📘 Lecture 20: Derivatives of Inverse Functions

🧩 Part 1: What Is $\log_b(x)$ — Really?

Let's begin with the definition:

> $\log_b(x) = y$ means that $b^y = x$

In words:
👉 "The power y that you raise b to — gives you x."

So if **b = 2** and **x = 8**,
then **$\log_2(8) = 3$**, because $2^3 = 8$.

That's it — the logarithm is just another way of asking:

> "What power of b gives me x?"

🧠 Part 2: What Are We Trying to Find?

We want **dy/dx** when
$y = \log_b(x)$

That is, how fast does $\log_b(x)$ change when x changes?

⚙️ Step 1: Rewrite in Exponential Form

From definition:
$\log_b(x) = y \Longrightarrow b^y = x$

Now we can use **implicit differentiation** on both sides with respect to x.

Differentiate both sides:

$d/dx(b^y) = d/dx(x)$

⚙️ Step 2: Differentiate Each Side

On the right:

- $d/dx(x) = 1$

On the left:

- b^y is a function of y, and y depends on x,
  so we'll use the chain rule.

Derivative of b^y with respect to y is:
$b^y \cdot \ln(b)$

Then multiply by dy/dx (because y depends on x):

So:
$b^y \cdot \ln(b) \cdot dy/dx = 1$

⚙️ Step 3: Solve for dy/dx

We want dy/dx, so divide both sides by $b^y \ln(b)$:

$dy/dx = 1 / [b^y \ln(b)]$

But from before, **$b^y = x$**. Substitute that in:

$dy/dx = 1 / [x \ln(b)]$

✅ That's our general derivative formula for $\log_b(x)$:

$d/dx [\log_b(x)] = 1 / (x \ln(b))$

💡 Step 4: The "Special Case" — Base e (Natural Log)

If the base happens to be **e** (Euler's number ≈ 2.718),
then ln(e) = 1.

So the formula becomes beautifully simple:

d/dx [ln(x)] = 1 / x

This is why **ln(x)** is the "natural" logarithm —
it's the cleanest version in calculus.

🔍 Step 5: What Does 1/x Actually Mean?

The derivative of ln(x) being **1/x** says this:

> As x increases, ln(x) increases — but more and more slowly.

When x is small (like 1 → 2),
the slope 1/x is large, so ln(x) grows quickly.

When x is large (like 100 → 101),
1/x is tiny — ln(x) barely changes.

That's why the ln(x) graph starts steep, then flattens out.

🧠 Step 6: Chain Rule Version — ln(u)

Now, what if instead of ln(x), we have ln of some *function*, like ln(2x + 3)?

Let y = ln(u), where u = u(x).

Then by the chain rule:

dy/dx = (1/u) · (du/dx)

So the general formula is:

d/dx [ln(u)] = (1/u) · (du/dx)

This works for any differentiable function u(x) > 0.

🧩 Step 7: Examples

Example 1:

$y = \ln(x^2 + 1)$

Then $u = x^2 + 1$
$du/dx = 2x$

So:
$dy/dx = (1 / (x^2 + 1)) \cdot 2x = 2x / (x^2 + 1)$

✅ Final Answer:
$dy/dx = 2x / (x^2 + 1)$

Example 2:

$y = \ln(\sin x)$

Let $u = \sin x$
$du/dx = \cos x$

So:
$dy/dx = (1/u)\cdot(du/dx) = (1/\sin x)\cdot\cos x = \cot x$

✅ Final Answer:
$dy/dx = \cot x$

💫 Step 8: Why This Matters (Feynman's View)

Feynman would say:

> "Logarithms and exponentials are mirror images.
> If e^x grows fast, $\ln(x)$ grows slow — they're two sides of the same coin."

Differentiating $\ln(x)$ gives $1/x$ —
which tells us that growth slows down exactly in proportion to size.

It's a *built-in balancing act* between growth and scale.

🧩 Step 9: Quick Reference Table

Function	Derivative
$\log_{(b)}(x)$	$1 / (x \ln b)$
$\ln(x)$	$1 / x$
$\ln(u)$	$(1/u) \cdot (du/dx)$
$\ln(x^2 + 1)$	$2x / (x^2 + 1)$
$\ln(\sin x)$	$\cot x$

🧩 The Derivative of Irrational Powers of x

"How can we differentiate x raised to a weird number?"

🧠 Step 1: The Story So Far

We already know the **Power Rule** works for:

- Positive integers (like x^2, x^3, etc.)
- Negative integers (like x^{-1}, x^{-2})
- Rational numbers (like $x^{1/2}$, $x^{3/4}$)

Now we'll see that it also works for **irrational powers** (like x^√2 or x^π).
We'll use **logarithms** — because they let us handle exponents *as multipliers*.

⚙️ Step 2: Let's Begin with the Function

Let
$y = x^r$,
where **r** is *any real number* (even irrational).

We want to find **dy/dx**.

⚙️ Step 3: Take the Natural Log of Both Sides

Taking ln on both sides gives:

$\ln(y) = \ln(x^r)$

Now use the log rule that moves powers to the front:

$\ln(y) = r \cdot \ln(x)$

⚙️ Step 4: Differentiate Both Sides (Implicitly)

Differentiate both sides with respect to x.

On the left:

- $d/dx\,[\ln(y)] = (1/y) \cdot dy/dx$ (by the chain rule)

On the right:

- $d/dx\,[r \ln(x)] = r \cdot (1/x)$

Now we have:

$(1/y) \cdot dy/dx = r / x$

⚙️ Step 5: Solve for dy/dx

Multiply both sides by y:

$dy/dx = y \cdot (r / x)$

And since $y = x^r$, substitute it back:

$dy/dx = x^r \cdot (r / x)$

Simplify:

$dy/dx = r \cdot x^{(r - 1)}$

✅ That's the **Power Rule** — and notice,
we never assumed r was an integer or rational.
We only used logarithms, which work for all real numbers.

🧩 Step 6: Feynman's Intuitive Explanation

Feynman would say:

> “If you understand why it works for integers,
> and you understand how logs turn exponents into multipliers —
> then the same logic has to work for every real exponent.”

That’s the beauty of the logarithm:
it’s the bridge between powers and multiplication.

So the **power rule** isn’t just a coincidence —
it’s a *universal truth* for all real exponents.

⚡ Derivatives of Exponential Functions

Now let’s flip the situation.

Instead of **x raised to a power**,
we’ll look at **a constant raised to a variable**.

So:
$y = b^x$, where $b > 0$

⚙️ Step 1: Take ln on Both Sides

$\ln(y) = \ln(b^x)$

Use the same rule:
$\ln(y) = x \cdot \ln(b)$

⚙️ Step 2: Differentiate Both Sides

$(1/y) \cdot dy/dx = \ln(b) \cdot (d/dx \text{ of } x) = \ln(b)$

Now multiply both sides by y:

$dy/dx = y \cdot \ln(b)$

Since $y = b^x$, substitute back:

✅ **$dy/dx = b^x \cdot \ln(b)$**

💡 Step 3: The Special Case — Base e

If the base happens to be **e**,
then $\ln(e) = 1$.

So the derivative simplifies beautifully:

$d/dx\,[e^x] = e^x$

This means **e^x is its own derivative!**

Feynman would call that "a perfect function" —
because its growth rate is exactly equal to its value.

🧠 Step 4: Chain Rule Extension

If we have a more general exponential function like:

$y = b^u$,
where $u = u(x)$

Then by the chain rule:

$dy/dx = b^u \cdot \ln(b) \cdot (du/dx)$

Special case (base e):

$d/dx\,[e^u] = e^u \cdot (du/dx)$

🔍 Step 5: Quick Summary Table

Function	Derivative
x^r	$r \cdot x^{(r-1)}$
b^x	$b^x \cdot \ln(b)$
e^x	e^x
e^u	$e^u \cdot (du/dx)$
b^u	$b^u \cdot \ln(b) \cdot (du/dx)$

🔑 Step 6: Feynman's Takeaway

> "Logs and exponentials are two sides of the same coin.
> One brings exponents down — the other pushes them up.
> When you understand one, the other is just a mirror."

That's why:

- $\ln(x)$ makes powers *linear* (easy to differentiate),
- e^x grows in a way that perfectly fits the natural rhythm of calculus.

Everything connects:
x^r → logs → power rule → exponentials → e^x → $\ln(x)$

It's one big loop — elegant, simple, complete. 🌱

🔄 Understanding Inverse Functions

We've already talked about functions — they're like machines that do something to an input and give an output.

You feed in a real number x, the function performs some action, and out pops a new number y.

But here's an interesting question:

> "If there's an action that turns x into y,
> is there another action that can turn y back into x?"

That second action, if it exists, is called the **inverse function**.
It's the "undo" button of mathematics.

🧩 Step 1: What does it mean for two functions to be inverses?

If a function f takes you from
$x \rightarrow y$,
then its inverse g takes you right back:
$y \rightarrow x$.

Formally, we say:

$f(g(x)) = x$
and
$g(f(x)) = x$

If both are true, then f and g are inverses of each other.

📘 Example:

Let's say:
$f(x) = 2x$
and
$g(x) = x / 2$

Then:

$f(g(x)) = 2 * (x / 2) = x$
$g(f(x)) = (2x) / 2 = x$

Perfect!
Each function undoes what the other one did.
They are inverse functions.

⚠️ Step 2: Do all functions have inverses?

No — not all of them.

To have an inverse, a function must be **one-to-one**.

That means:

- It never gives two different outputs for the same input,
- And it never gives the same output for two different inputs.

If you draw its graph, it must pass the **Horizontal Line Test** —
no horizontal line should cross the graph more than once.

Only such functions can have an inverse.

🔁 Step 3: The notation

If f has an inverse, we write it as f^{-1} (read as "f inverse").

Important:
$f^{-1}(x)$ is **not** the same as $1 / f(x)$.
It means "the function that undoes f."

We can write:

$y = f(x) \Leftrightarrow x = f^{-1}(y)$

⚙️ Derivative of an Inverse Function

Now, how fast does an inverse function change?
Let's find the derivative of $f^{-1}(x)$.

🧮 Step 1: Start from the basic relationship

If
 $y = f(x)$

then the inverse gives
 $x = f^{-1}(y)$

🧩 Step 2: Differentiate both sides with respect to x

From $y = f(x)$, we get:
 $dy/dx = f'(x)$

But from the inverse relationship $x = f^{-1}(y)$,
 we can think of y as the variable now and differentiate:
 $dx/dy = (f^{-1})'(y)$

🔁 Step 3: Flip the derivatives

These two are reciprocals of each other!

$(dy/dx) * (dx/dy) = 1$

So,
 $(f^{-1})'(y) = 1 / f'(x)$

💡 Step 4: Express in terms of x

If we replace y with f(x), we get the more useful version:

$(f^{-1})'(x) = 1 / f'(f^{-1}(x))$

That's the derivative of an inverse function.

🔍 Feynman's Intuition

Feynman would put it this way:

> “If one function stretches space by a certain factor,
> its inverse must compress it by exactly the opposite amount.”

So the slope of the inverse function is the **reciprocal** of the slope of the original function — they balance each other perfectly.

🧠 Example

If
$f(x) = 2x$,
then $f'(x) = 2$.

The inverse is
$f^{-1}(x) = x / 2$.

Then:
$(f^{-1})'(x) = 1 / f'(f^{-1}(x)) = 1 / 2$

Exactly what we expect — the inverse is **half as steep**.