

Model structures from transfer systems (Part 1)

Reference: "Model structures on finite total orders"

by Balchin, Ormsby, Osorno, Roitzheim.

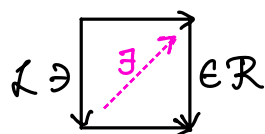
arXiv: 2109.07803

Last time:

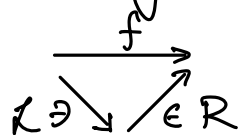
A weak factorization system (wfs) on a category \mathcal{C} is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms in \mathcal{C} s.th.

① Both \mathcal{L} and \mathcal{R} are closed under retracts

② $\mathcal{L} \leq^{\square} \mathcal{R}$



③ Every \xrightarrow{f} in \mathcal{C} can be factored as



For a finite lattice P , we have a 1:1 correspondence

$\{\text{transfer systems on } P\} \longleftrightarrow \{\text{wfs on } P\}$

$\mathcal{R} \rightsquigarrow (\square \mathcal{R}, \mathcal{R}).$

Defⁿ: A model category is a bicomplete category M equipped w/ 3 wide subcategories:

- W weak equivalences $(\xrightarrow{\sim})$
- C cofibrations $(\xrightarrow{\quad})$
- F fibrations (\twoheadrightarrow)

s.th.

we omit this
to get premodel
categories*

① W satisfies 2-out-of-3

② $(C \cap W, F)$ and $(C, F \cap W)$ are both
wfs on M .

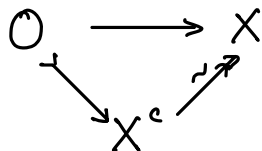
$C \cap W$ acyclic cofibrations $(\xrightarrow{\sim})$

$F \cap W$ acyclic fibrations (\twoheadrightarrow)

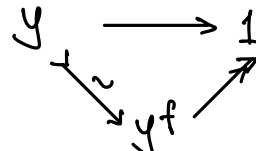
M_{cof} cofibrant objects $(0 \xrightarrow{!} X)$

M^{fib} fibrant objects $(Y \xrightarrow{!} 1)$

$M_{\text{cof}}^{\text{fib}}$ bifibrant objects



cofibrant replacement of X



fibrant replacement of Y .

$$\begin{array}{ccc}
 X \sqcup X & \xrightarrow{\quad} & X \\
 & \searrow & \nearrow \sim \\
 & CX &
 \end{array}$$

cylinder object on X

$$\begin{array}{ccc}
 Y & \xrightarrow{\quad} & Y \times Y \\
 & \searrow & \nearrow \\
 & PY &
 \end{array}$$

path object on Y .

Defⁿ: Given $f, g : X \longrightarrow Y$ in M ,

$$\begin{array}{ccc}
 X \sqcup X & \xrightarrow{[f, g]} & Y \\
 \downarrow & \nearrow H & \\
 CX & &
 \end{array}$$

left homotopy

$$f \sim_l g$$

$$\begin{array}{ccc}
 & & PY \\
 & \nearrow H & \downarrow \\
 X & \xrightarrow{[f, g]} & Y \times Y
 \end{array}$$

right homotopy

$$f \sim_r g$$

If $X \in M_{\text{cof}}$ and $Y \in M^{\text{fib}}$, \sim_l and \sim_r coincide and define an equivalence relation on $M(X, Y)$ that is compatible w/ composition.

Defⁿ: The homotopy category $\text{Ho } M$ of a model category M is given by:

$$\text{objects} = M_{\text{cof}}^{\text{fib}}$$

$$\text{morphisms} = M(X, Y) / \sim$$

Theorem: $Ho M \cong M[W^{-1}]$

where $M[W^{-1}]$ is obtained by formally inverting the weak equivalences in M .

Examples:

- ① Top $w|$ $W =$ weak htpy equivalences
 $\mathcal{C} =$ retracts of CW-inclusions
 $\mathcal{F} =$ Serre fibrations

$\forall n > 0,$

$$\begin{array}{ccc}
 D^n \times \{0\} & \longrightarrow & X \\
 \downarrow & \nearrow \mathcal{F} & \downarrow \\
 D^n \times [0, 1] & \longrightarrow & Y
 \end{array}
 \quad (\text{Serre fibrations}).$$

- ② Gpd $w|$ $W =$ equivalences of categories
 $\mathcal{C} =$ functors injective on objects
 $\mathcal{F} =$ isofibrations

$$\begin{array}{ccc}
 [0] & \longrightarrow & \mathcal{C} \\
 \downarrow & \nearrow \mathcal{F} & \downarrow F \\
 [1] & \longrightarrow & \mathcal{D}
 \end{array}
 \quad (\text{isofibrations}).$$

③ Set admits exactly 9 model structures.

Cofibrations	Fibrations	Weak equivalences	Homotopy category
bij	any	any	(-2)-types
surj	inj	any	(-2)-types
$\text{inj}_{\neq \emptyset} \cup \{\text{id}_{\emptyset}\}$	$\text{surj} \cup \text{inj}_{\emptyset}$	any	(-2)-types
$\text{any}_{\neq \emptyset} \cup \{\text{id}_{\emptyset}\}$	$\text{bij} \cup \text{inj}_{\emptyset}$	any	(-2)-types
inj	surj	any	(-2)-types
any	bij	any	(-2)-types
inj	$\text{surj} \cup \text{inj}_{\emptyset}$	$\text{any}_{\neq \emptyset} \cup \{\text{id}_{\emptyset}\}$	(-1)-types
any	$\text{bij} \cup \text{inj}_{\emptyset}$	$\text{any}_{\neq \emptyset} \cup \{\text{id}_{\emptyset}\}$	(-1)-types
any	any	bij	0-types

} contractible

} trivial

Source: Omar Antolin Camarena's blogpost.

Defⁿ: A model category $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$ is

- trivial if $\mathcal{W} = \text{core}(\mathcal{M})$ (isos)
- contractible if $\mathcal{W} = \mathcal{M}$ (all)

Note: Given a bicomplete category \mathcal{M} , we have a 1-1 correspondence:

$\{\text{wfs on } \mathcal{M}\} \longleftrightarrow \{\text{contractible model structures on } \mathcal{M}\}$

cofibs = \mathcal{L} , fibs = \mathcal{R}

Thus, for a finite lattice P , we have:

$$\{\text{transfer systems on } P\} \leftrightarrow \{\text{wfs on } P\} \leftrightarrow \{\text{contractible model structs. on } P\}$$
$$\mathcal{R} \hookrightarrow (\mathcal{R}, \mathcal{R}).$$

So, for instance, the # of contractible model structures on the lattice $[n] = \text{Sub}(C_p^n)$ is the n^{th} Catalan number.

Q: Can we enumerate AHL model structures on $[n]$?

Lemma [Droz-Zakharovich]:

$(P, \mathcal{W}, \mathcal{F}, \mathcal{C})$: model structure on a finite lattice P .

① If $x \leq y$ is a weak equivalence, so is $x \leq z$ and $z \leq y$ for any $x \leq z \leq y$.

② Given any $x \in P$, there exists a unique $\hat{x} \in P_{\text{cof}}^{\text{fib}}$ s.th. x & \hat{x} are weakly equivalent.

③ $H_0 P \simeq P_{\text{cof}}^{\text{fib}}$.

Thus, given any class \mathcal{W} of weak equivalences which is part of a model structure on $[n]$, we get a partition of $[n]$ into intervals

$$0 \xrightarrow{\sim} \dots \xrightarrow{\sim} a_1 \rightarrow a_{i_1+1} \xrightarrow{\sim} \dots \xrightarrow{\sim} a_{i_2} \rightarrow \dots \rightarrow a_{i_k+1} \xrightarrow{\sim} \dots \xrightarrow{\sim} n$$

$\underbrace{\hspace{10em}}_{[i_0]} \qquad \underbrace{\hspace{10em}}_{[i_1]} \qquad \underbrace{\hspace{10em}}_{[i_k]}$

If there are k such intervals, $Ho[n] \cong [k]$.

Propⁿ: A model structure on $[n]$ restricts to a contractible model structure on each interval in the partition.

Theorem: Any selection of contractible model structures on $[i_0], \dots, [i_k]$ uniquely determines a model structure on $[i_0 + \dots + i_k]$.

More on this next week!!

Q: Does the set $MS([n])$ of all model structures on $[n]$ carry any additional structure?

Defⁿ: ① An adjunction $L: \mathcal{M} \rightleftarrows \mathcal{N}: R$ b/w two model categories is a Quillen adjunction if

- L preserves cofibrations
(or equivalently, R preserves acyclic fibrations)
- L preserves acyclic cofibrations
(or equivalently, R preserves fibrations).

② A Quillen adjunction $L: \mathcal{M} \rightleftarrows \mathcal{N}: R$ is a Quillen equivalence if for every $X \in \mathcal{M}_{\text{cof}}$ and $Y \in \mathcal{N}^{\text{fib}}$, we have

$$LX \xrightarrow{f^{\#}} Y \hookrightarrow X \xrightarrow{f^b} RY$$

$f^{\#}$ is a weak equivalence in \mathcal{M}
iff f^b is a weak equivalence in \mathcal{N} .

Consider two model structures $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ & $(\mathcal{W}', \mathcal{C}', \mathcal{F}')$ on the same underlying category \mathcal{M} .

- Defⁿ: ① If $\mathcal{W} \subseteq \mathcal{W}'$ and $\mathcal{C} = \mathcal{C}'$, then $(\mathcal{W}', \mathcal{C}', \mathcal{F}')$ is a left Bousfield localization of $(\mathcal{W}, \mathcal{C}, \mathcal{F})$.
- ② If $\mathcal{W} \subseteq \mathcal{W}'$ and $\mathcal{F} = \mathcal{F}'$, then $(\mathcal{W}', \mathcal{C}', \mathcal{F}')$ is a right Bousfield localization of $(\mathcal{W}, \mathcal{C}, \mathcal{F})$.

$(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$: model category

S : class of morphisms in \mathcal{M} .

Suppose you want to construct a left Bousfield localization of $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$ s.th. every morphism in S becomes a weak equivalence.

- Defⁿ: ① An object $Z \in \mathcal{M}^{\text{fib}}$ is S -local if for every $s: A \rightarrow B$ in S , the induced map
- $$s^*: \text{Map}(B, Z) \rightarrow \text{Map}(A, Z)$$
- is a weak htpy equivalence in sSet .

② A morphism $f: X \rightarrow Y$ in \mathcal{M} is an S -local weak equivalence if for every S -local fibrant object Z , the induced map

$$f^*: \text{Map}(Y, Z) \rightarrow \text{Map}(X, Z)$$

is a weak htpy equivalence in \mathbf{sSet} .

$\mathcal{W}_S :=$ class of S -local weak equivalences

Remark: $S \subseteq \mathcal{W}_S$ and $\mathcal{W} \subseteq \mathcal{W}_S$.

Propⁿ: If $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$ is a left proper combinatorial model structure, then for any class S of morphisms in \mathcal{M} , there exists a model structure $L_S \mathcal{M} = (\mathcal{W}', \mathcal{C}', \mathcal{F}')$ on \mathcal{M} w/

- $\mathcal{W}' = \mathcal{W}_S$
- $\mathcal{C}' = \mathcal{C}$.

Def: A model category is left proper if the class of weak equivalences is stable under pushout along cofibrations.

Lemma: Every model structure on \mathcal{U} is left proper, right proper, and combinatorial.