Analysis comprehensive examination
Department of Mathematics

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This examination consists of three parts.

- Part A covers the core material. It has 8 questions worth a total possible score of 70 points.
- Part B covers the specialized material on abstract measure and integration. It has 3 questions worth 15 points each, of which you must attempt 2, for a total possible score of 30 points.
- Part C covers the specialized material on basic functional analysis. It has 3 questions worth 15 points each, of which you must attempt 2, for a total possible score of 30 points.

If you attempt more than the required number of questions in Part B or C, you must clearly indicate which questions are to be graded. If it is not clearly indicated, solutions to those appearing first in the booklet will be graded.

You need to achieve at least 97.5 points, which is 75% of the total possible 130 points, in order to pass the examination.

The total time of the examination is six hours. No books, notes, calculators or aids are allowed during the exam.

Part A: Core Material

Solve *all* of the following problems.

1. Let X be a set and let $C(X)$ denote the space of real-valued continuous functions on X, endowed with the norm
$$\|f\| = \sup_{x\in X} |f(x)|, \quad f \in C(X).$$
   (a) Show that $C(X)$ is complete. (5 pts)
   (b) Prove or disprove: any bounded sequence in $C(X)$ has a convergent subsequence. (5 pts)

2. For each integer $n \geq 1$, let $f_n : (0,1) \to \mathbb{R}$ be a differentiable function with the property that $f_n(1/2) = 0$ and
$$|f_n(x) - f_n(y)| \leq \frac{1}{n}|x-y|, \quad x, y \in (0,1).$$
Calculate
$$\lim_{y\to x} \lim_{n\to\infty} \frac{f_n(x) - f_n(y)}{x-y}$$
for each $x \in (0,1)$. (5 pts)

3. Let $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ be a function with continuous partial derivatives. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be distinct vectors in $\mathbb{R}^n$. Assume that there is a sequence $(\boldsymbol{x}_k)$ converging to $\mathbf{0} \in \mathbb{R}^n$ with the property that $\boldsymbol{f}(\boldsymbol{x}_k) = \boldsymbol{a}$ if k is odd, while $\boldsymbol{f}(\boldsymbol{x}_k) = \boldsymbol{b}$ if k is even. Find $\det \boldsymbol{f}'(\mathbf{0})$. [*Hint*: Can $\boldsymbol{f}$ have an inverse near the origin?] (5 pts)

4. Let $T \subset \mathbb{R}^3$ be the surface of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 4 - y$. Evaluate the surface integral
$$\iint_T \boldsymbol{r} \cdot \boldsymbol{\nu} \, dS,$$
where $\boldsymbol{r} = (x, y, z)$ and $\boldsymbol{\nu}$ is the unit outward normal vector to T. (10 pts)

5. Let $\mathcal{H}$ be a (real) Hilbert space of functions $f : \mathbb{R} \to \mathbb{R}$ with inner product defined as
$$\langle f, g \rangle = \int_{\mathbb{R}} f(x) g(x) e^{-x^2} \, dx, \qquad f, g \in \mathcal{H}.$$
Suppose $\{p_n, \ n \geq 0\}$ is an orthonormal basis for $\mathcal{H}$, where p_n is a polynomial of degree n.
   (a) Determine a choice of p_0, p_1 and p_2. (Simplify your expressions as much as you can, but you do not need to evaluate all integrals. A useful fact is $\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$.) (5 pts)
   (b) Let $(a_n)_{n\geq 0}$ be a sequence of real numbers. Define
$$f_n = \sum_{i=0}^{n} i a_i p_i, \qquad n \geq 0.$$
   Find necessary and sufficient conditions on the sequence (a_n) so that the sequence (f_n) converges in $\mathcal{H}$. (5 pts)

6. Let $a < b$ be real numbers and suppose that $f : [a, b] \to \mathbb{R}$ is a function with continuous derivative. Let $V(f)$ denote the total variation of f over $[a, b]$. Show that

$$V(f) = \int_a^b |f'(x)| \, dx.$$

(10 pts)

7. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc and let $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ denote the open unit ball. Let $f : \mathbb{D} \to \mathbb{C}$ be a function and define $F : B \to \mathbb{C}$ as

$$F(x, y) = f(x + iy), \quad (x, y) \in B.$$

Either prove or disprove the following statements.

(a) If f is holomorphic, then F is harmonic. (5 pts)

(b) If F is harmonic, then f is holomorphic. (5 pts)

8. Evaluate the following integral using residue theory:

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 9)^2} dx.$$

(10 pts)

Part B: Measure Theory

Solve 2 out of the following 3 problems. Throughout this section, a *measure space* is a triple $(X, \mathcal{A}, \mu)$ consisting of a set X, a σ-algebra $\mathcal{A}$ on X and a positive measure $\mu : \mathcal{A} \to [0, \infty]$.

1. Let $(X, \mathcal{A}, \mu)$ be any measure space and let $(\mathbb{R}, \mathcal{M}, \lambda)$ be the Lebesgue measure space on $\mathbb{R}$. For any $\mathcal{A}$-measurable function $f : X \to [0, \infty]$, define the *shadow of* f by

$$S(f) = \{(x, y) \in X \times \mathbb{R} : 0 \le y < f(x)\}.$$

   Let $(X \times \mathbb{R}, \mathcal{P}, \mu \times \lambda)$ denote the product measure space. Show that $S(f)$ is $\mathcal{P}$-measurable and that $(\mu \times \lambda)(S(f)) = \int_X f \, d\mu$. [*Hint*: Approximate by simple functions.]

2. Let $g : [0, 1] \to \mathbb{R}$ be an absolutely continuous and monotone function and let $E \subseteq [0, 1]$ be a set of Lebesgue measure 0. Show that the set $g(E) = \{g(x) \mid x \in E\}$ has Lebesgue measure 0.

3. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\{E_n\}_{n=1}^{\infty}$ be a countable collection of $\mathcal{A}$-measurable subsets. Using *only* the defining properties of a measure, show that

$$\mu\left(\cup_{n=1}^{\infty} E_n\right) = \lim_{N \to \infty} \mu\left(\cup_{n=1}^{N} E_n\right)$$

Part C: Functional Analysis

Solve 2 out of the following 3 problems. Throughout this section, we use the following notation: if X is a normed space, then X^* denotes the space of continuous linear functionals on X.

1. Let X be a normed space and let (x_n) be a sequence in X. Assume that there is $x \in X$ such that
$$\lim_{n\to\infty} \varphi(x_n) = \varphi(x)$$
for every $\varphi \in X^*$. Show that the sequence (x_n) is bounded in norm. [*Hint*: Use the Uniform Boundedness Principle.]

2. Let X be a normed space and let $M \subset X$ be a closed subspace. Consider the restriction map $\rho : X^* \to M^*$ defined as
$$\rho(\varphi) = \varphi|_M, \quad \varphi \in X^*.$$
Show that
$$\|\varphi + \ker\rho\|_{X^*/\ker\rho} = \|\varphi|_M\|_{M^*}$$
for every $\varphi \in X^*$. [*Hint*: Use the Hahn–Banach Theorem.]

3. Let X be a Banach space and let $T : X \to X$ be a linear operator. Assume that there is linear operator $S : X^* \to X^*$ with the property that
$$\varphi(Tx) = (S\varphi)(x)$$
for every $x \in X$ and every $\varphi \in X^*$. Show that T is bounded. [*Hint*: Use the Closed Graph Theorem.]