Model structures from transfer systems (Part 1)

Reference: "Model structures on finite total orders"
by Balchin, Ormsby, Osorno, Roitzheim.
arXiv: 2109.07803

Last time:

A weak factorization system (wfs) on a category $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms in $\mathcal{C}$ s.th.

① Both $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts

② $\mathcal{L} \subseteq {}^{\boxslash}\mathcal{R}$

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\mathcal{L} \ni"'] & \cdot \arrow[d, "\in \mathcal{R}"] \\ \cdot \arrow[r] \arrow[ur, dashed, "\exists"] & \cdot \end{tikzcd}$$

③ Every $\xrightarrow{f}$ in $\mathcal{C}$ can be factored as

$$\begin{tikzcd} \cdot \arrow[rr, "f"] \arrow[dr, "\mathcal{L} \ni"'] & & \cdot \\ & \cdot \arrow[ur, "\in \mathcal{R}"'] & \end{tikzcd}$$

For a finite lattice P, we have a 1:1 correspondence

$$\{\text{transfer systems on } P\} \longleftrightarrow \{\text{wfs on } P\}$$
$$\mathcal{R} \quad \rightsquigarrow \quad ({}^{\boxslash}\mathcal{R}, \mathcal{R}).$$

Def^n: A **model category** is a bicomplete category $\mathcal{M}$ equipped w/ 3 wide subcategories:

- $\mathcal{W}$ weak equivalences ($\xrightarrow{\sim}$)
- $\mathcal{C}$ cofibrations ($\rightarrowtail$)
- $\mathcal{F}$ fibrations ($\twoheadrightarrow$)

s.th.

we omit this to get premodel categories *

① $\mathcal{W}$ satisfies 2-out-of-3

② $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are both wfs on $\mathcal{M}$.

$\mathcal{C} \cap \mathcal{W}$	acyclic cofibrations	($\overset{\sim}{\rightarrowtail}$)
$\mathcal{F} \cap \mathcal{W}$	acyclic fibrations	($\twoheadrightarrow$)
$\mathcal{M}_{cof}$	cofibrant objects	($0 \overset{!}{\rightarrowtail} X$)
$\mathcal{M}^{fib}$	fibrant objects	($Y \overset{!}{\twoheadrightarrow} 1$)
$\mathcal{M}^{fib}_{cof}$	bifibrant objects	

$$\begin{tikzcd} 0 \arrow[rr] \arrow[dr, tail] & & X \\ & X^c \arrow[ur, "\sim"] & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[rr] \arrow[dr, tail, "\sim"'] & & 1 \\ & Y^f \arrow[ur] & \end{tikzcd}$$

cofibrant replacement of X **fibrant replacement of Y.**

$$\begin{tikzcd} X \sqcup X \arrow[rr] \arrow[dr, tail] & & X \\ & CX \arrow[ur, two heads, "\sim"'] & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[rr] \arrow[dr, tail, "\sim"] & & Y \times Y \\ & PY \arrow[ur, two heads] & \end{tikzcd}$$

cylinder object on X   **path object on Y.**

<u>Def^n</u>: Given $f, g : X \longrightarrow Y$ in $\mathcal{M}$,

$$\begin{tikzcd} X \sqcup X \arrow[r, "{[f,g]}"] \arrow[d, tail] & Y \\ CX \arrow[ur, "H"'] & \end{tikzcd} \qquad \begin{tikzcd} & PY \arrow[d, two heads] \\ X \arrow[ur, "H"] \arrow[r, "{[f,g]}"'] & Y \times Y \end{tikzcd}$$

left homotopy   **right homotopy**

$f \sim_\ell g$ $f \sim_r g$

If $X \in \mathcal{M}_{cof}$ and $Y \in \mathcal{M}^{fib}$, $\sim_\ell$ and $\sim_r$ coincide and define an equivalence relation on $\mathcal{M}(X, Y)$ that is compatible w/ composition.

<u>Def^n</u>: The <u>**homotopy category**</u> $\mathrm{Ho}\,\mathcal{M}$ of a model category $\mathcal{M}$ is given by:

$$\text{objects} = \mathcal{M}^{fib}_{cof}$$

$$\text{morphisms} = \mathcal{M}(X, Y)/\sim$$

<u>Theorem</u>: $\mathrm{Ho}\,\mathcal{M} \simeq \mathcal{M}[W^{-1}]$

where $\mathcal{M}[W^{-1}]$ is obtained by formally inverting the weak equivalences in $\mathcal{M}$.

<u>Examples</u>:

① Top w/ W = weak htpy equivalences

$\mathcal{C}$ = retracts of CW-inclusions

$\mathcal{F}$ = Serre fibrations

$$\forall\, n>0, \qquad \begin{tikzcd} D^n \times \{0\} \arrow[r] \arrow[d, hook] & X \arrow[d] \\ D^n \times [0,1] \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd} \qquad \text{(Serre fibrations).}$$

② Gpd w/ W = equivalences of categories

$\mathcal{C}$ = functors injective on objects

$\mathcal{F}$ = isofibrations

$$\begin{tikzcd} {[0]} \arrow[r] \arrow[d, hook] & \mathcal{C} \arrow[d, "F"] \\ {[1]} \arrow[r] \arrow[ur, dashed, "\exists"] & \mathcal{D} \end{tikzcd} \qquad \text{(isofibrations).}$$

③ Set admits exactly 9 model structures.

Cofibrations	Fibrations	Weak equivalences	Homotopy category
bij	any	any	(-2)-types
surj	inj	any	(-2)-types
$\mathrm{inj}_{\neq\emptyset} \cup \{\mathrm{id}_\emptyset\}$	$\mathrm{surj} \cup \mathrm{inj}_\emptyset$	any	(-2)-types
$\mathrm{any}_{\neq\emptyset} \cup \{\mathrm{id}_\emptyset\}$	$\mathrm{bij} \cup \mathrm{inj}_\emptyset$	any	(-2)-types
inj	surj	any	(-2)-types
any	bij	any	(-2)-types
inj	$\mathrm{surj} \cup \mathrm{inj}_\emptyset$	$\mathrm{any}_{\neq\emptyset} \cup \{\mathrm{id}_\emptyset\}$	(-1)-types
any	$\mathrm{bij} \cup \mathrm{inj}_\emptyset$	$\mathrm{any}_{\neq\emptyset} \cup \{\mathrm{id}_\emptyset\}$	(-1)-types
any	any	bij	0-types } trivial

contractible

Source: Omar Antolin Camarena's blogpost.

Def^n: A model category $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$ is

- **trivial** if $\mathcal{W} = \mathrm{core}(\mathcal{M})$ (isos)
- **contractible** if $\mathcal{W} = \mathcal{M}$ (all)

Note: Given a bicomplete category $\mathcal{M}$, we have a 1-1 correspondence:

$\{\text{wfs on } \mathcal{M}\} \longleftrightarrow \{\text{contractible model structures on } \mathcal{M}\}$

cofibs $= \mathcal{L}$, fibs $= \mathcal{R}$

Thus, for a finite lattice P, we have:

$$\{\text{transfer systems on } P\} \longleftrightarrow \{\text{wfs on } P\} \longleftrightarrow \left\{\begin{matrix}\text{contractible}\\ \text{model structs. on } P\end{matrix}\right\}$$

$$R \rightsquigarrow ({}^{\boxslash}R, R).$$

So, for instance, the # of contractible model structures on the lattice $[n] = \mathrm{Sub}(C_{p^n})$ is the n^{th} Catalan number.

<u>Q</u>: Can we enumerate <u>ALL</u> model structures on $[n]$?

<u>Lemma</u> [Droz–Zakharevich]:

$(P, \mathcal{W}, \mathcal{F}, \mathcal{C})$: model structure on a finite lattice P.

1. If $x \le y$ is a weak equivalence, so is $x \le z$ and $z \le y$ for any $x \le z \le y$.
2. Given any $x \in P$, there exists a unique $\hat{x} \in P^{fib}_{cof}$ s.th. x & $\hat{x}$ are weakly equivalent.
3. $\mathrm{Ho}\, P \simeq P^{fib}_{cof}$.

Thus, given any class $\mathcal{W}$ of weak equivalences which is part of a model structure on $[n]$, we get a partition of $[n]$ into intervals

$$\underbrace{0 \xrightarrow{\sim} \cdots \xrightarrow{\sim} a_1}_{[i_0]} \longrightarrow \underbrace{a_1+1 \xrightarrow{\sim} \cdots \xrightarrow{\sim} a_2}_{[i_1]} \longrightarrow \cdots \longrightarrow \underbrace{a_k+1 \xrightarrow{\sim} \cdots \xrightarrow{\sim} n}_{[i_k]}$$

If there are k such intervals, $\mathrm{Ho}\,[n] \simeq [k]$.

<u>Prop</u>^{n}: A model structure on $[n]$ restricts to a contractible model structure on each interval in the partition.

<u>Theorem</u>: Any selection of contractible model structures on $[i_0], \ldots, [i_k]$ uniquely determines a model structure on $[i_0 + \cdots + i_k]$.

More on this next week!!

<u>Q</u>: Does the set $MS([n])$ of all model structures on $[n]$ carry any additional structure?

Defn: (1) An adjunction $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ b/w two model categories is a <u>Quillen adjunction</u> if

- L preserves cofibrations (or equivalently, R preserves acyclic fibrations)
- L preserves acyclic cofibrations (or equivalently, R preserves fibrations).

(2) A Quillen adjunction $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ is a <u>Quillen equivalence</u> if for every $X \in \mathcal{M}_{cof}$ and $Y \in \mathcal{N}^{fib}$, we have

$$LX \xrightarrow{f^\#} Y \quad \rightsquigarrow \quad X \xrightarrow{f^b} RY$$

$f^\#$ is a weak equivalence in $\mathcal{M}$ iff f^b is a weak equivalence in $\mathcal{N}$.

Consider two model structures $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ & $(\mathcal{W}', \mathcal{C}', \mathcal{F}')$ on the same underlying category $\mathcal{M}$.

Def^n: ① If $\mathcal{W} \subseteq \mathcal{W}'$ and $\mathcal{C} = \mathcal{C}'$, then $(\mathcal{W}', \mathcal{C}', \mathcal{F}')$ is a left Bousfield localization of $(\mathcal{W}, \mathcal{C}, \mathcal{F})$.

② If $\mathcal{W} \subseteq \mathcal{W}'$ and $\mathcal{F} = \mathcal{F}'$, then $(\mathcal{W}', \mathcal{C}', \mathcal{F}')$ is a right Bousfield localization of $(\mathcal{W}, \mathcal{C}, \mathcal{F})$.

$(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$: model category

S : class of morphisms in $\mathcal{M}$.

Suppose you want to construct a left Bousfield localization of $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$ s.th. every morphism in S becomes a weak equivalence.

Def^n: ① An object $Z \in \mathcal{M}^{fib}$ is S-local if for every $s : A \to B$ in S, the induced map

$$s^* : Map(B, Z) \longrightarrow Map(A, Z)$$

is a weak htpy equivalence in sSet.

② A morphism $f: X \longrightarrow Y$ in $\mathcal{M}$ is an <u>S-local weak equivalence</u> if for every S-local fibrant object Z, the induced map

$$f^*: \mathrm{Map}(Y, Z) \longrightarrow \mathrm{Map}(X, Z)$$

is a weak htpy equivalence in sSet.

$\mathcal{W}_S :=$ class of S-local weak equivalences

<u>Remark</u>: $S \subseteq \mathcal{W}_S$ and $\mathcal{W} \subseteq \mathcal{W}_S$.

<u>Prop^n</u>: If $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$ is a left proper combinatorial model structure, then for any class S of morphisms in $\mathcal{M}$, there exists a model structure $L_S\mathcal{M} = (\mathcal{W}', \mathcal{C}', \mathcal{F}')$ on $\mathcal{M}$ w/

- $\mathcal{W}' = \mathcal{W}_S$
- $\mathcal{C}' = \mathcal{C}$.

<u>Def</u>

<u>Def^n</u>: A model category is <u>left proper</u> if the class of weak equivalences is stable under pushout along cofibrations.

<u>Lemma</u>: Every model structure on $[n]$ is left proper, right proper, and combinatorial.