Analysis Comprehensive Examination

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12h00 – 18h00

Instructions:

This examination consists of 3 parts, A,B and C:

- Part A covers the Core Material, it consists of 8 questions worth 8 marks each for a total of 64 marks. All questions must be answered for full marks.
- Part B covers the Advanced Material on Computational Mathematics. This section consists of 3 questions of which you must attempt 2. Each question is worth 16 marks for a total of 32 marks.
- Part C covers the Advanced Material on Differential Equations. This section consists of 3 questions of which you must attempt 2. Each question is worth 16 marks for a total of 32 marks.

Please take note of the following:

1. In Parts B and C, if you attempt more than 2 questions, you must clearly indicate which two questions you would like marked. Otherwise only the first 2 answers in the order they appear in your solutions will be graded.
2. The examination is worth a total of 128 marks. A total grade of 75% or 96 marks is required to pass the exam.
3. The examination length is 6 hours. No texts, reference books, calculators, cell phones, or other aids are permitted in the examination.

A. Core material

1. (a) Suppose f_n, $n \geq 1$, is a sequence of real-valued functions of bounded variation on $[0,1]$ which converges pointwise to a function f on $[0,1]$. Assume, in addition, that there exists $M > 0$ such that the variation of f_n on $[0,1]$ does not exceed M for any n, with M independent of n. Show that f has bounded variation on $[0,1]$.

   (b) Construct a sequence f_n, $n \geq 1$, of real-valued functions of bounded variation on $[0,1]$ which converges pointwise to a function f on $[0,1]$ such that f does not have bounded variation on $[0,1]$.

   (c) Suppose f_n, $n \geq 1$, is a sequence of real-valued continuous functions of bounded variation on $[0,1]$ which converges uniformly to a function f on $[0,1]$. Does it follow that f has bounded variation on $[0,1]$?

2. Let $\alpha(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$ Suppose f is of bounded variation on $[0,1]$ and $f \in R(\alpha)$ on $[0,1]$. Prove that f is a constant function.

3. Use residues to compute the value:
$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\, dx.$$

4. Prove **Dini's theorem**: If (f_n) is a sequence of real valued continuous functions converging pointwise to a continuous limit function f on a compact set $S \subset \mathbb{R}$, and if $f_n(x) \geq f_{n+1}(x)$, for each $x \in S$ and every $n \in N$, then $f_n \to f$ uniformly on S.

5. Find $f \in L^2([0, 2\pi])$ such that
$$f(x) \sim \sum_{n=1}^{\infty} \frac{\cos nx + \sin nx}{\sqrt{n}},$$
or prove that such function f does not exist.

6. Evaluate the following line integral:

$$\oint_C \big(zy\sin(xy) + (x+y)^2\big)\,dx + \big((x+y)^2 + zx\sin(xy)\big)\,dy + \big(yz^3 - \cos(xy)\big)\,dz,$$

where C is the curve of intersection of surfaces $z = \sqrt{x^2+y^2}$ and $(x-1)^2 + y^2 = 1$, directed counterclockwise when viewed from above.

7. Calculate

$$\sum_{n=0}^{\infty} \int_{-\frac{\pi}{2}}^{\pi} (-1)^n \frac{x^{2n+1}}{(2n)!} dx.$$

Make sure to justify each step.

8. (a) State Cauchy's formula for derivatives.

   (b) Suppose that h is an entire function which obeys $|h(z)| \leq \pi|z|^{11.71}$ for all $|z| \geq 3$. Prove that h is a polynomial of degree at most 11.

B. Computational Mathematics

1. Recall that Chebyshev polynomials form an orthogonal set on $[-1,1]$ with respect to the inner product $(f,g) := \int_{-1}^{1} f(t)g(t)w(x)dt$, where $w(x) = (1-x^2)^{-1/2}$. The first four Chebyshev polynomials (normalized so that their values at 1 are 1) are (you do not have to prove this):

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= 2x^2 - 1 \\ P_3(x) &= 4x^3 - 3x \end{aligned}$$

   Find $a_0, a_1, a_2, a_3 \in \mathbb{R}$ such that the polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ is the polynomial of best approximation to $f(x) = x^4$ from the space of algebraic polynomials of degree ≤ 3 in the $L^2_w[-1,1]$ norm.

   (Hint: you may use without proof that $(1, x^6) = 5\pi/16$.)

2. State Newton's method for the system of three nonlinear equations

$$\begin{aligned} &x_1^3 - 2x_2 - 2 = 0 \\ &x_1^3 - 5x_3^2 + 7 = 0 \\ &x_2 x_3^2 - 1 = 0, \end{aligned}$$

   and use it to calculate $\mathbf{x}^{(1)}$ assuming that $\mathbf{x}^{(0)} = [1,1,1]^T$.

3. (a) Define an 'admissible triangulation' of a polygonal domain $\Omega \subset R^2$.

   (b) Suppose that Ω_i is a convex polygon in R^2 with vertices v^1, v^2, ..., v^k. Define 'barycentric coordinates' $(\lambda_1, \ldots, \lambda_k)$ with respect to Ω. What is the representation in terms of barycentric coordinates of $conv\{v^i, v^j\}$?

   (c) Give a general definition of a finite element in R^n.

   (d) Suppose that K is a non-degenerate triangle in R^2 with vertices v^1, v^2 and v^3 (and midpoints of its edges denoted by m^1, m^2 and m^3), P is the space of algebraic quadratic polynomials in two variables, and $\Sigma_K = \{\sigma_1, \ldots, \sigma_6\}$ is the set of the following linear functionals: $\sigma_i(p) = p(v^i)$, $i = 1,2,3$, and $\sigma_i(p) = p(m^{i-3})$, $i = 4,5,6$. Prove that the set Σ_K is unisolvent.

C. Differential Equations

1. Let $f : I \to \mathbb{R}$ be a real-valued function defined on the open interval I. Consider the initial-value problem
$$\frac{dx}{dt} = f(x), \qquad x(0) = x_0 \in I. \tag{IVP}$$
A *solution* to (IVP) is a differentiable function $x = x(t)$ defined on a neighbourhood of $t = 0$.

   (a) State the existence and uniqueness theorem for (IVP).

   (b) Give an example of an IVP for which there are multiple distinct solutions.

   (c) Define the Picard iteration to approximate the solution to (IVP).

   (d) Consider
$$\frac{dx}{dt} = x, \qquad x(0) = 1.$$
Show that the n-th Picard iterate, $x_n(t)$, is the n-th degree Maclaurin polynomial of e^t when $x_0(t) \equiv 1$.

2. Consider the system of differential equations
$$\begin{aligned} \dot{x} &= y + a(x^2 + y^2 - 1)^2 x^3, \\ \dot{y} &= -x + a(x^2 + y^2 - 1)^2 y^3. \end{aligned}$$

   (a) Find all equilibria of the system.

   (b) Identify the linear stability of each equilibrium solution.

   (c) Prove that $(x, y) = (\cos(t), -\sin(t))$ is a periodic solution of the system.

   (d) Let $a > 0$. Use the Lyapunov function
$$V = \frac{1}{4}\left(x^2 + y^2 - 1\right)^2$$
to prove that:

      i. the equilibrium at the origin is repelling;

      ii. the periodic orbit in part (c) is attracting on one side and repelling on the other.

   (e) Sketch the phase portrait of the system.

3. Solve the one-dimensional heat equation for $u = u(t, x)$:

$$u_t = u_{xx},$$

with boundary conditions:

$$\begin{array}{rcr} u(t,0) = 1 & \text{and } u(t,1) = 0, & \text{for all } t > 0, \\ u(0,x) = 1 & & \text{for all } 0 \leq x \leq 1. \end{array}$$

Prove that the solution is unique.