

Lecture 23 – Absolute Extrema

The Big Idea

Before, we talked about relative maxima and relative minima — the local hills and valleys of a graph.

But what if we zoom out and look at the **entire mountain range**? Now we want to find the **tallest peak** and the **deepest valley** — no matter where they are.

That's what we call **absolute extrema**.

- Absolute Maximum → the highest point of the whole graph.
- Absolute Minimum → the lowest point of the whole graph.

Think of it like this:

 The Earth's surface is a function.

- Mt. Everest → Absolute Maximum (highest point).
- Mariana Trench → Absolute Minimum (deepest point).

Step 1: The Concept

When we say a function has an **absolute maximum**, we mean:
There is some point x_0 such that

$f(x_0) \geq f(x)$ for every x in the domain.

And for absolute minimum:

$f(x_0) \leq f(x)$ for all x in the domain.

So,

- Relative extrema = local hills and valleys
 - Absolute extrema = the highest and lowest in the entire journey
-

Step 2: A Simple Example

Consider

$$f(x) = 2x + 1$$

on the interval $[0, 3)$.

At $x = 0 \rightarrow f(0) = 1 \rightarrow$ this is the minimum value.

As x moves toward 3, $f(x)$ keeps increasing.

But the interval is $[0, 3)$, **not $[0, 3]$** ,
so $x = 3$ is **not included**.

That means even though we can get as close to 7 as we want,
we never actually reach 7.

 So there's **no absolute maximum**, only a **minimum at $x = 0$** .

That's the power of understanding **intervals** —
whether they're open or closed changes everything.

Step 3: What Makes a Function Have a Max or Min?

Mathematicians discovered something beautiful —
called the **Extreme Value Theorem**.

It says:

If a function is continuous on a closed interval $[a, b]$,
then it must have both an **absolute maximum** and an **absolute minimum**.

Plain English:

- Continuous \rightarrow no breaks or jumps.
- Closed interval \rightarrow both ends included.

Then your graph is guaranteed to have both highest and lowest points somewhere between a and b.

⚠ Step 4: When Extrema Don't Exist

If one of these conditions fails, extrema may disappear!

Example 1

If the interval is open, like $(0, 3)$,
the function might never reach the end value —
so no absolute max or min.

Example 2

If the function isn't continuous, it might jump —
so again, no extrema on that interval.

You can't find a highest or lowest if the graph breaks off!

▣ Step 5: The Procedure to Find Absolute Extrema

Here's the simple 3-step recipe ↴

Step 1:

Find the critical points — where
 $f(x) = 0$ or $f'(x)$ does not exist.

Step 2:

Evaluate $f(x)$ at:

- The critical points
- The endpoints of the interval

Step 3:

Compare all the values.

- The largest → Absolute Maximum
- The smallest → Absolute Minimum

That's it!

Example

Find the absolute extrema of

$$f(x) = 2x^3 - 15x^2 + 36x$$

on the interval [1, 5].

Step 1: Find derivative

$$f'(x) = 6x^2 - 30x + 36$$

$$f'(x) = 6(x - 2)(x - 3)$$

So, critical points are $x = 2$ and $x = 3$.

Step 2: Evaluate $f(x)$

$$\text{At } x = 1 \rightarrow f(1) = 23$$

$$\text{At } x = 2 \rightarrow f(2) = 28$$

$$\text{At } x = 3 \rightarrow f(3) = 27$$

$$\text{At } x = 5 \rightarrow f(5) = 55$$

Step 3: Compare

Minimum value = 23 at $x = 1$

Maximum value = 55 at $x = 5$

 So the function has:

Absolute Minimum at $x = 1$

Absolute Maximum at $x = 5$

Step 6: Extrema on Infinite Intervals

Sometimes, we study functions over the entire real line, like $(-\infty, \infty)$.

In that case, we look at how the function behaves as $x \rightarrow \pm\infty$.

Example:

$$f(x) = x^4 + 3x^3 - 2x + 1$$

As $x \rightarrow \pm\infty$, $f(x) \rightarrow +\infty$.

So, the function has **no maximum**,

but might have a **minimum** where the slope changes from negative to positive.

We find it by setting $f'(x) = 0$ and checking where the sign of slope changes — just like before!

Step 7: The Feynman Way to Think About It

Don't memorize — **imagine**.

Think of walking along a landscape described by your function:

- The slope ($f'(x)$) tells you whether you're going up or down.
- When slope = 0, you might be at a hilltop or a valley.
- If the road ends (endpoint of interval), check there too.

Then compare which is the highest and which is the lowest — that's literally what the function is doing.

Final Summary

Situation	What to Check	Meaning
$f'(x) = 0$ or DNE	Critical point	Possible peak or valley
Continuous on $[a, b]$	Guaranteed extrema	Theorem applies
Open interval / discontinuity	May have none	Edges missing or broken
Compare values at critical + endpoints	Find largest & smallest	Absolute max & min

In Feynman's Words

"You don't need to guess where the highest hill is — just measure the slopes.
When every path around you goes downhill, you're already standing at the top." 