Lecture # 35
Volume by Cylindrical Shells

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1. Big Picture: What is this lecture about?

In the previous lecture, we found volumes using:

• Disks
• Washers

Those methods slice the solid **perpendicular** to the axis of rotation.

Now we learn a **new method** called:

Cylindrical Shells

This method slices the region **parallel** to the axis of rotation.

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2. What is a Cylindrical Shell?

Start with a washer (a disk with a hole).

Now stretch it upward.

What do you get?

A **thin hollow cylinder** → this is called a **cylindrical shell**.

Geometrically:
A cylindrical shell is the space between two concentric cylinders.

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3. Volume of a Cylindrical Shell (Key Formula)

Consider a thin shell with:

• Inner radius = r_1
• Outer radius = r_2
• Height = h

Exact volume:

$$V = \pi (r_2^2 - r_1^2) h$$

Rewrite this using algebra:

$$r_2^2 - r_1^2 = (r_2 + r_1)(r_2 - r_1)$$

So,

$$V = 2\pi \text{ (average radius)} \times \text{(height)} \times \text{(thickness)}$$

This is the **core shell formula**.

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4. Shell Formula (Memory-Friendly Form)

Volume of a thin shell:

$$V = 2\pi \times \text{(radius)} \times \text{(height)} \times \text{(thickness)}$$

This is the formula we actually use in calculus.

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5. Why Use Cylindrical Shells?

Sometimes:

• Disk method becomes messy
• Washer method forces splitting the region
• Integrating with respect to x or y becomes hard

Shells often give:
• Simpler integrals
• One clean formula
• No splitting

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6. Shells When Rotating About the y-axis

Consider a region R:

- Top: $y = f(x)$
- Bottom: $y = 0$
- Left: $x = a$
- Right: $x = b$

Revolve this region about the **y-axis**.

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7. How the Shell is Formed

Take a thin vertical strip at position x.

When revolved about the y-axis:

- It forms a cylindrical shell
- Radius = x
- Height = $f(x)$
- Thickness = dx

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8. Volume of One Thin Shell

Using shell formula:

$dV = 2\pi \times (\text{radius}) \times (\text{height}) \times (\text{thickness})$

$dV = 2\pi\, x\, f(x)\, dx$

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9. Total Volume Using Shells

Add volumes of all shells:

$V = \int dV$
$V = 2\pi \int$ from a to b $x\, f(x)\, dx$

This is the **main cylindrical shell formula (y-axis)**.

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10. Important Observation

Shell method:
- Uses vertical strips
- Integrates with respect to x
- Best for rotation about y-axis

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11. Example 1 (Shells About y-axis)

Region bounded by:

$y = x$
$x = 1$
$x = 4$
$y = 0$

Revolved about the y-axis.

Here:
$f(x) = x$
$a = 1$
$b = 4$

Volume formula:

$V = 2\pi \int$ from 1 to 4 $x \cdot x\,dx$
$V = 2\pi \int$ from 1 to 4 $x^2\,dx$

After integration:

$V = (124/5)\pi$

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12. Why This Worked Easily

If we had used disks or washers:

• We would need y as a function of x
• Or solve for x in terms of y

Shells avoided all that.

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13. A Deeper Interpretation (Very Important)

Instead of thinking of thickness dx…

Think like this:

Each shell creates a **cylindrical surface** with:

Surface area = $2\pi \times$ radius $\times$ height
$= 2\pi x f(x)$

When we multiply this surface area by dx, we get volume.

So:

Volume by shells = Integral of surface areas

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14. Powerful Statement (Conceptual)

Volume by cylindrical shells is:

The integral of the surface area generated by a strip of the region taken parallel to the axis of rotation.

This viewpoint helps in more complex regions.

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15. Example 2 (More General Case)

Region R in first quadrant bounded by:

$y = x^2$
$y = x$

Revolved about the y-axis.

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16. Understanding the Geometry

At each x between 0 and 1:

- Height of shell = $x - x^2$
- Radius = x
- Thickness = dx

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17. Setting Up the Integral

Using shell formula:

$V = 2\pi \int$ from 0 to 1 $x (x - x^2)\, dx$

Simplify integrand:

$x (x - x^2) = x^2 - x^3$

So,

$V = 2\pi \int$ from 0 to 1 $(x^2 - x^3)\, dx$

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18. Final Answer

After evaluating the integral:

$V = (\pi / 6)$

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19. When to Use Cylindrical Shells (Exam Tip)

Use shells when:

- Axis of rotation is vertical (y-axis)
- Region is given in terms of x
- Disk/washer method requires solving for inverse functions
- Region would need splitting using washers

20. Final Takeaway (Lecture 35)

- Cylindrical shells slice parallel to the axis
- Shell volume formula:

$V = 2\pi \int (\text{radius})(\text{height})\, dx$

- Often simpler than disks or washers
- Interpretable as surface area being stacked

This method completes the **three major volume techniques**:

- Disks
- Washers
- Shells