LECTURE 30

FIRST FUNDAMENTAL THEOREM OF CALCULUS

BIG PICTURE: WHY THIS THEOREM IS SPECIAL

Before this theorem, we had TWO separate ideas:

1. Area under a curve (definite integrals)
2. Anti-derivatives (indefinite integrals)

They looked related, but no one had proven the connection.

The First Fundamental Theorem of Calculus does something magical:

It proves that **finding area** and **finding anti-derivatives** are actually the SAME story.

This is why calculus works.

STATEMENT OF THE FIRST FUNDAMENTAL THEOREM

If:

- f is continuous on the interval [a, b]
- F is any anti-derivative of f on [a, b]

Then:

Integral from a to b of f(x) dx
 = F(b) − F(a)

Meaning:

To compute area under f(x):

1. Find an anti-derivative F(x)

2. Plug in the limits
3. Subtract

That’s it.

WHAT THIS REALLY MEANS (INTUITION)

Instead of:

- slicing into rectangles
- taking limits
- adding infinitely many terms

We can:

- find ONE function (the anti-derivative)
- evaluate it at the endpoints

This is a HUGE shortcut — and it is mathematically justified.

WHY THIS WORKS (IDEA OF THE PROOF)

We won’t drown in symbols. Let’s understand the logic.

Step 1: Break the interval [a, b] into many small pieces.

$a = x_0 < x_1 < x_2 < \dots < x_n = b$

Each small piece has width:
delta $x_k = x_k - x_{k-1}$

Step 2: Use Mean Value Theorem on EACH small interval

Since:
$F'(x) = f(x)$

The Mean Value Theorem says:

On each small interval, there exists a point xk* such that:

$F(x_k) - F(x_{k-1}) = f(x_k^*) * \text{delta } x_k$

This looks EXACTLY like:
(height) × (width)

In other words:
Each small change in F equals the area of one rectangle.

Step 3: Add all the small pieces

Add all intervals together:

Left side:
$F(b) - F(a)$

Right side:
Sum of $f(x_k^*) * \text{delta } x_k$

This sum is a Riemann sum — an approximation of area.

Step 4: Take the limit

As rectangles get thinner:

- delta x → 0
- Riemann sum → exact area

So:

$F(b) - F(a)$
= Integral from a to b of f(x) dx

That completes the idea of the proof.

WHY CONTINUITY MATTERS

Continuity guarantees:

- Mean Value Theorem works
- Riemann sums converge
- Area behaves nicely

No continuity → no guarantee.

RELATIONSHIP BETWEEN DEFINITE AND INDEFINITE INTEGRALS

Important fact:

If F(x) is an anti-derivative of f(x),
then so is F(x) + C.

But when we compute:

F(b) − F(a)

The constant C cancels out:

[F(b) + C] − [F(a) + C] = F(b) − F(a)

So:

ANY anti-derivative works.

That’s why:

- constants do not matter in definite integrals
- we often set C = 0

VERY IMPORTANT RESULT

Integral from a to b of f(x) dx
= [F(x)] evaluated from a to b

This links:

- indefinite integrals (anti-derivatives)
- definite integrals (area)

They are two sides of the same coin.

EXAMPLE 1

Evaluate:

Integral from 1 to 2 of x dx

Anti-derivative of x is:
F(x) = x^2 / 2

Apply limits:

F(2) − F(1)
= (4 / 2) − (1 / 2)
= 2 − 0.5
= 1.5

So the area is 3/2.

EXAMPLE 2 (AREA UNDER COSINE)

Find area under:
y = cos(x)
from 0 to 2π

Anti-derivative of cos(x) is:
sin(x)

Evaluate:

sin(2π) − sin(0)
= 0 − 0
= 0

Why zero?

Because:

- area above x-axis
- equals area below x-axis

Signed areas cancel.

MEAN VALUE THEOREM FOR INTEGRALS (BIG IDEA)

For a continuous function f on [a, b]:

There exists some point x* in [a, b] such that:

Integral from a to b of f(x) dx
 = f(x*) * (b − a)

Meaning:

There is some height f(x*) such that:
 a rectangle of that height
 has the SAME area as the region under the curve.

INTUITION BEHIND THIS THEOREM

Let:
 M = maximum value of f on [a, b]
 m = minimum value of f on [a, b]

Then:
 m ≤ f(x) ≤ M

So area satisfies:
 m(b − a) ≤ area ≤ M(b − a)

That means:
 Average height lies somewhere between m and M

Because f is continuous,
 it must actually TAKE that average height somewhere.

That point is x*.

EXAMPLE (MEAN VALUE THEOREM FOR INTEGRALS)

Let:
f(x) = x^2
on [1, 4]

Compute area:

Integral from 1 to 4 of x^2 dx
= [x^3 / 3] from 1 to 4
= (64 − 1) / 3
= 63 / 3
= 21

Average height:

Average value = area / (b − a)
= 21 / 3
= 7

So:

f(x*) = 7
x*^2 = 7
x* = sqrt(7)

AVERAGE VALUE OF A FUNCTION

Definition:

If f is integrable on [a, b], then its average value is:

Average value of f
= (1 / (b − a)) * integral from a to b of f(x) dx

Think of it as:

“If the curve had constant height,
what height would give the same area?”

FINAL FEYNMAN INSIGHT

The First Fundamental Theorem of Calculus says:

- Area is accumulated change
- Anti-derivatives track accumulated change
- So evaluating area is just subtraction

Calculus is not magic.

It is bookkeeping — done perfectly.