Lecture 14: Tangent Lines and Rates of Change

We've already seen that a **secant line** connects two points on a curve. But if you slide one point closer and closer to the other, the secant line starts rotating until it reaches a **limiting position**. That limiting line is the **tangent line**.

Secant Line: The Starting Idea

Suppose we have a curve given by:
$y = f(x)$

Take two distinct points on it:
$P(x_0, f(x_0))$ and $Q(x_1, f(x_1))$

The slope of the **secant line** is:

$m\_sec = (f(x_1) – f(x_0)) / (x_1 – x_0)$

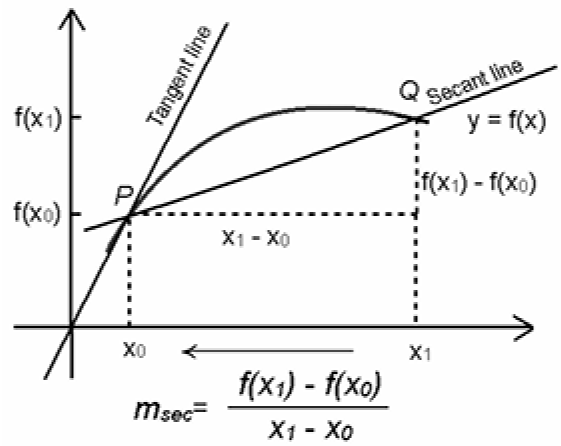
This slope measures the **average rate of change** between x_0 and x_1.

From Secant to Tangent

Now, let $x_1 \to x_0$ (that is, slide Q toward P).
Then the slope of the secant approaches the slope of the tangent line at P:

$m\_tan = \lim (x_1 \to x_0) (f(x_1) – f(x_0)) / (x_1 – x_0)$

That's our first big formula: the slope of a tangent line.

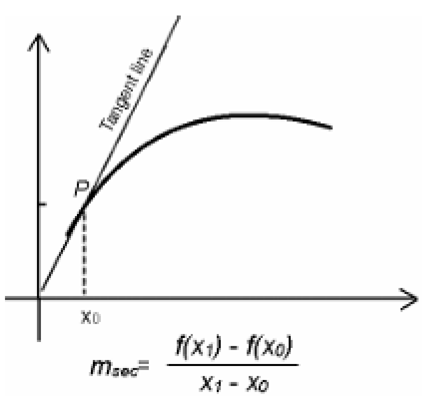


Why This Was Revolutionary

In the 17th century, mathematicians wanted to describe **instantaneous velocity**—not just average speed, but “how fast at this very instant.”

They realized: this problem is exactly the same as finding the slope of a tangent line.

- Average velocity = slope of secant line.
- Instantaneous velocity = slope of tangent line.



Average Velocity

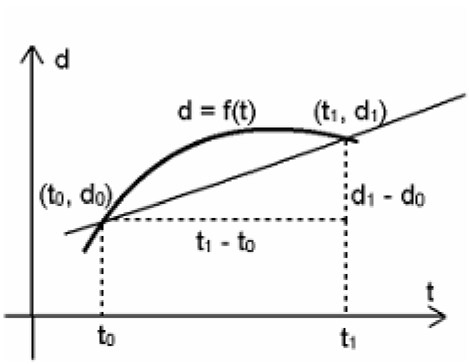
Formally:

Average Velocity = Distance traveled / Time elapsed

If distance = f(t), then over $[t_0, t_1]$:

$v\_ave = (f(t_1) – f(t_0)) / (t_1 – t_0)$

This is nothing more than the **slope of the secant line** through $(t_0, f(t_0))$ and $(t_1, f(t_1))$.



Instantaneous Velocity

Now suppose we want velocity exactly at time t_0.

We shrink the interval by letting $t_1 \to t_0$.
The secant slope becomes the tangent slope.

So instantaneous velocity is:

$v\_inst = \lim (t_1 \to t_0) (f(t_1) – f(t_0)) / (t_1 – t_0)$

That's the exact same formula we used for tangent slope—just with "time vs distance" instead of "x vs y."

Generalizing to Rate of Change

This idea works for everything, not just velocity:

- Rate of change of bacteria population w.r.t. time
- Rate of change of length of a rod w.r.t. temperature
- Rate of change of cost w.r.t. quantity produced

In general, if $y = f(x)$:

- Average rate of change = slope of secant line
- Instantaneous rate of change = slope of tangent line

Formula:

Rate of change = $\lim (\Delta x \to 0) (\Delta y / \Delta x)$

Where $\Delta y = f(x + \Delta x) - f(x)$.

✅ So what we did is **connect geometry (tangent line slope)** with **real-world physics (instantaneous velocity)** and then generalize it into the universal concept of **rate of change**.

Definition 3.1.1 – Average and Instantaneous Rate of Change

Suppose we have a function:
$y = f(x)$

Average Rate of Change

Take two points on the curve:
$(x_0, f(x_0))$ and $(x_1, f(x_1))$

The **average rate of change** of y with respect to x over $[x_0, x_1]$ is just the slope of the secant line:

$m\_sec = (f(x_1) – f(x_0)) / (x_1 – x_0)$

This tells us **how much y changes per unit change in x, on average, between x_0 and x_1.**

Instantaneous Rate of Change

Now, zoom in at a single point x_0.
The **instantaneous rate of change** of y with respect to x at x_0 is the slope of the tangent line:

$m\_tan = \lim (x_1 \to x_0) (f(x_1) – f(x_0)) / (x_1 – x_0)$

This tells us the **"speed of change at the exact instant x_0."**

Example

Let's take a simple function:
$y = f(x) = x^2 + 1$

(a) Average rate of change over [3, 5]

$f(3) = 3^2 + 1 = 10$
$f(5) = 5^2 + 1 = 26$

$m\_sec = (f(5) – f(3)) / (5 – 3)$
$= (26 – 10) / (2)$
$= 16 / 2$
$= 8$

So, on average, y increases **8 units** for every **1 unit** increase in x between 3 and 5.

(b) Instantaneous rate of change at $x_0 = –4$

We use the definition:

$m\_tan = \lim (x \to –4) (f(x) – f(–4)) / (x – (–4))$

First compute $f(–4)$:
$f(–4) = (–4)^2 + 1 = 16 + 1 = 17$

So:
$m\_tan = \lim (x \to –4) (x^2 + 1 – 17) / (x + 4)$
$= \lim (x \to –4) (x^2 – 16) / (x + 4)$
$= \lim (x \to –4) [(x – 4)(x + 4)] / (x + 4)$
$= \lim (x \to –4) (x – 4)$
$= –4 – 4 = –8$

So the slope = –8.

That means at x = –4, the function is **decreasing**, at a rate of 8 units down for every 1 unit across.

(c) Instantaneous rate of change at a general point x_0

$m\_tan = \lim (x \to x_0) (f(x) – f(x_0)) / (x – x_0)$

$= \lim (x \to x_0) (x^2 + 1 – (x_0^2 + 1)) / (x – x_0)$
$= \lim (x \to x_0) (x^2 – x_0^2) / (x – x_0)$
$= \lim (x \to x_0) [(x – x_0)(x + x_0)] / (x – x_0)$
$= \lim (x \to x_0) (x + x_0)$
$= 2x_0$

So the **general formula** is:
Instantaneous rate of change = $2x_0$

Quick Check

From part (c), plug in $x_0 = -4$:

m_tan = 2(–4) = –8 ✅ (matches part b).

Feynman-Style Intuition

- **Average rate** = like checking your speed by dividing “distance traveled / time taken.”
- **Instantaneous rate** = like looking at the **speedometer** at an exact moment.
- For $y = x^2 + 1$, the slope changes depending on where you are:
  - Positive slope when x is positive (function rising).
  - Negative slope when x is negative (function falling).
  - Zero slope at x = 0 (flat point).