Lecture 5

Distance; Circles; Equations of the form $y = ax^2 + bx + c$

In this lecture, we will do three main things:

1. Derive a formula for the distance between two points in the coordinate plane.
2. Use that formula to study equations and graphs of circles.
3. Study equations of the form $y = ax^2 + bx + c$ and their graphs.

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1. Distance Between Two Points

Warm-up: One Dimension

If A and B are points on a straight line with coordinates a and b, then the distance between them is:

$d = |b – a|$

Simple! Distance is just "how far apart they are on the number line."

Two Dimensions

Now suppose we have two points in the coordinate plane:
P1(x1, y1) and P2(x2, y2).

If we draw a right triangle with these points,

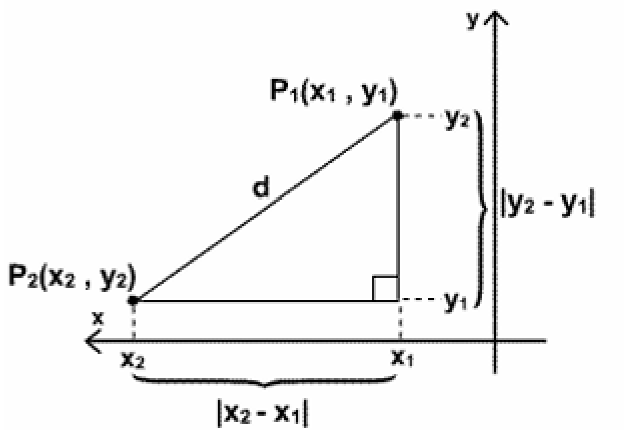
- the horizontal side has length $|x2 – x1|$
- the vertical side has length $|y2 – y1|$

By Pythagoras Theorem:

$d^2 = (x2 – x1)^2 + (y2 – y1)^2$

So, the distance formula is:

$d = \sqrt{(x2 – x1)^2 + (y2 – y1)^2}$



Example 1

Find the distance between the points (–2, 3) and (1, 7).

$d = \sqrt{(1 – (–2))^2 + (7 – 3)^2}$

$d = \sqrt{(3)^2 + (4)^2}$
$d = \sqrt{9 + 16}$
$d = \sqrt{25} = 5$

Notice: It doesn’t matter which point you call (x1, y1) and which (x2, y2). The result is the same.

Example 2: Right Triangle Test

Show that the points A(4, 6), B(1, –3), and C(7, 5) form a right triangle.

Step 1: Find the lengths of sides.

$AB = \sqrt{(1 – 4)^2 + (–3 – 6)^2} = \sqrt{9 + 81} = \sqrt{90}$
$AC = \sqrt{(7 – 4)^2 + (5 – 6)^2} = \sqrt{9 + 1} = \sqrt{10}$
$BC = \sqrt{(7 – 1)^2 + (5 – (–3))^2} = \sqrt{36 + 64} = \sqrt{100} = 10$

Step 2: Check Pythagoras.

$AB^2 + AC^2 = 90 + 10 = 100 = BC^2$

Therefore, $\triangle ABC$ is a right triangle with hypotenuse BC.

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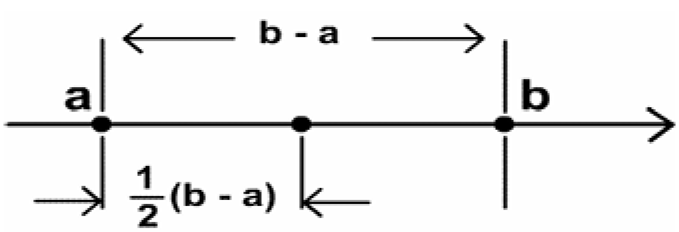
2. Midpoint Formula

Sometimes we need the **middle point** of a line segment.

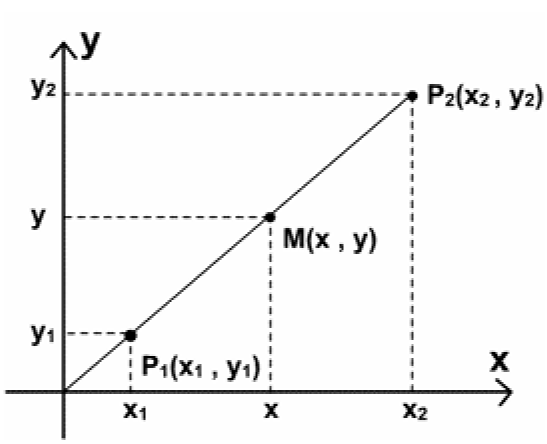
Start simple: On a number line, if the two points are a and b, the midpoint is the average:

midpoint = $(a + b) / 2$



Now extend this to two dimensions.
If $P1(x1, y1)$ and $P2(x2, y2)$ are endpoints, then the midpoint $M(x, y)$ has coordinates:

$x = (x1 + x2) / 2$
$y = (y1 + y2) / 2$



So,

Midpoint = ((x1 + x2)/2 , (y1 + y2)/2)

Example 3

Find the midpoint of (3, –4) and (7, 2).

x = (3 + 7)/2 = 10/2 = 5
y = (–4 + 2)/2 = –2/2 = –1

Midpoint = (5, –1)

✅ Up to here, we have the **distance formula** and the **midpoint formula** — two essential tools in coordinate geometry.

Circles

What is a Circle?

Think of a circle as the set of all points that stay at the same distance from a fixed center point.

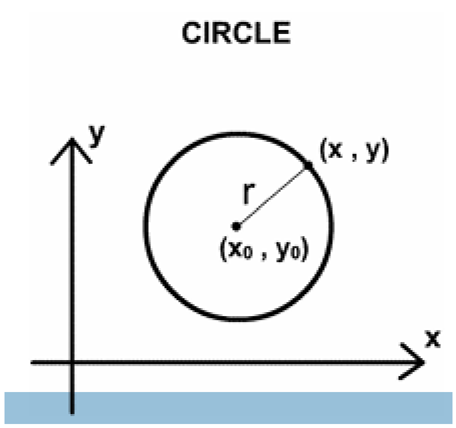
- If the center is $(x0, y0)$
- and the radius is r

then any point (x, y) lies on the circle **if its distance from $(x0, y0)$ equals r.**

So, using the distance formula:

$(x – x0)^2 + (y – y0)^2 = r^2$

This is called the **standard form of the equation of a circle.**



Example 1

Find the equation of the circle with center (–5, 3) and radius 4.

Here $x0 = –5$, $y0 = 3$, $r = 4$.

Equation:
$(x - (-5))^2 + (y - 3)^2 = 4^2$
$(x + 5)^2 + (y - 3)^2 = 16$

Expanded form:
$x^2 + 10x + 25 + y^2 - 6y + 9 = 16$
$x^2 + y^2 + 10x - 6y + 18 = 0$

Example 2

Find the equation of a circle with center (1, –2) that passes through (4, 2).

Step 1: Find radius.
$r = \sqrt{(4 - 1)^2 + (2 - (-2))^2}$
$r = \sqrt{(3)^2 + (4)^2}$
$r = \sqrt{9 + 16} = \sqrt{25} = 5$

Step 2: Write equation.
$(x - 1)^2 + (y + 2)^2 = 25$

Reading Center and Radius from Equation

If an equation is written as:
$(x - x0)^2 + (y - y0)^2 = r^2$

- Center = (x0, y0)
- Radius = r

Examples:

1. $(x - 2)^2 + (y - 5)^2 = 9 \rightarrow$ Center (2, 5), Radius 3
2. $(x + 7)^2 + (y + 1)^2 = 16 \rightarrow$ Center (–7, –1), Radius 4
3. $x^2 + y^2 = 25 \rightarrow$ Center (0, 0), Radius 5
4. $(x - 4)^2 + y^2 = 25 \rightarrow$ Center (4, 0), Radius 5

Special case:
$x^2 + y^2 = 1$ → Center (0, 0), Radius 1 → This is called the **unit circle.**

General Equation of a Circle

If you expand the standard form, you usually get something like:

$x^2 + y^2 + dx + ey + f = 0$

Or more generally:

$Ax^2 + By^2 + Dx + Ey + F = 0$
(where $A = B \neq 0$)

From such an equation, you can **complete the square** to get back the center and radius.

Example 3(a)

Find the center and radius of:
$x^2 + y^2 - 8x + 2y + 8 = 0$

Step 1: Group terms.
$(x^2 - 8x) + (y^2 + 2y) = -8$

Step 2: Complete the squares.

- For $x^2 - 8x$ → half of -8 is -4 → square is 16. Add 16.
- For $y^2 + 2y$ → half of 2 is 1 → square is 1. Add 1.
  Now add $16 + 1 = 17$ to both sides:
  $(x^2 - 8x + 16) + (y^2 + 2y + 1) = -8 + 16 + 1$

Step 3: Rewrite.
$(x - 4)^2 + (y + 1)^2 = 9$

Answer:
Center = $(4, -1)$
Radius = 3

Example 3(b)

Equation:
$2x^2 + 2y^2 + 24x - 81 = 0$

Step 1: Divide through by 2.
$x^2 + y^2 + 12x - 81/2 = 0$

Step 2: Complete the square (for x terms).

- For $x^2 + 12x$ → half of 12 is 6 → square is 36. Add 36.
  Now add 36 to both sides:
  $(x^2 + 12x + 36) + y^2 = 81/2 + 36$

Step 3: Rewrite.
$(x + 6)^2 + y^2 = 153/2$

Answer:
Center = $(-6, 0)$
Radius = $\sqrt{153/2}$

Degenerate Cases

Sometimes the equation looks like a circle but isn't really one.

$(x - x0)^2 + (y - y0)^2 = k$

- If $k > 0$ → A circle of radius $\sqrt{k}$
- If $k = 0$ → Just a single point (the center)
- If $k < 0$ → No solution (no real circle exists)

Example

(a) $(x - 1)^2 + (y + 4)^2 = -9$
No solutions (since squares cannot add to a negative number).

(b) $(x - 1)^2 + (y + 4)^2 = 0$
Only solution is $(1, -4)$. The circle shrinks to a point.

Theorem (General Equation of Circle)

An equation of the form:

$Ax^2 + By^2 + Dx + Ey + F = 0$

with $A = B \neq 0$
represents either:

- a circle,
- a single point,
- or no graph at all (degenerate case).

That's why it's often called the **general equation of a circle.**

Graph of $y = ax^2 + bx + c$

What is it?

An equation of the form:

$y = ax^2 + bx + c$, where $a \neq 0$

is called a **quadratic equation in x**.
Its graph is a curve called a **parabola**.

Shape of the Parabola

- If $a > 0$ → parabola opens **upwards** (like a smile 🙂).
- If $a < 0$ → parabola opens **downwards** (like a frown 🙁).

In both cases:

- The parabola is symmetric about a vertical line (its **axis of symmetry**).
- This line passes through a special point called the **vertex**.

The vertex is:

- The **lowest point** if the parabola opens upwards.
- The **highest point** if the parabola opens downwards.

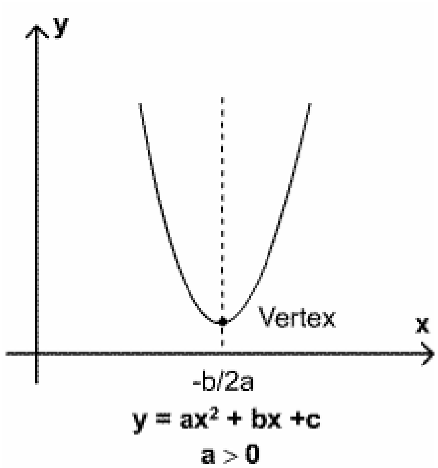
Formula for the Vertex

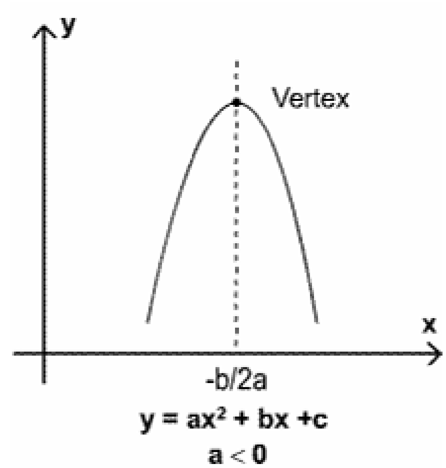
The x-coordinate of the vertex is:

$x = -b / 2a$

Once you know this x-value, substitute it back into the equation y = ax² + bx + c to find the corresponding y-value.

So, the vertex is (–b / 2a , y).





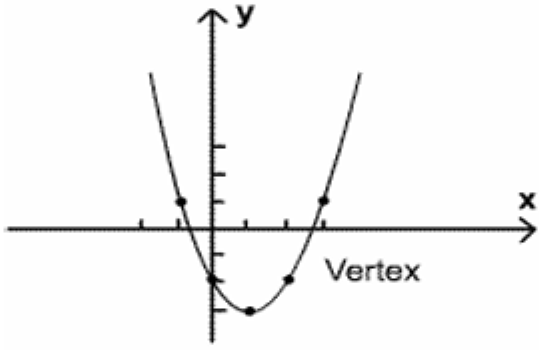
Example (a)

Sketch $y = x^2 - 2x - 2$.

Here: $a = 1$, $b = -2$, $c = -2$.

x-coordinate of vertex = $-(-2) / (2\cdot1) = 2 / 2 = 1$.
At $x = 1$, $y = (1)^2 - 2(1) - 2 = 1 - 2 - 2 = -3$.

So vertex = $(1, -3)$.
Since $a = 1 > 0$, the parabola opens upward.



$y = x^2 - 2x - 2$

$a > 0$

Example (b)

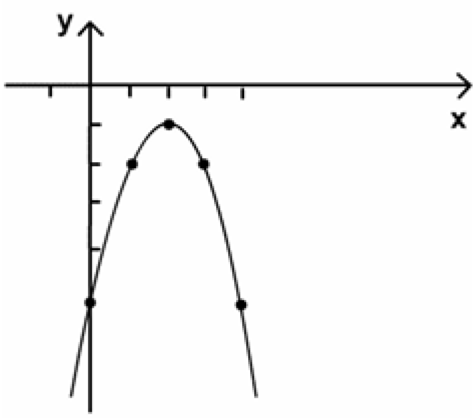
Sketch $y = -x^2 + 4x - 5$.

Here: $a = -1$, $b = 4$, $c = -5$.

x-coordinate of vertex $= -4 / (2\cdot -1) = -4 / -2 = 2$.
At $x = 2$, $y = -(2)^2 + 4(2) - 5 = -4 + 8 - 5 = -1$.

So vertex $= (2, -1)$.
Since $a < 0$, the parabola opens downward.

x	$y = -x^2 + 4x - 5$
0	-5
1	-2
2	-1
3	-2
4	-5



$y = -x^2 + 4x - 5$
$a < 0$

Intercepts of a Parabola

- **y-intercept**: Put x = 0 in the equation.
- **x-intercepts**: Put y = 0 and solve the quadratic equation $ax^2 + bx + c = 0$.

Example: Inequality

Solve $x^2 - 2x - 2 > 0$.

Step 1: Think of the parabola $y = x^2 - 2x - 2$.
It opens upward ($a = 1$).

Step 2: Find x-intercepts by solving $x^2 - 2x - 2 = 0$.

Using quadratic formula:

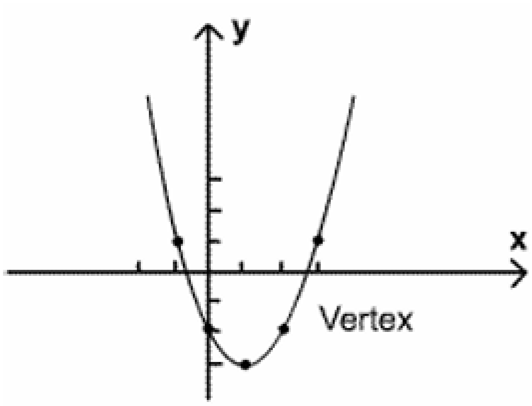
$x = (2 \pm \sqrt{(-2)^2 - 4(1)(-2)}) / 2$
$x = (2 \pm \sqrt{4 + 8}) / 2$
$x = (2 \pm \sqrt{12}) / 2$
$x = (2 \pm 2\sqrt{3}) / 2$
$x = 1 \pm \sqrt{3}$

So the parabola crosses the x-axis at $x = 1 - \sqrt{3}$ and $x = 1 + \sqrt{3}$.

Step 3: Where is $y > 0$?
Answer: When the parabola is above the x-axis → outside the roots.

So solution = $(-\infty , 1 - \sqrt{3}) \cup (1 + \sqrt{3} , \infty)$.

x	$y = x^2 - 2x - 2$
-1	1
0	-2
1	-3
2	-2
3	1



$y = x^2 - 2x - 2$
$a > 0$

Real-World Example: Throwing a Ball

A ball is thrown upward with velocity 24.5 m/s.
Its height above ground after t seconds is:

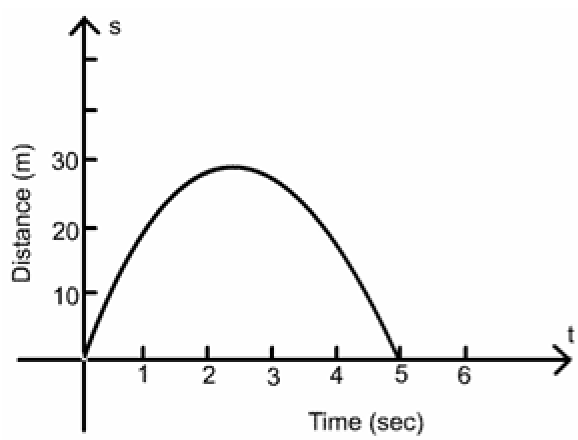
$s = 24.5t - 4.9t^2$

This is a parabola in t.

- Here $a = -4.9$, $b = 24.5$, $c = 0$.
- Vertex $t = -b / 2a = -24.5 / (-9.8) = 2.5$ seconds.
- At $t = 2.5$, height $s = 24.5(2.5) - 4.9(2.5)^2 = 30.6$ m.

So:

- The ball rises to a maximum height of ~30.6 m at t = 2.5 sec.
- It hits the ground again at t = 5 sec (since s = 0 there).



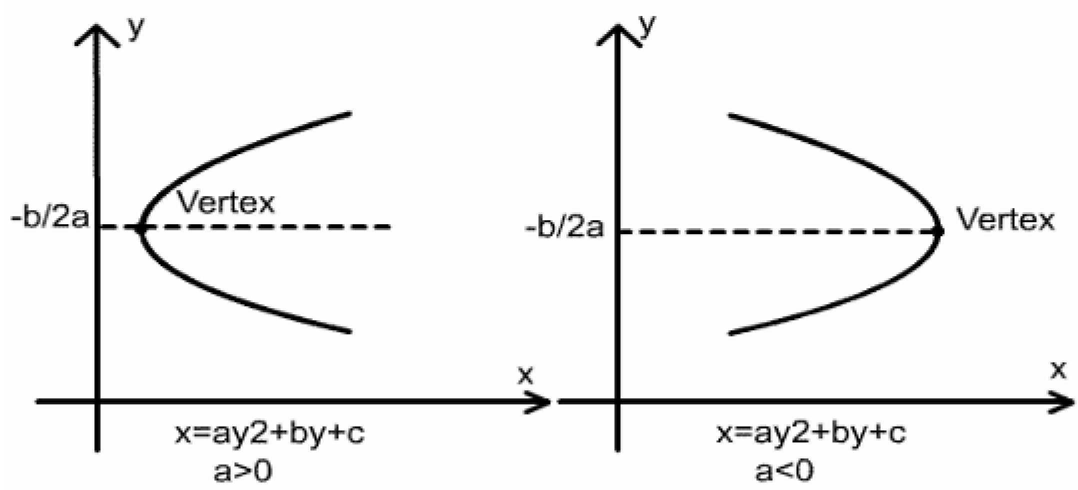
Parabolas in Terms of y

If we interchange x and y, we get:

$x = ay^2 + by + c$

This is a quadratic in y.
Its graph is also a parabola, but now the axis of symmetry is **horizontal** (parallel to x-axis).

The vertex has $y = -b / 2a$.


y
Vertex
-b/2a
x
x=ay2+by+c
a>0
y
-b/2a
Vertex
x
x=ay2+by+c
a<0