⭐ Lecture 26: Integration by Substitution

🧠 Big Picture First (Very Important)

Integration by substitution is **nothing new**.

It is simply the **reverse of the Chain Rule** you learned in differentiation.

Chain Rule (Differentiation)

When a function is **inside another function**, you use the chain rule.

Substitution (Integration)

When an integral contains a **function inside another function**, you undo the chain rule using substitution.

👉 Same idea, opposite direction.

🔍 Why Do We Need Substitution?

Some integrals look complicated because they are **compositions of functions**.

Example of a composition:

- Inner function: $(2x + 1)$
- Outer function: $(\ )^{50}$

Together:
$(2x + 1)^{50}$

The normal power rule **does NOT work directly** here.

So instead of fighting the expression, we **rename the inside**.

This is the key idea.

🎯 The Core Idea (Feynman Style)

If an integral looks messy, ask:

> “Is there something inside whose derivative is also present?”

If yes → **substitute it with a simpler name (u)**.

This turns a hard problem into an easy one.

📦 The Mathematics Behind It (Simple Logic)

Suppose:

$u = g(x)$

Then its derivative is:

$du/dx = g'(x)$
or
$du = g'(x)\,dx$

Now look at the integral:

$\int f(g(x)) \cdot g'(x)\,dx$

Since $g'(x)\,dx = du$, this becomes:

$\int f(u)\,du$

That’s it.
You have **removed x** completely.

🧪 Example 1 (Classic Power Example)

Evaluate:

$\int (2x + 1)^{50} \cdot 2x\,dx$

Step 1: Choose u

Let:

$u = 2x + 1$

Step 2: Differentiate u

$du = 2x\, dx$

Step 3: Substitute

The integral becomes:

$\int u^{50}\, du$

Step 4: Integrate

Using the power rule:

$\int u^{50}\, du = u^{51} / 51 + C$

Step 5: Replace u

Final answer:

$(2x + 1)^{51} / 51 + C$

⚠️ Important Warning (Very Common Mistake)

❌ Do NOT do this:

$\int (2x + 1)^{50}\, dx = (2x + 1)^{51} / 51$

This is **wrong**.

Why?

Because the power rule only works when x is **alone**, not trapped inside another function.

🧠 How to Choose u (Chess Analogy)

There is **no fixed rule** for choosing u.

But good choices:

- Inner expressions
- Inside brackets
- Expressions whose derivative appears nearby

Think like chess:

> Choose u so that the future becomes simple.

📋 General Procedure (Step-by-Step)

1. Choose $u = g(x)$
2. Compute $du = g'(x)\,dx$
3. Substitute u and du into the integral
4. Integrate with respect to u
5. Replace u back with $g(x)$

By Step 3, there should be **no x left**.

🧪 Trigonometric Examples

Example 2:

$\int \sin(x + 9)\,dx$

Let:
 $u = x + 9$
 $du = dx$

Integral becomes:

$\int \sin(u)\, du = -\cos(u) + C$

Final answer:

$-\cos(x + 9) + C$

Example 3:

$\int \cos(5x)\, dx$

Let:
 $u = 5x$
 $du = 5\, dx \rightarrow dx = du / 5$

Integral becomes:

$1/5 \int \cos(u)\, du$
 $= 1/5 \sin(u) + C$

Final answer:

$(1/5) \sin(5x) + C$

🧪 Product Example

Evaluate:

$\int 2 \sin(x) \cos(x)\, dx$

Let:
 $u = \sin(x)$
 $du = \cos(x)\, dx$

Integral becomes:

$\int 2u\,du$
$= u^2 + C$

Final answer:

$\sin^2(x) + C$

🧪 Another Example

Evaluate:

$\int \cos(x^2)\, x\, dx$

Let:
$u = x^2$
$du = 2x\,dx \rightarrow x\,dx = du / 2$

Integral becomes:

$1/2 \int \cos(u)\, du$
$= 1/2 \sin(u) + C$

Final answer:

$(1/2) \sin(x^2) + C$

🧪 Complicated Example (But Same Idea)

Evaluate:

$\int (3t^5 - 5)^4 \cdot t^4\, dt$

Choose:
$u = 3t^5 - 5$

Differentiate:
$du = 15t^4\, dt$

So:
$t^4\, dt = du / 15$

Integral becomes:

$1/15 \int u^4\, du$
$= 1/15 \cdot u^5 / 5 + C$
$= u^5 / 75 + C$

Replace u:

$(3t^5 - 5)^5 / 75 + C$

🧠 Final Insight (Feynman Style)

- Differentiation **breaks** a function into pieces
- Integration by substitution **glues them back together**

Whenever you see:

- a function inside another function
- and its derivative nearby

👉 **Use substitution**

With practice, your eyes will automatically spot u.