# University of Manitoba Department of Mathematics

Graduate Comprehensive Examination in Algebra

10 AM-4 PM 30 September, 2019.

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#### Instructions (Please read carefully):

- You have altogether 6 hours to complete the examination.
- Part A consists of 10 questions worth two marks each. Answer all questions in Part A on the question paper itself. Each of these questions can and should be answered in no more than three sentences.
- You have a choice of questions in each of Parts B and C. The questions in Part B are worth 5 marks each. Answer any 6 questions out of 10 in this part. The questions in Part C are worth 10 marks each. Answer any 4 questions out of 6 in this part.
- You may attempt as many questions as you like in Parts B and C; however, if you attempt more than the required number of questions, you must clearly indicate which answers you want evaluated. In the absence of any explicit indication, the first 6 questions for Part B, and the first 4 questions for Part C (in the order of their appearance in your answer booklets) will be evaluated.
- In order to pass this examination, you must obtain a score of at least 75% in total.

Be sure to keep in mind the following:

- Unless stated otherwise, vector spaces need not be finite dimensional.
- Unless stated otherwise, groups may be finite or infinite, abelian or non-abelian.
- Unless stated otherwise, rings may be commutative or non-commutative.
- Unless stated otherwise, rings R are assumed to have a multiplicative identity  $1 \in R$ .
- Unless stated otherwise, fields may be finite or infinite, of arbitrary characterstic.
- $S_n$  denotes the group of permutations on the set  $\{1, \ldots, n\}$ .

## PART A

Please answer each of the following 10 questions in the space provided. Each correct answer is worth two marks. Each question should be answered briefly; i.e., in no more than three sentences.

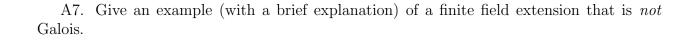
A1. Prove or disprove: For any prime number p, the group  $\mathbb{Z}_p \times \mathbb{Z}_p$  is a cyclic group.

A2. Prove or disprove: If a ring R is a principal ideal domain then the polynomial ring R[x] is also a principal ideal domain.

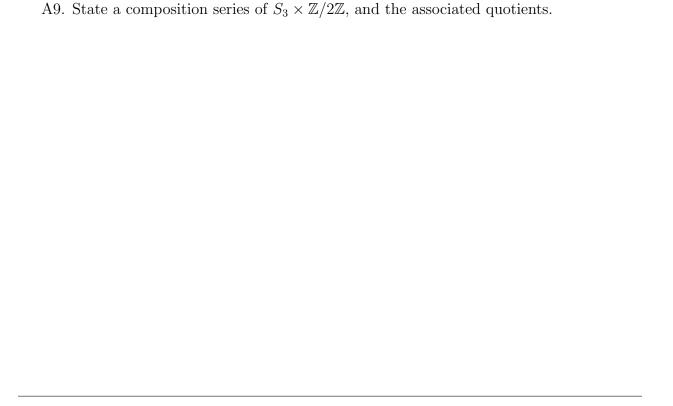
 $closure ext{ of } K.$ 

A5. Show that the only ideals of a field $F$ are $(0)$ and $F$ itself.	

A6. Define what it means for an extension of fields  $F \subset K$  to be normal.



A8. What are the possible Jordan canonical forms for a matrix A with characteristic polynomial  $p_A(t) = (t-3)^3(t+5)$ ?



A10. Suppose that V is a finite dimensional vector space. Write down an isomorphism  $T:V\to V^*$ , where  $V^*$  is the dual of V. Note that you need only describe the map T, you

do not need to prove it is an isomorphism.

### PART B

Please answer any 6 of the following 10 questions in your answer booklet. Each question is worth 5 marks. If you attempt more than 6 questions, then please indicate clearly which ones you want evaluated.

- B1. Let G be a group. Show that if |G| = 616 then G is not simple.
- B2. Let R be a ring such that  $a^2 = -a$  for all  $a \in R$ . Prove that R is abelian.
- B3. For a fixed element a in a group G let  $H_a = \{x \in G \mid xa = ax\}.$
- 1. Prove that  $H_a$  is a subgroup of G.
- 2. If  $G = S_5$  and a = (1 2 3), find two nontrivial elements of  $H_a$  beside a.
- B4. Let D be a Euclidean domain with norm  $\delta$ , and assume that  $\delta(x)\delta(y) = \delta(xy)$  for all  $x, y \in D$ . First show that  $\delta(1) = 1$ , and then prove that if  $\delta(a)$  is a prime number, then a is irreducible in D.
  - B5. Show that  $\mathbb{Q}(\sqrt[7]{10})$  has no proper subfields besides  $\mathbb{Q}$ .
  - B6. Give an example of a ring R and a projective R-module that is not free.
- B7. Suppose R is an integral domain, and M an R-module. A element  $m \in M$  is called torsion if there exists  $r \in R$  such that  $r \cdot m = 0$ . A module is called torsion-free if there are no non-zero torsion elements. Show that the set  $M_{tors}$  of torsion elements is a submodule of M, and that  $M/M_{tors}$  is torsion-free.
  - B8. Consider the symmetric group  $S_3$ .
  - 1. Prove that  $Inn(S_3) \cong S_3$ .
  - 2. Determine  $Aut(S_3)$ .

B9. State the Cayley-Hamilton theorem. Prove that it holds for matrices that are in Jordan canonical form.

B10. Let  $F \subset K$  be an extension of fields such that [K:F]=2 and suppose that  $char(F) \neq 2$ . Show that there exists  $a \in K$  such that K=F(a) and  $a^2 \in F$ .

### PART C

Please answer any 4 of the following 6 questions in your answer booklet. Each question is worth 10 marks. If you attempt more than 4 questions, then please indicate clearly which ones you want evaluated.

C1. Let G be a group.

- 1. Prove that if  $\frac{G}{Z(G)}$  is cyclic then G must be abelian.
- 2. Prove that if G is a non abelian group of order  $p^3$ , where p is a prime number, then |Z(G)| = p.
- C2. First define a principal ideal domain and a unique factorization domain, and then prove that any principal ideal domain is a unique factorization domain (Hint: You may use without proof the fact that principal ideal domains are Noetherian).
  - C3. Prove, using Zorn's lemma, that every vector space has a basis.

C4. Let 
$$f(x) = x^3 - 2 \in \mathbb{Q}[x]$$
.

- 1. Show that f(x) is irreducible.
- 2. Give an explicit presentation (with generators and relations) for the Galois group  $Gal(K/\mathbb{Q})$ , where K is the splitting field of f.
- C5. Suppose R is a commutative ring, and M is an R-module.
- 1. Define what it means for M to be flat.
- 2. Suppose  $f: R \to S$  is a flat morphism of commutative rings (i.e. S is flat as an R-module), and  $I \subset R$  is an ideal. Show that there is an isomorphism  $I \otimes_R S \simeq IS$ .
- 3. Find an example of rings R and S, a morphism  $f: R \to S$ , and an ideal  $I \subset R$  where conclusion of part (2) fails.
- C6. Suppose that R and S are commutative Artinian rings. Show that  $R \times S$  is also an Artinian ring.