The Fundamental Groupoid in Discrete Homotopy Theory

Talk 1: Covering graphs.

I. What is discrete homotopy theory?

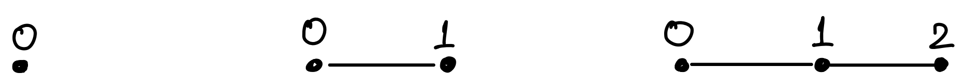
A *graph* is a set equipped w/ a reflexive, symmetric relation $\sim$.

A *graph map* is a function preserving this relation.

Graph := category of graphs & graph maps.

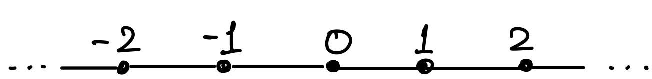
Examples:

① The n-interval I_n, for $n \in \mathbb{N}$.



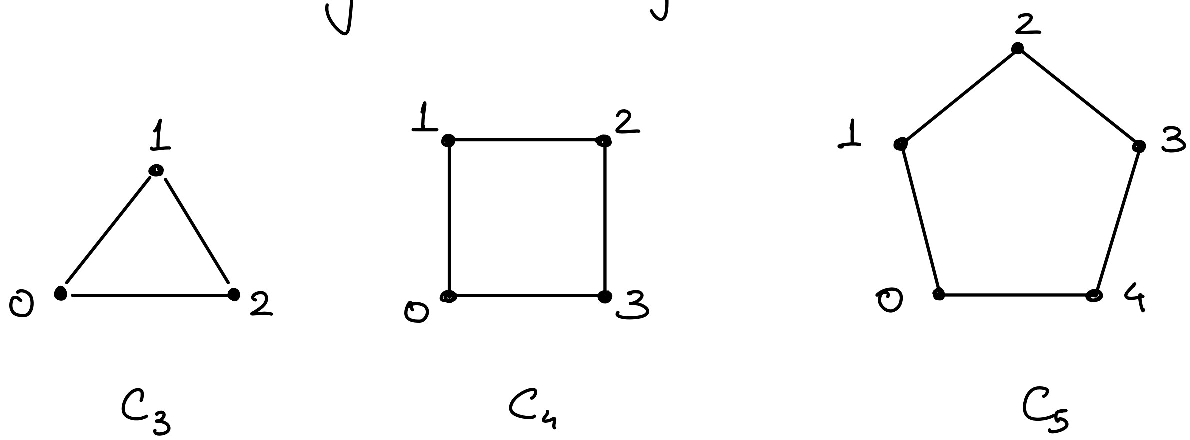
I_0 $\qquad$ I_1 $\qquad$ I_2

② The infinite interval I_∞



I_∞

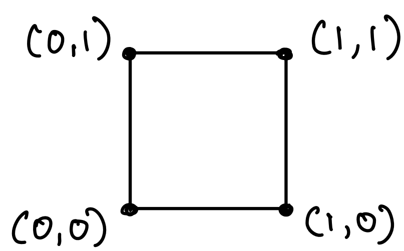
③ The n-cycle C_n. for $n \geqslant 3$.



The <u>box product</u> $X \square Y$ of X and Y is the set $X \times Y$ with the relation:

$$(x,y) \sim (x',y') \iff \begin{cases} x \sim x' \text{ and } y = y' \\ \quad \underline{\text{OR}} \\ x = x' \text{ and } y \sim y'. \end{cases}$$

<u>Example</u>:



$I_1 \square I_1$

An <u>A-homotopy</u> $H: f \Rightarrow g$ b/w $f, g : X \longrightarrow Y$ is a map

$$H : X \square I_n \longrightarrow Y \qquad \text{for some } n \in \mathbb{N}.$$

s.th.

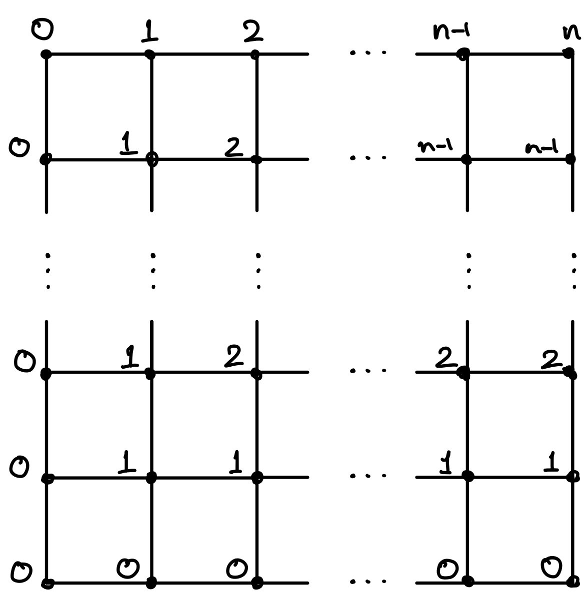
$$H(-,0) = f \qquad \text{and} \qquad H(-,n) = g$$

A map $f : X \longrightarrow Y$ is an <u>A-homotopy equivalence</u> if $\exists$ $g : Y \longrightarrow X$ together with A-htpies

$$g \circ f \Rightarrow id_X \qquad \text{and} \qquad f \circ g \Rightarrow id_Y .$$

<u>Examples</u> :

① $I_n \xrightarrow{!} I_0$ is an A-htpy equiv.



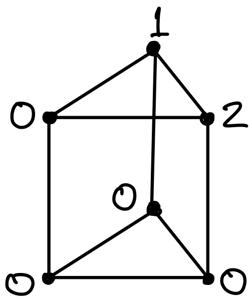
$$H : I_n \square I_n \longrightarrow I_n$$

$$(i,j) \longmapsto \min(i, n-j)$$

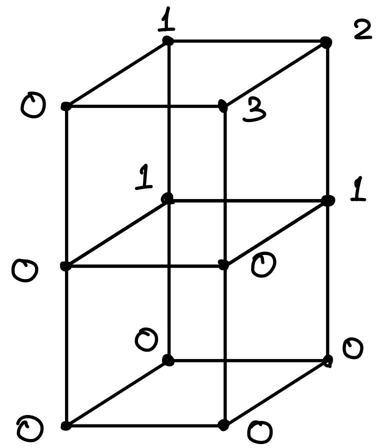
$$H(-,0) = id_{I_n} \; ; \; H(-,n) = c_0 .$$

<u>Remark</u>: $I_\infty \xrightarrow{!} I_0$ is NOT an A-htpy equivalence.

② $C_n \xrightarrow{!} I_0$ is an A-htpy equivalence for $n = 3, 4$.



$C_3 \square I_1 \longrightarrow C_3$



$C_4 \square I_2 \longrightarrow C_4$.

Fix a graph X w/ a base vertex $x_0 \in X$.

The <u>n^{th} A-homotopy group</u> $A_n(X, x_0)$ is given by:

$$A_n(X, x_0) = \left\{ f : I_\infty^{\square n} \longrightarrow X \;\middle|\; \begin{array}{l} f(t_1, \dots, t_n) = x_0 \\ \text{almost everywhere} \end{array} \right\} \Big/ \sim_*$$

where we are quotienting by A-htpies that also take the value x_0 almost everywhere.

The group operation is given by concatenating the finite non-constant regions.

II. Covering Graphs

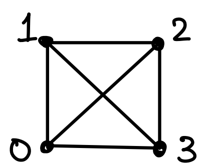
X : graph, $\quad x \in X$: vertex.

The <u>neighbourhood</u> N_x of x in X is a subgraph

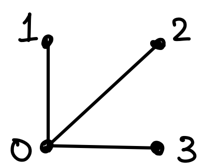
vertices : x and its neighbours

edges : $x \sim x'$ for each $x' \in N_x$, $x' \neq x$.

<u>Example</u>:



K_4



N_0

A map $p : Y \longrightarrow X$ is a <u>local isomorphism</u> if the restriction

$$p|_{N_y} : N_y \longrightarrow N_{p(y)}$$

is an isomorphism for every $y \in Y$.

Examples :

① $p : C_{nk} \longrightarrow C_n \quad ; \quad i \longmapsto i \pmod{n}$

is a local isomorphism for every $n \geq 3$ and $k \geq 1$.

② $p : I_\infty \longrightarrow C_n \quad ; \quad i \longmapsto i \pmod{n}$

is a local isomorphism for every $n \geq 3$.

X : graph , $x, x' \in X$: two vertices.

A path $\gamma : x \rightsquigarrow x'$ in X is a map $\gamma : I_\infty \longrightarrow X$ for which there exist integers $N_-, N_+ \in \mathbb{Z}$ s.th.

$$\gamma(i) = x \qquad \text{if} \quad i \leq N_-$$

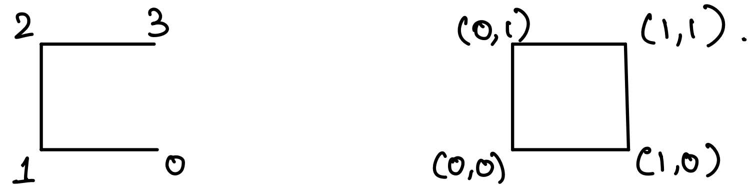
$$\gamma(i) = x' \qquad \text{if} \quad i \geq N_+ .$$

Lemma : Let $p : Y \longrightarrow X$ be a local isomorphism. For any path $\gamma : x \rightsquigarrow x'$ in X and any $y \in p^{-1}(x)$, there is a unique path $\tilde{\gamma}$ in Y starting at y s.th.

$p \circ \tilde{\gamma} = \gamma$. $\square$

Notation :

$$\sqsubset \; : \; I_3 \longrightarrow I_1 \square I_1$$



A map $p : Y \longrightarrow X$ is a <u>covering</u> if, in addition to being a local isomorphism, it has the RLP with respect to $\sqsubset : I_3 \longrightarrow I_1 \square I_1$.

$$\begin{tikzcd}
I_3 \arrow[r] \arrow[d, "\sqsubset"'] & Y \arrow[d, "p"] \\
I_1 \square I_1 \arrow[r] \arrow[ur, dashed] & X
\end{tikzcd}$$

<u>Remark</u>: If the base graph X has no 3- or 4-cycles, then every local isomorphism $p : Y \longrightarrow X$ is also a covering.

<u>Non-examples</u> :

① $p : C_{nk} \longrightarrow C_n \quad ; \quad i \longmapsto i \pmod{n}$

is NOT a covering if $n = 3, 4$ and $k > 1$.

② $p : I_\infty \longrightarrow C_n \quad ; \quad i \longmapsto i \pmod{n}$

is NOT a covering if $n = 3, 4$.

X : graph , $x, x' \in X$: two vertices.

$\gamma, \sigma : x \rightsquigarrow x'$: two paths in X

A <u>path homotopy</u> $H : \gamma \rightsquigarrow \sigma$ is a map

$$H : I_\infty \boxtimes I_n \longrightarrow X$$

s.th.

$$H(-,0) = \gamma \quad \text{and} \quad H(-,n) = \sigma$$

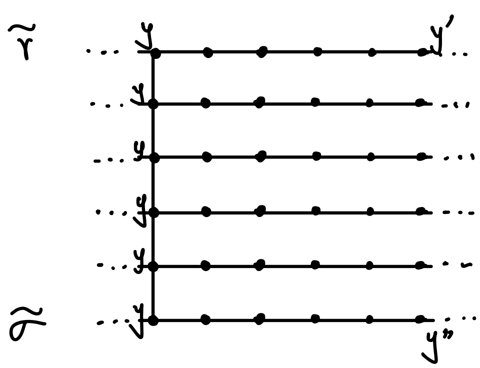
and s.th.

$$H(-,i) \text{ is a path } x \rightsquigarrow x' \text{ in } X.$$

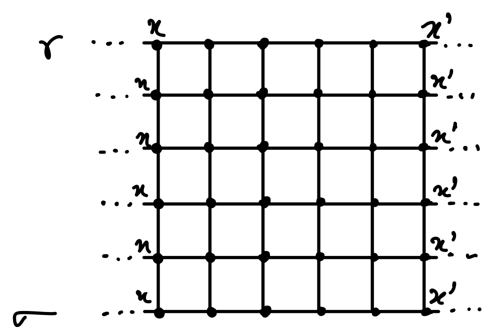
for each $i = 0, \ldots, n$.

Lemma: Let $p: \mathcal{Y} \longrightarrow X$ be a covering.
For two paths $\gamma, \sigma: x \rightsquigarrow x'$ in X,
and a vertex $y \in p^{-1}(x)$,
there exist paths $\tilde{\gamma}: y \rightsquigarrow y'$ and
$\tilde{\sigma}: y \rightsquigarrow y''$ in $\mathcal{Y}$ s.th. $p \circ \tilde{\gamma} = \gamma$ and
$p \circ \tilde{\sigma} = \sigma$.
If γ and σ are path-homotopic in X,
then $y' = y''$ and $\tilde{\gamma}$ and $\tilde{\sigma}$ are
path-homotopic in $\mathcal{Y}$.

Proof: Let $H: I_\infty \square I_n \longrightarrow X$ be a path-htpy $\gamma \Rightarrow \sigma$.



$\tilde{H}: I_\infty \square I_n \to \mathcal{Y}$



$H: I_\infty \square I_n \longrightarrow X$

$\square$

$p : Y \longrightarrow X$ covering map

$\gamma : x \rightsquigarrow x'$ path in X.

We can define a function:

$$\mathrm{unw}_{[\gamma]} : p^{-1}(x) \longrightarrow p^{-1}(x')$$
$$y \longmapsto \text{endpoint of } \tilde{\gamma}$$

where $\tilde{\gamma}$ is the unique lift of γ starting at y.

<u>Remark</u>: By path-homotopy lifting property, $\mathrm{unw}_{[\gamma]}$ only depends on the path-homotopy class $[\gamma]$ of γ.

<u>Def^n</u>: The <u>fundamental groupoid</u> $\Pi_1 X$ of a graph X

objects: vertices $x, x, \ldots$ of X

morphisms: path-homotopy classes $[\gamma] : x \longrightarrow x'$ of paths $\gamma : x \rightsquigarrow x'$ in X.

$p: Y \longrightarrow X$ covering map

We can define a functor $\mathrm{Fib}_x(p): \Pi_1 X \longrightarrow \mathrm{Set}$:

$$
\begin{array}{ccc}
x & \longmapsto & p^{-1}(x) \\
{\scriptstyle [r]}\downarrow & \mapsto & \downarrow {\scriptstyle \mathrm{unw}_{[r]}} \\
x' & \longmapsto & p^{-1}(x')
\end{array}
$$

$\mathrm{Cov}(X) :=$ category of coverings over X

objects: coverings $p: Y \longrightarrow X$

morphisms:
$$
\begin{tikzcd}
Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Y' \arrow[dl, "p'"] \\
& X &
\end{tikzcd}
$$

We get a functor $\mathrm{Fib}_x : \mathrm{Cov}(X) \longrightarrow \mathrm{Set}^{\Pi_1 X}$

$F : \Pi_1 X \longrightarrow Set$ functor

We can define a graph $Tot_X F$ on the set $\coprod_{x \in X} Fx$ as follows:

$$y \sim y' \iff \begin{cases} y \in Fx , \quad y' \in Fx' \\ x \sim x' \text{ is an edge in } X, \text{ and} \\ (F[e])(y) = y' \end{cases}$$

where $e : x \rightsquigarrow x'$ is the "edge path" in X.

We also have an obvious map $p : Tot_X F \longrightarrow X$.

<u>Prop^n</u>: $p : Tot_X F \longrightarrow X$ is a covering.

This gives a functor $Tot_X : Set^{\Pi_1 X} \longrightarrow Cov(X)$.

<u>Theorem</u>: For each $X \in Graph$, we have an equivalence of categories

$$Cov(X) \simeq Set^{\Pi_1 X} \qquad \square .$$

If (X, x_0) is a pointed, connected graph, then the inclusion $A_1(X, x_0) \hookrightarrow \Pi_1 X$ is part of an equivalence of categories.

<u>Corollary</u>: For a pointed, connected graph (X, x_0), we have an equivalence of categories:

$$\mathrm{Cov}(X) \simeq \mathrm{Set}^{A_1(X, x_0)}.$$

<u>Corollary</u>: For a pointed, connected graph (X, x_0), we have a Galois correspondence b/w:

$$\left\{ \begin{array}{l} \text{connected covers} \\ \text{of } (X, x_0) \end{array} \right\} \simeq \left\{ \begin{array}{l} \text{subgroups of } A_1(X, x_0) \\ \text{ordered by } \subseteq \end{array} \right\}^{op.}$$

$$p : (Y, y_0) \longrightarrow (X, x_0) \longmapsto p_*(A_1(Y, y_0)).$$

<u>Def^n</u>: A pointed covering $p : (Y, y_0) \longrightarrow (X, x_0)$ is a <u>universal cover</u> if it is initial in $\mathrm{Cov}(X, x_0)$.

If it exists, it is unique up to a unique iso.

<u>Theorem</u>: ① Every pointed graph (X, x) admits a universal cover $p : (\tilde{X}_x, [c_x]) \longrightarrow (X, x)$.

② A pointed covering $p : (Y, y) \longrightarrow (X, x)$ is a universal cover iff Y is simply connected.

③ $A_1(X, x) \cong \mathrm{Aut}_{\mathrm{Cov}(X)}(\tilde{X}_x)$.

<u>Examples</u>:

① The identity maps on C_3 and C_4 are their resp. universal covers

② The map $p : I_\infty \longrightarrow C_n$; $i \longmapsto i \pmod{n}$ is the universal cover for $n \geq 5$.

Thus,

$$A_1(C_n, *) \cong \begin{cases} 0 & \text{if } n = 3, 4 \\ \mathbb{Z} & \text{if } n \geq 5. \end{cases}$$