Lecture 15 – The Derivative

1. What we already know
   In the last lecture we said:
   The slope of a tangent line to the graph of a function $y = f(x)$ is

$$m_{tan} = \lim_{x_1 \to x_0} [f(x_1) - f(x_0)] / (x_1 - x_0)$$

---

2. A small trick (introducing h)
   Let $h = x_1 - x_0$
   Then $x_1 = x_0 + h$
   As $x_1 \to x_0$, this is the same as $h \to 0$.

Now the slope formula becomes:

$$m_{tan} = \lim_{h \to 0} [f(x_0 + h) - f(x_0)] / h$$

---

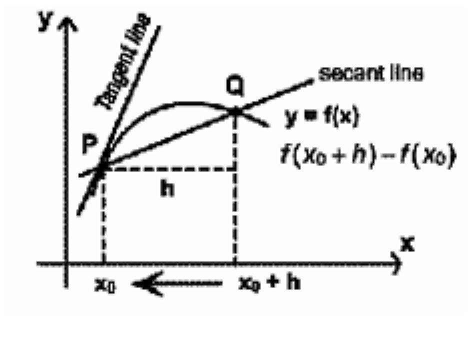
3. Definition of the Tangent Line
   If $P(x_0, y_0)$ is a point on the graph of f, the tangent line at P has slope

$$m_{tan} = \lim_{h \to 0} [f(x_0 + h) - f(x_0)] / h$$

Equation of tangent line:

$$y - y_0 = m_{tan} (x - x_0)$$

This definition only makes sense if the limit exists.



---

4. Example 1
   Find slope and tangent line of f(x) = x^2 at P(3, 9).

Step 1: Slope

m = lim (h → 0) [f(3+h) - f(3)] / h

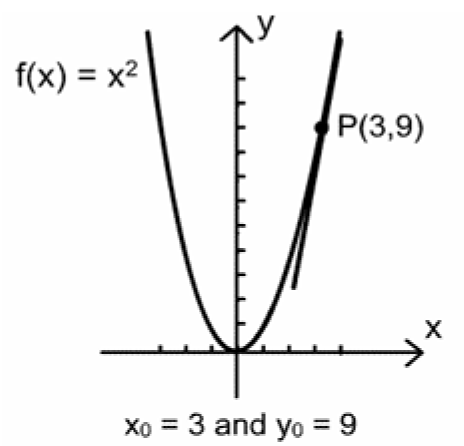
f(3+h) = (3+h)^2 = 9 + 6h + h^2
f(3) = 9

So, (f(3+h) - f(3)) / h = (6h + h^2) / h = 6 + h

Take limit h → 0: slope = 6

Step 2: Equation of tangent line

y - 9 = 6(x - 3)
y = 6x - 9



---

5. From slope to a new function
   The slope depends on where you are on the curve. At $x=3$ slope$=6$, at $x=2$ slope is different.
   So slope itself is a function of x. This new function is called the derivative.

General formula:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

---

6. Two interpretations of derivative
   a) Geometric: slope of tangent line at x
   b) Rate of change: instantaneous rate of change of y with respect to x

---

7. Example 2
   Let $f(x) = x^2 + 1$. Find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 + 1 - (x^2 + 1)}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$
$$= \lim_{h \to 0} (2x + h)$$
$$= 2x$$

So $f'(x) = 2x$

Check:
$x=2 \to$ slope$=4$
$x=0 \to$ slope$=0$
$x=-2 \to$ slope$=-4$

---

8. Example 3 (Straight Line)
   $f(x) = mx + b$

$$f'(x) = \lim_{h \to 0} \frac{m(x+h)+b - (mx+b)}{h}$$
$$= \lim_{h \to 0} \frac{mh}{h}$$
$$= m$$

So derivative of a straight line is just its slope m.

9. Example 4 ($f(x) = \text{sqrt}(x)$)
   $f(x) = \text{sqrt}(x)$

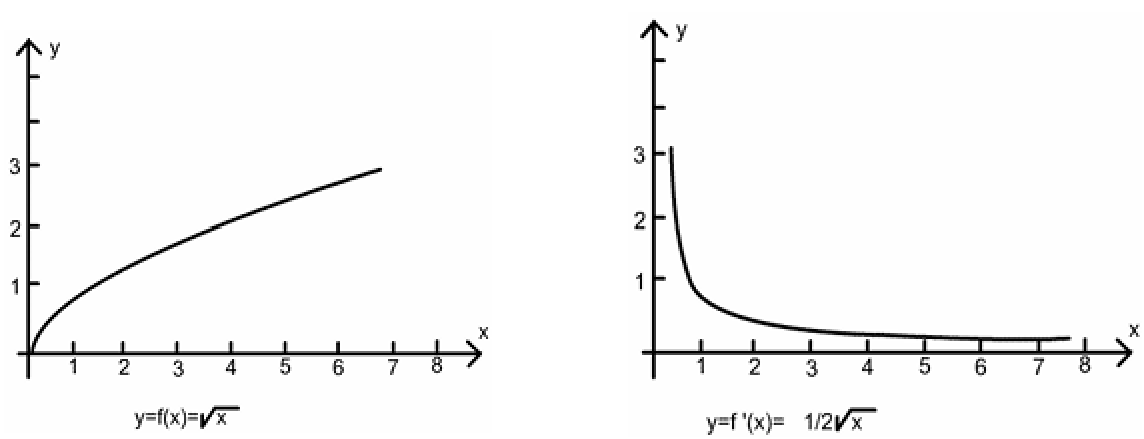
$f'(x) = \lim (h \to 0) [\text{sqrt}(x+h) - \text{sqrt}(x)] / h$

Multiply by conjugate:

$= \lim (h \to 0) [(x+h - x) / (h (\text{sqrt}(x+h) + \text{sqrt}(x)))]$
$= \lim (h \to 0) [h / (h (\text{sqrt}(x+h) + \text{sqrt}(x)))]$
$= \lim (h \to 0) [1 / (\text{sqrt}(x+h) + \text{sqrt}(x))]$
$= 1 / (2 \text{sqrt}(x))$

So $f'(x) = 1 / (2 \text{sqrt}(x))$

As $x \to 0+$, slope $\to$ infinity. Tangent line becomes vertical.



---

10. Big Picture

- Derivative = slope of tangent = instantaneous rate of change
- Formula: $f'(x) = \lim (h \to 0) [f(x+h) - f(x)] / h$
- Geometric meaning: slope
- Physical meaning: speed / rate of change
- Examples:

- $f(x) = x^2 \rightarrow f'(x) = 2x$
- $f(x) = mx + b \rightarrow f'(x) = m$
- $f(x) = \sqrt{x} \rightarrow f'(x) = 1 / (2 \sqrt{x})$

Derivative Notation

1. Differentiation = The Operation
   The process of finding the derivative is called **differentiation**.
   Think of it like an operation you perform on a function to create a new function.

Example: Addition is an operation. If you apply "+" to 3 and 5, you get 8.
Similarly, differentiation is an operation. If you apply it to f(x), you get a new function f'(x).

---

2. New Notation
   When the independent variable is x, we write the derivative in a few different ways.

The most common way:

d/dx [f(x)]

This is read as: "the derivative of f(x) with respect to x."

This is exactly the same as f'(x).

So,

d/dx [f(x)] = f'(x)

---

3. Example: f(x) = x^2
   We already know f'(x) = 2x.

In this notation:

d/dx [x^2] = 2x

At x = 1 → derivative = 2(1) = 2
At x = 2 → derivative = 2(2) = 4

---

4. If y = f(x)
   Sometimes we write the function as y = f(x).

Then the derivative can be written as:

$dy/dx = f'(x)$

So if $y = x^2$, then

$dy/dx = 2x$

This looks like a fraction “dy over dx” but right now it is just a **symbol**.
Later we will see it really does act like a ratio in some cases.

---

5. If the variable is different
   If the function is in terms of another variable, say u instead of x, then we adjust:

$dy/du = f'(u) = d/du\,[\, f(u)\,]$

Example: if $f(u) = u^2$, then

$dy/du = 2u$

---

6. Derivative at a specific point
   We can also ask for the value of the derivative at a specific point $x = x0$.

In notation:

$(d/dx\, f(x))$ at $x = x0$

or simply $f'(x0)$.

Example:

$d/dx\,[\, x^2\,]$ at $x = 1 = 2(1) = 2$
$d/dx\,[\, x^2\,]$ at $x = 0 = 0$

So the notation tells us both “the general formula for the derivative” and “the value at a point.”

Summary

- Differentiation = process of finding derivative.
- $d/dx [f(x)] = f'(x) = dy/dx$
- Example: $f(x) = x^2 \rightarrow d/dx [x^2] = 2x$
- dy/dx looks like a ratio but is really just a symbol (for now).
- If variable is u, then $dy/du = f'(u)$.
- Derivative at a point: $f'(x_0)$.

Existence of Derivatives

1. When does a derivative exist?
   From the definition:

$f'(x0) = \lim (h \to 0) [f(x0 + h) - f(x0)] / h$

This only makes sense if the limit exists.

- If the limit exists → function is **differentiable** at x0.
- If the limit does not exist → function is **not differentiable** at x0.

So the **domain of f'** is all the points where this limit exists.

---

2. Differentiability on an interval
   We say:

- f is differentiable on (a, b) if it is differentiable at every single point inside that interval.
- A function that is differentiable everywhere on the interval is called a **differentiable function**.

---

3. When does differentiability fail?
   There are three common situations:

a) **Corners**
Imagine the absolute value function $y = |x|$.

- From the right side slope = +1
- From the left side slope = –1
  The two limits don't match → derivative doesn't exist at x=0.

b) **Vertical Tangents**
If the tangent line is vertical, slope is infinite. Example: y = sqrt(x) at x=0. The slope goes to infinity.

c) **Discontinuities**
If the function jumps or breaks at a point, you can't even talk about slope. Example: step functions.

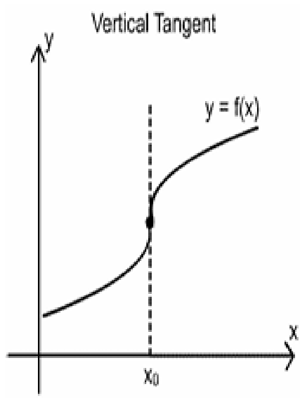


Fig 3.2.6(b)

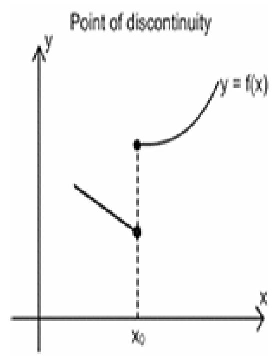


Fig 3.2.6(c)

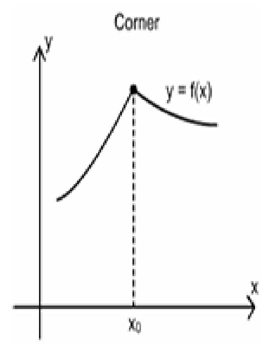
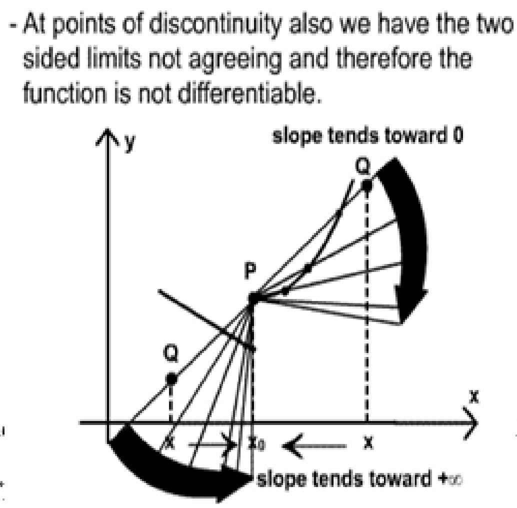


Fig 3.2.6(a)

-



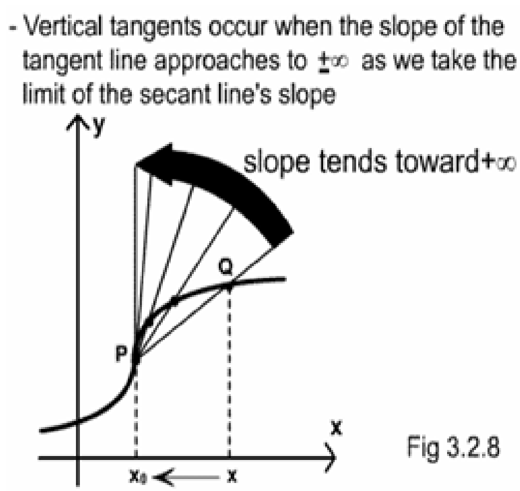
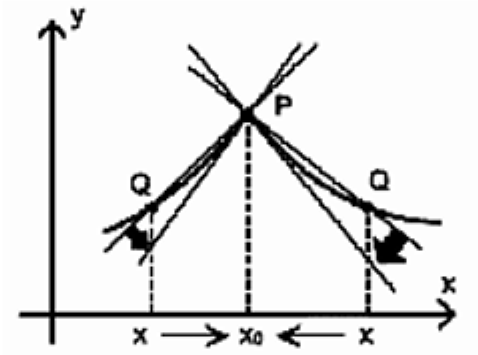


Fig 3.2.8

R
nuity
T



---

4. Relationship between Differentiability and Continuity
   Theorem:
   If a function is differentiable at x0, then it is also continuous at x0.

Intuition:

- Differentiability requires a very “tight” behavior: the secant slopes must settle down to one number.

- If the function were not continuous, the slopes could not settle, so no derivative.

So → differentiable ⇒ continuous.
But the opposite is not always true: continuous does not necessarily mean differentiable (like |x| at 0).

---

5. Example: Absolute Value Function
   f(x) = |x|

That means:

- f(x) = x if x ≥ 0

- f(x) = -x if x < 0

Now the derivative:

f'(x) = 1 if x > 0
f'(x) = -1 if x < 0

At x=0 → left slope = –1, right slope = +1. They do not match.
So derivative does not exist at x=0.

This is a **corner** example.

Summary

- Derivative exists at a point if the slope limit exists.
- Differentiable on (a,b) = differentiable at every point inside.
- Not differentiable at:
    - corners (|x| at 0),
    - vertical tangents (sqrt(x) at 0),
    - discontinuities (step functions).
- Differentiability implies continuity, but continuity does not guarantee differentiability.