RL - Function Approximations

Outline

1. Introduction
2. Bellman Equations
3. Temporal Difference (TD) Methods
4. Function Approximation for Value Functions
5. Actor-critic Methods
6. Deep Reinforcement Learning

Policy Function Approximation

Policy Gradient Methods

- Policy gradient methods learn a parameterized policy that can select actions without needing to compute a value function
- Policy π is parameterized with $\omega \in \mathbb{R}^n$

$$\pi(a \mid s, \omega) = p(a_t = a \mid s_t = s, \omega_t = \omega)$$

For instance, a Gibbs policy $\pi(a \mid s, \omega) = \dfrac{\exp(\omega^\top \psi(s, a))}{\sum_{a'} \exp(\omega^\top \psi(s, a'))}$ where $\psi(s, a)$ denotes the feature functions.

- Given a performance measure $J(\omega)$, the gradient is

$$\omega_{t+1} = \omega_t + \alpha \nabla_\omega J(\omega_t)$$

- $J(\omega)$ is typically $\displaystyle\sum_s d^\pi(s)\ V^{\pi(\omega)}(s)$.

Compared with Value-based Methods

- Pros: better convergence properties, effective in high-dim or continuous action space, can learn stochastic policies
- Cons: usually converge to local optimum, inefficient to evaluate, high variance

Policy Gradient Theorem

- Reward $J(\omega) = \sum_s d^{\pi_\omega}(s) V^{\pi_\omega}(s) = \sum_s d^{\pi_\omega}(s) \sum_a Q^{\pi_\omega}(s, a) \pi_\omega(a \mid s)$

where d^{π_ω} is the on-policy distribution under π, i.e., the stationary distribution of Markov chain for π_ω: $d^{\pi_\omega}(s) = \lim_{t \to \infty} P(s_t = s \mid s_0, \pi_\omega)$

Theorem $\nabla_\omega J(\omega) \propto \sum_s d^{\pi_\omega}(s) \sum_a Q^{\pi_\omega}(s, a) \nabla_\omega \pi_\omega(a \mid s)$

Provides a nice reformation of the derivative of the objective function to not involve the derivative of the state distribution or the Q

Policy Gradient Theorem: Proof Sketch

(1) A recursive form of derivative of the state value fn.

$$\nabla_w V^\pi(s) = \sum_{a\in A}\left(\nabla_w \pi_w(a|s) Q^\pi(s,a) + \pi_w(a|s)\sum_{s'} P(s'|s,a)\nabla_\theta V^\pi(s')\right)$$

(2) Unroll the recursive representation of $\nabla_w V^\pi(s)$

$$\nabla_w V^\pi(s) = \sum_{x\in S}\sum_{k=0}^{\infty} \rho^\pi(s\to x, k)\,\phi(x)$$

where $\phi(x) = \sum_{a\in A} Q^\pi(x,a)\,\nabla_w \pi_w(a|x)$

(3) A form excluding the derivative of Q-value fn

$$\nabla_w J(w) \propto \sum_s d^\pi(s)\,\phi(s)$$

Policy Gradient Theorem: Proof Details (1)

$$
\begin{aligned}
& \nabla_\theta V^\pi(s) \\
=& \nabla_\theta \Big(\sum_{a \in \mathcal{A}} \pi_\theta(a|s) Q^\pi(s,a) \Big) \\
=& \sum_{a \in \mathcal{A}} \Big(\nabla_\theta \pi_\theta(a|s) Q^\pi(s,a) + \pi_\theta(a|s) \nabla_\theta Q^\pi(s,a) \Big) && \text{; Derivative product rule.} \\
=& \sum_{a \in \mathcal{A}} \Big(\nabla_\theta \pi_\theta(a|s) Q^\pi(s,a) + \pi_\theta(a|s) \nabla_\theta \sum_{s',r} P(s',r|s,a)(r + V^\pi(s')) \Big) && \text{; Extend } Q^\pi \text{ with future state value.} \\
=& \sum_{a \in \mathcal{A}} \Big(\nabla_\theta \pi_\theta(a|s) Q^\pi(s,a) + \pi_\theta(a|s) \sum_{s',r} P(s',r|s,a) \nabla_\theta V^\pi(s') \Big) && P(s',r|s,a) \text{ or } r \text{ is not a func of } \theta \\
=& \sum_{a \in \mathcal{A}} \Big(\nabla_\theta \pi_\theta(a|s) Q^\pi(s,a) + \pi_\theta(a|s) \sum_{s'} P(s'|s,a) \nabla_\theta V^\pi(s') \Big) && \text{; Because } P(s'|s,a) = \textstyle\sum_r P(s',r|s,a)
\end{aligned}
$$

Policy Gradient Theorem: Proof Details (2)

★Def: k-step transition probability

the probability of transitioning from state s with policy π_w after k step: $\rho^{\pi}(s \to x, k)$

$$s \xrightarrow{a \sim \pi_w(\cdot|s)} s' \xrightarrow{a \sim \pi_w(\cdot|s')} s'' \quad \cdots\cdots$$

$$\rho^{\pi}(s \to x, k+1) = \sum_{s'} \rho^{\pi}(s \to s', k)\rho^{\pi}(s' \to x, 1)$$

Policy Gradient Theorem: Proof Details (2)

$$
\begin{aligned}
&\nabla_\theta V^\pi(s) \\
=&\phi(s) + \sum_a \pi_\theta(a|s) \sum_{s'} P(s'|s,a) \nabla_\theta V^\pi(s') \\
=&\phi(s) + \sum_{s'} \sum_a \pi_\theta(a|s) P(s'|s,a) \nabla_\theta V^\pi(s') \\
=&\phi(s) + \sum_{s'} \rho^\pi(s \to s', 1) \nabla_\theta V^\pi(s') \\
=&\phi(s) + \sum_{s'} \rho^\pi(s \to s', 1) \nabla_\theta V^\pi(s') \\
=&\phi(s) + \sum_{s'} \rho^\pi(s \to s', 1) [\phi(s') + \sum_{s''} \rho^\pi(s' \to s'', 1) \nabla_\theta V^\pi(s'')] \\
=&\phi(s) + \sum_{s'} \rho^\pi(s \to s', 1) \phi(s') + \sum_{s''} \rho^\pi(s \to s'', 2) \nabla_\theta V^\pi(s'') \text{ ; Consider } s' \text{ as the middle point for } s \to s'' \\
=&\phi(s) + \sum_{s'} \rho^\pi(s \to s', 1) \phi(s') + \sum_{s''} \rho^\pi(s \to s'', 2) \phi(s'') + \sum_{s'''} \rho^\pi(s \to s''', 3) \nabla_\theta V^\pi(s''') \\
=& \dots \text{; Repeatedly unrolling the part of } \nabla_\theta V^\pi(.) \\
=& \sum_{x \in \mathcal{S}} \sum_{k=0}^{\infty} \rho^\pi(s \to x, k) \phi(x)
\end{aligned}
$$

Policy Gradient Theorem: Proof Details (3)

$$\begin{aligned}
\nabla_\theta J(\theta) &= \nabla_\theta V^\pi(s_0) && \text{; Starting from a random state } s_0 \\
&= \sum_s \sum_{k=0}^{\infty} \rho^\pi(s_0 \to s, k)\phi(s) && \text{; Let } \eta(s) = \textstyle\sum_{k=0}^{\infty} \rho^\pi(s_0 \to s, k) \\
&= \sum_s \eta(s)\phi(s) && \\
&= \left(\sum_s \eta(s)\right) \sum_s \frac{\eta(s)}{\sum_s \eta(s)}\phi(s) && \text{; Normalize } \eta(s), s \in \mathcal{S} \text{ to be a probability distribution.} \\
&\propto \sum_s \frac{\eta(s)}{\sum_s \eta(s)}\phi(s) && \textstyle\sum_s \eta(s) \text{ is a constant} \\
&= \sum_s d^\pi(s) \sum_a \nabla_\theta \pi_\theta(a|s) Q^\pi(s, a) && d^\pi(s) = \tfrac{\eta(s)}{\sum_s \eta(s)} \text{ is stationary distribution.}
\end{aligned}$$

Policy Gradient Theorem: Proof Details (3)

In the episodic case, the constant of proportionality ($\sum_s \eta(s)$) is the average length of an episode; in the continuing case, it is 1 (Sutton & Barto, 2017; Sec. 13.2). The gradient can be further written as:

$$\begin{aligned}
\nabla_\theta J(\theta) &\propto \sum_{s\in\mathcal{S}} d^\pi(s) \sum_{a\in\mathcal{A}} Q^\pi(s,a) \nabla_\theta \pi_\theta(a|s) \\
&= \sum_{s\in\mathcal{S}} d^\pi(s) \sum_{a\in\mathcal{A}} \pi_\theta(a|s) Q^\pi(s,a) \frac{\nabla_\theta \pi_\theta(a|s)}{\pi_\theta(a|s)} \\
&= \mathbb{E}_\pi[Q^\pi(s,a)\nabla_\theta \ln \pi_\theta(a|s)] & \text{; Because } (\ln x)' = 1/x
\end{aligned}$$

Where $\mathbb{E}_\pi$ refers to $\mathbb{E}_{s\sim d_\pi, a\sim\pi_\theta}$ when both state and action distributions follow the policy π_θ (on policy).

REINFORCE

- Policy Gradient Theorem $\nabla_\omega J(\omega) = \sum_s d^\pi(s) \sum_a Q^\pi(s,a) \nabla_\omega \pi(a \mid s, \omega)$ gives us an exact expression for the gradient. We need a sampling method whose expectation equals or approximates this expression.
- We know that

$$\mathbb{E}_{\Pr(s_{t+1}, r_{t+1} \mid s_t), \pi(a_t \mid s_t, \omega), s_t}\left[\gamma^t R_t \frac{\nabla_\omega \pi(a_t \mid s_t, \omega)}{\pi(a_t \mid s_t, \omega)}\right] = \mathbb{E}_{s_t}\left[Q^\pi(s_t, \pi(a_t \mid s_t, \omega)) \frac{\nabla_\omega \pi(a_t \mid s_t, \omega)}{\pi(a_t \mid s_t, \omega)}\right]$$

$$= \sum_s d^\pi(s) \sum_a Q^\pi(s,a) \frac{\nabla_\omega \pi(a \mid s, \omega)}{\pi(a \mid s, \omega)} \pi(a \mid s, \omega) = \nabla_\omega J(\omega)$$

- Therefore, $\nabla_\omega J(\omega)$ can be evaluated through $\mathbb{E}_\pi[\gamma^t R_t \frac{\nabla_\omega \pi(a_t \mid s_t, \omega)}{\pi(a_t \mid s_t, \omega)}]$ ———— REINFORCE

REINFORCE

- Evaluate $\nabla_\omega J(\omega)$ through

$$\mathbb{E}_\pi[\gamma^t R_t \frac{\nabla_\omega \pi(a_t \mid s_t, \omega)}{\pi(a_t \mid s_t, \omega)}] = \mathbb{E}_\pi[\gamma^t R_t \nabla_\omega \log \pi(a_t \mid s_t, \omega)]$$

- Update the policy through

$$\omega_{t+1} = \omega_t + \alpha \gamma^t R_t \nabla_\omega \log \pi(a_t \mid s_t, \omega)$$

- In case π is a Gibbs policy

$$\nabla \log \pi(a_t \mid s_t, \omega) = \psi(s_t, a_t) - \sum_{a'} \pi(a' \mid s_t, \omega) \psi(s_t, a')$$

REINFORCE

Algorithm 3 REINFORCE

Input: a differentiable policy parameterization $\pi(a|s, \boldsymbol{\omega})$, $\alpha > 0$
Initialise $\boldsymbol{\omega}$
repeat
    Generate an episode $s_0, a_0, r_1, \cdots, s_T, a_T$ following $\pi(\cdot|\cdot, \boldsymbol{\omega})$
    for each step $t = 0, \cdots, T$ **do**
        $R_t \leftarrow$ return from step t
        $\boldsymbol{\omega} \leftarrow \boldsymbol{\omega} + \alpha\gamma^t R_t \nabla \log \pi(a|s_t, \boldsymbol{\omega})$
    end for
until convergence

- An episodic Monte-Carlo Policy-Gradient Method

Off-Policy Policy Gradient

1. The off-policy approach does not require full trajectories and can reuse any past episodes ("experience replay") for much better sample efficiency.
2. The sample collection follows a behavior policy different from the target policy, bringing better exploration.

Now let's see how off-policy policy gradient is computed. The behavior policy for collecting samples is a known policy (predefined just like a hyperparameter), labelled as $\beta(a|s)$. The objective function sums up the reward over the state distribution defined by this behavior policy:

$$J(\theta) = \sum_{s \in \mathcal{S}} d^\beta(s) \sum_{a \in \mathcal{A}} Q^\pi(s, a)\pi_\theta(a|s) = \mathbb{E}_{s \sim d^\beta} \Big[\sum_{a \in \mathcal{A}} Q^\pi(s, a)\pi_\theta(a|s) \Big]$$

where $d^\beta(s)$ is the stationary distribution of the behavior policy β; recall that $d^\beta(s) = \lim_{t \to \infty} P(S_t = s|S_0, \beta)$; and Q^π is the action-value function estimated with regard to the target policy π (not the behavior policy!).

Off-Policy Policy Gradient

Given that the training observations are sampled by $a \sim \beta(a|s)$, we can rewrite the gradient as:

$$\begin{aligned}
\nabla_\theta J(\theta) &= \nabla_\theta \mathbb{E}_{s\sim d^\beta}\Big[\sum_{a\in\mathcal{A}} Q^\pi(s,a)\pi_\theta(a|s)\Big] \\
&= \mathbb{E}_{s\sim d^\beta}\Big[\sum_{a\in\mathcal{A}} \big(Q^\pi(s,a)\nabla_\theta\pi_\theta(a|s) + {\color{red}\pi_\theta(a|s)\nabla_\theta Q^\pi(s,a)}\big)\Big] && \text{; Derivative product rule.} \\
&\overset{(i)}{\approx} \mathbb{E}_{s\sim d^\beta}\Big[\sum_{a\in\mathcal{A}} Q^\pi(s,a)\nabla_\theta\pi_\theta(a|s)\Big] && \text{; Ignore the red part: } {\color{red}\pi_\theta(a|s)\nabla_\theta Q^\pi(s,a)}\text{.} \\
&= \mathbb{E}_{s\sim d^\beta}\Big[\sum_{a\in\mathcal{A}} \beta(a|s)\frac{\pi_\theta(a|s)}{\beta(a|s)} Q^\pi(s,a)\frac{\nabla_\theta\pi_\theta(a|s)}{\pi_\theta(a|s)}\Big] \\
&= \mathbb{E}_\beta\Big[{\color{blue}\frac{\pi_\theta(a|s)}{\beta(a|s)}} Q^\pi(s,a)\nabla_\theta \ln\pi_\theta(a|s)\Big] && \text{; The blue part is the importance weight.}
\end{aligned}$$

where $\frac{\pi_\theta(a|s)}{\beta(a|s)}$ is the importance weight.

In summary, when applying policy gradient in the off-policy setting, we can simple adjust it with a weighted sum and the weight is the ratio of the target policy to the behavior policy,

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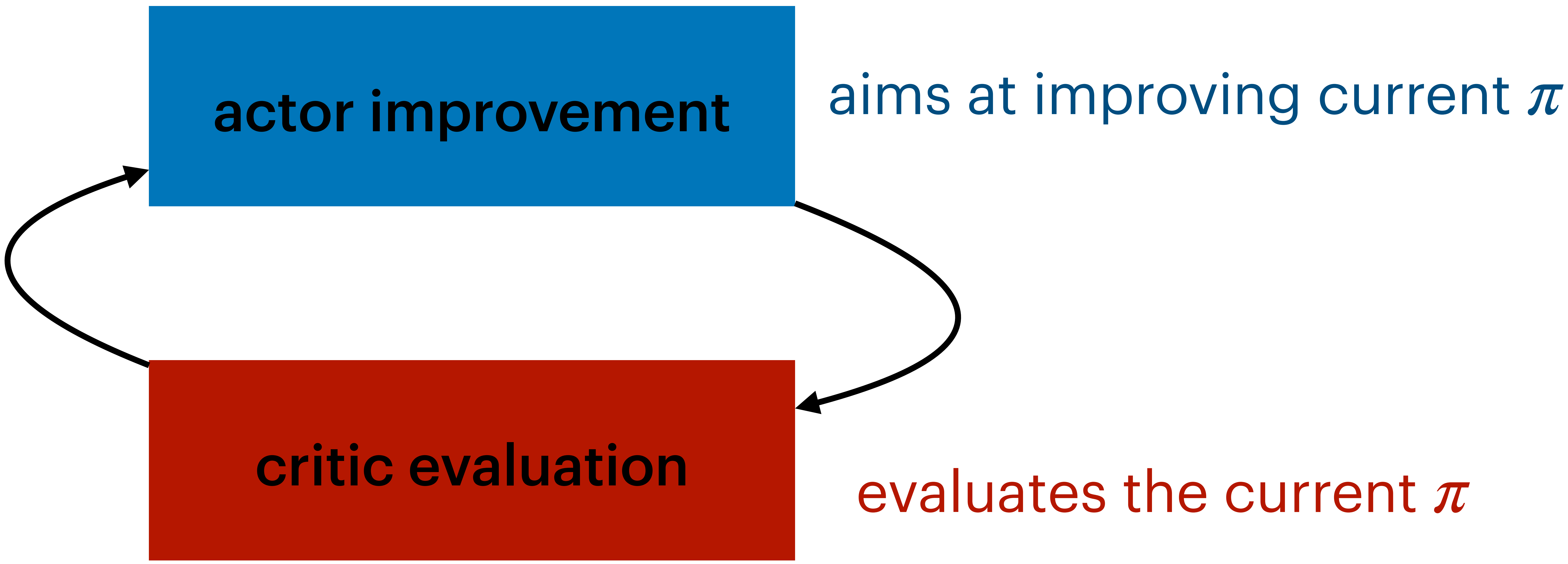
RL -Actor Critic & Intro to DRL

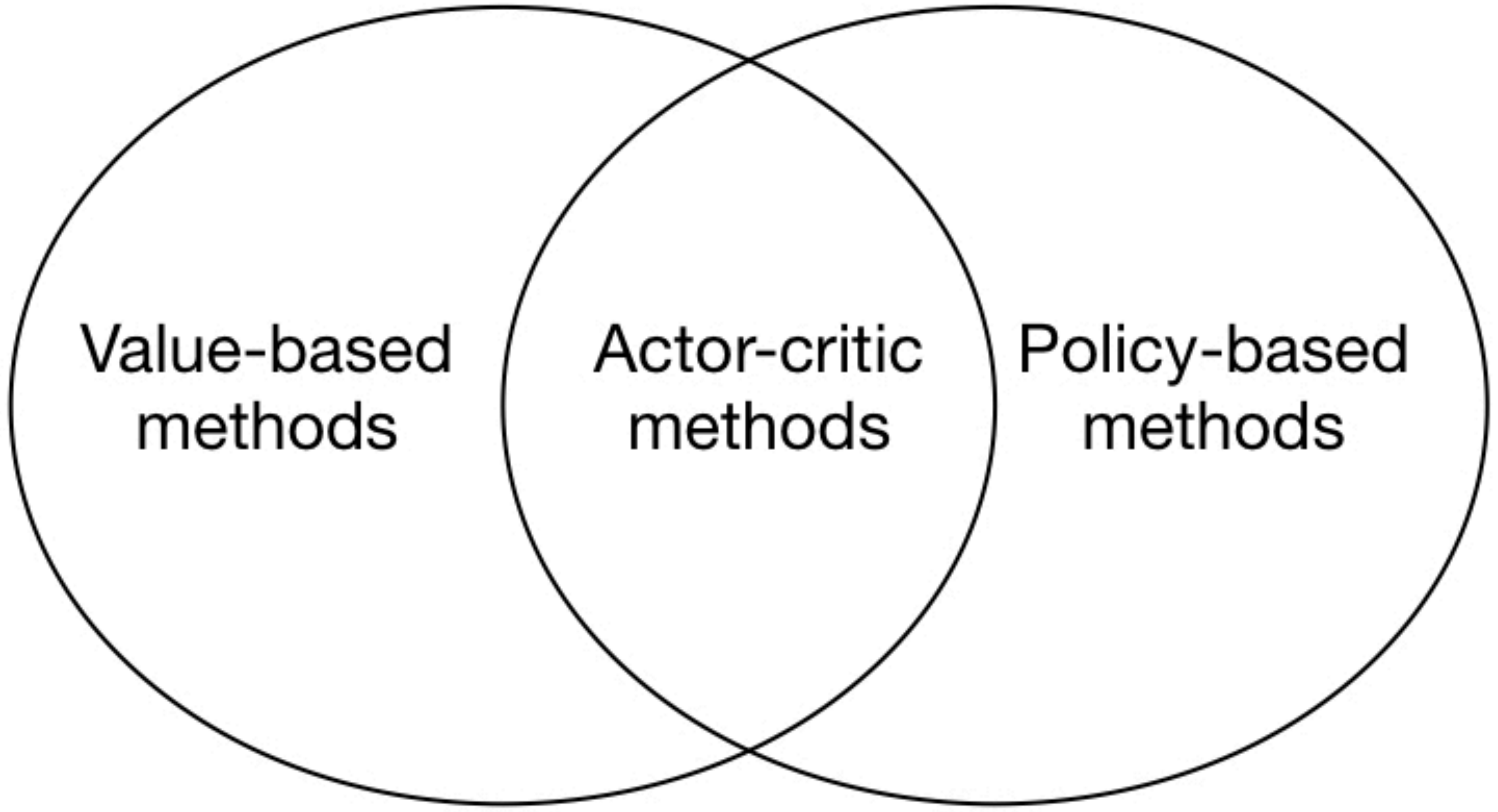
Actor-critic Methods

- A generalized policy iteration, **alternating** between a policy evaluation and a policy improvement step.



Relation to Other RL Methods

- Value-based methods
  - estimate the value function
  - policy is implicit (e.g., ϵ-greedy)
- Policy-based methods
  - estimate the policy
  - no value function
- Actor-critic methods
  - estimate the policy
  - estimate the value function



Implementing a Critic

- The critic estimates the value of the current policy - prediction problem
- Since the actor uses Q values to choose actions, the critic must estimate the Q function.
  - For **small state-spaces**, use tabular TD algorithms to estimate the Q function (SARSA, Q-learning, etc)
  - For **large state-spaces**, use LSTD to estimate the Q function.

Implementing an Actor

- Greedy improvement - move the policy toward the greedy policy underlying the Q function estimate obtained from the critic
  - For small state-action spaces: policy is greedy w.r.t. the obtained Q values
  - For large state-action spaces: policy is parameterized and the greedy action is computed on the fly
- Policy gradient - perform policy gradient directly on the performance surface $J(\omega)$ underlying the chosen parametric policy class

Variance Reduction in Policy gradient

- Instead of using $\nabla_\omega J(\omega) = \mathbb{E}_\pi[\gamma^t R_t \nabla_\omega \log \pi(a_t \mid s_t, \omega)]$ as in REINFORCE, we use $\nabla_\omega J(\omega) = \mathbb{E}_\pi[\gamma^t Q^\pi(s_t, a_t) \nabla_\omega \log \pi(a_t \mid s_t, \omega)]$ since we have estimated Q values in the Critic.

- Further we use an **advantage function** $A^\pi(s_t, a_t) = Q^\pi(s_t, a_t) - V^\pi(s_t)$ instead of $Q^\pi(s_t, a_t)$ since $\mathbb{E}_\pi[\gamma^t V^\pi(s_t) \nabla_\omega \log \pi(a_t \mid s_t, \omega)] = 0$

$$
\begin{aligned}
\mathbb{E}_{\tau\sim\pi_\theta}\left[\nabla_\theta \log \pi_\theta(a_t|s_t) b(s_t)\right] &= \mathbb{E}_{s_{0:t},a_{0:t-1}}\left[\mathbb{E}_{s_{t+1:T},a_{t:T-1}}\left[\nabla_\theta \log \pi_\theta(a_t|s_t) b(s_t)\right]\right] \\
&= \mathbb{E}_{s_{0:t},a_{0:t-1}}\left[b(s_t)\cdot \underbrace{\mathbb{E}_{s_{t+1:T},a_{t:T-1}}\left[\nabla_\theta \log \pi_\theta(a_t|s_t)\right]}_{E}\right] \\
&= \mathbb{E}_{s_{0:t},a_{0:t-1}}\left[b(s_t)\cdot \mathbb{E}_{a_t}\left[\nabla_\theta \log \pi_\theta(a_t|s_t)\right]\right] \\
&= \mathbb{E}_{s_{0:t},a_{0:t-1}}\left[b(s_t)\cdot 0\right] = 0
\end{aligned}
$$

$$
\mathbb{E}_{a_t}\left[\nabla_\theta \log \pi_\theta(a_t|s_t)\right] = \int \frac{\nabla_\theta \pi_\theta(a_t|s_t)}{\pi_\theta(a_t|s_t)} \pi_\theta(a_t|s_t) da_t = \nabla_\theta \int \pi_\theta(a_t|s_t) da_t = \nabla_\theta \cdot 1 = 0
$$

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Deep Reinforcement Learning

- Deep reinforcement learning refers to using a neural network to approximate the value function, the policy or the model.

- Nonlinear function approximate might be “rich”
- However, it may not give any interpretation or the estimate might be stuck at the local optima due to the non-convexity of the optimization landscape

Value-Based Algorithms

Policy Gradient Algorithms

Model-based Methods

A Brief Summary

Summary

- Neural networks can be used to approximate the value function, the policy or the model in reinforcement learning.
- Any algorithms that assumes a parametric approximation can be applied with neural networks
- However, vanilla versions might not always converge due to biased estimates and correlated samples
- With methods such as prioritised replay, double Q-network or duelling networks the stability can be achieved
- Neural networks can also be applied to actor-critic methods
- Using them for model-based method does not always work well due to compounding errors

Deep RL Methods

- Model-free
  - Value-based: DQN, Double DQN, Rainbow DQN, PER, Retrace, ...
  - Policy-based (usually actor-critic): A2C, A3C, DDPG, TRPO, PPO, ...
- Model-based
  - use dynamics model to simulate
  - use model to initialize model-free learners
  - use model to regularize learned representation