Lecture 16 — Techniques of Differentiation

🌿 The Core Idea

Remember:
A derivative tells us **how fast a function changes** — its *rate of change* or *slope*.

So if a function **never changes**, what's its rate of change?
👉 Zero.

That's the logic behind the first rule.

🔹 Theorem 3.3.1 — Derivative of a Constant Function

If

- `f(x) = c   (where c is a constant)`

then

- `f'(x) = 0`

✳️ Explanation

Think about what a constant function looks like.
For example:

- `f(x) = 5`

This means no matter what x is — 2, 10, or 1000 — the output is always 5.

That's a **flat, horizontal line** on the graph.
And what's the slope of a flat line?
Right — **0**.

So the rate of change (derivative) is 0 everywhere.

Mathematically:

- $f(x) = 5$
- $f'(x) = 0$

🔹 Theorem 3.3.2 — The Power Rule

If

- $f(x) = x^n$ (where n is a positive integer)

then

- $f'(x) = n \cdot x^{n-1}$

✳️ Explanation

Let's think step by step.

We start from the definition of the derivative:

- $f'(x) = \lim (h \to 0) [f(x + h) - f(x)] / h$

Now substitute f(x) = x^n:
f'(x) = lim (h -> 0) [(x + h)^n - x^n] / h

Using the Binomial Theorem

The Binomial Theorem expands (x + h)^n as:
(x + h)^n = x^n + n * x^(n-1) * h + (n(n-1)/2) * x^(n-2) * h^2 + ... + h^n

Plug this expansion into the derivative formula:
f'(x) = lim (h -> 0) [(x^n + n*x^(n-1)*h + h^2 terms) - x^n] / h

The x^n terms cancel, leaving:
f'(x) = lim (h -> 0) [n*x^(n-1)*h + h^2 terms] / h

Divide every term by h:
f'(x) = lim (h -> 0) [n*x^(n-1) + (terms containing h)]

As h -> 0 all terms that contain h go to 0, so we are left with:
Result: f'(x) = n * x^(n-1)

A quick example (n = 3)

Let f(x) = x^3.

1. Expand (x + h)^3:
   (x + h)^3 = x^3 + 3x^2*h* + *3x*h^2 + h^3

2. Subtract x^3:
   (x + h)^3 - x^3 = 3x^2*h* + *3x*h^2 + h^3

3. Divide by h:
   [(x + h)^3 - x^3] / h = 3x^2 + 3x*h + h^2

4. Take the limit as h -> 0:
   lim (h -> 0) (3x^2 + 3x*h + h^2) = 3x^2

So f'(x) = 3x^2, which matches the power rule.

Intuitive takeaway

The Binomial Theorem gives the algebraic expansion that reveals a single leading term containing h linearly: n * x^(n-1) * h. After dividing by h and taking the limit, all higher-power h terms disappear, leaving n * x^(n-1). In plain language: multiply by the exponent, then reduce the exponent by one.

🧮 Examples

1. If

- $f(x) = x^2$

then

- $f'(x) = 2x$

2. If

- $f(x) = x^3$

then

- $f'(x) = 3x^2$

3. If

- $f(x) = x^5$

then

- $f'(x) = 5x^4$

This simple rule — *multiply by the power and reduce the exponent by one* — is one of the most used rules in calculus.

🔹 Theorem 3.3.3 — Constant Multiple Rule

If **c** is a constant and **f(x)** is a differentiable function,
then the derivative of **c·f(x)** is:

- `d/dx [c·f(x)] = c · f'(x)`

✳️ Explanation

Constants don't affect the "shape" of a curve — they only scale it.
So multiplying a function by a constant just multiplies its slope by that same constant.

🧮 Example

Let's say

- `f(x) = x²`

then

- `d/dx [3·x²] = 3 · (d/dx [x²])`

By the power rule:

- `d/dx [x²] = 2x`

So:

- `d/dx [3·x²] = 3 · (2x) = 6x`

🔹 Geometric Intuition

- A **constant function** is perfectly flat → slope 0.
- A **power function** grows faster as n increases → slope gets steeper with higher x.
- A **constant multiple** just stretches or compresses the graph vertically, so the slope scales by that factor.

✅ **Summary Table**

Function Type	**Function f(x)**	**Derivative f′(x)**
Constant	c	0
Power	x^n	$n \cdot x^{n-1}$
Constant × Function	$c \cdot f(x)$	$c \cdot f'(x)$

DERIVATIVE OF SUMS AND DIFFERENCES OF FUNCTIONS

If f and g are differentiable functions at x, then their sum (f + g) is also differentiable, and

d/dx [f(x) + g(x)] = f'(x) + g'(x)

Similarly,
d/dx [f(x) – g(x)] = f'(x) – g'(x)

STEP-BY-STEP EXPLANATION

Let's start from the definition of the derivative:

f'(x) = lim (h -> 0) [f(x + h) – f(x)] / h

Now apply this to the sum (f + g):

d/dx [f(x) + g(x)]
= lim (h -> 0) [(f(x + h) + g(x + h)) – (f(x) + g(x))] / h

Simplify the numerator:

= lim (h -> 0) [(f(x + h) – f(x)) + (g(x + h) – g(x))] / h

We can split this fraction into two separate parts:

= lim (h -> 0) [f(x + h) – f(x)] / h + lim (h -> 0) [g(x + h) – g(x)] / h

Now, each of these is just the derivative of f and g respectively:

= f'(x) + g'(x)

That's it!
We've shown that the derivative of a sum is the **sum of the derivatives**.

EXAMPLE 1

Let's find the derivative of:

y = x^4 + x^3

Using the sum rule:
d/dx [x^4 + x^3] = d/dx [x^4] + d/dx [x^3]

Now apply the Power Rule:
$= 4x^3 + 3x^2$

So, $dy/dx = 4x^3 + 3x^2$

QUICK NOTE (FOR DIFFERENCE)

If you had $y = x^4 – x^3$,
then $d/dx [x^4 – x^3] = 4x^3 – 3x^2$
Same logic — subtraction works exactly the same way.

DERIVATIVE OF A PRODUCT

If f and g are differentiable at x, then their product f(x) * g(x) is also differentiable, and

$d/dx [f(x) * g(x)] = f(x) * g'(x) + g(x) * f'(x)$

INTUITIVE EXPLANATION

Think of f(x) and g(x) as two changing quantities.
When both change together, the rate of change of their product depends on **both changing at the same time**.

So the derivative is not just f'(x) * g'(x).
You need to account for the fact that while one changes, the other may also be changing.

That's why there are **two parts**:

1. The change due to f (keeping g fixed)

2. The change due to g (keeping f fixed)

EXAMPLE 2

Let's find the derivative of:
$y = x^2 * (x^3 + 1)$

Here $f(x) = x^2$ and $g(x) = (x^3 + 1)$

Using the product rule:
$\frac{d}{dx} [f(x) * g(x)] = f'(x) * g(x) + f(x) * g'(x)$

Compute each part:
$f'(x) = 2x$
$g'(x) = 3x^2$

Now substitute:
$= (2x)(x^3 + 1) + (x^2)(3x^2)$
$= 2x^4 + 2x + 3x^4$
$= 5x^4 + 2x$

So the derivative is:
$dy/dx = 5x^4 + 2x$

INTUITIVE TAKEAWAY

For sums or differences — derivatives act independently and simply add or subtract.
For products — the changes of both functions interact, so you apply the **product rule**.

In short:
(1) Derivative of sum/difference → add/subtract the derivatives
(2) Derivative of product → $fg' + gf'$

✳️ THEOREM 3.3.6 — DERIVATIVE OF A QUOTIENT

If f and g are differentiable at x, and g(x) ≠ 0,
then the derivative of their quotient is:

d/dx [f(x) / g(x)] = [g(x) * f'(x) – f(x) * g'(x)] / [g(x)]²

🧠 Step-by-Step Intuition

Let's think like Feynman — intuitively, not mechanically.

When you divide one function by another, like
y = f(x) / g(x),
both the top (f) and bottom (g) are changing.

You can't just differentiate the top or bottom separately —
you have to see how their changes interact.

The quotient rule tells us exactly how to balance those two motions.

🧩 How It Comes Together (Derivation)

We know the **Product Rule**:
d/dx [f(x) * g(x)] = f'(x) * g(x) + f(x) * g'(x)

Now, notice that a quotient can be written as:
f(x) / g(x) = f(x) * (1 / g(x))

Let's apply the product rule to that:
d/dx [f(x) * (1/g(x))] = f'(x) * (1/g(x)) + f(x) * d/dx (1/g(x))

So we just need to find the derivative of (1/g(x)).

✳️ THEOREM 3.3.7 — DERIVATIVE OF A RECIPROCAL

If $g(x)$ is differentiable and $g(x) \neq 0$, then:

$$\frac{d}{dx}\left[1 / g(x) \right] = -\, g'(x) / [\, g(x) \,]^2$$

💭 Intuitive Explanation

Think of $1/g(x)$ as "how much of g fits into 1."
If g increases, the reciprocal decreases — hence the negative sign.

If g changes fast, the reciprocal changes even faster in the opposite direction,
and the squared denominator keeps everything balanced.

🔁 Plugging It Back Into the Quotient Rule

Substitute this into our earlier product rule:

$$\frac{d}{dx}\left[f(x) \cdot (1/g(x)) \right] = f'(x) \cdot (1/g(x)) + f(x) \cdot \left[-\, g'(x) / g(x)^2 \right]$$

Simplify:

$$= [\, f'(x) \cdot g(x) - f(x) \cdot g'(x) \,] / [\, g(x) \,]^2$$

And that's our **Quotient Rule**!

✅ **Final Formula (Simplified)**

$$\frac{d}{dx}\left[f(x) / g(x) \right] = [\, g(x) \cdot f'(x) - f(x) \cdot g'(x) \,] / [\, g(x) \,]^2$$

💡 Intuitive Picture

Imagine $f(x)$ and $g(x)$ as two dancers —
both moving at different speeds.
When one moves up (increases) and the other down (decreases),
their ratio changes — and this rule perfectly captures that motion.

The minus sign shows they move in opposite directions,
and the square of the denominator keeps everything balanced.

🧮 Example

Let:
$f(x) = x^2$
$g(x) = x^3 + 1$

Then:
$f'(x) = 2x$
$g'(x) = 3x^2$

Now apply the Quotient Rule:

$d/dx\ [\ f(x) / g(x)\ ]$
$= [\ g(x)*f'(x) – f(x)*g'(x)\ ] / [\ g(x)\ ]^2$
$= [\ (x^3 + 1)(2x) – (x^2)(3x^2)\ ] / (x^3 + 1)^2$
$= [\ 2x^4 + 2x – 3x^4\ ] / (x^3 + 1)^2$
$= (\ –x^4 + 2x\ ) / (x^3 + 1)^2$

✅ Final Answer:
$dy/dx = (\ –x^4 + 2x\ ) / (x^3 + 1)^2$

🧠 Memory Trick (Rhyme)

“Low d High minus High d Low,
Square the bottom, and away we go!”

• “Low” → denominator (g)
• “High” → numerator (f)

✳️ THEOREM 3.3.8 — GENERALIZED POWER RULE (USING RECIPROCAL IDEA)

Now that we can differentiate reciprocals,
we can extend the Power Rule to **all integers**, even negative ones.

If n is any integer, then:
$d/dx [x^n] = n * x^{n-1}$

✅ Works for all n — positive, zero, or negative.

Example:

If $f(x) = x^{-2} = 1 / x^2$

Then by the rule:
$f'(x) = (-2) * x^{-3} = -2 / x^3$

It matches perfectly with what we'd get using the reciprocal rule.

🎯 In Summary

Rule	Formula	Meaning
Product Rule	$(fg)' = f'g + fg'$	Two functions multiplying
Quotient Rule	$(f/g)' = (g \cdot f' - f \cdot g') / g^2$	Two functions dividing
Reciprocal Rule	$(1/g)' = - g' / g^2$	Inverse of a function
General Power Rule	$(x^n)' = n \cdot x^{n-1}$	Works for all integers n