f is of the form $f(x, t) = \alpha(t)|x|^2 x$ where $1 \leq \alpha(t) \leq 2$ is smooth on $[0, \infty)$. Show for any nonzero x_0 there is no global solution of (1). Here you can assume a smooth solution exists on $[0, T)$ and then try and get bounds on T to show T must be finite.

3. [10 marks] Let B_R denote the open ball in $\mathbb{R}^N$ of radius R centered at the origin and suppose Ω is an open domain with smooth boundary in $\mathbb{R}^N$ with $\Omega \subset B_R$. Suppose $f \geq 0$ is smooth and bounded and u is a nonnegative smooth solution of

   $$\begin{cases} -\Delta u(x) = f(x) \geq 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4}$$

   Find an upper bound on $\sup_\Omega u$; of course the upper bound will involve f and R.

4. [10 marks] Let $u = u(x,t)$ denote a smooth solution of the heat equation

$$\begin{cases} u_t - \Delta u = 0, & \text{in } (x,t) \in (0,\pi) \times (0,\infty), \\ u(x,0) = \phi(x), & \text{in } (0,\pi), \\ u(0,t) = u(\pi,t) = 0 & \text{for } t > 0. \end{cases} \tag{5}$$

The following inequality will be useful for this question:

$$\int_0^\pi (\psi(x))^2 dx \le \int_0^\pi (\psi'(x))^2 dx,$$

for all $\psi \in X := \{\psi \in C^1[0,\pi] : \psi(0) = \psi(\pi) = 0\}$.

(a) [6 marks] Define $E(t) = \frac{1}{2}\int_0^\pi u(x,t)^2 dx$. Show one has $E'(t) \le -2E(t)$ for all $t > 0$. Use this to show that $E(t) \to 0$ as $t \to \infty$.

(b) [4 marks] Take $0 \le \phi \in C_c^\infty(0,\pi)$ (recall this is the set of smooth functions which are compactly supported in $(0,\pi)$). You can accept the fact that the maximum principle shows that $u(x,t) \ge 0$ in $(x,t) \in (0,\pi) \times (0,\infty)$. The point of this question is to show that $\sup_{x\in(0,\pi)} |u(x,t)| \to 0$ as $t \to \infty$. Since u is nonnegative we see its sufficient to show that $\sup_{x\in(0,\pi)} u(x,t) \to 0$.
Now for the question. Find some explicit function $v(x,t) \ge 0$, with $\sup_{x\in(0,\pi)} v(x,t) \to 0$ as $t \to \infty$ and such that you can apply the Maximum Principle (or the Comparison Principle) to show that $0 \le u(x,t) \le v(x,t)$ for $(x,t) \in (0,\pi) \times (0,\infty)$.

5. [10 marks] Suppose $\Omega = \{x \in \mathbb{R}^N : 1 < |x| < \infty\}$ where $N \ge 3$. Consider the equation for $u(x)$ given by

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

(a) [3 marks] Show there is an infinite number of solutions of (6). Recall for a radial function we have $\Delta u(x) = u_{rr}(r) + \frac{N-1}{r} u_r(r)$.

(b) [3 marks] Find a solution of (6) which also satisfies $\lim_{|x|\to\infty} u(x) = 4$.

(c) [4 marks] Show there is at most one solution of (6) which satisfies the extra condition from part (b).

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1. [10 marks] Let (x_k, f_k) for $k = 0, \ldots, n$ be a data set with distinct abscissæ, $x_i \ne x_j \ \forall i \ne j$, and positive ordinates, $f_k > 0$. For each statement below, prove it to be true or false.

(a) There exists a positive degree-$2n$ polynomial that interpolates the data; and,

(b) The positive degree-$2n$ interpolating polynomial is unique.

2. [10 marks] Consider the matrices A and $B \in \mathbb{R}^{n\times n}$, where:

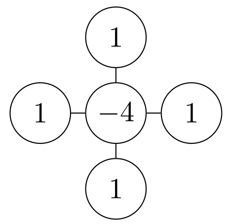
$$A = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}.$$

Use the Cholesky factorization to determine if the matrices are positive-definite.

3. [10 marks] Consider the Poisson equation in the square $\Omega = (0,1)^2$:

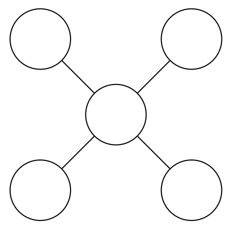
$$u_{xx} + u_{yy} = f, \qquad u(x,y) = 0, \quad \text{for} \quad (x,y) \in \partial\Omega.$$

(a) [4 marks] The classical five-point formula leads to the computational stencil:



$$u_{j,k} = h^2 f_{j,k}.$$

where $h = \Delta x = \Delta y$ and $u_{j,k} = u(jh, kh)$ and $f_{j,k} = f(jh, kh)$ for $j, k = 1, \ldots, n$. Fill in the computational stencil:



$$u_{j,k} = h^2 f_{j,k},$$

to ensure it has a local truncation error of the same order as that of the classical five-point formula.

(b) [3 marks] Is the discretization matrix you created symmetric? negative-definite?

(c) [3 marks] Describe a direct method for the solution of the discretized modified five-point formula. Choose an ordering to improve the complexity and state the improved complexity (in terms of n).

4. [10 marks] Consider the heat equation:

$$\begin{aligned} &u_t = u_{xx} + f(x,t), \quad x \in (0,1), \quad t > 0, \\ &u(x,t) = 0, \quad x = 0, \quad x = 1, \quad t > 0, \\ &u(x,0) = g(x), \quad x \in (0,1). \end{aligned}$$

(a) [3 marks] Write down the Crank–Nicolson scheme;

(b) [7 marks] Analyze the stability using either (i) eigenvalue, or (ii) Fourier (von Neumann) analysis.

5. [10 marks] Consider the conjugate gradient method to solve the linear system $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ is a symmetric positive-definite matrix and $x, b \in \mathbb{R}^n$.

Recall that after initializing $p^{(0)} := r^{(0)} := b - Ax^{(0)}$, one step ($k \to k+1$) in the conjugate gradient method is:

$$\begin{aligned} \alpha_k &= \frac{\langle r^{(k)}, r^{(k)} \rangle}{\langle p^{(k)}, Ap^{(k)} \rangle}, \\ x^{(k+1)} &= x^{(k)} + \alpha_k p^{(k)}, \\ r^{(k+1)} &= r^{(k)} - \alpha_k A p^{(k)}, \\ \beta_k &= -\frac{\langle r^{(k+1)}, Ap^{(k)} \rangle}{\langle p^{(k)}, Ap^{(k)} \rangle}, \\ p^{(k+1)} &= r^{(k+1)} + \beta_k p^{(k)}. \end{aligned}$$

Prove that:

(a) The residuals are orthogonal: $r^{(k)\top} r^{(\ell)} = 0$ for $k \neq \ell$; and,

(b) The search directions are A-conjugate: $p^{(k)\top} A p^{(\ell)} = 0$ for $k \neq \ell$.