📘 Lecture #8 — Graphs of Functions

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1. What is a graph of a function?

- A **graph of an equation** → all the points (x, y) in the plane that satisfy the equation.
- A **graph of a function f** → all the points (x, f(x)) in the plane.

👉 In short: the graph is the **picture** of how inputs (x) and outputs (f(x)) are related.

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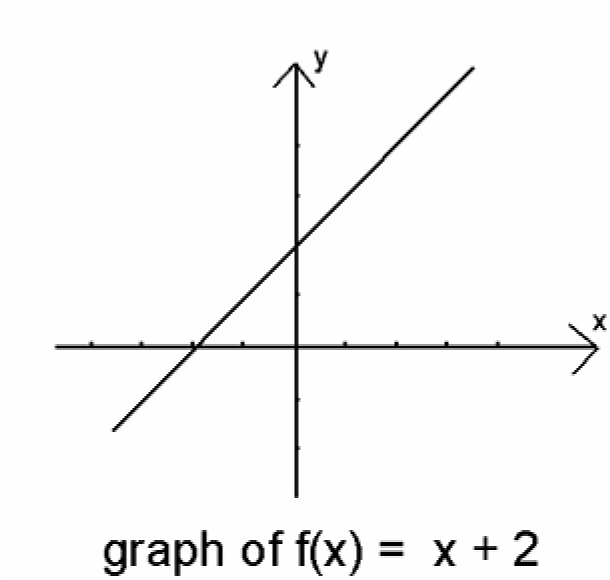
2. Example 1: Linear function

f(x) = x + 2

- Equation: `y = x + 2`
- Shape: Straight line.
- Slope: `1` (as x increases by 1, y increases by 1).
- y-intercept: `(0, 2)`
- x-intercept: Solve `0 = x + 2 → x = -2`. So point `(-2, 0)`.
- Behavior:
  - Goes **up to the right** (positive slope).
  - Extends infinitely in both directions.



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3. Example 2: Absolute value function

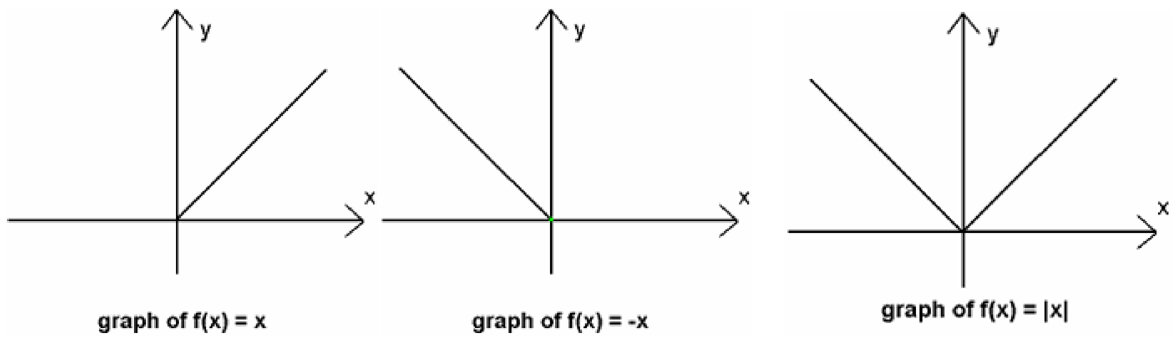
f(x) = |x|

Definition:

```
f(x) = x   if x ≥ 0
f(x) = -x  if x < 0
```

- Shape: V-shape with vertex at `(0,0)`.
- Slope:
  - Right side (x ≥ 0): slope = +1.
  - Left side (x < 0): slope = -1.
- Intercepts:
  - x-intercept = 0
  - y-intercept = 0

- Behavior:
    - Always non-negative ($y \geq 0$).
    - Symmetric about the y-axis.



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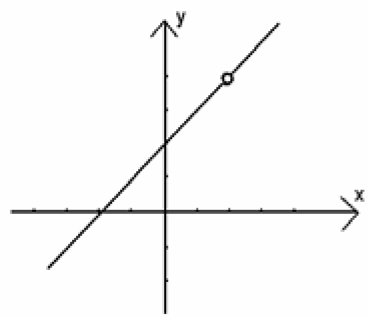
4. Example 3: Rational function with a hole

t(x) = (x² - 4) / (x - 2)

Step 1: Factor: $(x - 2)(x + 2) / (x - 2)$
Step 2: Simplify: $y = x + 2$ except $x \neq 2$.

- Shape: Straight line (same as Example 1 but shifted).
- y-intercept: $(0, 2)$
- x-intercept: Solve $0 = x + 2 \rightarrow x = -2$. So point $(-2, 0)$.
- Special behavior:
    - At $x = 2$, the function is **undefined** (division by zero).
    - So there is a **hole** at $(2, 4)$.

- Behavior:
  - Looks like a normal line, but you must remember the missing point.



graph of $y = x + 2 \quad x \neq 2$

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5. Example 4: Piecewise defined function

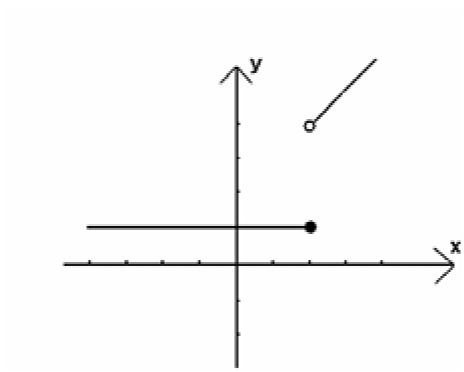
$g(x) = 1 \quad$ if $x \leq 2$

$g(x) = x + 2 \quad$ if $x > 2$

- For $x \leq 2$:
  - Graph is flat (horizontal line).
  - Equation: $y = 1$
  - Extends leftward to negative infinity.
- For $x > 2$:
  - Graph is slanted line with slope 1.
  - Starts just after $(2, 4)$ and goes up to the right.
- Intercepts:
  - For left piece: y-intercept = (0,1)

- For right piece: when $x=3$, $y=5$ (so it starts above).

- Special behavior:
  - Jump at $x = 2$:
    - Left-hand value = 1
    - Right-hand value = 4
    - So there is a **discontinuity** (the graph “jumps”).



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6. Why graphs matter?

- They let us **see** the behavior of functions.
- Simple graphs (lines, |x|, 1/x, etc.) become **building blocks**.
- We use these to create graphs of more complicated functions.

✅ Summary Table

Function	Graph Shape	Notes
f(x) = x + 2	Straight line (slope 1,	Linear

	intercept 2)	
$f(x) = \|x\|$	V-shape at origin	Piecewise ($y = x$ or $y = -x$)
$t(x) = (x^2 - 4)/(x-2)$	Line $y = x+2$ with hole at $x=2$	Rational, undefined at $x=2$
$g(x) = \{1 \text{ if } x \leq 2; x+2 \text{ if } x>2\}$	Horizontal then slanted line	Piecewise

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📌 1. The Idea of Translation

Translation = **sliding** the whole graph without changing its shape.

- No stretching, no bending, no rotating — just shifting.
- You take the original graph of $y = f(x)$ and move it **up, down, left, or right** depending on what is added or subtracted.

📌 2. The Four Types of Translations

Suppose $y = f(x)$ is known.

1. **Vertical shifts** (affect the output):
   - $y = f(x) + c$ → move UP by c units.
   - $y = f(x) - c$ → move DOWN by c units.
2. **Horizontal shifts** (affect the input):
   - $y = f(x + c)$ → move LEFT by c units.
   - $y = f(x - c)$ → move RIGHT by c units.

y = f(x) + c	graph of f(x) translates *UP* by c units
y = f(x) - c	graph of f(x) translates *DOWN* by c units
y = f(x + c)	graph of f(x) translates *LEFT* by c units
y = f(x - c)	graph of f(x) translates *RIGHT* by c units

⚠️ Notice the trick:

- Inside parentheses with `x` → direction feels **opposite**.
    - `+c` means left.
    - `-c` means right.
- Outside → direction is **same** as the sign.

📌 3. First Example

```
f(x) = x
Graph: y = x (a straight line through the origin, slope 1).
```

Now let's create a new function:

```
y = f(x - 3) + 2 = (x - 3) + 2
```

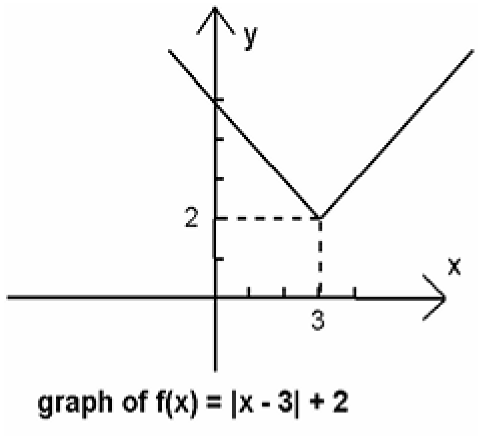
Step 1: **Shift Right by 3**

- Replace x by (x - 3).
- Whole line slides right.
- Point (0,0) moves to (3,0).

Step 2: **Shift Up by 2**

- Add +2 outside.
- Every y-value increases by 2.
- Point (3,0) becomes (3,2).

✅ Final graph: the line `y = x - 1`, but you got it by *moving* the original y = x.



📌 4. Second Example (Quadratic case)

Original:

```
y = x^2
(parabola opening upward with vertex at (0,0))
```

Given:

```
y = (x - 2)^2 + 1
```

Step 1: `(x - 2)` → shift **right by 2**.

- Vertex moves from (0,0) to (2,0).

Step 2: `+1` → shift **up by 1**.

- Vertex moves again to (2,1).

✅ Final graph: parabola same shape, vertex at (2,1).

📌 5. Example From Your Notes

`y = x^2 - 4x + 5`

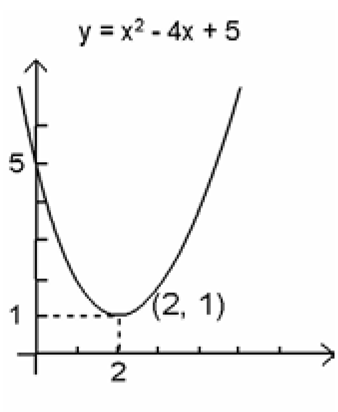
Complete the square:

`y = (x - 2)^2 + 1`

Now interpret:

- `(x - 2)` → shift parabola RIGHT 2.
- `+1` → shift UP 1.
- Vertex is at (2,1).
- Shape = same as y = x^2 (opens upward).

✅ Graph is parabola with vertex (2,1).



📌 6. Behavior to Notice

- Translations do **not change slope, curvature, or shape** — only the position.

- Domain usually stays the same (unless the shift moves a hole, asymptote, etc.).
- Range shifts with it. Example:
    - For y = x^2, range = [0, ∞).
    - For y = (x - 2)^2 + 1, range = [1, ∞).

📌 7. Summary

```
Translation Rules:

y = f(x) + c    → shift graph UP by c
y = f(x) - c    → shift graph DOWN by c
y = f(x + c)    → shift graph LEFT by c
y = f(x - c)    → shift graph RIGHT by c

Example 1:
y = (x - 3) + 2
Start: y = x
Step 1: shift RIGHT 3
Step 2: shift UP 2
Final: line through (3,2), slope = 1.

Example 2:
y = (x - 2)^2 + 1
Start: y = x^2
Step 1: shift RIGHT 2
Step 2: shift UP 1
Final: parabola with vertex (2,1), opening upward.
```

Reflections

A reflection just means "flipping" the graph across an axis.

Reflection about the y-axis:
Rule: Replace x with -x.
Geometrically: Point (x, y) becomes (-x, y).
Example:
$f(x) = x^3$
$f(-x) = (-x)^3 = -x^3$
The new graph is a mirror image of the original across the y-axis.

Reflection about the x-axis:
Rule: Multiply the whole function by -1.
So $f(x) \to -f(x)$.
Geometrically: Point (x, y) becomes (x, -y).
Example:
$f(x) = x^2$
$-f(x) = -x^2$
The parabola flips upside down.

Summary:
$y = f(-x) \to$ reflection about the y-axis
$y = -f(x) \to$ reflection about the x-axis

Example: $y = -(x+2)^3$

Step by step:

1. Start with the basic cube graph: $y = x^3$
2. Reflect about the x-axis: $y = (-x)^3 = -x^3$
3. Translate left by 2 units: $y = -(x + 2)^3$

Reflection + Translation combine to move/flip the cube graph into the right place.

Vertical Scaling

This means multiplying the function by a constant c:

$y = c * f(x)$

If $c > 1$ → Graph is stretched vertically (gets taller).
If $0 < c < 1$ → Graph is compressed vertically (gets shorter/flatter).

Example:
$y = \sin(x)$
$y = 2\sin(x)$
$y = (1/2)\sin(x)$

$y = \sin(x)$: amplitude = 1
$y = 2\sin(x)$: amplitude = 2 (stretched)
$y = (1/2)\sin(x)$: amplitude = 0.5 (compressed)

Summary:

- Reflections = flip graphs across x-axis or y-axis
- Scaling = stretch or compress graphs vertically

🔹 What's the issue?

We know the definition of a function:
👉 **Each input (x) must have exactly one output (y).**

That means: for a given x, you can't have two different y-values.

🔹 Vertical Line Test

Rule:
Draw vertical lines (x = constant) through the graph.

- If **any vertical line crosses the graph more than once**, then the graph is **not** a function.
- If **every vertical line crosses at most once**, then the graph **is** a function.

🔹 Example 1: Circle

Equation:
$\mathbf{x^2 + y^2 = 25}$

That's a circle of radius 5.
Now imagine drawing the vertical line $\mathbf{x = 3}$.

- It intersects the circle at two points: (3, +4) and (3, –4).
- So for input x = 3, we got **two outputs (y = +4, y = –4)**.

🚫 This breaks the rule of a function.
Conclusion: The circle is **not a function of x.**

🔹 Important Note

Even if it's not a function **y = f(x)**, sometimes the same graph *can* be a function the other way around: **x = g(y)**.

That's when we use the **Horizontal Line Test**:

- Each y should have only one x.
- For the circle, $x = \pm\sqrt{(25 - y^2)}$.
  Here, one y-value gives **two possible x-values**, so even in this sense, a circle fails to be a function unless we restrict to half the circle.

✅ Summary in plain words:

- A **function's graph** cannot stack two points vertically above the same x.
- Vertical Line Test = quick tool to check this.
- Example: lines, parabolas ($y = x^2$), exponentials ($y = e^x$) ✅ all pass.
- Circle, sideways parabolas ($x = y^2$), ellipses 🚫 fail.