Lecture 7: Operations on Functions

We already know what a function is: a rule that takes an input (x) and gives back an output (y).
Now, just like numbers can be added, subtracted, multiplied, or divided, we can do the same with **functions**.

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1. Arithmetic Operations on Functions

If we have two functions, f(x) and g(x), we can create **new functions** like this:

- **Addition**:
  (f + g)(x) = f(x) + g(x)
- **Subtraction**:
  (f – g)(x) = f(x) – g(x)
- **Multiplication**:
  (f · g)(x) = f(x) · g(x)
- **Division**:
  (f / g)(x) = f(x) / g(x), but only if g(x) ≠ 0

👉 The domain of the new function is usually the **intersection** of the domains of f and g.
👉 For division, we also remove points where g(x) = 0 (since division by zero is not allowed).

Example 1: Addition

Suppose:
$f(x) = x^2$
$g(x) = x$

Then:
$(f + g)(x) = f(x) + g(x) = x^2 + x$

✅ Domain of f(x) = all real numbers
✅ Domain of g(x) = all real numbers
👉 So domain of (f + g)(x) = all real numbers

Example 2: Subtraction

$f(x) = x^2 + 1$
$g(x) = x - 2$

Then:
$(f - g)(x) = (x^2 + 1) - (x - 2) = x^2 - x + 3$

Example 3: Multiplication

$f(x) = x$
$g(x) = \sqrt{x}$

Then:
$(f \cdot g)(x) = f(x) \cdot g(x) = x \cdot \sqrt{x} = x\sqrt{x}$

- Domain of $f(x) = (-\infty, \infty)$
- Domain of $g(x) = [0, \infty)$
  👉 Domain of $(f \cdot g)(x)$ = intersection = $[0, \infty)$

Example 4: Division

$f(x) = x^2 - 1$
$g(x) = x - 1$

Then:
$(f / g)(x) = (x^2 - 1) / (x - 1)$

Simplify numerator: $x^2 - 1 = (x - 1)(x + 1)$

So $(f / g)(x) = (x - 1)(x + 1) / (x - 1) = x + 1$, but only if $x \neq 1$

👉 Domain = all real numbers except $x = 1$

3. Special Notation

- $f^2(x)$ means $f(x) \cdot f(x)$
- $f^3(x)$ means $f(x) \cdot f(x) \cdot f(x)$
- In general: $f^n(x) = f(x)$ multiplied by itself n times

Example:
If $f(x) = \sin(x)$, then $f^2(x) = (\sin(x))^2 = \sin^2(x)$

✅ Summary:

- Functions can be added, subtracted, multiplied and divided
- Domains matter: new functions inherit the overlap of original domains, and division excludes points where denominator = 0.
- Special notation $f^2(x)$, $f^3(x)$, etc. means repeated multiplication of function values.

Composition of Functions

So far, we've seen **arithmetic operations** on functions (add, subtract, multiply, divide).
Now comes a **new type of operation**: **composition**.

👉 This has no analog in arithmetic — it's something special to functions.

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1. What is Composition?

Composition means: **apply one function, then feed its result into another function.**

Notation:

- $(f \circ g)(x) = f(g(x))$
- Read as "f composed with g of x"

Steps:

1. Take x (from the domain of g).
2. Compute g(x).
3. Plug g(x) into f(x).

Simple analogy:

- **Put your sock on first, then your shoe.**
- The order matters!

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2. Example 1

Let:
$f(x) = x^3$
$g(x) = x + 4$

Now, compute $(f \circ g)(x)$:

$(f \circ g)(x) = f(g(x))$
$= f(x + 4)$
$= (x + 4)^3$

👉 Domain:

- $g(x) = x + 4 \rightarrow$ domain = $(-\infty, \infty)$
- $f(x) = x^3 \rightarrow$ domain = $(-\infty, \infty)$
- So domain of $f \circ g = (-\infty, \infty)$

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3. Example 2

Let:
$f(x) = x^2 + 3$
$g(x) = \sqrt{x}$

Now:

$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 + 3 = x + 3$

👉 Domains:

- $g(x) = \sqrt{x} \rightarrow$ domain = $[0, \infty)$
- $f(x) = x^2 + 3 \rightarrow$ domain = $(-\infty, \infty)$
- But in composition, domain must be valid for both.
  👉 So domain of $(f \circ g)(x) = [0, \infty)$

Notice: if we switch the order $\rightarrow (g \circ f)(x) = g(f(x)) = g(x^2 + 3) = \sqrt{(x^2 + 3)}$, which is totally different.

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4. Key Idea: Order Matters

$(f \circ g)(x) \neq (g \circ f)(x)$ in general.

- First sock then shoe ≠ first shoe then sock.
- That's why order of composition is important.

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5. Decomposition of Functions

Sometimes a complicated function can be **broken into simpler functions**.
This is called **decomposition**.

Example:
$h(x) = (x + 1)^2$

We can write it as:

- $g(x) = x + 1$
- $f(x) = x^2$
- Then $h(x) = f(g(x))$

Another Example:
$h(x) = \sqrt{2x + 5}$

We can split it as:

- $g(x) = 2x + 5$
- $f(x) = \sqrt{x}$
- Then $h(x) = f(g(x))$

👉 Decomposition is useful because it makes complex functions easier to understand and work with.

✅ **Summary**:

- Composition = plugging one function into another.

- $(f \circ g)(x) = f(g(x))$
- Order matters → $(f \circ g)(x)$ is usually different from $(g \circ f)(x)$.
- Domain of $(f \circ g)$ = all x in domain of g for which g(x) is in domain of f.
- Complicated functions can be decomposed into simpler ones.

Function	g(x) Inside	f(x) Outside	composition
$(x^2+1)^{10}$	x^2+1	x^{10}	$(x^2+1)^{10}=f(g(x))$
$\sin^3 x$	$\sin x$	x^3	$\sin^3 x=f(g(x))$
$1/(x+1)$	$x+1$	$1/x$	$1/(x+1) = f(g(x))$
$\tan(x^5)$	x^5	$\tan x$	$\tan(x^5)=f(g(x))$

Classification of Functions

Functions can come in many types. Let's start with the basic ones you'll see most often.

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1. Constant Function

👉 A function that always gives the **same number**, no matter what x is.

Example:
$f(x) = 2$

- $f(1) = 2$
- $f(-7) = 2$
- $f(100) = 2$

✅ Output never changes.

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2. Monomial in x

👉 A monomial is of the form:
$f(x) = c \cdot x^n$

- where **c is a constant**
- **n is a nonnegative integer (0, 1, 2, 3, …)**

Examples of monomials:

- $f(x) = 5x^5$
- $f(x) = 2x$
- $f(x) = 3$ (because $x^0 = 1$, so constants are also monomials)

❌ Not monomials:

- $f(x) = x^{-2}$ (negative power)
- $f(x) = \sqrt{x} = x^{(1/2)}$ (fractional power)

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3. Polynomial in x

👉 A polynomial is a **sum of monomials**.

General form:
$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$

- where $a_0, a_1, a_2 \ldots a_n$ are constants (called coefficients)
- n is a nonnegative integer

Examples:

- $f(x) = 4x^2 + 3x – 1$
- $f(x) = 17x^3 + 4x^2 – 5$

✅ Polynomials are just combinations of terms like $c \cdot x^n$.

💡 **Quick Summary Table**

Type of Function	Formula	Example	Key Idea
Constant	$f(x) = c$	$f(x) = 2$	Always the same value
Monomial	$f(x) = c \cdot x^n$ $(n \geq 0)$	$f(x) = 5x^3$	Single power term
Polynomial	$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$	$f(x) = 3x^3 + 2x – 7$	Sum of monomials