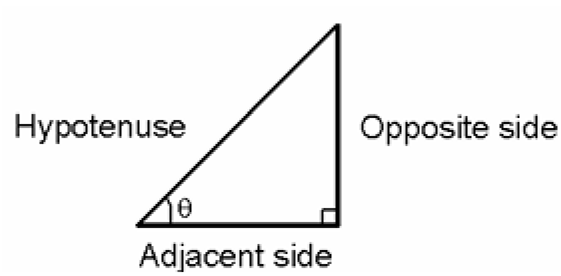
Continuity of Sine and Cosine

We all first meet **sine** and **cosine** inside a right triangle.

- **cos θ** = (adjacent side) / (hypotenuse)
- **sin θ** = (opposite side) / (hypotenuse)



So at first, they're just **ratios of sides** depending on an angle.

But later, we start thinking of them as **functions**:
you give them an angle (in radians), and they spit out a number.

That's why we now write them like:

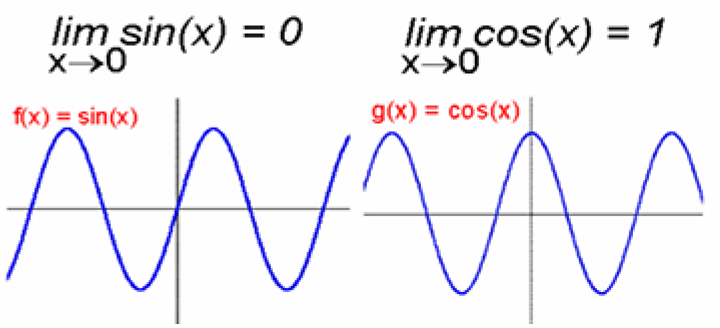
- f(x) = sin(x)
- f(x) = cos(x)

The Picture

Take the **unit circle** (a circle of radius 1). If you wrap a piece of string around it and then stretch that string out straight, you get the usual sine graph — the wavy one you've seen a million times.

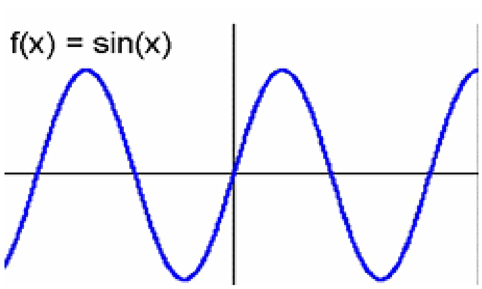
From that graph, two things are obvious:

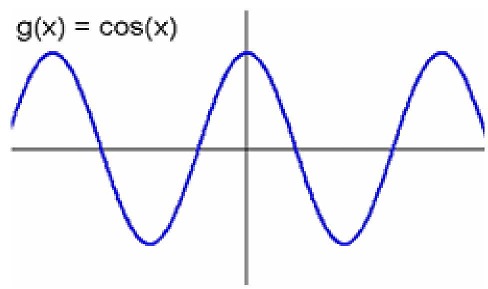
- As x → 0, sin(x) → 0
- As x → 0, cos(x) → 1



And we know exactly at x = 0:

- sin(0) = 0
- cos(0) = 1





So the **limit** and the **actual value** agree. That’s the whole idea of continuity!

Continuity: Reminder

A function f is continuous at some point c if three things hold:

1. f(c) is defined.

2. lim (x → c) f(x) exists.
3. lim (x → c) f(x) = f(c).

That's it. No jumps, no holes, no funny business.

Applying it to sin(x)

Let's check continuity of sine at some point `c`.
We want to show:

```
lim (x → c) sin(x) = sin(c).
```

Step 1: Change of perspective
Instead of working directly with `x → c`, we set

```
x = c + h
```

where `h → 0`.

Why do this?

- Because the real issue is not x itself, but *how far x is from c*.
- Writing `x = c + h` captures this "gap."
- As `x → c`, that gap `h → 0`.
  So this makes the limit easier to handle.

So the problem becomes:

```
lim (h → 0) sin(c + h).
```

Step 2: Use the sine addition formula

We know:

$$\sin(c + h) = \sin(c)\cos(h) + \cos(c)\sin(h).$$

This separates the “fixed part” ($\sin(c)$, $\cos(c)$) from the “tiny wiggle” ($\cos(h)$, $\sin(h)$).

Step 3: Take the limit as h → 0

- $\lim_{h \to 0} \cos(h) = 1$
- $\lim_{h \to 0} \sin(h) = 0$

So:

$$\lim_{h \to 0} \sin(c + h)$$

$$= \sin(c)\cdot(1) + \cos(c)\cdot(0)$$

$$= \sin(c).$$

Step 4: Wrap-up
That’s exactly what we needed:

$$\lim_{x \to c} \sin(x) = \sin(c).$$

✅ So sine is continuous everywhere.

👉 The substitution $x = c + h$ is just a clever way to zoom in near c and focus on the tiny change h.
It’s like saying: *“Forget the big picture — what happens if I nudge by a small step h?”*
And the trig identity shows that tiny step disappears smoothly, proving continuity.

What about cosine?

Same trick:
$\cos(c + h) = \cos(c)\cos(h) - \sin(c)\sin(h)$.

As $h \to 0$:

- $\cos(h) \to 1$
- $\sin(h) \to 0$

So:
$\lim (h \to 0) \cos(c + h)$
$= \cos(c)\cdot 1 - \sin(c)\cdot 0$
$= \cos(c)$. ✅

So cosine is continuous everywhere too.

Intuitive Analogy 🌊

Imagine sine and cosine as two smooth waves in the ocean. If you're standing at some point (say angle = c), the wave **doesn't suddenly break** or disappear under your feet. The limit as you approach that spot is exactly the height of the wave at that spot. That's continuity.

Theorem 2.8.1 (in plain words)

Both **sin(x)** and **cos(x)** are continuous functions — no holes, no jumps, no tears in their curves.

Continuity of Other Trigonometric Functions

We already know:

- **sin(x)** is continuous everywhere.
- **cos(x)** is continuous everywhere.

Now, what about the other trig functions?

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1. Tangent

By definition:
tan(x) = sin(x) / cos(x).

👉 Division rule: if f(x) and g(x) are continuous, then f(x)/g(x) is also continuous, **except where g(x) = 0** (because division by zero breaks the function).

Here, g(x) = cos(x).
So tan(x) is continuous **everywhere** except where cos(x) = 0.

Where is cos(x) = 0?

- At x = π/2, 3π/2, 5π/2, …
- More generally: x = (2n+1)π/2 for integers n.

So tan(x) is smooth everywhere, but it **jumps to infinity** at those vertical lines.

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2. Cotangent, Secant, Cosecant

- cot(x) = cos(x)/sin(x).
  👉 Continuous everywhere except sin(x) = 0 (multiples of π).
- sec(x) = 1/cos(x).
  👉 Continuous everywhere except cos(x) = 0 (odd multiples of π/2).

- csc(x) = 1/sin(x).
  👉 Continuous everywhere except sin(x) = 0 (multiples of π).

So each of these is “continuous on their allowed intervals,” but they break whenever the denominator vanishes.

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3. Why This Works

All of these are built from **sin(x)** and **cos(x)**, which we already know are continuous.
Then, by the theorem:
“If f(x) and g(x) are continuous, so is f(x)·g(x), and f(x)/g(x) (except where denominator = 0).”
That’s the backbone here.

The Squeeze Theorem and sin(x)/x

Here’s the famous limit:
lim (x → 0) [sin(x)/x] = 1.

This is a cornerstone result, used all the time in calculus.

Why is this tricky?

When x → 0:

- sin(x) → 0.
- x → 0.

So sin(x)/x looks like 0/0, an indeterminate form.
It’s like a **tug of war**: numerator and denominator are both shrinking to 0, so who wins?

The Idea of the Squeeze Theorem

If you can trap a function between two simpler functions, and those simpler ones have the same limit, then the trapped one must go there too.

Here's how it works for sin(x)/x.

Step 1: Geometric Setup

Imagine the unit circle (radius 1).
Take a small positive angle x (in radians).

- Arc length on the circle = x.
- sin(x) = vertical height.
- tan(x) = slope of the tangent line.

From geometry, we get:
$\sin(x) < x < \tan(x)$.

Step 2: Rewrite Inequalities

Divide everything by sin(x):
$1 < x/\sin(x) < 1/\cos(x)$.

Flip it around:
$\cos(x) < \sin(x)/x < 1$.

Step 3: Take Limits

As $x \to 0$:

- $\cos(x) \to 1$.
- The upper bound $\to 1$.

So sin(x)/x is squeezed to 1.

✅ Therefore:
$\lim (x \to 0) [\sin(x)/x] = 1$.

Feynman-style analogy

Think of sin(x)/x as a student caught between two teachers:

- Teacher 1 says “cos(x), come closer to 1.”
- Teacher 2 says “the constant 1.”

As $x \to 0$, both teachers agree on the value $\to 1$.
So the student (sin(x)/x) has no choice but to agree: it gets squeezed into 1.

The Squeeze Theorem (a.k.a. Sandwich Theorem)

Statement:

Suppose we have three functions g(x), f(x), h(x), and for all x near some point a:

$g(x) \leq f(x) \leq h(x)$.

If both the "outer" functions approach the same limit L as $x \to a$, i.e.

$\lim (x \to a)\ g(x) = \lim (x \to a)\ h(x) = L$,

then the "trapped" function f(x) must also go to L:

$\lim (x \to a)\ f(x) = L$. ✅

Feynman-style analogy

It's like you're squeezing a ball between two strong walls.
If the walls come together at the same spot L, the ball has nowhere else to go — it's forced to land at L too.

Example: Prove

$\lim (x \to 0)\ [\ (\sin(x))/x\ ] = 1$.

Step 1: A simpler warm-up

$\lim (x \to 0)\ (\sin^2(x))/x^2 = 0$

We know:
0 <= sin(x) <= 1

Squaring both sides:
0 <= sin^2(x) <= 1

Step 2: Build the fraction
We are interested in:
(sin^2(x)) / (x^2)

Since x^2 > 0 (near 0):
0 <= (sin^2(x)) / (x^2) <= 1 / (x^2)

Step 3: Refocus the inequality
We also know the key fact: |sin(x)| <= |x|
Squaring both sides:
sin^2(x) <= x^2

Divide through by x^2:
0 <= (sin^2(x)) / (x^2) <= 1

Step 4: Take the limit
As x -> 0:

- Left side -> 0
- Right side -> 1

So the middle is squeezed:
lim (x -> 0) (sin^2(x)) / (x^2) = 0

Why this works (plain words):

- Near 0, sin(x) is very small, even smaller than x.
- Squaring makes it shrink even faster.
- Compared to x^2, sin^2(x) is always smaller or equal.
- So the ratio (sin^2(x))/x^2 gets pushed down to 0. ✅

Step 2: The famous one — lim (x → 0) sin(x)/x

We want to prove:
$\lim_{x \to 0} \sin(x)/x = 1$.

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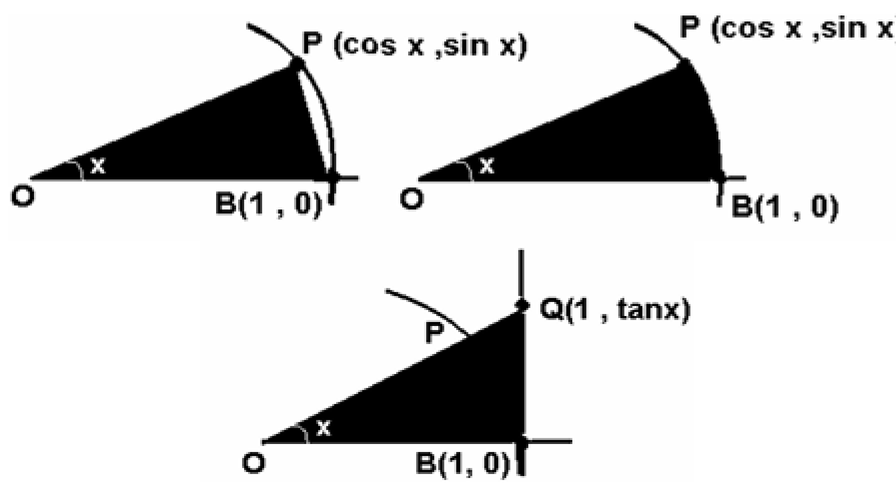
1. Why bring in geometry?

When algebra doesn't give you an easy handle (because sin(x)/x is a "0/0" indeterminate form), we use geometry as a guide.
The **unit circle** (radius = 1) is the perfect playground, because sin and cos are literally lengths on that circle.

Take an angle x (in radians). Then:

- The arc length that subtends angle x = **x**.
- The vertical height of that point on the circle = **sin(x)**.
- The tangent from that point to the x-axis = **tan(x)**.

So we have three "natural" lengths to compare: **sin(x), x, tan(x)**.



2. The geometric inequality

From the diagram (triangle inside the circle):
$\sin(x) < x < \tan(x)$, for $0 < x < \pi/2$.

That's intuitive:

- The vertical height (sin) is shortest.
- The arc (x) is in the middle.
- The tangent (tan) sticks out the most.

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3. Make it algebra-friendly

Take the inequality:
$\sin(x) < x < \tan(x)$.

Divide everything by sin(x):
$1 < x/\sin(x) < 1/\cos(x)$.

Now flip the inequality to put sin(x)/x in the middle (just take reciprocals carefully):
$\cos(x) < \sin(x)/x < 1$.

👉 This is the "squeeze setup": sin(x)/x is trapped between cos(x) and 1.

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4. Now take the limit

As $x \to 0$:

- $\cos(x) \to 1$.
- The upper bound $1 \to 1$.

So sin(x)/x has no escape — it must also go to 1. ✅

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5. Why this works (intuition)

Think of sin(x)/x as a little ratio comparing "how much height you gain" versus "how much angle you turn."

- At tiny angles, $\sin(x) \approx x$ (they both start off almost the same).

- The squeeze inequality formalizes this intuition: sin(x)/x never jumps out of the narrow cage between cos(x) and 1.
- As the cage closes to a single point (1), sin(x)/x is forced to land exactly there.

👉 So the whole proof is basically:
"Trap sin(x)/x between two simple friends (cos and 1), and as x → 0, those friends close in at 1."

Bonus: Behavior at infinity

As x → +∞ or x → –∞:

- sin(x) and cos(x) just keep oscillating between –1 and 1.
- So they don't settle to any single number.

👉 This means:
lim (x → ∞) sin(x) = DNE,
lim (x → ∞) cos(x) = DNE.

✅ Summary:
The **Squeeze Theorem** is the mathematical version of "if you box someone in from both sides, they have no freedom — they end up where you pin them."
It's the key to proving limits like sin(x)/x → 1.