References:

1. Cubical setting for discrete homotopy theory, revisited.
   Carranza, Kapulkin 2022.
   arXiv: 2202.03516

2. Homotopy n-types of cubical sets and graphs
   Kapulkin, M. 2024.
   arXiv: 2408.05289

Outline:

1) Fibration categories

2) Homotopy types of graphs

3) Homotopy n-types of graphs

4) Homotopy 1-types of graphs

Fibration categories

<u>Def^n</u>: A fib cat is a category $\mathcal{C}$ together w/ :

- a class of weak equivalences ($\xrightarrow{\sim}$)
- a class of fibrations ($\twoheadrightarrow$)

subject to certain axioms.

* Useful for computing htpy limits
* Ho $\mathcal{C}$ admits a nice description.

Suppose $\mathcal{C}$ & $\mathcal{D}$ are fib cats.

<u>Def^n</u>: A functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ is <u>exact</u> if it preserves

- fibrations
- acyclic fibs ($\overset{\sim}{\twoheadrightarrow}$)
- pullbacks along fibs
- terminal object.

* Preserves htpy limits
* Induces a functor Ho $\mathcal{C} \longrightarrow$ Ho$\mathcal{D}$.

Def^n: An exact functor $F: \mathcal{C} \to \mathcal{D}$ is a <u>weak equiv.</u> if the induced functor $\mathrm{Ho}\,\mathcal{C} \to \mathrm{Ho}\,\mathcal{D}$ is an equivalence of cats.

<u>Examples</u>:

① Top w/ $\cdot$ weak htpy equivs ($\xrightarrow{\sim}$)
$\cdot$ Serre fibs ($\twoheadrightarrow$) $\Big\}$ Top_∞

$\forall\, n > 0$.

$$\begin{tikzcd}
\partial[0,1]^n \setminus ([0,1]^{n-1} \times \{1\}) \arrow[r] \arrow[d, hook] & X \arrow[d] \\
{[0,1]^n} \arrow[r] \arrow[ur] & Y
\end{tikzcd}$$

② Top w/ $\cdot$ n-equivalences ($\xrightarrow{\sim}$)
$\cdot$ n-fibrations ($\twoheadrightarrow$) $\Big\}$ Top_n

$0 < k \leq n+1$

$$\begin{tikzcd}
\partial[0,1]^k \setminus ([0,1]^{k-1} \times \{1\}) \arrow[r] \arrow[d, hook] & X \arrow[d] \\
{[0,1]^k} \arrow[r] \arrow[ur] & Y
\end{tikzcd}$$

$k = n+2$

$$\begin{tikzcd} \partial[0,1]^k \setminus ([0,1]^{k-1} \times \{1\}) \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \partial[0,1]^k \arrow[r] \arrow[ur] & Y \end{tikzcd}$$

③ Gpd w/
- equivs of cats. ($\xrightarrow{\sim}$)
- isofibrations ($\twoheadrightarrow$)

$$\begin{tikzcd} {[0]} \arrow[r] \arrow[d, hook] & \mathcal{C} \arrow[d, "F"] \\ {[1]} \arrow[r] \arrow[ur] & \mathcal{D} \end{tikzcd}$$

<u>Theorem</u> (Classical): $\Pi_1 : \mathrm{Top}_1 \longrightarrow \mathrm{Gpd}$ is a weak equivalence of fib cats.

<u>Homotopy types of graphs</u>

<u>Def^n</u>: A graph map $f : X \longrightarrow Y$ is a <u>weak equiv</u> if it induces a bijection $f_* : \pi_0 X \longrightarrow \pi_0 Y$ and an isomorphism $f_* : A_n(X, x) \longrightarrow A_n(Y, fx)$ for all $n > 0$ and $x \in X$.

Examples:

① Every A-htpy equiv. is a weak eq.

② $I_\infty \xrightarrow{!} I_0$ is a weak eq.

③ $s: C_{n+1} \to C_n$ (collapse one edge) is a weak eq.

A-htpy equiv. $\rightsquigarrow$ Naive discrete htpy theory

"Nonexistence of colimits in naive discrete htpy theory"
Carranza, Kapulkin, Kim 2023.

Weak equiv. $\rightsquigarrow$ Discrete htpy theory.

Notation:

① For $n, m > 0$, the graph $I_m^{\square n}$ has

vertices: $(x_1, \dots, x_n)$ where each $x_i \in \{0, \dots, m\}$

edges: $(x_1, \dots, x_i, \dots, x_n) \sim (x_1, \dots, x_i \pm 1, \dots, x_n)$

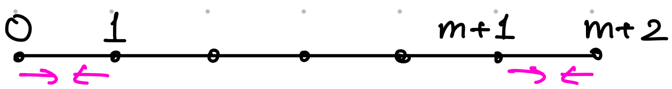
for any $i = 1, \dots, n$.

② $\partial I_m^{\square n} \underset{\text{full}}{\subseteq} I_m^{\square n}$ on vertices

$(x_1, \dots, x_n)$ s.th. $x_j = 0$ or m for at least one $j \in \{1, \dots, n\}$.

③ $\sqcap_m^n \subseteq \partial I_m^{\square n}$ on vertices
$(x_1, \dots, x_n)$ s.th. if $x_n = m$, then
$x_j = 0$ or m for at least one $j \in \{1, \dots, n-1\}$.

④ $c : I_{m+2} \longrightarrow I_m$ (collapse first & last edges)



$c^{\square n} : I_{m+2}^{\square n} \longrightarrow I_m^{\square n}$ restricts to maps

$$\partial I_{m+2}^{\square n} \longrightarrow \partial I_m^{\square n}$$

and $$\sqcap_{m+2}^n \longrightarrow \sqcap_m^n.$$

<u>Def^n</u>: A graph map $f : X \longrightarrow Y$ is a <u>fibration</u> if

$\forall\ n > 0,\ m > 0,$ and diagram

$$\begin{tikzcd} \sqcap^n_m \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ I_m^{\square n} \arrow[r] & Y \end{tikzcd}$$

$\exists \; M \geq m$ s.th.

$$\begin{tikzcd} \sqcap^n_M \arrow[r] \arrow[d, hook] & \sqcap^n_m \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ I_M^{\square n} \arrow[r] \arrow[urr] & I_m^{\square n} \arrow[r] & Y \end{tikzcd}$$

Theorem (Carranza – Kapulkin, '23):

Graph w/ • weak equivs
• fibrations

is a fib. cat, denoted Graph_∞.

Question: Is it weakly equivalent to Top_∞?

Homotopy n-types of graphs

Def^n: A graph map $f : X \to Y$ is an n-equivalence if it induces a bijection $f_* : \pi_0 X \to \pi_0 Y$ and an isomorphism $f_* : A_k(X, x) \to A_k(Y, fx)$

for $0 < k \le n$ and all $x \in X$.

<u>Def^n</u>: A graph map is an <u>n-fibration</u> if

for $0 < k \le n+1$, $m > 0$,

$$\begin{tikzcd} \sqcap^k_m \arrow[r] \arrow[d, hook] & X \arrow[d] \\ I^{\square k}_m \arrow[r] & Y \end{tikzcd} \quad \overset{\exists M \geq m}{\underset{\text{s.th.}}{\rightsquigarrow}} \quad \begin{tikzcd} \sqcap^k_M \arrow[r] \arrow[d, hook] & \sqcap^k_m \arrow[r] \arrow[d, hook] & X \arrow[d] \\ I^{\square k}_M \arrow[r] \arrow[urr] & I^{\square k}_m \arrow[r] & Y \end{tikzcd}$$

for $k = n+2$, $m > 0$

$$\begin{tikzcd} \sqcap^k_m \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \partial I^{\square k}_m \arrow[r] & Y \end{tikzcd} \quad \overset{\exists M \geq m}{\underset{\text{s.th.}}{\rightsquigarrow}} \quad \begin{tikzcd} \sqcap^k_M \arrow[r] \arrow[d, hook] & \sqcap^k_m \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \partial I^{\square k}_M \arrow[r] \arrow[urr] & \partial I^{\square k}_m \arrow[r] & Y \end{tikzcd}$$

<u>Theorem</u>: Graph w/ n-equivalences ($\xrightarrow{\sim}$)
and n-fibrations ($\twoheadrightarrow$)
is a fib. cat. denoted Graph_n.

<u>Revised Question</u>: Is Graph_n weakly equivalent to Top_n?

Homotopy 1-types of graphs

Theorem: The fundamental groupoid functor

$$\Pi_1 : \mathrm{Graph}_1 \longrightarrow \mathrm{Gpd}$$

is a weak equivalence of fib. cats.

Corollary: Graph_1 is weakly equivalent to Top_1.

Proving the theorem:

1. a graph map $f : X \to Y$ is a 1-equivalence iff $\Pi_1 f : \Pi_1 X \to \Pi_1 Y$ is an equiv. of cats.

2. Π_1 maps 1-fibrations to isofibrations

3. Π_1 preserves pullbacks along 1-fibs.

4. Given any graph Y and functor $F : \mathcal{G} \to \Pi_1 Y$ in Gpd, there exists a graph map $f : X \to Y$ and a commuting diagram as follows:

$$\begin{tikzcd} \mathcal{G}' \arrow[r, "\sim"] \arrow[d, "\sim"'] & \pi_1 X \arrow[d, "\pi_1 f"] \\ \mathcal{G} \arrow[r] & \pi_1 Y \end{tikzcd}$$

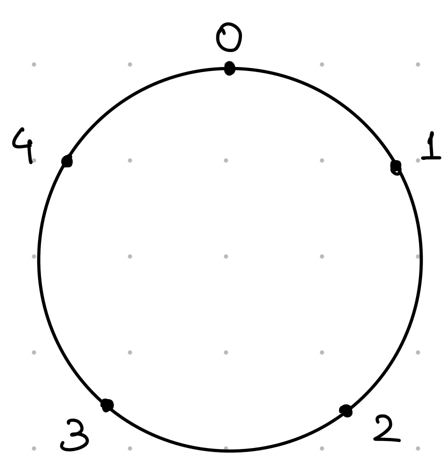
Proving 4 boils down to the following:

given a group $\mathcal{G}$, is there some pted, connected graph (X, x) s.th. $A_1(X, x) \cong \mathcal{G}$?

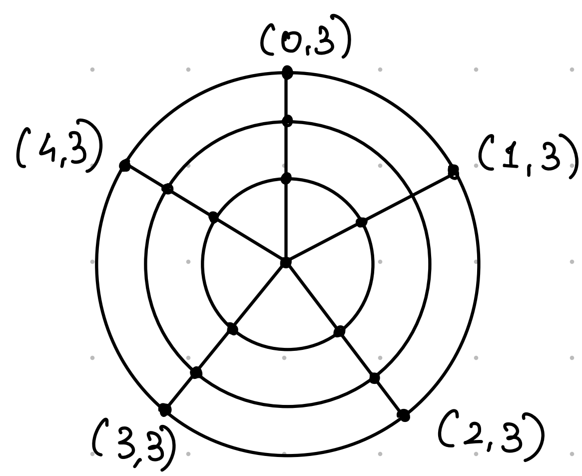
<u>Notation</u>:

$D_n = C_n \square I_3 / \sim \qquad (i,0) \sim (j,0)$ for all $i,j \in C_n$.

$\partial : C_n \hookrightarrow D_n \quad ; \quad i \mapsto (i,3)$



C_5



D_5

Let F_S be the free grp gen. by a set S.

Given any word $r = s_1^{d_1} \cdots s_k^{d_k}$ in F_S, define

$$\deg(r) = |d_1| + \cdots + |d_k|$$

and define $\omega_r : C_{5\deg(r)} \longrightarrow \bigvee_{s \in S} C_5$

first wrap d_1 times around C_5 corresp. to s_1
then wrap d_2 times around C_5 corresp. to s_2,
etc.

Given any group G w/ a presentation $G = \langle S \mid R \rangle$,
define a graph $X_{S,R}$ as follows

$$\begin{tikzcd}
\coprod_{r \in R} C_{5\cdot\deg(r)} \arrow[r, "\omega_r"] \arrow[d, hook, "\partial"'] & \bigvee_{s \in S} C_5 \arrow[d] \\
\coprod_{r \in R} D_{5\cdot\deg(r)} \arrow[r] & X_{S,R}.
\end{tikzcd}$$

Then, by Seifert–van Kampen, $\pi_1(X_{S,R}, *) \cong G$.