✳️ DERIVATIVE OF f(x) = sin(x)

We want to find the derivative of $f(x) = \sin(x)$.
In other words, we're asking:
"How fast is sine changing at any given angle x?"

🧩 Step 1: Start from the Definition of Derivative

By definition,
$f'(x) = \lim (h \to 0) [f(x + h) - f(x)] / h$

Since $f(x) = \sin(x)$, we can write:
$f'(x) = \lim (h \to 0) [\sin(x + h) - \sin(x)] / h$

🧠 Step 2: Expand Using the Trigonometric Formula

We know from trigonometry that:
$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

So,
$\sin(x + h) = \sin(x)\cos(h) + \cos(x)\sin(h)$

Substitute this into our derivative:

$f'(x) = \lim (h \to 0) [\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)] / h$

🧮 Step 3: Rearrange Terms

Group the sin(x) terms together:

$f'(x) = \lim (h \to 0) [\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)] / h$

Now, split the fraction into two parts:

$f'(x) = \sin(x) * [(\cos(h) - 1) / h] + \cos(x) * [\sin(h) / h]$

🧩 Step 4: Apply Two Famous Trig Limits

We know these standard limits:

1. lim (h → 0) [sin(h) / h] = 1
2. lim (h → 0) [(cos(h) – 1) / h] = 0

Now substitute these into the expression:

f'(x) = sin(x) * (0) + cos(x) * (1)

Simplify it:

f'(x) = cos(x)

✅ Final Result

The derivative of sin(x) is:

f'(x) = cos(x)

💡 Intuitive Explanation (Feynman Style)

Think of sin(x) as a **wave** that rises and falls smoothly.
At the start (x = 0), sin(x) begins to rise — its slope is **1**,
and that's exactly what **cos(x)** equals at x = 0.

Whenever sin(x) is at its highest point (like 1), it stops rising — its slope becomes **0**,
and cos(x) is also **0** there.

So cos(x) acts like the "speed" or "slope tracker" of sin(x).
Whenever sin(x) increases or decreases, cos(x) tells us how fast.

In other words:
cos(x) is the rate of change of sin(x).

🧮 Quick Verification Example

Let’s test at a few points:

x	sin(x)	f'(x) = cos(x)	Meaning
0	0	1	Sine starts rising fast
π/2	1	0	Sine stops increasing
π	0	–1	Sine falls sharply
3π/2	–1	0	Sine stops decreasing

You can literally “see” cos(x) describe how the slope of sin(x) changes over time.

🧠 Memory Trick

“Derivative of sin is cos — they’re dance partners.”

Whenever sin(x) leads, cos(x) follows —
one describes the motion, the other describes the speed.

✅ **Final Formula:**
If f(x) = sin(x), then
f′(x) = cos(x)

✳️ DERIVATIVE OF f(x) = cos(x)

We already found that the derivative of sin(x) is cos(x).
Now let's discover how the slope behaves for the **cosine** function.

🧩 Step 1: Start from the Definition

By definition,
$f'(x) = \lim_{h \to 0} [f(x + h) – f(x)] / h$

For f(x) = cos(x), we have:
$f'(x) = \lim_{h \to 0} [\cos(x + h) – \cos(x)] / h$

🧠 Step 2: Expand Using the Trig Formula

We know that:
$\cos(A + B) = \cos(A)\cos(B) – \sin(A)\sin(B)$

So,
$\cos(x + h) = \cos(x)\cos(h) – \sin(x)\sin(h)$

Substitute this into the derivative:

$f'(x) = \lim_{h \to 0} [\cos(x)\cos(h) – \sin(x)\sin(h) – \cos(x)] / h$

🧮 Step 3: Group and Simplify

Group the cos(x) terms together:

$f'(x) = \lim_{h \to 0} [\cos(x)(\cos(h) – 1) – \sin(x)\sin(h)] / h$

Split the terms:

$f'(x) = \cos(x) * [(\cos(h) – 1) / h] – \sin(x) * [\sin(h) / h]$

🧩 Step 4: Apply the Two Standard Limits

We already know:
lim (h → 0) [sin(h) / h] = 1
lim (h → 0) [(cos(h) – 1) / h] = 0

Substitute them in:

f'(x) = cos(x)(0) – sin(x)(1)

Simplify:
f'(x) = –sin(x)

✅ Final Result

If f(x) = cos(x),
then f′(x) = –sin(x)

💡 Intuitive Explanation (Feynman Style)

Think of sin(x) and cos(x) as **wave partners** —
when one rises, the other falls.

At x = 0, cos(x) = 1, but it starts decreasing immediately —
so the slope is **negative** there.
That's why we get the minus sign.

In short:
cos(x) tells how high we are,
–sin(x) tells how fast we're coming down.

🧠 Memory Trick

If derivative of sin(x) = cos(x),
then derivative of cos(x) = –sin(x).

They're the same pattern — just one step out of phase.

✅ **Final Formula:**
f'(x) = –sin(x)

✳️ DERIVATIVE OF $f(x) = \tan(x)$

Now let's find the derivative of $\tan(x)$.

We *could* start from the definition,
but that gets messy with expansions of sin and cos.
Instead, let's use a clever shortcut — the **Quotient Rule**.

🧩 Step 1: Write $\tan(x)$ as a Quotient

We know that:
$\tan(x) = \sin(x) / \cos(x)$

Let $f(x) = \sin(x)$ and $g(x) = \cos(x)$

Then,
$\tan(x) = f(x) / g(x)$

🧠 Step 2: Apply the Quotient Rule

The Quotient Rule says:

$d/dx\ [\ f(x) / g(x)\ ] = [\ g(x)f'(x) – f(x)g'(x)\ ] / [\ g(x)\ ]^2$

Now,
$f'(x) = \cos(x)$
$g'(x) = –\sin(x)$

Substitute these values:

$d/dx\ [\ \sin(x) / \cos(x)\ ]$
$= [\ \cos(x) * \cos(x) – \sin(x) * (–\sin(x))\ ] / [\ \cos(x)\ ]^2$

Simplify the numerator:

$= [\ \cos^2(x) + \sin^2(x)\ ] / \cos^2(x)$

🧩 Step 3: Use the Pythagorean Identity

We know:
$\sin^2(x) + \cos^2(x) = 1$

So:
$f'(x) = 1 / \cos^2(x)$

✅ Final Result

The derivative of tan(x) is:

$f'(x) = \sec^2(x)$

(because $\sec(x) = 1 / \cos(x)$)

💡 Intuitive Explanation (Feynman Style)

Think of tan(x) as **sine divided by cosine**.
As cosine gets smaller (approaching 0), tan(x) shoots up —
its slope grows incredibly fast near vertical asymptotes (like $x = \pi/2$).

That's why the derivative is **$\sec^2(x)$** —
it grows *much faster* when cos(x) becomes small.

🧠 Memory Trick

"Derivative of tan is $\sec^2$" —
like saying "tan stands tall — its slope squares up!"

You can also remember:
sin → cos
cos → –sin
tan → $\sec^2$
sec → sec·tan
cot → $-\csc^2$
csc → –csc·cot

✅ Final Formulas Summary

Function	Derivative
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$

DERIVATIVE OF $f(x) = \sec(x)$

Recall: $\sec(x) = 1 / \cos(x)$.

Use the reciprocal rule (or quotient rule). If $g(x) = \cos(x)$, then
$d/dx [1 / g(x)] = - g'(x) / [g(x)]^2$.

Here $g(x) = \cos(x)$ and $g'(x) = -\sin(x)$. So

$d/dx [\sec(x)] = d/dx [1 / \cos(x)]$
$= - (-\sin(x)) / [\cos(x)]^2$
$= \sin(x) / \cos^2(x)$.

Rewrite $\sin/\cos^2$ as $(1/\cos) * (\sin/\cos) = \sec(x) * \tan(x)$.

Therefore:
$d/dx [\sec(x)] = \sec(x) * \tan(x)$.

Intuition (Feynman-style): sec(x) is 1/cos(x). When cos decreases a bit, its reciprocal increases; the extra factor tan(x) appears because the slope depends on both the size of cos and how fast sin/cos is changing.

DERIVATIVE OF $f(x) = \csc(x)$

Recall: $\csc(x) = 1 / \sin(x)$.

Apply the reciprocal rule with $g(x) = \sin(x)$, $g'(x) = \cos(x)$:

$d/dx [\csc(x)] = d/dx [1 / \sin(x)]$
$= - g'(x) / [g(x)]^2$
$= - \cos(x) / \sin^2(x)$.

Rewrite as $-(1/\sin) * (\cos/\sin) = - \csc(x) * \cot(x)$.

Therefore:
$d/dx [\csc(x)] = - \csc(x) * \cot(x)$.

Intuition: same idea — reciprocal of sine decreases when sine grows, and the extra cot factor captures the relative rates.

DERIVATIVE OF $f(x) = \cot(x)$

Recall: $\cot(x) = \cos(x) / \sin(x) = 1 / \tan(x)$.

Use quotient rule or reciprocal rule. Using quotient rule with f = cos, g = sin:

d/dx [cot(x)] = [sin(x) * (−sin(x)) − cos(x) * cos(x)] / sin^2(x)
= [−sin^2(x) − cos^2(x)] / sin^2(x)
= − [sin^2(x) + cos^2(x)] / sin^2(x)
= − 1 / sin^2(x) (since sin^2 + cos^2 = 1)
= − csc^2(x).

Therefore:
d/dx [cot(x)] = − csc^2(x).

Intuition: cot is cos/sin; both numerator and denominator change and the result is always negative (because cot decreases where sine increases), with magnitude 1/sin^2 scaling the change.

SUMMARY OF TRIG DERIVATIVES (plain list)

d/dx [sin(x)] = cos(x)
d/dx [cos(x)] = −sin(x)
d/dx [tan(x)] = sec^2(x)
d/dx [sec(x)] = sec(x)·tan(x)
d/dx [csc(x)] = −csc(x)·cot(x)
d/dx [cot(x)] = −csc^2(x)

(Remember: these formulas are valid when the arguments are measured in RADIANS — see note below.)

WHY ANGLES MUST BE IN RADIANS (short, essential explanation)

The basic limit lim(h→0) sin(h)/h = 1 is true *only when h is measured in radians*. That limit is used in the derivative calculations for sin and cos. If you measure angles in degrees, the limit becomes sin(h°)/h° = (π/180) (as h°→0) and you get extra constant factors. In short:

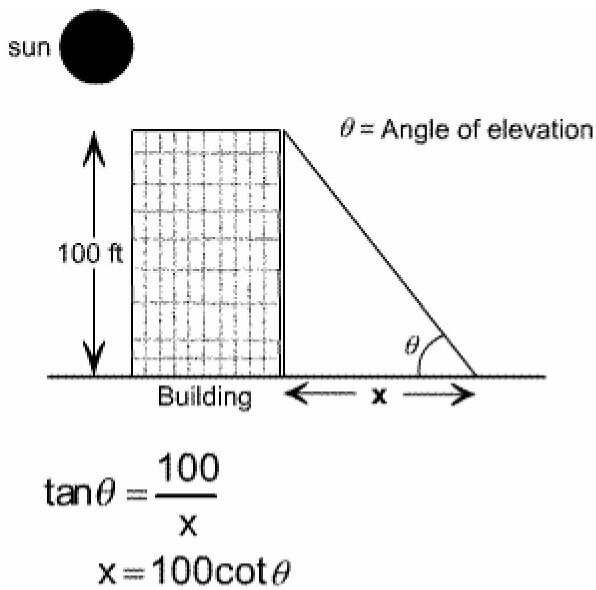
- Derivative formulas (like d/dx[sin x] = cos x) rely on sin(h)/h → 1 as h→0.
- That identity is true for radian measure, so trig derivatives above assume x is in radians.
- If x were in degrees, every trig derivative would include a constant factor (π/180) or its powers — messy and incorrect unless adjusted.

Bottom line: use radians when differentiating trig functions.

APPLICATION EXAMPLE — BUILDING SHADOW

Problem: A building is 100 feet high. Let θ be the sun's angle of elevation and x be the length of the building's shadow. Find dx/dθ (rate of change of shadow length with respect to θ) when θ = 45°, and give the answer in feet per degree.

Geometry relation:
tan(θ) = opposite/adjacent = 100 / x → x = 100 / tan(θ) = 100 · cot(θ).



Differentiate with respect to θ (use radians while differentiating):
dx/dθ = 100 · d/dθ [cot(θ)] = 100 · (−csc^2(θ)) = −100 · csc^2(θ). (units: feet per radian)

Evaluate at θ = 45° = π/4 radians:
sin(π/4) = √2/2 → csc(π/4) = 1 / sin(π/4) = √2 → csc^2(π/4) = 2.

So:
dx/dθ (at θ = π/4) = −100 · 2 = −200 feet per radian.

Convert to feet per degree:
1 degree = π/180 radians, so
dx/d(degree) = dx/d(radian) * (radian per degree)
= (−200 ft/radian) * (π / 180) rad/degree
= −200π / 180 ft/degree
= −10π / 9 ft/degree (exact form)

Numeric approx: −10π/9 ≈ −3.49 ft/degree.

Interpretation: at 45°, as the sun's elevation angle increases by 1 degree, the shadow shortens by about 3.49 feet (negative sign = shadow length decreasing).