Solving Congruences

Theorem. Let $d = (a, m)$, and consider the equation

$$ax = b \pmod{m}.$$

(a) If $d \nmid b$, there are no solutions.

(b) If $d \mid b$, there are exactly d distinct solutions mod m.

Proof. Observe that

$$ax = b \pmod{m} \quad \Leftrightarrow \quad ax + my = b \quad \text{for some} \quad y.$$

Hence, (a) follows immediately from the corresponding result on linear Diophantine equations. The result on linear Diophantine equations which corresponds to (b) says that there are infinitely many integer solutions

$$x = x_0 + \frac{m}{d}t,$$

where x_0 is a particular solution. I need to show that of these infinitely many solutions, there are exactly d distinct solutions mod m.

Suppose two solutions of this form are congruent mod m:

$$x_0 + \frac{m}{d}t_1 = x_0 + \frac{m}{d}t_2 \pmod{m}.$$

Then

$$\frac{m}{d}t_1 = \frac{m}{d}t_2 \pmod{m}.$$

Now $\dfrac{m}{d}$ divides both sides, and $\left(\dfrac{m}{d}, m\right) = \dfrac{m}{d}$, so I can divide this congruence through by $\dfrac{m}{d}$ to obtain

$$t_1 = t_2 \pmod{d}.$$

Going the other way, suppose $t_1 = t_2 \pmod{d}$. This means that t_1 and t_2 differ by a multiple of d:

$$t_1 - t_2 = kd.$$

So

$$\frac{m}{d}t_1 - \frac{m}{d}t_2 = \frac{m}{d} \cdot kd = km.$$

This implies that

$$\frac{m}{d}t_1 = \frac{m}{d}t_2 \pmod{m}$$

so

$$x_0 + \frac{m}{d}t_1 = x_0 + \frac{m}{d}t_2 \pmod{m}.$$

Let me summarize what I've just shown. I've proven that two solutions of the above form are equal mod m if and only if their parameter values are equal mod d. That is, if I let t range over a complete system of residues mod d, then $x_0 + \dfrac{m}{d}t$ ranges over all possible solutions mod m. To be very specific,

$$x_0 + \frac{m}{d}t \pmod{m} \quad \text{for } t = 0, 1, 2, \ldots, d - 1$$

are all the solutions mod m. $\square$

Example. $6x = 7 \pmod 8$. Since $(6, 8) = 2 \not| 7$, there are no solutions. $\square$

Example. $3x = 7 \pmod 4$. Since $(3, 4) = 1 \mid 7$, there will be 1 solutions mod 4. I'll find it in three different ways.

Using linear Diophantine equations.

$$3x = 7 \pmod 4 \text{ implies } 2x + 4y = 7 \text{ for some } y.$$

By inspection $x_0 = 1$, $y_0 = 1$ is a particular solution. $(3, 4) = 1$, so the general solution is

$$x = 1 + 4t, \quad y = 1 - 3t.$$

The y equation is irrelevant. The x equation says

$$x = 1 \pmod 4.$$

Using the Euclidean algorithm. Since $(3, 4) = 1$, some linear combination of 3 and 4 is equal to 1. In fact,

$$(-1) \cdot 3 + 1 \cdot 4 = 1.$$

This tells me how to juggle the coefficient of x to get $1 \cdot x$:

$$\begin{array}{rrcll} & 4x & = & 0 & \pmod 4 \\ - & 3x & = & 7 & \pmod 4 \\ \hline & x & = & 1 & \pmod 4 \end{array}$$

(I used the fact that $7 = -1 \pmod 4$).

Using inverses mod 4. Here is a multiplication table mod 4:

$*$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

I see that $3 \cdot 3 = 1 \pmod 4$, so I multiply the equation by 3:

$$3x = 7 \pmod 4, \quad x = 21 = 1 \pmod 4. \quad \square$$

Theorem. Let $d = (a, b, m)$, and consider the equation

$$ax + by = c \pmod m.$$

(a) If $d \not| c$, there are no solutions.

(b) If $d \mid c$, there are exactly md distinct solutions mod m.

I won't give the proof; it follows from the corresponding result on linear Diophantine equations just like the previous proof.

Example. Consider the equation

$$2x + 6y = 4 \pmod{10}.$$

$(2, 6, 10) = 2 \mid 4$, so there are $2 \cdot 10 = 20$ solutions mod 10. I'll solve the equation using a reduction trick similar to the one I used to solve two variable linear Diophantine equations.

The given equation is equivalent to

$$2x + 6y + 10z = 4 \text{ for some } z.$$

Set

$$w = \frac{2}{(2,6)}x + \frac{6}{(2,6)}y.$$

Then

$$(2,6)w + 10z = 4, \quad 2w + 10z = 4, \quad w + 5z = 2.$$

$w_0 = -3$, $z_0 = 1$, is a particular solution. The general solution is

$$w = -3 + 5s, \quad z = 1 - s.$$

Substitute for w:

$$\frac{2}{(2,6)}x + \frac{6}{(2,6)}y = -3 + 5s, \quad x + 3y = -3 + 5s.$$

$x_0 = 5s$, $y_0 = -1$, is a particular solution. The general solution is

$$x = 5s + 3t, \quad y = -1 - t.$$

$t = 0, 1, \ldots, 9$ will produce distinct values of y mod 10. Note, however, that s and $s + 2r$ produce $5s$ and $5s + 10r$, which are congruent mod 10. That is, adding a multiple of 2 to a given value of s makes the $5s$ term in x repeat itself mod 10. So I can get all possibilities for x mod 10 by letting $s = 0, 1$.

All together, the distinct solutions mod 10 are

$$x = 5s + 3t, \quad y = -1 - t, \quad \text{where} \quad s = 0, 1 \quad \text{and} \quad t = 0, 1, \ldots, 9. \quad \square$$