

## denntorch: (DEN)sity of (N)on-(M)arkovian phenomena with Libtorch

The computational framework for **denntorch** is derived using the formalism from the publicly-available book: Diffusing Ideas. Let's begin by writing the generalised non-Markovian master equation for stochastic phenomena which was given in this chapter of the book

$$P_{t+1}(x|z) = \int_{\Omega_t} dX' P_{t+1}(X|z) = \int_{\Omega_t} dX' P_t(X'|z) P_{(t+1)t}(x|X', z).$$

For now, let's imagine that  $x$  is just a scalar (as opposed to a row vector) for simplicity in the expressions. We can then write down a kind of Kramers-Moyal expansion of the equation above

$$P_{t+1}(x|z) = \int_{\Omega_t} dX' P_t(X'|z) \left[ \delta(x - x') + \sum_{t'=0}^t \partial_x \delta(x - x') [\mu_1]_{t'}(X'', z) \right. \\ \left. + \sum_{t'=0}^t \partial_x \delta(x - x') \sum_{t''=0}^{t'} \partial_x \delta(x - x'') [\mu_2]_{t't''}(X'', z) \dots \right],$$

where  $\partial_x \delta$  in some sense denotes the formal derivative of a Dirac delta function and the  $\mu_i$  terms denote the  $i$ -th conditional moments with respect to the distribution over  $x$  at all of the previous timesteps (which are indicated by the timestep indices).

Truncating the expansion terms up to second order results in the following difference equation

$$P_{t+1}(x|z) - P_t(x|z) \simeq - \sum_{t'=0}^t \frac{\partial}{\partial x} \left\{ P_{t'}(X|z) [\mu_1]_{t'}(X, z) \right\} \Big|_{X_{t'}=x} \\ + \sum_{t'=0}^t \sum_{t''=0}^{t'} \frac{\partial^2}{\partial x^2} \left\{ P_{t'}(X|z) [\mu_2]_{t't''}(X, z) \right\} \Big|_{X_{t'}=x \wedge X_{t''}=x}.$$