

Lecture 31

Evaluating Definite Integrals by Substitution

1. Big Picture: What's New in This Lecture?

So far, you know how to:

- Find **indefinite integrals** using substitution
- Find **definite integrals** using the Fundamental Theorem of Calculus
- Approximate area using **Riemann sums**

Now we combine these ideas.

This lecture answers two questions:

1. How do we use **substitution** when limits are involved?
 2. What do we do when **exact integration is impossible**?
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2. Evaluating a Definite Integral by Substitution

We want to evaluate an integral of the form:

Integral from a to b of $f(x) dx$

But $f(x)$ is complicated — usually a **composition of functions**.

So we use **substitution**, just like before.

There are **two valid methods**.

3. Method 1: Substitute First, Apply Limits Later

Idea (Very Important)

1. Ignore the limits for now
2. Evaluate the **indefinite integral** using substitution
3. Apply the limits at the end

This method relies on the fact:

$$\begin{aligned} & \text{Integral from } a \text{ to } b \text{ of } f(x) \, dx \\ &= [\text{Antiderivative of } f(x)] \text{ evaluated from } a \text{ to } b \end{aligned}$$

Example (Method 1)

Evaluate:

$$\text{Integral from 0 to 2 of } x(1 + x^2)^3 \, dx$$

Step 1: Choose substitution

$$\text{Let } u = 1 + x^2$$

$$\text{Then } du = 2x \, dx$$

Step 2: Rewrite the integral

$$x \, dx = du / 2$$

So the integral becomes:

$$(1/2) \int u^3 \, du$$

Step 3: Integrate

$$(1/2) * (u^4 / 4) = u^4 / 8$$

Step 4: Substitute back

$$= (1 + x^2)^4 / 8$$

Step 5: Apply limits

$$\text{At } x = 2 \rightarrow (1 + 4)^4 / 8 = 625 / 8$$

$$\text{At } x = 0 \rightarrow (1 + 0)^4 / 8 = 1 / 8$$

Final answer:

$$(625 - 1) / 8 = 78$$

4. Method 2: Change the Limits (Cleaner & Smarter)

Big Idea

Instead of switching back to x at the end, we:

- Change x -limits into u -limits
- Integrate **entirely in u**

This avoids unnecessary algebra.

How to Change Limits

If:

$$u = g(x)$$

Then:

- Lower limit: $u = g(a)$
 - Upper limit: $u = g(b)$
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Same Example (Method 2)

Integral from 0 to 2 of $x(1 + x^2)^3 \, dx$

Step 1: Substitution

$$u = 1 + x^2$$

$$du = 2x \, dx$$

Step 2: Change limits

When $x = 0 \rightarrow u = 1$

When $x = 2 \rightarrow u = 5$

Step 3: Rewrite integral

(1/2) integral from 1 to 5 of $u^3 du$

Step 4: Integrate

(1/2) * ($u^4 / 4$) from 1 to 5

= (1/8)(625 - 1)

= 78

Same answer, less work.

5. Why Method 2 Is Often Better

- No need to substitute back
- Cleaner algebra
- Fewer chances for mistakes
- Preferred in exams

Use Method 1 when limits are simple.

Use Method 2 when substitution is obvious.

6. Trigonometric Example

Evaluate:

Integral from 0 to $\pi/4$ of $\cos(\pi - x) dx$

Step 1: Substitution

Let $u = \pi - x$

Then $du = -dx$

Step 2: Change limits

When $x = 0 \rightarrow u = \pi$

When $x = \pi/4 \rightarrow u = 3\pi/4$

Step 3: Rewrite integral

$$\begin{aligned}- & \text{integral from } \pi \text{ to } 3\pi/4 \text{ of } \cos(u) du \\&= \text{integral from } 3\pi/4 \text{ to } \pi \text{ of } \cos(u) du\end{aligned}$$

Step 4: Integrate

$$\begin{aligned}&= \sin(u) \text{ evaluated from } 3\pi/4 \text{ to } \pi \\&= 0 - (\sqrt{2}/2) \\&= -\sqrt{2}/2\end{aligned}$$

(Sign matches geometry.)

7. Approximation by Riemann Sums

When Do We Need This?

Sometimes:

- Integral has no elementary anti-derivative
- Function is too complicated

So we **approximate**.

8. Reminder: Riemann Sum

A Riemann sum looks like:

Sum of $f(x_k^*) * \Delta x$

This approximates the area using rectangles.

As number of rectangles increases:

- Approximation improves
 - Error decreases
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9. Approximation Formula

For large n:

Integral from a to b of $f(x) dx$
≈ Sum from k = 1 to n of $f(x_k^*) * \Delta x$

Where:

- $\Delta x = (b - a) / n$

- x_k^* can be:

- left endpoint
 - right endpoint
 - midpoint
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10. Types of Approximations

Left Endpoint

Uses left side of each subinterval
Tends to **underestimate** if function increases

Right Endpoint

Uses right side
Tends to **overestimate** if function increases

Midpoint

Uses center of each subinterval
Usually **most accurate**

11. Example: Numerical Approximation

Approximate:

Integral from 0 to 1 of $(1 / (1 + x^2)) dx$

Exact value is:

$$\pi / 4 \approx 0.7854$$

Using:

- $n = 10$
- $n = 20$
- $n = 50$
- $n = 100$

We observe:

- Left sum < exact value
- Right sum > exact value
- Midpoint sum converges fastest

As n increases, all approximations approach $\pi / 4$.

12. Final Feynman Insight

Substitution is just **changing perspective**:

- Same area
- New variable
- Simpler expression

Riemann sums remind us:

- Integrals are limits of sums
- Exact answers are ideal
- Approximations are practical

Calculus is not about formulas.

It's about **thinking flexibly** about change and accumulation.