Analysis Comprehensive Examination

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Instructions:

This examination consists of 3 parts, A, B, and C:

- Part A covers the Core Material. It consists of 7 questions worth 10 marks each for a total of 70 marks. All questions must be answered for full marks.
- Part B covers the Advanced Material on Basic Functional Analysis. This section consists of 3 questions of which you must attempt 2. Each question is worth 15 marks for a total of 30 marks.
- Part C covers the Advanced Material on Abstract Measure and Integration Theory. This section consists of 3 questions of which you must attempt 2. Each question is worth 15 marks for a total of 30 marks.

Please take note of the following:

1. In Parts B and C, if you attempt more than 2 questions, you must clearly indicate which two questions you would like marked. Otherwise only the first 2 answers in the order they appear in your solutions will be graded.
2. The examination is worth a total of 130 marks. A total grade of 75% or 97.5 marks is required to pass the exam. Complete, detailed justifications are required for full marks.
3. The examination length is 6 hours. No texts, reference books, calculators, cell phones, or other aids are permitted in the examination.

A. Core material

1. (a) (6 points) Suppose $f : [0, 1] \to [1, \infty)$ is a function of bounded variation on $[0, 1]$. Show that $g(x) = \frac{1}{f(x)}$ is of bounded variation on $[0, 1]$.

   (b) (4 points) Suppose $f : [0, 1] \to (0, 1)$ is a function of bounded variation on $[0, 1]$. Does it follow that $g(x) = \frac{1}{f(x)}$ is of bounded variation on $[0, 1]$? If so, prove it. If not, give a counterexample, with an explanation.

2. (a) (6 points) Suppose $\{\phi_n\}_{n=1}^{\infty}$ is an orthonormal system of complex-valued functions on $[0, 1]$. Show that for any interval $I \subset [0, 1]$ we have

$$\lim_{n\to\infty} \int_I \phi_n(x)\, dx = 0.$$

   (b) (4 points) Give an example of a complex-valued orthogonal system of functions $\{\phi_n\}_{n=1}^{\infty}$ on $[0, 1]$ such that

$$\lim_{n\to\infty} \int_I \phi_n(x)\, dx = 1.$$

3. (a) (4 points) Find all values of $z \in \mathbb{C}$ such that $e^{iz^2} = 1$.

   (b) (6 points) Let γ be the unit circle $\{z : |z| = 1\}$ traced once counter-clockwise. Evaluate

$$\frac{1}{2\pi i} \int_\gamma \frac{z}{e^{iz^2} - 1}\, dz.$$

4. Let

$$f(z) = \frac{z^2}{z - 5}.$$

   (a) (5 points) Find the power series of f around $z = 0$ on $\{z \in \mathbb{C} : |z| < 5\}$.

   (b) (5 points) Find the Laurent series of f around $z = 0$ on $\{z \in \mathbb{C} : 5 < |z|\}$.

5. (10 points) Let H be a real Hilbert space with inner product $(\cdot, \cdot)$. Let Y be a subspace of H and let $x \in H$. Suppose that the distance from x to Y is equal to d, i.e. $d = \inf_{y \in Y} \|x - y\|$. Prove the following refinement of Cauchy-Schwarz inequality:

$$(x, y) \leq \sqrt{\|x\|^2 - d^2} \cdot \|y\|, \quad y \in Y.$$

(Hint: consider $\|x - \lambda y\|^2$ for $\lambda \in \mathbb{R}$.)

6. (10 points) Let A be the set of those real numbers in $[0, 1]$ which have at least one decimal representation containing only the digits 0 and 9 (for example, $1 = 0.\overline{9}$ belongs to A). Prove that A is Lebesgue measurable, and compute its Lebesgue measure.

   (Hint: Consider $[0, 1] \setminus A$. Start by removing the set whose first digit is different from 0 and 9, then consider the second digit, etc.)

7. For real numbers a, b such that $0 \leq a < b \leq 1$ define the function $S_{[a,b]}$ on $[0, 1]$ as follows:

$$S_{[a,b]}(x) = \begin{cases} \frac{x-a}{b-a}, & \text{if } x \in [a, b] \\ 0, & \text{if } x \in [0, 1] \setminus [a, b] \end{cases}.$$

   (a) (3 points) Prove that the series $\sum_{n=1}^{\infty} (-1)^n S_{\left[\frac{1}{2^{n+1}}, \frac{1}{2^{n-1}}\right]}$ converges pointwise to some function f on $[0, 1]$.

   (b) (7 points) Use the dominated convergence theorem to compute $\int_0^1 f \, dm$, where m is the Lebesgue measure on $[0, 1]$.

B. Functional Analysis

1. (a) (9 points) Prove that the identity operator $I : f \mapsto f$ is continuous from $L^2[0,1]$ into $L^1[0,1]$ and compute the norm of this map. (Hint: use Hölder's inequality).

   (b) (6 points) Give an example of a measurable function f that belongs to $L^1[0,1]$ but not to $L^2[0,1]$. (Hint: f should be unbounded.)

2. For any Banach space Y let S_Y be its unit sphere, i.e. $S_Y = \{y \in Y : \|y\| = 1\}$.

   (a) (9 points) Let X be a Banach space and X^* its dual. Use the Hahn-Banach theorem to prove that for any $x \in S_X$ there is a linear functional $f \in S_{X^*}$ such that $f(x) = 1$.

   (b) (6 points) Let X be a reflexive Banach space. Prove that for any linear functional $f \in S_{X^*}$ there is an element $x \in S_X$ such that $f(x) = 1$.

3. Let X and Y be Banach spaces. A sequence of operators $\{T_n : X \to Y\}_{n=1}^{\infty}$ converges pointwise to an operator $T : X \to Y$ if $\lim\limits_{n\to\infty} T_n(x) = T(x)$ for every $x \in X$. Denote by $L(X,Y)$ the space of all bounded linear operators from X to Y and let Z be another Banach space.

   (a) (5 points) Suppose that $\{T_n\}_{n=1}^{\infty} \subset L(X,Y)$ converges pointwise to $T \in L(X,Y)$ and $S \in L(Y,Z)$.

   Prove that $\{S \circ T_n\}_{n=1}^{\infty}$ converges pointwise to $S \circ T$.

   (b) (10 points) Suppose that $\{T_n\}_{n=1}^{\infty} \subset L(X,Y)$ converges pointwise to $T \in L(X,Y)$ and $\{S_n\}_{n=1}^{\infty} \subset L(Y,Z)$ converges pointwise to $S \in L(Y,Z)$.

   Show that $\{S_n \circ T_n\}_{n=1}^{\infty}$ converges pointwise to $S \circ T$. (Hint: use the uniform boundedness principle.)

C. Measure and Integration Theory

1. (a) (3 points) State the definition of mutually singular measures.

   (b) (5 points) Suppose ν_1, ν_2 and μ are measures on a measure space $(X, \mathfrak{B})$. If $\nu_1 + \nu_2$ and μ are mutually singular, show that ν_1 and μ are mutually singular and that ν_2 and μ are mutually singular.

   (c) (7 points) Suppose ν_1, ν_2 and μ are measures on a measure space $(X, \mathfrak{B})$. Assume that ν_1 and μ are mutually singular, and that ν_2 and μ are mutually singular. Does it follow that $\nu_1 + \nu_2$ and μ are mutually singular?

2. Suppose ν and μ are measures on a σ-finite measure space $(X, \mathfrak{B})$ such that ν is absolutely continuous with respect to μ ($\nu \ll \mu$), and let $\left[\frac{d\nu}{d\mu}\right]$ be the Radon-Nikodym derivative of ν with respect to μ.

   (a) (6 points) Show that for any simple function f on X

$$\int f \, d\nu = \int f \left[\frac{d\nu}{d\mu}\right] d\mu.$$

   (b) (9 points) Show that for any nonnegative measurable function f on X

$$\int f \, d\nu = \int f \left[\frac{d\nu}{d\mu}\right] d\mu.$$

3. (15 points) Suppose $0 < h < 1$ and m is the one-dimensional Lebesgue measure on $[0, 1]$. For any real-valued Lebesgue-integrable function f on $[0, 1]$, define

$$\phi(x) := \frac{1}{2h} \int_{[0,1] \cap [x-h, x+h]} f \, dm, \quad x \in [0, 1].$$

   Use Tonelli's theorem to prove that

$$\int_{[0,1]} |\phi| \, dm \le \int_{[0,1]} |f| \, dm.$$

   You can use without a proof that the set $D = \{(x, y) \in [0, 1]^2 : |x - y| \le h\}$ is measurable as a subset of $[0, 1]^2$ equipped with the product measure $m \times m$.