# 📘 Lecture #11 – Limits: A Rigorous Approach

## 1. Why do we even need this formal stuff?

We've been casually saying:

$\lim (x \to a) f(x) = L$

means: as x gets closer to a, f(x) gets closer to L.

That's fine in everyday language. But mathematics doesn't like "close enough" — it wants an exact recipe for what "close" means.
So the question is:

👉 How do we measure closeness in a way that's airtight and leaves no wiggle room?

That's why we bring in the epsilon (ε) – delta (δ) definition.

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## 2. The Big Idea (ε and δ)

Imagine you set a tolerance window around L, say within $\varepsilon = 0.01$.
This means: "I want f(x) to be within 0.01 of L."

The definition says:

- No matter how strict your ε-window is,
- I can always find a δ-window around $x = a$,
- such that whenever x is within δ of a (but not exactly equal to a),
- the output f(x) will land inside that ε-window.

Formally written:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

It's like a promise: you challenge me with how close you want $f(x)$ to $L$, and I can always adjust $\delta$ to keep the function inside that target.

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## 3. Left-hand vs Right-hand

Sometimes, approaching from the left or right gives different answers.

- Left-hand limit ($x \to a^-$): values just less than a.
- Right-hand limit ($x \to a^+$): values just greater than a.

If they match → the two-sided limit exists.
If not → the limit doesn't exist.

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## 4. Example 1: A Simple Linear Function

Let's test:

$\lim (x \to 2) (3x - 5) = 1$

## 1. What are we really checking?

We want to guarantee:
If x is close to 2, then $f(x) = 3x - 5$ is close to 1.

But "close" must be measured exactly:

- closeness of $f(x)$ to 1 is controlled by **ε**
- closeness of x to 2 is controlled by **δ**

So the game is:
ε sets the challenge → δ is our response.

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## 2. Work out the difference

Start with:
$f(x) - 1 = (3x - 5) - 1 = 3(x - 2)$

So:
$|f(x) – 1| = 3|x – 2|$

This is the key line.

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### 3. What does it mean?

It means: the "error" in f(x) (distance from 1) is exactly **3 times the error in x** (distance from 2).

Think of it like this in plain life terms:

- Suppose your friend is standing at point **2** on the number line.
- Every step you take away from 2, your function (f(x)) moves 3 steps away from 1.

So if you're off by 0.1 in x, f(x) is off by 0.3.
If you're off by 0.01 in x, f(x) is off by 0.03.

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### 4. Connecting ε and δ

Now, the definition says:
We want $|f(x) – 1| < \varepsilon$.

But we know $|f(x) – 1| = 3|x – 2|$.

So, for this to be true:
$3|x – 2| < \varepsilon$

Divide by 3:
$|x – 2| < \varepsilon / 3$

💡 **Boom:** That's the relation!
δ must be chosen as ε/3, so that the error in x never gets magnified too much in f(x).

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### 5. Intuition with an Example

Imagine you're bargaining at a shop in Lahore.
The shopkeeper says: *"For every 1 rupee you change in the price of x, the function f(x) changes by 3 rupees."*

Now, if you want the final price ($f(x)$) to stay within **$\varepsilon = 6$ rupees** of the target (1 rupee), you must keep x within **$\delta = \varepsilon/3 = 2$ rupees** of 2.

That's why δ is tied to ε by this ratio.

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✅ So the relation is:

- Output error = 3 × Input error
- To control output error (ε), we shrink input error (δ) proportionally: $\delta = \varepsilon / 3$.

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## 5. Example 2: A Broken Case

Define:

$f(x) = 1$, if $x > 0$
$f(x) = -1$, if $x < 0$

Now check $\lim (x \to 0) f(x)$.

- From the right ($x \to 0^+$), $f(x) \to 1$
- From the left ($x \to 0^-$), $f(x) \to -1$

They don't match.
👉 So the two-sided limit does not exist.

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## 6. Analogy 🎯

Think of L as the bullseye on a dartboard.

- ε = how tight your ring is around the bullseye (maybe 1 cm, maybe 0.001 cm).
- δ = how close you stand to the board (distance from x = a).

No matter how strict your ε, if I can always adjust δ so that every dart lands in that ε-ring, the limit exists.

But if darts from the left and right land in different bullseyes, the limit fails.

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✅ **Summary:**
The rigorous definition makes limits bulletproof. It removes fuzziness and gives us a way to prove, with certainty, whether a limit exists and what it equals.