📘 Lecture 19: Implicit Differentiation

Topics Covered

- The method of implicit differentiation
- Derivatives of rational powers of x
- Differentiability of implicit functions

🌱 1. What is Implicit Differentiation?

Let's start simple.

Sometimes, we have equations where **y is mixed up with x** — it's not neatly isolated on one side.
For example:

$$xy = 1$$

We want **dy/dx**, but y isn't alone.
How can we differentiate something like this?

✳️ The Easy Way (When You Can Solve for y)

Let's first solve this one normally.

$xy = 1$
$\rightarrow y = 1/x$

That's simple algebra — we've isolated y.

Now we can differentiate both sides with respect to x.

On the left, the derivative of y is just **dy/dx**.
On the right, we have **1/x**, which we can rewrite as $\mathbf{x^{-1}}$.

So,
$d/dx\ (x^{-1}) = -1 \times x^{-2} = -1/x^2$

Therefore,
$dy/dx = -1/x^2$

That's it!
We found the derivative using the **power rule** — because the exponent is –1, we multiply by it and reduce the power by one.

⚙️ The Implicit Way (When You Can't Solve for y Easily)

But what if we don't solve for y?

Let's start again with the same equation:
$xy = 1$

This time, we'll find the derivative **without** isolating y first.
This method is called **Implicit Differentiation** — it's used when y is tangled up with x, and we can't (or don't want to) solve for it directly.

So, we'll differentiate both sides **with respect to x**.
But there's one important idea here:
Since y itself depends on x, we'll treat y as a *function of x*.

That means whenever we take the derivative of y, we must multiply by **dy/dx** — that's the **Chain Rule** in action.

Now, let's differentiate both sides:

$d/dx\ (x \cdot y) = d/dx\ (1)$

On the right side, the derivative of 1 is 0.

On the left side, we have a product of two functions — x and y — so we'll apply the **Product Rule**:

$(x)(dy/dx) + (y)(1) = 0$

Let's simplify that:

$x(dy/dx) + y = 0$

Now we can solve for **dy/dx** (just basic algebra):

$x(dy/dx) = -y$
$dy/dx = -y/x$

✅ And there it is — the derivative, found **without ever solving for y**.

Notice something neat?
It's the **same result** we got before, when we did it the "normal" way ($dy/dx = -1/x^2$, and since $y = 1/x$, that's the same thing).

So what's the big idea here?
Implicit differentiation lets you find **dy/dx** even when y is "stuck" inside an equation with x.
You don't have to untangle it — you just differentiate *everything* while remembering that y secretly depends on x.

That's the heart of implicit differentiation — a way to see how things change **together**, even when one variable hides inside the other.

💡 Why It's Useful

Implicit differentiation helps when:

- It's **hard or impossible** to separate y.
- You have **curves defined by equations** involving both x and y (like circles, ellipses, or trigs).

🧩 Example 1

Find dy/dx if:
$y^2 + 5\sin(y) = x^2$

We can't solve this easily for y, so let's differentiate implicitly.

Differentiate both sides:

$d/dx(y^2) + d/dx(5\sin(y)) = d/dx(x^2)$

$\rightarrow (2y)(dy/dx) + 5\cos(y)(dy/dx) = 2x$

Factor out dy/dx:

$dy/dx\ [2y + 5\cos(y)] = 2x$

So:

$dy/dx = 2x / [2y + 5\cos(y)]$

That's your derivative — it includes both **x** and **y**, and that's okay!
Sometimes you **can't separate** them algebraically.

🧮 Example 2 — Slope of a Tangent

Find the slope of the tangent line at (4, 0) on the curve:
$y^4 + 3xy^3 + 7x^4 = 4$

Differentiate both sides implicitly:

$d/dx(y^4) + d/dx(3xy^3) + d/dx(7x^4) = 0$

$\rightarrow (4y^3)(dy/dx) + [3y^3 + 9xy^2(dy/dx)] + 28x^3 = 0$

Group dy/dx terms:

$dy/dx\ (4y^3 + 9xy^2) + 3y^3 + 28x^3 = 0$

Solve for dy/dx:

$dy/dx = -(3y^3 + 28x^3) / (4y^3 + 9xy^2)$

Now substitute the point (x, y) = (4, 0):

$dy/dx = -(0 + 28\times64) / (0 + 0)$
$\rightarrow dy/dx = -(1792) / (0)$
→ Slope = **undefined**

Since the **denominator is zero** and the numerator is non-zero, the slope is **undefined** ∞.

🌀 Example 3

Find dy/dx if:
$x^2 - 2xy^2 = 9$

Differentiate both sides with respect to x:

$2x - 2[(y^2) + x(2y)(dy/dx)] = 0$

Simplify:

$2x - 2y^2 - 4xy(dy/dx) = 0$

Rearrange for dy/dx:

$-4xy(dy/dx) = -2x + 2y^2$
$\rightarrow dy/dx = (x - y^2) / (2xy)$

That's the derivative using **implicit differentiation**.

🧠 2. Derivatives of Rational Powers of x

You already know the **power rule** works for integers:

$d/dx\,(x^n) = n \cdot x^{n-1}$

But it also works for **rational powers** (like ½, ⅔, etc.):

$d/dx\,(x^r) = r \cdot x^{r-1}$

For example:
$d/dx\,(x^{½}) = (½)x^{-½} = 1 / (2\sqrt{x})$

So yes — **the power rule works for fractions too!**

🔍 3. Differentiability of Implicit Functions

If an equation relates x and y in a smooth way (no sharp corners, breaks, or undefined regions), then y can often be treated as a **differentiable function** of x — even if you can't write it explicitly.

That’s why implicit differentiation works — it’s just the **Chain Rule in disguise.**

🎯 Summary

- When y is tangled up with x, differentiate **implicitly**.
- Always apply **Chain Rule** when differentiating y.
- Solve for **dy/dx** at the end.
- Works for circles, trigs, logs, exponentials — anything with x and y mixed.

👉 Think of it like this:

> “If I can’t isolate y, I’ll still track how y moves when x moves — that’s what implicit differentiation does.”

🧩 Feynman Insight

When x moves a little, y must adjust to keep the equation true.
Implicit differentiation is just writing down **how that adjustment happens**.

It’s not magic — it’s just being honest about what’s changing.