Analysis Comprehensive Examination

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Instructions:

This examination consists of 3 parts, A,B and C:

- Part A covers the Core Material, it consists of 8 questions worth 10 marks each for a total of 80 marks. All questions must be answered for full marks.
- Part B covers the Advanced Material on Basic Functional Analysis. This section consists of 3 questions of which you must attempt 2. Each question is worth 15 marks for a total of 30 marks.
- Part C covers the Advanced Material on Abstract Measure and Integration Theory. This section consists of 3 questions of which you must attempt 2. Each question is worth 15 marks for a total of 30 marks.

Please take note of the following:

1. In Parts B and C, if you attempt more than 2 questions, you must clearly indicate which two questions you would like marked. Otherwise only the first 2 answers in the order they appear in your solutions will be graded.
2. The examination is worth a total of 140 marks. A total grade of 75% or 105 marks is minimally required to pass the exam.
3. The examination length is 6 hours. No texts, reference books, calculators, cell phones, or other aids are permitted in the examination.

A. Core material

1. Let $f(x)$ be a real-valued function defined on a set D of real numbers.

   (a) State what it means by saying that f is uniformly continuous on D. [2]

   (b) Is the function $f(x) = \frac{1}{x}$ uniformly continuous on $(0, 1]$? Justify your answer. [8]

2. Let $f\colon [a, b] \to \mathbb{R}$ be Lebesgue integrable. Define $F(x) = \displaystyle\int_{[a,x]} f$.

   (a) Show that the function $F(x)$ is of bounded variation on $[a, b]$. [5]

   (b) Suppose that $f(x) \geq 0$ a.e. on $[a, b]$. Find the total variation of F on $[a, b]$. Justify your answer. [5]

3. Let C be the curve formed by the intersection of the cone $z = \sqrt{x^2 + y^2}$ and the cylinder $x^2 + (y-1)^2 = 1$ oriented in the counterclockwise direction looking down from the positive z-axis high above the xy-plane. If

$$\mathbf{F} = e^{x^2}\mathbf{i} + [x + \sin(y^2)]\mathbf{j} + z\mathbf{k},$$

   evaluate

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

   [10]

4. Prove or disprove the following statement. [10]

   Let $\{z_0, \ldots, z_m\}$ be distinct points in the complex plane. There exists a unique analytic function $r : \mathbb{C} \setminus \{z_1, \ldots, z_m\} \to \mathbb{C}$ that satisfies:

   (a) $\lim\limits_{z \to \zeta} |(z - \zeta) r(z)| < \infty$ for all $\zeta \in \mathbb{C}$;

   (b) r is bounded at ∞;

   (c) $r(z_0) = c_0 \in \mathbb{C}$; and,

   (d) $\operatorname*{Res}\limits_{z = z_k} r(z) = A_k$ for $k = 1, \ldots, m$.

5. Evaluate the integral $\displaystyle\int_{\mathbb{R}} \frac{\cos x}{x^2 + 1}\, \mathrm{d}x$. [10]

6. Determine the number of solution(s) in $D = \{z \in \mathbb{C} : |z| < 1\}$ to the equation

$$\exp\left(\frac{z^4 + z^{-1}}{2}\right) = 2021.$$

   Justify your answer. [10]

7. Consider the function $f(x) = |x|$ on $[-\pi, \pi]$.

   (a) Find the Fourier series of $f(x)$ on $[-\pi, \pi]$. [6]

   (b) At what points in $[-\pi, \pi]$ does the Fourier series of $f(x)$ converge to $f(x)$? Justify your answer. [2]

   (c) Show that

$$\frac{\pi^2}{16} = -\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\left(\frac{(2n-1)3\pi}{4}\right). \qquad [2]$$

   Hint: Evaluate the Fourier series of $f(x)$ at a certain point.

8. Let $(x_n)_{n=1}^{\infty}$ be a bounded, non-decreasing real sequence, $x_n \in \mathbb{R}$.

   (a) Prove that x_n converges. [4]

   (b) Define a sequence (x_n) by $x_1 = \frac{1}{2}$ and $x_{n+1} := -(x_n - 1)^2 + 1$ for $n \in \mathbb{N}$.

      i. Show that $\frac{1}{2} \leq x_n \leq 1$ for all $n \in \mathbb{N}$ and that (x_n) is a non-decreasing sequence. [4]

      ii. What is $\sup x_n$? [2]

B. Functional Analysis

1. (a) State the Closed Graph Theorem for linear operators. [2]

   (b) Let $\mathcal{H}$ be a Hilbert space of complex-valued functions on a set X. Suppose that the point evaluation functional ℓ_x: $\mathcal{H} \to \mathcal{H}$ defined by $\ell_x(h) = h(x)$ is bounded for each $x \in X$. Let $\mathrm{Mult}(\mathcal{H})$ be the set of all complex-valued functions on X which 'multiply' $\mathcal{H}$ into itself. That is, if $F \in \mathrm{Mult}(\mathcal{H})$ and $h \in \mathcal{H}$, then the pointwise product $F \cdot h$ belongs to $\mathcal{H}$. Prove that the multiplication operator M_F: $h \mapsto F \cdot h$ is a bounded linear operator on $\mathcal{H}$ for each $F \in \mathrm{Mult}(\mathcal{H})$. [13]

2. Let $AC[0,1]$ be the set of all absolutely continuous complex-valued functions on $[0,1]$. Consider the linear subspace

$$\mathrm{Dom}(D) := \left\{ h \in AC[0,1] \,\middle|\, h' \in L^2[0,1] \text{ and } h(0) = 0 \right\}$$

   of $L^2[0,1]$, and define the linear map $D : \mathrm{Dom}(D) \to L^2[0,1]$ by $Dh = h'$ for $h \in \mathrm{Dom}(D)$, where h' denotes the derivative of h.

   (a) Prove that D is a closed operator, *i.e.* prove that its graph, $G(D)$, is a closed subspace of $L^2[0,1] \times L^2[0,1]$. [13]

   (b) Does the closed graph theorem imply that D is bounded? Why or why not? [2]

3. Let $\mathcal{H}$ be a complex Hilbert space with inner product denoted by $\langle \cdot, \cdot \rangle$. A sequence $(x_n)_{n=1}^{\infty} \subset \mathcal{H}$ is said to *converge weakly* to $x \in \mathcal{H}$ if for each $y \in \mathcal{H}$,

$$\langle x_n - x, y \rangle \to 0.$$

   Similarly, (x_n) is *weakly Cauchy* if for each $y \in \mathcal{H}$ and each $\epsilon > 0$ there is an $N \in \mathbb{N}$ so that $n, m > N$ implies that

$$|\langle x_n - x_m, y \rangle| < \epsilon.$$

   Prove that $\mathcal{H}$ is *weakly complete*, *i.e.* that every weakly Cauchy sequence in $\mathcal{H}$ converges weakly to an element of $\mathcal{H}$. Show further that every weakly Cauchy sequence is uniformly bounded in the Hilbert space norm. [15]

C. Measure and Integration Theory

1. (a) State the Monotone Convergence Theorem for integrals. [2]

   (b) Use the Monotone Convergence Theorem to show the following Fatou's Lemma: Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f_n \colon X \to [0, \infty]$, $n = 1, 2, \ldots$, be a sequence of measurable functions. Define $f(x) = \liminf_{n\to\infty} f_n(x)$ $(x \in X)$. Then

$$\int f d\mu \leq \liminf_{n\to\infty} \int f_n d\mu. \qquad [13]$$

2. Let μ and ν be measures on the measurable space $(X, \mathcal{A})$. Suppose that $\mu \geq \nu$, i.e. $\mu(E) \geq \nu(E)$ for all $E \in \mathcal{A}$. Consider the set function λ defined by

$$\lambda(E) = \sup\{\mu(A) - \nu(A) : A \in \mathcal{A}, A \subset E \text{ and } \nu(A) < \infty\}.$$

   Show that λ is a measure on $(X, \mathcal{A})$. [15]

3. Let μ and ν be two finite signed measures on $(X, \mathcal{A})$. Define $\mu \wedge \nu = \frac{1}{2}(\mu + \nu - |\mu - \nu|)$. Show that $\mu \wedge \nu \leq \mu$, $\mu \wedge \nu \leq \nu$, and $\mu \wedge \nu \geq \eta$ for any signed measure η such that $\eta \leq \mu$ and $\eta \leq \nu$. [15]