Analysis Comprehensive Examination

Department of Mathematics

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This examination consists of three parts.

– Part A covers the core material. It has 8 questions worth 10 points each, of which you must attempt 6, for a total possible score of 60 points.

– Part B covers the specialized material on computational mathematics. It has 3 questions worth 15 points each, of which you must attempt 2, for a total possible score of 30 points.

– Part C covers the specialized material on differential equations. It has 3 questions worth 15 points each, of which you must attempt 2, for a total possible score of 30 points.

If you attempt more than the required number of questions in Parts A, B, or C, you must clearly indicate which questions are to be graded. If it is not clearly indicated, solutions to those appearing first in the booklet will be graded.

You need to achieve at least 90 points, which is 75% of the total possible 120 points, in order to pass the examination.

The total time of the examination is six hours. No books, notes, calculators or aids are allowed during the exam.

Part A: Core Questions

1. You are given two functions on $[0, 2\pi)$:

$$f_1(x) = \frac{1}{5 - 4\cos x}, \quad \text{and} \quad f_2(x) = |x - \pi|.$$

   They correspond to two Fourier series:

$$g_1(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{k=1,2}^{\infty} \frac{\cos(kx)}{k^2}, \quad \text{and} \quad g_2(x) = \frac{1}{3} + \frac{2}{3} \sum_{k=1}^{\infty} \frac{\cos(kx)}{2^k}.$$

   Here, the notation $k = 1, 2$ denotes that the summation runs over the odd integers.

   (a) Which Fourier series correspond to which functions and why?

   (b) Evaluate $\|f_1\|_2$ and $\|f_2\|_2$.

2. Use the Lebesgue dominated convergence theorem to show that the function $F : [0, \infty) \to \mathbb{R}$ defined by

$$F(t) = \int_0^\infty \frac{e^{-x}}{1 + tx} \, \mathrm{d}x,$$

   is continuous. (The integral is understood as a Lebesgue integral. There is no need to justify that F is well-defined; i.e., you may assume that the integrand is Lebesgue integrable for each fixed value of $t \in [0, \infty)$.)

3. Let f be a function of bounded variation on the interval $[a, b]$ and let α be an increasing function on $[a, b]$.

   (a) Denote $M(f) = \sup\{f(x) : x \in [a, b]\}$, $m(f) = \inf\{f(x) : x \in [a, b]\}$, and denote the total variation of f on $[a, b]$ by $V_f[a, b]$. Show that $M(f) - m(f) \leq V_f[a, b]$.

   (b) State Riemann's condition for the Riemann-Stieltjes integrability of f with respect to α on $[a, b]$.

   (c) Assume further that α is continuous on $[a, b]$. Show that f satisfies Riemann's condition and so $\int_a^b f \, d\alpha$ exists.

4. Let (f_n) be a sequence of Riemann integrable (bounded) functions on $[0, 1]$. Suppose that $f_n \to f$ uniformly on $[0, 1]$.

   (a) Show that f is Riemann integrable on $[0, 1]$. (Hint: Use Lebesgue's criterion for Riemann integrability.)

   (b) Show that $\lim_{n\to\infty} \int_0^1 f_n = \int_0^1 f$.

5. Evaluate the following integrals, and fully justify your answer.

(a)
$$\frac{1}{2\pi i}\int_{|z|=4}\frac{e^z}{z^2+\pi^2}\,dz$$
where the curve is traced once counter-clockwise around 0.

(b)
$$\frac{1}{2\pi i}\int_{|w|=2}\frac{e^{w^2}}{(w-z)^3}\,dw.$$
where z is a fixed complex number such that $|z|<2$, and the curve is traced once counter-clockwise around 0.

6. Let $\mathbb{D}=\{z\in\mathbb{C}\,:\,|z|<1\}$ and let $f:\mathbb{D}\to\mathbb{D}$ be a complex analytic function. Assume that f has a zero of order two at 0.

   (a) Prove that $|f(z)|\le|z|$ for all $z\in\mathbb{D}$.

   (b) Using part (a), improve this to $|f(z)|\le|z|^2$ for all $z\in\mathbb{D}$. *Hint*: consider a new function $g(z)=f(z)/z$.

7. Let $\varphi:\mathbb{R}\to[0,1]$ be a continuous, periodic function of period 1 that satisfies
$$\varphi(0)=0,\qquad \varphi(1/2)=1.$$
   Define the series
$$f(t)=\sum_{k=1}^{\infty}2^{-k}\varphi(3^{2k}t).$$

   (a) Prove that f is uniformly convergent.

   (b) Prove that for each $\xi\in[0,1]$, there exists $t\in[0,1]$ such that $f(t)=\xi$.

8. In the following question, a hypersurface is always closed and bounds a compact set that contains 0 in its interior.
   Consider the vector field
$$\mathbf{F}(\mathbf{x})=\lambda(\mathbf{x})\cdot\mathbf{x},\qquad \mathbf{x}\in\mathbb{R}^n\backslash\{0\},$$
   where λ is a smooth, scalar function.

   (a) Find λ so that the vector field is divergence-free and λ is identically 1 on the unit sphere centred at 0.

   (b) Let S_1,S_2 be two non-intersecting hypersurfaces that bound compact regions D_1,D_2 respectively. Assume that $D_1\subset D_2$ and 0 lies in the interior of D_1. Prove that
$$\oint_{S_1}\mathbf{F}\cdot d\mathbf{S}_1=\oint_{S_2}\mathbf{F}\cdot d\mathbf{S}_2,$$
   where $d\mathbf{S}_i$ is the outward-pointing element of oriented surface area of S_i, $i=1,2$.

   (c) Assume $n=3$. Compute $\oint_S\mathbf{F}\cdot d\mathbf{S}$ where S is any surface that bounds a region containing 0 in its interior.

Part B: Computational Mathematics

1. Consider $n+1$ data points $(x_0, f_0), \ldots, (x_n, f_n)$ with distinct abscissæ; that is, $x_i \neq x_j \ \forall i \neq j$. Prove that there exists a unique degree-n polynomial that interpolates the data.

2. Let $a \in \mathbb{R}^{n-1}$, $b \in \mathbb{R}^{n-1}$ and $c \in \mathbb{R}$. For a symmetric arrowhead matrix $A \in \mathbb{R}^{n \times n}$, defined by

$$A = \begin{bmatrix} a_1 & & & b_1 \\ & \ddots & & \vdots \\ & & a_{n-1} & b_{n-1} \\ b_1 & \cdots & b_{n-1} & c \end{bmatrix},$$

use the Cholesky factorization to determine conditions for which the matrix A is positive-definite.

3. Let $\Omega = (0,1)^2$ be the unit square, and consider the partial differential equation:

$$u_{xy} = f, \qquad u(x,y) = 0, \quad \text{for} \quad (x,y) \in \partial\Omega.$$

   (a) Find a finite difference scheme that has second-order local truncation error.

   (b) Describe a direct method for the solution of the finite difference scheme. Choose an ordering to improve the complexity and state the improved complexity (in terms of n).

   (c) Are there any problematic discretization sizes?

Part C: Differential Equations

1. Let $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$\mathbf{f}(t, \mathbf{x}) = (|x_1| + |x_2|, \sqrt{|t|}|x_1| + \sin(x_2)), \qquad \mathbf{x} = (x_1, x_2).$$

   (a) Assume that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ solve the initial value problem

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \qquad \mathbf{x}(1) = (1, 1). \tag{IVP}$$

   Prove that on their common domain, $\mathbf{x}_1(t) = \mathbf{x}_2(t)$.

   (b) Prove that the maximal solution to (IVP) is defined for all $t \in \mathbb{R}$.

2. Let

$$\dot{x} = y, \qquad \dot{y} = -x(1 - x). \tag{DE}$$

   (a) Show that this system is equivalent to

$$\frac{dx}{y} = \frac{dy}{x(x-1)}.$$

   Find an implicit solution $H(x, y)$ to this differential equation.

   (b) Sketch the phase portrait of (DE).

   (c) Show that there are maximal solutions to (DE) that:

      i. are defined for all time; and

      ii. are defined for only a finite amount of time.

3. Let $\Omega \subset \mathbb{R}^n$ be a domain and $u \in C^2(\Omega)$ be a harmonic function.

   (a) Prove that for any $x \in \Omega$ and any $r > 0$ such that the closed ball of radius r centred at x, $B_r(x)$, is contained in Ω, then

$$u(x) = \frac{1}{v(r)} \int_{B_r(x)} u(y)\ dy,$$

   where dy is the volume form and $v(r) = \int_{B_r(x)} dy$.

   (b) Prove that if $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and u attains an extreme value in Ω, then u is constant.