LAST TIME :

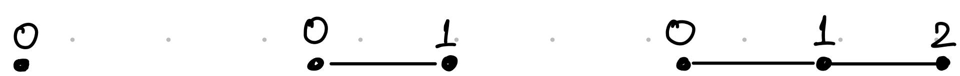
A graph is a set equipped w/ a reflexive, symmetric relation $\sim$

A graph map is a function preserving this relation.

Graph := category of graphs & graph maps.

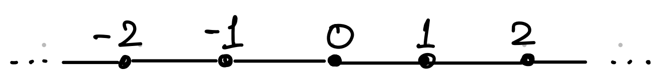
Examples :

① The n-interval I_n



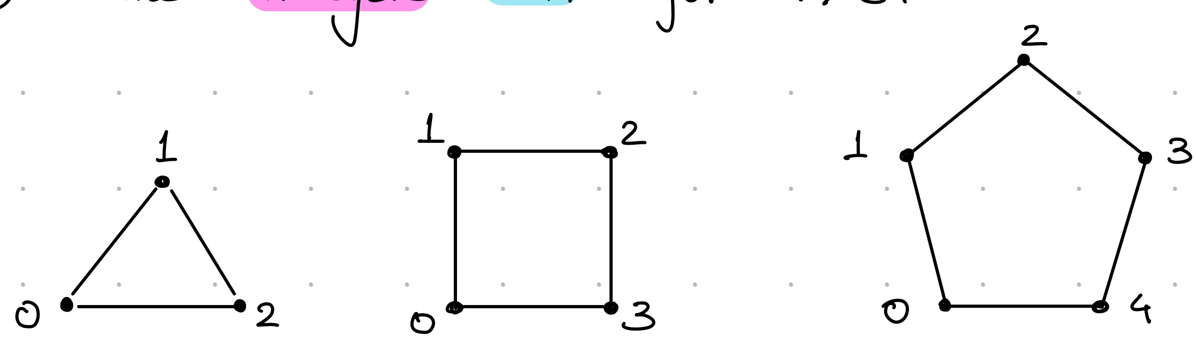
I_0 I_1 I_2

② The infinite interval I_∞



I_∞

③ The n-cycle C_n. for $n \geq 3$.



C_3 C_4 C_5

The box product $X \square Y$ of X and Y :

$$(X \square Y)_V = X_V \times Y_V$$

$$(X \square Y)_E = \left\{ (x,y) \sim (x',y') \;\middle|\; \begin{array}{c} x = x' \;\&\; y \sim y' \\ \text{OR} \\ x \sim x' \;\&\; y = y' \end{array} \right\}$$

<u>Example</u>: $I_1 \square I_1 \cong C_4$

$$\begin{tikzcd} (0,1) \arrow[r, dash] \arrow[d, dash] & (1,1) \arrow[d, dash] \\ (0,0) \arrow[r, dash] & (1,0) \end{tikzcd}$$

An <u>A-homotopy</u> $H : f \Rightarrow g$ b/w $f, g : X \longrightarrow Y$ is a map

$$H : X \square I_n \longrightarrow Y$$

s.th.

$$H(-,0) = f \quad \text{and} \quad H(-,n) = g$$

for some $n \in \mathbb{N}$.

A map $f : X \longrightarrow Y$ is an <u>A-homotopy equivalence</u> if $\exists$ $g : Y \longrightarrow X$ and A-htpies

$$g \circ f \Rightarrow id_X \quad \text{and} \quad f \circ g \Rightarrow id_Y .$$

Examples: $C_3 \xrightarrow{!} I_0$ and $C_4 \xrightarrow{!} I_0$ are A-htpy equivalences.

The n^{th} A-htpy grp of a pted graph (X, x)

$$A_n(X,x) := \left\{ f : I_\infty^{\square n} \longrightarrow X \mid f(\cdots) = x \text{ almost everywhere} \right\} \Big/ \sim_*$$

References:

1. H. Barcelo, X. Kramer, R. Laubenbacher, C. Weaver "Foundations of a connectivity theory for simplicial complexes", 2001.

2. E. Babson, H. Barcelo, M. de Longueville, R. Laubenbacher "Homotopy theory of graphs", 2006.

Computing the fundamental group

⤳ Covering graphs:

$$A_1(C_n, *) \cong \begin{cases} 0 & \text{if } n = 3, 4 \\ \mathbb{Z} & \text{if } n \geq 5. \end{cases}$$

⤳ Seifert – van Kampen theorem.

Q: When is a pushout square

$$\begin{tikzcd} (X_0, x) \arrow[r] \arrow[d] & (X_1, x) \arrow[d] \\ (X_2, x) \arrow[r] & (X, x) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

in Graph_* preserved by the functor

$A_1 : \mathrm{Graph}_* \longrightarrow \mathrm{Grp}$?

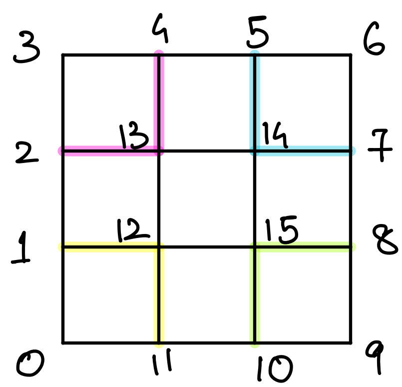
Theorem (Barcelo, Kramer, Laubenbacher, Weaver 2001):

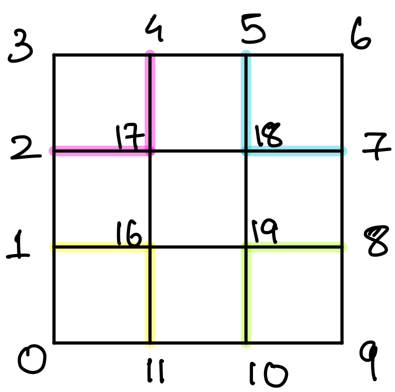
1. If X_0, X_1 and X_2 are connected, and
2. If every $h : I_1 \square I_1 \longrightarrow X$ factors through X_1 or X_2,

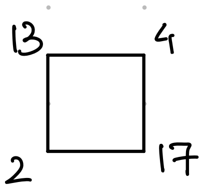
Then, $A_1 : \mathrm{Graph}_* \longrightarrow \mathrm{Grp}$ preserves the pushout.

Non-example:

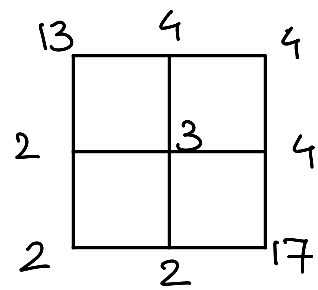
$$\begin{tikzcd} C_{2m+2n} \arrow[r, hook, "\partial"] \arrow[d, hook, "\partial"'] & I_m \square I_n \arrow[d] \\ I_m \square I_n \arrow[r] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad m, n \geq 3.$$







$\rightsquigarrow$



A map $s : I_m \longrightarrow I_n$ is <u>shrinking</u> if it is order-preserving and surjective.

<u>Notation</u>:

① $\partial_{m,n} : I_{2m+2n} \longrightarrow I_m \square I_n$. (boundary).

② $e_i : I_1 \longrightarrow I_n$ (i^{th} edge)

<u>Theorem</u> (Kapulkin – M.): <u>arXiv: 2303.06029</u>

① If X_0, X_1 and X_2 are connected, and

② For every $h : I_1 \square I_1 \longrightarrow X$, there exists a map $H : I_m \square I_n \longrightarrow X$ together w/ a shrinking map $s : I_{2m+2n} \longrightarrow I_4$ s.th.

$$\begin{tikzcd}
I_{2m+2n} \arrow[r, "\partial_{m,n}"] \arrow[d, "s"'] & I_m \square I_n \arrow[r, "H"] & X \arrow[d, equal] \\
I_4 \arrow[r, "\partial_{1,1}"] & I_1 \square I_1 \arrow[r, "h"] & X
\end{tikzcd}$$

and s.th. for each $1 \leq i \leq m$ & $1 \leq j \leq n$

$$I_1 \sqcup I_1 \xrightarrow{e_i \sqcup e_j} I_m \sqcup I_n \xrightarrow{H} X$$

factors through X_1 or X_2,

Then, $A_1 : \mathrm{Graph}_* \longrightarrow \mathrm{Grp}$ preserves the pushout.

Proving the theorem:

① Finite formulation of $A_1(X, x)$:

A <u>path</u> $\gamma : x \rightsquigarrow x'$ of length $n \in \mathbb{N}$ in X is a map $\gamma : I_n \longrightarrow X$ s.th. $\gamma(0) = x$ and $\gamma(n) = x'$.

$P_n X(x, x')$

<u>vertices</u>: paths $x \rightsquigarrow x'$ of length n.

<u>edges</u>: $\gamma \sim \gamma'$ if $\exists$ $H : I_n \sqcup I_1 \longrightarrow X$ s.th. $H(-,0) = \gamma$ and $H(-,1) = \gamma'$.

If

$\gamma \in P_n(x, x')$

$s : I_m \longrightarrow I_n$ shrinking map

then, γ and $\gamma \circ s$ are <u>reparametrizations</u> of each other.

We write $\gamma \sim_s \gamma \circ s$

"s" for shrinking.

$$P_{\mathbb{N}} X(x, x') := \coprod_{n \in \mathbb{N}} P_n X(x, x') \Big/ \sim_s$$

<u>Theorem</u> (Kapulkin – M.):

$$\boxed{A_1(X, x) \cong \pi_0 P_{\mathbb{N}} X(x, x).}$$

② Multi-object generalization of $A_1(X, x)$

Fundamental groupoid $\Pi_1 X$ of a graph X

<u>objects</u>: vertices of X

<u>morphisms</u>: $\Pi_1 X(x, x') := \pi_0 P_{\mathbb{N}} X(x, x')$.

③ Characterizing functors $F : \Pi_1 X \longrightarrow \mathcal{G}$

$$\begin{tikzcd}
\Pi_1 X_0 \arrow[r] \arrow[d] & \Pi_1 X_1 \arrow[d] \arrow[ddr, bend left, "F_1"] & \\
\Pi_1 X_2 \arrow[r] \arrow[drr, bend right, "F_2"'] & \Pi_1 X \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, "\exists !"] & \\
& & \mathcal{G}
\end{tikzcd}$$

Suppose you have a functor $F : \Pi_1 X \longrightarrow \mathcal{G}$.

① $F : X_v \longrightarrow \operatorname{ob} \mathcal{G}$

②
$$
\begin{array}{c}
F_{x,x'} : \Pi_1 X(x,x') \longrightarrow \mathcal{G}(Fx, Fx') \\
\hline
F_{x,x'} : \pi_0 P_{\mathbb{N}} X(x,x') \longrightarrow \mathcal{G}(Fx, Fx') \\
\hline
F_{x,x'} : P_{\mathbb{N}} X(x,x') \longrightarrow \mathcal{G}(Fx, Fx')_{\text{discrete}} \\
\hline
\{ F^{(n)}_{x,x'} : P_n X(x,x') \longrightarrow \mathcal{G}(Fx, Fx')_{\text{discrete}} \}_{n \in \mathbb{N}} \\
\hline
F^{(1)}_{x,x'} : \underset{\text{either } I_0 \text{ or } \phi}{\underset{\shortparallel}{P_1 X(x,x')}} \longrightarrow \mathcal{G}(Fx, Fx')_{\text{discrete}}
\end{array}
$$

Given arbitrary data

$$F : X_v \longrightarrow \operatorname{ob} \mathcal{G}$$

$$F^{(1)}_{x,x'} : P_1 X(x,x') \longrightarrow \mathcal{G}(Fx, Fx')_{\text{discrete}}$$

does it define a functor $F : \Pi_1 X \longrightarrow \mathcal{G}$?

Lemma: Given the data:

$$F : X_v \longrightarrow \mathrm{ob}\, \mathcal{G}$$

$$F^{(1)}_{x,x'} : P_1 X(x,x') \longrightarrow \mathcal{G}(Fx, Fx')_{\text{discrete}}$$

s.th.

① $F^{(1)}_{x,x}(c_x) = id_{Fx}$

② $F^{(1)}_{x,x'}(\overline{e}) = F^{(1)}_{x',x}(e)^{-1}$

// **Notation:** Given $\gamma \in P_n X(x,x')$, define

$$F^{(n)}_{x,x'}(\gamma) = F^{(1)}(e_n(\gamma)) \circ \cdots \circ F^{(1)}(e_1(\gamma)) //$$

Then:

① $F^{(m+n)}(\gamma * \sigma) = F^{(n)}(\sigma) \circ F^{(m)}(\gamma)$

② $F^{(n)}(\overline{\gamma}) = (F^{(n)}(\gamma))^{-1}$

③ $F^{(m)}(\gamma \circ s) = F^{(n)}(\gamma)$

④ If for every $h : I_1 \sqcup I_1 \longrightarrow X$ we have $F^{(4)}(h \circ \partial_{1,1}) = id$, then whenever $\gamma \sim \gamma'$ in $P_n X(x,x')$, we have $F^{(n)}_{x,x'}(\gamma) = F^{(n)}_{x,x'}(\gamma')$.

(5) Given $H: I_m \square I_n \longrightarrow X$ s.th. for each $1 \leq i \leq m$ and $1 \leq j \leq n$ we have

$$F^{(4)}\left(H \circ (e_i \square e_j) \circ \partial_{1,1}\right) = id$$

then

$$F^{(2m+2n)}\left(H \circ \partial_{m,n}\right) = id.$$

<u>Proof of Seifert – van Kampen:</u>

For every $h: I_1 \square I_1 \longrightarrow X$, there exists a map $H: I_m \square I_n \longrightarrow X$ together w/ a shrinking map $s: I_{2m+2n} \longrightarrow I_4$ s.th.

$$h \circ \partial_{1,1} \circ s = H \circ \partial_{m,n}$$

and for every $1 \leq i \leq m$ and $1 \leq j \leq n$, $H \circ (e_i \square e_j)$ factors through X_1 or X_2.

$\Rightarrow \quad F^{(4)}\left(H \circ (e_i \square e_j) \circ \partial_{1,1}\right) = id \quad \forall\ 1 \leq i \leq m,\ 1 \leq j \leq n.$

$\Rightarrow \quad F^{(2m+2n)}\left(H \circ \partial_{m,n}\right) = id$

$\Rightarrow \quad F^{(2m+2n)}\left(h \circ \partial_{1,1} \circ s\right) = id$

$$\Longrightarrow \quad F^{(a)}(h \circ \partial_{1,1}) = \mathrm{id}.$$