LECTURE 29 — THE DEFINITE INTEGRAL

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1. BIG PICTURE: WHAT ARE WE REALLY DOING?

So far, we learned how to find **area under a curve** by cutting the region into rectangles and adding their areas.

Now we want a **clean, universal definition** of area that works for:

- equal-width rectangles
- unequal-width rectangles
- curves above the x-axis
- curves below the x-axis
- curves that cross the x-axis
- even some discontinuous functions

This final, powerful idea is called the **Definite Integral**.

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2. WHY DO WE NEED A NEW DEFINITION?

Earlier, we divided [a, b] into n equal parts.

Delta x = (b − a) / n

As n → infinity, delta x → 0
This worked nicely.

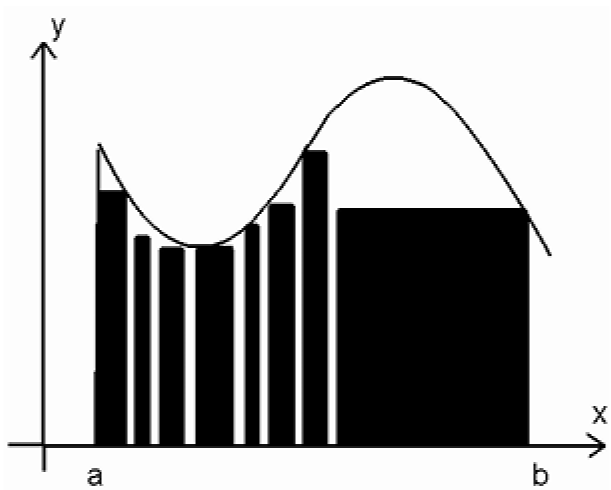
But what if rectangle widths are NOT equal?

Example:

- Left half keeps getting divided into smaller rectangles
- Right half stays one big rectangle

Then even if n → infinity:

- Left widths → 0
- Right width DOES NOT → 0



That's a problem.

So we fix it.

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3. THE KEY FIX: MESH SIZE

Instead of caring about "n", we care about the **largest rectangle width**.

We divide [a, b] into subintervals with widths:
delta x1, delta x2, delta x3, …, delta xk

Now define:

Mesh size = maximum(delta xk)

If the **largest width goes to zero**, then:
ALL rectangle widths go to zero.

That guarantees accuracy.

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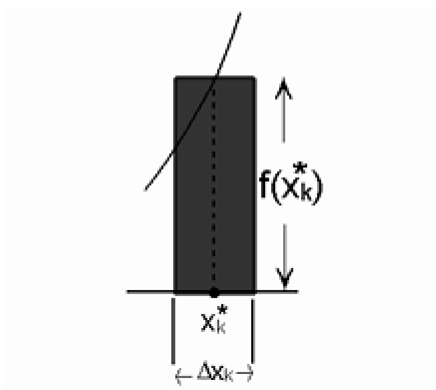
4. BUILDING THE AREA STEP BY STEP

For each subinterval:

- pick ANY point inside it (left, right, midpoint — doesn't matter)
- call it xk*

Height of rectangle = f(xk*)
Width of rectangle = delta xk

Area of one rectangle = f(xk*) * delta xk



Area of k^{th} rectangle = $f(x_k^*) \,.\, \Delta x_k$

Total area approximation:

Sum from k = 1 to n of:
f(xk*) * delta xk

This is called a **Riemann Sum**.

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5. THE DEFINITION OF AREA (IMPORTANT)

If f is continuous and f(x) ≥ 0 on [a, b]:

Area under y = f(x) from a to b is:

Limit as mesh size → 0 of
Sum of f(xk*) * delta xk

This limit is what we mean by **area**.

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6. THIS LIMIT HAS A NAME

This limit is written as:

Integral from a to b of f(x) dx

Read as:
"Definite integral of f from a to b"

So:

Integral from a to b of f(x) dx
= Area under the curve y = f(x) on [a, b]

This is NOT an antiderivative.
This is a NUMBER.

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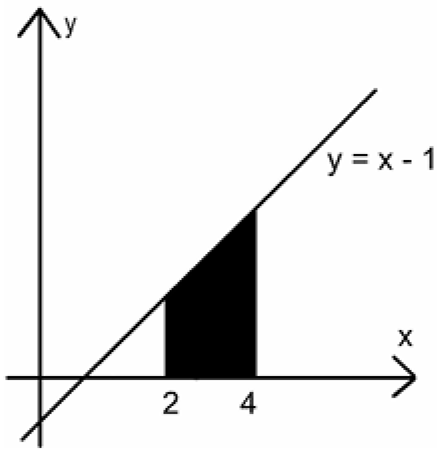
7. SIMPLE GEOMETRIC EXAMPLE

Integral from 2 to 4 of (x − 1) dx



This represents area under y = x − 1 from x = 2 to x = 4.

That region is a trapezoid.

Height = 2
Bases = 1 and 3

Area = (1/2) * height * (sum of bases)
Area = (1/2) * 2 * (1 + 3) = 4

So the definite integral equals 4.

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8. WHAT IF THE CURVE GOES BELOW THE X-AXIS?

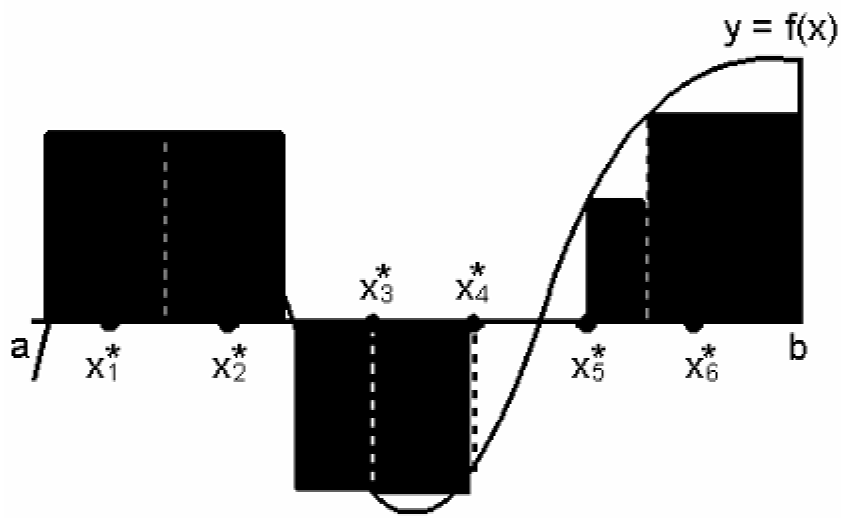
Now things get interesting.

Rectangles above x-axis → positive area
Rectangles below x-axis → negative area

So the definite integral gives **signed area**:

Signed area = (area above x-axis) − (area below x-axis)

If more area is below, the integral becomes negative.



9. EXAMPLE WITH NEGATIVE AREA

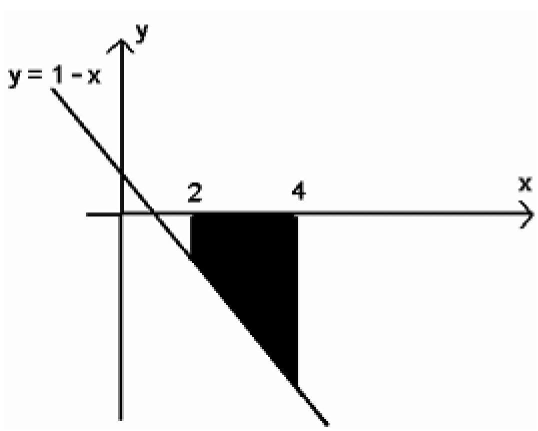
Integral from 2 to 4 of (1 − x) dx

The graph lies below the x-axis.

Geometric area = 4
But since it is below the x-axis:

Definite integral = −4

Negative sign just tells us the region is below the axis.



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10. FUNCTIONS WITH DISCONTINUITIES

If a function is:

- continuous → always integrable
- bounded with only finitely many discontinuities → integrable
- unbounded → NOT integrable

If the Riemann sum limit exists, the function is called:
Riemann integrable

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11. SPECIAL CASES OF LIMITS

Integral from a to a of f(x) dx = 0

Integral from b to a of f(x) dx
= − integral from a to b of f(x) dx

Changing direction flips the sign.

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12. BASIC PROPERTIES OF DEFINITE INTEGRALS

If c is a constant:

Integral of c f(x) dx = c * integral of f(x) dx

Integral of (f + g) dx = integral of f dx + integral of g dx

Integral of (f − g) dx = integral of f dx − integral of g dx

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13. SPLITTING INTERVALS

If c lies between a and b:

Integral from a to b of f(x) dx
= integral from a to c + integral from c to b

This works no matter the order of a, b, c.

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14. INEQUALITIES WITH INTEGRALS

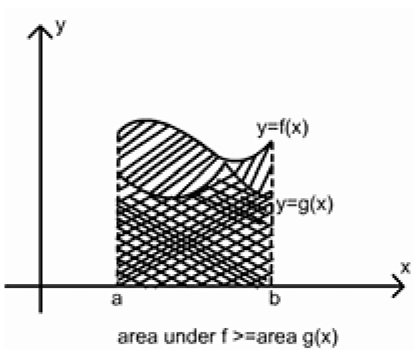
If f(x) ≥ 0 on [a, b]:

Integral from a to b of f(x) dx ≥ 0

If f(x) ≥ g(x) on [a, b]:

Integral of f(x) dx ≥ integral of g(x) dx

Integrals respect inequalities.



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15. BOUNDED FUNCTIONS

A function is bounded on [a, b] if:

There exists M such that:
−M ≤ f(x) ≤ M for all x in [a, b]

Geometrically:
The graph fits between two horizontal lines.

Bounded + finite discontinuities → integrable.

16. FINAL FEYNMAN INSIGHT

A definite integral is NOT magic.

It is simply this idea:

“Add up infinitely many tiny rectangles,
and let the widest one shrink to zero.”

That single idea:

- defines area
- handles negatives
- handles discontinuities
- builds the foundation of calculus

Once you understand that,
everything else is just technique.