Lecture # 33
Application of the Definite Integral

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1. What this lecture is about (Big Picture)

So far, we used definite integrals to find:

• Area under ONE curve

Now we extend this idea to find:

• Area BETWEEN two curves

This lecture covers:

• Area between $y = f(x)$ and $y = g(x)$
• Area when curves cross the x-axis
• Area when integration is done with respect to y

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2. Area Between Two Curves (First Area Problem)

Suppose we have two functions:

$y = f(x)$ (upper curve)
$y = g(x)$ (lower curve)

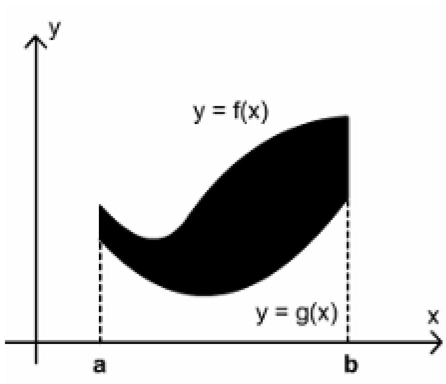
Both are continuous on an interval $[a, b]$.

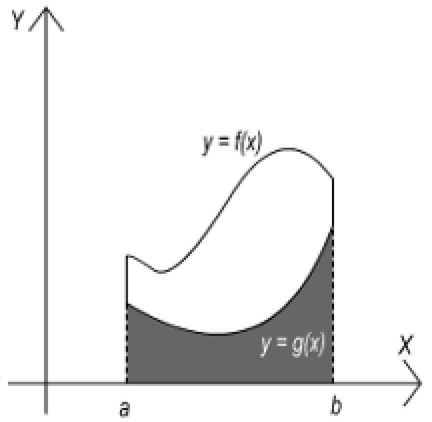
Also assume:

$f(x) \geq g(x)$ for all x in $[a, b]$

This means:

• $f(x)$ is always above $g(x)$
• The curves may touch but do NOT cross





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3. Idea Behind the Formula

Think of this region as:

Area between curves
= Area under top curve
− Area under bottom curve

So,

A = (Area under f) − (Area under g)

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4. Area Formula (Using Integrals)

In integral form:

A = ∫ from a to b [f(x) − g(x)] dx

This formula works whether the curves are above or below the x-axis.

Why?

Because we are subtracting the curves directly —
not measuring distance from the x-axis.

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5. What If f(x) and g(x) Are Negative?

If both curves go below the x-axis:

• We can shift both curves upward by a constant
• This does NOT change the area between them
• The constant cancels during subtraction

So the same formula still works:

A = ∫ from a to b [f(x) − g(x)] dx

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6. Important Things to Remember

When solving area-between-curves problems:

• Always identify the TOP curve
• Always identify the BOTTOM curve
• Decide the correct limits of integration
• Sometimes the region must be split

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7. Step-by-Step Procedure

1. Sketch the curves (very important)
2. Identify upper curve and lower curve
3. Find intersection points if limits are not given
4. Write integrand: (top − bottom)
5. Integrate and simplify

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8. Example 1 (Simple Vertical Slices)

Find the area bounded above by:

$y = x + 6$

and below by:

$y = x^2$

with vertical boundaries:

$x = 0$ and $x = 2$

Solution:

Upper curve: $f(x) = x + 6$
Lower curve: $g(x) = x^2$

Limits: $a = 0$, $b = 2$

Area formula:

$A = \int$ from 0 to 2 $[(x + 6) - x^2]\, dx$

Simplify integrand:

$= \int$ from 0 to 2 $(-x^2 + x + 6)\, dx$

Evaluate the integral to get the area.

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9. Example 2 (Limits Found by Intersection)

Find the area enclosed between:

$y = x^2$
$y = x + 6$

(No vertical boundary lines given)

Step 1: Find intersection points

Set equations equal:

$x^2 = x + 6$
$x^2 - x - 6 = 0$
$(x - 3)(x + 2) = 0$

So,

$x = -2$ and $x = 3$

Step 2: Identify curves

Upper curve: $y = x + 6$
Lower curve: $y = x^2$

Limits: -2 to 3

Step 3: Set up integral

$A = \int$ from -2 to 3 $[(x + 6) - x^2]\, dx$

Evaluate to get total area.

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10. When the Region Is Complicated

Sometimes:

• The top curve changes
• The bottom curve changes

Then:

• We split the region into parts
• Compute area of each part
• Add them together

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11. Example with Splitting the Region

Find the area enclosed by:

$x = y^2$
$y = x - 2$

Step 1: Convert both into same variable

From $y = x - 2 \rightarrow x = y + 2$

Now compare:

$x = y^2$
$x = y + 2$

Step 2: Find intersection points

$y^2 = y + 2$
$y^2 - y - 2 = 0$
$(y - 2)(y + 1) = 0$

So:

$y = -1$ and $y = 2$

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12. Why Integration with Respect to y Is Better Here

Using x = functions of y:

- Left boundary: $x = y^2$
- Right boundary: $x = y + 2$

Limits: $y = -1$ to $y = 2$

So we avoid splitting the region.

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13. Area Formula (Integration w.r.t y)

Area between $x = w(y)$ and $x = v(y)$:

$A = \int$ from c to d $[\, w(y) - v(y) \,]\, dy$

Where:

$w(y)$ = right boundary
$v(y)$ = left boundary

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14. Final Example (Clean Setup)

Find the area enclosed by:

$x = y^2$
$x = y + 2$

Limits: $y = -1$ to $y = 2$

Right boundary: $x = y + 2$
Left boundary: $x = y^2$

Area:

$A = \int$ from -1 to 2 $[(y + 2) - y^2]\, dy$

Evaluate to get the final area.

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15. Key Takeaway of Lecture 33

- Area between curves = difference of integrals
- Always subtract bottom from top (or left from right)
- Choose x or y wisely to avoid splitting
- Sketching saves time and prevents mistakes

This lecture shows how definite integrals solve real geometric problems —
not just abstract formulas.