

Model structures from transfer systems (Part 1)

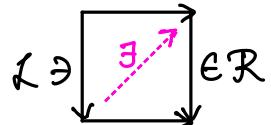
Reference: "Model structures on finite total orders"
by Balchin, Ormsby, Osorno, Roitzheim.
arXiv: 2109.07803

Last time:

A weak factorization system (wfs) on a category \mathcal{C} is a pair (L, R) of classes of morphisms in \mathcal{C} s.t.

① Both L and R are closed under retracts

② $L \subseteq {}^\square R$



③ Every \xrightarrow{f} in \mathcal{C} can be factored as

$$\xrightarrow{f} \xrightarrow[L \ni]{eR}$$

For a finite lattice P , we have a 1:1 correspondence
 $\{\text{transfer systems on } P\} \leftrightarrow \{\text{wfs on } P\}$
 $R \rightsquigarrow ({}^\square R, R)$.

Def": A model category is a bicomplete category

\mathcal{M} equipped w/ 3 wide subcategories:

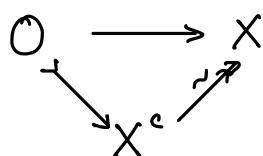
- \mathcal{W} weak equivalences ($\xrightarrow{\sim}$)
- \mathcal{C} cofibrations (\rightarrowtail)
- \mathcal{F} fibrations (\rightarrowtail)

s.t.h.

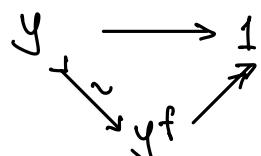
we omit this
to get premodel
categories*

- ① \mathcal{W} satisfies 2-out-of-3
- ② $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are both wfs on \mathcal{M} .

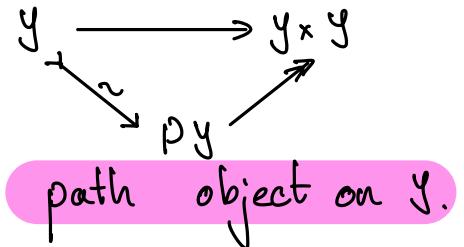
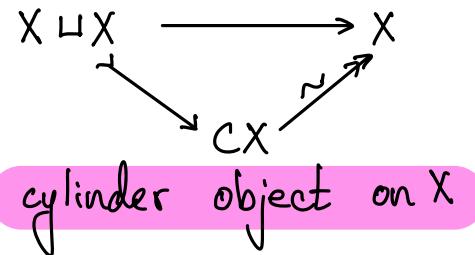
$\mathcal{C} \cap \mathcal{W}$	acyclic cofibrations	(\rightarrowtail^{\sim})
$\mathcal{F} \cap \mathcal{W}$	acyclic fibrations	(\rightarrowtail)
\mathcal{M}_{cof}	cofibrant objects	$(0 \rightarrowtail ! \rightarrow X)$
\mathcal{M}^{fib}	fibrant objects	$(y \rightarrowtail ! \rightarrow 1)$
$\mathcal{M}_{\text{cof}}^{\text{fib}}$	bifibrant objects	



cofibrant replacement of X



fibrant replacement of y .



Defⁿ: Given $f, g : X \rightarrow Y$ in M ,

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{[f, g]} & Y \\ \downarrow & \nearrow H & \\ & CX & \end{array}$$

left homotopy

$$f \sim_L g$$

$$\begin{array}{ccc} & PY & \\ & \nearrow H & \downarrow \\ X & \xrightarrow{[f, g]} & Y \times Y \end{array}$$

right homotopy

$$f \sim_R g$$

If $X \in M_{\text{cof}}$ and $Y \in M^{\text{fib}}$, \sim_L and \sim_R coincide and define an equivalence relation on $M(X, Y)$ that is compatible w/ composition.

Defⁿ: The homotopy category $\text{Ho } M$ of a model

category M is given by:

$$\text{objects} = M_{\text{cof}}^{\text{fib}}$$

$$\text{morphisms} = M(X, Y)/\sim$$

Theorem: $H_0 M \cong M[W^{-1}]$

where $M[W^{-1}]$ is obtained by formally inverting the weak equivalences in M .

Examples:

① Top w/ W = weak htpy equivalences

C = retracts of CW-inclusions

F = Serre fibrations

$$\begin{array}{ccc} D^n \times \{0\} & \longrightarrow & X \\ \downarrow & \nearrow \exists & \downarrow \\ D^n \times [0, 1] & \longrightarrow & Y \end{array} \quad (\text{Serre fibrations}).$$

② Gpd w/ W = equivalences of categories

C = functors injective on objects

F = isofibrations

$$\begin{array}{ccc} [0] & \longrightarrow & e \\ \downarrow & \nearrow \exists & \downarrow F \\ [1] & \longrightarrow & \emptyset \end{array} \quad (\text{isofibrations}).$$

③ Set admits exactly 9 model structures.

Cofibrations	Fibrations	Weak equivalences	Homotopy category
bij	any	any	(-2)-types
surj	inj	any	(-2)-types
$\text{inj}_{\neq 0} \cup \{\text{id}_0\}$	$\text{surj} \cup \text{inj}_0$	any	(-2)-types
$\text{any}_{\neq 0} \cup \{\text{id}_0\}$	$\text{bij} \cup \text{inj}_0$	any	(-2)-types
inj	surj	any	(-2)-types
any	bij	any	(-2)-types
inj	$\text{surj} \cup \text{inj}_0$	$\text{any}_{\neq 0} \cup \{\text{id}_0\}$	(-1)-types
any	$\text{bij} \cup \text{inj}_0$	$\text{any}_{\neq 0} \cup \{\text{id}_0\}$	(-1)-types
any	any	bij	0-types } trivial

contractible

Source: Omar Antolin Camarena's blog post.

Defⁿ: A model category (M, W, C, F) is

- trivial if $W = \text{core}(M)$ (isos)
- contractible if $W = M$ (cell)

Note: Given a bicomplete category M , we have a 1-1 correspondence:

$$\{ \text{wfs on } M \} \longleftrightarrow \{ \text{contractible model structures on } M \}$$

$$\text{cofibs} = L, \text{fibs} = R$$

Thus, for a finite lattice P , we have:

$\{\text{transfer systems on } P\} \leftrightarrow \{\text{wfs on } P\} \leftrightarrow \{\begin{matrix} \text{contractible} \\ \text{model structs. on } P \end{matrix}\}$
R and $(^R R, R)$.

So, for instance, the # of contractible model structures on the lattice $[n] = \text{Sub}(C^{[n]})$ is the n^{th} Catalan number.

Q: Can we enumerate AhL model structures on $[n]$?

Lemma [Droz-Zakharevich]:

(P, W, \mathcal{F}, e) : model structure on a finite lattice P .

- ① If $x \leq y$ is a weak equivalence, so is $x \leq z$ and $z \leq y$ for any $x \leq z \leq y$.
- ② Given any $x \in P$, there exists a unique $\hat{x} \in P_{\text{cof}}^{\text{fib}}$ s.t. $x \not\sim \hat{x}$ are weakly equivalent.
- ③ $\text{Ho } P \simeq P_{\text{cof}}^{\text{fib}}$.

Thus, given any class \mathcal{W} of weak equivalences which is part of a model structure on $[n]$, we get a partition of $[n]$ into intervals

$$0 \xrightarrow{\sim} \cdots \xrightarrow{\sim} i_1 \xrightarrow{\sim} i_1 + 1 \xrightarrow{\sim} \cdots \xrightarrow{\sim} i_2 \xrightarrow{\sim} \cdots \xrightarrow{\sim} i_k + 1 \xrightarrow{\sim} \cdots \xrightarrow{\sim} n$$

$[i_0]$ $[i_1]$ $[i_k]$

If there are k such intervals, $H_0 [n] \simeq [k]$.

Prop: A model structure on $[n]$ restricts to a contractible model structure on each interval in the partition.

Theorem: Any selection of contractible model structures on $[i_0], \dots, [i_k]$ uniquely determines a model structure on $[i_0 + \dots + i_k]$.

More on this next week!!

Q: Does the set $MS([n])$ of all model structures on $[n]$ carry any additional structure?

Defⁿ: ① An adjunction $L: M \rightleftarrows N: R$ b/w two model categories is a Quillen adjunction if

- L preserves cofibrations
(or equivalently, R preserves acyclic fibrations)
- L preserves acyclic cofibrations
(or equivalently, R preserves fibrations).

② A Quillen adjunction $L: M \rightleftarrows N: R$ is a Quillen equivalence if for every $X \in M_{\text{cof}}$ and $Y \in N^{\text{fib}}$, we have

$$LX \xrightarrow{f^\#} Y \iff X \xrightarrow{f^b} RY$$

$f^\#$ is a weak equivalence in M
iff f^b is a weak equivalence in N .

Consider two model structures (W, C, F) & (W', C', F') on the same underlying category \mathcal{M} .

Def" : ① If $W \subseteq W'$ and $C = C'$, then (W', C', F') is a left Bousfield localization of (W, C, F) .
 ② If $W \subseteq W'$ and $F = F'$, then (W', C', F') is a right Bousfield localization of (W, C, F) .

(\mathcal{M}, W, C, F) : model category

S : class of morphisms in \mathcal{M} .

Suppose you want to construct a left Bousfield localization of (\mathcal{M}, W, C, F) s.t. every morphism in S becomes a weak equivalence.

Def" : ① An object $Z \in \mathcal{M}^{\text{fib}}$ is S -local if for every $s : A \rightarrow B$ in S , the induced map
 $s^* : \text{Map}(B, Z) \longrightarrow \text{Map}(A, Z)$
 is a weak htpy equivalence in $s\text{Set}$.

② A morphism $f: X \rightarrow Y$ in M is an S -local weak equivalence if for every S -local fibrant object Z , the induced map $f^*: \text{Map}(Y, Z) \rightarrow \text{Map}(X, Z)$ is a weak htpy equivalence in $s\text{Set}$.
 $w_S :=$ class of S -local weak equivalences

Remark: $S = w_S$ and $W = w_S$.

Prop: If (M, W, C, F) is a left proper combinatorial model structure, then for any class S of morphisms in M , there exists a model structure $L_S M = (W', C', F')$ on M w/

- $W' = w_S$
- $C' = C$.

Def: A model category is left proper if the class of weak equivalences is stable under pushout along cofibrations.

Lemma: Every model structure on InJ is left proper, right proper, and combinatorial.