What is Calculus?

Imagine you're watching a car drive down the road. You don't just want to know how far it went after an hour—you want to know **how fast it's moving at every single moment**. That's what calculus is all about: understanding **how things change, continuously**.

In other words, calculus studies the rate at which one quantity changes with respect to another. For example:

- How does **distance** change w.r.t **time**?
- How does **time** change w.r.t **speed**?
- How does **the amount of water flowing** change w.r.t **time**?

It's about zooming in, moment by moment, to see **the continuous flow of change**. And don't worry—we'll also see exactly what we mean by "continuously" in a way that actually makes sense, not just in a textbook.

Let's start with **Real Numbers**. Don't worry about **complex or imaginary numbers** for now — those are like a side adventure you can read about later. Today, we'll focus on numbers that actually appear on the number line, the ones we use in everyday life.

First, we'll take a little journey through **history**: how humans discovered these numbers, why they were a big deal, and how they slowly became part of our mathematical world. Then, we'll explore the **different types of real numbers** — fractions, whole numbers, decimals, and more.

From now on, we'll just call them **"Reals"** for short. Ready? Let's jump in and see how these numbers grew from simple counting into the foundation of all modern math.

Natural Numbers

1, 2, 3, 4, 5, …

These are called **natural numbers** because they were the very first numbers humans ever thought about. Think about it: if you wanted to count goats, or apples, or steps, what numbers would you use? Exactly — 1, 2, 3, 4…

Our ancestors discovered these numbers naturally, just by **counting things around them**. That's why we call them "natural numbers." They're the first numbers our minds met, and they still form the foundation of all math today.

Integers, Sets, and Subsets

The **natural numbers** — 1, 2, 3, … — are just the beginning. They belong to a bigger family called **integers**, which also includes zero and negative numbers.

Now, let's talk about **sets**. A set is just a **collection of things**. It could be anything: apples, cars, or even politicians.

For example, imagine this set of politicians:

{Donald Trump, Tony Blair, Ronald Reagan} B

A **subset** is simply a part of that collection. Like:

{Donald Trump, Tony Blair} A

Every member of a subset is also a member of the bigger set. Mathematicians write this relationship as:

$$A \subseteq B$$

Here's how to **read it aloud**: "A is a subset of B." It means **everything in A is also in B**.

So, in our politician example, the smaller set {Donald Trump, Tony Blair} is a subset of the bigger set {Donald Trump, Tony Blair, Ronald Reagan}.

Notice the **curly brackets { }** — we always use them to denote a set. Later, we'll explore more **notations and rules for sets**, but for now, just remember:

- **Set = collection of things**
- **Subset = part of that collection**
- **$A \subseteq B$ = everything in A is also in B**

Integers

Let's go back to the **integers**:

… -4, -3, -2, -1, 0, 1, 2, 3, 4 …

Think of them as **natural numbers**, plus **zero**, plus the **negatives** of the natural numbers.

Now, you might wonder: why wasn't zero part of the natural numbers? Zero has a fascinating story. In **ancient Greece**, it was sometimes called the **"number of the devil"**. In **Indian philosophy**, it was embraced more easily, and later **Arab and European mathematicians** promoted it for commerce and science. Today, we happily include it in the **set of integers**.

And what about the **negative numbers**? These weren't "discovered" in the usual sense — they were a kind of **artificial invention**. But they solved problems. For example, take the simple equation:

$$X + 2 = 0$$

To make it true, we need $x = -2$. Suddenly, negative numbers aren't so scary — they're useful!

So integers are basically: **all natural numbers, zero, and their negatives**. And each of these numbers has a story that shaped the way humans think about math.

Rational Numbers

So far, we've got the naturals and the integers. But there's more. Integers themselves belong to an even bigger family of numbers called the *rational numbers*.

What are rationals? Simple: they're just ratios of integers — fractions. As long as we don't divide by zero, we're safe.

Examples:
2/3, 7/5, 6/1, -5/2

Notice something interesting: every integer is also a rational number. Why? Because you can always write an integer as a fraction. For example, 5 can be written as 5/1. So integers are automatically part of the rationals.

Now, you might ask: why can't we divide by zero?

Good question. Let's test it.

- If $x \neq 0$, then an equation like $x/0 = y$ leads to contradictions — it just doesn't make sense.

- If $x = 0$, then $0/0$ could equal *any number* y. It has infinitely many answers. And that's a disaster in math, because we want clear, unique values.

  $0/0 = 1$, $0/0 = 2$ and so on

So division by zero is undefined. It's like asking, "What's the color of a sound?" — the question itself doesn't fit.

That's why we avoid dividing by zero. It keeps our number system consistent and logical.

So, to sum up: **Rational numbers = all ratios of integers (except dividing by zero). Integers are part of them.**

Irrational Numbers

Now we come to a fascinating chapter in the story of numbers: the discovery of *irrational numbers*.

Let’s go back to Pythagoras, the ancient Greek philosopher and mathematician. He loved numbers so much that he studied them for their own sake — not just for solving practical problems. In fact, he treated mathematics almost like a religion.

Pythagoras believed that every physical quantity could be expressed neatly as a whole number plus some fraction m/n. In other words, he thought **all numbers were rational**. And rational numbers had a nice property: when you write them in decimal form, the digits either end, or they repeat in a clear pattern.

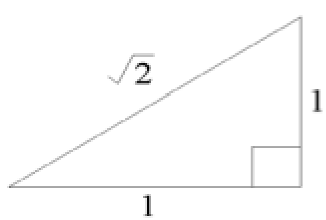
Examples:

- 1/2 = 0.5 (ends)
- 1/3 = 0.3333… (repeats forever)

This fit perfectly with Pythagoras’ belief that the universe was orderly and made of rational numbers.

But then came a shock. In the 5th century B.C., Hippasus of Metapontum showed that not every number could be written as a ratio of integers. Using geometry, he proved that the length of the diagonal (hypotenuse) of a right triangle with both legs equal to 1 could not be expressed as a fraction. That diagonal has length √2 — and √2 is **irrational**.

√2 = 1.4142135



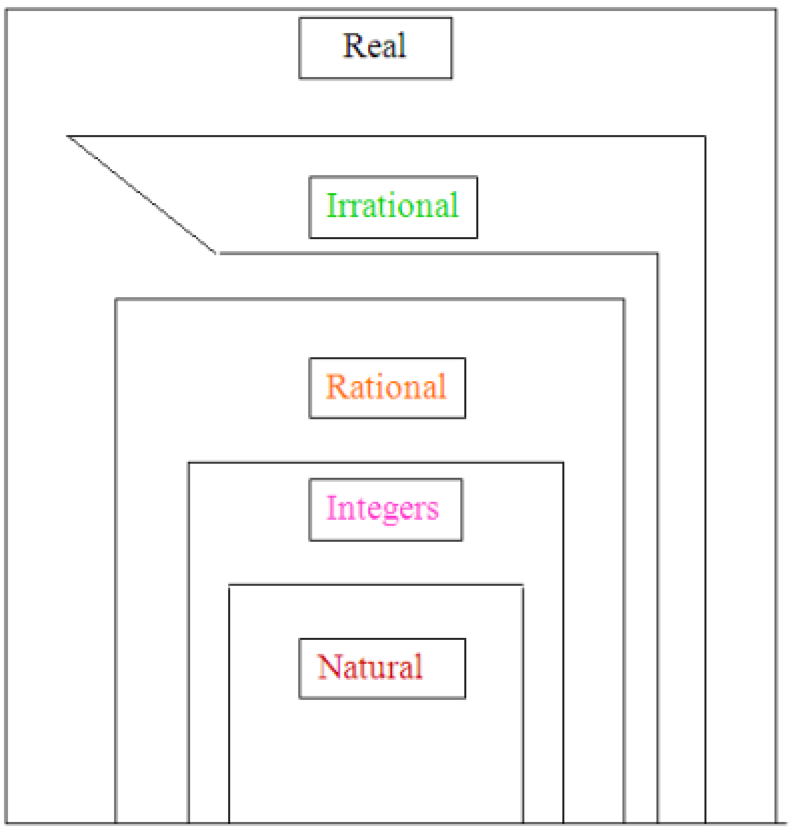
An irrational number is one that **cannot be written as a ratio of integers**. Its decimal expansion goes on forever without repeating.

Other famous irrational numbers include $\sqrt{3}$, $\sqrt{5}$, and even π (pi).

So, the rational numbers and the irrational numbers together make up a bigger family: the **Real Numbers**.

That’s why we sometimes call this whole system the *Real Number Line* — it’s the complete collection of rationals and irrationals stretching endlessly in both directions.

Pictorial summary of the hierarchy of REAL NUMBNERS



Coordinate Line

In the 1600s, something revolutionary happened in math: the birth of *analytic geometry*. This gave us a way to connect two worlds:

- You could **draw a picture of an equation**.
- And you could **write an equation for a picture**.

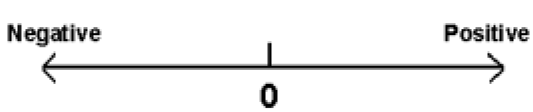
In other words, algebra and geometry shook hands.

The idea came from the French mathematician René Descartes. There’s even a story about him. One day, he wanted to figure out what really makes us human. He shut himself away in a cold, dark furnace room (don’t worry, it wasn’t burning!) and cut himself off from the world. Without his senses, he realized he could still think. From that came his famous line: *“I think, therefore I am.”*

Now, back to geometry. Descartes’ key insight was to connect **real numbers with points on a line**. Here’s how it works:

1. Draw a straight line.
2. Choose one direction as **positive** (usually to the right) and the other as **negative** (to the left). We could have chosen the opposite, but by cultural convention right = positive, left = negative. This is now the standard.

3. Pick one special point on the line and call it the **origin**. We mark it with the number 0.
4. Choose a unit of length — say 1 cm — and mark equal steps in both directions. To the right we get +1, +2, +3 … and to the left we get -1, -2, -3 …



Now we’ve built something powerful: the **coordinate line** (also called the real line).

Here’s the rule:

- The origin corresponds to 0.
- Every positive number corresponds to a point that many units to the right of the origin.
- Every negative number corresponds to a point that many units to the left of the origin.

And the number that matches each point is called its **coordinate**.

So now, every real number lives on this line. And every point on this line has a real number. Numbers and geometry have finally merged into one picture.

Example 1

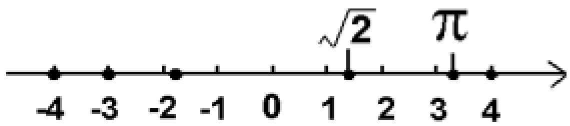
Let’s put some points on our coordinate line. Imagine we’ve marked the positions of these numbers:

-4, -3, -1.75, -0.5, π, √2, and 4.

Now, π (pi) and √2 aren’t “nice” fractions, so we can’t place them exactly. Instead, we use their decimal approximations:

- $\pi \approx 3.14$
- $\sqrt{2} \approx 1.41$

So, we put π a little past 3, and √2 a little past 1.



Here’s the key idea: **every real number has a home on the line**. If you pick any real number — even something strange like -1.75 or π — there is exactly one point that belongs to it.

And the reverse is also true: **every point on the line represents exactly one real number**.

This is what mathematicians mean when they say: *“The real numbers and the points on the coordinate line are in one-to-one correspondence.”*

In plain English: numbers and points match up perfectly, one by one.

Order Properties

In mathematics, there’s an important idea called the *order* of a set. Don’t worry — we won’t dive into deep set theory here. Instead, we’ll look at how order works for the set of real numbers.

Here’s the rule:

For any two real numbers, say *a* and *b*:

- If $b - a$ is positive, then $b > a$ (we say “b is greater than a”) or, equivalently, $a < b$ (“a is less than b”).

You’ve seen these symbols before:

- “<” means *less than*
- “>” means *greater than*

A statement that uses < or > is called an **inequality**. Notice: saying $a < b$ is the same thing as saying $b > a$.

So, what does this really mean? The order of the real numbers tells us how to compare their “size.” But size only makes sense when we compare two numbers. For example, saying “3 is larger” is incomplete. Larger than what? But saying “$3 > 2$” makes sense.

Now let’s add a bit more:

- The inequality $a \leq b$ means *a is less than or equal to b*. So it’s true if $a < b$ **or** $a = b$.
  Example: $2 \leq 6$. This is true because 2 is less than 6.
- Similarly, $a \geq b$ means *a is greater than or equal to b*.

What about something like $a < b < c$? This means:

- $a < b$ and $b < c$.
  In words: b is between a and c.

Here’s how this connects to the coordinate line:

- As you move to the **right**, numbers get bigger.

- As you move to the **left**, numbers get smaller.

So on the number line:

- If $a < b$, then a lies to the left of b.
- If $a < b < c$, then a is to the left of b, and b is to the left of c.

In short: inequalities are just a way to describe the *order* of numbers on the number line.

Example: Correct and Incorrect Inequalities

Let’s look at some examples of inequalities and see which ones are correct.

Correct inequalities:

- $3 < 8$
- $7 > 1.5$
- $12 \geq 5$
- $0 \leq 2 \leq 4$
- $\pi \leq 12$
- $5 \geq 3$
- $0 < 1 < 3$

Notice how each statement makes sense when you compare the numbers. Each inequality correctly describes the *order* of the numbers.

Incorrect inequalities:

- $2 \geq 4$ (false, because 2 is less than 4)
- $0 \leq 5 < 3$ (false, because 5 is not less than 3)

- $-\pi > 0$ (false, because $-\pi$ is negative)

Remark:

Sometimes we want to be precise about numbers that are greater than or equal to zero versus strictly greater than zero:

- A number is called **nonnegative** if it satisfies $a \geq 0$. This means it can be zero or positive.
- A number is called **positive** if it satisfies $a > 0$.

So, every positive number is nonnegative, but not every nonnegative number is positive — zero is the exception.

These basic properties of inequalities are used a lot in calculus. We won't go through formal proofs here, but later we'll look at examples that show how these rules work in practice.

Theorem 1.1.1: Properties of Inequalities

Let's go through some important rules about inequalities. Don't worry — we'll keep it intuitive rather than formal.

a. Transitivity:
If $a < b$ and $b < c$, then $a < c$.

- In words: if a is less than b, and b is less than c, then a is definitely less than c.

b. Adding or subtracting the same number:
If $a < b$ and you add or subtract the same number c, the inequality still holds:

- $a + c < b + c$
- $a - c < b - c$

c. Multiplying by a number:

- If $c > 0$ (positive), then multiplying preserves the inequality:
  - If $a < b$, then $a \cdot c < b \cdot c$
- If $c < 0$ (negative), multiplying **flips the inequality**:
  - If $a < b$, then $a \cdot c > b \cdot c$

d. Adding inequalities:
If $a < b$ and $c < d$, then:

- $a + c < b + d$
- In other words, you can combine inequalities by adding corresponding sides.

e. Ratios of numbers with the same sign:
If a and b are **both positive** or **both negative**, and $a < b$, then:

- a / b < 1 (if positive)
- Or equivalently, certain proportional relationships hold — the main idea is that **inequalities behave predictably when numbers share the same sign**.

Remark:
All these rules also work if we replace < and > with ≤ and ≥.

Intervals

Earlier, we talked a little about sets. Now, let's focus on a special kind of set that is **very important in calculus**: *intervals*.

First, a quick refresher on some set notation:

- **a** ∈ **A** means *a is an element of the set A*
- **a** ∉ **A** means *a is NOT in A*
- ∅ represents the *empty set*, a set that contains nothing
- **A** ∪ **B** is the set of all elements in A or B (or both)
    - Example: A = {1,2,3,4}, B = {1,2,3,4,5,6,7} → A ∪ B = {1,2,3,4,5,6,7}
- **A ∩ B** is the set of elements in both A and B
    - Example: A ∩ B = {1,2,3,4}
- **A = B** means the sets are exactly the same
- **A** ⊂ **B** means A is a *subset* of B

You can list the members of a set using braces { }. For example:

- The set of positive integers less than 5: {1,2,3,4}
- The set of positive even integers: {2,4,6,…} — the dots mean the pattern continues.

Sometimes a set is too big (or infinite) to list every element. Then we use **set-builder notation**:

- Example: $\{x : x \text{ is a real number and } 2 < x < 3\}$
- Read this as: *"the set of all x such that x is a real number and 2 < x < 3."*
- In simpler terms, it's all the numbers between 2 and 3.

Now, here's where **intervals** come in:

- Geometrically, an interval is just a **segment on the coordinate line**.
- If a and b are real numbers with $a < b$, an interval is the line connecting a and b.

But there's a catch: sometimes the endpoints are included, sometimes they're not. So we define:

1. **Closed Interval [a, b]**
   - Includes the endpoints a and b
   - Set notation: $[a, b] = \{x : a \leq x \leq b\}$
   - Geometrically, draw a solid line with **solid dots** at a and b
2. **Open Interval (a, b)**
   - Excludes the endpoints a and b
   - Set notation: $(a, b) = \{x : a < x < b\}$
   - Geometrically, draw a line with **open dots** at a and b

These ideas let us describe all kinds of intervals:

- Closed, open, half-open, infinite intervals, etc.
- The notation tells you whether the endpoints are included (square brackets) or not (parentheses).

Intervals are like **slices of the number line**, and they are everywhere in calculus — from defining domains to limits and integrals.

INTERVAL NOTATION	SET NOTATION	GEOMETRIC PICTURE	CLASSIFICATION
(a,b)	$\{x : a < x < b\}$	a b	Finite ; open
[a,b]	$\{x : a \leq x \leq b\}$	a b	Finite ; closed
[a,b)	$\{x : a \leq x < b\}$	a b	Finite ; half-open
(a,b]	$\{x : a < x \leq b\}$	a b	Finite ; half open
(- ∞,b]	$\{x : x \leq b\}$	b	Infinite ; closed
(-∞,b)	$\{x : x < b\}$	b	Infinite ; open

Infinite Intervals

Intervals don't always have to stop at a number. Some intervals **stretch forever** in one direction or both.

To describe this, we use the symbols:

- **−∞** (read "negative infinity") → the interval goes endlessly to the left
- **+∞** (read "positive infinity") → the interval goes endlessly to the right

Important: infinity is **not a number**. It just tells us that the interval **never ends**.

An interval that goes on forever in either direction is called an **infinite interval**. For example:

- [a, +∞) → starts at a and goes forever to the right
- (−∞, b] → starts from negative infinity and stops at b

Intervals that start and end with **finite numbers** are called **finite intervals**.

Sometimes, a finite interval includes **one endpoint but not the other**. This is called **half-open** (or half-closed):

- [a, b) → includes a, but not b
- (a, b] → includes b, but not a

Some infinite intervals can also be considered “closed” if they include their finite endpoint:

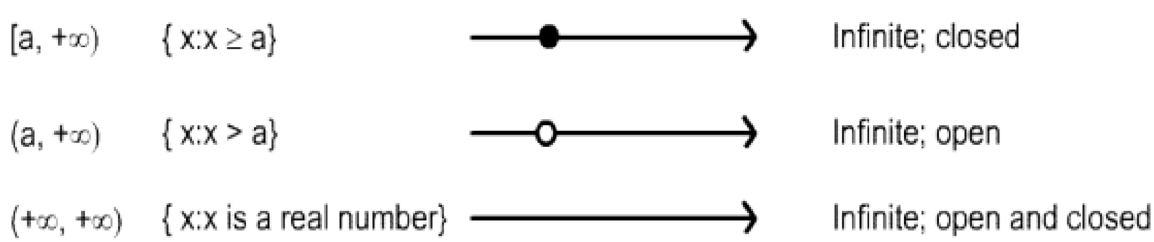
- [a, +∞) or (−∞, b]

Other infinite intervals with no endpoints at infinity, like (−∞, +∞), are considered **both open and closed**.

As one of my topology instructors liked to say:

> “A set is not a door! It can be OPEN, it can be CLOSED, and it can be OPEN and CLOSED!!”

Take a moment to picture this on the number line. Imagine the arrows stretching off forever, solid dots for included endpoints, open dots for excluded ones. Let that sink in.



Solving Inequalities

We've talked about inequalities before, but now let's see how to **solve them** — that is, how to find all numbers that make the inequality true.

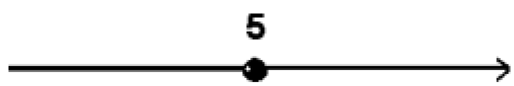
Consider a simple inequality:

$x < 5$

- If $x = 1$, it works, because $1 < 5$. ✅
- If $x = 7$, it doesn't work, because 7 is not less than 5. ❌

The **solution set** of an inequality is the set of all numbers that make it true.

- For $x < 5$, the solution set is all numbers less than 5.



Important Fact:

If you **don't multiply both sides by zero or an expression containing the unknown**, then the rules from **Theorem 1.1.1** (adding, subtracting, multiplying/dividing by positive/negative numbers) **do not change the solution set**.

The process of finding the solution set is called **solving the inequality**.

Example 1

Solve:
$3x - 7 \leq 2x + 9$

Solution:
We want to isolate x on one side:

1. Subtract 2x from both sides:
   $3x - 2x - 7 \leq 9$
   $\rightarrow x - 7 \leq 9$

2. Add 7 to both sides:
   $x \leq 16$

Notice: we didn't multiply by anything involving x, so the solution set remains valid.

✅ **Solution set:** $x \leq 16 \rightarrow$ interval: $(-\infty, 16]$

Example 2

Solve:
$-7 \leq 25 - 9x < 9$

This is a **compound inequality**, meaning it combines two inequalities. We can solve it step by step.

1. Subtract 25 from all parts:
   $-32 \leq -9x < -16$

2. Divide by −9. **Important:** dividing by a negative number **reverses the inequality signs**:
   $32/9 \geq x > 16/9$

3. Rewrite with smaller number first:
   $16/9 < x \leq 32/9$

✅ **Solution set:** $x \in (16/9, 32/9]$

Notice how solving inequalities is similar to solving equations — we just follow the rules carefully, **paying attention when multiplying or dividing by negative numbers**.

You can solve more inequalities in the same way and write the solution as **intervals on the number line**.