Continuity

Think of a function like drawing a curve on paper.

👉 If you can **draw the curve without lifting your pencil**, the function is continuous there.
👉 If at some point you must lift your pencil (a hole, a jump, or a break), then the function is **discontinuous** at that point.

That's the big picture. Now let's go step by step.

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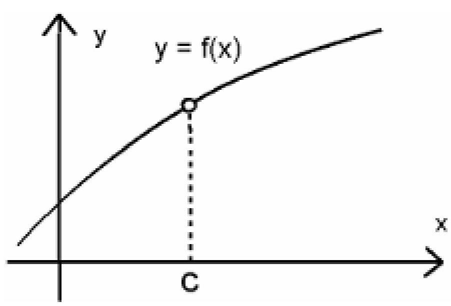
1. When is a function discontinuous at $x = c$?

Three ways things can go wrong:

1. **Function not defined at c**
   Example: $f(x) = (x^2 - 4) / (x - 2)$.
   At $x = 2$, denominator = 0 → undefined → discontinuous.



2. **Limit doesn't exist**
   If left side ≠ right side, or function wiggles without settling (like $\sin(1/x)$ near 0).
   Then $\lim f(x)$ as $x \to c$ does not exist.

3. **Limit exists, but doesn't match the actual value**
   Function is defined at c, and limit exists, but they're not equal.
   Example: step functions (jump discontinuity).

So — a break/discontinuity happens if ANY of these conditions fail.

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2. Formal Definition of Continuity at a Point

A function $f(x)$ is continuous at $x = c$ if ALL 3 conditions hold:

(a) $f(c)$ is defined
(b) $\lim (x \to c) f(x)$ exists
(c) $\lim (x \to c) f(x) = f(c)$

👉 If even one fails, discontinuity at c.
👉 If true for every c in an interval (a, b), then f is continuous on (a, b).
👉 If true for all real numbers, we simply call it a "continuous function."

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3. Examples

Example 1:

$f(x) = (x^2 - 4) / (x - 2)$

At $x = 2$:
$f(2)$ undefined → discontinuous.

Example 2: Piecewise function

$g(x) =$

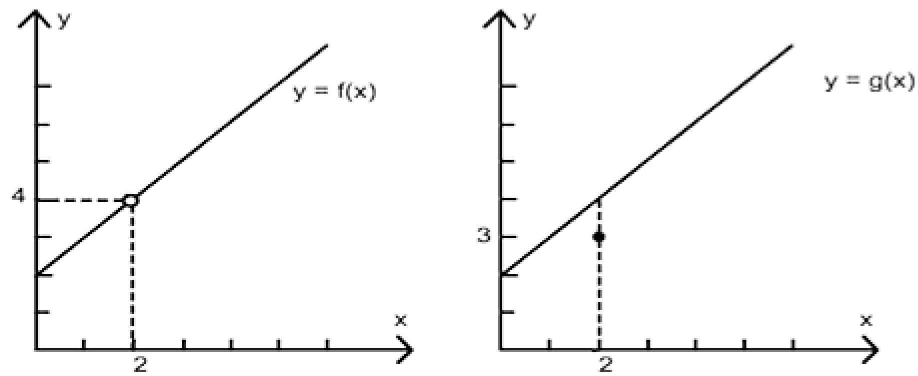
- $(x^2 - 4) / (x - 2)$, if $x \neq 2$
- 3, if $x = 2$

Check at $x = 2$:

$\lim (x \to 2) (x^2 - 4)/(x - 2) = \lim (x \to 2) (x - 2)(x + 2)/(x - 2) = \lim (x \to 2) (x + 2) = 4$

But $g(2) = 3$

So condition (c) fails → discontinuous.



Example 3: Polynomial

$f(x) = x^2 - 2x + 1$

At any c:
$\lim (x \to c) f(x) = f(c)$

So polynomial functions are always continuous.

Example 4: Absolute Value Function

$f(x) = |x| =$

- $-x$, if $x < 0$
- x, if $x \geq 0$

Take any c.

If $c \geq 0 \to f(c) = c$, and $\lim (x \to c) f(x) = c$.
If $c < 0 \to f(c) = -c$, and $\lim (x \to c) f(x) = -c$.

So $|x|$ is continuous everywhere.

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4. The Pencil Test

To quickly test continuity:

- If the graph has **no holes, no jumps, no vertical asymptotes**, it's continuous.
- If you need to lift your pencil → discontinuous.

✅ **Summary in Plain Words**
Continuity means: the function doesn't suddenly disappear, jump, or give a different answer when you zoom in on a point.

Properties of Continuous Functions

Think of **continuity** like smoothness: no jumps, no holes, no breaks in the graph.
Now, what happens if we take two smooth (continuous) functions and combine them? Do we still get smoothness?

👉 The answer is **yes** — under certain rules.

Theorem 2.7.3

If f and g are continuous at some point c, then:

a) f + g is continuous at c
b) f – g is continuous at c
c) f × g is continuous at c
d) f / g is continuous at c, as long as g(c) ≠ 0.
(If g(c) = 0, then dividing by zero makes it discontinuous at that point.)

Why is this true? (Proof idea)

If f and g are continuous at c, it means:
lim (x → c) f(x) = f(c)
lim (x → c) g(x) = g(c)

Now, by the **limit rules** we already know:

- lim (f(x) + g(x)) = lim f(x) + lim g(x)
- lim (f(x) – g(x)) = lim f(x) – lim g(x)
- lim (f(x) × g(x)) = (lim f(x)) × (lim g(x))
- lim (f(x) / g(x)) = (lim f(x)) / (lim g(x)), provided lim g(x) ≠ 0

So plugging in continuity:

- $(f + g)(c) = f(c) + g(c)$
- $(f - g)(c) = f(c) - g(c)$
- $(f \times g)(c) = f(c) \times g(c)$
- $(f / g)(c) = f(c) / g(c)$, if $g(c) \neq 0$

✅ That shows continuity is preserved under addition, subtraction, multiplication, and (careful) division.

Special Case: Rational Functions

A **rational function** is just one polynomial divided by another.
Example: $h(x) = (x^2 - 9) / (x^2 - 5x + 6)$

- The numerator $(x^2 - 9)$ is a polynomial → continuous everywhere.
- The denominator $(x^2 - 5x + 6)$ is also a polynomial → continuous everywhere.

So the only danger is when the denominator = 0.

Solve:
$x^2 - 5x + 6 = 0$
$(x - 2)(x - 3) = 0$
So denominator = 0 when $x = 2$ or $x = 3$.

👉 Therefore:
$h(x)$ is continuous **everywhere except at $x = 2$ and $x = 3$** (where it blows up → vertical asymptotes).

Feynman-style summary 🎯

- Smooth + Smooth = Smooth.
- Smooth – Smooth = Smooth.
- Smooth × Smooth = Smooth.

- Smooth ÷ Smooth = Smooth (as long as you don’t divide by zero).

That’s why rational functions are continuous everywhere except where their denominator vanishes.

📘 Continuity of Composition of Functions

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1. The Big Idea

Imagine two machines:

- Machine **g** takes an input x and gives you some output.
- Machine **f** then takes that output and gives you a final result.

So the combo (composition) looks like:
f(g(x)) → input goes through g first, then f.

Now, what happens when x gets close to some number c?

- If **g(x)** settles to some value **L** as $x \to c$,
- and **f** is continuous at that value L (meaning it behaves smoothly there),

👉 then the whole combo will also settle smoothly:
$\lim_{x \to c} f(g(x)) = f(\lim_{x \to c} g(x))$

In plain words: *limits pass through continuous functions.*

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2. Example

Take:

- $f(x) = x^2$
- $g(x) = 5 – x$

Then: $f(g(x)) = (5 – x)^2$

Now, what's $\lim_{x \to 3} f(g(x))$?

Step 1: First find lim (x → 3) g(x).
g(3) = 5 – 3 = 2.

Step 2: f is continuous at L = 2.
So we just plug in: f(2) = 2^2 = 4.

✅ Therefore, lim (x → 3) f(g(x)) = 4.

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3. Theorem 2.7.6 (Simplified)

If g is continuous at c
and f is continuous at g(c),
then the combo f ∘ g is continuous at c.

Think of it like this: if both machines run smoothly, then chaining them together also runs smoothly.

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4. Continuity from Left and Right

Sometimes you're standing at the edge of a domain (like at the starting point of a road).
At an endpoint, you can't approach from both sides — only from one.

So we define:

- **Continuous from the left at c** → the limit as x → c^- equals f(c).
- **Continuous from the right at c** → the limit as x → c^+ equals f(c).

That's why at boundaries we only check from one side.

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5. Continuity on a Closed Interval [a, b]

For a function to be continuous on [a, b]:

1. It must be continuous everywhere inside (a, b).
2. It must be continuous from the right at a.

3. It must be continuous from the left at b.

So basically: smooth inside, no jumps at the edges.

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6. Example: $f(x) = 9 - x^2$

Check continuity on [-3, 3].

- Inside (–3, 3): it's a polynomial, so always continuous.
- At –3: only check right-hand limit.
 $\lim (x \to -3^+) (9 - x^2) = 0 = f(-3)$.
- At 3: only check left-hand limit.
 $\lim (x \to 3^-) (9 - x^2) = 0 = f(3)$.

✅ So the function is continuous on the entire interval [–3, 3].

🎯 Analogy

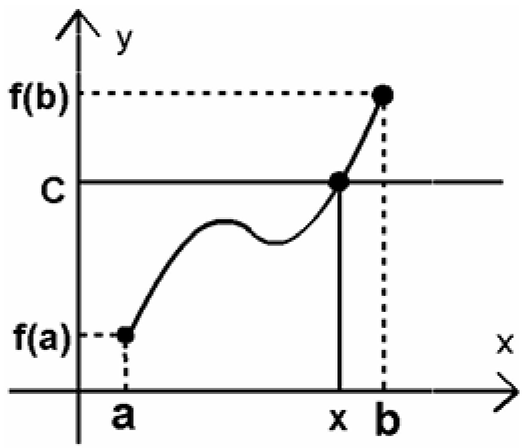
Think of roads:

- Inside the city (a, b), roads are smooth everywhere.
- At the city gates (a and b), you only check from one side.
 If cars can enter and exit without hitting a bump, the road is continuous.

👉 That's the essence of these theorems:
Smooth functions stay smooth when you combine them, and continuity at edges just means no bumps when you enter or exit.

Intermediate Value Theorem (IVT)

Theorem 2.7.9
If a function *f* is continuous on a closed interval [a, b], and *C* is any number between f(a) and f(b),
then there exists at least one x in [a, b] such that f(x) = C.



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1. What does this really mean?

Think of a continuous function like a **pen drawing a curve without lifting your hand**.

- If at x = a, the pen is at height f(a).
- At x = b, the pen is at height f(b).
- Then, as your pen travels smoothly from f(a) to f(b), it **must cross every height in between**.

That's the essence of IVT: no gaps, no teleportation — smooth travel means you pass through everything in between.

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2. Real-life Analogy 🌡️

Imagine at 8 AM the temperature is **20°C** and at noon it is **30°C**.
The thermometer rose continuously (no sudden jumps).
👉 Then at **some time between 8 AM and 12 PM**, the temperature **must have been 25°C**.

That’s IVT in real life. Continuous change guarantees all in-between values appear.

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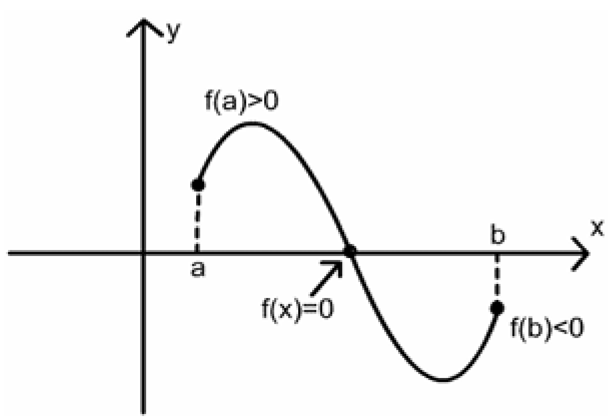
3. Special Case: Crossing Zero

Theorem 2.7.10
If f(a) and f(b) have **opposite signs** (one is positive, the other is negative), then the function must cross the x-axis ($f(x) = 0$) somewhere between a and b.

👉 This is often called the **Zero Theorem** or **Root Existence Theorem**.

Why?
Because if a curve starts below the x-axis (negative) and ends above it (positive), it must cross the axis somewhere in between (by continuity).



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4. Example

Solve (in principle):
$f(x) = x^3 – x – 1 = 0$

This is not easy to factor. But let’s test endpoints:

- At x = 1:
  $f(1) = 1^3 – 1 – 1 = –1$ (negative).

- At x = 2:
  $f(2) = 2^3 – 2 – 1 = 8 – 3 = 5$ (positive).

So:
$f(1) < 0$ and $f(2) > 0$.

By IVT, since $f(x)$ is continuous (a polynomial), there must be **at least one solution between $x = 1$ and $x = 2$**.

We don't know exactly where yet, but we're sure it's there.

✅ **Summary :**
The Intermediate Value Theorem is like saying:

> "If you walk smoothly from the ground floor ($f(a)$) to the 5th floor ($f(b)$), you must pass the 1st, 2nd, 3rd, and 4th floors on the way."

And if your journey starts below ground (negative) and ends above ground (positive), you must pass the ground floor ($f(x) = 0$).