Lecture # 34
Volume by Slicing; Disks and Washers

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1. Big Picture: What is this lecture about?

So far we used definite integrals to find:

• Area (2D)

Now we move one dimension higher and find:

• Volume (3D)

Key idea:
Volume is just **“area stacked along a direction.”**

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2. From 0D to 3D (Intuition First)

Think of building shapes step by step:

• Point → no size (0D)
• Line → length (1D)
• Rectangle → area (2D)
• Solid → volume (3D)

A **solid** can be thought of as many very thin **2D slices stacked together**.

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3. Cylinders (The Simplest Solid)

If you take a 2D shape and move it straight in one direction, you get a cylinder.

Examples:
• Move a circle → solid cylinder
• Move a washer (circle with a hole) → hollow cylinder

For a right cylinder:

Volume = (Area of cross section) × (Height)

$V = A \cdot h$

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4. Why We Need a New Method

Not all solids are perfect cylinders.

Many solids:
• Change shape along their length
• Have different cross-sectional areas at different points

So we cannot use one simple formula.

Solution:
Use **the method of slicing**.

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5. The Method of Slicing (Core Idea)

Imagine slicing the solid into **very thin slices**.

Each slice:
• Has thickness Δx (or Δy)
• Looks almost like a cylinder
• Has a cross-sectional area $A(x)$

Volume of one thin slice:

$V \approx A(x^*) \cdot \Delta x$

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6. Adding All the Slices

If we add volumes of all slices:

$V \approx A(x_1^*)\Delta x + A(x_2^*)\Delta x + \dots + A(x_n^*)\Delta x$

This is a Riemann sum.

When slices become infinitely thin:

$V = \int$ from a to b $A(x)\,dx$

This is the **general volume formula by slicing**.

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7. Cross Sections Perpendicular to x-axis

If slices are perpendicular to the x-axis:

- Thickness = dx
- Area = $A(x)$

Volume formula:

$V = \int A(x)\,dx$

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8. Cross Sections Perpendicular to y-axis

If slices are perpendicular to the y-axis:

- Thickness = dy
- Area = $A(y)$

Volume formula:

$V = \int A(y)\,dy$

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9. Solids of Revolution (Disks Method)

Now suppose a region is revolved around an axis.

This creates a **solid of revolution**.

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10. Disks Perpendicular to the x-axis

Let $y = f(x) \geq 0$ on $[a, b]$

Revolve the region under $y = f(x)$ about the x-axis.

Each cross section:
- Is a disk
- Radius = $f(x)$

Area of disk:

$A(x) = \pi [f(x)]^2$

Volume formula:

$V = \pi \int$ from a to b $[f(x)]^2 \, dx$

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11. Example: Disk Method

Find the volume when $y = x$ is revolved about the x-axis from $x = 1$ to $x = 4$.

Here:
$f(x) = x$

$V = \pi \int$ from 1 to 4 $x^2 \, dx$

After evaluation:

$V = (15/2)\pi$

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12. Deriving Volume of a Sphere

A sphere of radius r can be formed by revolving:

$y = \sqrt{r^2 - x^2}$

about the x-axis, from $x = -r$ to r.

Using disk method:

$V = \pi \int$ from $-r$ to r $(r^2 - x^2) \, dx$

Evaluating gives:

$V = (4/3)\pi r^3$

This is the standard sphere formula.

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13. Washers Perpendicular to the x-axis

Now consider a region between two curves:

Upper curve: $y = f(x)$
Lower curve: $y = g(x)$
with $f(x) \geq g(x)$

Revolve around the x-axis.

Each cross section:
- Is a washer (disk with a hole)
- Outer radius = $f(x)$
- Inner radius = $g(x)$

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14. Area of a Washer

Area = π [outer radius² − inner radius²]

$A(x) = \pi [f(x)^2 - g(x)^2]$

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15. Washer Volume Formula (x-axis)

$V = \pi \int$ from a to b $[f(x)^2 - g(x)^2]\, dx$

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16. Example: Washer Method

Given:
$f(x) = x^2 + 1$
$g(x) = x$
Interval: $[0, 2]$

Revolved about the x-axis.

$V = \pi \int$ from 0 to 2 $[(x^2 + 1)^2 - x^2]\, dx$

Evaluate to get the volume.

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17. Disks Perpendicular to the y-axis

Sometimes it's easier to rotate around the y-axis.

If $x = u(y)$ and region is revolved about y-axis:

Disk radius = $u(y)$

Volume:

$V = \pi \int$ from c to d $[u(y)]^2\, dy$

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18. Washers Perpendicular to the y-axis

If region lies between:

Right boundary: $x = u(y)$
Left boundary: $x = v(y)$

Washer area:

$A(y) = \pi [u(y)^2 - v(y)^2]$

Volume formula:

$V = \pi \int$ from c to d $[u(y)^2 - v(y)^2]\, dy$

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19. How to Choose the Correct Method

Ask these questions:

- Are we revolving around x-axis or y-axis?
- Are cross sections disks or washers?
- Is dx or dy easier?
- Which setup avoids splitting the region?

20. Final Takeaway (Lecture 34)

- Volume = sum of infinitely thin slices
- Slicing turns 3D problems into integrals
- Disks → solid (no hole)
- Washers → solid with hole
- Choose dx or dy wisely

This lecture shows how **integration builds 3D solids** from 2D ideas.