Analysis Comprehensive Examination

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Instructions:

This examination consists of 3 parts, A, B, and C:

- Part A covers the Core Material. It consists of 7 questions worth 10 marks each for a total of 70 marks. All questions must be answered for full marks.
- Part B covers the Advanced Material on Basic Functional Analysis. This section consists of 3 questions of which you must attempt 2. Each question is worth 15 marks for a total of 30 marks.
- Part C covers the Advanced Material on Complex Analysis. This section consists of 3 questions of which you must attempt 2. Each question is worth 15 marks for a total of 30 marks.

Please take note of the following:

1. In Parts B and C, if you attempt more than 2 questions, you must clearly indicate which two questions you would like marked. Otherwise only the first 2 answers in the order they appear in your solutions will be graded.
2. The examination is worth a total of 130 marks. A total grade of 75% or 97.5 marks is required to pass the exam. Complete, detailed justifications are required for full marks.
3. The examination length is 6 hours. No texts, reference books, calculators, cell phones, or other aids are permitted in the examination.

A. Core material

1. (a) (6 points) Suppose $f : [0,1] \to [1,\infty)$ is a function of bounded variation on $[0,1]$. Show that $g(x) = \frac{1}{f(x)}$ is of bounded variation on $[0,1]$.

   (b) (4 points) Suppose $f : [0,1] \to (0,1)$ is a function of bounded variation on $[0,1]$. Does it follow that $g(x) = \frac{1}{f(x)}$ is of bounded variation on $[0,1]$? If so, prove it. If not, give a counterexample, with an explanation.

2. (a) (6 points) Suppose $\{\phi_n\}_{n=1}^{\infty}$ is an orthonormal system of complex-valued functions on $[0,1]$. Show that for any interval $I \subset [0,1]$ we have

$$\lim_{n\to\infty} \int_I \phi_n(x)\, dx = 0.$$

   (b) (4 points) Give an example of a complex-valued orthogonal system of functions $\{\phi_n\}_{n=1}^{\infty}$ on $[0,1]$ such that

$$\lim_{n\to\infty} \int_I \phi_n(x)\, dx = 1.$$

3. (a) (4 points) Find all values of $z \in \mathbb{C}$ such that $e^{iz^2} = 1$.

   (b) (6 points) Let γ be the unit circle $\{z : |z| = 1\}$ traced once counter-clockwise. Evaluate

$$\frac{1}{2\pi i} \int_\gamma \frac{z}{e^{iz^2} - 1}\, dz.$$

4. Let

$$f(z) = \frac{z^2}{z-5}.$$

   (a) (5 points) Find the power series of f around $z = 0$ on $\{z \in \mathbb{C} : |z| < 5\}$.

   (b) (5 points) Find the Laurent series of f around $z = 0$ on $\{z \in \mathbb{C} : 5 < |z|\}$.

5. (10 points) Let H be a real Hilbert space with inner product $(\cdot,\cdot)$. Let Y be a subspace of H and let $x \in H$. Suppose that the distance from x to Y is equal to d, i.e. $d = \inf_{y\in Y} \|x-y\|$. Prove the following refinement of Cauchy-Schwarz inequality:

$$(x,y) \leq \sqrt{\|x\|^2 - d^2} \cdot \|y\|, \quad y \in Y.$$

(Hint: consider $\|x-\lambda y\|^2$ for $\lambda \in \mathbb{R}$.)

6. (10 points) Let A be the set of those real numbers in $[0,1]$ which have at least one decimal representation containing only the digits 0 and 9 (for example, $1 = 0.\overline{9}$ belongs to A). Prove that A is Lebesgue measurable, and compute its Lebesgue measure.

   (Hint: Consider $[0,1]\setminus A$. Start by removing the set whose first digit is different from 0 and 9, then consider the second digit, etc.)

7. For real numbers a,b such that $0 \leq a < b \leq 1$ define the function $S_{[a,b]}$ on $[0,1]$ as follows:

$$S_{[a,b]}(x) = \begin{cases} \frac{x-a}{b-a}, & \text{if } x \in [a,b] \\ 0, & \text{if } x \in [0,1]\setminus[a,b] \end{cases}.$$

   (a) (3 points) Prove that the series $\sum_{n=1}^{\infty}(-1)^n S_{\left[\frac{1}{2n+1},\frac{1}{2n-1}\right]}$ converges pointwise to some function f on $[0,1]$.

   (b) (7 points) Use the dominated convergence theorem to compute $\int_0^1 f\,dm$, where m is the Lebesgue measure on $[0,1]$.

B. Functional Analysis

1. (a) (9 points) Prove that the identity operator $I : f \mapsto f$ is continuous from $L^2[0,1]$ into $L^1[0,1]$ and compute the norm of this map. (Hint: use Hölder's inequality).

   (b) (6 points) Give an example of a measurable function f that belongs to $L^1[0,1]$ but not to $L^2[0,1]$. (Hint: f should be unbounded.)

2. For any Banach space Y let S_Y be its unit sphere, i.e. $S_Y = \{y \in Y : \|y\| = 1\}$.

   (a) (9 points) Let X be a Banach space and X^* its dual. Use the Hahn-Banach theorem to prove that for any $x \in S_X$ there is a linear functional $f \in S_{X^*}$ such that $f(x) = 1$.

   (b) (6 points) Let X be a reflexive Banach space. Prove that for any linear functional $f \in S_{X^*}$ there is an element $x \in S_X$ such that $f(x) = 1$.

3. Let X and Y be Banach spaces. A sequence of operators $\{T_n : X \to Y\}_{n=1}^{\infty}$ converges pointwise to an operator $T : X \to Y$ if $\lim_{n\to\infty} T_n(x) = T(x)$ for every $x \in X$. Denote by $L(X,Y)$ the space of all bounded linear operators from X to Y and let Z be another Banach space.

   (a) (5 points) Suppose that $\{T_n\}_{n=1}^{\infty} \subset L(X,Y)$ converges pointwise to $T \in L(X,Y)$ and $S \in L(Y,Z)$.

   Prove that $\{S \circ T_n\}_{n=1}^{\infty}$ converges pointwise to $S \circ T$.

   (b) (10 points) Suppose that $\{T_n\}_{n=1}^{\infty} \subset L(X,Y)$ converges pointwise to $T \in L(X,Y)$ and $\{S_n\}_{n=1}^{\infty} \subset L(Y,Z)$ converges pointwise to $S \in L(Y,Z)$.

   Show that $\{S_n \circ T_n\}_{n=1}^{\infty}$ converges pointwise to $S \circ T$. (Hint: use the uniform boundedness principle.)

C. Complex Analysis

1. (a) (6 points) Let f be a meromorphic function on $\mathbb{C}$, which omits a point $p \in \mathbb{C}$. That is, for all $z \in \mathbb{C}$, either z is a pole of f or $f(z) \neq p$. Show that $1/(f(z) - p)$ extends to an entire function h (that is, a holomorphic function $h : \mathbb{C} \to \mathbb{C}$).

   (b) (3 points) State the Riemann mapping theorem.

   (c) (6 points) Fix points p and q in $\mathbb{C}$ such that $p \neq q$ and let L be the line segment $L = \{tp + (1-t)q : t \in [0,1]\}$.

   Let f be a meromorphic function on $\mathbb{C}$ which omits L, that is, for all $z \in \mathbb{C}$, either z is a pole or $f(z) \notin L$. Show that f is constant. (Hint: What can you say about $h(\mathbb{C})$?)

2. Let $\mathcal{H}(\mathbb{D})$ denote the class of holomorphic functions with domain $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For $f \in \mathcal{H}(\mathbb{D})$, the Bloch norm of f is defined to be the (possibly infinite) quantity

$$\|f\| = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)|.$$

   (a) (6 points) Fix $M > 0$ and define $\mathcal{F}_M = \{f \in \mathcal{H}(\mathbb{D}) : \|f\| \leq M\}$. Show that $\mathcal{F}_M$ is equicontinuous at each point $z_0 \in \mathbb{D}$.

   (b) (3 points) Prove or disprove: $\mathcal{F}_M$ is a normal family in $\mathcal{H}(\mathbb{D})$.

   (c) (6 points) Now let $\mathcal{F}_M^0 = \{f \in \mathcal{F}_M : f(0) = 0\}$. Prove or disprove: $\mathcal{F}_M^0$ is a normal family.

3. Let G be the region in $\mathbb{C}$ given by

$$G = \{re^{i\theta} : r \in (0, \infty) \ \text{ and } \ 0 < \theta < \pi/3\}.$$

   Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

   (a) (6 points) Find a bijection $F : \operatorname{cl}\mathbb{D} \to \operatorname{cl} G$ from the closure of $\mathbb{D}$ to the closure of G in the Riemann sphere $\mathbb{C}_\infty$, which satisfies

   (I) F is continuous with respect to the topology of the Riemann sphere;

   (II) F is a holomorphic map from $\mathbb{D}$ to G, and

   (III) $F(-i) = 0$, $F(i) = \infty$; and $F(1) = 1$.

   (Hint: first find a map from the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ onto G.)

   (b) (3 points) Let $f : \partial_\infty G \to \mathbb{R}$ be a continuous function with respect to the topology of the Riemann sphere. (Recall that $\partial_\infty G$ is the boundary of G in $\mathbb{C}_\infty$.) Show that if v is a solution to the Dirichlet problem on $\mathbb{D}$ with respect to the boundary values $f \circ F$ on $\{z : |z| = 1\}$, then $u = v \circ F^{-1}$ is a solution to the Dirichlet problem on G with boundary values f on $\partial_\infty G$.

(c) (6 points) Recall that the Poisson kernel of the disk $\mathbb{D}$ is given by

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2} = \mathrm{Re}\left(\frac{1+z}{1-z}\right)$$

for $0 \leq r < 1$ and $-\infty < \theta < \infty$, where $z = re^{i\theta}$.
For $z \in \partial_\infty G$ define

$$f(z) = \begin{cases} 1/(|z|+1) & z \neq \infty \\ 0 & z = \infty. \end{cases}$$

and let u be the solution to the Dirichlet problem on G with boundary values f on $\partial_\infty G$. Show that

$$u(z) = \mathrm{Re}\left(\frac{1}{2\pi i}\int_{|w|=1} \frac{w + F^{-1}(z)}{w(w - F^{-1}(z))} \cdot \frac{1}{|F(w)|+1}\, dw\right)$$

(where the integral is taken once counter-clockwise).