Lecture # 32
Second Fundamental Theorem of Calculus

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1. What this lecture is really about (Big Picture)

Until now:

• We used integrals to find areas
• We used derivatives to find rates of change

This lecture shows something powerful:

Integration and differentiation are NOT opposites only in theory —
they are directly connected in practice.

This lecture explains:

• What a dummy variable is
• What happens when an integral has a variable limit
• The Second Fundamental Theorem of Calculus
• Why every continuous function has an antiderivative
• How functions can be defined using integrals

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2. Dummy Variable (Very Important Concept)

Look at these three integrals:

Integral from a to b of f(x) dx
Integral from a to b of f(t) dt
Integral from a to b of f(y) dy

All three have the SAME value.

Why?

Because the letter inside the integral does not matter —
it only tells us which variable we are integrating with respect to.

The limits (a and b) matter.
The function form matters.
The letter does NOT.

That's why the variable inside an integral is called a
DUMMY VARIABLE.

It is just a placeholder.

Example (Dummy Variable)

Integral from 1 to 2 of x^3 dx
Integral from 1 to 2 of t^3 dt
Integral from 1 to 2 of y^3 dy

All give the same number.

Changing x to t or y does NOT change the value.

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3. Definite Integrals with Variable Upper Limit

Now we look at integrals like:

Integral from a to x of f(t) dt

Notice two things:

- The upper limit is NOT a number — it is x
- The integration variable is t (not x)

This is done on purpose to avoid confusion.

Important idea:

When the upper limit is x,
the final answer will be a FUNCTION of x.

This is just like differentiation:

Derivative at a number → gives a number
Derivative at x → gives a function

Same idea here.

Example

Evaluate:

Integral from 2 to x of t^2 dt

Step 1: Integrate normally

Antiderivative of t^2 is $t^3 / 3$

Step 2: Apply limits

$= (x^3 / 3) - (2^3 / 3)$
$= x^3 / 3 - 8 / 3$

Final answer is a FUNCTION of x.

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4. Intuition Behind the Second Fundamental Theorem

Imagine a function f(x).

Now define a new function A(x) as:

A(x) = area under f(t) from a to x

So A(x) is changing as x changes.

Key question:

What is the derivative of A(x)?

Answer (Surprisingly simple):

The derivative of the area function
is just the original function f(x).

In symbols:

If
A(x) = Integral from a to x of f(t) dt

Then
$A'(x) = f(x)$

This is the Second Fundamental Theorem of Calculus.

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5. Second Fundamental Theorem of Calculus (Formal Statement)

If f is continuous on an interval, then:

d/dx [Integral from a to x of f(t) dt] = f(x)

Read it in words:

The derivative of a definite integral
with respect to its upper limit
is equal to the integrand evaluated at that limit.

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6. Why This Theorem Is Powerful

It tells us:

• Differentiation undoes integration
• Area functions automatically produce derivatives
• Continuous functions always have antiderivatives

This was NOT obvious before this theorem.

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7. Example to Verify the Theorem

Let f(x) = x^3 (continuous everywhere)

Consider:

Integral from 1 to x of t^3 dt

Step 1: Evaluate the integral

Antiderivative of t^3 is t^4 / 4

So the integral becomes:

= x^4 / 4 − 1 / 4

Step 2: Differentiate the result

$d/dx (x^4 / 4 - 1 / 4) = x^3$

Which is exactly f(x).

The theorem works.

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8. Existence of Antiderivatives (Very Important Result)

Before this theorem, we had a problem:

Some functions looked like they should have antiderivatives,
but we could not find them using formulas.

Now the theorem tells us:

If a function f is continuous on an interval,
then the function

F(x) = Integral from a to x of f(t) dt

IS an antiderivative of f.

So:

Every continuous function has an antiderivative.

Even if we cannot write it using elementary formulas.

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9. Functions Defined by Integrals

Sometimes we cannot simplify an integral into a known formula.

But that's okay.

We can still define a function using an integral.

Example

Consider:

Integral from 1 to x of (1 / t) dt

The function 1/t is continuous for $t \geq 1$
So an antiderivative exists.

But we cannot express it using polynomials or simple algebra.

So we define:

F(x) = Integral from 1 to x of (1 / t) dt

This IS a valid function.

Even if we don't know a simple formula for it.

Later, this function will be called ln(x).

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10. Why This Matters Conceptually

This lecture teaches us:

- Integration can define functions
- Derivatives can be recovered instantly
- Area, change, and accumulation are deeply connected

Calculus is not about memorizing formulas.

It is about understanding how:

Accumulation → Area → Integral
Change → Slope → Derivative

And the Fundamental Theorems connect them perfectly.