Analysis Comprehensive Examination

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April 24, 2023
10 – 16 (CDT)

Instructions:

This examination consists of 3 parts labeled A,B and C.

- Part A covers the core material, it consists of 8 questions worth 5 marks each for a total of 40 marks. All questions must be answered for full marks.
- Part B covers the specialized material on Basic Functional Analysis. This section consists of 3 questions of which you must attempt 2. Each question is worth 15 marks for a total of 30 marks.
- Part C covers the specialized material on Differential Equations. This section also consists of 3 questions of which you must attempt 2. Each question is worth 15 marks for a total of 30 marks.

Please take note of the following:

1. In Parts B and C, if you attempt more than 2 questions, you must clearly indicate which two questions you would like marked. Otherwise only the first 2 answers in the order they appear in your solutions will be graded.
2. The examination is worth a total of 100 marks. A total score of 75%, or 75 marks, is minimally required to pass the exam.
3. The examination length is 6 hours. No texts, reference books, calculators, cell phones, or other aids are permitted during the examination.

A. Core material

1. Let α be an increasing function on $[a, b]$, and let f be a continuous function on $[a, b]$.
   (a) Show that f is Riemann-Stieltjes integrable on $[a, b]$.
   (b) Show that there is $c \in [a, b]$ such that $\int_a^b f d\alpha = f(c)\,(\alpha(b) - \alpha(a))$.

2. Let I and J be two intervals and f a two-variable function defined on the rectangle $I \times J$. Suppose that $f(x, y)$ is continuous in the variable y for each $x \in I$ and, for each $y \in J$, the function f_y defined by $f_y(x) = f(x, y)$ is measurable on I and $|f_y| \leq g$ a.e. on I for all $y \in J$, where g is a Lebesgue integrable function on I. Show that the function $F(y) = \int_I f(x, y)dx$ is continuous on J. *Hint:* Note that F is continuous at y if and only if $\lim_{n\to\infty} F(y_n) = F(y)$ for each sequence (y_n) from the domain such that $y_n \to y$.

3. Let $f(x)$ be a continuous function on the interval $[0, 1]$. Suppose that $\int_0^1 x^n f(x)dx = 0$ for all integers $n \geq 0$. Show that $f(x) = 0$ for all $x \in [0, 1]$. *Hint*: Apply the Weierstrass approximation theorem.

4. Find the Fourier series of the function $f(x) = x$ on the interval $[0, 2\pi]$. At what point in $[0, 2\pi]$ does the Fourier series converge to f? Justify your answer with some appropriate theorem(s) from the textbook.

5. (a) State Cauchy's formula for derivatives.
   (b) Suppose that h is an entire function which obeys $|h(z)| \leq \pi|z|^{17.11}$ for all $|z| \geq 2$. Prove that h is a polynomial of degree at most 17. *Hint:* Write out the Taylor series for h centred at $z_0 = 0$. Apply Cauchy's inequalities for derivatives.

6. How many roots (counting multiplicities) does the polynomial,
$$p(z) := 6z^4 + z^3 - 2z^2 + z - 1,$$
   have in the complex unit disk, $\mathbb{D} := \{z|\ |z| < 1\}$? Justify your answer.

7. Let $\alpha \in \mathbb{R}\backslash\{-1\}$ and consider $F(x, y) = x^2 + xy + y + \alpha \sin y$. Show that there exist a $\delta > 0$ and a function $x \mapsto y(x)$ defined on $|x| < \delta$ such that $y(0) = 0$ and $F(x, y(x)) = 0$ for $|x| < \delta$.

8. In this question $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$.
   (i) For $t \neq 0$ show that $\nabla|x|^t = t|x|^{t-2}x$ for $x \in \mathbb{R}^N\backslash\{0\}$.
   (ii) Take $F(x) = \frac{-x}{4\pi|x|^3}$ and take $0 \in \Omega \subset \mathbb{R}^3$ a smooth bounded open set. Let ϕ be smooth in Ω with $\phi = 0$ on $\partial\Omega$.
   Show
$$\int_\Omega F(x) \cdot \nabla\phi(x)dx = \phi(0),$$
   where you should interpret this integral as
$$\lim_{\varepsilon\searrow 0} \int_{\Omega\backslash\overline{B_\varepsilon}} F(x) \cdot \nabla\phi(x)dx,$$
   where $B_r := \{x \in \mathbb{R}^3 : |x| < r\}$ where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^3$.

B. Functional Analysis

Choose two from the following three questions to answer.

1. Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $(e_n)_{n=1}^{\infty}$, and let $(\lambda_n)_{n=1}^{\infty}$ be a bounded sequence of complex numbers. For each $x \in \mathcal{H}$, define

$$Kx := \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n.$$

Show that K is compact if and only if $\lambda_n \to 0$. *Hints:* Show that K can be approximated in operator–norm by finite–rank operators. Prove that e_n converges weakly to 0.

2. A (separable, complex) Hilbert space, $\mathcal{H}$, of functions on a set, X, is called a *reproducing kernel Hilbert space* (RKHS), if for each $x \in X$, the linear functional on $\mathcal{H}$ defined by $f \mapsto f(x)$ is bounded. The multiplier algebra, $\mathrm{Mult}(\mathcal{H})$, of $\mathcal{H}$ is then the set of all functions $h : X \to \mathbb{C}$ so that for any $f \in \mathcal{H}$, $h \cdot f \in \mathcal{H}$.

   (a) Show that if $h \in \mathrm{Mult}(\mathcal{H})$, then the linear map $M_h : \mathcal{H} \to \mathcal{H}$ defined by $M_h f := h \cdot f$, is bounded.

   (b) Prove that if $h \in \mathrm{Mult}(\mathcal{H})$ and $x \in X$, that $M_h^* - \overline{h(x)} I$ is not invertible and that

$$\|M_h\| \geq \sup_{x \in X} |h(x)|.$$

3. Let X be a complex normed linear space.

   (a) Show that X is a Banach space, *i.e.* complete if and only if given $(x_n)_{n=1}^{\infty} \subseteq X$, the series $\sum_n x_n$ converges to an element of X whenever $\sum_{n=1}^{\infty} \|x_n\|$ converges. *Hint:* To show that this second condition implies completeness, assume $(x_n) \subseteq X$ is Cauchy, and without loss in generality, that $\|x_n - x_{n+1}\| \leq 2^{-n}$. Let $y_n = x_n - x_{n+1}$ and consider $\sum y_n$.

   (b) Use the previous part to show that if X is a Banach space and $S \subseteq X$ is a closed subspace, then the quotient space X/S is complete. *Hint:* Given $(x_n + S) \subseteq X/S$ so that $\sum \|x_n + S\| < +\infty$, argue that you can choose $y_n \in S$ so that $\|x_n + y_n\| \leq \|x_n + S\| + 2^{-n}$ and apply (a).

C. Differential Equations

The following principles are provided to you.

Theorem. *(Maximum Principle)*

- *(Elliptic maximum principle) Suppose $u = u(x)$ is a smooth solution of*

$$-\Delta u(x) = f(x) \geq 0 \qquad \text{in } \Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$. Then $\min_{\overline{\Omega}} u = \min_{\partial\Omega} u$.

- *(Parabolic maximum principle). Let Ω be a bounded domain in $\mathbb{R}^N$ and for $0 < T < \infty$ set $\Omega_T := \Omega \times (0, T]$ and*

$$\Gamma_T := (\Omega \times \{t = 0\}) \cup (\partial\Omega \times (0, T)) = \text{“ bottom and sides of cylinder } \Omega_T\text{”}.$$

Then if $u = u(x, t)$ is a smooth solution of

$$u_t - \Delta u = f(x, t) \geq 0 \quad \text{in } \Omega_T,$$

then $\min_{\overline{\Omega_T}} u = \min_{\Gamma_T} u$.

Choose two from the following three problems to answer.

Problem 1. In this question let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^N$ and let $\|\cdot\|$ denote the usual operator norm induced by $|\cdot|$.
Let $f : \mathbb{R}^N \times [0, \infty) \to \mathbb{R}^N$ smooth with the property for all $R > 0$ there is some C_R such that

$$\sup_{t \geq 0, |x| \leq R} \{|f(x, t)| + \|D_x f(x, t)\|\} \leq C_R.$$

Show for all $x_0 \in \mathbb{R}^N$ there is a local solution of the following ODE

$$\begin{cases} x'(t) = f(x(t), t), & t > 0, \\ x(0) = x_0. \end{cases} \tag{1}$$

Problem 2. Let $\Omega = \{(x, y) : x \in \mathbb{R}, 0 < y < \pi\}$. Show the only smooth u which satisfies $\Delta u = 0$ in Ω with $u = 0$ on $\partial\Omega$ and $|u(x, y)| \leq 1$ is $u = 0$.
Hint. Write $u(x, y)$ as a Fourier series in y and use the boundedness of u. You don't need to justify the use of the Fourier series.

Problem 3. Let D denote a bounded open smooth region in $\mathbb{R}^N$ and set $\Omega = \mathbb{R}^N \backslash \overline{D}$. Suppose u is a smooth function with $\Delta u = 0$ in Ω and $u = 0$ on $\partial\Omega$, and suppose that $\lim_{|x| \to \infty} u(x) = 0$. Show $u = 0$.
Hint. Cut the domain off into a bounded set and then apply a maximum principle.