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Section = CS-5
Subject = Math-II

Unit = 4

**Q1:** Show that $u = e^{-2xy}\sin(x^2-y^2)$ is harmonic function:

**Sol:** $u = e^{-2xy}\sin(x^2-y^2)$

$$\frac{\partial u}{\partial x} = -2ye^{-2xy}\sin(x^2-y^2) + 2xe^{-2xy}\cos(x^2-y^2)$$

$$\frac{\partial u}{\partial y} = -2xe^{-2xy}\sin(x^2-y^2) - 2ye^{-2xy}\cos(x^2-y^2)$$

$$\frac{\partial^2 u}{\partial x^2} = 4y^2e^{-2xy}\sin(x^2-y^2) - 4xe^{-2xy}\cos(x^2-y^2) - 4xye^{-2xy}\cos(x^2-y^2) - 4x^2e^{-2xy}\sin(x^2-y^2) + 2e^{-2xy}\cos(x^2-y^2)$$

$$\frac{\partial^2 u}{\partial y^2} = 4x^2e^{-2xy}\sin(x^2-y^2) + 4xye^{-2xy}\cos(x^2-y^2) - 2e^{-2y}\cos(x^2-y^2) + 4xye^{-2xy}\cos(x^2-y^2) - 4y^2e^{-2xy}\sin(x^2-y^2)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

u is a harmonic function.

**Q2** Prove that -

$$\int_c \frac{e^{2z}}{(z+1)^4}\,dz = \frac{8\pi i}{3e^2}$$ where c is the circle

$|z| = 3$

P.T.O

Sol→ Here $f(z) = e^{2z}$ is analytic inside the circle $C: |z| = 3$ and the point $a = -1$ lies within $C$.

Hence by the Cauchy's integral formula, we have

$$\int_C \frac{f(z)\,dz}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^n(a)$$

$$\therefore \int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{3!} f'''(a) = \frac{2\pi i}{3!} [f'''(z)]_{z=a}$$

$$= \frac{2\pi i}{6} \left[\frac{d^3}{dz^3} e^{2z}\right]_{z=-1}$$

$$= \frac{\pi i}{3} [8e^{2z}]_{z=-1}$$

$$= \frac{8\pi i}{3} [e^{-2}]$$

$$= \frac{8\pi i}{3e^2}$$ Ans., Proved.

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Q3→ Use calculus of residues to show that

$$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta = \frac{\pi}{6}$$

Let

putting $z = e^{i\theta}$, so that

$$d\theta = \frac{dz}{iz}$$

also $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$ and $\cos 2\theta = \frac{1}{2}\left(z^2+\frac{1}{z^2}\right)$

Thus $$I = \int_C \frac{\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)\frac{dz}{iz}}{5+2\left(z+\frac{1}{z}\right)}.$$

$$= \int_C \frac{(z^4+1)\,dz}{2iz^2(5z+2z^2+2)}$$

$$= \frac{1}{2i}\int_C \frac{(z^4+1)\,dz}{z^2(2z^2+5z+2)} = \frac{1}{2i}\int_C f(z)\,dz$$

where C is the unit circle $|z|=1$

then $f(z) = \frac{z^4+1}{z^2(2z^2+5z+2)}$

$$= \frac{z^4+1}{z^2(2z+1)(z+2)}$$

The poles of $f(z)$ are given $z=0,0,-1/2,-2$ ie $z=0$ is a double poles and $z=-\frac{1}{2}$, $z=-2$ are single poles $f(z)$ since $z=-2$ lies outside the circle $|z|=1$

Res $f(0) = \frac{1}{1!}\lim_{z\to 0}\left[\frac{d}{dz}\{z^2 f(z)\}\right]$.

$$= \frac{1}{1!}\lim_{z\to 0}\left[\frac{d}{dz}\left\{\frac{z^4+1}{2z^2+5z+2}\right\}\right]$$

$$= \lim_{z\to 0}\left[\frac{(2z^2+5z+2)4z^3-(z^4+1)(4z+5)}{(2z^2+5z+2)^2}\right]$$

$$= \frac{-5}{4}$$

and Res $f(-1/2) = \lim_{z\to -\frac{1}{2}}\left[\left(z+\frac{1}{2}\right)f(z)\right]$

$$= \lim_{z\to -1/2}\frac{z^4+1}{2z^2(z+2)}$$

$$= \frac{(1/16)+1}{2\cdot\frac{1}{4}\left(-\frac{1}{2}+2\right)}$$

$$= \frac{17}{16}\cdot\frac{4}{3}$$

$$= \frac{17}{12}$$

Thus by residue theorem, we get

$$I = \frac{1}{2i}\times 2\pi i\{\text{Res } f(0) + \text{Res } f(-1/2)\}$$

$$= \pi\left(-\frac{5}{4}+\frac{17}{12}\right)$$

$$= \frac{\pi}{6} \text{ proved.}$$

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Q4) Define the following singularity with example

1) Isolated singularity
2) Removable singularity
3) Essential singularity

Ans) (1) Isolated singularity:- A point $z=a$ is said to be singularity of the function $f(z)$ if $f(z)$ is analytic atleast at each point in some neighbourhood of the point is defined by $|z-a|<\delta$.

Example:- → consider the function $f(z) = \frac{z+1}{z(z+2)}$
for singularity
$z(z+2)=0$
$z=0,-2$ which is isolated singularity

(2) Removal of singularity:- All pn's are zero i.e. the no terms in principle part of $f(z)$ is called Removal of singularity of

$$f(z) = \frac{z^2 + z + 1)}{z(z-1)}$$

$$f(z) = z + (z+1)(z)^{-1}(z-1)^{-1}$$

(*) Essential Singularity:- If the Principal part of $f(z)$ contain an no. of terms then $z=A$ is called the Isolated ~~of f(z)~~ essential Singularity $F(z)$. ~~f(z) =~~.

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Q5) Find the residue of $f(z) = \frac{1-e^{2z}}{z^4}$ at its Poles.

Solve:) Here

$$f(z) = \frac{1-e^{2z}}{z^4}$$

$$= \frac{1 - \left[1 + \frac{2z}{1!} + \frac{(2z)^2}{2!} + \frac{(2z)^3}{3!} + \frac{(2z)^4}{4!} + \cdots\right]}{z^4}$$

$$= \frac{\left(-2 - 2z - \frac{4}{3}z^2 - \frac{2}{3}z^3 - \cdots\right)}{z^3} \quad (1)$$

Here $f(z)$ has Poles of order 3 at $z=0$

$$\therefore Res_{z=0} f(z) = \frac{1}{(3-1)!} \lim_{z\to 0} \frac{d^2}{dz^2}\left\{(z-0)^3 \cdot \frac{(1-e^{2z})}{z^4}\right\}$$

$$= \frac{1}{2} \lim_{h\to 0} \frac{d^2}{dz^2}\left\{-2 - 2z - \frac{4}{3}z^2 - \frac{2}{3}z^3 - \cdots\right]$$

$$= \frac{1}{2} \lim_{h\to 0}\left(-\frac{8}{3} - \frac{2}{3}\cdot 6z - \cdots\right\} = -\frac{4}{3} \quad \text{Ans.}$$

# Unit = 5

Enrol. no.:- 0206CS191191

Q1) Show that the vector $\vec{A} = (-x^2+yz)\hat{i} + (4y+z^2x)\hat{j} + (2xz-4z)\hat{k}$ is solenoidal?

Sol:) Given $\vec{A} = (-x^2+yz)\hat{i} + (4y+z^2x)\hat{j} + (2xz-4z)\hat{k}$

Now $\text{div}\vec{A} = \nabla\cdot\vec{A}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left\{(-x^2+yz)\hat{i} + (4y+z^2x)\hat{j} + (2xz-4z)\hat{k}\right\}$$

$$= \frac{\partial}{\partial x}(-x^2+yz) + \frac{\partial}{\partial y}(4y+z^2x) + \frac{\partial}{\partial z}(2xz-4z)$$

$$= -\cancel{2x} + \cancel{4} + \cancel{2x} - \cancel{4}$$

$$= 0 \quad \text{Proved.}$$

Hence the given vector is solenoidal.

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Q2) Using Stoke's theorem, Evaluate-

$\int_c [(x+y)dx + (2x-z)dy + (z+y)dz]$ where, c is the boundary of the triangle with vertices (2,0,0), (0,3,0) and (0,0,6)

Sol:) Let $\vec{F} = (x+y)\hat{i} + (2x-z)\hat{j} + (z+y)\hat{k}$

We know that by Stoke's theorem

$$\int_c \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds \quad (1)$$

where 'c' is the boundary of $\Delta ABC$, 's' be the surface of ABC and $\hat{n}$ be unit vector normal to surface

S of $\Delta ABC$ in outward direction.

Now $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$

$$= 2\hat{i} + \hat{k}$$

The equation of triangular plane is

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1 \quad \text{i.e,} \quad 3x + 2y + z = 6$$

Suppose $\phi(x, y, z) = 3x + 2y + z - 6$

$$\nabla \phi = 3\hat{i} + 2\hat{j} + \hat{k}$$

$\therefore \quad \hat{n} = \frac{\nabla \phi}{|\nabla \phi|}$

$$= \frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{9+4+1}}$$

$$= \frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}}$$

$\therefore \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_S (2\hat{i} + \hat{k}) \cdot \left( \frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}} \right) ds$

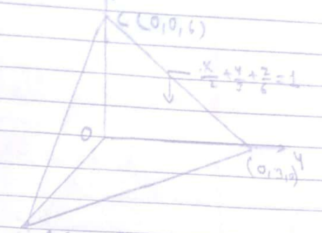
$$= \iint_S \frac{7}{\sqrt{14}}\, ds \quad \text{--- (2)}$$

consider projection R of surface S on xy-plane, which is $\Delta AOB$.

$\therefore \quad ds = \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$

$$= \frac{dx\,dy}{1/\sqrt{14}}$$

from equation (i) & (ii) we get




$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds$

$= \iint_R \frac{7}{\sqrt{14}} \cdot \frac{dx\,dy}{1/\sqrt{14}}$

$= 7 \iint_R dx\,dy$

$= 7(\text{area of } \Delta AOB)$

$\therefore \int_C \vec{F} \cdot d\vec{r} = 7 \times \frac{1}{2} \times 2 \times 3$

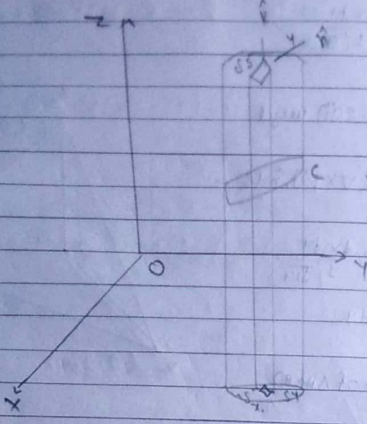
$\boxed{\int_C \vec{F} \cdot d\vec{r} = 21}$ Ans,

(*) Prove that Stoke's theorem

St~~atement: If S~~ be.

Let $z = \phi(x, y)$ be the equation of the surface of S. Now let R be the orthogonal projection of S on the xy-plane and let C be its boundary which is oriented as shown in the fig. we may write the line integral over C' as the integral C, Thus.

$$\therefore \oint_C f_1(x, y, z)\, dx = \oint_{C'} f_1[x, y, \phi(x, y)]\, dx$$




$$= \oint_C \{F_1[x, y, \phi(x,y)]\,dx + 0\,dy\}$$

$$= -\iint_R \frac{\partial f_1}{\partial y}(x, y, \phi)\,dx\,dy \quad (\text{by Green's th})$$

$$\therefore \oint_C f_1(x,y,z)\,dx = -\iint_R \left(\frac{\partial f_1}{\partial y} + \frac{\partial f}{\partial z}\cdot\frac{\partial \phi}{\partial y}\right)dx\,dy \quad \text{— (11)}$$

The direction cosines of the normal to the surface

$z = \phi(x,y)$ are $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, -1$, $\frac{\cos\alpha}{\partial\phi/\partial x} = \frac{\cos\beta}{\partial\phi/\partial y} = \frac{\cos\gamma}{-1}$

Moreover, $dx\,dy$ = projection of $ds$ on the xy-plane $= ds\cos\gamma$ i.e.

$ds = dx\,dy / \cos\gamma$

from eq (11) we have

$$\oint_C f_1(x,y,z)\,dx = -\iint_S \left[\frac{\partial f_1}{\partial y} - \frac{\partial f_1}{\partial z}\cdot\frac{\cos\beta}{\cos\gamma}\right]\cos\gamma\,ds$$

$$= -\iint_S \left[\frac{\partial f_1}{\partial y}\cos\gamma - \frac{\partial f_1}{\partial z}\cdot\cos\beta\right]ds$$

$$\oint_C f_1(x,y,z)\,dx = \iint_S [\nabla \times (f_1\hat{i})]\cdot\hat{n}\,ds \quad \text{— (iii)}$$

$$\oint_C f_2(x,y,z)\,dy = \iint_S \left[\frac{\partial f_2}{\partial x}\cos\gamma - \frac{\partial f_2}{\partial z}\cos\alpha\right]ds$$

$$= \iint_S [\nabla \times f_2\hat{j}]\,\hat{n}\,ds \quad \text{— (4)}$$

$$\oint_C f_3(x,y,z)\,dz = \iint_S \left[\frac{\partial f_3}{\partial y}\cos\alpha - \frac{\partial f_3}{\partial x}\cos\beta\right]ds$$

$$= \iint_S [\nabla \times f_3\hat{k}]\,\hat{n}\,ds \quad \text{— (5)}$$

Adding equation (iii) (4) & (5) we get

$$\oint_C (f_1dx + f_2dy + f_3dz) = \iint_S [\nabla \times (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})]\cdot\hat{n}\,ds$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$ proved.

**Q3** Verify Stoke's theorem for –

$$\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$$

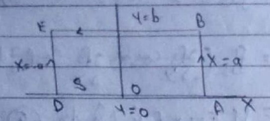
taken around the rectangle bounded by the lines $x = \pm a$, $y = 0$, $y = b$

**Sol:** we have

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= (-2y - 2y)\,\hat{k}$$

$= -4y\hat{k}$




Also $\hat{n} = \hat{k}$

$$\therefore \iint_S (\text{curl}\,\vec{F}).\hat{n}\,ds = \int_{y=0}^{b}\int_{x=-a}^{a} (-4y\hat{k}).\hat{k}\,dx\,dy$$

$$= -4\int_{y=0}^{b}\int_{x=-a}^{a} y\,dx\,dy$$

$$= -4\int_{y=0}^{b} [xy]_{x=-a}^{a}\,dy$$

$$= -4\int_{y=0}^{b} 2ay\,dy$$

$$= -4[ay^2]_0^b$$

$$= -4ab^2$$

Also $\oint_C \vec{F}.d\vec{r} = \oint_C [(x^2+y^2)\hat{i} - 2xy\hat{j}].(dx\hat{i}+dy\hat{j})]$

$$= \oint_C [(x^2+y^2)dx - 2xy\,dy]$$

$$= \int_{DA} [(x^2+y^2)\,dx - 2xy\,dy] + \int_{AB} + \int_{BE} + \int_{ED}$$

Along DA, $y=0$ and $dy=0$, Along AB, $x=a$ and $dx=0$
Along BE, $y=b$ and $dy=0$

Along ED, $x=-a$ and $dx=0$

$$\oint_C \vec{F}.d\vec{r} = \int_{x=-a}^{a} x^2 dx + \int_{y=0}^{b} -2ay\,dy + \int_{x=a}^{-a} (x^2+b^2)\,dx + \int_{y=b}^{0} 2ay\,dy$$

$$= \int_{-a}^{a} x^2 dx - \int_{-a}^{a} (x^2+b^2)\,dx - 4a\int_0^b y\,dy$$

$$= -\int_{-a}^{a} b^2 dx - 4a\int_0^b y\,dy$$

$$= -2ab^2 - 4a\left[\frac{y^2}{2}\right]_0^b$$

$$= -4ab^2$$

Hence $\oint_C \vec{F}.d\vec{r} = \iint_S (\text{curl}\,\vec{F}).\hat{n}\,ds$ proved.

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**Q42** Show that vector field

$\vec{V} = (\sin y + z)\hat{i} + (x\cos y - z)\hat{j} + (x-y)\hat{k}$ is irrotational

**Sol** Here $\vec{V} = (\sin y+z)\hat{i} + (x\cos y - z)\hat{j} + (x-y)\hat{k}$

to show that $\vec{V}$ is irrotational, we shall show that curl $\vec{V} = \vec{0}$.

$\therefore$ curl $\vec{V} = \nabla \times \vec{V}$

$$= \cancel{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\sin y+z) & & \end{vmatrix}}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\sin y+z) & (x\cos y-z) & (x-y) \end{vmatrix}$$

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$$= \hat{i}\left\{\frac{\partial}{\partial y}(x-y) - \frac{\partial}{\partial z}(x\cos y - z)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(x-y) - \frac{\partial}{\partial z}(\sin y + z)\right\}$$

$$+ \hat{k}\left\{\frac{\partial}{\partial x}(x\cos y - z) - \frac{\partial}{\partial y}(\sin y + z)\right\}$$

$$= \hat{i}\{(-1)-(-1)\} - \hat{j}(1-1) + \hat{k}(\cos y - \cos y)$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(0)$$

$$= \vec{0}$$ Proved

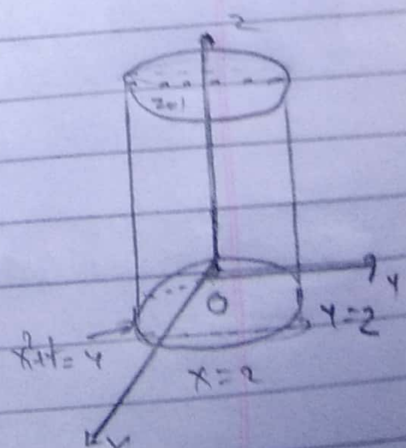
Q5) Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where $\vec{F} = x\hat{i} - y\hat{j} + (z^2-1)\hat{k}$ and S is closed ~~surface~~ bonded ~~by~~ the planes $z=0$, $z=1$ and the cylinder $x^2+y^2=4$.

Sol: By divergence theorem, we have

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dv = \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(z^2-1)\right]dv$$

$$= \iiint_V (1-1+2z)\,dV$$

$$= \iiint_V 2z\,dV$$




~~Clearly z varies~~ $= \iiint_V 2z\,dx\,dy\,dz$

Clearly z varies from 0 to 1

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \left[\int_0^1 2z\,dz\right]dx\,dy$$

(where ~~by~~ ~~st~~. R is the region bounded by circle $x^2+y^2=1$)

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iint_R [z^2]_0^1\, dx\, dy$$

$$= \iint_R dx\, dy$$

$$= (\text{Area of a circle } x^2+y^2 = 4)$$

$$= 4\pi$$

or $$\iint_S \vec{F} \cdot \hat{n}\, ds = \pi(2)^2$$

$$= 4\pi \quad \text{Ans}$$

~~computer~~ computing