Newton’s Method – Finding Roots the Smart Way

Let’s start simple.

We already know that in algebra, if we have an equation like
$ax + b = 0$,
its solution is **$x = -b/a$**.

That’s easy — a straight line crosses the x-axis at that point.

But what if we face something more complex, like:
$\cos(x) - x = 0$

Now we can’t solve this by any simple algebraic trick.
There’s no neat formula for x.

So what do we do?
We **approximate** — we find values of x that *get really close* to the real solution.
One of the smartest ways to do this is called **Newton’s Method**.

What Are We Really Trying to Do?

When we say **$f(x) = 0$**, we’re basically asking:

> “At what value of x does the graph of f(x) cross the x-axis?”

Because wherever the function equals zero, that’s where its graph touches or crosses the x-axis.

So our goal is to find that special x — the **root** of the equation.

The Idea Behind Newton’s Method

Imagine you’re standing on the graph of the function at some point — say **x_1**, your first guess.

From that point, draw a **tangent line** — a line that just touches the curve at that point.

Now, extend that tangent line until it meets the x-axis.
That point of intersection will be your next guess, **x_2**.

Here's the magic:
x_2 is usually a much better approximation to the real root than your first guess **x_1**.

Then, you repeat the process again:
draw another tangent at **x_2**, find where it hits the x-axis, call it **x_3**, and so on.

Each time, you get closer and closer to the true solution.

That's Newton's Method — a beautiful cycle of guess, correct, repeat.

The Formula (Where the Magic Happens)

Let's build the formula step by step.

At point **x_1**, the tangent line has the equation:

$$y - f(x_1) = f'(x_1)(x - x_1)$$

Now, this tangent line crosses the x-axis when **$y = 0$**, so we plug in $y = 0$:

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

Simplify this to find **x_2**:

$$x_2 = x_1 - f(x_1) / f'(x_1)$$

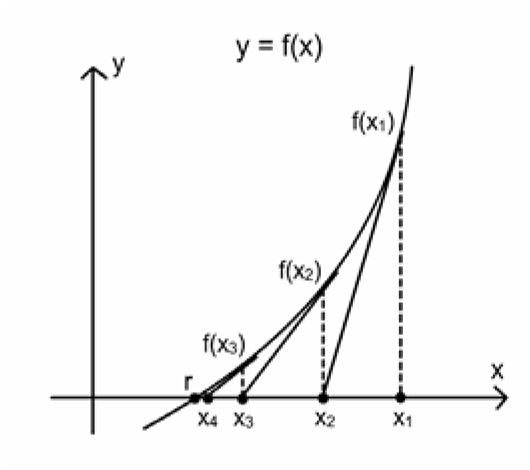
That's the heart of Newton's Method.

We can write this as a general formula:

$$x_{n+1} = x_n - f(x_n) / f'(x_n)$$

This means:

- Take your current guess (**x_n**),
- Subtract the ratio of the function value to its derivative,
- You'll get a new, improved guess (**x_{n+1}**).



Example: Solving cos(x) = x

We can rewrite this as

$f(x) = x - \cos(x)$

So for **$f(x) = x - \cos(x)$**, we get:

$f'(x) = 1 + \sin(x)$

Now, plug this into the Newton formula:

$x_{n+1} = x_n - (x_n - \cos(x_n)) / (1 + \sin(x_n))$
$x_{n+1} = (\cos(x_n) + x_n * \sin(x_n)) / (1 + \sin(x_n))$ — simplified version

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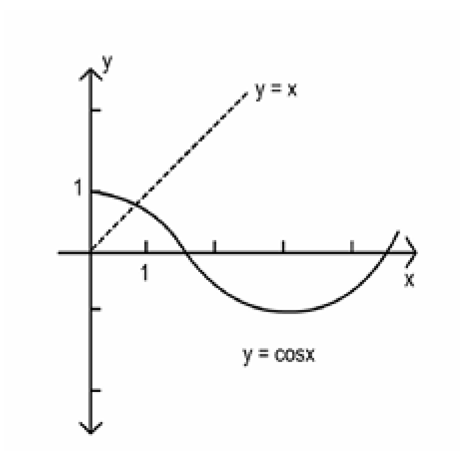


Step-by-step Calculation

Let’s take our first guess x_1 **= 1** (since from the graph, the solution is near 1).

Now calculate:

1. $x_2 = x_1 – (\cos(1) – 1) / (1 + \sin(1))$
   = 1 – (0.5403 – 1) / (1 + 0.8415)
   = 1 – (–0.4597) / 1.8415
   = 1 + 0.2497
   x_2 **≈ 1.2497** (that’s our next guess)

2. $x_3 = x_2 – (\cos(x_2) – x_2) / (1 + \sin(x_2))$
   Plug in x_2 = 0.7503 (alternate approach from lecture):
   = 0.7503 – (cos(0.7503) – 0.7503) / (1 + sin(0.7503))
   = 0.7503 – (0.7317 – 0.7503) / (1 + 0.6816)
   = 0.7503 – (–0.0186) / 1.6816
   = 0.7503 + 0.011
   x_3 **≈ 0.7391**

That’s close enough!
So, the root is approximately **x ≈ 0.7391**.

What’s Happening Visually

Each tangent line acts like a little “bridge” connecting your current guess to the x-axis — it keeps nudging you closer and closer to the true root.

It’s like playing a “hot and cold” game with math — every tangent gets you *hotter*, i.e., closer to the real answer.

When Newton’s Method Fails

It’s powerful, but not perfect.
Here are some cases when it breaks down:

1. **Division by Zero:**
   If f′(x□) = 0 at any step, the formula divides by zero — the tangent line is horizontal, and

it never meets the x-axis. Game over.

2. **No Convergence:**
 Sometimes, instead of getting closer, your guesses start bouncing around or even diverging. That means your initial guess was too far, or the function behaves badly there.

3. **Example: f(x) = x^(1/3)**
 The derivative here becomes infinite or zero at x = 0, and the guesses don't converge.

So, Newton's method **works beautifully when conditions are right** — but it's not foolproof.

In Summary

- The goal: Solve f(x) = 0
- The method: Use tangent lines iteratively
- The formula: **$x_{n+1} = x_n - f(x_n)/f'(x_n)$**
- The benefit: Each step gets you closer to the real root
- The risk: If slope = 0 or function behaves oddly, it fails

Final Thought

Newton's Method is like a conversation between calculus and geometry.
You ask the curve, "Hey, where do you cross the x-axis?"
And the curve replies,

> "Not here, but if you follow my slope, you'll find out soon."

Each tangent line is a clue.
Follow enough of them — and the truth reveals itself.

Rolle’s Theorem – The “Zero Slope” Guarantee

Let’s start with the big idea.

Rolle’s Theorem says something beautifully simple:

> If a smooth curve starts and ends at the same height,
> then somewhere in between, it must have a flat (horizontal) tangent.

In math language, that means:
If a function **f(x)** crosses the x-axis at two points (or has the same value at two points),
then *somewhere between those two points,* its derivative **f′(x)** is **zero**.

The Logic Behind Rolle’s Theorem

Here’s what the theorem actually says:

If a function **f(x)** satisfies three conditions:

1. **Continuous on [a, b]**
   → The graph has no breaks, jumps, or holes between a and b.

2. **Differentiable on (a, b)**
   → The graph is smooth — no sharp corners or cusps.

3. **f(a) = f(b)**
   → The function starts and ends at the same height.

Then, there exists at least one point **c** between **a** and **b**
such that **f′(c) = 0.**

That’s Rolle’s Theorem.

Feynman’s Intuitive Way to See It

Imagine you’re walking up and down a small hill.

- You start at ground level (f(a)).

- You climb, maybe descend, and after some distance, you return to the same ground level ($f(b)$).

At some point on that journey, you must have been *neither climbing up nor going down* — you were *flat* for a brief moment.
That's the point where the slope = 0, or mathematically, **$f'(c) = 0$.**

That's what Rolle's Theorem guarantees — that somewhere between two equal heights, the slope must flatten out.

Example: $f(x) = \sin(x)$

Let's test the theorem with a real function.

Take **$f(x) = \sin(x)$**
on the interval **$[0, 2\pi]$**

Now check the three conditions:

1. **Continuous?**
   Yes, $\sin(x)$ is continuous everywhere.

2. **Differentiable?**
   Yes, $\sin(x)$ is smooth everywhere.

3. **Equal endpoints?**
   $f(0) = \sin(0) = 0$
   $f(2\pi) = \sin(2\pi) = 0$
   ✅ They're equal!

So, by Rolle's Theorem, there must exist some point **c** between 0 and 2π where **$f'(c) = 0$**.

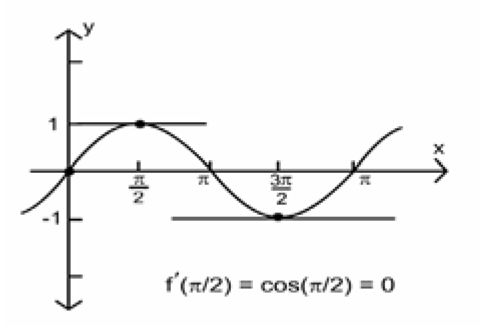
Now find the derivative:

$$f'(x) = \cos(x)$$

Set this equal to 0:

$$\cos(c) = 0$$

That happens at **c = π/2** and **c = 3π/2**.
So indeed, there are two points in (0, 2π) where the slope of sin(x) is zero — exactly as Rolle's Theorem promised!



A Real-Life Analogy (Lahore to Islamabad)

Imagine this:

> You start your car in **Lahore**, velocity = 0 (you're not moving yet).
> You drive all the way to **Islamabad**, and when you stop there, your velocity = 0 again.

Let's assume:

- Your velocity changes smoothly (no teleporting or jumps).
- It's defined at every moment (you can always measure how fast you're going).

Now, according to Rolle's Theorem,
there must be at least one moment during your trip where your acceleration = 0.

That's the moment when your speed stops increasing and hasn't started decreasing yet — in other words, your velocity graph has a flat tangent.

That's exactly the "f′(c) = 0" moment.

Mathematical Visualization

If we represent your motion mathematically:

- Distance traveled → $f(x)$
- Velocity → $f'(x)$
- Acceleration → $f''(x)$

Then Rolle's Theorem says:

> If $f(a) = f(b)$, then there exists a c between a and b where $f'(c) = 0$.

In the car analogy:

- a = time when you start (velocity = 0)
- b = time when you stop (velocity = 0)
- c = some time in between where acceleration = 0

It's a **guarantee**, not a guess.

A Little Twist — What If You Keep Accelerating?

Someone might ask:

> "But what if I just keep accelerating all the way?"

That would mean your velocity graph never flattens —
it keeps rising.

But in that case, your **velocity at the end wouldn't be zero** —
you'd still be moving fast when you reach Islamabad!

So the condition **f(a) = f(b)** breaks down,
and Rolle's Theorem doesn't apply.

It's only true when you start and end at the same level.

Geometric View

Think of a curve that starts and ends at the same height:
like the top of a hill, a wave, or a smile curve (a parabola).

Somewhere in between those two equal points,
the slope must *flatten out*.

That's Rolle's Theorem in action —
the guarantee of a **horizontal tangent** between equal heights.

Summary

Condition	Meaning	Why It Matters
f(x) is continuous on [a, b]	No jumps or holes	Curve is unbroken
f(x) is differentiable on (a, b)	Smooth graph	Tangent exists
f(a) = f(b)	Same start and end level	Ensures a "turn" happens

Then → ∃ c ∈ (a, b) such that **f′(c) = 0**

Example Recap

f(x) = sin(x), on [0, 2π]

✅ Continuous
✅ Differentiable
✅ f(0) = f(2π) = 0

⇒ There exists c = π/2, 3π/2 where f′(c) = 0.

Feynman-Style Closing Thought

Rolle's Theorem isn't just about numbers —
it's about *movement* and *change*.

It says:

If you start and end at the same height,
you must have been perfectly flat at least once in between.

It’s nature’s quiet way of saying —
every journey that begins and ends at the same point
must have a moment of calm in the middle.

Mean Value Theorem (MVT) — The Secret of the "Average Slope"

Let's start with the core idea in simple words:

> The Mean Value Theorem says that for a smooth curve,
> there exists at least one point where the tangent has exactly the same slope
> as the line joining the two endpoints of the curve.

In other words:
If you take two points on a smooth, continuous curve and draw a straight line between them (a *secant line*),
then somewhere between those points, the curve itself will have a *tangent line* parallel to that secant.

What the Theorem Really Says

If a function **f(x)** satisfies:

1. **Continuous on [a, b]**
   → The graph has no breaks between a and b.

2. **Differentiable on (a, b)**
   → The graph is smooth — no corners or sharp turns.

Then there exists at least one point **c** in (a, b) such that:

> $$f'(c) = [f(b) - f(a)] / (b - a)$$

This means:

> The instantaneous rate of change (the tangent's slope)
> equals the average rate of change (the secant's slope).

Feynman's Intuition

Imagine you're driving from **Lahore** to **Islamabad** again.

- Lahore → Islamabad = total distance: say, 376 km.
- Total time: 4 hours.

Your **average speed** = total distance ÷ total time = 376 ÷ 4 = 94 km/h.

Now here's the key question:

> Was there ever a moment during your drive when your *instantaneous speed* was exactly **94 km/h**?

The Mean Value Theorem says:
✅ Yes, **there must have been!**

At least once during that trip, your speedometer read exactly 94 km/h.
That's your **point c**, where the tangent slope (instantaneous rate) equals the secant slope (average rate).

The Math Behind the Magic

Let's understand why that's true.

We'll use **Rolle's Theorem** to prove it — because the Mean Value Theorem is basically Rolle's Theorem wearing new clothes.

Step 1: Construct a New Function

We define a helper function **v(x)** as:

$$v(x) = f(x) - [\ (f(b) - f(a)) / (b - a)\ ] * (x - a) - f(a)$$

Don't get scared by the formula — it's just **f(x)** with the secant line subtracted out.

So, $v(x)$ is basically the **vertical distance** between the curve and the straight line joining points $A(a, f(a))$ and $B(b, f(b))$.

Step 2: Check What Happens at the Endpoints

At **x = a**:

$$v(a) = f(a) - [(f(b) - f(a)) / (b - a)] * (a - a) - f(a) = 0$$

At **x = b**:

$$v(b) = f(b) - [(f(b) - f(a)) / (b - a)] * (b - a) - f(a)$$
$$v(b) = f(b) - (f(b) - f(a)) - f(a) = 0$$

So, **v(a) = v(b) = 0**

That means v(x) satisfies all the conditions of **Rolle's Theorem**:

- It's continuous and differentiable (since f(x) is),
- v(a) = v(b) = 0.

Step 3: Apply Rolle's Theorem

By Rolle's Theorem, there exists at least one **c** in (a, b) such that:

$$v'(c) = 0$$

Step 4: Differentiate v(x)

Differentiate v(x):

$$v'(x) = f'(x) - [(f(b) - f(a)) / (b - a)]$$

Now at the special point **c**, v′(c) = 0, so:

$$f'(c) - [(f(b) - f(a)) / (b - a)] = 0$$

Simplify it:

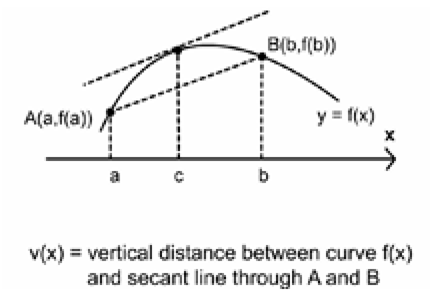
$$\mathbf{f'(c) = [f(b) - f(a)] / (b - a)}$$

And that's exactly the **Mean Value Theorem**.

Visual Understanding

- Draw two points A(a, f(a)) and B(b, f(b)) on a smooth curve.
- Connect them with a straight line — that's your *secant line*.
- Now look for a tangent line somewhere between them that's *parallel* to that secant. The MVT guarantees that such a point **c** always exists.

So, **the tangent and the secant share the same slope** — that's the geometric meaning.



Example

Let's take **f(x) = x²** on the interval [1, 3].

1. **Find f(b) – f(a):**
   f(3) – f(1) = 9 – 1 = 8
2. **Find b – a:**
   3 – 1 = 2
3. **Secant slope:**
   (f(b) – f(a)) / (b – a) = 8 / 2 = 4

Now, by the Mean Value Theorem, there exists c in (1, 3) such that f′(c) = 4.

f′(x) = 2x → 2c = 4 → c = 2

✅ So at **x = 2**, the tangent has slope 4 — exactly matching the secant's slope.
That's your MVT in action!

Another Real-Life Analogy

Suppose your friend says:

> "I averaged 60 km/h on my trip."

The MVT says there must be at least one instant where your **actual speed** was **exactly 60 km/h** — not slower, not faster — exactly equal to the average.

This is why MVT is sometimes called the **"There-must-have-been-a-moment" theorem.**

Summary

Concept	Meaning
Secant line slope	$(f(b) – f(a)) / (b – a)$ → average rate of change
Tangent line slope	$f'(c)$ → instantaneous rate of change
Theorem statement	There exists a $c \in (a, b)$ where $f'(c) = [f(b) – f(a)] / (b – a)$
Real meaning	At some instant, your speed (instantaneous) equals your average speed

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Feynman-Style Closing Thought

The Mean Value Theorem connects two worlds:

- The **average** world — what happens between two points.
- The **instantaneous** world — what happens at a single point.

It tells us that no matter how curvy or unpredictable a smooth path may look,
there's always at least one point where the "momentary slope" matches the "overall slope."

It's nature's way of saying:

“Between any two moments in motion, there must be a moment that perfectly represents the whole trip.”