CMSC 471:
Reasoning with Bayesian Belief Network

Chapters 12 & 13

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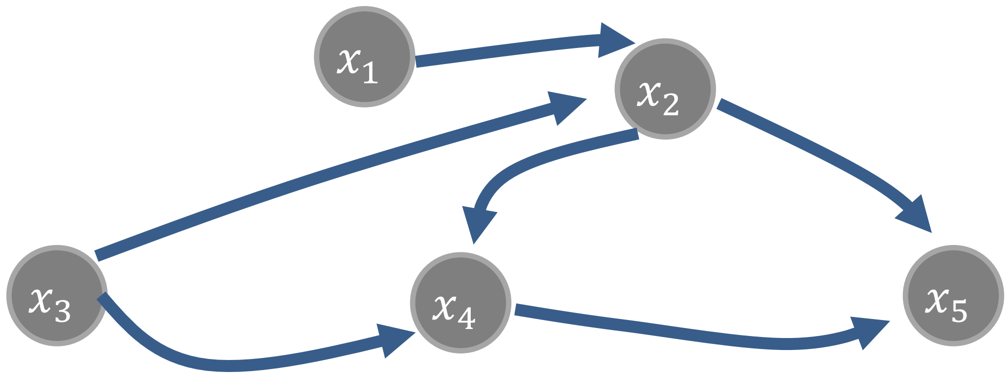
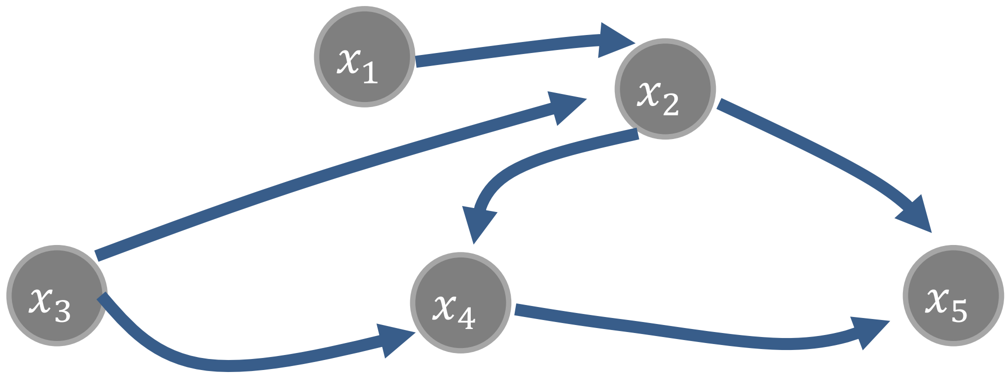
Some slides courtesy Tim Finin and Frank Ferraro

Bayesian Networks: Directed Acyclic Graphs



$$p(x_1, x_2, x_3, x_4, x_5) =$$
$$p(x_1)p(x_3)p(x_2|x_1, x_3)p(x_4|x_2, x_3)p(x_5|x_2, x_4)$$

Bayesian Networks: Directed Acyclic Graphs

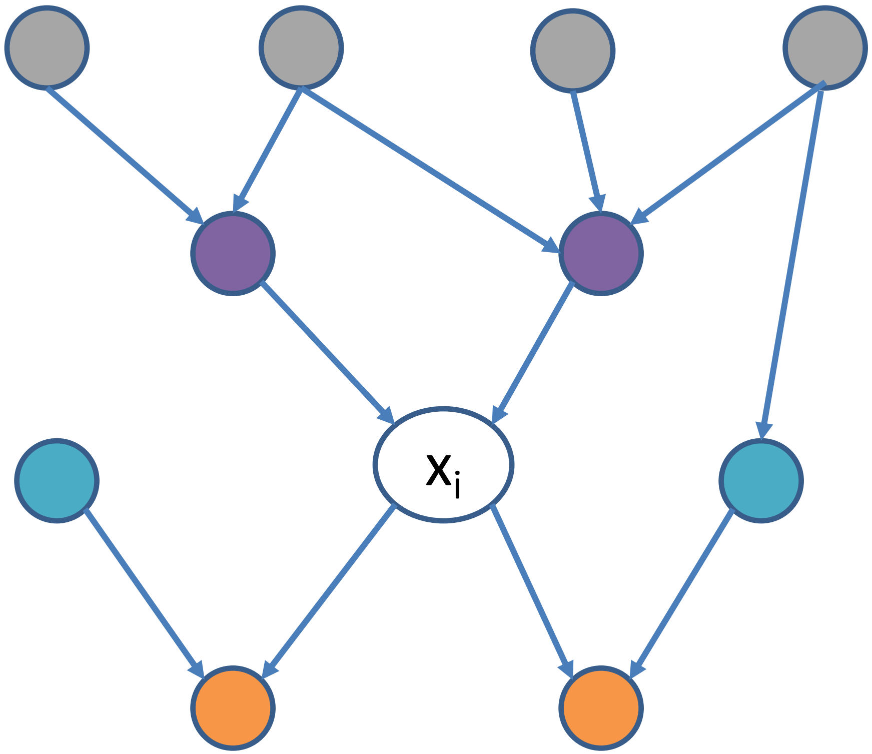


$$p(x_1, x_2, x_3, \dots, x_N) = \prod_i p(x_i \mid \pi(x_i))$$

exact inference in general DAGs is NP-hard

inference in trees can be exact

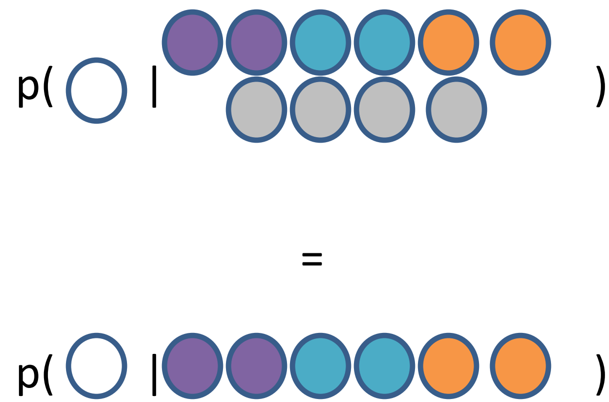
Markov Blanket



Markov blanket of a node x is its parents, children, and children's parents

(in this example, shading does not show observed/latent)

The **Markov Blanket** of a node x_i the set of nodes needed to form the complete conditional for a variable x_i



Given its Markov blanket, a node is conditionally independent of all other nodes in the BN

Fundamental Inference & Learning Question

- Compute posterior probability of a node given some other nodes

$$p(Q|x_1, \dots, x_j)$$

- Some techniques
  - MLE (maximum likelihood estimation)/MAP (maximum a posteriori) [covered 2nd]
  - Variable Elimination [covered 1st]
  - (Loopy) Belief Propagation ((Loopy) BP)
  - Monte Carlo
  - Variational methods
  - …

Advanced topics

Variable Elimination

- Inference: Compute posterior probability of a node given some other nodes

$$p(Q|x_1, \dots, x_j)$$

- Variable elimination: An algorithm for exact inference
  – Uses dynamic programming
  – Not necessarily polynomial time!

Variable Elimination (High-level)

Goal: $p(Q|x_1, \dots, x_j)$

(The word "factor" is used for each CPT.)

1. Pick one of the non-conditioned, MB variables
2. Eliminate this variable by marginalizing (summing) it out from all factors (CPTs) that contain it
3. Go back to 1 until no (MB) variables remain
4. Multiply the remaining factors and normalize.

Variable Elimination: Example

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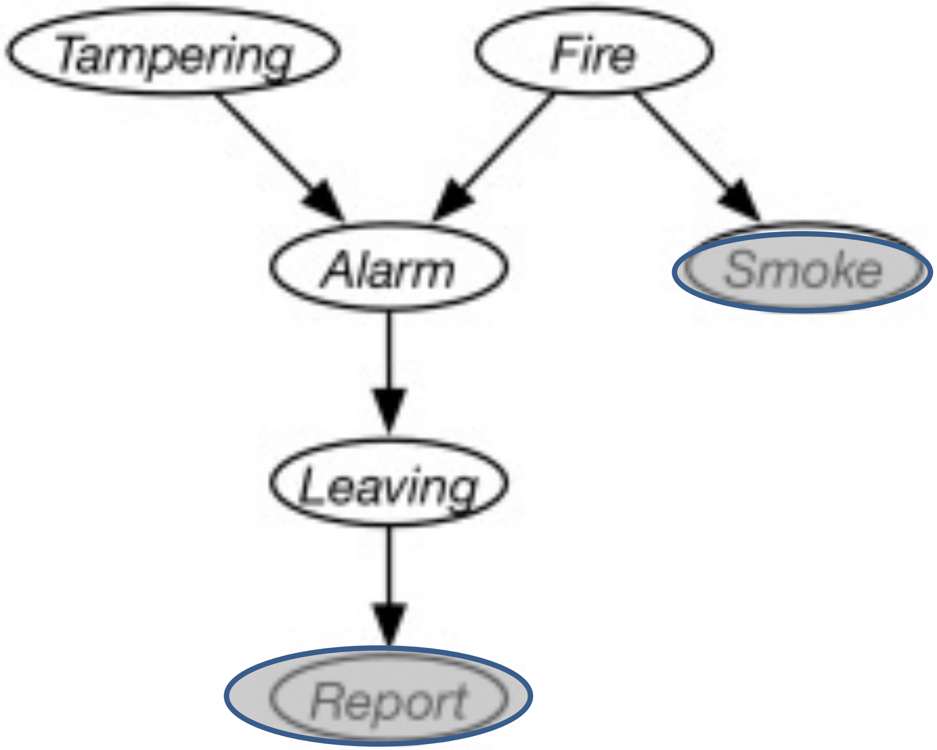

Goal: P(Tampering | Smoke=true ∧ Report=true)

Variable Elimination: Example

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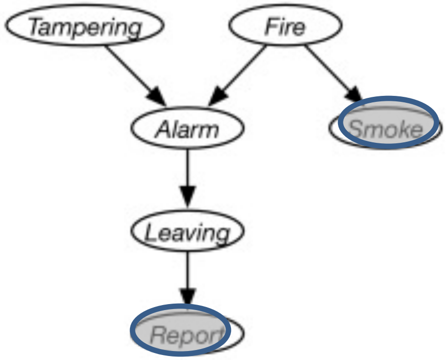
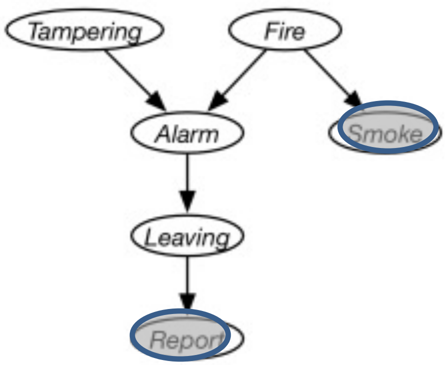


Goal: P(Tampering | Smoke=true ∧ Report=true)

$Conditional Probability$	$Factor$
$P(Tampering)$	$f_0(Tampering)$
$P(Fire)$	$f_1(Fire)$
$P(Alarm \mid Tampering, Fire)$	$f_2(Tampering, Fire, Alarm)$
$P(Smoke = yes \mid Fire)$	$f_3(Fire)$
$P(Leaving \mid Alarm)$	$f_4(Alarm, Leaving)$
$P(Report = yes \mid Leaving)$	$f_5(Leaving)$

Variable Elimination: Example



Goal: P(Tampering | Smoke=true ∧ Report=true)

Task: Eliminate Fire

(The word "factor" is used for each CPT.)

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Variable Elimination: Example

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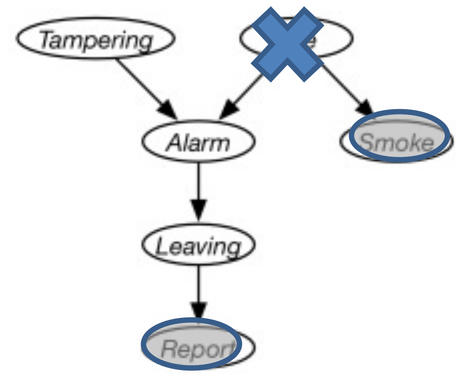

Goal: P(Tampering | Smoke=true ∧ Report=true)

f1(Fire)

f2(Tampering, Fire, Alarm)

f3(Fire)

f6(Tampering, Alarm) =

$$= \sum_{u} f_1(\text{Fire} = u) f_2(T, F = u, A) f_3(F = u)$$

$$= \sum_{u} p(\text{Fire} = u) p(A \mid T, F = u) p(S = y \mid F = u)$$

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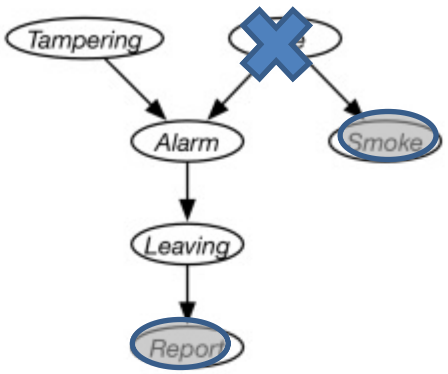

Goal: P(Tampering | Smoke=true ∧ Report=true)

f6(Tampering, Alarm) =

$$= \sum_u p(\text{Fire} = u) p(A \mid T, F = u) p(S = y \mid F = u)$$

$$= p(\text{Fire} = y) p(A \mid T, F = y) p(S = y \mid F = y) + p(\text{Fire} = n) p(A \mid T, F = n) p(S = y \mid F = n)$$

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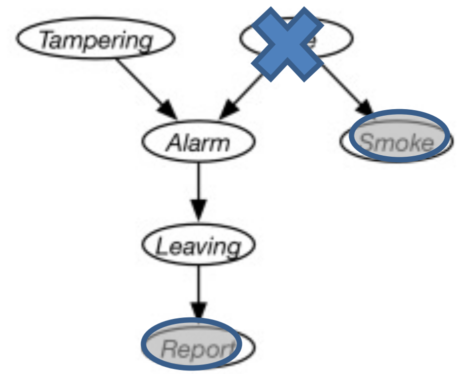
Variable Elimination: Example



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Goal: P(Tampering | Smoke=true ∧ Report=true)

f6(Tampering, Alarm) =

$$= \sum_u p(\text{Fire} = u) p(A \mid T, F = u) p(S = y \mid F = u)$$

Tamp.	Alarm	f6
Yes	Yes	$p(\text{Fire} = y)p(A = y \mid T = y, F = y)p(S = y \mid F = y) +$ $p(\text{Fire} = n)p(A = y \mid T = y, F = n)p(S = y \mid F = n)$
Yes	No	...
No	No	...
No	Yes	...

Variable Elimination: Example

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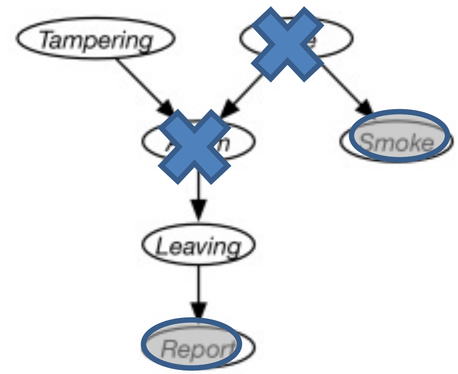

Goal: P(Tampering | Smoke=true ∧ Report=true)

Task: Eliminate Alarm

Variable Elimination: Example

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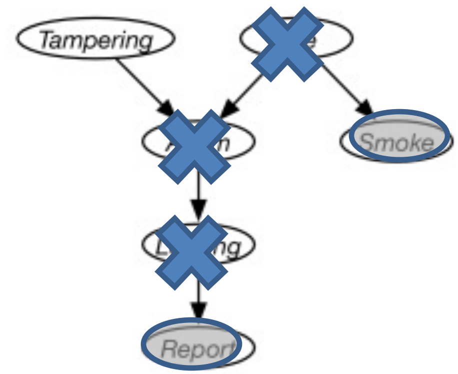

Goal: P(Tampering | Smoke=true ∧ Report=true)

…other computations not shown---see the book or lecture…

PM example 9.27

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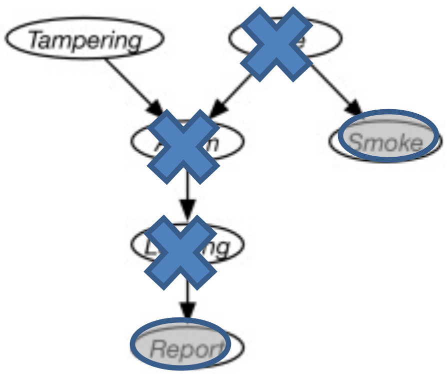

Goal: P(Tampering | Smoke=true ∧ Report=true)

Task: Normalize in order to compute p(Tampering)

We'll have a single factor f8(Tampering):

$$p(T = u) = \frac{f_8(T = u)}{\sum_v f_8(T = v)}$$

$ConditionalProbability$	$Factor$
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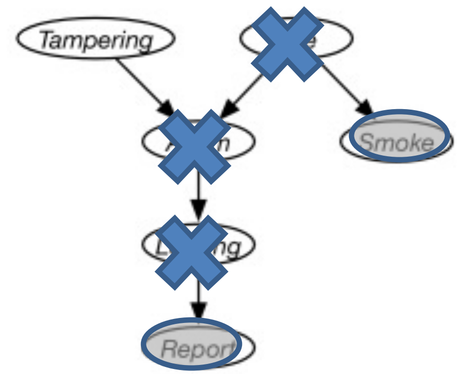


Goal: P(Tampering | Smoke=true ∧ Report=true)

Task: Normalize in order to compute ***p(Tampering)***

We'll have a single factor f8(Tampering):

$$p(T = yes) = \frac{f_8(T = yes)}{f_8(T = yes) + f_8(T = no)}$$

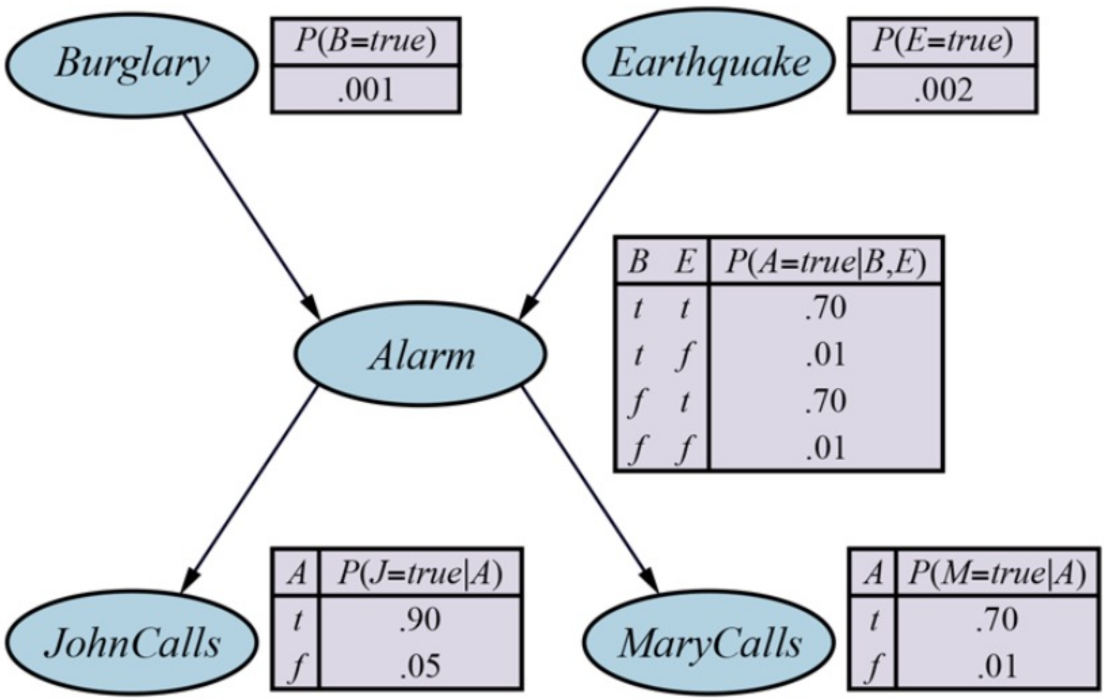
Variable Elimination: Example

- The posterior distribution over *Tampering* is given by

$$\frac{P(Tampering = u)\, f_8(Tampering = u)}{\sum_v P(Tampering = v)\, f_8(Tampering = v)}$$

Another example

Figure 13.2



$$\mathbf{P}(Burglary|JohnCalls = true, MaryCalls = true) = \langle 0.284, 0.716 \rangle.$$

$$\mathbf{P}(B|j,m) = \alpha\,\mathbf{P}(B,j,m) = \alpha \sum_e \sum_a \mathbf{P}(B,j,m,e,a).$$

$$P(b|j,m) = \alpha \sum_e \sum_a P(b)P(e)P(a|b,e)P(j|a)P(m|a).$$

$$P(b|j,m) = \alpha\,P(b) \sum_e P(e) \sum_a P(a|b,e)P(j|a)P(m|a).$$

$$\mathbf{P}(B|j,m) = \alpha \underbrace{\mathbf{P}(B)}_{\mathbf{f}_1(B)} \sum_e \underbrace{P(e)}_{\mathbf{f}_2(E)} \sum_a \underbrace{\mathbf{P}(a|B,e)}_{\mathbf{f}_3(A,B,E)} \underbrace{P(j|a)}_{\mathbf{f}_4(A)} \underbrace{P(m|a)}_{\mathbf{f}_5(A)}.$$

$$\mathbf{P}(B|j,m) = \alpha\,\mathbf{f}_1(B) \times \sum_e \mathbf{f}_2(E) \times \sum_a \mathbf{f}_3(A,B,E) \times \mathbf{f}_4(A) \times \mathbf{f}_5(A).$$

$$\begin{aligned}\mathbf{f}_6(B,E) &= \sum_a \mathbf{f}_3(A,B,E) \times \mathbf{f}_4(A) \times \mathbf{f}_5(A) \\ &= (\mathbf{f}_3(a,B,E) \times \mathbf{f}_4(a) \times \mathbf{f}_5(a)) + (\mathbf{f}_3(\neg a,B,E) \times \mathbf{f}_4(\neg a) \times \mathbf{f}_5(\neg a)).\end{aligned}$$

Now we are left with the expression

$$\mathbf{P}(B|j,m) = \alpha\, \mathbf{f}_1(B) \times \sum_e \mathbf{f}_2(E) \times \mathbf{f}_6(B,E).$$

- Next, we sum out E from the product of $\mathbf{f}_2$ and $\mathbf{f}_6$:

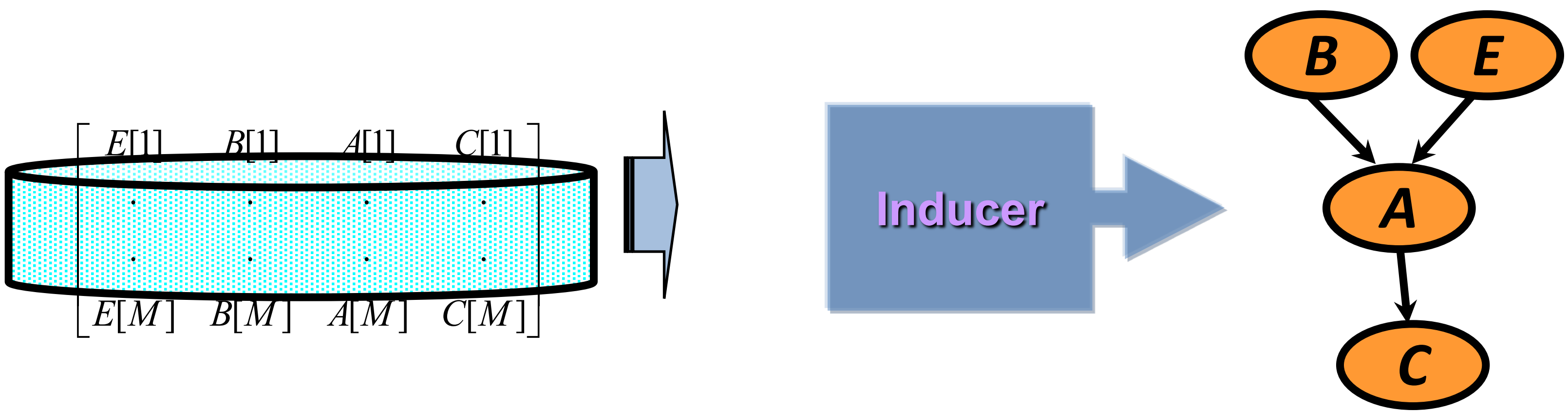
$$\begin{aligned}\mathbf{f}_7(B) &= \sum_e \mathbf{f}_2(E) \times \mathbf{f}_6(B,E) \\ &= \mathbf{f}_2(e) \times \mathbf{f}_6(B,e) + \mathbf{f}_2(\neg e) \times \mathbf{f}_6(B,\neg e).\end{aligned}$$

This leaves the expression

$$\mathbf{P}(B|j,m) = \alpha\, \mathbf{f}_1(B) \times \mathbf{f}_7(B)$$

Learning Bayesian networks

- Given training set $\boldsymbol{D} = \{\boldsymbol{x}[1], \ldots, \boldsymbol{x}[M]\}$
- Find graph that best matches ***D***
  - – model selection
  - – parameter estimation



Data D

Learning Bayesian Networks

- Describe a BN by specifying its (1) structure and (2) conditional probability tables (CPTs)
- Both can be learned from data, but
  - learning structure much harder than learning parameters
  - learning when some nodes are hidden, or with missing data harder still
- Four cases:

Structure	*Observability*	*Method*
Known	Full	Maximum Likelihood Estimation
Known	Partial	EM (or gradient ascent)
Unknown	Full	Search through model space
Unknown	Partial	EM + search through model space

Variations on a theme

- **Known structure, fully observable**: only need to do parameter estimation
- **Unknown structure, fully observable:** do heuristic search through structure space, then parameter estimation
- **Known structure, missing values:** use expectation maximization (EM) to estimate parameters
- **Known structure, hidden variables:** apply adaptive probabilistic network (APN) techniques
- **Unknown structure, hidden variables:** too hard to solve!

Fundamental Inference Question

- Compute posterior probability of a node given some other nodes

$$p(Q|x_1, \dots, x_j)$$

- Some techniques
  - MLE (maximum likelihood estimation)/MAP (maximum a posteriori) [covered 2nd]
  - Variable Elimination [covered 1st]
  - (Loopy) Belief Propagation ((Loopy) BP)
  - Monte Carlo
  - Variational methods
  - …

Advanced topics

Parameter estimation

- Assume known structure
- Goal: estimate BN parameters θ
  - entries in local probability models, P(X | Parents(X))
- A parameterization θ is good if it is likely to generate the observed data:

$$L(\theta : D) = P(D \mid \theta) = \prod_m P(x[m] \mid \theta)$$

i.i.d. samples

- Maximum Likelihood Estimation (MLE) Principle: Choose θ^* so as to maximize *L*

Parameter estimation II

- The likelihood **decomposes** according to the structure of the network
  → we get a separate estimation task for each parameter
- The MLE (maximum likelihood estimate) solution for **discrete** data & RV values:
  – for each value *x* of a node *X*
  – and each instantiation ***u*** of *Parents(X)*

$$\theta^*_{\boldsymbol{x}|\boldsymbol{u}} = \frac{N(\boldsymbol{x}, \boldsymbol{u})}{N(\boldsymbol{u})}$$

sufficient statistics

  – Just need to collect the counts for every combination of parents and children observed in the data
  – MLE is equivalent to an assumption of a uniform prior over parameter values

Estimating Probability of Heads

X=1 X=0

- I show you the above coin X, and hire you to estimate the probability that it will turn up heads $(X = 1)$ or tails $(X = 0)$
- You flip it repeatedly, observing
  - it turns up heads α_1 times
  - it turns up tails α_0 times
- Your estimate for $P(X = 1)$ is....?

$$\hat{P}(X=1) \approx \frac{\alpha_1}{\alpha_1 + \alpha_0}$$

Estimating θ = P(X=1)

X=1 X=0

Test A:

100 flips: 51 Heads (X=1) α_1, 49 Tails (X=0) α_0

$$\frac{\alpha_1}{\alpha_1+\alpha_0} = \frac{51}{100} \rightarrow \hat{P}(X=1) = 0.51$$

Test B:

3 flips: 2 Heads (X=1) α_1, 1 Tails (X=0) α_0

$$= \frac{2}{2+1} = 0.666$$

Maximum Likelihood Estimation

X=1 X=0

P(X=1) = θ P(X=0) = (1-θ)

Data D: $= \{1 \quad 0 \quad 0 \quad 1\} \quad 1$

$P(D|\theta) = \theta \cdot (1-\theta) \cdot (1-\theta) \cdot \theta \cdot \theta = \theta^{\alpha_1} (1-\theta)^{\alpha_0}$

Flips produce data D with α_1 heads, α_0 tails

- flips are independent, identically distributed 1's and 0's (Bernoulli)
- α_1 and α_0 are counts that sum these outcomes (Binomial)

$$P(D|\theta) = P(\alpha_1, \alpha_0|\theta) = \theta^{\alpha_1}(1-\theta)^{\alpha_0}$$

Maximum Likelihood Estimate for Θ

$$\begin{aligned}\hat{\theta} &= \arg\max_{\theta} \ \ln P(\mathcal{D} \mid \theta) \\ &= \arg\max_{\theta} \ \ln \theta^{\alpha_H}(1-\theta)^{\alpha_T}\end{aligned}$$

- Set derivative to zero: $\boxed{\frac{d}{d\theta} \ln P(\mathcal{D} \mid \theta) = 0}$

[C. Guestrin]

$$\hat{\theta} = \arg\max_{\theta} \ \ln P(D|\theta)$$

$$= \arg\max_{\theta} \ \ln \underbrace{[\theta^{\alpha_1}(1-\theta)^{\alpha_0}]}$$

- Set derivative to zero: $\boxed{\frac{d}{d\theta} \ln P(\mathcal{D} \mid \theta) = 0}$

$$\boxed{\text{hint: } \frac{\partial \ln \theta}{\partial \theta} = \frac{1}{\theta}}$$

$$\frac{\partial}{\partial \theta} \ \alpha_1 \ln \theta + \alpha_0 \ln(1-\theta)$$

$$\alpha_1 \frac{1}{\theta} + \alpha_0 \underbrace{\frac{\partial \ln(1-\theta)}{\partial \theta}}$$

$$\underbrace{\frac{\partial \ln(1-\theta)}{\partial(1-\theta)}}_{\frac{1}{1-\theta}} \cdot \underbrace{\frac{\partial(1-\theta)}{\partial\theta}}_{-1}$$

$$\boxed{0 = \alpha_1 \frac{1}{\theta} - \frac{\alpha_0}{1-\theta}}$$

$$\theta = \frac{\alpha_1}{\alpha_1 + \alpha_0}$$

Summary:
Maximum Likelihood Estimate

X=1 X=0

P(X=1) = θ
P(X=0) = 1-θ
(Bernoulli)

- Each flip yields boolean value for X

$$X \sim \text{Bernoulli: } P(X) = \theta^X (1-\theta)^{(1-X)}$$

- Data set D of independent, identically distributed (iid) flips produces α_1 ones, α_0 zeros (Binomial)

$$P(D|\theta) = P(\alpha_1, \alpha_0|\theta) = \theta^{\alpha_1}(1-\theta)^{\alpha_0}$$

$$\hat{\theta}^{MLE} = \text{argmax}_\theta \, P(D|\theta) = \frac{\alpha_1}{\alpha_1+\alpha_0}$$

Learning: Maximum Likelihood Estimation (MLE)

Core concept in intro statistics:

- Observe some data $\mathcal{X}$
- Compute some distribution $g(\mathcal{X})$ to {predict, explain, generate} $\mathcal{X}$
- Assume g is controlled by parameters ϕ, i.e., $g_{\phi}(\mathcal{X})$
  - Sometimes written $g(\mathcal{X}; \phi)$
- Learning appropriate value(s) of ϕ allows you to **GENERALIZE** about $\mathcal{X}$

Learning: Maximum Likelihood Estimation (MLE)

Central to machine learning:

- Observe some data $(\mathcal{X}, \mathcal{Y})$
- Compute some function $f(\mathcal{X})$ to {predict, explain, generate} $\mathcal{Y}$
- Assume f is controlled by parameters θ, i.e., $f_\theta(\mathcal{X})$
  - Sometimes written $f(\mathcal{X}; \theta)$

Learning Parameters for the Die Model

$$p(w_1, w_2, \ldots, w_N) = p(w_1)p(w_2)\cdots p(w_N) = \prod_i p(w_i)$$

maximize (log-) likelihood to learn the probability parameters

Q: Why is maximizing log-likelihood a reasonable thing to do?

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Q: Why is maximizing log-likelihood a reasonable thing to do?

A: Develop a good model for what we observe

Learning Parameters for the Die Model: Maximum Likelihood (Intuition)

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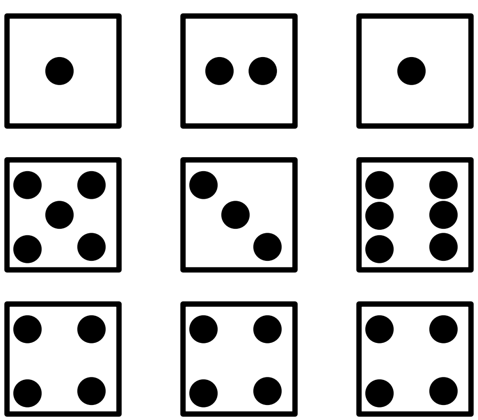
If you observe these 9 rolls…



…what are “reasonable” estimates for p(w)?

p(1) = ?

p(2) = ?

p(3) = ?

p(4) = ?

p(5) = ?

p(6) = ?

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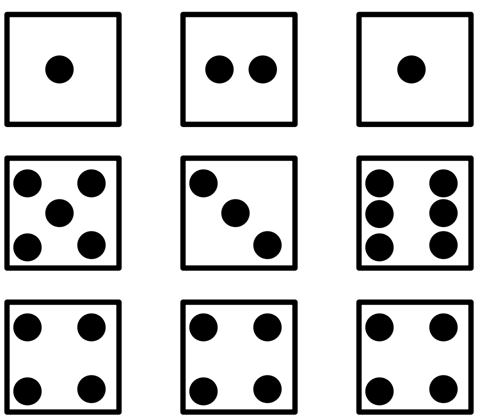

…what are "reasonable" estimates for p(w)?

p(1) = 2/9 p(2) = 1/9

p(3) = 1/9 p(4) = 3/9

p(5) = 1/9 p(6) = 1/9

maximum likelihood estimates

Learning: Maximum Likelihood Estimation (MLE)

Core concept in intro statistics:

- Observe some data $\mathcal{X}$
- Compute some distribution $g(\mathcal{X})$ to {predict, explain, generate} $\mathcal{X}$
- Assume g is controlled by parameters ϕ, i.e., $g_\phi(\mathcal{X})$
  - Sometimes written $g(\mathcal{X}; \phi)$
- Learning appropriate value(s) of ϕ allows you to **GENERALIZE** about $\mathcal{X}$

How do we "learn appropriate value(s) of ϕ?"

Many different options: a common one is **maximum likelihood estimation (MLE)**

- Find values ϕ s.t. $g_\phi(\mathcal{X} = \{x_1, \dots, x_N\})$ is maximized
- Independence assumptions are very useful here!
- Logarithms are also useful!

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Example: How much does it snow?

- $\mathcal{X} = \{x_1, x_2, \dots, x_N\}$ are snowfall values from the previous N storms
- Goal: learn ϕ such that g correctly models, as accurately as possible, the amount of snow likely

Advanced topic

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$$\max_{\phi} \sum_{i=1}^{N} \log g_\phi(x_i)$$

Advanced topic

MLE Snowfall Example

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Advanced topic

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Q: Why is taking logarithms okay?

Q: What other assumptions, or decisions, do we need to make?

x_i is positive, real-valued. What's a faithful probability distribution for x_i?

- Normal? ✗
- Gamma? ✓
- Exponential? ✓
- Bernoulli? ✗
- Poisson? ✗

Advanced topic

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x_i is positive, real-valued. What's a faithful probability distribution for x_i?

- Normal? ✗
- Gamma? ✓ $p(X = x) = \frac{x^{k-1}\exp(\frac{-k}{\theta})}{\theta^k \Gamma(k)}$
- Exponential? ✓
- Bernoulli? ✗
- Poisson? ✗

Advanced topic

MLE Snowfall Example

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Q: Why is taking logarithms okay?

Q: What other assumptions, or decisions, do we need to make?

x_i is positive, real-valued. What's a faithful/nice-to-compute-and-good-enough probability distribution for x_i?

- Normal? ✗ ✓
- Gamma? ✓ ?
- Exponential? ✓ ?
- Bernoulli? ✗ ✗
- Poisson? ✗ ✗

$$p(X = x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

Advanced topic

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$$\max_{\phi} \sum_{i=1}^{N} \log g_{\phi}(x_i)$$

$$x_i \sim \mathrm{Normal}(\mu, \sigma^2)$$

$$\max_{(\mu,\sigma^2)} \sum_{i=1}^{N} \log \mathrm{Normal}_{\mu,\sigma^2}(x_i) =$$

Advanced topic

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Advanced topic

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Advanced topic

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Q: How do we find μ, σ^2?

A: Differentiate and find that

$$\hat{\mu} = \frac{\sum_i x_i}{N}$$

$$\sigma^2 = \frac{\sum_i (x_i - \hat{\mu})^2}{N}$$

Learning: Maximum Likelihood Estimation (MLE)

Central to machine learning:

- Observe some data $(\mathcal{X}, \mathcal{Y})$
- Compute some function $f(\mathcal{X})$ to {predict, explain, generate} $\mathcal{Y}$
- Assume f is controlled by parameters θ, i.e., $f_\theta(\mathcal{X})$
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  - Sometimes written $f(\mathcal{X}; \theta)$
- Parameters are learned to $\underbrace{\text{minimize error (loss) } \ell}_{\text{Advanced topic}}$

Learning: Maximum Likelihood Estimation (MLE)

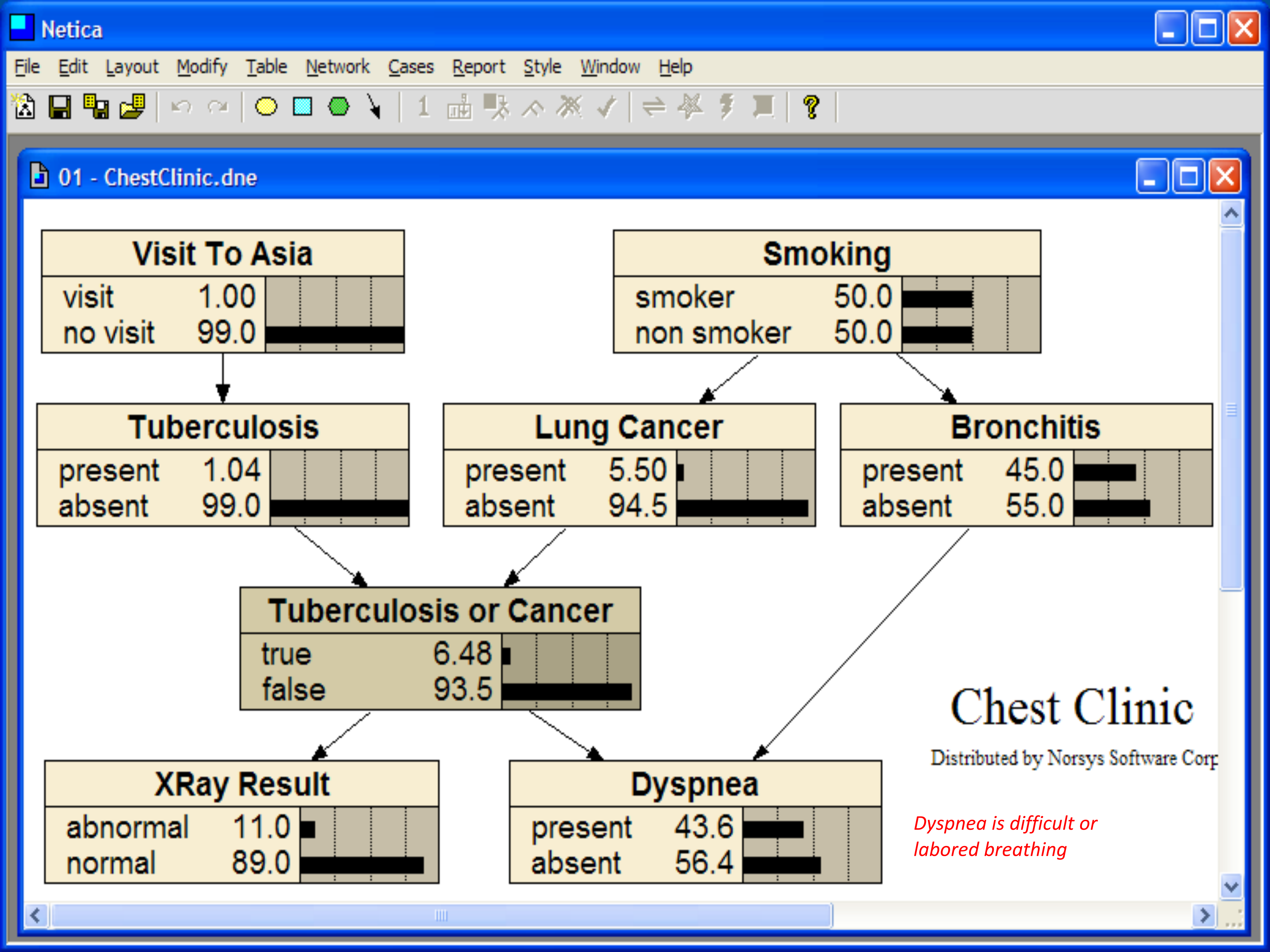
Example: Can I sleep in the next time it snows/is school canceled?

- $\mathcal{X} = \{x_1, x_2, \dots, x_N\}$ are snowfall values from the previous N storms
- $\mathcal{Y} = \{y_1, y_2, \dots, y_N\}$ are closure results from the previous N storms
- Goal: learn θ such that f correctly predicts, as accurately as possible, if UMBC will close in the next storm:
  - y^*_{n+1} from x_{n+1}

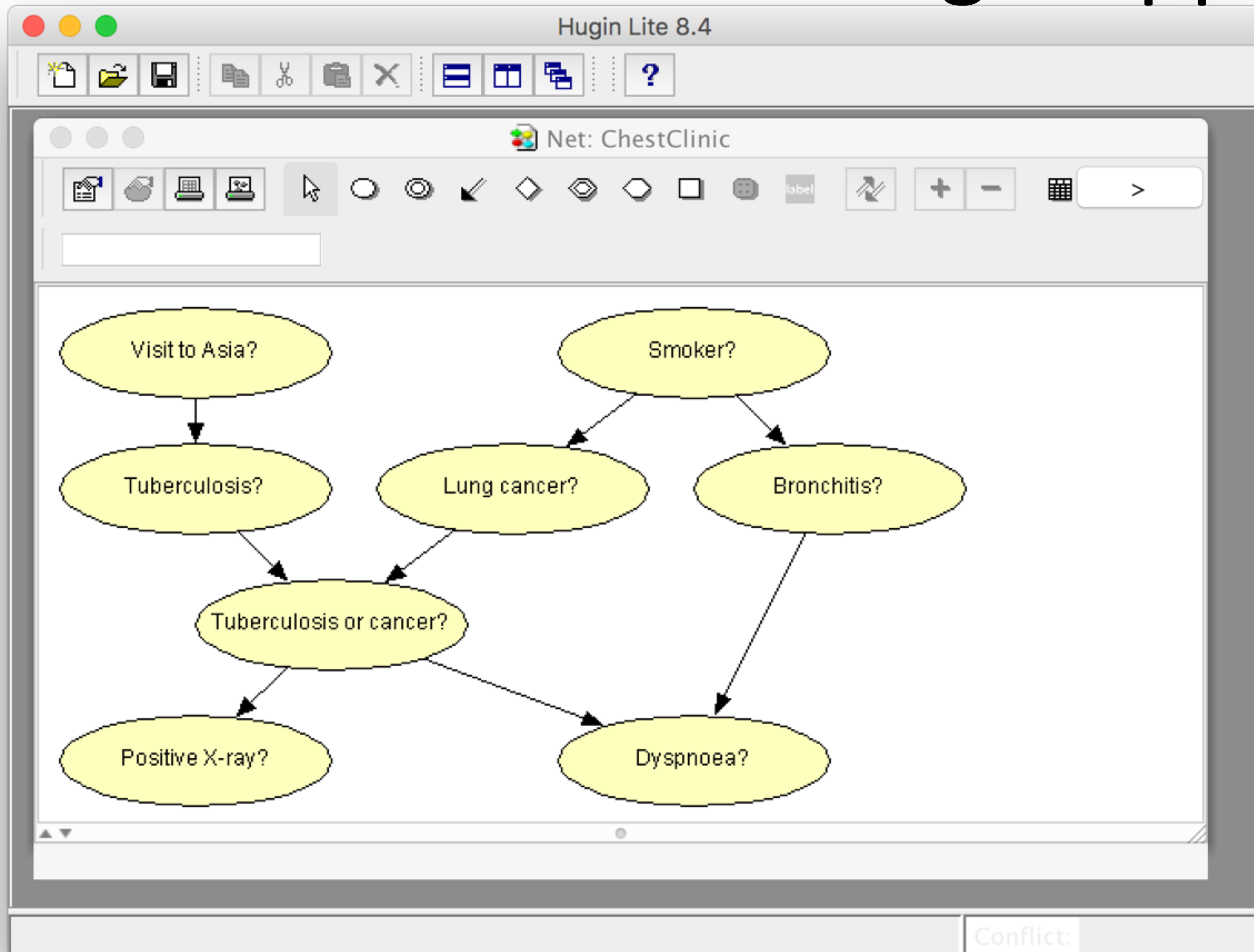
- If we assume the output of f is a *probability distribution* on $\mathcal{Y}|\mathcal{X}$…
  - $f(\mathcal{X}) \rightarrow \{p(\text{yes}|\mathcal{X}), p(\text{no}|\mathcal{X})\}$
- Then re: θ, {predicting, explaining, generating} $\mathcal{Y}$ means… *what*?

Some software tools

- Netica: Windows app for working with Bayes-ian belief networks and influence diagrams
  - Commercial product, free for small networks
  - Includes graphical editor, compiler, inference engine, etc.
  - To run in OS X or Linus you need Wire or Crossover
- Hugin: free demo versions for Linux, Mac, and Windows are available
- BBN.ipynb based on an AIMA notebook


Netica
File Edit Layout Modify Table Network Cases Report Style Window Help
01 - ChestClinic.dne
Visit To Asia
visit 1.00
no visit 99.0
Smoking
smoker 50.0
non smoker 50.0
Tuberculosis
present 1.04
absent 99.0
Lung Cancer
present 5.50
absent 94.5
Bronchitis
present 45.0
absent 55.0
Tuberculosis or Cancer
true 6.48
false 93.5
XRay Result
abnormal 11.0
normal 89.0
Dyspnea
present 43.6
absent 56.4
Chest Clinic
Distributed by Norsys Software Corp
Dyspnea is difficult or labored breathing

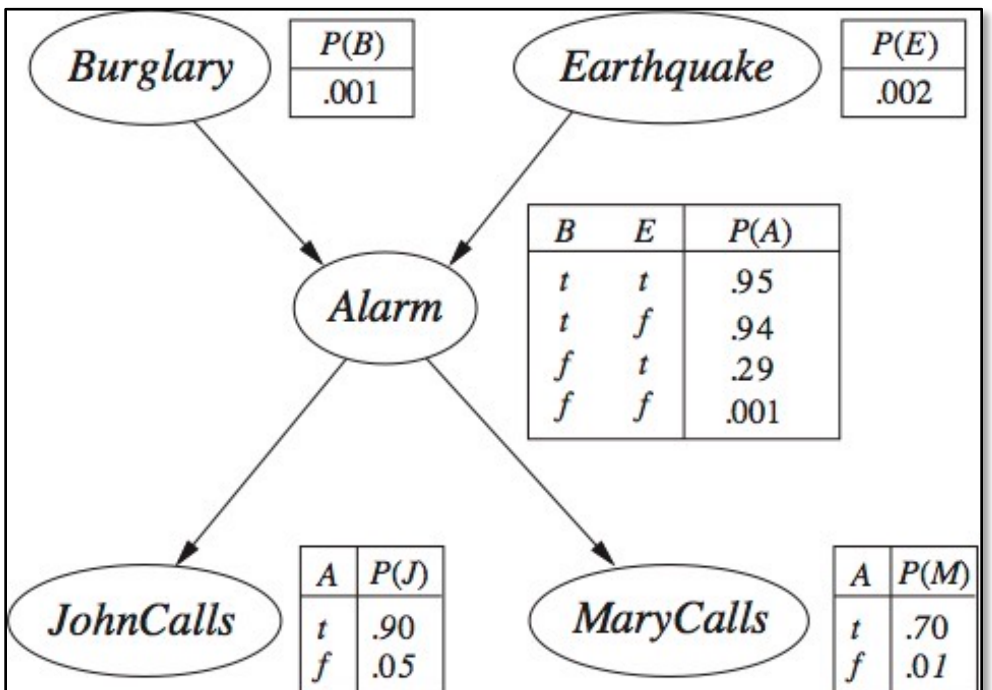
Same BBN model in Hugin app



See the 4-minute HUGIN Tutorial on YouTube

Python Code

See this [AIMA notebook]() on colab showing how to construct this BBN Network in Python



Judea Pearl example

There's is a house with a burglar alarm that can be triggered by a burglary or earthquake. If it sounds, one or both neighbors John & Mary, might call the owner to say the alarm is sounding.