Analysis Comprehensive Examination

Department of Mathematics, University of Manitoba

Examiners: Craig Cowan, Kang-Ling Liao, Andriy Prymak, Richard M. Slevinsky

September 25, 2023
9:00 AM – 3:00 PM

Instructions

This examination consists of three parts labeled A, B and C.

- Part A covers the core real and complex analysis material. It consists of one question worth three marks, two questions worth five marks each, one question worth seven marks, one question worth 8 marks, and one question worth 12 marks for a grand total of 40 marks. All questions are marked.
- Part B covers the specialized material on differential equations. This section consists of four questions of which you must attempt three. Each question is worth 10 marks for a total of 30 marks.
- Part C covers the specialized material on computational mathematics. This section consists of four questions of which you must attempt three. Each question is worth 10 marks for a total of 30 marks.

Please take note of the following:

1. In Parts B and C, if you attempt more than three questions, you must clearly indicate which three questions you would like marked. Otherwise only the first three answers in the order they appear in your solutions will be graded.
2. The examination is worth a total of 100 marks. At least 75 marks, or 75%, are required to pass the exam.
3. The examination length is six hours. No texts, reference books, calculators, cell phones, or other aides are permitted during the examination.

A Core material

1. (5 marks) Give an example (with justification) of a function $f : [0,1] \to [0,1]$ such that the Riemann-Stieltjes integral $\int_0^1 f(x)d(x^2)$ does not exist.

2. (8 marks) Use the Lebesgue dominated convergence theorem to show that the function $F : [0,\infty) \to \mathbb{R}$ defined by
$$F(t) = \int_0^\infty e^{-x}\sin(tx)\,dx$$
is continuous. (The integral is understood as a Lebesgue integral. There is no need to justify that F is well-defined.)

3. (5 marks) Give an example (with justification) of a function $f : [0,1] \to \mathbb{R}$ which is continuous on $[0,1]$ but is not of bounded variation on $[0,1]$.

4. (12 marks: 4 each part) (a) Find the Fourier series (with respect to the standard trigonometric system) of the 2π-periodic function given by
$$f(x) = \begin{cases} -1, & x \in [-\pi, 0), \\ 0, & x = 0, \\ 1, & x \in (0, \pi). \end{cases}$$
(b) Find the sum of this Fourier series, where exists.
(c) Do the partial sums of this Fourier series converge to their limit uniformly on $[-1,1]$?

5. (7 marks) Using complex analysis evaluate the integral
$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}dx.$$

6. (3 marks) Let $n \geq 2$ denote an integer and consider $f(z) = z^n + \frac{1}{2} + \frac{e^z}{100}$. How many zeros does f have in $D = \{z \in \mathbb{C} : |z| < 1\}$?

B Differential Equations

Answer three of the following four questions.

1. (10 marks) [LaSalle's invariance principle] For the system

$$\begin{cases} x' = f(x), \ f : G \subseteq \mathbb{R}^n \to \mathbb{R}^n, \ G: \text{ open set} \\ x(0) = x_0 \end{cases}$$

If $V(x)$ is a Lyapunov function on $G \subseteq \mathbb{R}^n$ and $\gamma^+(x_0)$ is a bounded orbit of $x' = f(x)$ such that $\gamma^+(x_0) \subseteq G$. Let

$$\begin{aligned} S &:= \{x \in \overline{G} : \ \dot{V}(x) = 0\} \\ M &:= \text{ the largest invariant set in } S \text{ with respect to the flow } x' = f(x) \end{aligned}$$

Then the $\omega-$limit set $\omega(x_0) \subseteq M$, i.e., $\lim_{t\to\infty} dist(\varphi(t, x_0), M) = 0$.

Consider the following predator-prey system

$$\begin{cases} x' &= rx(1 - \dfrac{x}{k}) - \dfrac{mx}{a+x}y \\ y' &= (\dfrac{mx}{a+x} - d)y \end{cases} \tag{1}$$

If there exist a unique positive equilibrium (x^*, y^*) with $y^* < ra/m$ of the system (1), then use the LaSalle's invariance principle to prove

$$\lim_{t\to\infty}(x(t), y(t)) = (x^*, y^*),$$

for any $(x(0), y(0))$ with $x(0) > 0$ and $y(0) > 0$.

2. (10 marks) Let $f : R \to \ R$ be a continuous function and $g : R \to \ R$ be a Lipschitz function. Prove that the system

$$\begin{aligned} x'(t) &= g(x(t)), \ x(t_0) = x_0 && (2) \\ y'(t) &= f(x(t))y(t), \ y(t_0) = y_0 && (3) \end{aligned}$$

has at most one solution on any closed interval.

3. (10 marks) Let $\Omega := \{x \in \mathbb{R}^N : |x| > 1\}$ where $N \geq 3$. Consider the following equation

$$\Delta u(x) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{4}$$

It is helpful to remember that for a function $u(x)$ if the function is radial one can write $\Delta u(x) = u''(r) + \frac{(N-1)u'(r)}{r}$ where N the dimension and $r = |x|$.

(i) (3 pts) Find a nonzero solution of (4).

(ii) (3 pts) Find a solution of (4) with the further requirement that

$$\lim_{|x| \to \infty} u(x) = 1.$$

(iii) (4 pts) Show there is at most one solution of (4) that satisfies this additional assumption of $\lim_{|x|\to\infty} u(x) = 1$.

4. (10 marks) Let B_R denote the open ball of radius R in $\mathbb{R}^N$ centered at the origin. Suppose $0 < R_1 < R_2 < \infty$ and that Ω is a open set in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Suppose $f(x) > 0$ is smooth and bounded in Ω with $B_{R_1} \subset \Omega \subset B_{R_2}$ and suppose u is a solution of

$$\begin{cases} -\Delta u &= f & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

(i) (5 pts) Show that

$$u(x) \leq \left\{ \sup_{\Omega} f(x) \right\} \frac{(R_2^2 - |x|^2)}{2N} \qquad \forall x \in \Omega.$$

(ii) (5 pts) Show that

$$u(x) \geq \left\{ \inf_{B_{R_1}} f(x) \right\} \frac{(R_1^2 - |x|^2)}{2N}, \quad \forall x \in B_{R_1}.$$

C Computational Mathematics

Answer three of the following four questions.

1. (10 marks) If we use the following five functions

$$g_1(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{2},\ g_2(x) = \sqrt{\frac{2x^3 - 2x + 3}{3}},\ g_3(x) = \frac{1}{2}\,[\frac{2x-3}{x^2} + 3],$$
$$g_4(x) = x - \frac{2x^3 - 3x^2 - 2x + 3}{6x^2 - 6x - 2},\ g_5(x, y) = x - \frac{(2x^3 - 3x^2 - 2x + 3)(x - y)}{(2x^3 - 3x^2 - 2x + 3) - (2y^3 - 3y^2 - 2y + 3)}$$

   to generate iteration sequences $\{x_i\}_{i=1}^{\infty}$ to converge to the root of

$$f(x) = 2x^3 - 3x^2 - 2x + 3 = 0$$

   then rank these five functions in order, based on their speed of convergence, with initial iteration at $x_0 = 1.0001$ for $g_1(x),\ g_2(x),\ g_3(x)$, and $g_4(x)$ and $x_0 = 1.0001$ and $x_1 = 1.00011$ for $g_5(x, y)$. You need to justify the rank by appropriate theorems or explanation.

2. (10 marks) Let $a \in \mathbb{R}^{n-1}$, $b \in \mathbb{R}^{n-1}$ and $c \in \mathbb{R}$. For a symmetric arrowhead matrix $A \in \mathbb{R}^{n \times n}$, defined by

$$A = \begin{bmatrix} a_1 & & & b_1 \\ & \ddots & & \vdots \\ & & a_{n-1} & b_{n-1} \\ b_1 & \cdots & b_{n-1} & c \end{bmatrix},$$

   use the Cholesky factorization to determine conditions for which the matrix A is positive-definite.

3. (10 marks) Consider the linear system

$$\begin{bmatrix} 1 & & 1 \\ & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

   (a) (1 mark) Explain why the method of conjugate gradients will converge to the solution of this linear system.

   (b) (7 marks) Using the initial guess $x^{(0)} = [0, 1, 0]^\top$, solve the linear system by the method of conjugate gradients. *Recall: after initializing* $p^{(0)} := r^{(0)} := b - Ax^{(0)}$, *one step* $(k \to k+1)$ *in the conjugate gradient method is*

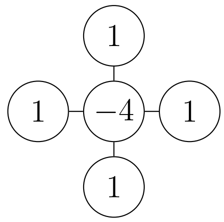
$$\begin{aligned} \alpha_k &= \frac{\langle r^{(k)}, r^{(k)} \rangle}{\langle p^{(k)}, Ap^{(k)} \rangle}, \\ x^{(k+1)} &= x^{(k)} + \alpha_k p^{(k)}, \\ r^{(k+1)} &= r^{(k)} - \alpha_k Ap^{(k)}, \\ \beta_k &= \frac{\langle r^{(k+1)}, r^{(k+1)} \rangle}{\langle r^{(k)}, r^{(k)} \rangle}, \\ p^{(k+1)} &= r^{(k+1)} + \beta_k p^{(k)}. \end{aligned}$$

(c) (2 marks) State a property of the method that proves that you have attained the solution.

4. (10 marks) Consider the Poisson equation in the square $\Omega = (0,1)^2$:

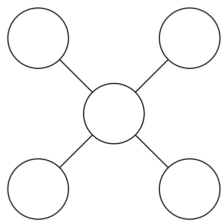
$$u_{xx} + u_{yy} = f, \qquad u(x,y) = 0, \quad \text{for} \quad (x,y) \in \partial\Omega.$$

(a) (4 marks) The classical five-point formula leads to the computational stencil



$u_{j,k} = h^2 f_{j,k}.$

where $h = \Delta x = \Delta y$ and $u_{j,k} = u(jh, kh)$ and $f_{j,k} = f(jh, kh)$ for $j, k = 1, \dots, n$. Fill in the computational stencil



$u_{j,k} = h^2 f_{j,k},$

to ensure it has a local truncation error of the same order as that of the classical five-point formula.

(b) (3 marks) Is the discretization matrix you created symmetric? negative-definite?

(c) (3 marks) Describe a direct method for the solution of the discretized modified five-point formula. Choose an ordering to improve the complexity and state the improved complexity (in terms of n).