🚀 Absolute Value — Explained Simply

Imagine you're walking on a straight road — like a number line.
You start at 0. Move right → you hit positive numbers. Move left → you hit negative numbers.

Now, here's the key idea:
👉 Absolute value = distance from 0.
And distance is always non-negative. You can't say "I'm –5 meters away." Distance ignores direction.

So:

- $|5| = 5$ (5 units away from 0, on the right)
- $|-4| = 4$ (4 units away from 0, on the left)
- $|0| = 0$ (you're standing at 0, no distance at all)

That's it. Absolute value is just "how far from zero, no matter left or right."

⚖️ Formal Definition (in plain words)

For any real number a:

$|a| = a$, if $a >= 0$
$|a| = -a$, if $a < 0$

In English:

- If a is already positive or zero → leave it as is.
- If a is negative → flip the sign.

🔍 Watch out (Common Misunderstanding)

When you see “+a” or “–a,” don’t automatically assume +a means positive and –a means negative.
Why? Because a itself could be negative.

Example:

- If $a = -3 \rightarrow$ then $-a = -(-3) = +3$ (positive)
- If $a = -3 \rightarrow$ then $+a = -3$ (still negative)

So the plus and minus in front are operations, not “labels” of positive/negative.

🧩 Examples

1. $|5| = 5 \rightarrow$ 5 is already non-negative
2. $|-4/7| = 4/7 \rightarrow$ strip the minus
3. $|0| = 0 \rightarrow$ zero is neutral

👉 Rule of thumb: Absolute value = number without the minus sign.

🧮 Solving Equations with Absolute Value

Example 1
Solve: $|x - 3| = 4$

Meaning: “distance of x from 3 is 4”

Case 1: $x - 3 = 4 \rightarrow x = 7$
Case 2: $x - 3 = -4 \rightarrow x = -1$

Solutions: $x = 7$ or $x = -1$

Example 2
Solve: |3x – 2| = |5x + 4|

Rule: If two numbers have the same absolute value, then they are either equal OR exact opposites.

Case 1: 3x – 2 = 5x + 4
→ –2 – 4 = 5x – 3x
→ –6 = 2x
→ x = –3

Case 2: 3x – 2 = –(5x + 4)
→ 3x – 2 = –5x – 4
→ 3x + 5x = –4 + 2
→ 8x = –2
→ x = –1/4

Solutions: x = –3 or x = –1/4

✨ Feynman Takeaway

Absolute value is nothing mysterious.
It's just **distance from zero**.
Every problem is asking: "How far is this number (or expression) from zero?"

🌱 Relationship Between Square Roots and Absolute Values

First, let's recall the basics:

👉 A *square root* of a number "a" is a number which, when squared, gives you "a."

Example:

- The square roots of 9 are +3 and –3, because
  $3^2 = 9$ and $(-3)^2 = 9$

So we say:

- Positive square root of 9 = 3
- Negative square root of 9 = –3

By definition, the symbol $\sqrt{a}$ always means **the positive square root** (sometimes called the *principal square root*).

Thus:
$\sqrt{9} = 3$
$\sqrt{0} = 0$

⚠️ Common Mistake

A very common error is to assume:
$\sqrt{(a^2)} = a$

This is **not always true.**
It only works when $a \geq 0$.

Example:
If $a = -4$, then
$a^2 = (-4)^2 = 16$
$\sqrt{(a^2)} = \sqrt{16} = 4$

But $a = -4$, not $+4$
So $\sqrt{a^2} \neq a$ in this case.

✅ The Correct Relationship

To fix this problem, we use **absolute value**.

The real rule is:
$\sqrt{a^2} = |a|$

Why? Because absolute value always gives the non-negative version.

Examples:

1. If $a = 5 \rightarrow \sqrt{a^2} = \sqrt{25} = 5 = |5|$
2. If $a = -5 \rightarrow \sqrt{a^2} = \sqrt{25} = 5 = |-5|$

So the absolute value guarantees that the square root always matches the **non-negative root.**

✨ Feynman Takeaway

- Every positive number has two square roots (one positive, one negative).
- The symbol $\sqrt{a}$ always refers to the *positive* root.
- $\sqrt{a^2}$ is **not equal to a**, it is equal to $|a|$.

👉 Think of it like this:
Squaring a number hides the sign.
Taking a square root gives back the distance from zero.
That distance is exactly what absolute value means.

📐 Theorem: Square Root and Absolute Value

Statement:
For any real number a,

$\sqrt{a^2} = |a|$

🧠 Proof (explained simply)

1. Start with a^2.
    - If $a = +a$, then $a^2 = (+a)^2$
    - If $a = -a$, then $a^2 = (-a)^2$
      Either way, squaring makes the result positive.

So both $+a$ and $-a$ are square roots of a^2.

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2. Now recall: $\sqrt{a^2}$ means **the non-negative square root of a^2.**

- If $a \geq 0$, then $+a$ is already non-negative.
  So $\sqrt{a^2} = +a$.
- If $a < 0$, then $-a$ is the non-negative choice (because a is negative, but $-a$ flips it positive).
  So $\sqrt{a^2} = -a$.

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3. Put it all together:

$\sqrt{a^2} = a$, if $a \geq 0$
$\sqrt{a^2} = -a$, if $a < 0$

And this is exactly the definition of absolute value.

✅ Final Result

$\sqrt{(a^2)} = |a|$

✨ Feynman Takeaway

Squaring wipes out the sign (both +a and –a collapse into the same result).
Taking a square root asks: “Which root is non-negative?”
That’s why $\sqrt{(a^2)}$ is not “a” but “absolute a.”

👉 In one line:
Square root of a square is the **absolute value** of the number.

📘 Properties of Absolute Value

Theorem: If a and b are real numbers, then

(a) $|-a| = |a|$ → a number and its negative have the same absolute value.
(b) $|ab| = |a|\,|b|$ → the absolute value of a product is the product of the absolute values.
(c) $|a/b| = |a| / |b|$ → the absolute value of a ratio is the ratio of the absolute values.

🧠 Proofs Explained

(a) $|-a| = |a|$

Why?
Squaring wipes out the sign.

$(-a)^2 = a^2$
So $\sqrt{(a^2)} = |a|$

Thus $|-a| = |a|$

(b) $|ab| = |a|\,|b|$

Think of squaring again:

$(ab)^2 = (a^2)(b^2)$

Take square root:
$|ab| = |a|\,|b|$

This works for any number of factors:

$|a1\ a2\ a3 \ldots an| = |a1|\,|a2|\,|a3| \ldots |an|$

Special case:
If all numbers are the same (say all equal to a), then

$|a^n| = |a|^n$

(c) $|a/b| = |a| / |b|$

For division, the same rule applies.

$(a/b)^2 = (a^2) / (b^2)$

Take square root:
$|a/b| = |a| / |b|$ (as long as $b \neq 0$).

✅ Examples

1. $|-4| = |4| = 4$
2. $|(2)(-3)| = |-6| = 6 = |2||-3| = (2)(3) = 6$
3. $|5/4| = 5/4 = |5| / |4| = 5/4$

✨ Feynman Takeaway

- Absolute value ignores signs — flipping negative to positive when needed.
- Multiplication and division keep the same rule: you can “pull out” the absolute value across factors or fractions.
- Think of it as a *distance rule*: distance doesn’t care about direction, even when you multiply or divide.

📏 Geometric Interpretation of Absolute Value

Absolute value comes up naturally in distance problems — because **distance is always non-negative.**

Imagine a number line (a straight road).
Pick two points:

- Point A at coordinate a
- Point B at coordinate b

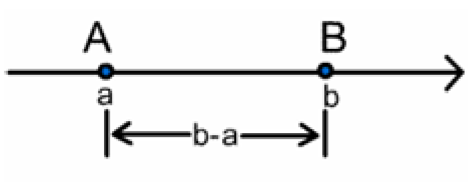
The distance **d** between A and B is:

d = |a – b|

🧠 Why does this work?

Case 1: If a < b

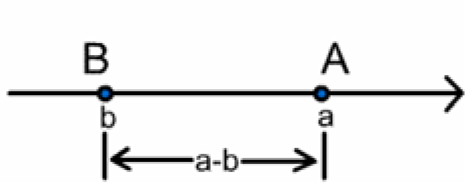
- Then b – a is positive
- Distance d = b – a = |b – a|



Case 2: If a > b

- Then b – a is negative

- But distance can't be negative, so we flip it:
  a – b = –(b – a) = |b – a|



Case 3: If a = b

- Then the two points overlap
- Distance d = 0 = |b – a|

✅ Key Formula

For any two real numbers a and b:

Distance between a and b = |a – b|

🌍 Feynman Takeaway

Absolute value is really just **distance on the number line.**

- Between 2 and 7 → |7 – 2| = 5
- Between –3 and 4 → |4 – (–3)| = |4 + 3| = 7
- Between –5 and –2 → |–2 – (–5)| = |3| = 3

👉 Whether you go left or right, absolute value measures **how far apart** two points are.

📐 Theorem: Distance Formula

If A and B are points on a coordinate line with coordinates **a** and **b**, then the distance **d** between them is:

$d = |b - a|$

👉 In short: Distance = absolute value of the difference.

This is why absolute value has such a natural **geometric meaning** — it always measures distance.

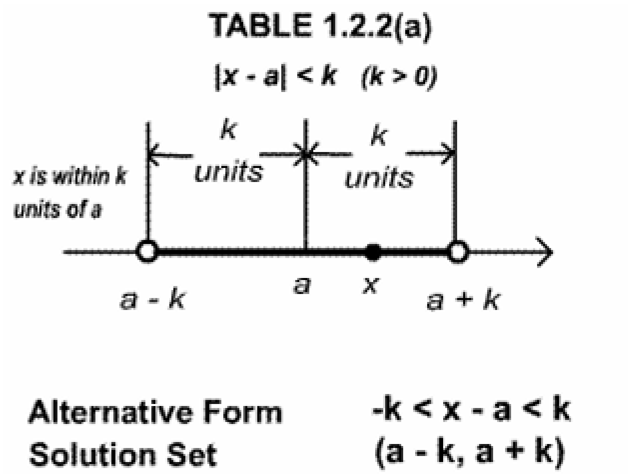
🗂️ Table of Common Expressions

EXPRESSION	GEOMETRIC INTERPRETATION ON A COORDINATE LINE
$\|x-a\|$	The distance between x and a
$\|x+a\|$	The distance between x and -a
$\|x\|$	The distance between x and origin

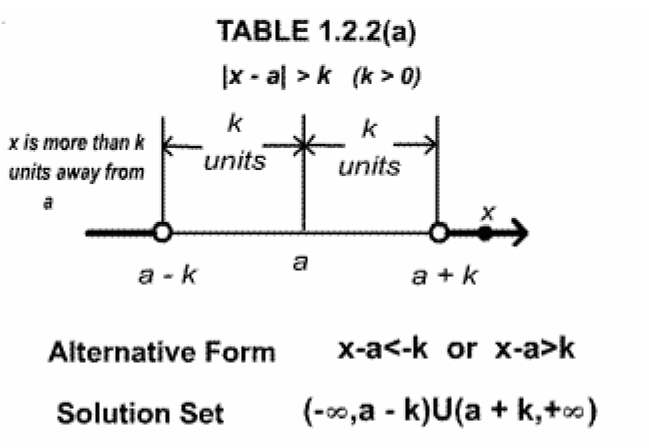
🧠 Why is this powerful?

These forms show up everywhere in algebra, especially in inequalities.
For example:

- $|x - a| < k$ → "x is less than k units away from a"



- |x – a| > k → “x is more than k units away from a”



So absolute value inequalities are really **distance statements.**

✅ Examples

1. |x – 5| < 2
   → x is within 2 units of 5
   → Interval: 3 < x < 7

2. |x + 3| > 4
   → Distance from –3 is greater than 4

→ $x < -7$ or $x > 1$

✨ Feynman Takeaway

Absolute value isn't just a symbol.
It's a **distance measure on the number line.**

- $|x - a|$ = how far x is from a
- $|x|$ = how far x is from 0
- Inequalities like $|x - a| < k$ are just saying "stay close," while $|x - a| > k$ says "stay far."

🧮 Example

Problem: Solve $|x - 3| < 4$

🧠 Step 1: Translate the inequality into distance language

$|x - 3| < 4$ means:
"The distance between x and 3 is less than 4."

So x must stay within 4 units of 3.

🧠 Step 2: Rewrite as a double inequality

$-4 < x - 3 < 4$

🧠 Step 3: Solve for x

Add 3 to all sides:

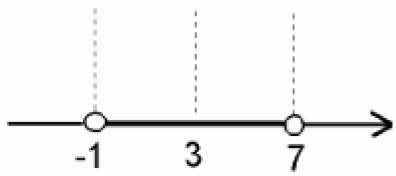
$-4 + 3 < x - 3 + 3 < 4 + 3$

$-1 < x < 7$

✅ Final Answer

The solution set is all x values between –1 and 7.

In **interval notation**:
$(-1, 7)$



✨ Feynman Takeaway

Absolute value inequalities are really just distance rules.

- $|x - 3| < 4$ means: “x is less than 4 units away from 3.”
- That gives a clean interval around 3: from –1 up to 7.

👉 Always think: absolute value = distance, then translate into a range.

Example
Solve:
$|x + 2| \geq 4$

Step 1: Recall the definition of absolute value.
For any expression inside | |, we split into two cases:

$|x + 2| \geq 4$ means either

1. $x + 2 \geq 4$
   or
2. $x + 2 \leq -4$

Step 2: Solve each case.

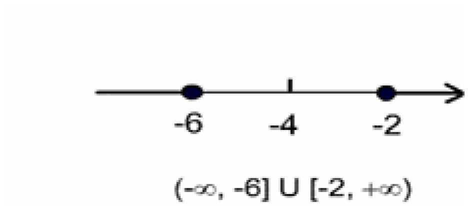
Case 1:
x + 2 ≥ 4
Subtract 2:
x ≥ 2

Case 2:
x + 2 ≤ -4
Subtract 2:
x ≤ -6

Step 3: Combine the two results.
So the solution is:
x ≥ 2 OR x ≤ -6

Step 4: Write in set/interval notation.
(-∞, -6] ∪ [2, ∞)

✅ **Final Answer:**
The solution set is
(-∞, -6] ∪ [2, ∞)



The Triangle Inequality

It is **not always true** that
$|a + b| = |a| + |b|$

For example:
Take $a = 2$, $b = -3$

Then
$a + b = 2 + (-3) = -1 \rightarrow |a + b| = |-1| = 1$

But
$|a| + |b| = |2| + |-3| = 2 + 3 = 5$

So, $|a + b| \neq |a| + |b|$

✅ **The Correct Statement (Theorem 1.2.5):**

For any real numbers a and b,
$|a + b| \leq |a| + |b|$

This is called the **Triangle Inequality**.
(It is so important that it even connects to the famous *Heisenberg Uncertainty Principle* in quantum physics.)

Proof (Step by Step)

We know from earlier properties:

- $|a| \geq a$ and $|a| \geq -a$
- $|b| \geq b$ and $|b| \geq -b$

That means:

- $-|a| \leq a \leq |a|$
- $-|b| \leq b \leq |b|$

Now add these two inequalities:
(-|a| ≤ a ≤ |a|)
+
(-|b| ≤ b ≤ |b|)

= -(|a| + |b|) ≤ (a + b) ≤ (|a| + |b|)

This tells us:

- (a + b) is trapped between -(|a| + |b|) and (|a| + |b|)

So by definition of absolute value:
|a + b| ≤ |a| + |b|

✅ **Final Result:**
|a + b| ≤ |a| + |b|

This is the **Triangle Inequality**.