Lecture 6: Functions

1. What is a Function?

A function is simply a **rule** that connects one value to another.

- You give the rule an input (x), and it gives you **exactly one output (y)**.
- Think of it like a vending machine: press button x → get snack y.
  If pressing the same button gives two different snacks, it's **not a function**.

Notations for Functions

1. Where Did the Notation Come From?

In the 1700's, the Swiss mathematician **Euler** introduced the notation:

$y = f(x)$

This is read as: **"y equals f of x."**

It simply means:

- y depends on x.
- f is just the label of the rule (not a number, not multiplication).

---

2. Independent vs Dependent

- The variable **inside** the brackets (x in f(x)) is the **independent variable**.

- The variable on the left side (y) is the **dependent variable**.
- Why? Because the value of y "depends" on what x is.

So:
$y = f(x) \rightarrow$ y depends on x.

---

3. Reading f(x)

Important points:

- $f(x)$ is read as **"f of x"**, not "f times x."
- f is not a number, it's just the "name tag" for the function.
- Functions are useful because they tell us clearly: **which input produces which output**.

---

4. Simple Examples

Example 1:
$y = f(x) = x^2$

Then:
$f(3) = (3)^2 = 9$
$f(-2) = (-2)^2 = 4$

So $f(3) = 9$ and $f(-2) = 4$.

---

5. Using Other Letters

We don't always need f and x. Any letters can be used:

$y = g(x)$
$y = h(x)$
$s = f(t)$

Example:
$s = f(t) \rightarrow$ means "s is a function of t."

If $f(t) = t^2$, then:
$f(2) = 4$, $f(3) = 9$, $f(5) = 25$.

---

6. Example with Different Function

Suppose we define a function $\varphi(x)$:

$\varphi(x) = 1 / (x - 1)$

Now try values:
$\varphi(5) = 1 / (5 - 1) = 1/4$
$\varphi(1) = 1 / (1 - 1) = 1/0 \rightarrow$ undefined

So $\varphi(5)$ is fine, but $\varphi(1)$ does not exist.
This shows how the notation helps us check which inputs are valid (domain).

---

7. Replacing x with Other Symbols

Functions work with any symbol, not just x.

Example:
$F(x) = 2x^2 - 1$

Then we can replace x with something else:
$F(d) = 2d^2 - 1$
$F(t - 1) = 2(t - 1)^2 - 1 = 2t^2 - 4t + 1$

This shows we can plug in *expressions* too, not just numbers.

---

8. Same Function, Different Symbols

If two functions have the same formula, they are really the same function, no matter what letters are used.

Example:
$g(c) = c^2 - 4c$
$g(x) = x^2 - 4x$

These two are **the same function**.
Why? Because the structure (formula) is the same, only the variable name is different.

✅ Summary (Feynman Style)

- Euler gave us y = f(x).
- Inside the bracket = independent variable.
- Left side = dependent variable.
- f(x) is read as “f of x,” not “f times x.”
- f is just a label for the rule, not a number.
- Variables can be changed, but the formula defines the function.

Domain of a Function

1. What is the Domain?

- The **domain** is the set of all inputs (x-values) you are **allowed** to use in a function.
- Not every number always works. Some values are forbidden — like dividing by zero, or taking square roots of negative numbers (in real numbers).
- Sometimes real-life meaning also restricts the values.

Think of the domain as the **menu of valid buttons** on your vending machine.

---

2. Example with Cardboard

Suppose we have a cardboard square of size 10 cm × 10 cm.
We cut out small squares of side length x cm from each corner.
The area that remains is:

$y = 100 - 4x^2$

Now:

- x cannot be negative (length can't be negative).
- x cannot be larger than 5 (otherwise corners overlap).

So:
$0 \le x \le 5$

Therefore, the **domain = [0, 5]**.

👉 The physical meaning of x (length) tells us the domain.

---

3. Two Types of Domains

(a) Natural Domain

- This comes from the formula itself.
- You look at the function and ask: “Which x-values make sense here?”
- If x makes the formula undefined (like dividing by zero), exclude it.

Example:
$h(x) = 1 / ((x - 1)(x - 3))$

- If $x = 1 \rightarrow$ denominator $= 0 \rightarrow$ undefined
- If $x = 3 \rightarrow$ denominator $= 0 \rightarrow$ undefined

So domain = all real numbers **except 1 and 3**.
In interval notation: $(-\infty, 1) \cup (1, 3) \cup (3, \infty)$

(b) Restricted Domain

- Sometimes, in algebra, we simplify formulas by canceling factors.
- But this can change the domain if we’re not careful.

Example:
$h(x) = (x^2 - 4) / (x - 2)$

Factorize numerator:
$h(x) = (x - 2)(x + 2) / (x - 2)$

Cancel $(x - 2)$:
$h(x) = x + 2$

But wait! Originally, when $x = 2$, denominator $= 0 \rightarrow$ undefined.
So even though the simplified version looks valid for all x, the **original domain** excluded $x = 2$.

Correct way:
$h(x) = x + 2$, but $x \neq 2$

👉 Always respect the original domain before simplifying.

✅ Summary (Feynman Style)

- **Domain** = all valid inputs for x.
- Physical meaning or formula rules decide it.
- **Natural domain**: comes directly from the formula (exclude undefined cases).
- **Restricted domain**: happens when we simplify functions but must remember the original restrictions.
- Always check what inputs are really allowed.

Range of a Function

1. What is the Range?

- When you put all possible x-values (from the domain) into a function, you get a bunch of y-values.
- The **range** is the set of all those y-values.
- In other words:
  Domain = what you can put in
  Range = what comes out

Think of a juice machine:

- Fruits you put in = domain
- Juices you get out = range

---

2. Techniques for Finding the Range

(a) By Inspection (looking directly at the function)

Example 1:
$f(x) = x^2$
So $y = x^2$.

- Squares are never negative.
- As x varies over all real numbers, $y \geq 0$.

Range = $[0, \infty)$

Example 2:
$g(x) = \sqrt{(x - 1)} + 2$
So $y = \sqrt{(x - 1)} + 2$

- First, domain: $x \geq 1$ (inside the root must be ≥ 0).
- Then, $\sqrt{(x - 1)} \geq 0$.
- So $y \geq 2$.

Range = $[2, \infty)$

(b) By Algebra (solving for x in terms of y)

Example 3:
$y = (x + 1) / (x - 1)$

Domain: all real numbers except $x = 1$.
But what about the range? Let's solve for x:

$y = (x + 1) / (x - 1)$
Multiply both sides: $y(x - 1) = x + 1$
$yx - y = x + 1$
$yx - x = y + 1$
$x(y - 1) = y + 1$
$x = (y + 1) / (y - 1)$

This shows:

- The formula works for all y, except $y = 1$ (since denominator would be 0).

Range = $(-\infty, 1) \cup (1, \infty)$

---

3. Piecewise Functions

Sometimes functions are defined in parts.

Example (Taxi Fare):
A cab ride costs 1.75 $ for the first mile.
After 1 mile, it costs 0.50 $ per extra mile.

f(x) = {
1.75 , if $0 \leq x \leq 1$

1.75 + 0.50(x - 1), if x > 1
}

Here:

- For rides ≤ 1 mile, cost is fixed at 1.75.
- For rides > 1 mile, cost increases linearly.

So the **range** starts from 1.75 and goes up without limit: [1.75, ∞)

---

4. Reversing Roles of x and y

Usually, x = independent, y = dependent.
But sometimes, it's easier to flip the roles.

Example:
3x + 2y = 6

We can write this as:
y = -3/2 x + 3
or
x = -2/3 y + 2

Both are valid. It depends on which variable we want to treat as independent.

✅ Summary (Feynman Style)

- **Range** = all possible outputs (y-values).
- Find it by:
    - Looking directly (inspection)
    - Solving algebraically for x in terms of y
- Piecewise functions may give "step-like" ranges.
- Sometimes it's easier to flip x and y depending on the problem.