# Lecture # 21 Applications of Differentiation

## 🔄 Related Rates

We've already learned that **derivatives** measure *how fast* something changes.
But in the real world, things are connected — one thing changes *because* another thing does.

That's what **related rates** are all about.

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### 🧠 The Big Idea

A related rate problem is when **two or more quantities are linked**,
and we know how one of them is changing, but want to find how the other is changing.

For example:

- How fast is the **area of a circle** increasing when its **radius** is growing?
- How fast is the **shadow of a man** moving when he walks away from a streetlight?
- How fast is the **top of a ladder** sliding down when the **bottom** moves away?

All these are *rates that depend on other rates* — hence, "related rates."

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### 💡 Step 1: The Key Principle

If two variables are related by an equation,
then their *rates of change* are related by the **derivative** of that equation.

So, if
**A = f(r)**
then by differentiating both sides with respect to **time (t)**, we relate **dA/dt** and **dr/dt**.

That's the heart of every related rates problem.

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# 🌊 Example 1 — The Oil Spill Problem

Oil is spreading in a circular shape.
The radius **r** of the spill increases at a constant rate of **2 ft/sec**.

We want to find **how fast the area A is increasing** when **r = 60 ft**.

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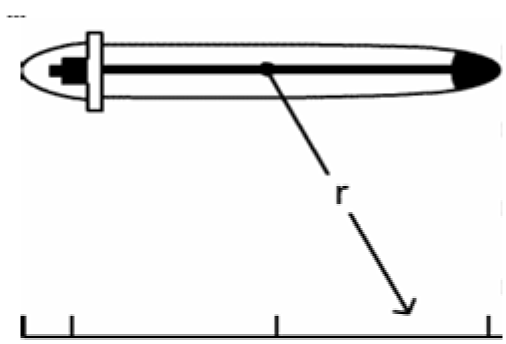
## 🧩 Step-by-Step Reasoning

1. **Draw the situation** (in your mind or on paper):
    - A big circle spreading outward.
    - Radius **r** is growing over time **t**.
    - Area **A** is also growing as the circle expands.




2. **Write down what you know:**
    - **dr/dt = 2 ft/sec**
    - **r = 60 ft**
    - We want **dA/dt = ?**
3. **Find a formula that links A and r:**
   Since the spill is circular:
   **$A = \pi r^2$**

4. **Differentiate both sides with respect to time (t):**
   **dA/dt = 2πr (dr/dt)**

   Notice we used the **chain rule** here, because A changes as r changes, and r changes with time.

5. **Plug in the known values:**
   - **r = 60 ft**
   - **dr/dt = 2 ft/sec**

6. So:
   **dA/dt = 2π(60)(2) = 240π ft²/sec**

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### 🎯 Final Answer

When the radius is 60 ft,
the area is increasing at **240π ft² per second**
(≈ **754 ft²/sec** if you multiply it out).

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### 🧠 Feynman's Insight

Feynman would say:

> "When the radius grows a little, the area doesn't just grow by that little — it grows *in proportion to the circumference!*"

That's why the rate of change of area depends on **r**:
as the circle gets bigger, each small increase in radius adds a *wider ring* of area than before.

So the bigger the circle → the faster the area increases.

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## 🪜 Example 2 — The Sliding Ladder Problem

A 5-foot ladder is leaning against a wall.
The bottom slides **away from the wall** at **2 ft/sec**.
How fast is the **top sliding down** when the bottom is **4 ft** from the wall?

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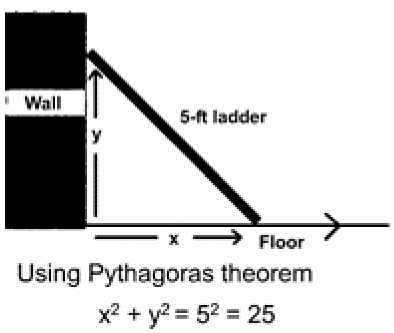
## 🧩 Step-by-Step Reasoning

Let's assign variables:

- **x** = distance from the wall to the base of the ladder (horizontal)
- **y** = height of the ladder top on the wall (vertical)
- **t** = time in seconds

We know:

- The ladder length is constant: $\mathbf{x^2 + y^2 = 5^2}$
- **dx/dt = 2 ft/sec** (base moving away)
- **x = 4 ft**
- We need to find **dy/dt** (top moving down)

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## 🧮 Step 1: Differentiate with respect to time

Differentiate $\mathbf{x^2 + y^2 = 25}$ with respect to **t**:

**2x (dx/dt) + 2y (dy/dt) = 0**

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## 🧮 Step 2: Simplify

Divide through by 2:

**$x\,(dx/dt) + y\,(dy/dt) = 0$**

Now solve for **$dy/dt$:**

**$dy/dt = -(x / y)(dx/dt)$**

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## 🧮 Step 3: Find y when x = 4

From the Pythagoras relation:

**$x^2 + y^2 = 25 \rightarrow 4^2 + y^2 = 25 \rightarrow y^2 = 9 \rightarrow y = 3$ ft**

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## 🧮 Step 4: Substitute values

**$dy/dt = -(4 / 3)(2) = -8/3$ ft/sec**

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## 🎯 Final Answer

The top of the ladder is sliding **down the wall** at
**8/3 ft/sec (≈ 2.67 ft/sec)** when the base is 4 ft from the wall.

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## 🧠 Feynman's Insight

Feynman would describe it like this:

> "As the bottom moves out, the top must move down to keep the ladder the same length.
> When the bottom is close to the wall, a small outward slide makes the top drop quickly.
> But when it's already far out, the same slide barely moves the top."

That's why the rate of the top falling depends on the *ratio* between x and y —
the geometry itself controls the relationship between their speeds.

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## ⚙️ General Strategy for Related Rates

1. **Draw the situation** and label changing quantities.
2. **Write down what rates** are known and what is being asked.
3. **Find an equation** that links those quantities.
4. **Differentiate with respect to time (t)**.
5. **Plug in known values** and solve for the unknown rate.

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### 🎯 The Core Idea

Think of a function as a road that climbs and dips as you move along it (from left to right). You're walking on this road — the x-axis — and the road's height (the y-value) changes as you move.

Now, calculus gives us a *slope detector* — the derivative, written as f prime of x — that tells whether you're going uphill, downhill, or flat at any moment.

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### 🧩 Step 1: What "Increasing" and "Decreasing" Mean

A function is **increasing** when, as x moves right, y also moves up.
👉 Example: if you take two x-values, say $x_1$ is less than $x_2$, and $f(x_1)$ is less than $f(x_2)$, the function is increasing.

A function is **decreasing** when, as x moves right, y goes down.
👉 That means if $x_1$ is less than $x_2$, and $f(x_1)$ is greater than $f(x_2)$, the function is decreasing.

So in plain English:

- Increasing = climbing up
- Decreasing = sliding down

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### 🧠 Step 2: What Derivatives Tell Us

The derivative f prime of x is the slope of the tangent line — your road's incline at that point.

Now check this logic:

- If f prime of x is greater than 0, the tangent slope is positive, meaning you're going uphill → the function is increasing.
- If f prime of x is less than 0, the tangent slope is negative, meaning you're going downhill → the function is decreasing.
- If f prime of x equals 0, the road is flat → the function is constant (no change).

So, the **sign of f prime of x** literally tells you the behavior of the graph.

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### 🪜 Step 3: Visual Intuition

Imagine three roads:

- **Positive slope:**
  Your car is climbing — derivative is positive.
  Every next point on the road is higher than the last.

- **Negative slope:**
  Your car is descending — derivative is negative.
  Every next point is lower than the last.

- **Zero slope:**
  You're on a perfectly flat surface — derivative is zero.

So, derivatives act like a **speedometer of height** — showing whether you're moving up, down, or steady.

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### 🧮 Step 4: How to Find Increasing and Decreasing Intervals

Here's the process:

1. Find f prime of x — that's your slope function.
2. Set f prime of x equal to zero to find the *critical points* (possible flat or turning points).
3. Test the intervals around those points to see where f prime of x is positive or negative.
4. Draw your conclusion:
   - If f prime of x > 0 → the function is increasing
   - If f prime of x < 0 → the function is decreasing

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### 🧠 Example

Let's say:
$f(x) = (x - 2)^2$

**Step 1: Find derivative**
f prime of x = 2(x - 2)

**Step 2: Find where it's zero**
2(x - 2) = 0 → x = 2

**Step 3: Test intervals**

- For x < 2 → f prime of x = 2(x - 2) < 0 → decreasing
- For x > 2 → f prime of x = 2(x - 2) > 0 → increasing

So the function decreases up to x = 2, then increases after x = 2.
At x = 2, the slope is zero — that's the **turning point** (bottom of the curve).

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### 🌄 Step 5: What's Really Happening

You can now see how the math reflects the shape:

- Negative slope = going downhill
- Zero slope = valley or peak
- Positive slope = going uphill

That's how calculus lets you "read" the shape of any curve — even without drawing it.

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### 💬 In Feynman's Words

> "If you want to know whether something is rising or falling, you don't look at the thing itself — you look at its rate of change. The derivative tells you not where you are, but where you're heading."

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### 🎯 The Core Idea — What Is Concavity?

If the *slope* (derivative) tells you whether a function is going *uphill or downhill*,
then the *concavity* tells you **how** it's curving —
whether it's bending *upward like a smile* 😊 or *downward like a frown* ☹️*.*

You can think of concavity as the **shape of the curve's bend** —
how the slope itself is changing.

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### 🧩 Step 1: The Logic Behind Concavity

Remember, the derivative f′(x) tells us the *slope* of the curve.
Now imagine watching that slope change as you move along x.

If the slope is **increasing**, the curve bends *upward* → this is called **concave up.**
If the slope is **decreasing**, the curve bends *downward* → this is called **concave down.**

So concavity describes the *behavior of the slope.*

👉 Increasing slope → curve bends upward
👉 Decreasing slope → curve bends downward

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### 🧠 Step 2: The Second Derivative — f″(x)

To measure how the slope changes, we take the derivative *again!*
That's called the **second derivative**, written as f double prime of x.

- If f″(x) > 0 → slope is increasing → the curve is **concave up.**
  (Think: a bowl facing up — it can "hold" water.)

- If f″(x) < 0 → slope is decreasing → the curve is **concave down.**
  (Think: an upside-down bowl — water would slide off.)

So, the **sign** of f″(x) tells you the direction of the curve's bend.

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### 📈 Step 3: Visual Intuition

Imagine driving on a hilly road:

- **Concave up:**
  You start going downhill but the slope is getting less steep — soon you'll go uphill.
  That's why it *curves upward.*

- **Concave down:**
  You start going uphill but the slope is getting less steep — soon you'll go downhill.
  That's why it *curves downward.*

So, concavity describes whether your "road" is **bending toward the sky or toward the ground.**

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### 🪜 Step 4: The Inflection Point

Sometimes, the curve *switches* from bending up to bending down (or vice versa).
That point is called an **inflection point.**

At an inflection point, the second derivative changes sign:

- From positive to negative → curve goes from concave up to concave down.
- From negative to positive → curve goes from concave down to concave up.

It's like a roller coaster changing from a dip to a hill.

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### 🧮 Step 5: Example

Let's take a simple function:
$f(x) = x^3$

**Step 1: Find the first derivative**
$f'(x) = 3x^2$

**Step 2: Find the second derivative**
$f''(x) = 6x$

Now test its sign:

- When $x < 0 \rightarrow f''(x) = 6x < 0 \rightarrow$ concave down
- When $x > 0 \rightarrow f''(x) = 6x > 0 \rightarrow$ concave up

So the curve bends downward for $x < 0$, and upward for $x > 0$.
At $x = 0$, $f''(x) = 0$ → that's the **inflection point.**

The graph of $x^3$ has that classic "S" shape — bending down, flattening, then bending up.

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### 🌄 Step 6: How It All Connects

- First derivative ($f'$) → tells you *direction* (uphill or downhill).
- Second derivative ($f''$) → tells you *curvature* (how that direction changes).

In short:

- $f'(x) > 0$ → function is increasing
- $f'(x) < 0$ → function is decreasing
- $f''(x) > 0$ → curve is concave up
- $f''(x) < 0$ → curve is concave down

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### 💬 In Feynman's Words

> "If the first derivative tells you where the car is going — up or down —
> then the second derivative tells you whether you're pressing the accelerator or the brake."