mains-jee2

1. Considering only the principal values of the inverse trigonometric functions, the value of

$$\tan\left(\sin^{-1}\left(-\frac{3}{5}\right) - 2\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$$

is

(A) $\frac{7}{24}$
(B) $-\frac{7}{24}$
(C) $-\frac{5}{24}$
(D) $\frac{5}{24}$

2. Let

$S = \{(x, y) \in \mathbb{R}\times\mathbb{R} \mid x \geq 0, y \geq 0, y^2 \leq 4x, y^2 \leq 12-2x, \text{ and } 3y+\sqrt{8}x \leq 5\sqrt{8}\}$.

If the area of the region S is $\alpha\sqrt{2}$, then α is equal to

(A) $\frac{17}{2}$
(B) $\frac{17}{3}$
(C) $\frac{17}{4}$
(D) $\frac{17}{5}$

3. Let $k \in \mathbb{R}$. If

$$\lim_{x\to 0^+} (\sin(\sin kx) + \cos x + x)^{\frac{2}{x}} = e^6,$$

then the value of k is

(A) 1
(B) 2

(C) 3

(D) 4

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{\pi}{x^2}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then which of the following statements is TRUE?

(A) $f(x) = 0$ has infinitely many solutions in the interval $\left(\frac{1}{10^{10}}, \infty\right)$.

(B) $f(x) = 0$ has no solutions in the interval $\left(\frac{1}{\pi}, \infty\right)$.

(C) The set of solutions of $f(x) = 0$ in the interval $\left(0, \frac{1}{10^{10}}\right)$ is finite.

(D) $f(x) = 0$ has more than 25 solutions in the interval $\left(\frac{1}{\pi^2}, \frac{1}{\pi}\right)$.

5. Let S be the set of all $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ such that

$$\lim_{x \to \infty} \frac{\sin(x^2)(\log x)^\alpha \sin\left(\frac{1}{x^2}\right)}{x^\alpha (\log(1+x))^\beta} = 0.$$

Then which of the following is (are) correct?

(A) $(-1, 3) \in S$

(B) $(-1, 1) \in S$

(C) $(1, -1) \in S$

(D) $(1, -2) \in S$

6. A straight line drawn from the point $P(1, 3, 2)$, parallel to the line

$$\frac{x-2}{1} = \frac{y-4}{2} = \frac{z-6}{1}$$

intersects the plane $L_1 : x - y + 3z = 6$ at the point Q. Another straight line which passes through Q and is perpendicular to the plane L_1 intersects the plane $L_2 : 2x - y + z = -4$ at the point R. Then which of the following statements is (are) TRUE?

(A) The length of the line segment PQ is $\sqrt{6}$.

(B) The coordinates of R are $(1, 6, 3)$.

(C) The centroid of the triangle PQR is $\left(\frac{4}{3}, \frac{5}{3}, \frac{3}{3}\right)$.

(D) The perimeter of the triangle PQR is $\sqrt{2} + \sqrt{6} + \sqrt{11}$.

7. Let A_1, B_1, C_1 be three points in the xy-plane. Suppose that the lines A_1C_1 and B_1C_1 are tangents to the curve $y^2 = 8x$ at A_1 and B_1, respectively. If $O = (0, 0)$ and $C_1 = (-4, 0)$, then which of the following statements is (are) TRUE?

   (A) The length of the line segment OA_1 is $4\sqrt{3}$.

   (B) The length of the line segment A_1B_1 is 16.

   (C) The orthocenter of the triangle $A_1B_1C_1$ is $(0, 0)$.

   (D) The orthocenter of the triangle $A_1B_1C_1$ is $(1, 0)$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$, and $g : \mathbb{R} \to (0, \infty)$ be a function such that $g(x + y) = g(x)g(y)$ for all $x, y \in \mathbb{R}$. If

$$f\left(-\frac{3}{5}\right) = 12 \quad \text{and} \quad g\left(-\frac{1}{3}\right) = 2,$$

   then the value of

$$f\left(\frac{1}{4}\right) + g(-2) - 8g(0)$$

   is ____.

9. A bag contains N balls out of which 3 balls are white, 6 balls are green, and the remaining balls are blue. Assume that the balls are identical otherwise. Three balls are drawn randomly one after the other without replacement. For $i = 1, 2, 3$, let W_i, G_i, and B_i denote the events that the ball drawn in the i^{th} draw is a white ball, green ball, and blue ball, respectively. If the probability

$$P(W_1 \cap G_2 \cap B_3) = \frac{2}{5N}$$

   and the conditional probability

$$P(B_3 \mid W_1 \cap G_2) = \frac{2}{9},$$

   then N equals ____.

10. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \frac{\sin x}{e^x} \cdot \frac{x^{2023} + 2024x + 2025}{(x^2 - x + 3)} + \frac{x^{2023} + 2024x + 2025}{e^x(x^2 - x + 3)}.$$

   Then the number of solutions of $f(x) = 0$ in $\mathbb{R}$ is ____.

11. Let $\mathbf{p} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{q} = \mathbf{i} - \mathbf{j} + \mathbf{k}$. If for some real numbers α, β, and γ, we have

$$15\mathbf{i} + 10\mathbf{j} + 6\mathbf{k} = \alpha(2\mathbf{p} + \mathbf{q}) + \beta(\mathbf{p} - 2\mathbf{q}) + \gamma(\mathbf{p} \times \mathbf{q}),$$

then the value of γ is _____.

12. A normal with slope $\frac{1}{\sqrt{6}}$ is drawn from the point $(0, -\alpha)$ to the parabola $x^2 = -4\alpha y$, where $a > 0$. Let L be the line passing through $(0, -\alpha)$ and parallel to the directrix of the parabola. Suppose that L intersects the parabola at two points A and B. Let r denote the length of the latus rectum and s denote the square of the length of the line segment AB. If $r : s = 1 : 16$, then the value of $24a$ is _____.

13. Let the function $f : [1, \infty) \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} (-1)^{n+1} 2, & \text{if } t = 2n - 1, n \in \mathbb{N}, \\ \frac{(2n+1)-t}{2} f(2n-1) + \frac{(t-(2n-1))}{2} f(2n+1), & \text{if } 2n - 1 < t < 2n + 1, n \in \mathbb{N}. \end{cases}$$

Define $g(x) = \int_1^x f(t)\,dt, x \in (1, \infty)$. Let α denote the number of solutions of the equation $g(x) = 0$ in the interval $(1, 8]$ and

$$\beta = \lim_{x \to 1^+} \frac{g(x)}{x - 1}.$$

Then the value of $\alpha + \beta$ is _____.

PARAGRAPH "I"

Let $S = \{1, 2, 3, 4, 5, 6\}$ and X be the set of all relations R from S to S that satisfy both the following properties:

- R has exactly 6 elements.
- For each $(a, b) \in R$, we have $|a - b| \geq 2$.

Let $Y = \{R \in X :$ The range of R has exactly one element$\}$ and $Z = \{R \in X : R$ is a function from S to $S\}$.

Let $n(A)$ denote the number of elements in a set A.

14. If $n(X) = \binom{m}{6}$, then the value of m is _____.

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15. If the value of $n(Y) + n(Z)$ is k^2, then $|k|$ is _____.