🔹 Lecture 9: Limits

1. Why Limits?

Calculus was invented to answer **two big problems**:

1. **Tangent Problem** → How do we find the slope of a curve at a single point?
2. **Area Problem** → How do we find the exact area under a curve?

Both of these ideas lead to the same underlying concept: the **Limit**.

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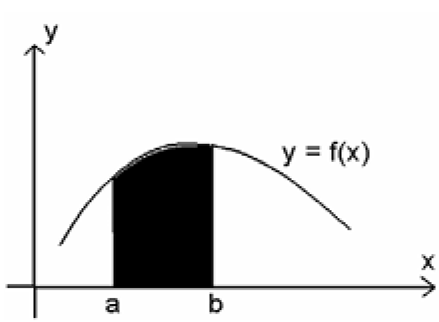
2. Area Problem → Integral Calculus

- Suppose you want the area under a curve y = f(x) between x = a and x = b.
- Easy if f(x) is a rectangle or triangle, but curves are tricky.
- The trick: chop the region into **many skinny rectangles**, add their areas, and then make the rectangles infinitely skinny.
- That "infinite process" is handled using **limits**.



👉 This becomes the foundation of **Integral Calculus**.

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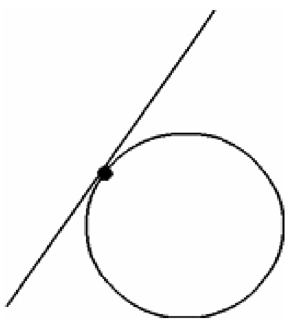
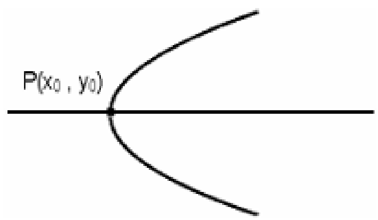
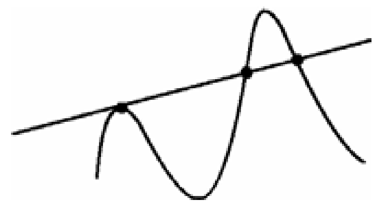
3. Tangent Problem → Differential Calculus

- In geometry, the tangent to a circle is a line that touches it at exactly one point.
- But for other curves (like sideways parabolas), just “touching once” is not enough.
  - Sometimes a line touches once but isn’t really tangent (wrong slope).
  - Sometimes a tangent line can touch at more than one point.







So we need a **better definition** of tangent.

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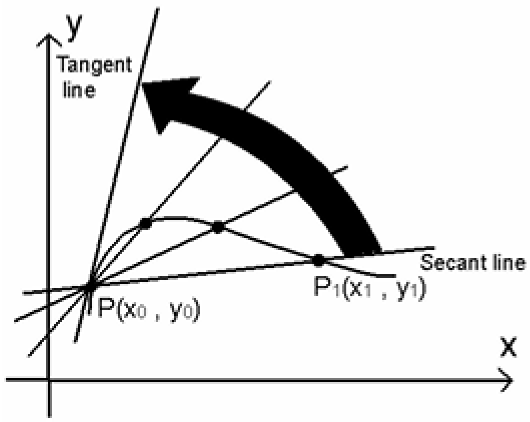
4. Secant Line → Tangent Line (using Limits)

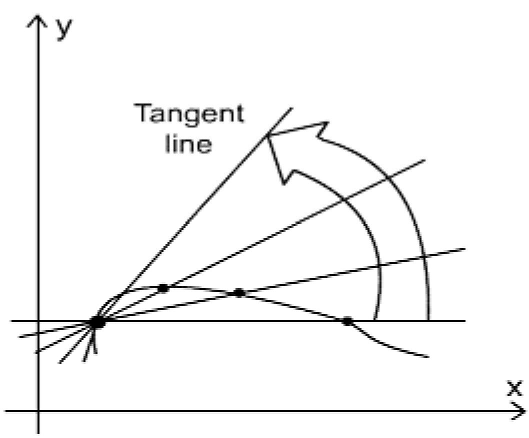
- Pick a point **P** on the curve.
- Pick another point **P1** on the same curve (not equal to P).
- Draw the line through P and P1 → this is called a **Secant Line**.



Now, slide P1 closer and closer to P:

- The secant line keeps changing slope.
- As P1 → P, the secant line approaches a **limiting position**.
- That limiting line is the **Tangent Line** at P.

👉 This works for **all curves**, not just circles.



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5. Why Limits are Essential

Without the idea of a limit, we can't:

- Define tangent slope at a single point.
- Define area under a curve using infinitely many rectangles.

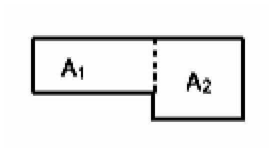
So, **Limits are the foundation** of all of calculus.

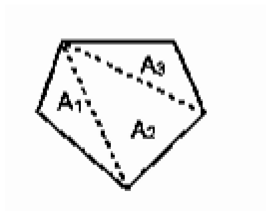
✅ **Summary (Plain words):**

- Limits let us handle “approaching” behavior.
- Tangents = limit of slopes of secant lines.
- Areas = limit of sums of many skinny rectangles.
- Differential Calculus = tangent problem.
- Integral Calculus = area problem.
- Both come from the same seed: the **Limit**.

Area as a Limit

For easy shapes like rectangles, triangles, and circles, we already know formulas for area. Why? Because we can cut them up into neat little pieces (like rectangles and triangles) that fit perfectly.





But when we look at a curved shape — for example, the area under y = f(x) from x = a to x = b — there is no simple "cut and paste" trick. The curve bends, so rectangles and triangles never fit exactly.

So what do we do? We **cheat smartly.**

1. **Approximation with rectangles**
    - Take the interval [a, b] on the x-axis.
    - Chop it into equal parts (like slicing bread).
    - Over each slice, stand a rectangle whose height is given by the curve.
    - Add up all these rectangles. That sum is an **approximate area**.
2. At this stage, it's not exact — it's like drawing a jagged staircase under a smooth curve.

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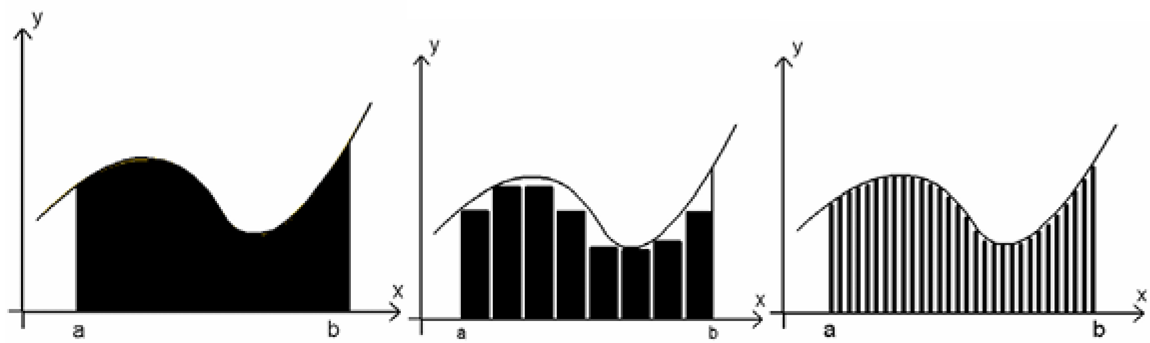
2. **Make it better by adding more rectangles**
    - If we use 4 rectangles, the picture is rough.
    - If we use 10 rectangles, the approximation is better.
    - If we use 100 rectangles, the rectangles fit so tightly that the "stairs" almost disappear.

3. In other words, the more slices we use, the closer we get to the true area.



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3. **The magic of the limit**
    - Imagine pushing this to infinity: infinitely many rectangles, each infinitely thin.
    - Now, the jagged edges vanish completely.
    - What we're left with is the **exact area under the curve.**

This final value — the one we approach as the number of rectangles goes to infinity — is called the **limit**.

⚡ **Feynman Intuition**

- First, we guess with clumsy blocks.
- Then, we improve by slicing thinner.
- Finally, we let the slices become infinitely thin — and boom, that's the exact area.

So the limit is just a clever way of adding infinitely many "tiny pieces" to measure something curved that we can't handle with simple geometry.

Understanding Limits

1. The Big Idea

*A **limit** is basically a question:*

👉 *"As x gets closer and closer to some number, what does f(x) get closer and closer to?"*

Notice: we're not asking what happens at the number itself.
We're asking what happens nearby.

It's like sneaking up to someone's house:

- *You don't step inside the door.*
- *You just keep walking closer and closer.*
- *The limit is: Where does your path lead?*

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2. Reading the Notation

- **lim (x → a) f(x)**
  Read this as: *"The value f(x) approaches, as x approaches a."*
- **x → a⁺** means "x approaches a from the right side" (bigger than a, sliding down).
- **x → a⁻** means "x approaches a from the left side" (smaller than a, sliding up).

So:

- **lim (x → 0⁻) f(x)** → limit from the left.
- **lim (x → 0⁺) f(x)** → limit from the right.

If both sides agree → the limit exists.
If they don't → no limit.

2. Example: f(x) = sin(x)/x

We look at:
$f(x) = \sin(x) / x$

⚠️ Problem: At $x = 0$ → denominator = 0, so the function is **not defined**.

Question: *But what happens as x gets very close to 0?*

3. Approaching from Both Sides

- If x comes from **negative side** (left of 0), we call it the **left-hand limit**:
  $\lim (x \to 0^-) \sin(x)/x$
- If x comes from **positive side** (right of 0), we call it the **right-hand limit**:
  $\lim (x \to 0^+) \sin(x)/x$

Both mean: we're sliding toward 0, but from opposite directions.

4. What We See

If you plug small values:

- For $x = 0.1 \to \sin(0.1)/0.1 \approx 0.998$
- For $x = 0.01 \to \sin(0.01)/0.01 \approx 0.9999$
- For $x = -0.1 \to \sin(-0.1)/(-0.1) \approx 0.998$
- For $x = -0.01 \to \sin(-0.01)/(-0.01) \approx 0.9999$

From both left and right, the value is squeezing toward **1**.

So:
$\lim (x \to 0^-) \sin(x)/x = 1$
$\lim (x \to 0^+) \sin(x)/x = 1$

Since they match →
lim (x → 0) sin(x)/x = 1

X	$f(x) = \frac{\sin(x)}{x}$
1.0	0.84147
0.8	0.89670
0.6	0.94107
0.4	0.97355
0.2	0.99335
0.01	0.99998

X	$f(x) = \frac{\sin(x)}{x}$
-1.0	0.84147
-0.8	0.89670
-0.6	0.94107
-0.4	0.97355
-0.2	0.99335
-0.01	0.99998

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5. The Principle

👉 If **left-hand limit** = **right-hand limit**, the limit exists.
👉 If they're different, the limit does not exist.

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6. Why It Matters

This idea of "approaching a value" is the **engine of calculus**:

- To define **tangent lines**, we need limits.
- To define **areas under curves**, we need limits.

Without limits, calculus would fall apart.

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7. A Warning: Limits Can Mislead

The Setup

Take this limit:
lim (x → 0) sin(1/x)

At first, it looks innocent. "What's the harm? Just plug in values close to 0 and see what happens."
But here's the trap.

Step 1: What happens when x → 0?

If x is very close to 0, say 0.001, then **1/x = 1000.**
If x is even closer, say 0.000001, then **1/x = 1,000,000.**

So as x → 0, the inside of the sine (1/x) doesn't go to 0…
👉 It shoots off toward infinity.

Step 2: Now feed that into sin(1/x)

What does sine do when its input gets larger and larger?

- sine is periodic → it keeps cycling between –1 and +1, no matter how big the input gets.
- So sin(1/x) bounces: up, down, up, down … forever.

When x is near 0, 1/x is gigantic, so sine is cycling insanely fast.

Step 3: Why this kills the limit

Normally, when you zoom in on a limit, the function values start to "settle" toward some number.

- Example: sin(x)/x near 0 → it squeezes toward 1.
- But here, sin(1/x) never settles.

Zoom in as much as you like, the graph still bounces between –1 and +1.

Step 4: Why numbers alone mislead

If you test just a few numbers, you might think it has a limit:

- Try x = 0.1 → sin(10) ≈ –0.544
- Try x = 0.01 → sin(100) ≈ –0.506

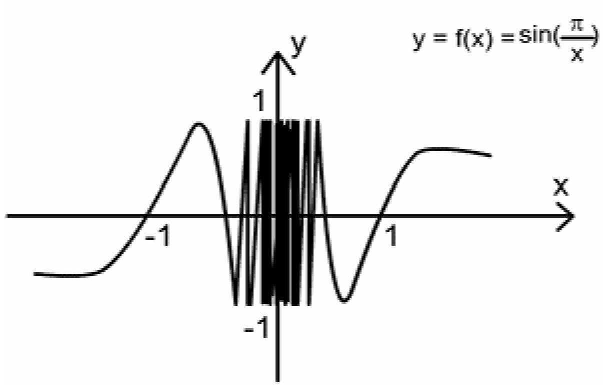
- Try x = 0.001 → sin(1000) ≈ 0.826

See the problem?
The numbers don't "move closer" to anything — they jump around.
If you only checked one or two values, you could mistakenly think it's heading somewhere.

Step 5: The truth

Because sin(1/x) oscillates infinitely fast as x → 0, the function has **no single value to settle on.**

So we say:
👉 The limit does not exist (DNE).



✅ **Summary in Plain Words**:

- A limit is about "where the function is heading" as x approaches something.
- If both left and right sides head to the same spot → limit exists.
- If they disagree or oscillate → no limit.
- sin(x)/x near 0 → goes to 1.
- sin(1/x) near 0 → never settles → no limit.

TABLE of Limit Notations and situations

NOTATION	HOW TO READ THE NOTATION
$\lim_{x \to x_0^+} f(x) = L_1$	The limit of f(x) as x approaches x_0 from the right is equal to L_1
$\lim_{x \to x_0^-} f(x) = L_2$	The limit of f(x) as x approaches x_0 from the left is equal to L_2
$\lim_{x \to x_0} f(x) = L$	The limit of f(x) as x approaches x_0 is equal to L

Existence (and Non-Existence) of Limits1.

The Core Idea

A **limit exists** if, as x creeps closer and closer to some number, the function's output (y-values) settle near one single value.

👉 Think of it like walking toward a door:

- If you always end up at the same door no matter how you approach, the limit exists.
- If the hallway stretches to infinity, you never reach a door — that's like $+\infty$ or $-\infty$.
- If sometimes you end up at one door, and sometimes at a different one, then there's no clear destination. That's **DNE (Does Not Exist)**.

So: **limits are about destinations, not the exact journey.**

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2. Why Limits Can Fail

Two main "troublemakers":

1. **Oscillation** → the function bounces around endlessly without settling.
   Example: $\sin(1/x)$ near $x = 0$. It wiggles between -1 and $+1$ infinitely fast.
2. **Unbounded Growth** → the function shoots off toward $+\infty$ or $-\infty$.
   Example: $1/x^2$ near 0. The closer you get to 0, the higher it climbs.

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3. Case 1: Limit Blows Up to $+\infty$

Take **$f(x) = 1/x^2$** as $x \to 0$.

- From the right ($x \to 0^+$), numbers like 0.1, 0.01, 0.001 make $1/x^2$ grow bigger: 100, 10,000, 1,000,000…
- From the left ($x \to 0^-$), negative numbers squared become positive, so same thing happens.

Both sides agree: the function rockets upward forever.

We write:
$\lim_{x \to 0} 1/x^2 = +\infty$

⚠️ But careful → $+\infty$ is **not a number**, it's just a way of saying "no ceiling, it grows without bound."

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4. Case 2: Limit Blows Down to –∞

Now flip the sign: **$g(x) = -1/x^2$**.

- As $x \to 0$ from either side, $1/x^2$ shoots up… but the minus sign flips it downward.
- So values become –100, –10,000, –1,000,000…

We write:
$\lim_{x \to 0} -1/x^2 = -\infty$

Again, $-\infty$ isn't a number — it's just "falling without end."

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5. Case 3: Left & Right Don't Match

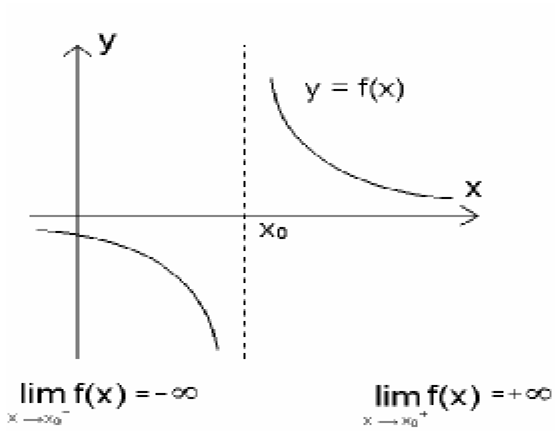
Now imagine a function like:

- As $x \to 0^+$, $f(x) \to +\infty$
- As $x \to 0^-$, $f(x) \to -\infty$

One side climbs, the other side dives. They disagree.

👉 If the two sides don't match, the **two-sided limit does not exist (DNE)**.

Think: you're walking to a meeting place. From one road, you end up on the rooftop; from another, you end up in the basement. No agreement → no single destination.



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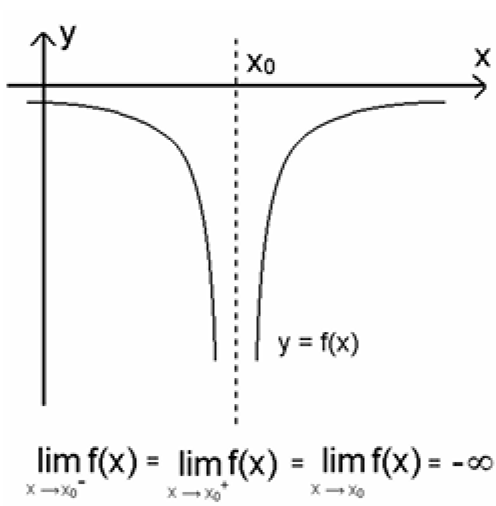
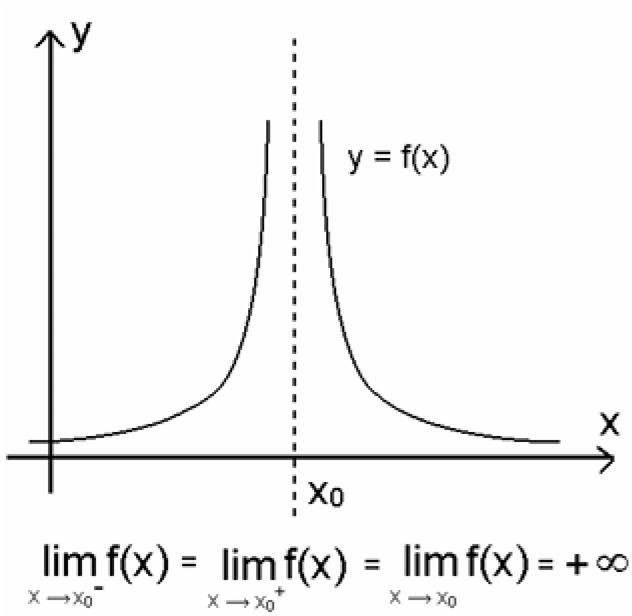
6. Limits at Infinity

So far we looked at x approaching a point. What if **x itself goes to infinity**?

- **x** → **+∞** means x grows larger and larger forever.
- **x** → **–∞** means x goes more and more negative without bound.





Example:
$f(x) = (2x + 3)/(x + 1)$

- For huge x, the "+3" and "+1" barely matter.
- Numerator ≈ 2x, denominator ≈ x.
- So f(x) ≈ 2.

We say:
$\lim (x \to +\infty)\ (2x + 3)/(x + 1) = 2$

That's like the graph flattening out near a horizontal line (called a **horizontal asymptote**).

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7. Oscillation but Still Settling

Sometimes oscillation doesn't ruin everything.

Take $f(x) = \sin(x)/x$ as $x \to +\infty$.

- sin(x) keeps bouncing between –1 and +1 forever.
- But dividing by x squashes the swings smaller and smaller.

- Eventually, the oscillations fade out to 0.

So:
lim (x → +∞) sin(x)/x = 0

Here, the “noise” dies out in the distance.

✅ Summary in Plain Words

- A limit exists if the function approaches **one single destination**.
- +∞ and −∞ describe direction, not actual numbers.
- If left and right don’t match → no limit.
- At infinity, limits tell us about the “end behavior” of a curve.
- Oscillations can break a limit (sin(1/x)), but sometimes they calm down (sin(x)/x).

⚡ Think of limits like **watching where a car is heading**:

- If both roads lead to the same garage → limit exists.
- If the car drives upward forever → +∞.
- If it drives downward forever → −∞.
- If one road goes up and the other down → no agreement, no limit.
- If it wiggles but eventually parks → the limit is where it parked.