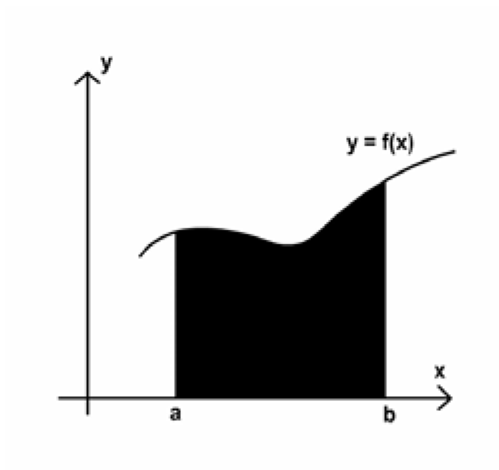
⭐ The Area Problem

Imagine you draw a curve on a graph — something smooth and continuous, like a hill.
Now imagine you want to find the area under that curve, starting from some point **a** on the x-axis up to some point **x**.

This "area under a curve" is what we call:

A(x) = area under f(x) from a to x

Why does A(x) depend on x?
Because the further you move to the right, the more area you're collecting.



🔍 The Big Idea (Newton–Leibniz Trick)

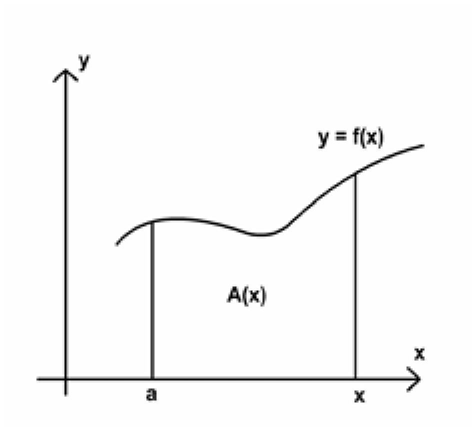
Newton and Leibniz realized something clever:

Instead of trying to calculate the area directly…
first look at **how the area changes** when x changes a little bit.

In other words:

Step 1: Find A′(x) — the derivative of the area
Step 2: Use it to recover A(x)

This sounds backwards, but it's genius.



🎯 Step 1: What is A′(x)?

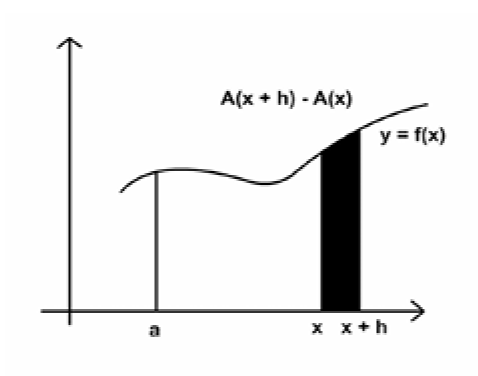
Use the definition of derivative:

A′(x) = lim (h → 0) [A(x + h) – A(x)] / h

This means:

“How much extra area do I get when I extend x to x + h?”

The difference A(x + h) – A(x) is the small strip of area between x and x + h.



📦 Step 2: Approximate that Small Strip

Between x and x + h, the curve looks almost straight (because h is tiny).

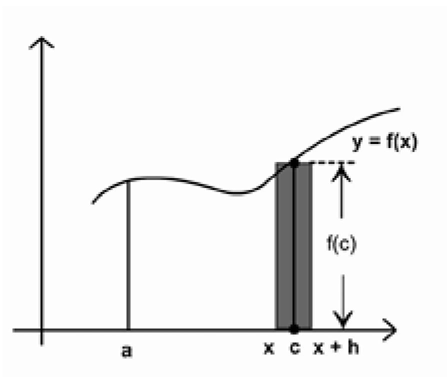
So the small curved slice is almost a **rectangle** with:

- Width = h
- Height ≈ f(c) for some point c between x and x + h

So:

A(x + h) – A(x) ≈ f(c) * h

As h gets smaller, this becomes exact.



🎯 Step 3: Plug into the Derivative Formula

A′(x)
= lim (h → 0) [f(c) * h] / h
= lim (h → 0) f(c)

As h → 0, the midpoint c → x.
Since f is continuous:

lim (c → x) f(c) = f(x)

So finally we get:

⭐ A′(x) = f(x)

This is the heart of the Area Problem.

The derivative of the area under f(x) is simply the height of the graph.
This is why integration is the opposite of differentiation.

🧪 Example

Find the area under:

$y = x^2$
from 0 to 1.

Step 1: On the interval [0, x], we know:

$A'(x) = x^2$

Step 2: Find a function whose derivative is x^2.
This is an anti-derivative.

We know:

$d/dx\ (x^3 / 3) = x^2$

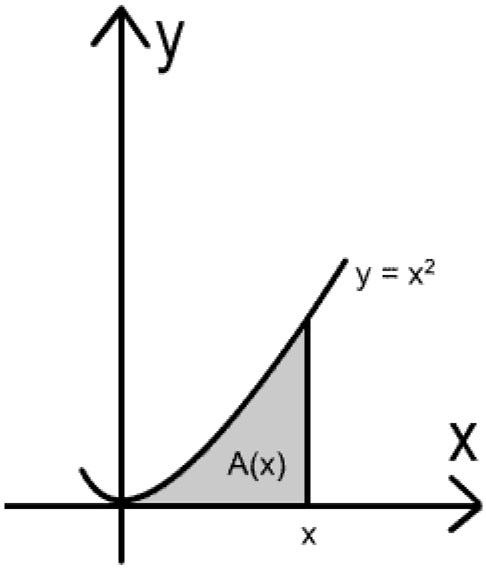
So:

$A(x) = x^3 / 3$

Step 3: Evaluate at x = 1:

$A(1) = 1^3 / 3 = 1/3$

✔ So the area under $y = x^2$ from 0 to 1 is **1/3 square units**.



📚 Anti-Derivatives

A function F(x) is an anti-derivative of f(x) if:

$F'(x) = f(x)$

For $f(x) = x^2$, all of these are anti-derivatives:

$x^3 / 3$
$x^3 / 3 + 1$
$x^3 / 3 + \pi$
$x^3 / 3 + C$

Because when you differentiate, the constant disappears.

🧠 Final Insight

- Differentiation tells you how fast something is changing.
- Integration tells you how much total has accumulated.

- The Area Problem connects them:

Rate of change of area = height of the function.

This idea becomes the **Fundamental Theorem of Calculus**.
