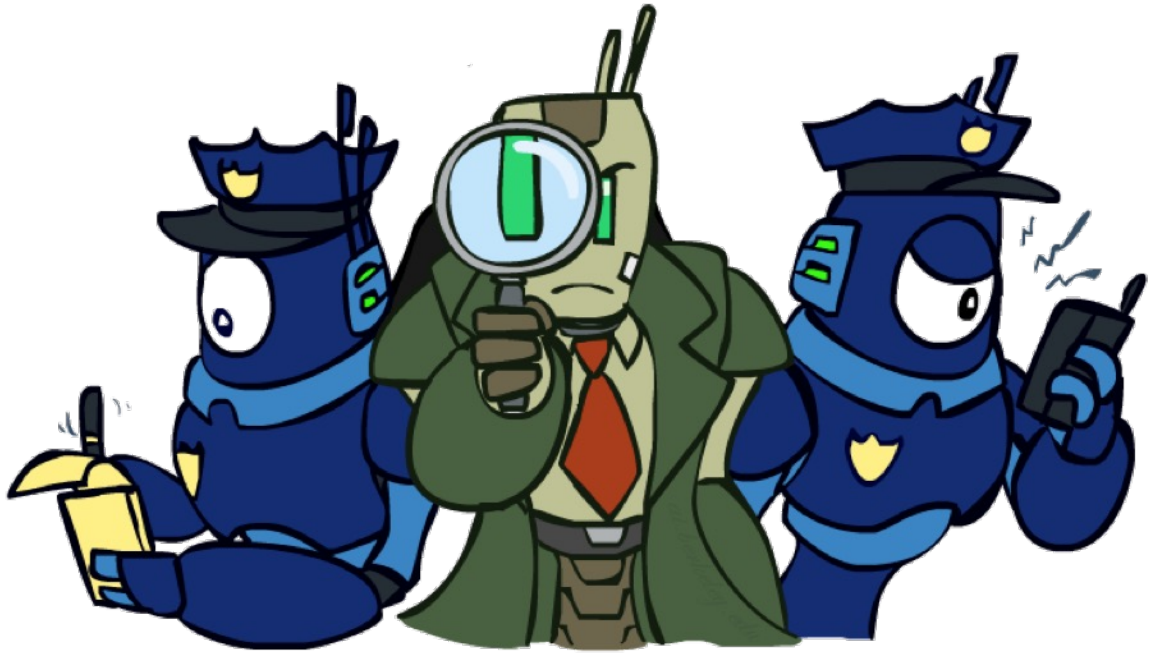
# CMSC 471

## Constraint Satisfaction Problems

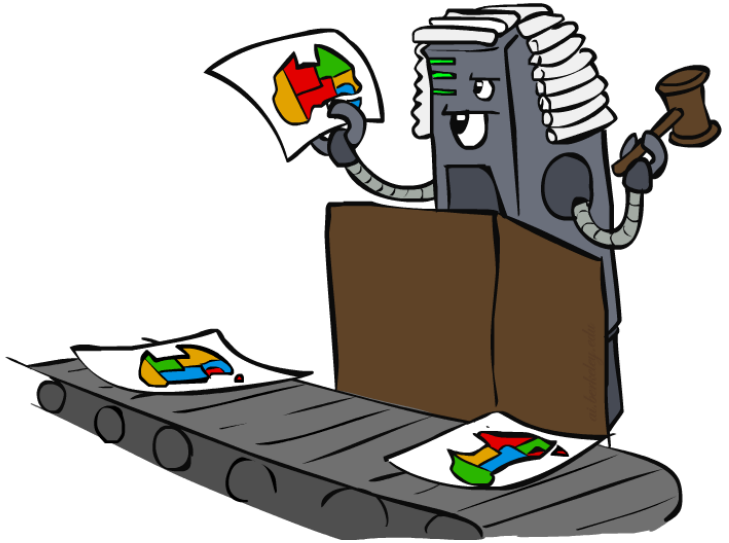


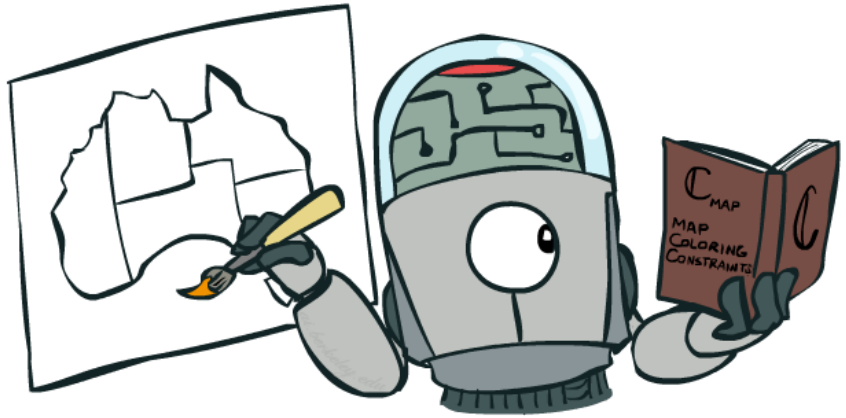
Instructor: KMA Solaiman

These slides were created by Dan Klein and Pieter Abbeel at UC Berkeley. [ai.berkeley.edu]

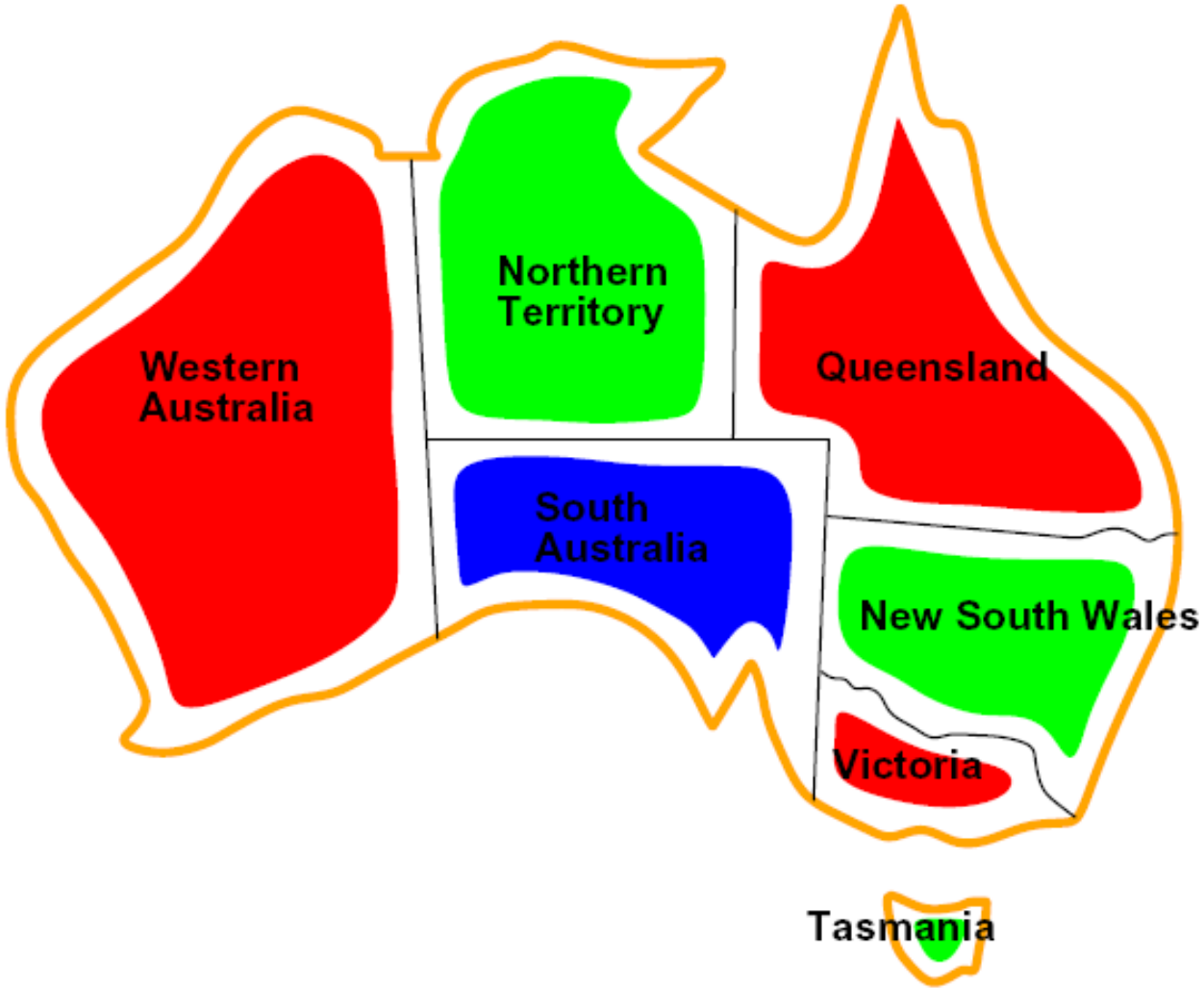
# Constraint Satisfaction Problems

- Standard search problems:
  - State is a "black box": arbitrary data structure
  - Goal test can be any function over states
  - Successor function can also be anything
- Constraint satisfaction problems (CSPs):
  - A special subset of search problems
  - State is defined by variables $X_i$ with values from a domain $D$ (sometimes $D$ depends on $i$)
  - Goal test is a set of constraints specifying allowable combinations of values for subsets of variables
- Simple example of a *formal representation language*
- Allows useful general-purpose algorithms with more power than standard search algorithms





# CSP Examples



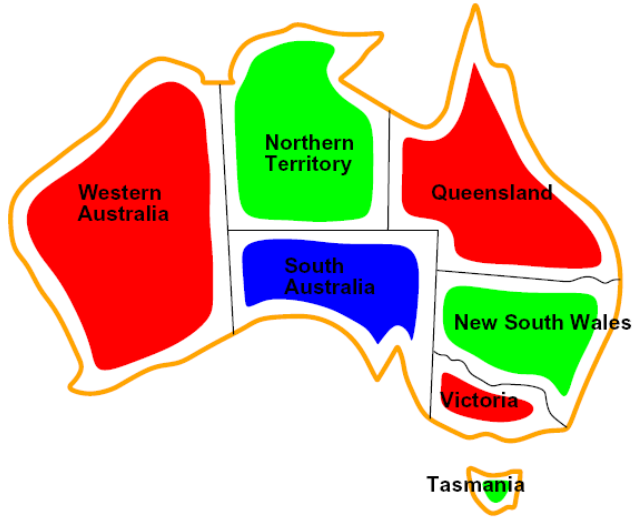
# Example: Map Coloring

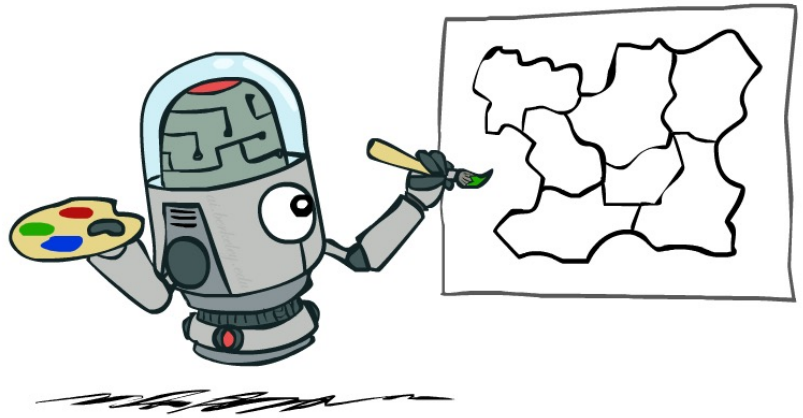
- Variables: $\mathrm{WA},\ \mathrm{NT},\ \mathrm{Q},\ \mathrm{NSW},\ \mathrm{V},\ \mathrm{SA},\ \mathrm{T}$
- Domains: $\mathrm{D} = \{\mathrm{red}, \mathrm{green}, \mathrm{blue}\}$
- Constraints: adjacent regions must have different colors

  Implicit: $\mathrm{WA} \neq \mathrm{NT}$

  Explicit: $(\mathrm{WA}, \mathrm{NT}) \in \{(\mathrm{red}, \mathrm{green}), (\mathrm{red}, \mathrm{blue}), \ldots\}$
- Solutions are assignments satisfying all constraints, e.g.:

  $\{\mathrm{WA}{=}\mathrm{red},\ \mathrm{NT}{=}\mathrm{green},\ \mathrm{Q}{=}\mathrm{red},\ \mathrm{NSW}{=}\mathrm{green},$
  $\mathrm{V}{=}\mathrm{red},\ \mathrm{SA}{=}\mathrm{blue},\ \mathrm{T}{=}\mathrm{green}\}$

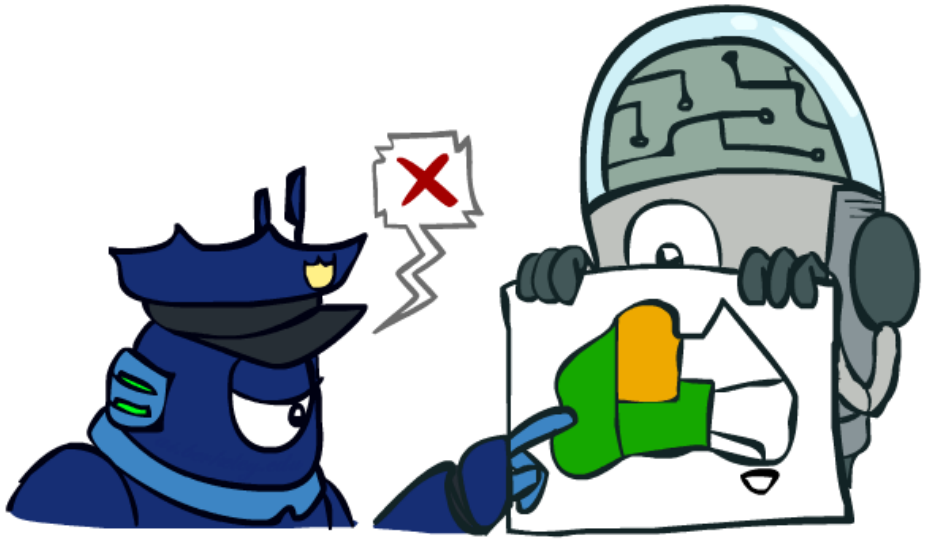


# Solving CSPs

# Backtracking Search

- Backtracking search is the basic uninformed algorithm for solving CSPs
- Idea 1: One variable at a time
  - Variable assignments are commutative, so fix ordering
  - I.e., [WA = red then NT = green] same as [NT = green then WA = red]
  - Only need to consider assignments to a single variable at each step
- Idea 2: Check constraints as you go
  - I.e. consider only values which do not conflict previous assignments
  - Might have to do some computation to check the constraint
  - "Incremental goal test"
- Depth-first search with these two improvements is called *backtracking search* (not the best name)



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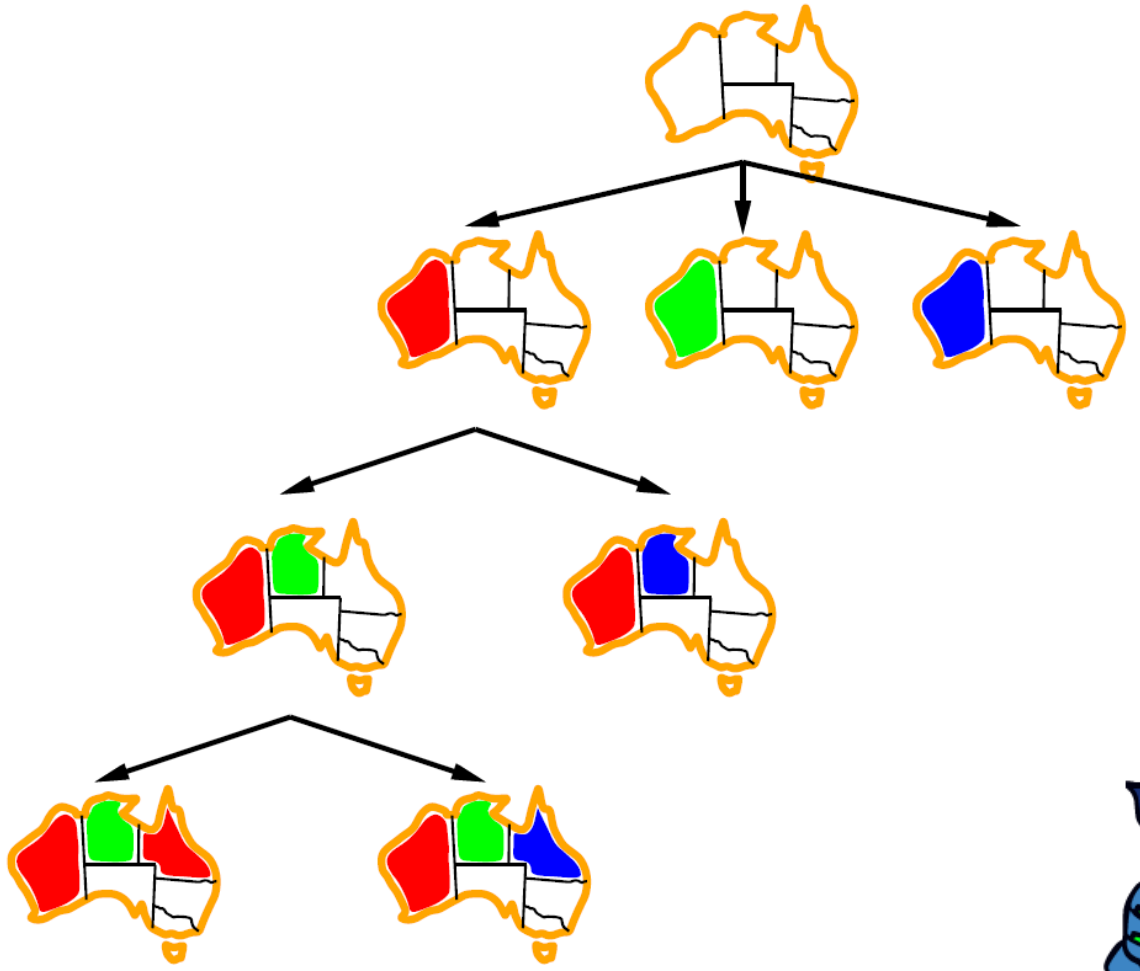
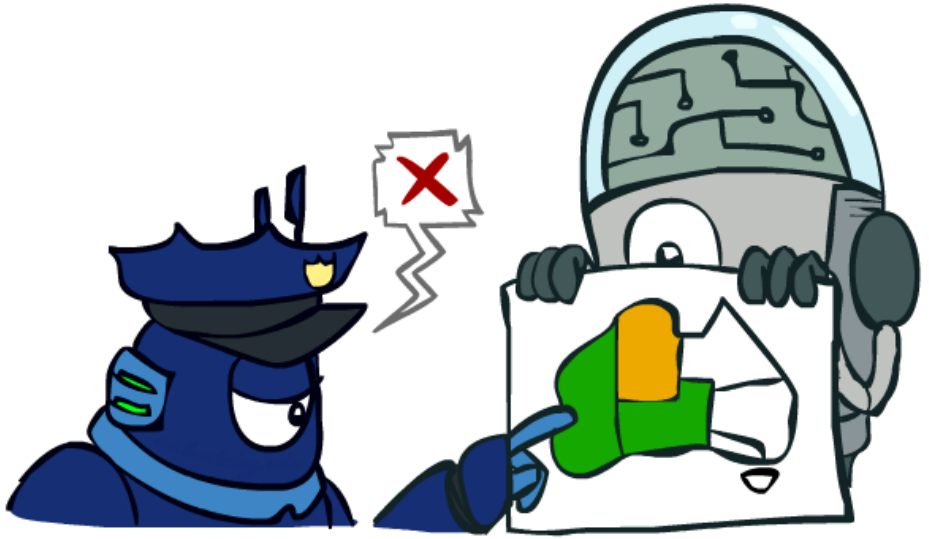
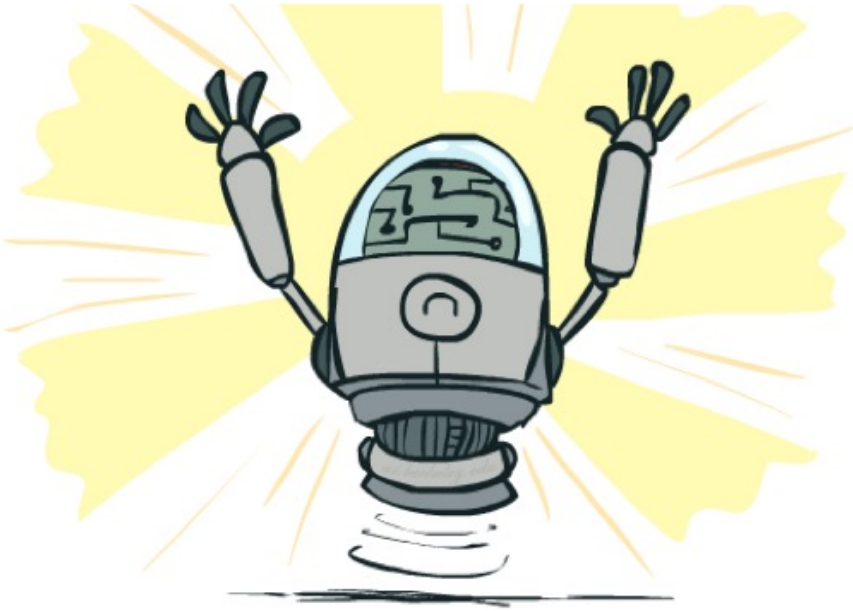




# Improving Backtracking

- General-purpose ideas give huge gains in speed
- Ordering:
  - Which variable should be assigned next?
  - In what order should its values be tried?
- Filtering: Can we detect inevitable failure early?
- Structure: Can we exploit the problem structure?



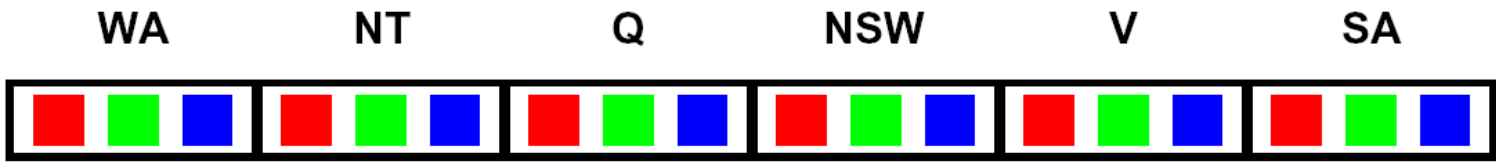
# Filtering

# Filtering

Keeping track of domains for unassigned variables and cross off bad options

# Filtering: Forward Checking

- Filtering: Keep track of domains for unassigned variables and cross off bad options
- Forward checking: Cross off values that violate a constraint when added to the existing assignment




[Demo: coloring -- forward checking]

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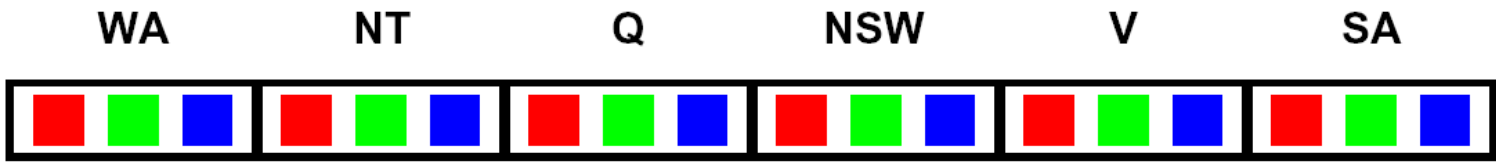


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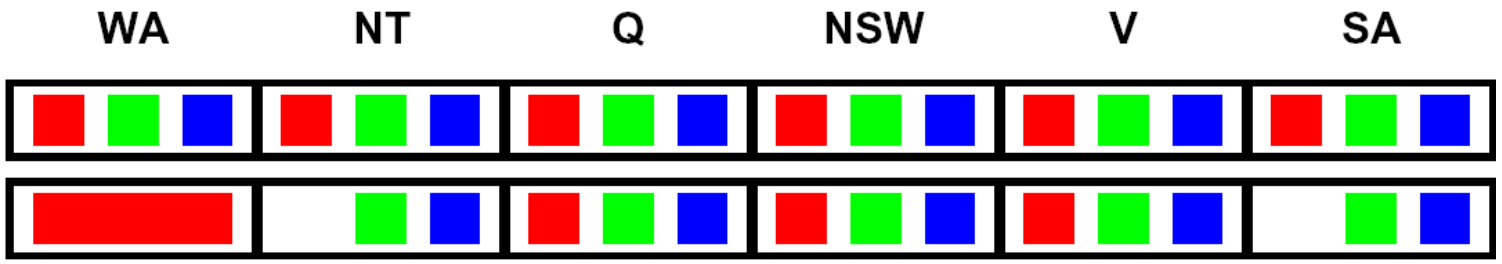


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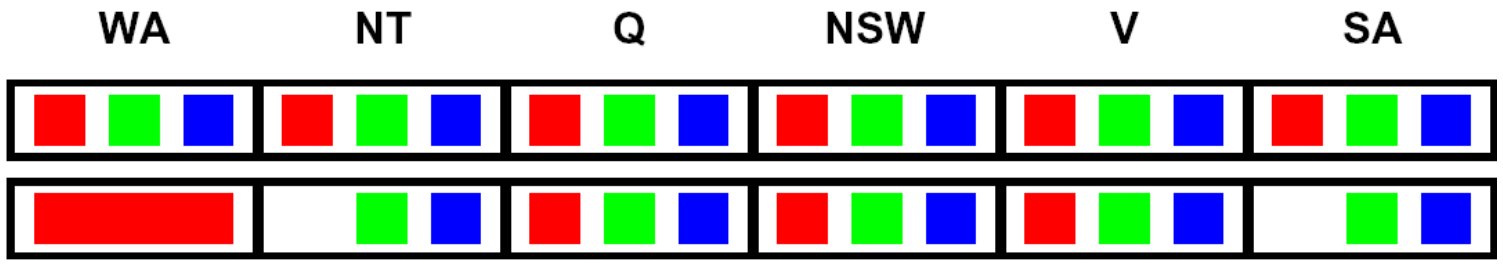


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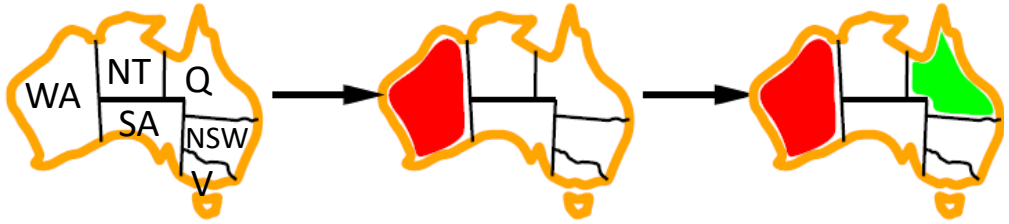


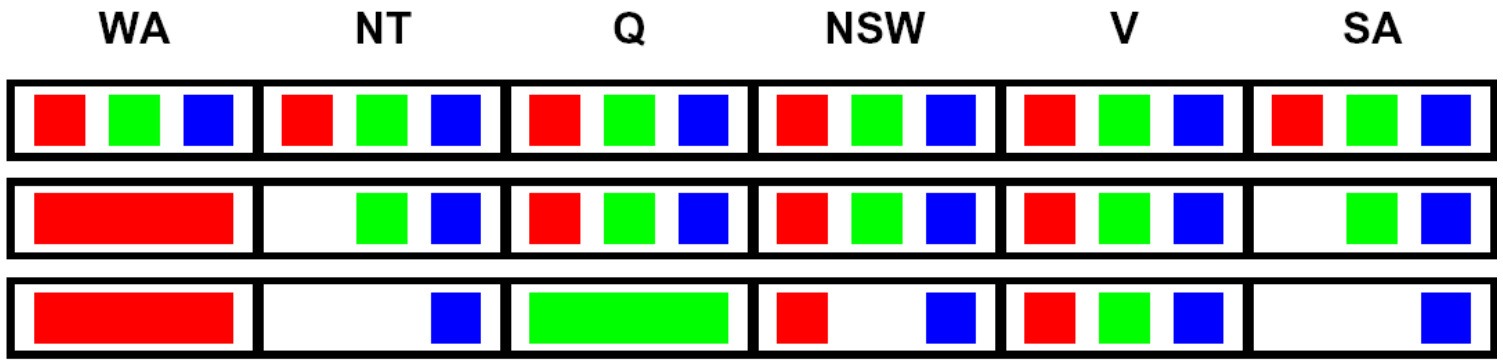


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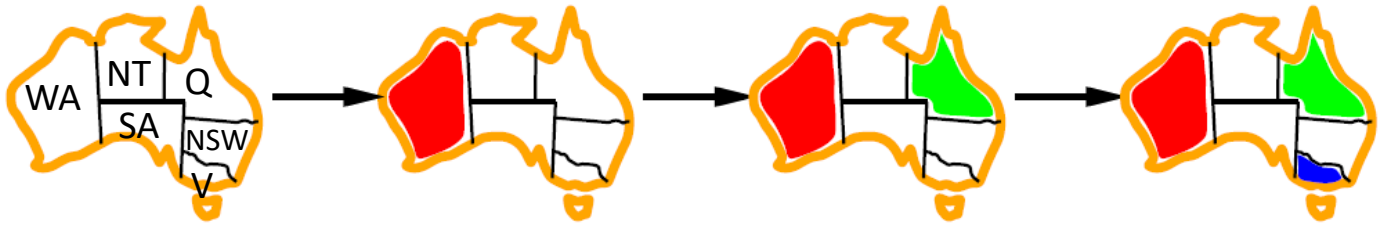


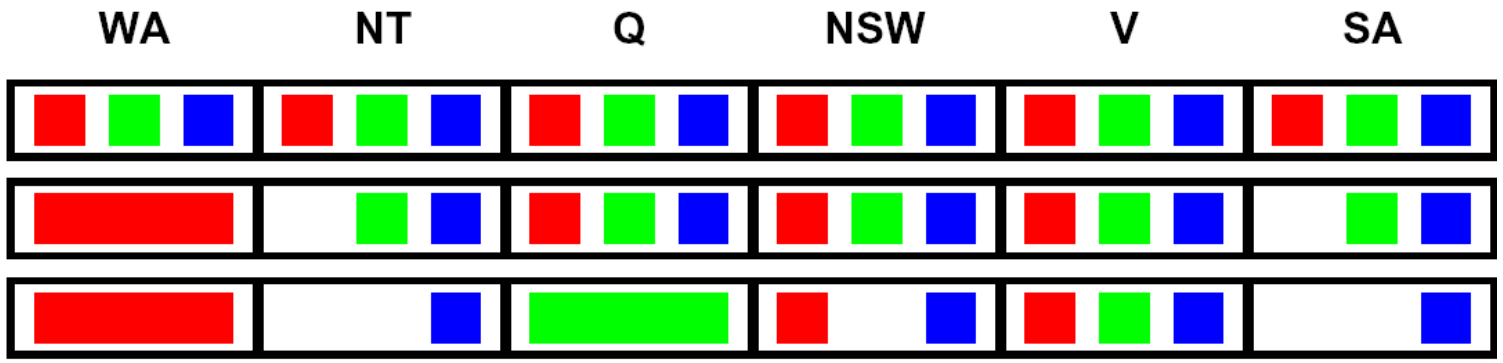


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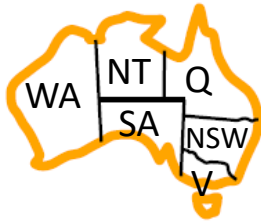
WA
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Q
SA
NSW
V

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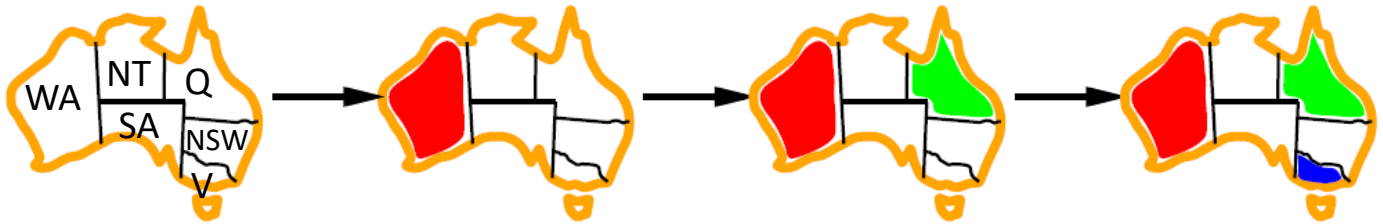


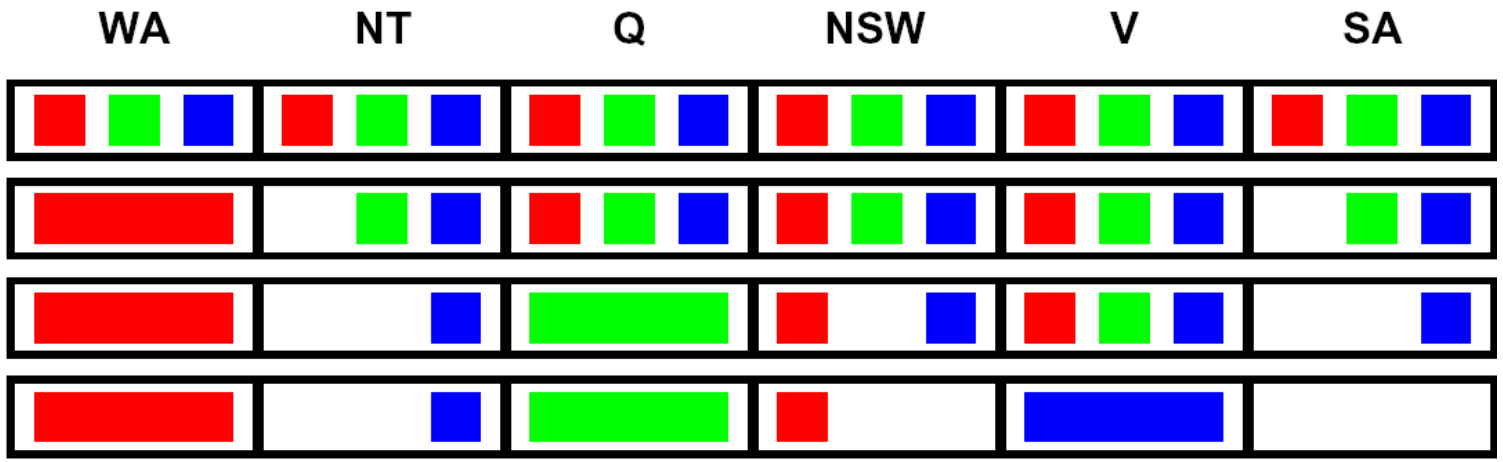


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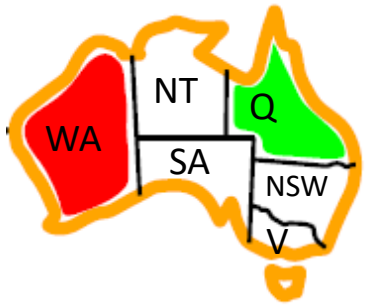
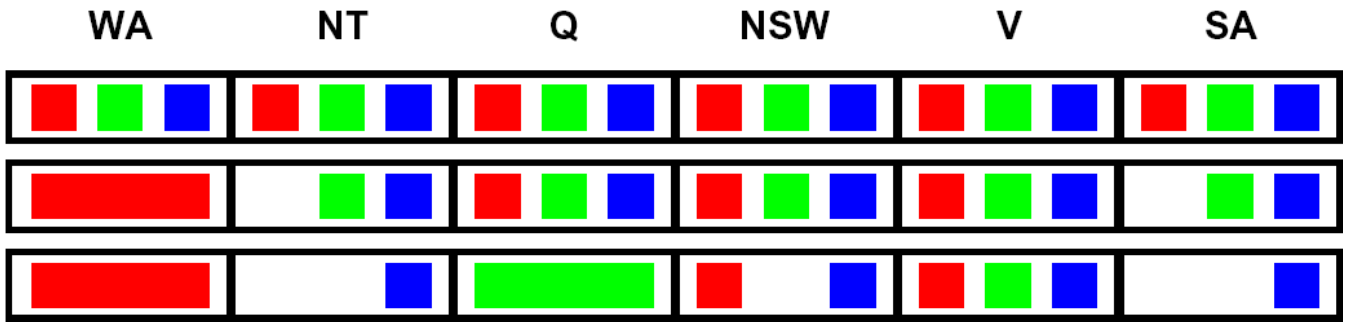
# Video of Demo Coloring – Backtracking with Forward Checking

# Filtering: Constraint Propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



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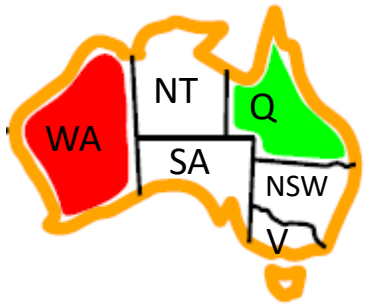


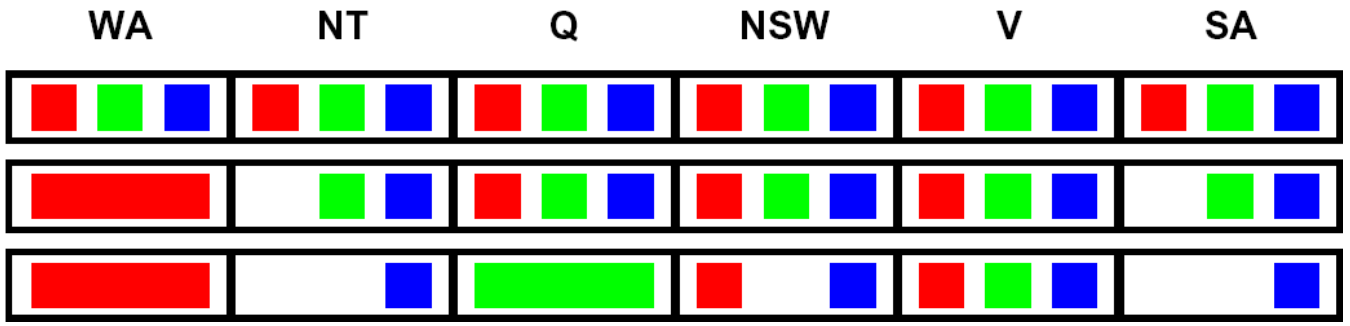


- NT and SA cannot both be blue!

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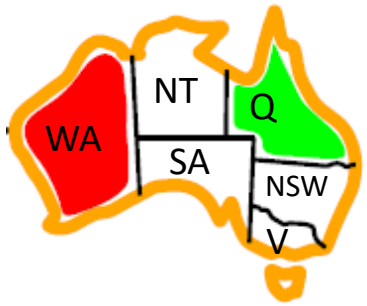


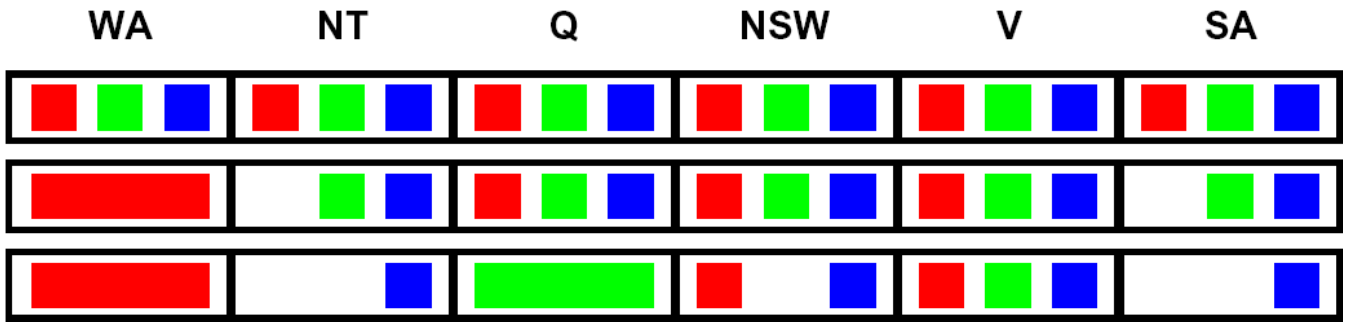


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- Why didn’t we detect this yet?

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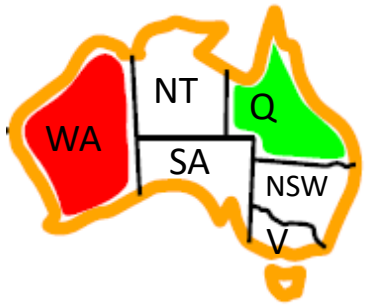
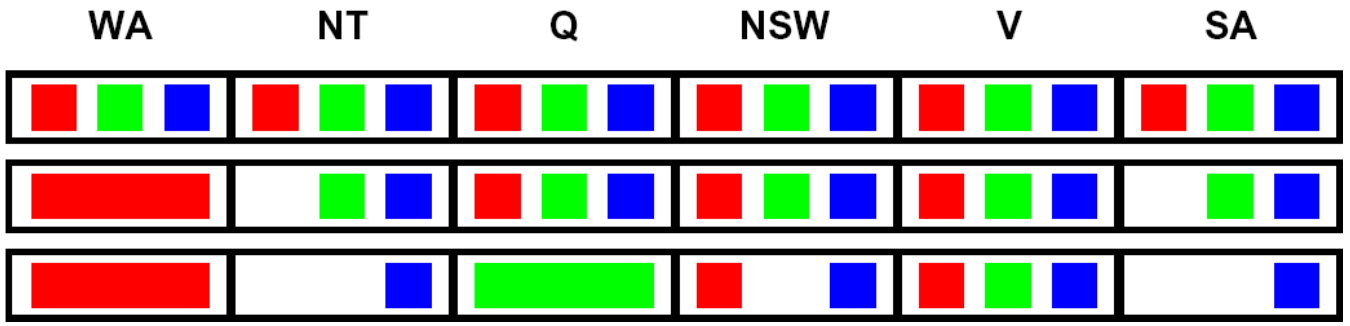
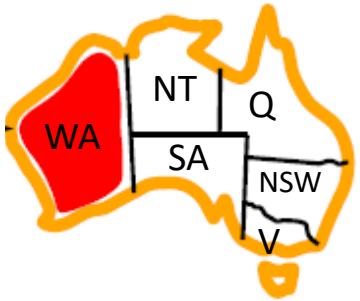


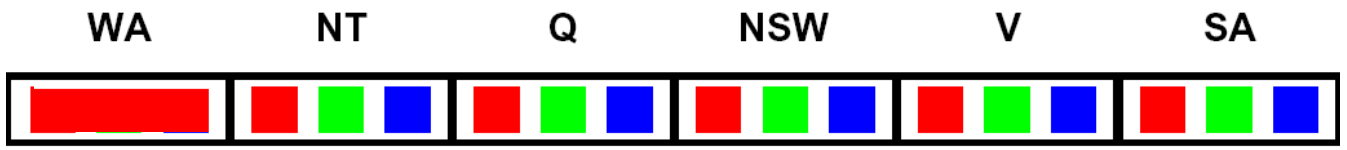




- NT and SA cannot both be blue!
- Why didn’t we detect this yet?
- *Constraint propagation:* reason from constraint to constraint

# Consistency of A Single Arc

- An arc X → Y is consistent iff for *every* x in the tail there is *some* y in the head which could be assigned without violating a constraint



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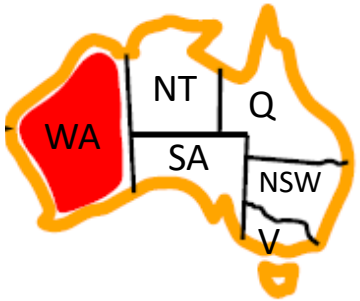


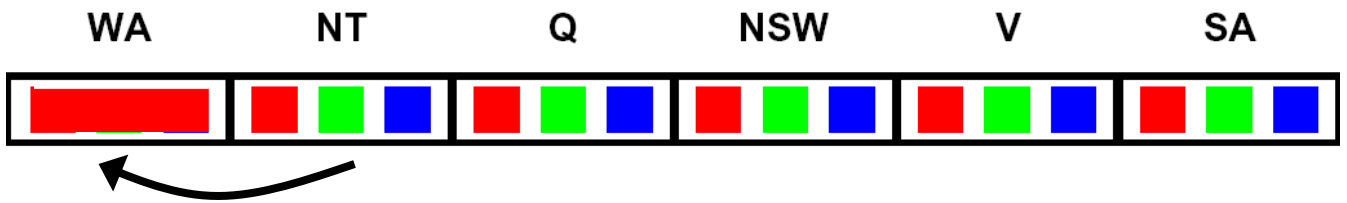

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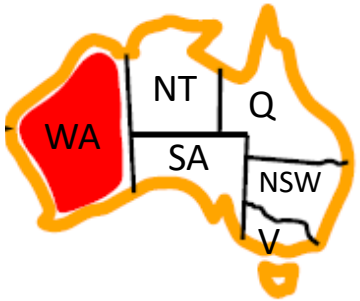


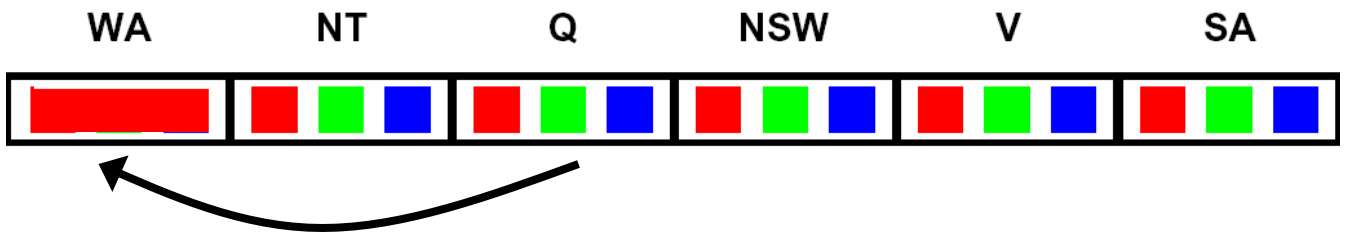

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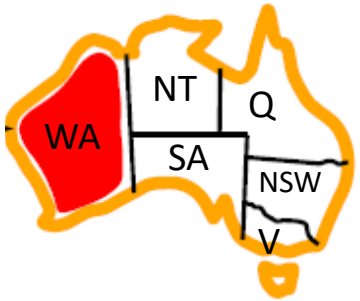


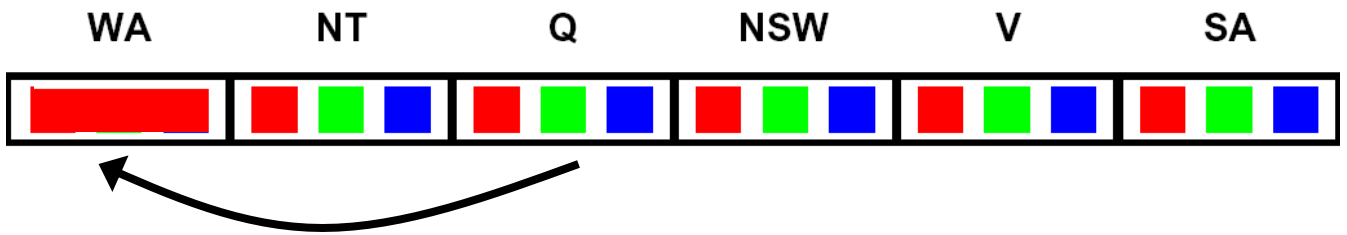


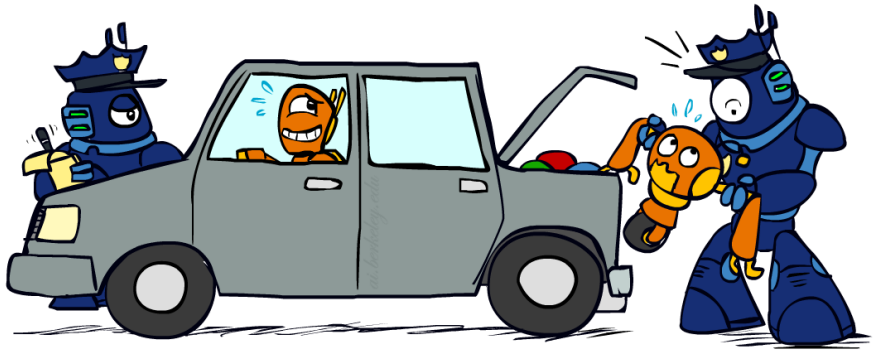

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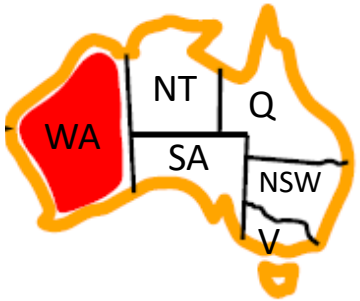


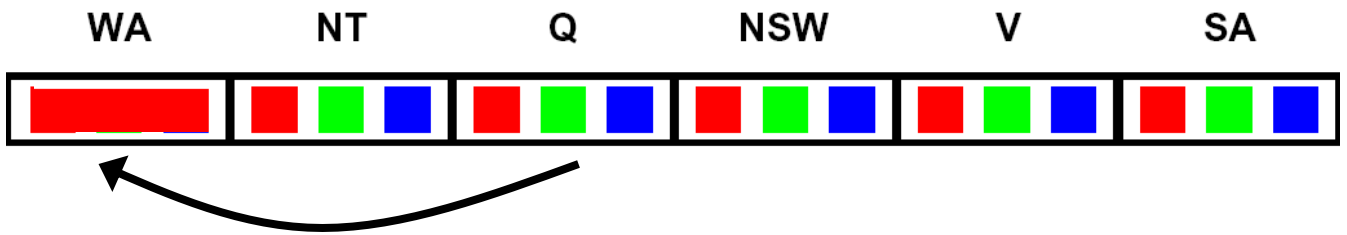


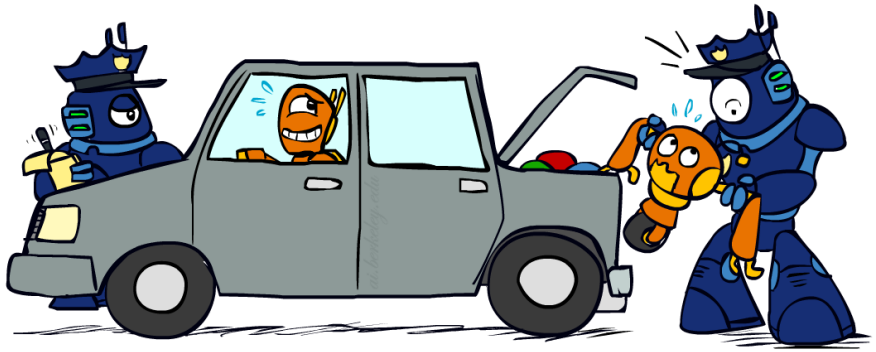

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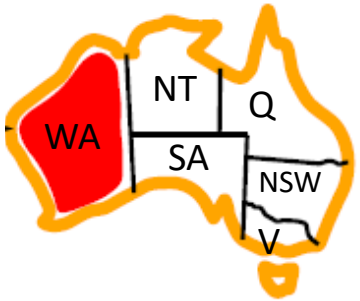
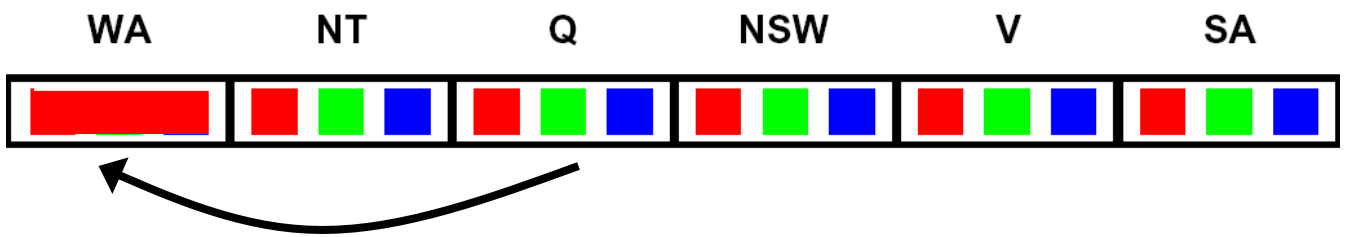
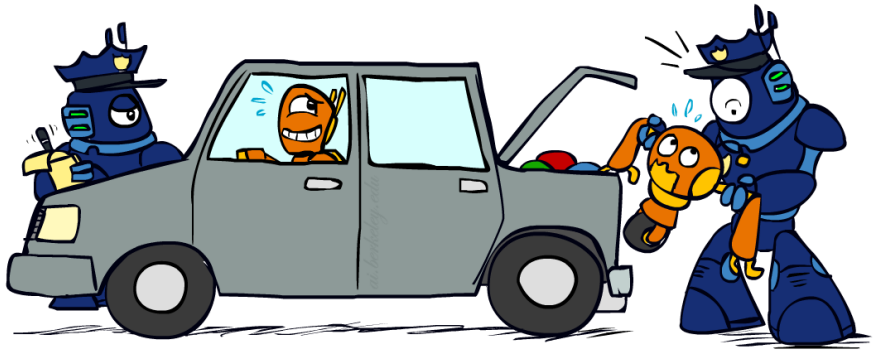


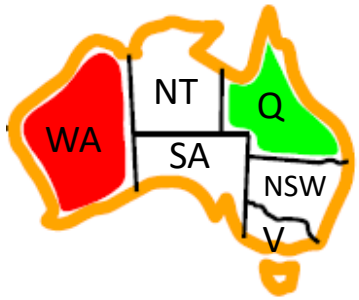



*Delete from the tail!*

- Forward checking: Enforcing consistency of arcs pointing to each new assignment

# Arc Consistency of an Entire CSP

- A simple form of propagation makes sure all arcs are consistent:



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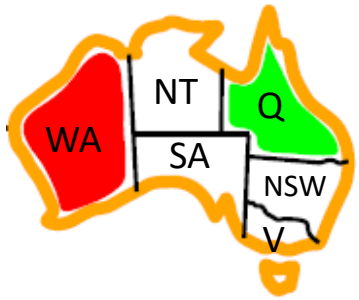


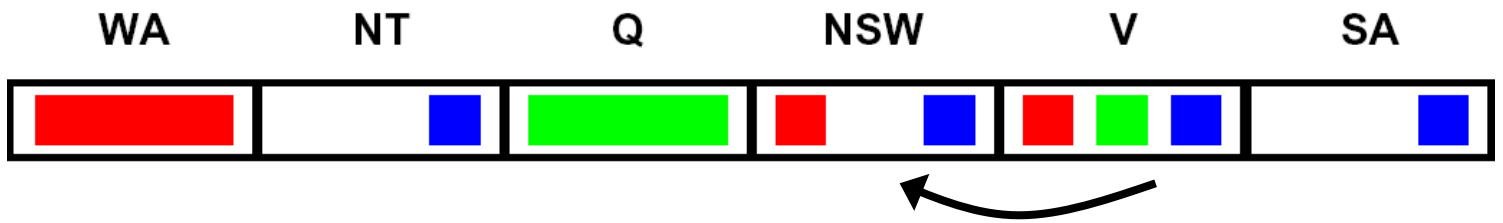


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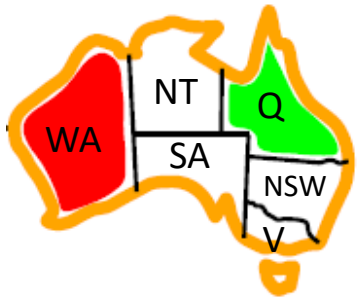


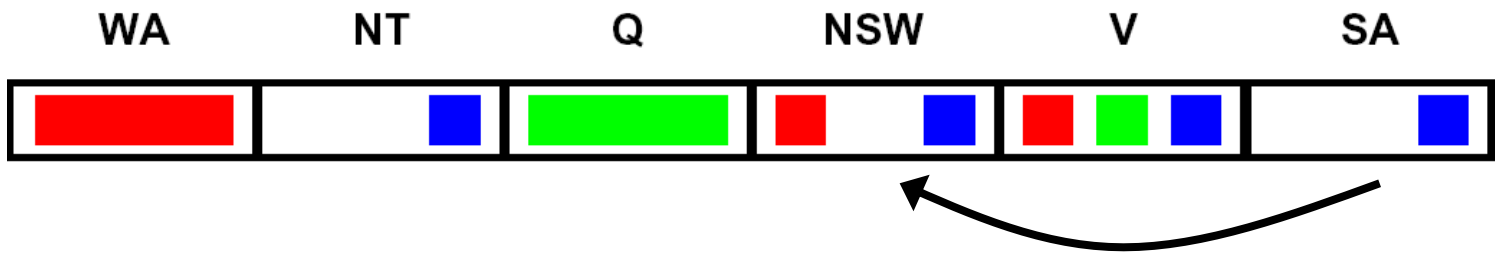


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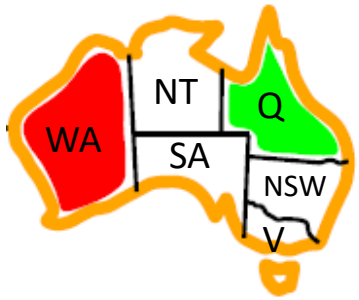


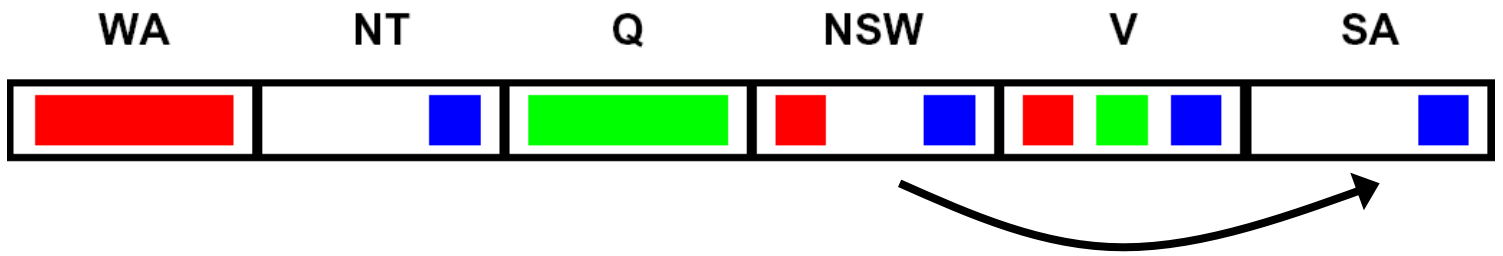


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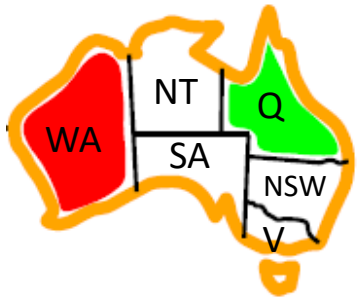


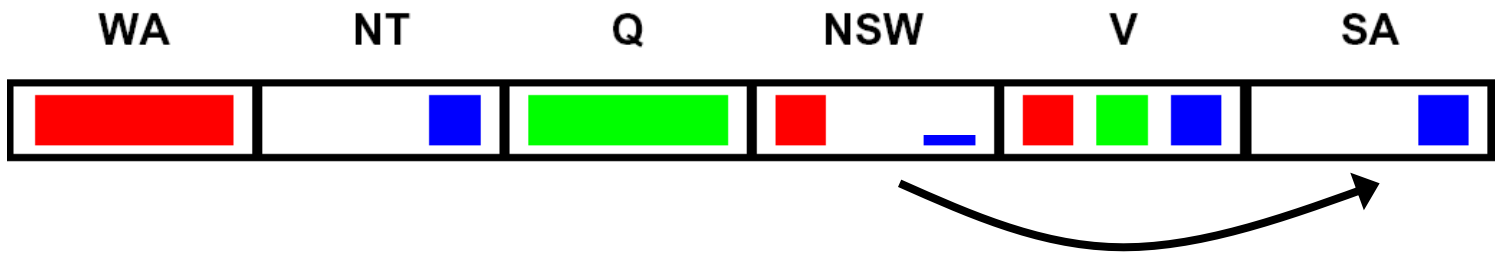


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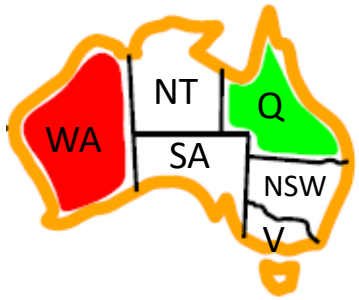


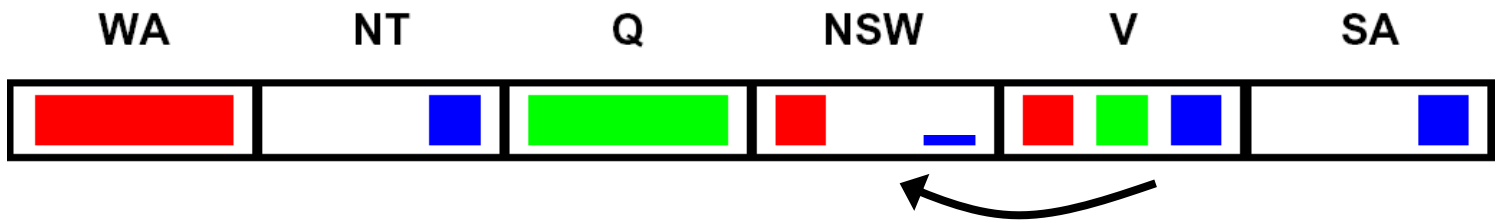


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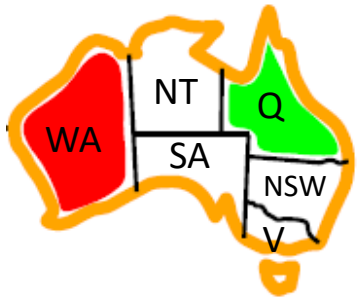


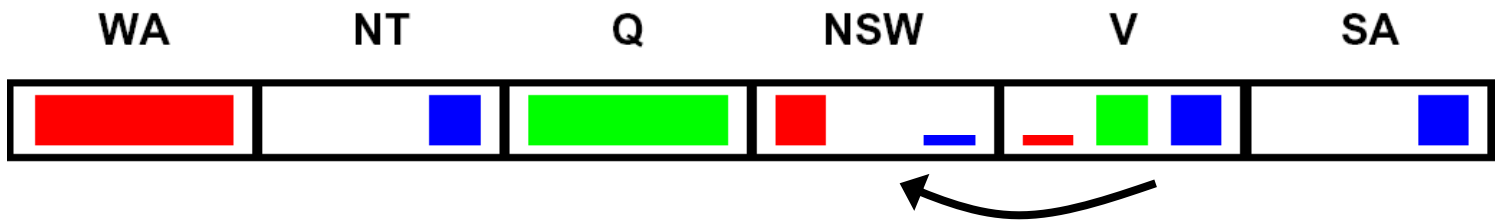


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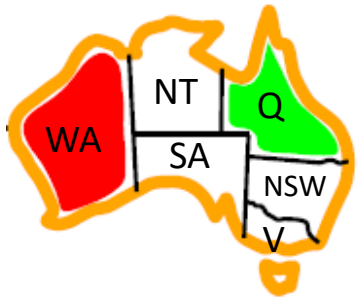


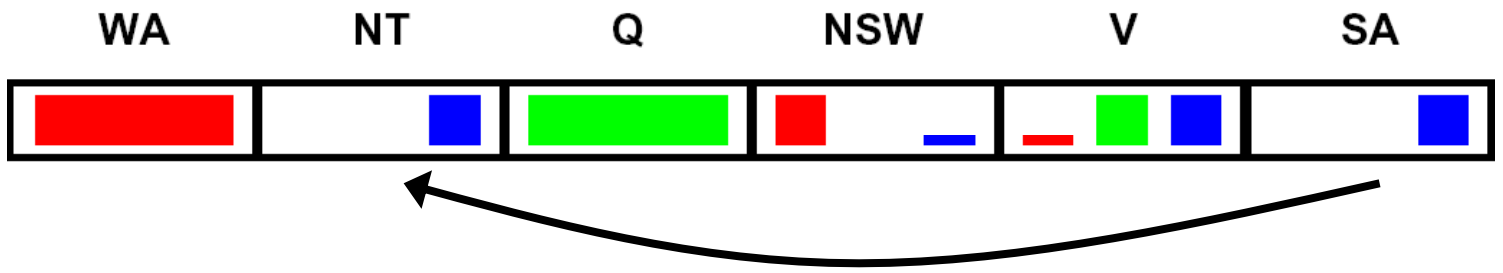


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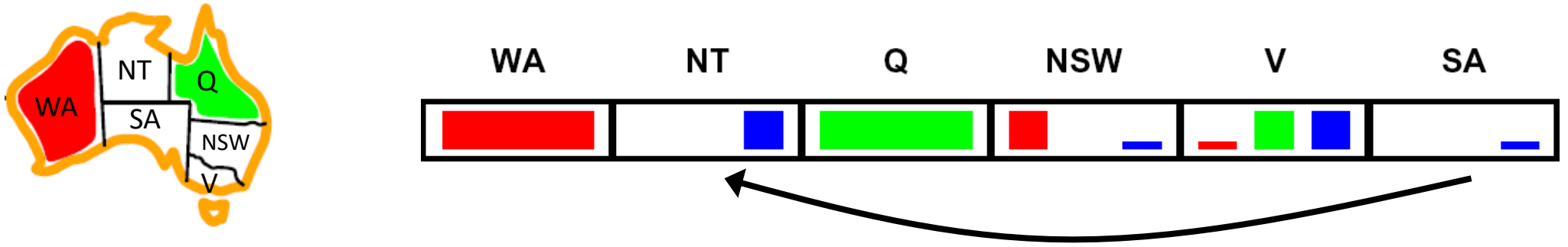



- Important: If X loses a value, neighbors of X need to be rechecked!
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment
- What's the downside of enforcing arc consistency?

*Remember: Delete from the tail!*

# Enforcing Arc Consistency in a CSP

```
function AC-3( csp) returns the CSP, possibly with reduced domains
  inputs: csp, a binary CSP with variables {X_1, X_2, ..., X_n}
  local variables: queue, a queue of arcs, initially all the arcs in csp

  while queue is not empty do
    (X_i, X_j) ← REMOVE-FIRST(queue)
    if REMOVE-INCONSISTENT-VALUES(X_i, X_j) then
      for each X_k in NEIGHBORS[X_i] do
        add (X_k, X_i) to queue

---

function REMOVE-INCONSISTENT-VALUES( X_i, X_j) returns true iff succeeds
  removed ← false
  for each x in DOMAIN[X_i] do
    if no value y in DOMAIN[X_j] allows (x,y) to satisfy the constraint X_i ↔ X_j
      then delete x from DOMAIN[X_i]; removed ← true
  return removed
```

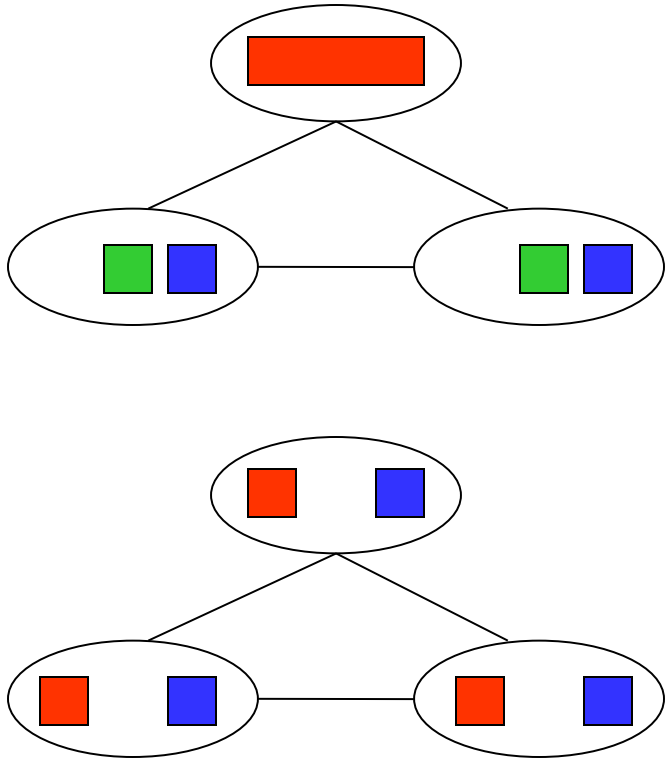
- Runtime: $O(n^2d^3)$, can be reduced to $O(n^2d^2)$
- … but detecting all possible future problems is NP-hard – why?

# Video of Demo Coloring – Backtracking with Forward Checking – Complex Graph

# Video of Demo Coloring – Backtracking with Arc Consistency – Complex Graph

# Limitations of Arc Consistency

- After enforcing arc consistency:
  - Can have one solution left
  - Can have multiple solutions left
  - Can have no solutions left (and not know it)



*What went wrong here?*

[Demo: coloring -- forward checking]
[Demo: coloring -- arc consistency]

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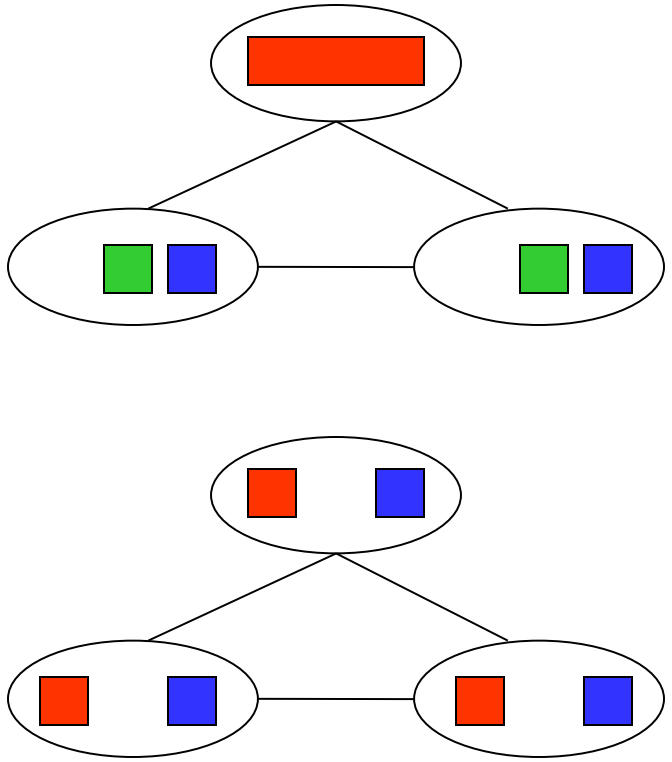
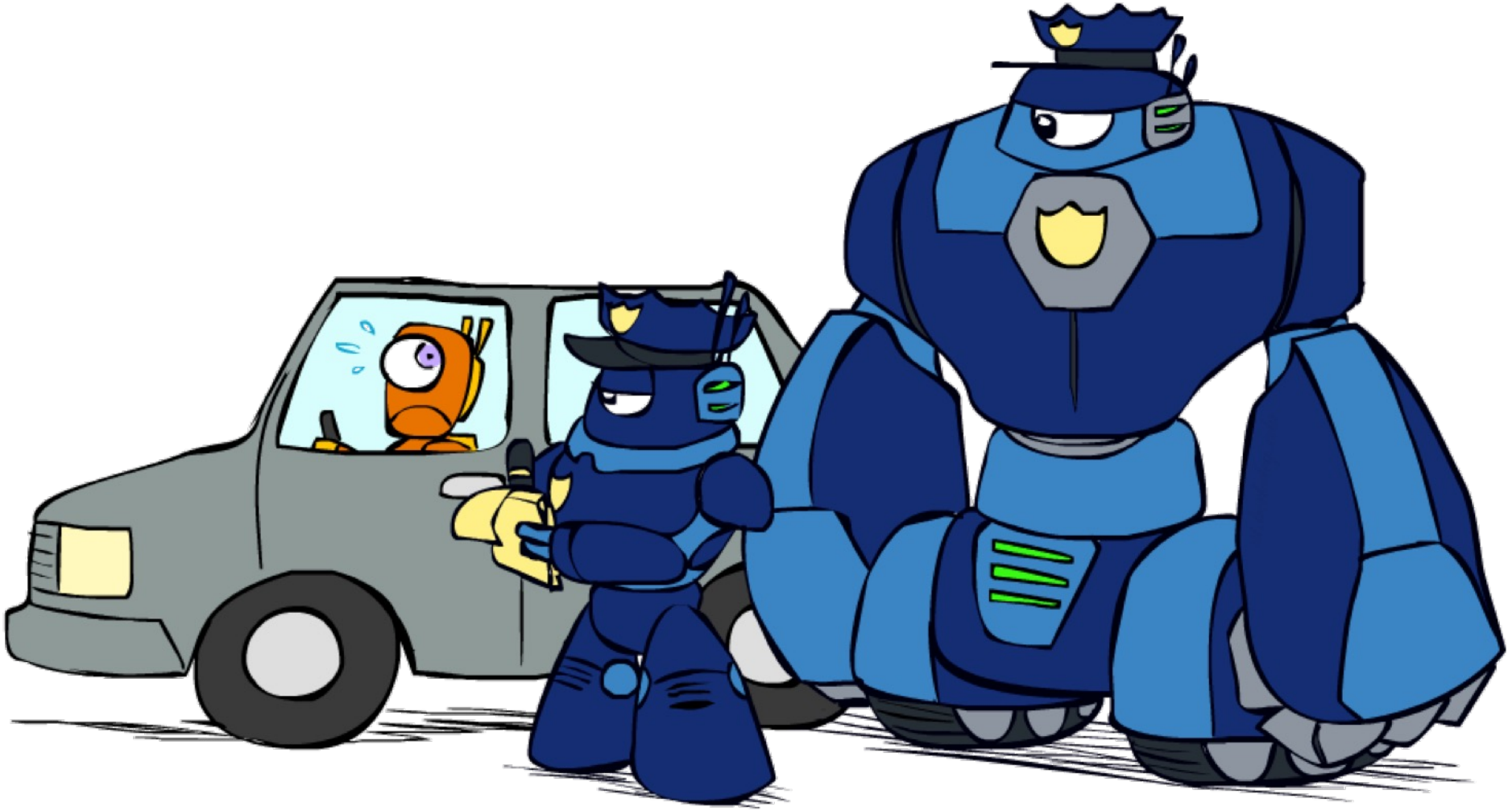


*What went wrong here?*

- Arc consistency still runs inside a backtracking search!

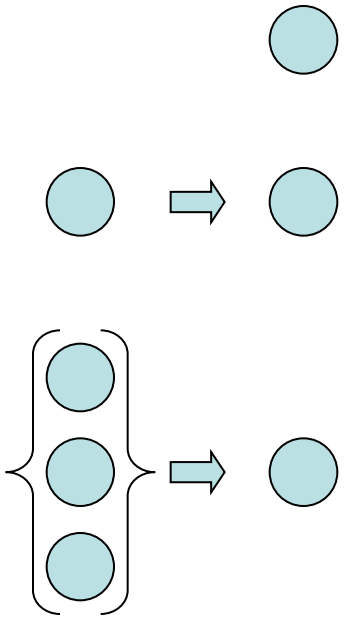
[Demo: coloring -- forward checking]
[Demo: coloring -- arc consistency]

# K-Consistency

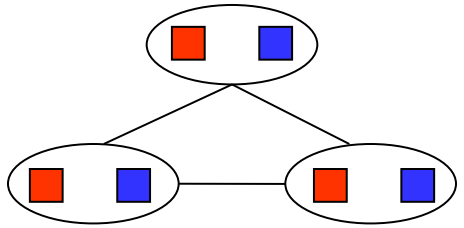


# K-Consistency

- Increasing degrees of consistency
  - 1-Consistency (Node Consistency): Each single node's domain has a value which meets that node's unary constraints
  - 2-Consistency (Arc Consistency): For each pair of nodes, any consistent assignment to one can be extended to the other
  - K-Consistency: For each k nodes, any consistent assignment to k-1 can be extended to the $k^{th}$ node.



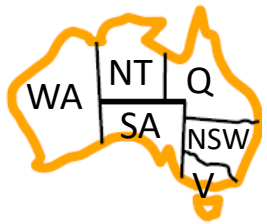
- Higher k more expensive to compute
- (You need to know the k=2 case: arc consistency)



# Ordering

# Ordering: Minimum Remaining Values

- Variable Ordering: Minimum remaining values (MRV):
  - Choose the variable with the fewest legal left values in its domain



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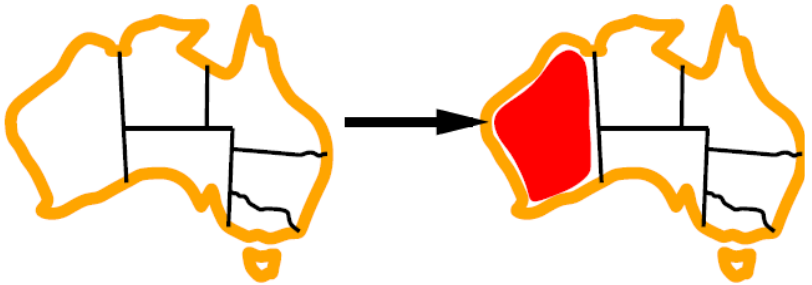

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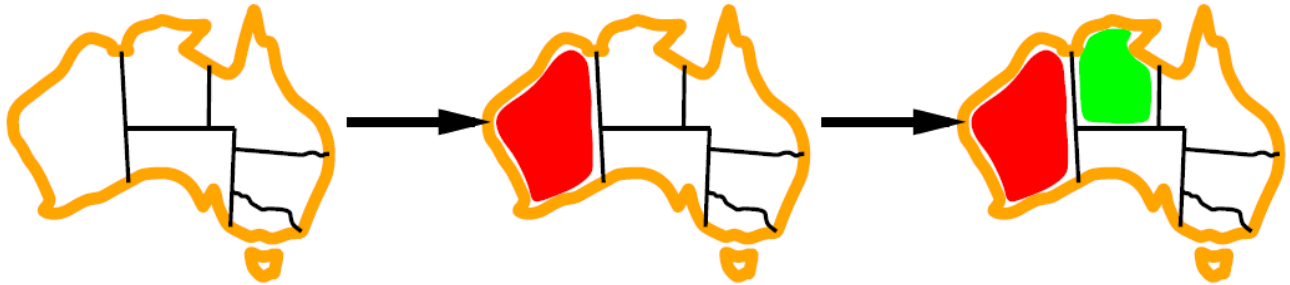

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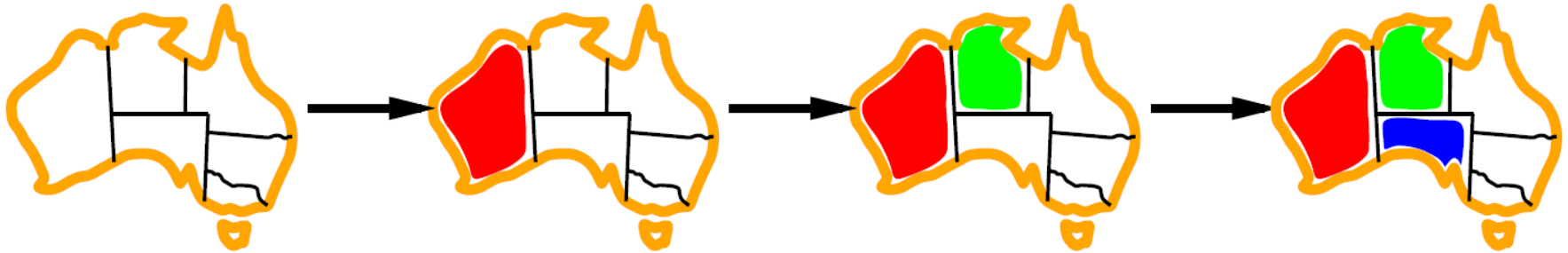

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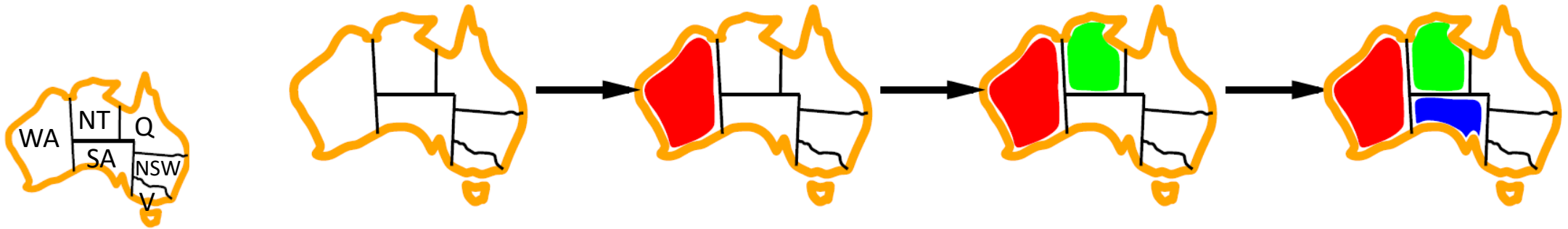


- Why min rather than max?

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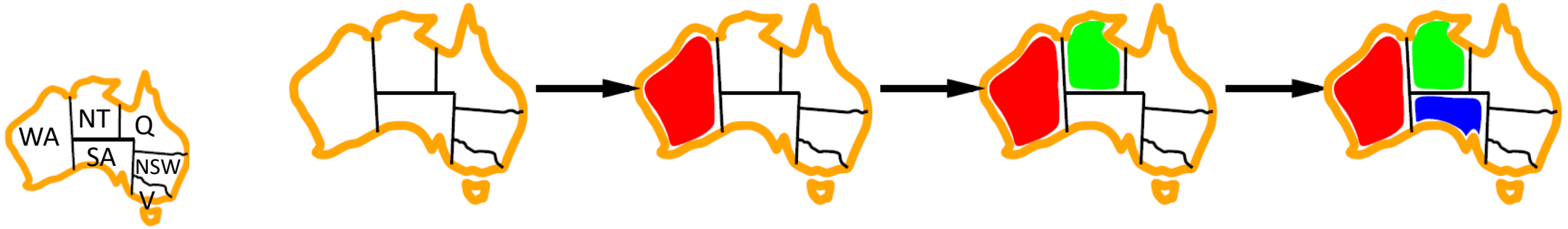


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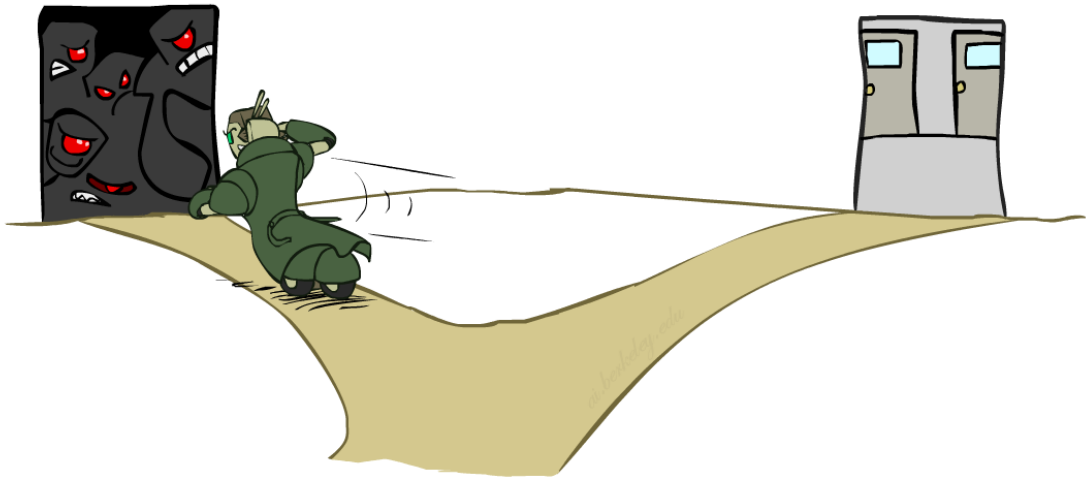

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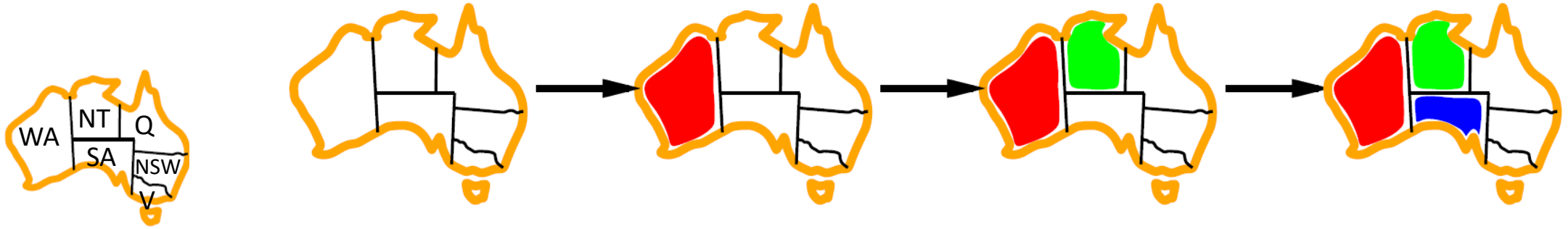
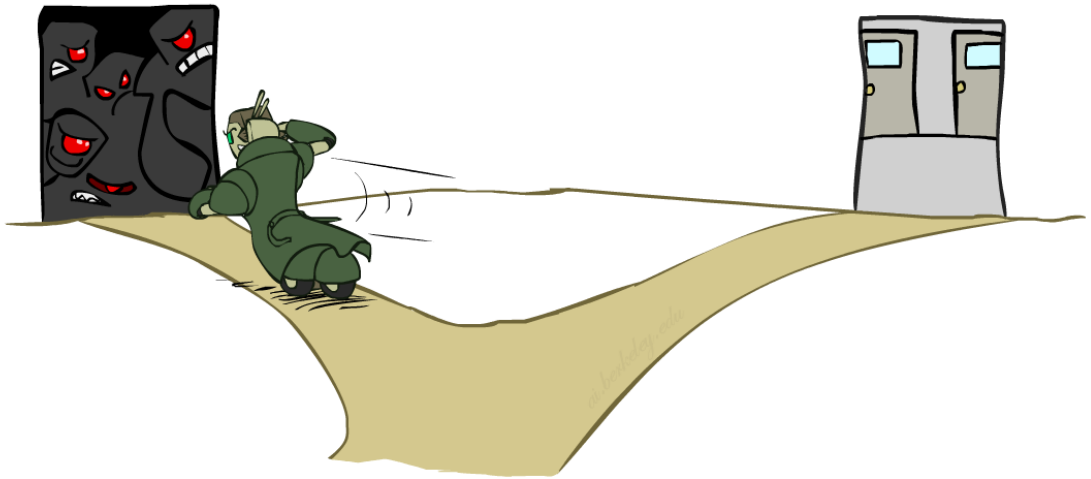



- Why min rather than max?
- Also called “most constrained variable”
- “Fail-fast” ordering



# Ordering: Least Constraining Value

- Value Ordering: Least Constraining Value
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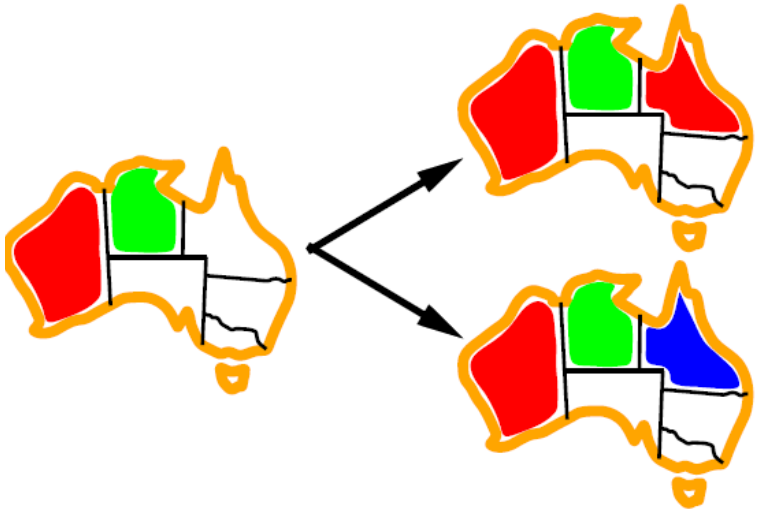
  - Note that it may take some computation to determine this!  (E.g., rerunning filtering)



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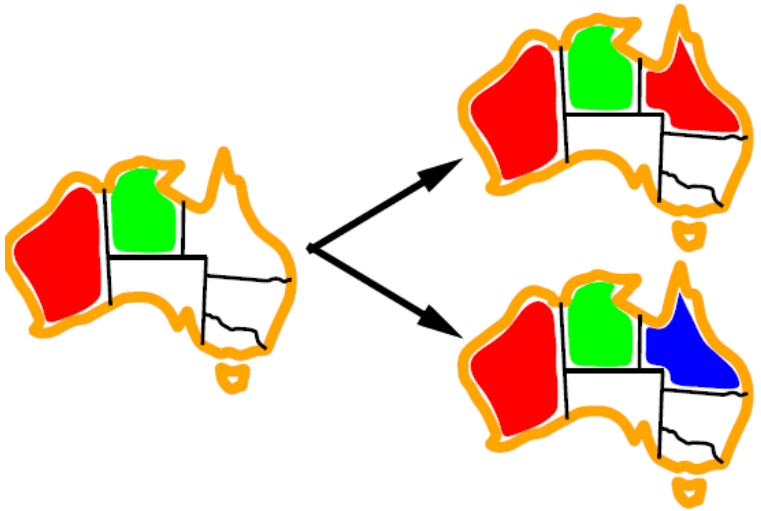

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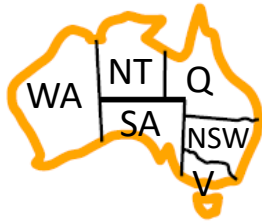

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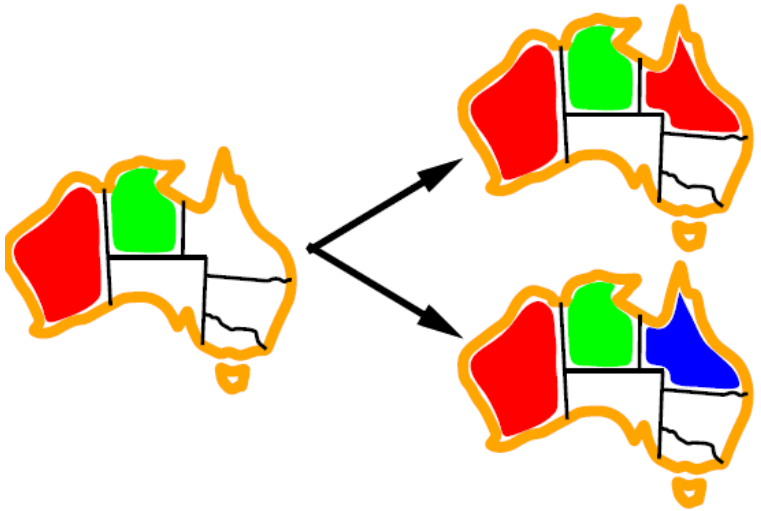

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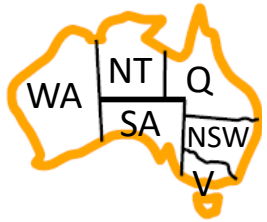


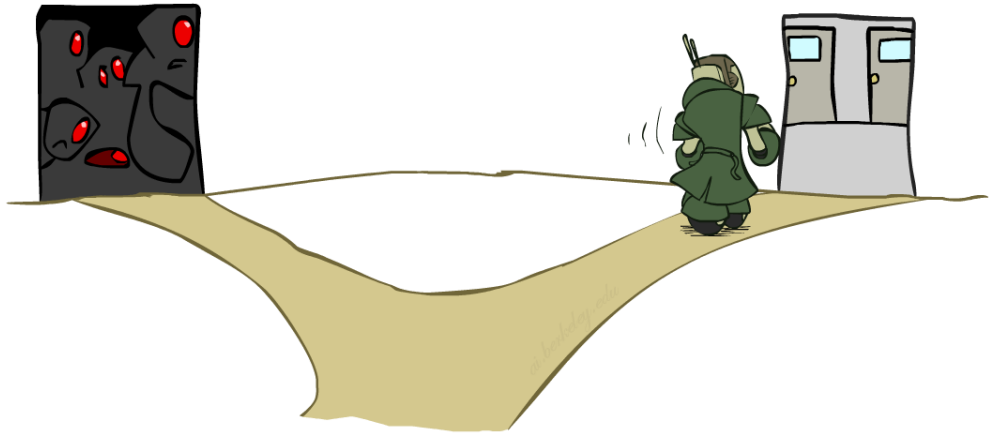

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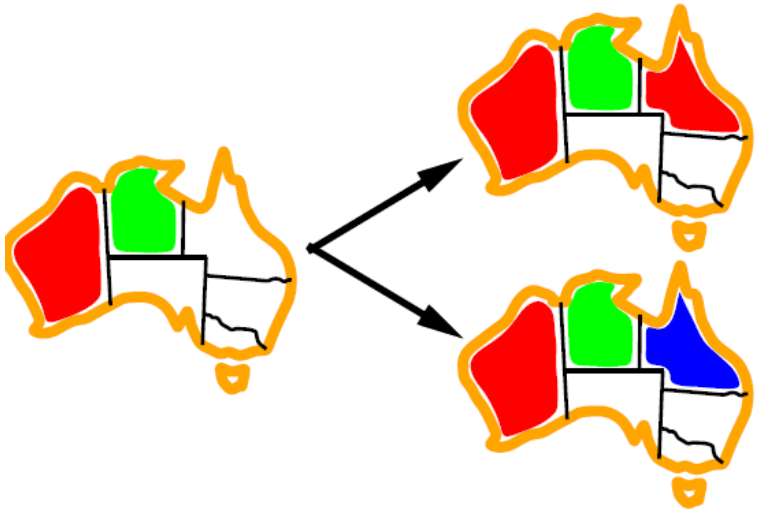
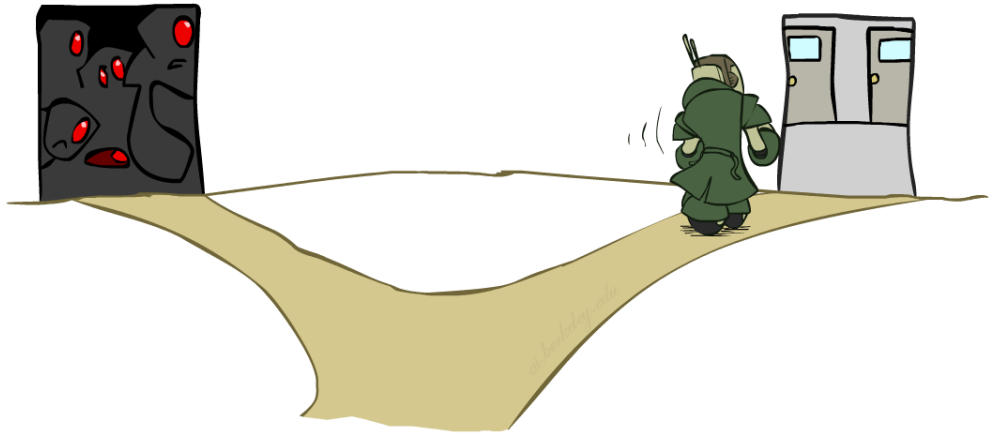


- Why least rather than most?

- Combining these ordering ideas makes 1000 queens feasible



# Demo: Coloring -- Backtracking + Forward Checking + Ordering