University of Manitoba
Department of Mathematics

Graduate Comprehensive Exam in Algebra

Specialized topics: Combinatorics and Linear Algebra

29 April 2022 10:00AM – 4:00PM CDT

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Instructions

- You have 6 hours to write the exam. At the conclusion of the exam, you will be given an additional 15 minutes in order to scan your exam solutions; no writing of solutions is allowed in this time.
- This is a *closed book* exam. No notes, textbooks, or any other resources are permitted.
- This exam has 3 parts:
  - **Part A**: Solve all 5 questions. This part will consist of 1 short answer question worth 24 marks and 4 long answer questions worth 14 marks each. There are 80 marks available in this part.
  - **Part B**: Solve 5 of the 7 available questions. Each question is worth 12 marks. There are 60 marks available in this part.
  - **Part C**: Solve 5 of the 7 available questions. Each question is worth 12 marks. There are 60 marks available in this part.

  The total number of marks available is 200; to pass the exam, a minimum score of 75% (= 150/200) is required.
- Be sure to provide adequate justification for your answers.
- Write your solutions on clean letter paper. If your solution to a problem takes more than one page, please clearly indicate the problem, page number, and total number of pages, on each page of the solution.
- At the conclusion of the exam, submit your solutions via:
  `https://forms.office.com/r/1FWTAm5tN4`

  In the submission form, indicate precisely which problems in Parts B and C are being submitted for grading. If this indication is absent for either part, then the first 5 of Part B and/or the first 5 of Part C submitted will be graded.

Notation and conventions:

- Unless otherwise indicated, vector spaces may be finite or infinite dimensional.
- Unless otherwise stated, groups may be finite or infinite, and are not assumed to be abelian.
- Unless otherwise stated, "ring" is assumed to mean a ring with unit.
- Unless otherwise stated, "ring" includes both commutative and noncommutative rings.
- Fields may be finite or infinite, of any characteristic.
- Permutations are assumed to act on their arguments from the left, as in usual functional notation, and cycles are displayed in left-to-right order, so if $\sigma = (1, 2, 3)$ then we write $\sigma(1) = 2$, $\sigma(2) = 3$, and $\sigma(3) = 1$, and $\sigma\tau$ is defined by $(\sigma\tau)(x) = \sigma(\tau(x))$.
- $\mathcal{S}_n$ denotes the group of permutations of the set $\{ 1, \ldots, n \}$.
- Unless otherwise noted, all graphs are simple.
- In Part C, results of one question (even those not attempted) can be used in the solution of another question.

Part A

This section is worth 40% of the total. Solve all 5 questions. This part will consist of 1 short answer question (with multiple sub-parts) worth 24 marks and 4 long answer questions worth 14 marks each. There are 80 marks available in this part.

[24] A1. State definitions, theorems, or an example. No explanations are required but answers must be stated with enough detail to make it clear that you understand the content.

(a) List all abelian groups of order 72, up to isomorphism.
(You may use any form of describing these groups that you choose).

(b) Define *normal subgroup* of a group, and state an example of a subgroup which is *not* normal.

(c) One of the classes "Principal Ideal Domains", "Unique Factorization Domains" properly contains the other. State a simple example that separates them.

(d) Define "prime ideal" and "maximal ideal" in a commutative ring, and state an example showing that the two concepts are different.

(e) State any example of an algebraic field extension $k \supseteq \mathbb{Q}$ of degree 2; and any example of a transcendental field extension $K \subseteq \mathbb{Q}$ of transcendence degree 2.

(f) What are the possible cardinalities (sizes) of finite fields? Up to isomorphism, how many are there of each finite size?

(g) Let $T : V \to W$ be linear transformation of vector spaces over a field k, with V of finite dimension n. Define the rank and nullity of T and state the Dimension Theorem.

(h) Define "inner product" on a complex vector space V.

[14] A2. Let G be a group and $H \leq G$ be a subgroup. Define 'left coset of H in G'.

Prove that the cosets of H in G partition G.

Show that if xH and yH are left cosets of H, then the map $\varphi : xh \mapsto yh$ is well-defined and a bijection $xH \to yH$.

State and prove *Lagrange's Theorem.*

State an example of $H < G$ and a left coset which is not a right coset.

But nonetheless show that "$\psi : aH \mapsto Ha^{-1}$" is a well-defined bijection between the set of left cosets of H and the set of right cosets of H.

[14] A3. Let $q(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with integer coefficients.

(a) State *Eisentein's Irreducibility Criterion* for $q(x)$.

(b) Prove that $q(x)$ is irreducible over the rationals iff $\bar{q}(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$ is irreducible over the rationals.

(c) Prove that if $0 \neq a \in \mathbb{Q}$, then for any $b \in \mathbb{Q}$, $q(x)$ is irreducible over the rationals iff $q(ax + b)$ is irreducible over the rationals.

[14] A4. Show first that if M is a maximal ideal in a commutative integral domain D, then D/M is a field.

Let $\mathbb{F}$ be a field, $p(x)$ a non-constant irreducible polynomial over $\mathbb{F}$. Explain how to construct an extension field $\mathbb{F}' > \mathbb{F}$ (with proofs) in which $p(x)$ has a root.

[14] A5. Describe the *Jordan normal form* (also called *Jordan canonical form*) of a square complex matrix M. To what extent is this unique? Under what circumstances does a real square matrix have a real Jordan normal form?

Part B

This section is worth 30% of the total. Answer any 5 of the 7 questions available. Each question is worth 12 marks. Clearly indicate which problems are to be graded. There are 60 marks available in this part.

[12] B1. Recall that for positive integers n, k, the *Stirling number of the second kind*, $S(n,k)$, is the number of partitions of a set of cardinality n into k non-empty (unordered) parts. Prove that

$$S(n,k) = \frac{1}{k!}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}(k-\ell)^{n}.$$

[12] B2. (a) Define a chain and antichain in a poset.

(b) In the poset $(\mathcal{P}(\{1,2,\ldots,n\}),\subseteq)$, how many different maximal chains are there?

(c) Prove Sperner's lemma: If the sets $A_1, A_2, \ldots, A_m \subseteq \{1,2,\ldots,n\}$ form an antichain in the poset $(\mathcal{P}(\{1,2,\ldots,n\}),\subseteq)$, then $m \le \binom{n}{\lfloor n/2 \rfloor}$.

[12] B3. For every $n \ge 0$, let a_n be the number of integer solutions to $s_1 + s_2 + s_3 = n$ with $s_1, s_2 \ge 1$ and $s_3 \ge 0$.

(a) Give the generating function for the sequence $(a_n)_{n\ge 0}$ and express it as a rational function.

(b) Use the generating function to give a closed-form expression for a_n.

(c) Give a direct counting argument for the closed-form expression for a_n found in the previous part.

[12] B4. Let $G = (V, E)$ be a graph of order n. Let $\overline{G}$ be the complement of G: the graph with vertex set V and edge set $\{\{u,v\} \mid u, v \in V,\ u \ne v\} \setminus E$. Prove that:

(a) $\chi(G) + \chi(\overline{G}) \le n + 1$, and

(b) $n \le \chi(G)\chi(\overline{G})$

[12] B5. For any graph G, let $\alpha(G)$ denote the independence number, $\alpha'(G)$ denote the matching number (or edge independence number), $\beta(G)$ denote the vertex covering number (minimum number of vertices to cover all edges), and $\beta'(G)$ denote the edge covering number (minimum number of edges to cover all vertices).

Prove Gallai's theorem: If $G = (V, E)$ is a graph with n vertices, none of which are isolated, then:

(a) $\alpha(G) + \beta(G) = n$

(b) $\alpha'(G) + \beta'(G) = n$.

[12] B6. (a) Define *orthogonal Latin squares.*

(b) Prove that if there is a family of r mutually orthogonal $n \times n$ Latin squares, then $r \leq n - 1$.

(c) Let p be prime. Give a construction for a family of $p - 1$ mutually orthogonal $p \times p$ Latin squares.

[12] B7. Recall that a (b, v, r, k, λ)-design is one in which there are v varieties (or points), b blocks, every block has exactly k varieties, every variety appears in exactly r blocks, and every pair of varieties appears simultaneously in exactly λ blocks.

(a) Let $\mathcal{B} = (V, B)$ be a (b, v, r, k, λ)-design. Prove that $bk = vr$ and $r(k-1) = \lambda(v-1)$.

(b) Prove Fisher's inequality: Let $\mathcal{B} = (V, B)$ be a (b, v, r, k, λ)-design. Then, $b \geq v$.

Part C

This section is worth 30% of the total. Answer any 5 of the 7 questions available. Each question is worth 12 marks. Clearly indicate which problems are to be graded. There are 60 marks available in this part.

[12] C1. Recall that $A, B \in \mathcal{M}_n$ are *similar* if there is $S \in \mathcal{M}_n$ nonsingular such that $B = S^{-1}AS$. Suppose $A, B \in \mathcal{M}_n$ are similar. Show:

(a) A and B have the same characteristic polynomial.

(b) A and B have the same eigenvalues.

[12] C2. $A \in \mathcal{M}_n$ is a *square root* of $B \in \mathcal{M}_n$ if $A^2 = B$. Show that every diagonalisable matrix $B \in \mathcal{M}_n$ has a square root.

[12] C3. Let $|||\cdot|||$ be a matrix norm on $\mathcal{M}_n$ and $S \in \mathcal{M}_n$ be nonsingular. Show that the function

$$|||A|||_S = \left|\left|\left|SAS^{-1}\right|\right|\right|, \quad \forall A \in \mathcal{M}_n$$

is a matrix norm and that, furthermore, if $|||\cdot|||$ is induced by the norm $\|\cdot\|$ on $\mathbb{C}^n$, then the matrix norm $|||\cdot|||_S$ is induced by the norm $\|\cdot\|_S$ on $\mathbb{C}^n$ defined, for $x \in \mathbb{C}^n$, by $\|x\|_S = \|Sx\|$.

[12] C4. Let $|||\cdot|||$ be a matrix norm on $\mathcal{M}_n$, $A \in \mathcal{M}_n$ and λ an eigenvalue of A. Denoting $\rho(A)$ the spectral radius of A, show that

$$|\lambda| \leq \rho(A) \leq |||A||| \tag{1}$$

and, if A nonsingular,

$$\rho(A) \geq |\lambda| \geq \left|\left|\left|A^{-1}\right|\right|\right|^{-1}. \tag{2}$$

(It may be useful to consider the matrix $X = \mathbf{x}\mathbf{e}^T$, where $\mathbf{x} \neq \mathbf{0}$ is an eigenvector associated to λ and $\mathbf{e} = (1, \dots, 1)^T$.)

[12] C5. We admit the following result: $A \in \mathcal{M}_n$ is nonsingular if there is a matrix norm $|||\cdot|||$ such that $|||\mathbb{I} - A||| < 1$.

Show that if $A \in \mathcal{M}_n$ is such that $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for all $i = 1, \dots, n$, then A is nonsingular. (You may otherwise use the Gershgorin Disk Theorem if you wish.)

[12] C6. Let $A \in \mathcal{M}_n$ be nonnegative. Show that $\rho(A) \leq |||A|||_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n a_{ij}$ and $\rho(A) \leq |||A|||_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n a_{ij}$ and that, if all row sums of A are equal, then $\rho(A) = |||A|||_\infty$ and if all column sums of A are equal, then $\rho(A) = |||A|||_1$.

[12] C7. Let $A \in \mathcal{M}_n$ be nonnegative. Show that if there is a positive vector $\mathbf{x}$ and a nonnegative real number λ such that either $A\mathbf{x} = \lambda\mathbf{x}$ or $\mathbf{x}^T A = \lambda\mathbf{x}^T$, then $\lambda = \rho(A)$.