`dennm`: (DEN)sity of (N)on-(M)arkovian phenomena

The computational framework for `dennm` is derived using the formalism from the publicly-available book: Diffusing Ideas. Let's begin by writing the generalised non-Markovian master equation for stochastic phenomena which was given in this chapter of the book

$$P_{\mathsf{t}+1}(x|z) = \int_{\Omega_{\mathsf{t}}} \mathrm{d}X' P_{\mathsf{t}+1}(X|z) = \int_{\Omega_{\mathsf{t}}} \mathrm{d}X' P_{\mathsf{t}}(X'|z) P_{(\mathsf{t}+1)\mathsf{t}}(x|X', z).$$

For now, let's imagine that x is just a scalar (as opposed to a row vector) for simplicity in the expressions. We can then write down a kind of Kramers-Moyal expansion of the equation above

$$\begin{aligned} P_{\mathsf{t}+1}(x|z) = \int_{\Omega_{\mathsf{t}}} \mathrm{d}X' P_{\mathsf{t}}(X'|z) \bigg[\delta(x-x') &+ \sum_{\mathsf{t}'=0}^{\mathsf{t}} \partial_x \delta(x-x')[\mu_1]_{\mathsf{t}'}(X'', z) \\ &+ \sum_{\mathsf{t}'=0}^{\mathsf{t}} \partial_x \delta(x-x') \sum_{\mathsf{t}''=0}^{\mathsf{t}'} \partial_x \delta(x-x'')[\mu_2]_{\mathsf{t}'\mathsf{t}''}(X'', z) \dots \bigg], \end{aligned}$$

where $\partial_x \delta$ in some sense denotes the formal derivative of a Dirac delta function and the μ_i terms denote the i-th conditional moments with respect to the distribution over x at all of the previous timesteps (which are indicated by the timestep indices).

Truncating the expansion terms up to second order results in the following difference equation

$$\begin{aligned} P_{\mathsf{t}+1}(x|z) - P_{\mathsf{t}}(x|z) \simeq &- \sum_{\mathsf{t}'=0}^{\mathsf{t}} \frac{\partial}{\partial x} \Big\{ P_{\mathsf{t}'}(X|z)[\mu_1]_{\mathsf{t}'}(X, z) \Big\} \Big|_{X_{\mathsf{t}'}=x} \\ &+ \sum_{\mathsf{t}'=0}^{\mathsf{t}} \sum_{\mathsf{t}''=0}^{\mathsf{t}'} \frac{\partial^2}{\partial x^2} \Big\{ P_{\mathsf{t}'}(X|z)[\mu_2]_{\mathsf{t}'\mathsf{t}''}(X, z) \Big\} \Big|_{X_{\mathsf{t}'}=x \, \wedge \, X_{\mathsf{t}''}=x} . \end{aligned}$$