⭐ Lecture 28: Area as Limits

🧠 Big Picture First

Earlier, we learned how to find areas using **anti-derivatives**.

But now we ask a deeper question:

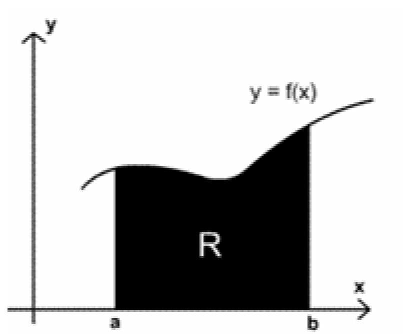
> **What IS area, really?**
> How do we define it precisely, without guessing formulas?

The answer uses the same powerful idea we used for derivatives:

👉 **Limits**

Just like a tangent line comes from secant lines,
area comes from rectangles.

🎯 The Problem We Are Solving



We are given:

- A continuous, non-negative function **y = f(x)**
- An interval **[a, b]**
- A region bounded by:

- the x-axis (below)
- $x = a$ and $x = b$ (sides)
- the curve $y = f(x)$ (above)

We want the **area of this region**.

🔍 The Core Idea (Feynman Style)

We don't know the area exactly.

So we do what physicists always do:

> **Approximate it… then refine the approximation… forever.**

We:

1. Break the region into rectangles
2. Add their areas
3. Let the rectangles become infinitely thin

The limit of this process **defines the area**.

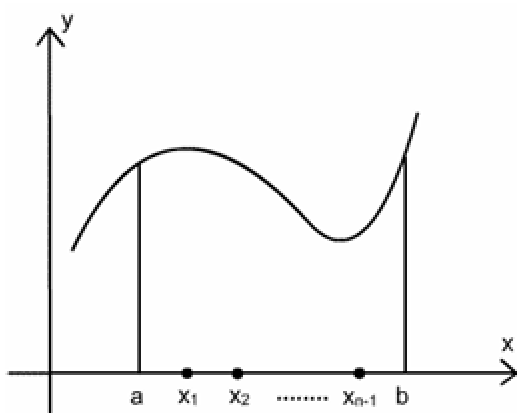
📦 Step 1: Divide the Interval

Take the interval **[a, b]** and divide it into **n equal pieces**.

Each piece has width:

$\Delta x = (b - a) / n$

These division points form a **regular partition**.



📐 Step 2: Build Rectangles

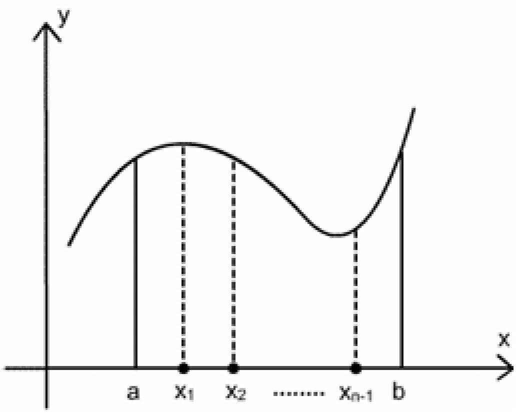
In each small subinterval:

- Choose a point (any point!)
- Measure the height **f(x*)**
- Build a rectangle:
  - Width = Δx
  - Height = f(x*)

Area of one rectangle:

f(x*) · Δx

Do this **n times**.



➕ Step 3: Add All Rectangle Areas

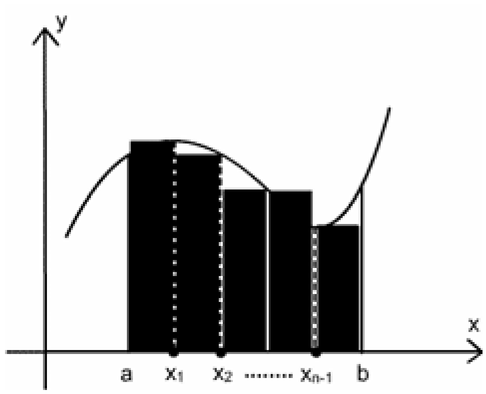
Total approximate area:

$\text{Area}_n = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x$

Using sigma notation:

$\text{Area}_n = \sum_{i=1}^{n} f(x_i^*)\, \Delta x$

This is **not exact yet** — it’s an approximation.



♾️ Step 4: Take the Limit

Now comes the magic.

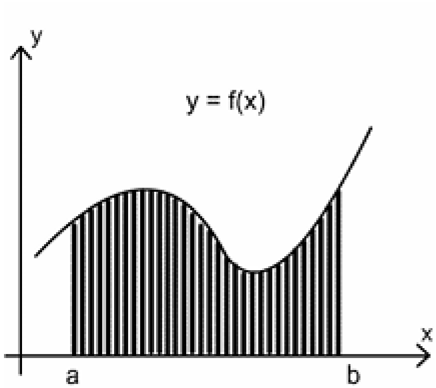
As **n** → ∞:

- Rectangles become thinner
- Gaps disappear
- Approximation becomes exact

So we **define area** as:

$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$

⭐ **This is the precise definition of area**



🧠 Important Technical Insight

You might worry:

> "What if I choose different points x* inside each interval?"

Good question.

Because **f(x) is continuous**, it turns out:

- Left endpoints
- Right endpoints
- Midpoints

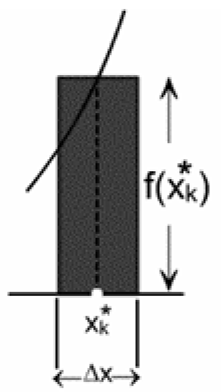
👉 **All give the same limit**

So the definition is **well-defined**.

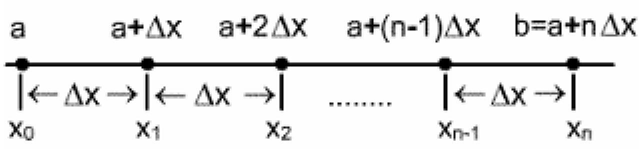
📍 Common Choices for x***

- **Left endpoint:**
  x□* = a + (k − 1)Δx
- **Right endpoint:**
  x□* = a + kΔx
- **Midpoint:**
  x□* = a + (k − ½)Δx

Each gives a different approximation —
but the **same final area**.



Area of k^{th} rectangle = $f(x_k^*)\,.\,\Delta x$



This figure shows that

$$x_k = a + k\Delta x \ \text{ for } k = 0, 1, 2 \ldots n$$

🧪 Example 1: Area under y = x on [1, 2]

(Right endpoints)

$\Delta x = (2 - 1) / n = 1 / n$

Right endpoint:
$x_k^* = 1 + k/n$

Area of kth rectangle:
$f(x_k^*)\Delta x = (1 + k/n)(1/n)$

Sum of areas:
$\sum_{k=1}^{n} (1 + k/n)(1/n)$

Take the limit as $n \to \infty$:

$A = 3/2$

✔ This matches the **trapezoid area** from geometry.

🧪 Example 2: Same Problem (Left endpoints)

Same steps, different x^*.

You again get:

$A = 3/2$

👉 Different rectangles, **same limit**.

🧪 Example 3: Area under $y = 9 - x^2$ on $[0, 3]$

Exact computation using limits is **long and painful**.

So instead, we use:

- Left endpoint approximation
- Right endpoint approximation
- Midpoint approximation

With large **n**, computers do the work.

📊 Numerical Approximations (Key Insight)

As n increases:

- Left approximation ↓
- Right approximation ↑
- Midpoint approximation → true value fastest

Example results:

n	Left	Right	Midpoint
10	19.305	16.605	18.0225
20	18.664	17.314	18.0056
50	18.268	17.728	18.0009

⭐ **Midpoint is the best practical method**

🧠 Final Feynman Insight

Area is **not a formula**.

Area is a **process**:

- Approximate
- Improve
- Take a limit

Integration works because:

> **Infinite thin rectangles perfectly fill space**

This idea leads directly to:

- Riemann sums
- Definite integrals
- The Fundamental Theorem of Calculus