Analysis Comprehensive Examination

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Instructions:

This examination consists of 3 parts, A,B and C:

- Part A covers the Core Material, it consists of 6 questions worth 10 marks each for a total of 60 marks. All questions must be answered for full marks.
- Part B covers the Advanced Material on Abstract Measure and Integration Theory. This section consists of 3 questions of which you must attempt 2. Each question is worth 15 marks for a total of 30 marks.
- Part C covers the Advanced Material on Basic Functional Analysis. This section consists of 3 questions of which you must attempt 2. Each question is worth 15 marks for a total of 30 marks.

Please take note of the following:

1. In Parts B and C, if you attempt more than 2 questions, you must clearly indicate which two questions you would like marked. Otherwise only the first 2 answers in the order they appear in your solutions will be graded.
2. The examination is worth a total of 120 marks. A total grade of 75% or 90 marks is required to pass the exam.
3. The examination length is 6 hours. No texts, reference books, calculators, cell phones, or other aids are permitted in the examination.

A. Core material

1. Consider the vector field $\mathbf{F}(x,y,z) = [-y+z\sqrt{\sin(xy)}]\mathbf{i}+xe^z\mathbf{j}+\cos(xz)\mathbf{k}$ in $\mathbb{R}^3$, where $\mathbf{i}=(1,0,0)$, $\mathbf{j}=(0,1,0)$, $\mathbf{k}=(0,0,1)$ are the standard unit coordinate vectors. Let M be the half-sphere $x^2+y^2+z^2=9$, $z>0$, oriented with unit normal vector $\mathbf{n}$ pointing upwards (i.e. in particular $\mathbf{n}(0,0,3)=(0,0,1)$). State Stokes' theorem and apply it to evaluate the surface integral
$$\iint_M (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$$
of the vector field $\nabla\times\mathbf{F}$.

2. Consider the series
$$f(x) = \sum_{n=1}^{\infty} e^{-nx}\left(\cos(nx)-\sin(nx)\right).$$
Show that this series converges on $[\pi,\infty)$, and that for all $x\in[\pi,\infty)$
$$\int_\pi^x f(t)\,dt = \sum_{n=1}^{\infty}\frac{e^{-nx}\sin(nx)}{n}.$$

3. Let $\lfloor y\rfloor$ denote the greatest integer which is less than or equal to y. Let
$$f(x) = \begin{cases} 1/\left\lfloor \frac{1}{x}\right\rfloor - x^2 & x\in(0,1] \\ 0 & x=0. \end{cases}$$
Show that f is of bounded variation.

4. (a) State the Cauchy-Riemann Theorem on analyticity of functions.

   (b) Fix $z_0\in\mathbb{C}$ and let $D_r(z_0)$ be the open disk:
$$D_r(z_0) := \{z\in\mathbb{C}\mid |z-z_0|<r\}.$$
   Use part (a) to show that if f is analytic in $D_r(z_0)$ and $|f(z)|=1$ for all $z\in D_r(z_0)$, then $f(z)=c\in\mathbb{C}$ is constant in $D_r(z_0)$.

5. Let C be the positively oriented triangle in the complex plane defined by the points $z_1 = -1$, $z_2 = 1$ and $z_3 = 2i$.

   (a) State Cauchy's Theorem and use it to calculate

   $$I_a = \int_C 3e^{2z+i} dz$$

   .

   (b) State Cauchy's Integral Formula for derivatives and use it to calculate

   $$I_b = \int_C \frac{3e^{2z+i}}{(z-i)^3} dz$$

   .

   (c) State the Residue Theorem and use it to calculate

   $$I_c = \int_C \frac{3e^{2z+i}}{(z-i)^2} dz$$

   .

6. Let $\mathbb{Q}$ denote the rational numbers, and $\mathbb{Q}^c := \mathbb{R} \backslash \mathbb{Q}$ its complement in $\mathbb{R}$, the irrational numbers.

   (a) Consider the indicator function $f := \chi_{[0,1]\cap\mathbb{Q}^c}$. (This is defined to be 1 on $[0,1]\cap\mathbb{Q}^c$ and 0 everywhere else.) Is this function Riemann integrable on $[0,1]$? Is it Lebesgue integrable? Compute these integrals if they exist.

   (b) Determine whether or not a monotone convergence theorem holds for the Riemann integral. That is, if f_n is a monotone non-decreasing sequence of positive semi-definite and Riemann integrable functions in $[0,1]$ which converges pointwise to some function f, is it true that

   $$\lim_{n\uparrow\infty} \int_0^1 f_n(x) dx = \int_0^1 \lim_{n\uparrow\infty} f_n(x) dx?$$

   Hint: Let $(q_k)_{k=1}^{\infty}$ be an enumeration of $\mathbb{Q} \cap [0,1]$ and consider the indicator functions χ_n of the finite sets $Q_n := \{q_1, ..., q_n\}$.

B. Measure and Integration Theory

1. Let (X, Σ, μ) be a measure space. That is, X is a set, Σ is a $\sigma-$algebra of subsets of X, and $\mu : \Sigma \to [0, +\infty) \cup \{+\infty\}$ is a positive measure.

   (a) Write down the definition of the integral of a non-negative $\mu-$integrable function, f. If $0 \leq g \leq f$ are $\mu-$integrable, show that

   $$\int_X g d\mu \leq \int_X f d\mu.$$

   (b) State Fatou's Lemma and prove it using the Monotone Convergence Theorem.

   (c) Show by example that the inequality in Fatou's Lemma can be strict.

2. (a) Compute:

   $$\sum_{k=0}^{\infty} \int_0^{1/2} x^{2k+1} dx,$$

   by showing that the order of the summation and integration can be interchanged.

   (b) Consider the integrals:

   $$I_n := \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{-(x_1^2+\cdots+x_n^2)} dx_1 dx_2 ... dx_n.$$

   Compute I_n for any $n \in \mathbb{N}$. *Hint*: First compute I_2 by changing variables to polar co-ordinates.

3. (a) State the Lebesgue dominated convergence theorem.

   (b) Prove that

   $$\lim_{n\to\infty} \int_0^{\infty} n \sin(x/n) \frac{1}{x(1+x^2)} dm(x)$$

   exists and evaluate it. Here, m denotes Lebesgue measure on $\mathbb{R}$.

C. Functional Analysis

1. (a) State Hölder's inequality.

   (b) Assume that $f \in L^p([0,1]) \cap L^r([0,1])$ where $1 < p < r < \infty$. Show that $f \in L^q([0,1])$ for any $q \in (p,r)$, and

   $$\|f\|_q \leq \|f\|_p^{\lambda} \|f\|_r^{1-\lambda}$$

   where $\lambda \in (0,1)$ is the unique number such that

   $$\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}.$$

   Hint: $p/\lambda q$ and $r/(1-\lambda)q$ are conjugate exponents.

   (c) Let $X := \{1, ..., N\}$, $\Sigma = 2^X$, the set of all subsets of X, and μ be counting measure on Σ. That is, given $\Omega \in \Sigma$, $\mu(\Omega)$ is the number of elements in Ω, and $\mu(\emptyset) = 0$. Write out Hölder's inequality for finite sequences.
   Hint: View elements of $L^p(X, \mu)$ as finite sequences.

   (d) Suppose that $a, b, c > 0$ are positive real numbers so that $a + b + c = 3$. What is the smallest possible value of

   $$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}?$$

   Hint: Apply Hölder's inequality to

   $$\left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}\right)^{2/3} (a+b+c)^{1/3}.$$

2. (a) State the open mapping theorem for linear operators.

   (b) Let X and Y be Banach spaces, $T : X \to Y$ a bounded linear map. Show that there is a constant $c > 0$ such that for each $x \in X$, $\|Tx\| \geq c\|x\|$ if and only if $\operatorname{Ker}(T)$, the kernel of T, is $\{0\}$ and $T(X)$, the range of T, is closed in Y.

3. A (separable, complex) Hilbert space of complex-valued functions, $\mathcal{H}$, on a set $X \subseteq \mathbb{C}^d$, is said to be a *reproducing kernel Hilbert space* (RKHS), if for each $x \in X$, the linear functional of point evaluation at x is bounded: Given $h \in \mathcal{H}$,

$$h \overset{\ell_x}{\mapsto} h(x); \qquad \|\ell_x\| < +\infty.$$

The *multiplier algebra*, $\mathrm{Mult}(\mathcal{H})$, of $\mathcal{H}$ is the set of all complex-valued functions, $F : X \to \mathbb{C}$, which 'multiply' $\mathcal{H}$ into itself. That is, $F \in \mathrm{Mult}(\mathcal{H})$ and $h \in \mathcal{H}$ imply that $F \cdot h \in \mathcal{H}$.

(a) State the closed graph theorem.

(b) Prove that the map $h \mapsto Fh$ defines a bounded linear operator on $\mathcal{H}$.

(c) if $F \in \mathrm{Mult}(\mathcal{H})$, and $M_F : \mathcal{H} \to \mathcal{H}$ is the corresponding linear multiplication operator, show that $\overline{F(x)}$ is an eigenvalue of M_F^*, for any $x \in X$. *Hint:* Apply the Riesz representation lemma to each ℓ_x.