🎓 Lecture 22 — Relative Extrema

Understanding Maxima, Minima, and Critical Points

🌄 The Big Picture

Every function's graph is like a landscape — full of **hills** and **valleys**.

- The **hills** are called **relative maxima**.
- The **valleys** are called **relative minima**.

We call them "*relative*" because — just like not every hill on Earth is Mount Everest — a local hill might be the highest **in its area**, even if it's not the highest overall. Same with valleys: a local dip might not be the deepest point of all, but still a *local minimum.*

So, relative extrema are about **highs and lows within an interval**, not necessarily the absolute ones.

🧩 Step 1: What Are Relative Extrema?

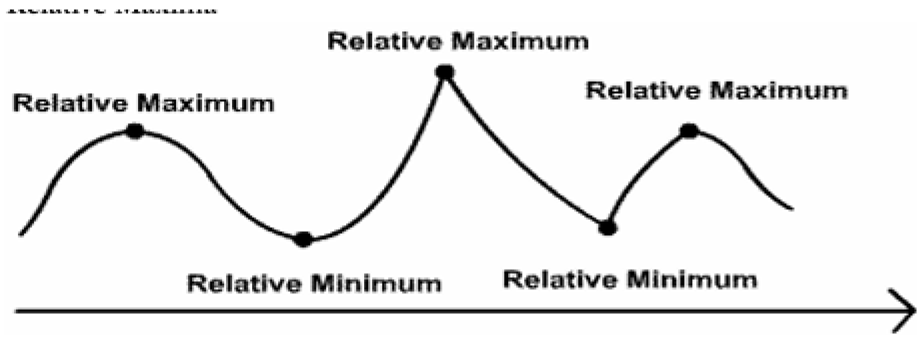
Formally:

- A function has a **relative maximum** at x_0 if $f(x_0) \geq f(x)$ for all x near x_0.
- A function has a **relative minimum** at x_0 if $f(x_0) \leq f(x)$ for all x near x_0.

In simpler words:

- At a **maximum**, the curve peaks — it stops rising and starts falling.
- At a **minimum**, the curve dips — it stops falling and starts rising.

So these points are *turning points* of the graph.



🧠 Step 2: The Role of Derivatives — Finding Critical Points

How do we find those peaks and valleys?
We look at the **slope** — the derivative, f′(x).

- When f′(x) changes from positive to negative → you hit a **maximum**.
- When f′(x) changes from negative to positive → you hit a **minimum**.

These points where the slope *changes direction* are called **critical points**.

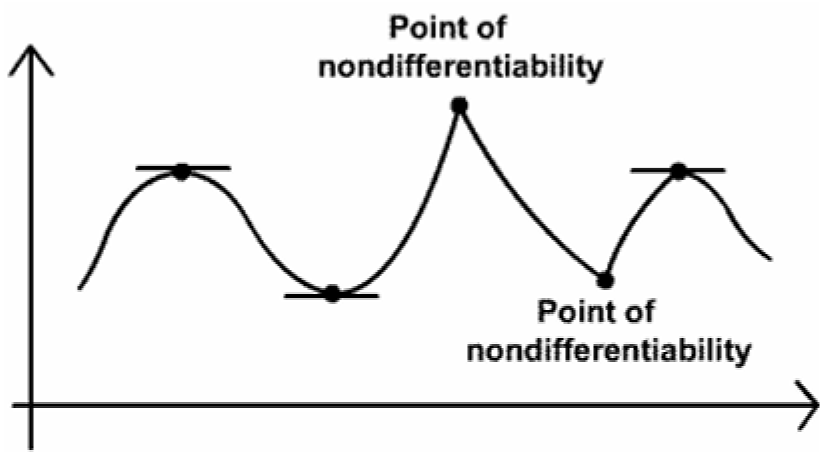
👉 A **critical point** is where either:

1. f′(x) = 0 (flat slope → horizontal tangent), or
2. f′(x) does not exist (corner or cusp).

So:

> “Relative extrema occur at **critical points**.”

That’s your golden rule.


Point of
nondifferentiability
Point of
nondifferentiability

⚙️ Step 3: The First Derivative Test

Let's apply this rule more systematically.

1️⃣ Find $f'(x)$.
2️⃣ Set $f'(x) = 0$ to find **critical points**.
3️⃣ Check where $f'(x)$ changes sign (positive ↔ negative).

If:

- $f'(x)$ changes from + to – → **Relative Maximum**
- $f'(x)$ changes from – to + → **Relative Minimum**
- $f'(x)$ doesn't change sign → no extremum at that point.

This is called the **First Derivative Test.**

📘 Example 1

Let
$f(x) = 3x^3 - 5x^2$

Step 1: Find $f'(x)$
$f'(x) = 9x^2 - 10x = x(9x - 10)$

Step 2: Set $f'(x) = 0$
$x = 0$ or $x = 10/9$

So, there are two critical points.

Step 3: Test sign changes of $f'(x)$ around these points.

- Before $x=0$ → $f'(x)$ negative
- Between 0 and 10/9 → $f'(x)$ positive
- After 10/9 → $f'(x)$ negative

So:

- At $x=0$ → f' changes (– to +) → **Relative Minimum**
- At $x=10/9$ → f' changes (+ to –) → **Relative Maximum**

✅ **Result:**
Relative Min at $x=0$
Relative Max at $x=10/9$

🧮 Step 4: The Second Derivative Test

Sometimes the sign test feels messy. There's a shortcut.

1️⃣ Find $f'(x)$ and set it to zero to find critical points.
2️⃣ Then find $f''(x)$ — the second derivative.

Now check $f''(x)$ at those points:

- If $f''(x_0) > 0$ → curve bends *upward* → **Minimum**
- If $f''(x_0) < 0$ → curve bends *downward* → **Maximum**
- If $f''(x_0) = 0$ → test fails; use first derivative test instead.

It's basically checking the *concavity* at that point.

📘 Example 2

Let
$f(x) = x^4 – 2x^2$

Step 1: $f'(x) = 4x^3 – 4x = 4x(x^2 – 1)$
→ critical points at $x = 0, \pm 1$

Step 2: $f''(x) = 12x^2 – 4$

Now test:

- f″(0) = –4 → concave down → **Maximum**
- f″(1) = 8 → concave up → **Minimum**
- f″(–1) = 8 → concave up → **Minimum**

✅ So:

- Relative Max at x=0
- Relative Mins at x=±1

Beautiful symmetry!

🪜 Step 5: Why This Works

> "At the top of a hill, the slope is zero — you've stopped climbing.
> But what makes it a *hill* and not a flat road is that, right after that, the slope starts turning downward.
> That's what the second derivative tells you — whether your road is bending up or bending down."

So:

- f′(x) = 0 tells you *something is happening* (flat spot).
- f″(x) tells you *what kind* of something it is (hill or valley).

📈 Step 6: Extrema and Graph Sketching

When graphing complex functions — especially **polynomials** or **rational functions** — you can find their shape without plotting hundreds of points by analyzing:

1. **Critical points** → where the slope is zero or undefined.
2. **Increasing/decreasing intervals** → from sign of $f'(x)$.
3. **Concavity** → from sign of $f''(x)$.
4. **Inflection points** → where $f''(x)$ changes sign.

Once you combine all this, the function's *personality* — its "shape" — becomes clear.

🧭 Example (Polynomial)

Let's sketch
$f(x) = x^3 - 3x^2 + 2$

1️⃣ $f'(x) = 3x^2 - 6x = 3x(x - 2)$
→ critical points at $x=0$ and $x=2$

2️⃣ $f''(x) = 6x - 6$

- $f''(0) = -6$ → concave down → **Max at x=0**
- $f''(2) = +6$ → concave up → **Min at x=2**

That gives you the shape — a hill at 0 and a valley at 2.

🧮 Step 7: Rational Functions & Extrema

For rational functions (fractions of polynomials),
you apply the *same rules* — but be careful about where the denominator = 0.
Those are **discontinuities** or **vertical asymptotes** — breaks in the graph.

Example:

$f(x) = (x^2 – 2) / (x^2 – 1)$

- Denominator = 0 at $x = \pm 1$ → vertical asymptotes.
- Find $f'(x)$ normally to locate possible maxima/minima **where the function is defined.**

💬 In Feynman's Words

> "The graph of a function is like a living landscape.
> Derivatives tell you how the land rises or falls;
> second derivatives tell you whether you're standing on a hilltop, in a valley,
> or at the turning point between the two."

✅ **Summary:**

Concept	What It Means	Test
Critical Point	$f'(x) = 0$ or undefined	Potential extremum
Maxima	Highest point locally	$f''(x) < 0$
Minima	Lowest point locally	$f''(x) > 0$
Inflection Point	Curve changes concavity	$f''(x) = 0$, sign changes