Analysis comprehensive examination
Department of Mathematics
University of Manitoba

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9AM–3PM

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This examination consists of three parts.

- Part A covers the core material. It has 8 questions worth 10 points each, and you must attempt all questions in this part for a total possible score of 80 points.
- Part B covers the specialized material on abstract measure and integration. It has 3 questions worth 15 points each, of which you must attempt 2, for a total possible score of 30 points.
- Part C covers the specialized material on basic functional analysis. It has 3 questions worth 15 points each, of which you must attempt 2, for a total possible score of 30 points.

If you attempt more than the required number of questions in Part B or C, you must clearly indicate which questions are to be graded. If it is not clearly indicated, solutions to those appearing first in the booklet will be graded.

You need to achieve at least 105 points, which is 75% of the total possible 140 points, in order to pass the examination.

The total time of the examination is six hours. No books, notes, calculators or aids are allowed during the exam.

Part A

Solve *all* of the following problems.

Problem 1. (10 marks) Let

$$g_n(x) = \frac{e^{-n^4x^4}}{n} \qquad \text{for } x \in \mathbb{R}, \quad n = 1, 2, \dots,$$

and let $g(x) \equiv 0$ for all $x \in \mathbb{R}$.

(a) Show that the sequence (g_n) converges uniformly on $\mathbb{R}$ to the constant function g.

(b) Show that the sequence (g_n') converges pointwise on $\mathbb{R}$ to g.

(c) Show that the sequence (g_n') does not converge uniformly on any neighbourhood of 0 to g.

Problem 2. (10 marks) In this problem, a *simple closed contour* is a piece-wise continuously differentiable closed curve in the plane without self-intersection.

(a) Let $D \subset \mathbb{C}$ be an open connected subset, $\zeta \in D$ and $g : D \to \mathbb{C}$ a function that is holomorphic on $D \setminus \{\zeta\}$ and continuous on D. Prove that for any simple closed contour $\gamma \subset D \setminus \{\zeta\}$, we have

$$\oint_\gamma g(z)\, dz = 0.$$

(b) Let $U \subset \mathbb{C}$ be an open disk, $\zeta \in U$ and $f : U \to \mathbb{C}$ be holomorphic. Show that

$$f(\zeta) = \frac{1}{2\pi i} \oint_{\gamma_\zeta} \frac{f(z)}{z - \zeta}\, dz,$$

where $\gamma_\zeta \subset U \setminus \{\zeta\}$ is a simple closed contour that encircles ζ once. [*Hint*: Apply part (a) to an appropriately chosen function.]

Problem 3. (10 marks) Let α be a non-zero complex number and n a positive integer. Compute the value of

$$\oint_\gamma \frac{\bar{z}\exp(\alpha z)}{z^n}\, dz,$$

where $\gamma = \{z \in \mathbb{C} \mid |z| = 1\}$ is positively oriented.

Problem 4. (10 marks) In each part, either give an example (and prove it is an example) or prove that no such example exists.

(a) A sequence (x_n) of real numbers such that the series $\sum_n x_n$ converges but is not absolutely convergent.

(b) A real-valued function f on a metric space X such that f is continuous but not uniformly continuous.

(c) A metric space X and a Cauchy sequence (x_n) in X that has no limit in X.

Problem 5. (10 marks) Let f, α and α_n $(n = 1, 2, \dots)$ be real-valued functions of bounded variation on $[0, 1]$. Assume that f is continuous on $[0, 1]$. For each $x \in [0, 1]$, define

$$g_n(x) = \int_0^x f(t)\, d\alpha_n(t) \quad \text{and} \quad g(x) = \int_0^x f(t)\, d\alpha(t).$$

Assuming that the sequence (α_n) converges to α uniformly on $[0,1]$, show that (g_n) converges to g uniformly on $[0,1]$.

Problem 6. (10 marks) Prove that there exist a neighborhood D of the point $(0,0) \in \mathbb{R}^2$ and functions $f, g : D \to \mathbb{R}$ such that $f(0,0) = 1$, $g(0,0) = -1$,

$$(f(x,y))^{2021} + xg(x,y) + (g(x,y))^{2022} = 2 \quad \text{and} \quad \sin(f(x,y) + g(x,y)) = 0$$

for any $(x,y) \in D$. Moreover, find the value of $\frac{\partial f}{\partial x}$ at $(x,y) = (0,0)$.

Problem 7. (10 marks) For each $n = 0, 1, 2, \ldots$, the Legendre polynomial P_n is defined by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

Show that the set $\{P_n\}$ of Legendre polynomials forms an orthogonal system of functions in $L^2([-1,1])$. [*Hint*: Use (without proof) that $\frac{d^k}{dx^k}(x^2-1)^n$ is zero at $x = \pm 1$ for $0 \leq k \leq n-1$.]

Problem 8. (10 marks) Suppose $f : [0,\infty) \to \mathbb{R}$ is a real-valued function that is Lebesgue integrable on $[0,n]$ for every positive integer n.

(a) If f is Lebesgue integrable on $[0,\infty)$, show that

$$\lim_{n\to\infty} \frac{1}{n} \int_{[0,n]} f = 0.$$

(b) Assume that

$$\lim_{n\to\infty} \frac{1}{n} \int_{[0,n]} f = 0$$

and that $\lim_{x\to\infty} f(x) = 0$. Does it follow that f is Lebesgue integrable on $[0,\infty)$? Fully justify your answer.

Part B

Solve 2 out of the following 3 problems.

Problem 1. (15 marks) Let ω be an outer measure on some set X. Let E and F be subsets of X that are measurable with respect to ω. Show that $E \cup F$ is also measurable with respect to ω.

[*Hint*: For any subset $A \subset X$, we have $A \cap (E \cup F) = (A \cap E) \cup ((A \setminus E) \cap F)$.]

Problem 2. (15 marks) Let $(X, \mathfrak{T}, \mu)$ be a finite measure space and let $f : X \to [0, \infty)$ be a bounded measurable function. Let $R = \sup_{x \in X} f(x)$ and assume that

$$\lim_{n\to\infty} \frac{1}{R^n} \int_X f^n d\mu > 0.$$

Show that f attains its maximum on X.

Problem 3. (15 marks) Let $(X, \mathfrak{T}, \lambda)$ be a finite measure space and let μ be another finite positive measure on $\mathfrak{T}$. Show that the following statements are equivalent.

(i) The measure μ is absolutely continuous with respect to λ.

(ii) For every bounded sequence $f_n : X \to \mathbb{R}$ of measurable functions converging [λ]-almost everywhere on X to 0, we have $\lim_{n\to\infty} \int_X f_n d\mu = 0$.

Part C

Solve 2 out of the following 3 problems.

Problem 1. (15 marks) Let M be a closed subspace of a normed vector space X. Let

$$M^\perp = \{\varphi \in X^* : M \subset \ker \varphi\}.$$

Show that there is an isometric isomorphism between M^* and $X^*/M^\perp$.

Problem 2. (15 marks) Let X and Y be Banach spaces. Let $T : X \to Y$ be a bounded linear operator with closed range. Let (x_n) be a sequence in X such that (Tx_n) converges to 0 in norm. Show that there is a sequence (ξ_n) in X converging to 0 in norm such that $\xi_n - x_n \in \ker T$.

Problem 3. (15 marks) Let λ denote the Lebesgue measure on $[0,1]$. Let $1 < p, q < \infty$ be numbers chosen such that $1/p+1/q = 1$. Let (f_n) be a sequence in $L^p([0,1], \lambda)$. Show that the following statements are equivalent.

(i) For each function $g \in L^q([0,1], \lambda)$, the sequence $\left(\int_{[0,1]} f_n g d\lambda\right)$ is convergent.

(ii) There is a function $f \in L^p([0,1], \lambda)$ such that

$$\lim_{n\to\infty} \int_{[0,1]} f_n g d\lambda = \int_{[0,1]} f g d\lambda$$

for every function $g \in L^q([0,1], \lambda)$.