✳️ THE CHAIN RULE

Let’s start with a question:
What happens when one quantity depends on another, and *that* one depends on yet another?

In other words —
what if **y depends on u**, and **u depends on x**?
How does y change when x changes?

That’s the *core mystery* the Chain Rule solves.

🧠 Step 1: The Setup — A Function Inside Another

We have a composition like
y = f(g(x))
(read as “f of g of x”).

That means:

- x changes → g(x) changes → f(g(x)) changes.
 There’s a whole *chain of cause and effect*.

So, the total rate of change of y with respect to x must take into account **both links** in this chain.

🔍 Step 2: The Key Idea

Let’s write it more visually:

```
x → g(x) → f(g(x))
```

Now, when x changes a little bit (Δx), g(x) also changes a bit (Δg), and because g changed, f changes too (Δf).

So the overall rate is:

```
(change in y) / (change in x)
= (change in y / change in u) × (change in u / change in x)
```

In calculus language:

```
dy/dx = (dy/du) × (du/dx)
```

That's it — the whole Chain Rule is hidden inside this simple reasoning.

🧩 Step 3: Writing the Formal Theorem

If

- g(x) is differentiable at x, and
- f(u) is differentiable at u = g(x),

then the composition y = f(g(x)) is differentiable, and

```
dy/dx = f'(g(x)) · g'(x)
```

This is the **Chain Rule**.

In short:
👉 Differentiate the *outer function* (f),
👉 keep the *inside* (g(x)) as it is,
👉 then multiply by the derivative of the *inside* (g′(x)).

💡 Step 4: Intuitive Analogy

Imagine a speedometer on a car.
The car's **speed (y)** depends on the **engine's rotation (u)**,
and the engine's rotation depends on **how far you press the pedal (x)**.

So if you press the pedal slightly,
the engine speeds up (du/dx),
and then that engine speed makes the car move faster (dy/du).

The total change in car speed (dy/dx) is the *product* of these two effects.

That's why we multiply the derivatives.

🧮 Step 5: Example

Find the derivative of

$$y = [4\cos(x)]^3$$

Step 5.1 — Identify the composition

We can see:

- Outer function: $f(u) = u^3$
- Inner function: $g(x) = 4\cos(x)$

Then

$$y = f(g(x)) = (4\cos(x))^3$$

Step 5.2 — Differentiate using the Chain Rule

According to the rule:

$$dy/dx = f'(g(x)) \cdot g'(x)$$

Compute each part:

- $f'(u) = 3u^2$
- $g'(x) = 4 \times (-\sin(x)) = -4\sin(x)$

Now substitute $u = 4\cos(x)$:

$$dy/dx = 3\,[4\cos(x)]^2 \times (-4\sin(x))$$

Simplify step-by-step:

```
= 3 × 16 × 4 x cos²(x) × (-sin(x))
= -192 cos²(x) sin(x)
```

✅ **Final Answer:**

```
dy/dx = -192 cos²(x) sin(x)
```

🔁 Step 6: Memory Trick

You can think of the Chain Rule as "canceling the du" in

```
dy/dx = (dy/du) × (du/dx)
```

It's not real algebra — it's just a visual way to remember how everything connects:

```
dy    dy    du
—— = —— × ——
dx    du    dx
```

The *du* in the middle is like the "bridge" between f and g.
That's why it's called the **chain** rule — it connects two rates of change into one.

🌈 Step 7: Feynman's Takeaway

The Chain Rule is not just a formula —
it's a story of **how one change triggers another**.

Whenever something depends on something else,
and that something depends on *yet another thing*,
the total rate of change is the **product** of each small link in the chain.

That's all calculus really is —
understanding how motion, growth, and change travel through these invisible connections.

🔹 Summary Table

Concept	**Formula**	**Meaning**
Chain Rule	$(f(g(x)))' = f'(g(x)) \cdot g'(x)$	Multiply outer derivative by inner derivative
Visualization	$dy/dx = (dy/du)(du/dx)$	Links rates through the “chain” variable u
Example	$y = (4\cos(x))^3 \rightarrow dy/dx = -192 \cos^2(x) \sin(x)$	Composition of functions

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🧩 The Generalized Derivative Formula — “Chain Rule Made Simple”

Let’s start with a thought:

When something changes, and that thing depends on another thing which also changes...
how do we measure the total change?

We’ve seen this before — that’s exactly the Chain Rule.

But this time, we’ll write it in a way that’s even simpler and more powerful.

🧠 Step 1: Recall the Chain Rule

The chain rule says:

$dy/dx = (dy/du) \times (du/dx)$

It’s like saying:

“How y changes with x” = “How y changes with u” × “How u changes with x.”

Each piece measures part of the story —
you just multiply them to see the full effect.

🧩 Step 2: Express dy/du using f(u)

Suppose $y = f(u)$.

Then when we differentiate with respect to u, we get:

$dy/du = f'(u)$

Now, put this back into the chain rule:

$dy/dx = f'(u) \times (du/dx)$

That’s our **Generalized Derivative Formula** — short, neat, and super practical.

💡 Step 3: What It Really Means

Think of it like this:

You have a machine that does two steps:

1. First, x goes into a box and becomes u.
2. Then u goes into another box and becomes y.

So, when x changes slightly, both boxes react —
and their combined sensitivity is the **product** of both derivatives.

That's the intuition behind $f'(u) \times (du/dx)$.

⚙️ Step 4: Example 1 — Polynomial Composition

Let's say:
$f(x) = (x^2 - 1)^3$

We can let:
$u = x^2 - 1$
Then $f(x) = u^3$.

Now apply the formula:
$dy/dx = f'(u) \times (du/dx)$

Compute each part:
$f'(u) = 3u^2$
$du/dx = 2x$

Substitute back $u = x^2 - 1$:
$dy/dx = 3(x^2 - 1)^2 \times (2x)$

Simplify:
$dy/dx = 6x(x^2 - 1)^2$

✅ Simple, elegant, and exactly what you'd get using the long version of the chain rule — but in one smooth step.

🧮 Step 5: Example 2 — Trig Function Inside Another

Find: $d/dx\,[\sin(2x)]$

Let:
$u = 2x$
Then $y = \sin(u)$

So:
$dy/dx = (dy/du) \times (du/dx)$

Compute:
$dy/du = \cos(u)$
$du/dx = 2$

Substitute back $u = 2x$:
$dy/dx = \cos(2x) \times 2 = 2\cos(2x)$

✅ Clean and fast.

🧩 Step 6: Example 3 — Combination of Algebra and Trig

Find: $d/dx\,[\tan(x^2 + 1)]$

Let:
$u = x^2 + 1$
Then $y = \tan(u)$

Now:
$dy/dx = (dy/du) \times (du/dx)$

Compute:
$dy/du = \sec^2(u)$
$du/dx = 2x$

Substitute $u = x^2 + 1$:
$dy/dx = \sec^2(x^2 + 1) \times 2x$

✅ So:
$dy/dx = 2x \sec^2(x^2 + 1)$

🧠 Step 7: Why It's "Generalized"

Because this formula works no matter what u is —
it could be a polynomial, a trig function, a logarithm, or an exponential.

It doesn't care what's inside —
you just take the **outer derivative (f′)** and multiply by the **inner derivative (du/dx)**.

It's like a universal rule for nested functions.

🧩 Step 8: Quick Reference Table

Function	Let u =	Result
sin(2x)	2x	2cos(2x)
$\tan(x^2 + 1)$	$x^2 + 1$	$2x \sec^2(x^2 + 1)$
$(x^2 - 1)^3$	$x^2 - 1$	$6x(x^2 - 1)^2$
e^(3x)	3x	3e^(3x)
$\ln(x^4)$	x^4	4/x

🔑 Step 9: Feynman's Takeaway

You can think of the **generalized derivative** as a "chain rule in disguise."
It's the same idea — just written in a simpler, more compact way.

Whenever you see something *inside something else* —
don't panic, don't expand — just **differentiate the outside**,
then **multiply by the derivative of the inside.**

That's it.
No magic, no shortcuts — just *cause and effect* written in calculus form.

🧩 How Do We Know What to Let *u* Equal?

Let's start with the big question students always ask:
"How do I know what to choose for u in the chain rule?"

Here's the simple rule of thumb:

> **Pick u to simplify the problem — so that what remains is something you already know how to differentiate.**

That's it.
You're not guessing blindly; you're making a *smart substitution*.

🧠 Step 1: The Goal of Substitution

The idea is to spot a part of your function that's **nested** inside something else.
That inner part becomes *u.*

You choose *u* so that when you replace it,
the remaining expression turns into something you can easily differentiate —
like sin(u), cos(u), tan(u), e^u, ln(u), etc.

⚙️ Step 2: Let's See an Example

We want to find:

$\frac{d}{dx}[\cos^3(x + e^x)]$

This means:
$\cos(x + e^x)$ is raised to the 3rd power.

Looks messy, right?
That's our clue that we should use substitution.

🧩 Step 3: Choose *u* Wisely

Inside the big cube, there's $\mathbf{\cos(x + e^x)}$.
So, let's choose:

$u = \cos(x + e^x)$

Now, rewrite the original expression:

$y = u^3$

Ah — that's much simpler!
Now we can differentiate step by step.

🧮 Step 4: Differentiate Step by Step

We know:

$dy/dx = (dy/du) \times (du/dx)$

Compute each part:

- $dy/du = 3u^2$
- du/dx = derivative of $\cos(x + e^x)$

Let's handle du/dx carefully:

$du/dx = -\sin(x + e^x) \times (1 + e^x)$

(We used the chain rule again inside, since $x + e^x$ is also composite.)

Now, multiply them together:

$dy/dx = 3u^2 \times [-\sin(x + e^x) \times (1 + e^x)]$

🔁 Step 5: Substitute *u* Back

Replace u with $\cos(x + e^x)$:

$dy/dx = 3[\cos(x + e^x)]^2 \times [-\sin(x + e^x)(1 + e^x)]$

Simplify:

$dy/dx = -3[\cos(x + e^x)]^2 \sin(x + e^x)(1 + e^x)$

✅ Done.

We used substitution to make a messy function clean, and the chain rule to handle the layers.

💡 Step 6: Why That *u* Worked

If we had picked something else — like $u = x + e^x$ —
we'd still have a function inside a cube and another cosine sitting on top.

That wouldn't simplify much.

But by choosing $u = \cos(x + e^x)$,
the outer part (the cube) became **simple** — u^3 —
and we could differentiate it easily.

So:

> Always pick *u* that "collapses" the structure and makes the outer layer simple.

🧩 Step 7: The Shortcut Version (Without Explicit *u*)

Sometimes, you don't even need to name *u.*

Just follow the mental pattern of the chain rule:

> "Differentiate the **outer** function,
> then multiply by the derivative of the **inner** function."

⚙️ Example 2: d/dx [cos(3x + 1)]

Outer function: cos(·)
Inner function: (3x + 1)

Step 1 — derivative of the outer (cos): -sin(·)
Step 2 — multiply by derivative of the inner (3x + 1): 3

So:

$\frac{d}{dx}[\cos(3x + 1)] = -\sin(3x + 1) \times 3$
Simplify → $-3 \sin(3x + 1)$

✅ That's it.
We didn't even have to explicitly say $u = 3x + 1$,
because we already know how the chain rule pattern works.

🧠 Step 8: Feynman's Takeaway

If you ever feel lost about what to let *u* be,
ask yourself this simple question:

> "What substitution will make the rest look like something I already know how to differentiate?"

That's your *u.*

You're not guessing —
you're simplifying the world, layer by layer, until what's left is something familiar.

That's the essence of calculus —
turning something complex into something simple, step by step.