Lecture 10: Limits and Computational Techniques

1. Quick Recap

In the last lecture, we saw the **graphical view of limits**. We drew curves and observed how functions behave as x gets closer to some point.
Now, we move one step forward: **algebraic techniques** to compute limits.

👉 Idea: Instead of only looking at the graph, we will now use formulas and rules.

Proofs will come later when we define LIMIT rigorously. For now, intuition and calculation are enough.

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2. The Setup

When we write:

`Lim x → a f(x)`

we assume two things:

1. The left-hand limit and right-hand limit are the same.
2. So the limit exists.

That means we don't need to write LHL or RHL separately here.

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3. Two Basic Functions

We begin with two simple but powerful cases.

- Case 1: `f(x) = k` (a constant function)
- Case 2: `g(x) = x` (the identity function)

4. Limits of Basic Functions

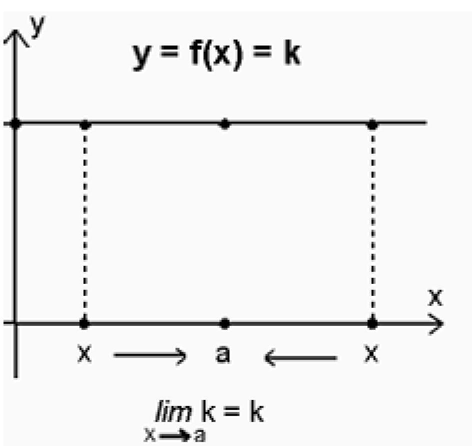
1. **Constant Function**
   If `f(x) = k`, then as x → a:

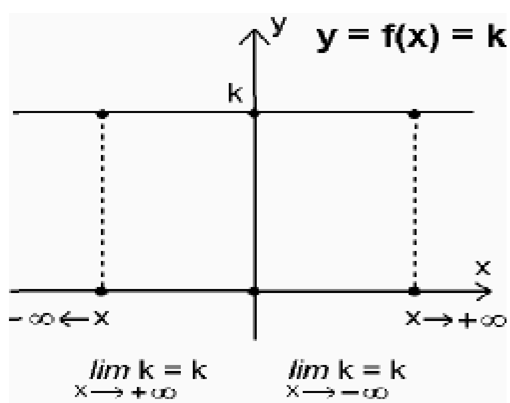
   `Lim x → a f(x) = Lim x → a k = k`

   👉 Meaning: No matter where you are, the function is always stuck at k.
   Example: Think of a water tank that is always filled at 100 liters. Whether you check today, tomorrow, or in 10 years — it's always 100.

Limit	Example
$\lim_{x \to a} k = k$	$\lim_{x \to 2} 3 = 3$, $\lim_{x \to -2} 3 = 3$
$\lim_{x \to +\infty} k = k$	$\lim_{x \to +\infty} 3 = 3$, $\lim_{x \to +\infty} 0 = 0$
$\lim_{x \to -\infty} k = k$	$\lim_{x \to -\infty} 3 = 3$, $\lim_{x \to -\infty} 0 = 0$

Limit
$\lim_{x \to a} x = a$
$\lim_{x \to +\infty} x = +\infty$
$\lim_{x \to -\infty} x = -\infty$

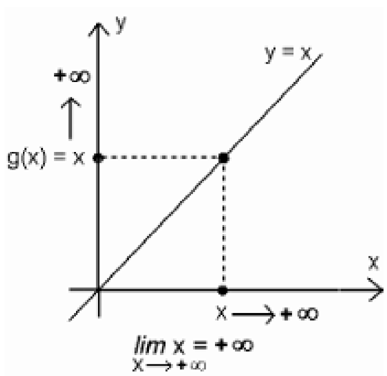


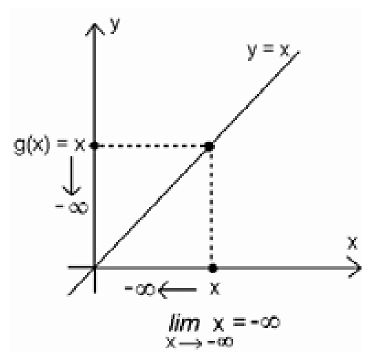


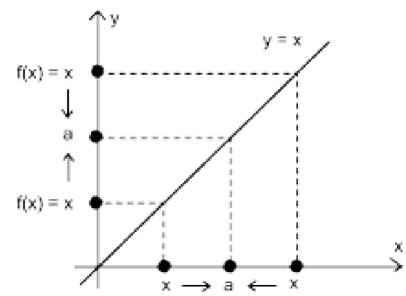
2. **Identity Function**
   If `g(x) = x`, then as x → a:

   `Lim x → a x = a`

   👉 Meaning: As x gets closer to a, the output also gets closer to a.
   Example: Imagine a car driving straight on GT Road. If the car is at kilometer 10, then the reading is exactly 10. If it's approaching kilometer 20, the reading approaches 20. Nothing fancy.







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5. A Useful Theorem (Limit Laws)

We won't prove this right now (proofs are in Appendix C of your book), but here's the **toolbox** you need.

Suppose limits exist for f(x) and g(x) as x → a. Then:

1. **Sum Rule**:
   `Lim x → a [f(x) + g(x)] = Lim f(x) + Lim g(x)`

2. **Difference Rule**:
   `Lim x → a [f(x) − g(x)] = Lim f(x) − Lim g(x)`

3. **Product Rule**:
   `Lim x → a [f(x) * g(x)] = (Lim f(x)) * (Lim g(x))`

4. **Quotient Rule**:
   `Lim x → a [f(x) / g(x)] = (Lim f(x)) / (Lim g(x))`
   (as long as denominator ≠ 0)

👉 Think of these like "shortcuts" in cricket scoring. If a batsman makes 30 runs and another makes 40, the team total is just 70. You don't need to re-calculate the whole thing — just add. Limits behave similarly.

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6. Why This Matters

These rules let us handle complicated functions by breaking them into simple parts.

- If you can handle constants and x,
- and you know how limits add, subtract, multiply, and divide,
- then you can compute limits of polynomials, rationals, and much more.

👉 It's like cooking. If you know how to cook rice and lentils separately, then **khichdi** is just combining both.

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7. Summary

- Constant function → Limit is that constant.
- Identity function → Limit is the approaching value.
- Limit laws → Add, subtract, multiply, divide limits easily.
- Big idea → Break down a complex function into small pieces and compute limit step by step.

Theorem 2.5.1 – Limit Laws

This theorem basically tells us how limits behave when you add, subtract, multiply, or divide functions.

Think of it like **rules of arithmetic** but applied to limits.

Statement of Theorem

Suppose

```
Lim x → a f(x) = L1
 Lim x → a g(x) = L2
```

(both limits exist). Then:

a) **Sum Rule**
```
 Lim x → a [f(x) + g(x)] = L1 + L2
```

👉 Limit of the sum = sum of the limits.

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b) **Difference Rule**
```
 Lim x → a [f(x) − g(x)] = L1 − L2
```

👉 Limit of the difference = difference of the limits.

---

c) **Product Rule**
```
 Lim x → a [f(x) * g(x)] = L1 * L2
```

👉 Limit of the product = product of the limits.

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d) **Quotient Rule**
```
 Lim x → a [f(x) / g(x)] = L1 / L2 , provided L2 ≠ 0
```

👉 Limit of the quotient = quotient of the limits.

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e) **Power Rule**
$\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$

👉 Limit of a power = power of the limit.

Intuitive Explanation (Feynman Style)

- **Sum Rule**: Imagine two street hawkers at Anarkali Bazaar. One sells samosas, the other sells jalebi. If the first guy consistently sells 20 pieces per hour, and the second sells 30 pieces per hour, then together they will sell 50 pieces per hour. The total is just the sum.

- **Difference Rule**: If Pakistan scores 280 runs and India scores 270 runs, then the winning margin is just 10. Again, limit of the difference is simply the difference of the limits.

- **Product Rule**: Suppose one farmer grows 5 kg of mangoes per tree, and he has 10 trees. The product (5 * 10 = 50 kg) tells you the total. Same with limits — multiply the outcomes.

- **Quotient Rule**: If one rickshaw carries 12 passengers in an hour and you have 4 rickshaws, then per rickshaw it's 12/4 = 3 passengers. But warning 🚨: this rule only works if the denominator isn't zero (otherwise it's nonsense).

- **Power Rule**: If your monthly pocket money tends to 5000 PKR, then the square of your pocket money tends to (5000)^2. Simple: raise the limit to that power.

Extensions

1. **Many Functions (Sum Rule extended)**
   $\lim_{x \to a} [f_1(x) + f_2(x) + \dots + f_n(x)] = \lim f_1(x) + \lim f_2(x) + \dots + \lim f_n(x)$

👉 Example: If 3 bowlers in cricket bowl consistently 2, 3, and 4 overs each, the total overs = 9.

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2. **Many Functions (Product Rule extended)**
   $\lim_{x \to a} [f_1(x) * f_2(x) * \dots * f_n(x)] = (\lim f_1(x)) * (\lim f_2(x))$

```
* … * (Lim fn(x))
```

👉 Example: If 3 workers produce 2, 3, and 5 units respectively, the total combined productivity multiplies.

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3. **Special Case: Constant Factor**
   If f(x) = k (a constant), then

```
Lim x → a [k * g(x)] = k * Lim x → a g(x)
```

👉 Example: If each plate of biryani costs 200 rupees, and the number of plates tends to 50, then the total cost tends to 200 * 50. The constant factor (200) just moves outside.

Why This Theorem is Useful

This theorem is like your **calculator for limits**.

- You don't need to re-invent the wheel for every new function.
- Break functions into small pieces, apply the rules, and rebuild.
- Later, when we face complicated polynomials, rationals, or trigonometric functions, these rules make computation quick and reliable.

✨ Summary:

- Sum → sum of limits
- Difference → difference of limits
- Product → product of limits
- Quotient → quotient of limits (denominator ≠ 0)
- Power → power of the limit

- Constants can be factored out

Limits of Polynomials

1. What is a Polynomial?

A polynomial looks like this:

```
f(x) = a_n x^n + a_(n-1) x^(n-1) + ... + a_1 x + a_0
```

where all the a's (coefficients) are real numbers.

👉 Think of a polynomial as a **recipe**:

- Some amount of x^2
- Some amount of x^3
- Some constant sugar sprinkled at the end

When x changes, the whole recipe output changes.

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2. Theorem 2.5.2 – Limit of a Polynomial

The limit of a polynomial at x = a is just the polynomial evaluated at a.

In other words:

```
Lim x → a f(x) = f(a)
```

👉 This is the easiest case: just plug in the value of x.

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3. Example

Find:

```
Lim x → 5 (x^2 − 4x + 3)
```

Step 1: Break it down using limit laws:

```
= Lim x → 5 (x^2) − Lim x → 5 (4x) + Lim x → 5 (3)
```

Step 2: Compute each limit:

```
= (5^2) − (4*5) + 3
```

Step 3: Simplify:

```
= 25 − 20 + 3 = 8
```

👉 Done! No fancy trick. For polynomials, “limit = direct substitution.”

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4. Intuition (Feynman Style)

Imagine you’re buying mangoes in Anarkali Bazaar. If price = 2x + 5, and x = 5 kg, just plug in 5 and you know the cost. That’s exactly what polynomial limits do — direct substitution.

Limits Involving 1/x

Now comes the interesting case: functions like

```
f(x) = 1/x
```

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1. Behavior near Zero

Look at what happens as x approaches 0:

- As $x \to 0^+$ (from the right, small positive numbers):
  `1/x → +∞`

👉 Example: If you divide 1 rupee among 0.01 friends (tiny positive number), each gets 100 rupees. The fewer friends, the bigger the share — it blows up.

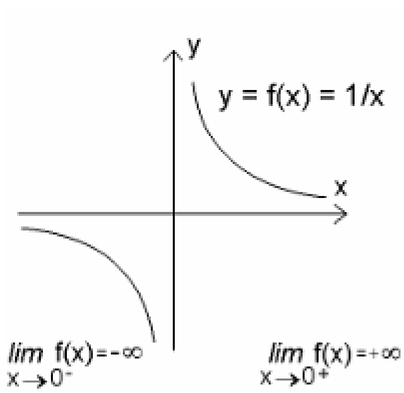
- As $x \to 0^-$ (from the left, small negative numbers):
  `1/x → −∞`

👉 Example: If you divide 1 rupee among –0.01 (impossible negative) “friends,” the math explodes negative.

So:

$\lim_{x \to 0^+} (1/x) = +\infty$
$\lim_{x \to 0^-} (1/x) = -\infty$

👉 The left-hand and right-hand limits are not equal, so **the limit at x = 0 does not exist.**



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2. Behavior as x → ∞ or –∞

- As x → +∞, $1/x \to 0$
- As x → –∞, $1/x \to 0$

👉 Example: If you divide 1 rupee among 10,000 friends, each friend's share is almost nothing. Same if it's –10,000. So the graph flattens toward zero.

	Values	Conclusion
x 1/x	1 .. .01 .. .001 ... 1 .. 100 .. 1000 ...	$x \longrightarrow 0^+$ $1/x \longrightarrow +\infty$
x 1/x	-1 .. -.01 .. -.001 ... -1 .. -100 .. -1000 ...	$x \longrightarrow 0^-$ $1/x \longrightarrow -\infty$
x 1/x	1 .. 100 .. 1000 ... 1 .. .01 .. .001 ...	$x \longrightarrow +\infty$ 1/x decreases towards 0
x 1/x	-1 .. -100 .. -1000 ... -1 .. -.01 .. -.001 ...	$x \longrightarrow -\infty$ 1/x increases towards 0

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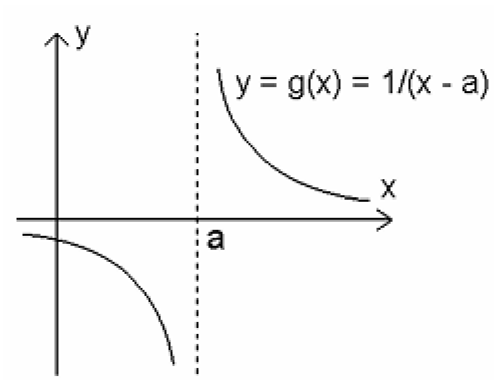
3. Translation (1 / (x – a))

If `g(x) = 1 / (x − a)` then it's just the same graph, shifted by a units.

- As $x \to a^+ \to +\infty$
- As $x \to a^- \to -\infty$
- As $x \to \pm\infty \to 0$

👉 Example: If x – a is close to zero, you again get the same "explosion" behavior, just shifted at a different point.



Summary

1. **Polynomial Functions:**
   - Limit at x = a is just f(a).
   - Direct substitution works.
2. **1/x Functions:**
   - As $x \to 0^+ \to +\infty$
   - As $x \to 0^- \to -\infty$
   - As $x \to \pm\infty \to 0$

- For $g(x) = 1/(x - a)$, same story, just shifted to $x = a$.

Limits of Polynomials as $x \to +\infty$ and $-\infty$

1. Big Idea

When x becomes **very large** (positively or negatively), the behavior of a polynomial is controlled by its **highest power term**.

👉 Why?
Because the highest power grows fastest, while smaller powers become negligible.

Example:
$f(x) = x^5 + 2x^3 + 7$
As $x \to \infty$,

- x^5 grows the fastest
- $2x^3$ and 7 become tiny compared to x^5
  So the limit depends only on x^5.

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2. General Rules

- For x^n as $x \to +\infty$:
  - If n is even $\to +\infty$
  - If n is odd $\to +\infty$
- For x^n as $x \to -\infty$:
  - If n is even $\to +\infty$ (negative raised to an even power is positive)
  - If n is odd $\to -\infty$ (negative raised to an odd power stays negative)

So:

$\lim_{x \to +\infty} x^n = +\infty$ (for any n = 1,2,3,...)

$\lim_{x \to -\infty} x^n = +\infty$ if n is even, and $= -\infty$ if n is odd.

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3. Examples

1. `Lim x → +∞ (2x^5)`
   = +∞
   (Because x^5 → +∞, and multiplied by 2 still → +∞)

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2. `Lim x → +∞ (−7x^6)`
   = −∞
   (Because x^6 → +∞, but multiplied by −7 flips the sign)

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3. `Lim x → −∞ (x^5)`
   = −∞
   (Odd power, negative input stays negative, and it grows without bound)

---

4. `Lim x → −∞ (x^6)`
   = +∞
   (Even power, negative input squared into positive, grows without bound)

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4. Polynomials in General

Suppose:

`P(x) = c_n x^n + c_(n−1) x^(n−1) + ... + c_1 x + c_0`

As x → ±∞:

- The **leading term** `c_n x^n` decides the behavior.
- Lower-order terms vanish in comparison.

So:

$$\text{Lim } x \to \pm\infty \; P(x) = \text{Lim } x \to \pm\infty \; (c_n x^n)$$

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5. Intuitive Explanation (Feynman Style)

Think of Lahore traffic:

- A bicycle (constant term)
- A motorbike (x term)
- A Suzuki van (x^2 term)
- A Daewoo bus (x^3 term)
- A high-speed train (x^5 term)

When the road stretches to infinity, the **train dominates**. The bicycle and motorbike are still there, but compared to the train's speed, they're negligible.

That's exactly what happens in polynomials: the **highest power of x dominates** at infinity.

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6. Motivation with Factorization

Take $P(x) = c_n x^n + c_{(n-1)} x^{(n-1)} + ... + c_0$

Factor out x^n:

$$P(x) = x^n [c_n + (c_{(n-1)}/x) + (c_{(n-2)}/x^2) + ... + (c_0/x^n)]$$

As $x \to \infty$ or $-\infty$,

- All fractions like $(c_{(n-1)}/x)$, $(c_{(n-2)}/x^2)$, ... $\to 0$
- What's left is $x^n * c_n$

So the leading term decides everything.

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7. Summary

- For polynomials, **limits at infinity are controlled by the highest degree term**.
- Even powers → always positive at ±∞.
- Odd powers → positive at +∞, negative at –∞.
- Coefficient c_n decides whether the whole polynomial goes to +∞ or –∞.

📘 Limits of Rational Functions

👉 A rational function is just a fraction of two polynomials:

$f(x) = P(x) / Q(x)$

where $P(x)$ and $Q(x)$ are polynomials.

Now, limits of rational functions depend on what's happening with the denominator when $x \to a$.

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1. Case 1: Denominator ≠ 0 at x = a

Example:
$\lim (x\to 2)\ (3x + 5) / (x - 4)$

Step 1: Substitute directly.
Top → $3(2) + 5 = 11$
Bottom → $2 - 4 = -2$

So,
$\lim (x\to 2)\ (3x + 5) / (x - 4) = 11 / -2 = -11/2$

Analogy: If the bottom is not zero, it's like normal division: nothing blows up, so just plug in.

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2. Case 2: Both numerator and denominator → 0 (the "0/0" case)

This is an indeterminate form.
Here, numerator and denominator share a common factor (like $x - a$) which cancels out.

Example:
$\lim (x\to 2)\ (x^2 - 4) / (x - 2)$

Factorize numerator:
$x^2 - 4 = (x - 2)(x + 2)$

So,
$(x^2 - 4) / (x - 2) = (x + 2)$, for $x \neq 2$

Now substitute:
$\lim (x\to 2)\ (x + 2) = 4$

⚠️ Note: $f(2)$ itself doesn't exist (denominator was 0). But the limit exists, because as x gets very close to 2, the function behaves like $x + 2$.

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3. Case 3: Denominator → 0 but numerator ≠ 0

This is like $1/x$ near 0.
The numerator stays fixed, but denominator shrinks → result blows up.

- From right side (denominator positive small) → $+\infty$
- From left side (denominator negative small) → $-\infty$

Example:
$\lim_{x \to 4^+} (x - 2) / [(x - 4)(x + 2)]$

Numerator → $4 - 2 = 2$ (finite)
Denominator → very small positive (right side) or negative (left side)

As $x \to 4^+ \to +\infty$
As $x \to 4^- \to -\infty$

So, the two-sided limit does not exist in finite sense, but one-sided limits are $+\infty$ and $-\infty$.

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4. Case 4: $x \to +\infty$ or $x \to -\infty$

Trick: Divide numerator and denominator by the highest power of x.

Example:
$\lim_{x \to \infty} (2x^2 - 3) / (5x^2 + 4x)$

Divide by x^2:
$= \lim_{x \to \infty} (2 - 3/x^2) / (5 + 4/x)$

As $x \to \infty$, $3/x^2 \to 0$ and $4/x \to 0$.

So result = $2/5$

🚀 Quick Rules (as $x \to \pm\infty$)

Let deg(P) = n and deg(Q) = m

- If n < m → limit = 0
- If n = m → limit = ratio of leading coefficients
- If n > m → limit = ±∞ (sign depends on coefficients and direction)

🌟 Analogy Wrap-up

Think of denominator as the “floor”:

- If floor is solid (nonzero) → just plug in
- If floor disappears (0/0) → cancel common holes, then plug in
- If floor collapses but roof stays (nonzero/0) → you fall into infinity
- If building grows infinitely tall (x → ±∞) → only the top floor (highest degree term) matters