SIAM J. CONTROL OPTIM.
Vol. 50, No. 1, pp. 490–506

A CONVEX CONDITION FOR ROBUST STABILITY ANALYSIS VIA POLYHEDRAL LYAPUNOV FUNCTIONS*

R. AMBROSINO†, M. ARIOLA†, AND F. AMATO‡

Abstract. In this paper we study the robustness analysis problem for linear continuous-time systems subject to parametric time-varying uncertainties making use of piecewise linear (polyhedral) Lyapunov functions. A given class of Lyapunov functions is said to be "universal" for the uncertain system under consideration if the robust stability of the system is equivalent to the existence of a Lyapunov function belonging to the class. In the literature it has been shown that the class of polyhedral functions is universal, while, for instance, the class of quadratic functions is not. This fact justifies the effort of developing efficient algorithms for the construction of polyhedral Lyapunov functions. In this context, we provide a low computational cost procedure, based on a novel convex condition, for the construction of a polyhedral Lyapunov function. In the section on the numerical examples, we consider some benchmark problems for the robust stability analysis and we show that the proposed low computational cost approach, though only sufficient, is less conservative than all the other approaches presented so far in the literature.

Key words. linear uncertain systems, robust stability, convex relaxation, polyhedral Lyapunov functions

AMS subject classifications. 93D05, 93D09, 93D20

DOI. 10.1137/110796479

1. Introduction.

Robust control was widely investigated by the automatic control community especially during the 1980s and 1990s. Several different approaches have been proposed both for robust stability analysis and for the design of robust control systems (see [1] and [16] for a survey). In this paper we focus on the robust stability analysis problem for linear continuous-time systems subject to parametric time-varying uncertainties. Typically, this problem is tackled by means of quadratic Lyapunov functions (see, for instance, [4, 10, 20]). This approach has been shown to be conservative with respect to approaches using other types of Lyapunov functions [27]. Different classes of nonquadratic Lyapunov function have been proposed in the literature in the last few years: piecewise quadratic Lyapunov functions have been considered for both linear systems with time-varying perturbations [26] and switching linear systems [19]; in [27] the use of homogeneous polynomial Lyapunov functions (HPLFs) to prove robust stability of linear systems with time-varying uncertainties has been considered. The use of polyhedral Lyapunov functions for robust stability analysis was first proposed in [11, 12]. Polyhedral Lyapunov functions are positively homogeneous functions whose level surfaces are the boundary of a polytope. In [21] it is proved that this class of functions is universal for the robustness analysis problem involving linear systems subject to parametric uncertainties, in the sense that robust stability of the system is equivalent to the existence of a polyhedral Lyapunov function belonging to this class. In [8] it is also shown that these functions are universal for the

*Received by the editors May 25, 2010; accepted for publication (in revised form) November 17, 2011; published electronically February 14, 2012.
http://www.siam.org/journals/sicon/50-1/79647.html
†Dipartimento per le Tecnologie, Università degli Studi di Napoli Parthenope, Centro Direzionale di Napoli, Isola C4, 80143 Napoli, Italy (ambrosino@uniparthenope.it, marco.ariola@uniparthenope.it).
‡School of Computer Science and Biomedical Engineering, Università degli Studi Magna Græcia di Catanzaro, Via T. Campanella 115, 88100 Catanzaro, Italy (amato@unicz.it).

stabilization problem. More recently, in [6], it has been proved that the HPLFs, or "smooth" polyhedral Lyapunov functions, are a universal class both for the stability and the stabilization problems.

One problem concerning the use of both polyhedral functions and HPLFs in the robust stability context consists of the development of an efficient numerical approach to find, for a given uncertain system, a Lyapunov function belonging to these classes. In [7], the author proposes a procedure for constructing a polyhedral Lyapunov function for a linear discrete-time system; this procedure is then extended [8, 5] to continuous-time systems via a suitable "Euler approximating system." More recently some convex or quasiconvex formulations of this problem have been proposed. In [22, 23, 24] the author provides a procedure for constructing a polyhedral Lyapunov function for linear continuous-time systems based on linear programming techniques. In [15, 13, 14], the authors propose a linear matrix inequality (LMI) formulation for the construction of HPLFs for linear systems with time-varying structured uncertainties. However, both these formulations provide only sufficient conditions for the robust stability of uncertain systems (except in some particular cases, as shown in [13]), even if these classes of Lyapunov functions are universal.

The starting point of this paper is the necessary and sufficient condition for the existence of a polyhedral Lyapunov function proposed in [2]. This condition is nonlinear and in [2] it is implemented via a high computational cost procedure, based on nonconvex optimization methods. The novel contribution of this paper consists of a convex relaxation of the necessary and sufficient condition in [2]. Basing on this relaxation, we propose an algorithm for the construction of polyhedral Lyapunov functions consisting of the iterative solution of a linear optimization problem. As in [13, 22, 23, 24], the proposed procedure introduces some conservativeness in the robustness analysis process. Hence, to show the effectiveness of the proposed methodology, our approach is used in the same context as the examples presented in [15], [23], and [24]; in all cases, we show that it leads to less conservative results.

The paper is organized as follows. In section 2 we give some preliminary definitions and results concerning polytopes, and the problem we deal with is precisely stated. In section 3 the main results of our work are provided. In section 4 some numerical examples, concerning second- and third-order linear uncertain systems, illustrate the effectiveness of the proposed approach; moreover the computational burden of our methodology is discussed. Finally, some conclusions are drawn in section 5.

2. Preliminaries. In this paper we deal with the stability of a linear system subject to uncertain parameters

$$\dot{x}(t) = A(p)x(t), \tag{2.1}$$

where $A(\cdot) : \mathcal{R} \subset \mathbb{R}^q \to \mathbb{R}^{n\times n}$ and $\mathcal{R}$ is a box, i.e.,

$$\mathcal{R} := [\underline{p}_1, \overline{p}_1] \times [\underline{p}_2, \overline{p}_2] \times \cdots \times [\underline{p}_q, \overline{p}_q].$$

In what follows we shall assume the following:

- The vector-valued function

$$p(\cdot) = \begin{pmatrix} p_1(\cdot) & p_2(\cdot) & \cdots & p_q(\cdot) \end{pmatrix}^T$$

is any Lebesgue measurable function $p(\cdot) : [0, +\infty] \to \mathcal{R}$.

- The matrix-valued function $A(\cdot)$ depends multiaffinely on the parameter vector p, that is,

$$A(p) = \sum_{i_1,\dots,i_q \in \{0,1\}} A_{i_1,\dots,i_q} p_1^{i_1} \cdots p_q^{i_q}. \tag{2.2}$$

  Note that the matrix $A(p)$ is an affine matrix-valued function of the components p_i of p when all other components of p are held fixed.

Remark 1. The structure assumed for the system matrices in (2.2) captures many cases of practical interest and, in particular, the affine dependence on parameters; for more details see [1].

DEFINITION 1 (robust stability). *System* (2.1) *is said to be robustly stable if for any Lebesgue measurable vector-valued function* $p(\cdot) : [0, +\infty] \to \mathcal{R}$*, the resulting linear time-varying system*

$$\dot{x}(t) = A\big(p(t)\big)x(t)$$

is exponentially stable.

We focus on the problem of determining some conditions guaranteeing the robust stability of system (2.1). In order to study these problems, we will make use of the class of (symmetrical) polyhedral Lyapunov functions, which are piecewise linear functions of the form

$$V(x) = \|Q^T x\|_\infty, \tag{2.3}$$

where $Q \in \mathbb{R}^{n\times m}$ is a full row rank matrix and, given a vector $v \in \mathbb{R}^n$, $\|v\|_\infty := \max_{i=1,\dots,n} |v_i|$ denotes the infinity norm of v. In [12] and [21] it has been shown that the class of symmetric polyhedral Lyapunov function is universal for the robust stability analysis problem.

2.1. Notions on polytopes. In the following we provide some preliminary definitions and results on linear algebra and polytopes which will be useful for stating the main results of the paper.

If we deal with a finite set of points, say, $K = \{x^{(1)}, \dots, x^{(l)}\} \subset \mathbb{R}^n$, the *convex hull* of K turns out to be a *polytope*, whose *dimension* [28, p. 5] is given by the dimension of the affine hull of K, i.e.,

$$\operatorname{rank} \begin{bmatrix} x^{(2)} - x^{(1)} & x^{(3)} - x^{(1)} & \dots & x^{(l)} - x^{(1)} \end{bmatrix}.$$

Moreover, as stated in the next lemma, the set of vertices of a given polytope $\mathcal{P}$ is a subset of K.

LEMMA 1 (see [28]). *Given a polytope defined as convex hull of* $K = \{x^{(1)} \dots x^{(l)}\} \subset \mathbb{R}^n$*, the vertices of the polytope are the points* $x^{(i)} \in K$ *which satisfy the following property:*

$$x^{(i)} \notin \operatorname{conv}\left(K - \{x^{(i)}\}\right).$$

Remark 2. Note that given a collection of symmetric points $K = \{x^{(1)}, \dots, x^{(2l)}\}$, $x^{(i)} = -x^{(l+i)}, i = 1, \dots, l$, if $x^{(i)}$ is a vertex of $\operatorname{conv}(K)$, then also $x^{(l+i)} = -x^{(i)}$ is a vertex of $\operatorname{conv}(K)$.

In this paper we will focus on polytopes symmetrical with respect to the origin of $\mathbb{R}^n$. To this regard note that given any symmetrical polytope $\mathcal{P} \subset \mathbb{R}^n$, there always exists a full row rank matrix $Q \in \mathbb{R}^{n\times m}$, $m \geq n$, such that the polytope $\mathcal{P}$ can be alternatively defined as (see [25, p. 6])

$$\mathcal{P} = \wp(Q) := \{x \in \mathbb{R}^n : \|Q^T x\|_\infty \leq 1\}. \tag{2.4}$$

Therefore a given symmetric polytope $\mathcal{P}$ admits two different equivalent descriptions: the first as a convex hull of its vertices, the other in the matrix form (2.4). The algorithm in [3], implemented in the MATLAB routine *convhulln*, enables us to find the matrix Q defining a polytope starting from the polytope vertices.

In the following, given a symmetric polytope $\wp(Q)$, we denote by $x_Q^{(i)}$, $i = 1\dots 2l$, the vertices of $\wp(Q)$ and assume that $x_Q^{(i)} = -x_Q^{(i+l)}$ for $i = 1\dots l$. Moreover we denote by

- $q_{i,h}$ with $h = 1\dots s_i$, the s_i columns of Q such that $q_{i,h}^T x_Q^{(i)} = 1$; these represent the hyperfaces of the polytope where the vertex $x_Q^{(i)}$ lies;
- $x_{q_i}^{(j)}$ with $j = 1\dots k_i$, the k_i vertices of $\wp(Q)$ such that $q_i^T x_{q_i}^{(j)} = 1$; these represent the vertices of the polytope that lie on the hyperplane q_i.

Using this notation, we can now define the following concepts.

DEFINITION 2 (set of neighbors of a vertex). *Given a polytope $\wp(Q)$ we define the set of neighbors of a vertex $x_Q^{(i)}$ belonging to the hyperface $q_{i,h}$ with $h = 1\dots s_i$ as*

$$\mathcal{N}(x_Q^{(i)}, q_{i,h}) = \left\{ x_Q^{(j)}\ j \neq i : q_{i,h}^T x_Q^{(j)} = 1 \right\}.$$

The set of neighbors of a vertex $x_Q^{(i)}$ is

$$\mathcal{N}(x_Q^{(i)}) = \bigcup_{h=1,\dots,s_i} \mathcal{N}(x_Q^{(i)}, q_{i,h}).$$

In the following we denote with $x_Q^{(i,h,j)}$ the jth element of $\mathcal{N}(x_Q^{(i)}, q_{i,h})$.

Remark 3. Note that the cardinality of $\mathcal{N}(x_Q^{(i)}, q_{i,h})$ for $h = 1\dots s_i$ is greater than or equal to $n-1$. However, we assume, without loss of generality, that the number of elements in each set $\mathcal{N}(x_Q^{(i)}, q_{i,h})$ is always $n-1$. This is the minimum number of vertices that together with $x_Q^{(i)}$ allows us to define the hyperface $q_{i,h}$. This assumption is always true in the two-dimensional case and, given a vertex $x_Q^{(i)}$, we denote the element of $\mathcal{N}(x_Q^{(i)}, q_{i,h})$ with $x_Q^{(i,h)}$.

DEFINITION 3 (isolation set of a polytope). *Given a polytope $\wp(Q)$, we define the isolation set $\mathcal{I}(\wp(Q))$ of a polytope $\wp(Q)$ as any possible subset of the vertices of $\wp(Q)$ containing for each hyperplane q_i with v_i vertices at least $v_i - 1$ of them. An isolation set is said to be minimal iff*

$$\forall q_i \quad \exists!\, x_{q_i}^{(j)} \notin \mathcal{I}(\wp(Q)), \quad j = 1,\dots,v_i.$$

DEFINITION 4 (isolated vertex of a polytope). *Given a polytope $\wp(Q)$ and an isolation set $\mathcal{I}(\wp(Q))$, we define the isolated vertex of $\wp(Q)$ as any*

$$x_Q^{(i)} \notin \mathcal{I}(\wp(Q)).$$

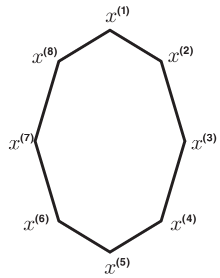

FIG. 2.1. *The isolation sets for the polytope* $\wp(Q)$ *are* $\mathcal{I}(\wp(Q)) = \{x^{(1)}, x^{(3)}, x^{(5)}, x^{(7)}\}$ *and* $\mathcal{I}(\wp(Q)) = \{x^{(2)}, x^{(4)}, x^{(6)}, x^{(8)}\}$.

Remark 4. Given a polytope $\wp(Q)$ and an isolation set $\mathcal{I}(\wp(Q))$, all the neighbors of an isolated vertex $x_Q^{(i)}$ are in $\mathcal{I}(\wp(Q))$. Indeed, for each neighbor $x_Q^{(j)}$ of a vertex $x_Q^{(i)}$, there exists a hyperface $\tilde{q}$ of the polytope that contains both the vertices, i.e., such that $\tilde{q}\,x_Q^{(i)} = \tilde{q}\,x_Q^{(j)} = 1$. Then, our claim is proved noting that according to Definitions 3 and 4, each hyperface of the polytope contains 0 or 1 isolated vertex.

Remark 5. It is difficult in general to find a minimal isolation set in the n-dimensional case. However, for the two-dimensional case, the minimal isolation set is univocally determined imposing a vertex as an isolated vertex. For example, given the polytope in Figure 2.1, if we choose x_1 as an isolated vertex, all its neighbors, i.e., the vertices $x^{(2)}$ and $x^{(8)}$, are inside the isolation set (see Remark 4). Then, according to the definition of minimal isolation set, the vertices $x^{(3)}$ and $x^{(7)}$ are isolated vertices. Going on with this constructive method, we conclude that the isolation set is $\mathcal{I}(\wp(Q)) = \{x^{(2)}, x^{(4)}, x^{(6)}, x^{(8)}\}$ and the isolated vertices are $x^{(1)}, x^{(3)}, x^{(5)}, x^{(7)}$.

It is straightforward to notice that the minimal isolated sets are two, they are disjoined, and their union contains all the vertices of the polytope.

2.2. Quadratic and polyhedral stability. Let us consider the following definitions.

DEFINITION 5 (quadratic stability [20, 4, 10]). *System* (2.1) *is said to be quadratically stable if there exists a quadratic Lyapunov function in the form* $x^T Q x$*, with* Q *symmetric positive definite, such that its derivative along the solutions of system* (2.1) *is negative definite for all* $p \in \mathcal{R}$.

In order to state the definition of polyhedral stability, given a generic system in the form $\dot{x} = f(x)$ and a Lyapunov function $V(x)$, we recall the definition of the Dini (upper) derivative [18] of $V(x)$ along the solutions of the system

$$\dot{V}(x) = \lim_{\tau \to 0^+} \sup \left. \frac{V(x + \tau \dot{x}) - V(x)}{\tau} \right|_{\dot{x} = f(x)} .$$

Such a definition returns the classical derivative when $V(x)$ is continuously differentiable but also enables us to treat the more general case in which the Lyapunov function is not differentiable everywhere (as is the case of polyhedral functions).

DEFINITION 6 (polyhedral stability [8]). *System* (2.1) *is said to be polyhedrally stable if there exists a polyhedral Lyapunov function in the form* (2.3) *such that its Dini derivative along the solutions of system* (2.1) *is negative definite for all* $p \in \mathcal{R}$.

Both quadratic stability and polyhedral stability guarantee robust stability of system (2.1); however, an uncertain system can be robustly stable even if it is not quadratically stable. Hence, if we focus only on the class of quadratic Lyapunov functions we make a conservative choice.

To clarify this point, recall that according to [8], a given class of Lyapunov functions is said to be "universal" for system (2.1) if the existence of a Lyapunov function which proves robust stability of the uncertain system implies the existence of a Lyapunov function belonging to the class which does the same job.

Following this definition, the class of quadratic Lyapunov functions is not universal for systems in the form (2.1). For example in [9, p. 73], an uncertain system depending on one parameter is shown to be robustly stable by using a *piecewise quadratic* Lyapunov function but not quadratically stable. For an interesting discussion on this issue see the seminal paper [11].

Conversely, as shown in [12, 8], the class of polyhedral Lyapunov functions is universal. This consideration justifies the effort of developing efficient algorithms for the construction of polyhedral Lyapunov functions, which is the topic discussed in section 3.

Finally, the following result provides a necessary and sufficient condition for polyhedral stability of system (2.1).

THEOREM 1 (polyhedral stability theorem [2]). *System* (2.1) *is polyhedrally stable iff there exists a polytope $\wp(Q)$ of dimension n such that the following condition holds for all $i = 1, \ldots, l$: $h = 1, \ldots, s_i$:*

$$\max_{p \in \mathrm{vert}(\mathcal{R})} q_{i,h}^T \, A(p) \, x_Q^{(i)} < 0, \tag{2.5}$$

where $\mathrm{vert}(\mathcal{R})$ *denotes the set of all vertices of* $\mathcal{R}$*.*

3. Main results. In [2], an optimization procedure is proposed in order to find a polytope $\wp(Q)$ which satisfies the condition of Theorem 1. In this procedure, starting from an initial polytope $\wp(Q)$, an optimization problem over the vertices (and the hyperfaces) of the polytope is implemented in order to verify the condition (2.5). This problem is not guaranteed to converge to a solution, even if one exists, due to the nonconvexity of the function in (2.5). However, in the example section in [2], the authors show how their method ensures good performance in terms of robust stability analysis even if the computational burden may be severe, especially when the order of the system increases.

To this regard, the main goal of this paper is to relax the conditions stated in Theorem 1 to make them convex with respect to a new set of optimization variables and, in this way, to reduce the computational burden. To this aim, given again an initial polytope $\wp(Q)$, we implement an optimization problem over the vertices of the polytope imposing some constraints for the movement of the vertices. In this way we are able to verify condition (2.5) with the aid of the optimization problem based on linear programming techniques. The final outcome will be a procedure for the construction of polyhedral functions which turns out to be not only faster but also less conservative than the algorithm proposed in [2], as the benchmark examples provided at the end of the paper will illustrate.

3.1. Relaxation of the constraints. For the sake of presentation simplicity, the relaxation procedure is first developed in the two-dimensional case and then it is extended to the general n-dimensional case.

THEOREM 2 (two-dimensional case). *Let us consider a polytope $\wp(Q)$ of dimension 2 and a minimal isolation set $\mathcal{I}(\wp(Q))$. Assume that the vertices of the polytope are ordered so that $\pm x_Q^{(i)}$, $i = 1 \ldots r$, are the isolated vertices. Then, a suffi-*

cient condition for the polyhedral stability of system (2.1) *is that there exists a vector* $\alpha = [\alpha_1,\, \alpha_2 \ldots \alpha_r]$, *with* $\alpha_i > 0$ *for* $i = 1, \ldots, r$, *such that the following conditions hold:*

$$\alpha_i\,(q_{i,h} + C_{i,h})^T\, A(p)\, x_Q^{(i)} < C_{i,h}^T\, A(p)\, x_Q^{(i)} \tag{3.1a}$$

$$\alpha_i\,(q_{i,h} + C_{i,h})^T\, A(p)\, x_Q^{(i,h)} < C_{i,h}^T\, A(p)\, x_Q^{(i,h)} \tag{3.1b}$$

$$\alpha_i \le 1, \tag{3.1c}$$

$$\alpha_i\, \underline{e}_i^T x_Q^{(i)} > 1 \tag{3.1d}$$

for $i = 1 \ldots r$, $h = 1, 2$, *and* $p \in \mathrm{vert}(\mathcal{R})$, *where*

$$C_{i,h} = -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_Q^{(i,h)} & x_Q^{(i)} \end{bmatrix}^{-1} \tag{3.2a}$$

$$\underline{e}_i : \begin{cases} e_i^T x_Q^{(i,1)} = 1 \\ e_i^T x_Q^{(i,2)} = 1. \end{cases} \tag{3.2b}$$

Proof. The idea behind the theorem can be sketched as follows. Starting from the polypote $\wp(Q)$, we try to optimize its shape so that it verifies the condition of Theorem 1, under the following assumptions:

1. The points in $\mathcal{I}(\wp(Q))$ are fixed, so they are not optimization variables.
2. Each isolated points $x_Q^{(i)}$, $i = 1 \ldots r$, can move along the line connecting it with the origin.

Taking into account that $\mathcal{I}(\wp(Q))$ is a minimal isolation set and therefore on each hyperface there is one isolated point, the matrix Q defining the polytope can be rewritten as

$$Q = [q_{1,1}\;\; q_{1,2}\;\; q_{2,1}\;\; q_{2,2}\;\; \ldots q_{r,1}\;\; q_{r,2}].$$

Let us denote by $\wp(\hat{Q})$ a generic polytope that we can construct according to assumptions 1 and 2. The hyperfaces of this polytope will verify the conditions

$$\hat{q}_{i,h}(\alpha_i) : \begin{cases} \hat{q}_{i,h}^T x_Q^{(i,h)} = 1, \\ \hat{q}_{i,h}^T \alpha_i x_Q^{(i)} = 1 \end{cases}$$

with $\alpha_i \in \mathbb{R}^+$ for $i = 1 \ldots r$ and $h = 1, 2$.

As consequence, the robust stability conditions of Theorem 1 for system (2.1) can be formulated as

$$\max_{p \in \mathrm{vert}(\mathcal{R})} \hat{q}_{i,h}^T(\alpha_i)\, A(p)\, x_Q^{(i)} \alpha_i < 0, \tag{3.3a}$$

$$\max_{p \in \mathrm{vert}(\mathcal{R})} \hat{q}_{i,h}^T(\alpha_i)\, A(p)\, x_Q^{(i,h)} < 0 \tag{3.3b}$$

for $h = 1, 2$, $i = 1, \ldots, r$.

It is important to recognize that according to this formulation, the optimization variables are the elements of the vector $\alpha = [\alpha_1,\, \alpha_2 \ldots \alpha_r]$ and the column vectors $\hat{q}_{i,h}(\alpha_i)$, with $i = 1 \ldots r$ and $h = 1, 2$, while both $x_Q^{(i,h)}$ and $x_Q^{(i)}$ are fixed.

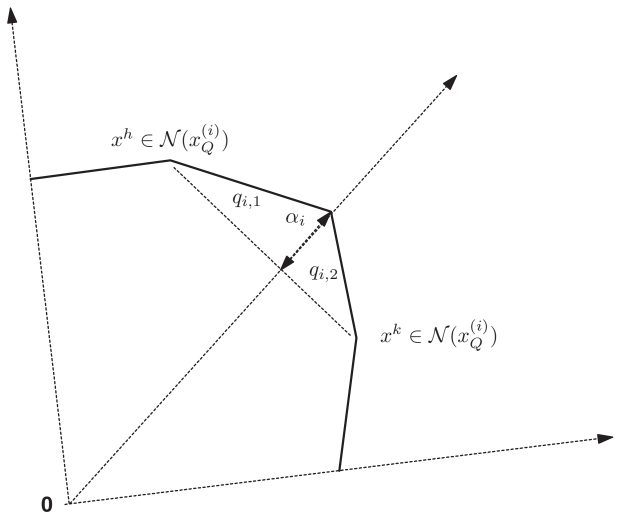

FIG. 3.1. *Range of variation of* α_i.

In the appendix we also prove that

$$\hat{q}_{i,h}(\alpha_i) = q_{i,h} + \frac{\alpha_i - 1}{\alpha_i} C_{i,h}, \quad i = 1 \dots r, \quad h = 1, 2, \tag{3.4}$$

where $C_{i,h}$ is defined in (3.2a). Hence, taking into account (3.4) and multiplying both the terms in (3.3b) by $\alpha_i > 0$, conditions (3.3a)–(3.3b) can be rewritten as in (3.1a)–(3.1b).

Finally, to ensure the convexity of the polytope, each vertex $\alpha_i x_Q^{(i)}$ of $\wp(\hat{Q})$, $i = 1 \dots r$, can move only from its initial position ($\alpha_i = 1$) to the intersection with the hyperplane $\underline{e}_i$ passing through all its neighbors (see Figure 3.1). The definition of $\underline{e}_i$ is in (3.2b) and the convexity conditions are

$$\begin{cases} \underline{e}_i^T(\alpha_i x_Q^{(i)}) > 1, \\ \alpha_i \leq 1 \end{cases} \tag{3.5}$$

for $i = 1, \dots, r$. The convexity condition (3.5) for the polytope $\wp(\hat{Q}(\alpha))$ fixes a conservative range for the optimization variables α_i, but it makes them independent of each other. This last point guarantees that $\wp(\hat{Q})$ is convex. $\square$

In the following theorem we generalize the previous result to the n-dimensional case.

THEOREM 3 (n-dimensional case). *Let us consider a polytope $\wp(Q)$ of dimension n and a minimal isolation set $\mathcal{I}(\wp(Q))$. Assume that the vertices of the polytope are ordered so that $\pm x_Q^{(i)}$, $i = 1 \dots r$, are the isolated vertices. Then, a sufficient condition for the polyhedral stability of system* (2.1) *is that there exists a vector $\alpha = [\alpha_1, \alpha_2 \dots \alpha_r]$, with $\alpha_i > 0$ for $i = 1, \dots, r$, such that the following conditions hold:*

$$\alpha_i \,(q_{i,h} + C_{i,h})^T A(p)\, x_Q^{(i)} < C_{i,h}^T A(p)\, x_Q^{(i)}, \tag{3.6a}$$

$$\alpha_i \,(q_{i,h} + C_{i,h})^T A(p)\, x_Q^{(i,h,j)} < C_{i,h}^T A(p)\, x_Q^{(i,h,j)}, \tag{3.6b}$$

$$\alpha_i \leq 1, \tag{3.6c}$$

$$\alpha_i\, \underline{e}_i^T x_Q^{(i)} > 1, \quad \underline{e}_i \in \underline{E}_i, \tag{3.6d}$$

for $i = 1 \dots r$, $h = 1, \dots s_i$, $j = 1 \dots n-1$, *and* $p \in \mathrm{vert}(\mathcal{R})$, *where*

$$C_{i,h} = -\left[\begin{array}{cccc} 0 & \dots & 0 & 1 \end{array}\right] \left[\begin{array}{cccc} x_Q^{(i,h,1)} & \dots & x_Q^{(i,h,n-1)} & x_Q^{(i)} \end{array}\right]^{-1}, \tag{3.7a}$$

$$\underline{E}_i = \{\textit{All the hyperplanes } e_i \textit{ connecting } n \textit{elements in } \mathcal{N}(x_Q^{(i)})\}. \tag{3.7b}$$

Proof. The proof follows the same guidelines of that of Theorem 2. The only noticeable difference concerns the convexity condition (3.6d). While in Theorem 2 the lower bound for each α_i is given by the single inequality (3.1d), in Theorem 3 we deal with a set of inequalities to be satisfied, each for all possible hyperplanes connecting n elements in $\mathcal{N}(x_Q^{(i)})$. In particular the number of inequalities in (3.6d) is $\binom{card(\mathcal{N}(x_Q^{(i)}))}{n}$. □

Remark 6. As clearly stated in Remark 5, finding a minimal isolation set in the n-dimensional case can be difficult or even impossible. For this reason, it is important to recognize that Theorem 3 can also be applied with a nonminimal isolation set. The only restriction is that if the isolation set is not minimal, some hyperfaces $\tilde{q}_i$ of the polytope $\wp(Q)$ will be defined only by vertices belonging to the isolation set $\mathcal{I}(\wp(Q))$, and hence they will remain in their initial position. Hence, to guarantee the robust stability of system (2.1) we also need to check that for each fixed hyperface $\tilde{q}_i$

$$\max_{p \in \mathrm{vert}(\mathcal{R})} \tilde{q}_i^T\, A(p)\, x_{\tilde{q}_i}^{(j)} < 0,$$

where $x_{\tilde{q}_i}^{(j)}$ with $j = 1 \dots k_i$, the k_i vertices of $\wp(Q)$ such that $q_i^T x_{q_i}^{(j)} = 1$.

Finding a suitable solution α for Theorem 3 implies that Theorem 1 admits a feasible solution and hence the uncertain linear system is robustly stable. However, our approach may introduce conservativeness for the following reasons:

- Finding a feasible solution depends on the initial condition, i.e., the shape and the number of vertices of the initial polytope $\wp(Q)$.
- We assume that only a subset of vertices, i.e., the isolated vertices, can move and their movement is strongly limited by the convexity conditions.

In the following section we shall reduce the conservativeness by introducing an iterative procedure to check the conditions of Theorem 3.

3.2. Optimization procedure. First, we convert the feasibility conditions (3.6a)–(3.6d) of Theorem 3 into a minimization problem, where the objective function is related to the maximum value, for $p \in \mathrm{vert}(\mathcal{R})$, of the Dini derivative of the polyhedral Lyapunov function along the trajectories of the system. Then our aim is to reduce the value of the objective function at each iteration of the procedure until it possibly becomes less than zero. We can readily derive the following result.

COROLLARY 1. *Let us consider a polytope* $\wp(Q)$ *of dimension* n *and an isolation set* $\mathcal{I}(\wp(Q))$. *Assume that the vertices of the polytope are ordered so that* $\pm x_Q^{(i)}$, $i = 1 \dots r$, *are the isolated vertices, and consider the minimization problem*

$$\begin{aligned} &\min_{y,\,\alpha} && y \\ &s.t. && y > \alpha_i\,(q_{i,h} + C_{i,h})^T\, A(p)\, x_Q^{(i)} - C_{i,h}^T\, A(p)\, x_Q^{(i)}, \\ &&& y > \alpha_i\,(q_{i,h} + C_{i,h})^T\, A(p)\, x_Q^{(i,h,j)} - C_{i,h}^T\, A(p)\, x_Q^{(i,h,j)}, \\ &&& \alpha_i \le 1, \\ &&& \alpha_i\, \underline{e}_i^T x_Q^{(i)} > 1, \quad \underline{e}_i \in \underline{E}_i, \end{aligned} \tag{3.8}$$

for $i = 1, \dots, r$, $h = 1, \dots s_i$, $j = 1 \dots n-1$, *and* $p \in \mathrm{vert}(\mathcal{R})$, *where* $C_{i,h}$ *and* $\underline{E}_i$ *are defined, respectively, in* (3.7a) *and* (3.7b).

Then, if the optimal value of y *is negative, the uncertain linear system* (2.1) *is polyhedrally stable.*

The following procedure can be conceptually divided into two parts:

- Given a polytope $\wp(Q)$ we try to optimize its shape according to the minimization problem (3.8) alternating the isolation sets $\mathcal{I}(\wp(Q))$ until the improvement is lower than a certain bound β.
- Then, we increase the number of vertices and we repeat the previous optimization strategy. We do that until the improvement due to the increase of the vertices is lower than a certain bound γ.

At the end of the procedure, if the value of the cost function y is negative, we can conclude that system (2.1) is polyhedrally stable.

PROCEDURE 1.

1. *Fix an initial number* $2l \geq 2n$ *of symmetric points* $x_Q^{(i)} \in \mathbb{R}^n$ *on a hypersphere of unit radius and sort them in such a way that* $x_Q^{(i)} = -x_Q^{(i+l)}$. *Let us denote by* $\wp(Q)^1$ *the polytope defined by the vertices* $x_Q^{(i)}$.
   Fix a desired value for the parameters β *and* γ. *Moreover, let* y_{old} *be the value of* y *in the optimization problem* (3.8) *when* $\alpha_i = 1$, $i = 1, \dots, r$.
2. *Let us denote by* $\mathcal{I}(\wp(Q))^i$, $i = 1 \dots z$ *and* $z \geq 2$, *the minimal number of isolation sets for the polytope such that* $\bigcap \mathcal{I}(\wp(Q))^i = \emptyset$, *i.e., for any admissible vertex* $x_Q^{(j)}$ *there exists at least one isolation set* $\mathcal{I}(\wp(Q))^i$ *such that* $x_Q^{(j)} \notin \mathcal{I}(\wp(Q))^i$.
   For $i = 1 \dots z$
   (a) *Solve the minimization problem* (3.8) *over the polytope* $\wp(Q)^i$ *with the isolation set* $\mathcal{I}(\wp(Q))^i$. *Denote by* $(y^{i,*}, \alpha^*)$ *the corresponding optimal values of* (y, α).
   (b) *Let us define the polytope* $\wp(Q)^{i+1}$ *as*
   $$\wp(Q)^{i+1} = \mathrm{conv}\{\pm\alpha_1^* x_Q^1, \dots, \pm\alpha_r^* x_Q^r, \\ \pm x_Q^{r+1}, \dots, \pm x_Q^l\}$$
   end
3. ***If*** $y^{z+1,*} < 0$ go to *step* 4.
   else if $y^{1,*} - y^{z+1,*} > \beta$
   $$\wp(Q)^1 \leftarrow \wp(Q)^{z+1}$$
   and go to *step* 2.
   else if $y^{old} - y^{z+1,*} > \gamma$, *define the set of points*[1]
   $$\mathcal{S} = \left\{ \bar{x}_k = \frac{\sum_{i=1}^n x_{q_k}^{(i)}}{n} \quad \textit{with } q_k, \textit{ for } k = 1, \dots, m, \right. \\ \left. \textit{the hyperfaces of } \wp(Q)^{z+1} \right\}. \tag{3.9}$$
   Let us denote by $\wp(Q)^1$ *the polytope with vertices*
   $$\mathrm{vert}(\wp(Q)^1) = \mathrm{vert}(\wp(Q)^{z+1}) \bigcup S,$$

[1]The set $\mathcal{S}$ contains the vertices we want to add to the polytope $\wp(Q)^{z+1}$. In particular, we define a new vertex in the middle of each hyperface of $\wp(Q)^{z+1}$.

let $y^{old} = y^{z+1,*}$ *and* go to *step* 2.
else *the procedure failed.*

4. *The polyhedral Lyapunov function which proves the polyhedral stability of system* (2.1) *is*

$$V(x) = \|Q^T x\|_\infty, \tag{3.10}$$

where Q *describes the polytope* $\wp(Q)^{z+1}$.

Remark 7. To solve the minimization problem (3.8) we used `CVX`, a package for specifying and solving convex programming [17].

Procedure 1 does not guarantee finding a polyhedral Lyapunov function satisfying the conditions of Theorem 1 even if system (2.1) is polyhedrally stable. Indeed, on one hand we have relaxed, at the price of some conservativeness, the conditions of Theorem 1 in a convex form; on the other hand the convergence of the procedure still depends on the choice of the initial polytope and on the sequence of the isolation sets in Step 2.

However, as we shall show in the benchmark examples illustrated in the next section, the proposed procedure improves the performance with respect to the approach given in [2] in terms of computational speed and, more generally, can obtain less conservative results than any other available method.

4. Numerical examples.

4.1. Example 1. This example was first presented in [27] and has been used by many authors as a benchmark for comparing the various robust stability approaches proposed in the literature.

Let us consider the following linear uncertain system [27]:

$$\dot{x}(t) = A(p)x(t)$$

with

$$A(p) := \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix} + p \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \tag{4.1}$$

where $0 \leq p \leq \alpha$.

This system is shown to be quadratically stable for $\alpha < 3.82$ [27]. In [5], a polyhedral Lyapunov function has been constructed guaranteeing robust stability for $\alpha \leq 6$. In [26], the authors constructed a piecewise quadratic Lyapunov function via a sequence of LMI optimizations achieving robust stability for $\alpha < 6.2$; the robust stability bound $\alpha \leq 6.8649$ is achieved by means of a homogeneous polynomial Lyapunov function of degree 20 [14]. Finally, in [2] the authors managed to demonstrate that the system is robustly stable up to $\alpha = 6.8$ and it becomes unstable for $\alpha = 7.2$. The results are summarized in Table 4.1.

We applied Procedure 1 to this example in order to find a polyhedral Lyapunov function satisfying the conditions of Theorem 3. In this way we managed to demonstrate that system (4.1) is robustly stable up to $\alpha = 6.87$, which definitely improves the value $\alpha = 6.0$ found in [5] making use of the same class of Lyapunov functions and also is slightly better than any other proposed result. Moreover, it improves the result previously presented in [2], where condition (2.5) is implemented without any relaxation and hence with a heavier computational effort, as explained in section 4.5.

We proved the polyhedral (and hence robust) stability of system (4.1) for $\alpha = 6.87$ using the polytope $\wp(Q)$ of 9694 vertices ($l = 4847$) shown in Figure 4.1.

TABLE 4.1
Maximum value α for the parameter p in system (4.1) for which robust stability is demonstrated by using various approaches. In [2], it is proved that the system is unstable for $\alpha \geq 7.2$.

Method	Max. value of α
Quadratic Lyapunov functions [27]	3.82
Polyhedral Lyapunov functions [5]	6.0
Piecewise quadratic Lyapunov functions [26]	6.2
Polyhedral Lyapunov functions [2]	6.8
HPLFs of degree 20 [14]	6.8649
Polyhedral Lyapunov functions (Procedure 1)	6.87

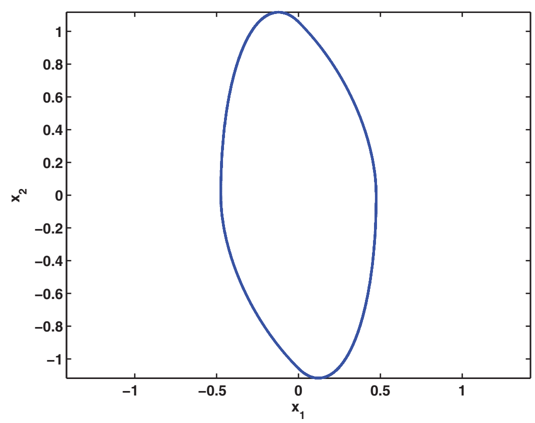

FIG. 4.1. *Level curve of the polyhedral Lyapunov function which proves the robust stability of system* (4.1) *for* $\alpha = 6.87$.

4.2. Example 2. Let us consider an example proposed in [14, section 3.5.3]. In this example the authors evaluate the maximum value of κ such that the solution of the differential equation

$$\ddot{x}(t) + \dot{x}(t) + k(t)x(t) = 0, \quad k(t) \in [0, \kappa], \tag{4.2}$$

remains bounded. In order to show the effectiveness of their method, the authors first calculate analytically the maximum admissible value of κ, that is, $\kappa^* = 3.0448$. Then, by means of an HPLF of degree 24, they find an estimate of the maximum value of κ that is equal to $\tilde{\kappa} = 2.79$. They also point out that increasing the degree of the Lyapunov function, the estimated value *exhibits a growth toward* κ^*.

Using our approach, we managed to demonstrate that the solution of the differential equation (4.2) remains bounded for κ up to $\tilde{\kappa} = 3.01$, which is definitely better than the value $\tilde{\kappa} = 2.79$ found in [14] and is also very close to the theoretical maximum value of 3.0448. We managed to prove the polyhedral stability of system (4.2) for $\tilde{\kappa} = 3.01$ using a polytope $\wp(Q)$ of 676 vertices ($l = 338$).

4.3. Example 3. In this example we compare our approach with the one presented in [23], where the construction of a polyhedral Lyapunov function is also based on a linear programming technique.

Let us consider the following linear uncertain system [23]:

$$\dot{x} = \left(\sum_{i=1}^{4} p_i A_i \right) x, \quad p_i \geq 0 \ : \sum_{i=1}^{4} p_i = 1, \tag{4.3}$$

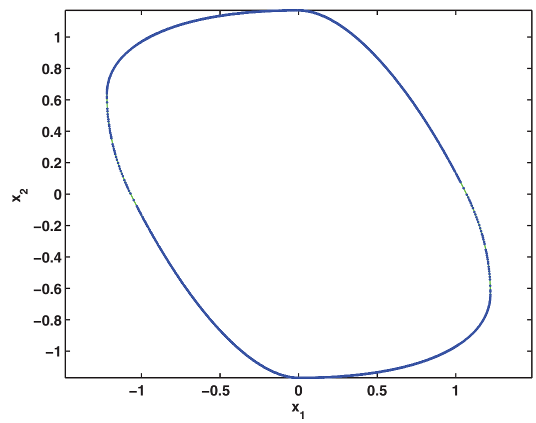

FIG. 4.2. *Level curve of the polyhedral Lyapunov function which proves the robust stability of system* (4.3) *for* $\beta = 1.9$.

with

$$A_1 = \begin{pmatrix} -0.5 & -0.25 \\ 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -0.5 & -0.5\beta \\ 1 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -\beta & -0.5\beta \\ 1 & 0 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} -\beta & -\beta^2 \\ 1 & 0 \end{pmatrix}. \tag{4.4}$$

In [23] the author proves that system (4.3) is

- Quadratically stable for $0 < \beta \leq 1.3$,
- Polyhedrally stable for $0 < \beta \leq 1.8$, and
- Unstable for $\beta \geq 2$.

In [2] the authors managed to prove that system (4.3) is

- Polyhedrally stable for $0 < \beta \leq 1.87$ and
- Unstable for $\beta \geq 1.95$.

Our approach allows us to improve the above robust stability bounds; indeed we managed to prove that system (4.3) is polyhedrally stable for $0 < \beta \leq 1.9$.

In particular, the bound we found on β for the robust stability of system (4.3) is better than the one proposed in [23] by using the same class of Lyapunov functions and it is not far from the exact one.

We proved the polyhedral stability of system (4.3) for $\beta = 1.9$ using the polytope $\wp(Q)$ of 2260 vertices ($l = 1130$) shown in Figure 4.2.

4.4. Example 4. Let us consider the feedback loop studied in [24] (see Figure 4.3) composed by a double integrator with a phase lead element $\frac{1+s}{1+0.1s}$ and a nonlinear time-dependent function $\sigma(e,t)$ which is assumed to satisfy the sector condition

$$\gamma \leq \frac{\sigma(e,t)}{e} \leq \delta. \tag{4.5}$$

The problem is to determine for a fixed value $\gamma = 0.2$ the bound δ such that the feedback system is absolutely stable.

In [24] the author showed that the absolute stability of this system is equivalent to the robust stability of the linear uncertain system

$$\dot{x} = (p_1 A_1 + p_2 A_2)\, x, \quad p_1, p_2 \geq 0 : p_1 + p_2 = 1, \tag{4.6}$$

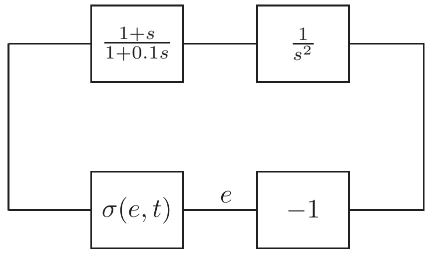

FIG. 4.3. *The feedback system studied in Example* 3. *The nonlinear time-dependent function* $\sigma(e,t)$ *is assumed to satisfy the sector condition* (4.5).

TABLE 4.2

Application of Procedure 1 *to system* (4.6). *The robust stability of the system for* δ *up to the value specified in the left column is proved using a polyhedral Lyapunov function with a number of vertices specified in the right column. For the sake of comparison, the author in* [24] *proves that the system is robustly stable for* $\delta \leq 1$ *using a polyhedral Lyapunov function with* 2×3201 *vertices.*

Value for δ	Number of vertices ($2 \times l$)
0.9	2×242
1.00	2×276
1.05	2×330
1.07	2×459

with

$$A_1 = \begin{pmatrix} -10 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -10 & -10\delta & -10\delta \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and he verified the robust stability of (4.6) for $\delta \leq 1$ using a polyhedral Lyapunov function with 6402 vertices (while the maximum value applying the circle criterion is $\delta \leq 0.5467$). Moreover, he claimed that with this number of vertices, his method is not able to prove the robust stability of the system for $1.00 \leq \delta \leq 1.05$. This result was improved in [2], where we managed to prove the robust stability of the system for $\delta \leq 1.03$.

Applying Procedure 1 we proved the robust stability of system (4.6) for different values of δ, as shown in Table 4.2, where the number of vertices refers to the polyhedral Lyapunov function necessary to prove the robust stability of system (4.6). The polyhedral Lyapunov function $\wp(Q)$ proving the polyhedral (and hence robust) stability of system (4.6) for $\delta = 1.07$ is shown in Figure 4.4.

4.5. Computational burden. Theorem 1, first proposed in [2], consists of a necessary and sufficient condition for polyhedral (and hence robust) stability of the uncertain linear system (2.1). However, the practical implementation of Theorem 1 in [2] introduces a certain degree of conservatism due to the nonconvex structure of the optimization problem. This also brings a high computational burden that increases both with the order n of the system and with the number of the vertices $2l$ of the polytope.

In this paper we relax the conditions stated in Theorem 1 to make them convex with respect to a new set of optimization variables, and we provide Procedure 1 to construct in the general n-dimensional case a polyhedral Lyapunov function solving iteratively the linear optimization problem (3.8).

In the following we propose a computation burden analysis of the convex optimization problem (3.8) to better illustrate the benefits of this approach with respect to the one in [2]. Given a symmetric polytope $\wp(Q)$ of $2l$ vertices and a minimal isolation

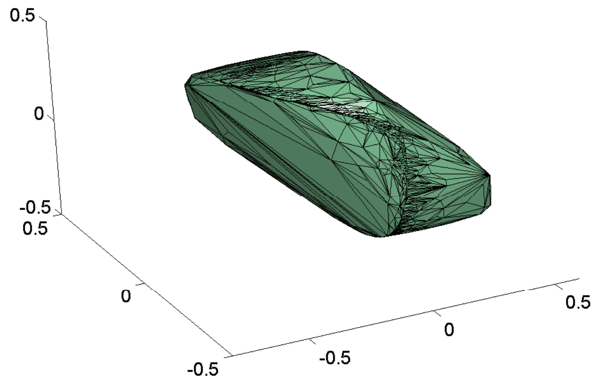

FIG. 4.4. *Level curve of the polyhedral Lyapunov function which proves the robust stability of system* (4.6) *for* $\delta = 1.07$.

set $\mathcal{I}(\wp(Q))$, the number of isolated vertices is equal to the number k of hyperfaces of the polytope and so, considering the symmetry of the polytope and the problem formulation, the number of optimization variables is $m = \frac{k}{2} + 1$. It is important to notice that m is related to the number of isolated vertices and it does not depend on the dimension of the system. Hence, this method is scalable with respect to n. On the other hand, the complexity of the problem grows linearly with respect to the number of optimization variables and hence with respect to the number of isolated vertices. This is due to the fact that the optimization variables α_i are decoupled and, as consequence, the problem (3.8) can be turned out into $\frac{k}{2}$ different optimization problems for each α_i with $i = 1 \dots \frac{k}{2}$.

It is important to note that the proposed relaxation method not only strongly reduces the computational burden respect to [2] but also demonstrates better performance in all the proposed examples.

5. Conclusions. In this paper we have considered the robustness analysis problem for a linear uncertain system subject to parametric time-varying uncertainties. Our main contribution consists of a novel procedure to directly construct, in the general n-dimensional case, a polyhedral Lyapunov function, which allows us to prove robust stability of the system under consideration, by solving iteratively a linear optimization problem.

With respect to our previous paper [2], this work reduces the computational burden for the computation of the polyhedral function. Moreover, some benchmark examples show that our methodology can perform much better in terms of degree of conservativeness than earlier methods based both on polyhedral functions and other classes of nonquadratic Lyapunov functions.

6. Appendix.

6.1. Definition of the matrix $C_{i,h}$ for the n-dimensional case. Let us consider a polytope $\wp(Q)$ of dimension n and a hyperface q of vertices $x_1, x_2, \dots x_n$. It implies

$$q^T X = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}, \tag{6.1}$$

where

$$X = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}.$$

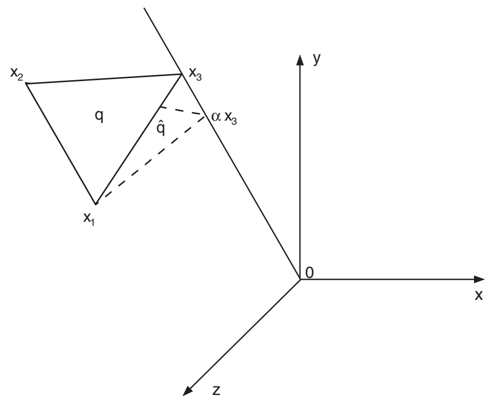

FIG. 6.1. *Movement of the vertex x_3 along the line connecting it to the origin in the three-dimensional case.*

Assume that a vertex x_i moves along the line connecting it to the origin and indicate with $\hat{q}$ the hyperface defined by the vertices $x_1, \ldots, \alpha x_i, \ldots x_n$, as in Figure 6.1. It implies

$$\hat{q}^T \hat{X} = \begin{bmatrix} 1 & \ldots & 1 & \ldots & 1 \end{bmatrix}, \tag{6.2}$$

where

$$\hat{X} = \begin{bmatrix} x_1 & \ldots & \alpha x_i & \ldots & x_n \end{bmatrix}.$$

Note that both the square matrices X and $\hat{X}$ are full rank matrices because they are composed by the minimal number of vertices defining a hyperplane.
From (6.1) and (6.2) we have

$$q^T X - \hat{q}^T \hat{X} = \begin{bmatrix} 0 & \ldots & 0 \end{bmatrix} \tag{6.3}$$

and hence

$$q^T = \hat{q}^T \hat{X} X^{-1}. \tag{6.4}$$

Taking into account that

$$\hat{X} = \begin{bmatrix} \mathbf{0} \ldots \mathbf{0} & (\alpha - 1)x_i & \mathbf{0} \ldots \mathbf{0} \end{bmatrix} + X,$$

where $\mathbf{0}$ is the zeros vector, we have

$$\begin{aligned} q^T &= \hat{q}^T (\begin{bmatrix} \mathbf{0} \ldots \mathbf{0} & (\alpha - 1)x_i & \mathbf{0} \ldots \mathbf{0} \end{bmatrix} + X) X^{-1} \\ &= \hat{q}^T (\begin{bmatrix} \mathbf{0} \ldots \mathbf{0} & (\alpha - 1)x_i & \mathbf{0} \ldots \mathbf{0} \end{bmatrix} X^{-1} + I) \\ &= \hat{q}^T + \frac{\alpha - 1}{\alpha} \hat{q}^T \begin{bmatrix} \mathbf{0} \ldots \mathbf{0} & \alpha x_i & \mathbf{0} \ldots \mathbf{0} \end{bmatrix} X^{-1} \\ &= \hat{q}^T + \frac{\alpha - 1}{\alpha} \delta_i X^{-1}, \end{aligned} \tag{6.5}$$

where δ_i is the vector with the value of 1 in the i-position while all the other elements are zero.

Finally, if we define

$$C_i = -\delta_i X^{-1}$$

we have

$$\hat{q}^T = q^T + \frac{\alpha - 1}{\alpha} C_i.$$

REFERENCES

[1] F. Amato, *Robust Control of Linear Systems Subject to Uncertain Time-Varying Parameters*, Springer-Verlag, Berlin, 2006.

[2] R. Ambrosino, F. Amato, and M. Ariola, *Stability and instability conditions using polyhedral Lyapunov functions*, IET Control Theory Appl., 43 (2010), pp. 1179–1187.

[3] C. B. Barber, D. P. Dobkin, and H. T. Huhdanpaa, *The quickhull algorithm for convex hulls*, ACM Trans. Math. Software, 22 (1996), pp. 469–483.

[4] B. R. Barmish, *Stabilization of uncertain systems via linear control*, IEEE Trans. Automat. Control, 28 (1983), pp. 848–850.

[5] F. Blanchini and S. Miani, *On the transient estimate for linear systems with time-varyinguncertain parameters*, IEEE Trans. Circuits Systems, 43 (1996), pp. 592–596.

[6] F. Blanchini and S. Miani, *A new class of universal Lyapunov functions for the control of uncertain linear systems*, IEEE Trans. Automat. Control, 44 (1999), pp. 641–647.

[7] F. Blanchini, *Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions*, IEEE Trans. Automat. Control, 39 (1994), pp. 428–433.

[8] F. Blanchini, *Nonquadratic Lyapunov functions for robust control*, Automatica, 31 (1995), pp. 451–461.

[9] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, 1994.

[10] S. Boyd and Q. Yang, *Structured and simultaneous Lyapunov functions for system stability problems*, Internat. J. Control, 49 (1989), pp. 2215–2240.

[11] R. K. Brayton and C. H. Tong, *Stability of dynamical systems: A constructive approach*, IEEE Trans. Automat. Control, 26 (1979), pp. 224–234.

[12] R. K. Brayton and C. H. Tong, *Constructive stability and asymptotic stability of dynamical systems*, IEEE Trans. Circuits Systems, 27 (1980), pp. 1121–1130.

[13] G. Chesi, A. Garulli, A. Tesi, and A. Vicino, *Homogeneous Lyapunov functions for systems with structured uncertainties*, Automatica, 39 (2003), pp. 1027–1035.

[14] G. Chesi, A. Garulli, A. Tesi, and A. Vicino, *Homogeneous Polynomial Forms for Robustness Analysis of Uncertain Systems*, Springer-Verlag, Berlin, 2009.

[15] G. Chesi, *Robust stability of time-varying uncertain systems with rational dependence on the uncertainty*, IEEE Trans. Automat. Control, 45 (2010), pp. 2353–2357.

[16] P. Dorato, R. Tempo, and G. Muscato, *Bibliography on robust control*, Automatica, 29 (1993), pp. 201–213.

[17] M. Grant and S. Boyd, *CVX: Matlab Software for Disciplined Convex Programming*, http://stanford.edu/~boyd/cvx (2009).

[18] W. Hahn, *Stability of Motion*, Springer-Verlag, Berlin, 1967.

[19] M. Johansson and A. Rantzer, *Computation of piecewise quadratic Lyapunov functions for hybrid systems*, IEEE Trans. Automat. Control, 43 (1998), pp. 555–559.

[20] G. Leitmann, *Guaranteed asymptotic stability for some linear systems with bounded uncertainties*, ASME J. Dynam. Syst. Meas. Control, 101 (1979), pp. 212–216.

[21] A. P. Molchanov and E. S. Pyatnitskii, *Lyapunov functions specifying necessary and sufficient conditions of absolute stability of nonlinear nonstationary control system*, Autom. Remote Control, 47 (1986).

[22] A. Polański, *Lyapunov function construction by linear programming*, IEEE Trans. Automat. Control, 42 (1997), pp. 1113–1116.

[23] A. Polański, *Destabilizing strategies for uncertain linear systems*, IEEE Trans. Automat. Control, 45 (2000), pp. 2378–2382.

[24] A. Polański, *On absolute stability analysis by polyhedral Lyapunov functions*, Automatica, 36 (2000), pp. 573–578.

[25] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1972.

[26] L. Xie, S. Shishkin, and M. Fu, *Piecewise Lyapunov functions for robust stability of linear time-varyingsystems*, Systems Control Lett., 31 (1997), pp. 165–171.

[27] A. L. Zelentsowsky, *Nonquadratic Lyapunov function for robust stability analysis of linear uncertain systems*, IEEE Trans. Automat. Control, 39 (1994), pp. 135–138.

[28] G. M. Ziegler, *Lectures on Polytopes*, Springer, New York, 1998.