JEE MAINS 2024-2

1. Considering only the principal values of the inverse trigonometric functions, the value of

$$\tan\left(\sin^{-1}\left(-\frac{3}{5}\right) - 2\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$$

is

   A. $\frac{7}{24}$
   B. $-\frac{7}{24}$
   C. $-\frac{5}{24}$
   D. $\frac{5}{24}$

2. Let $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \geq 0, y \geq 0, y^2 \leq 4x, y^2 \leq 12 - 2x \text{ and } 3y + \sqrt{8}x \leq 5\sqrt{8}\}$. If the area of the region S is $\alpha\sqrt{2}$, then α is equal to

   A. $\frac{17}{2}$
   B. $\frac{17}{3}$
   C. $\frac{17}{4}$
   D. $\frac{17}{5}$

3. Let $k \in \mathbb{R}$. If

$$\lim_{x \to 0^+} (\sin(\sin kx) + \cos x + x)^{\frac{2}{x}} = e^6,$$

then the value of k is

   A. 1
   B. 2
   C. 3
   D. 4

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{\pi}{x^2}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then which of the following statements is TRUE?

A. $f(x) = 0$ has infinitely many solutions in the interval $\left(\frac{1}{10^{10}}, \infty\right)$.
B. $f(x) = 0$ has no solutions in the interval $\left(\frac{1}{\pi}, \infty\right)$.
C. The set of solutions of $f(x) = 0$ in the interval $\left(0, \frac{1}{10^{10}}\right)$ is finite.
D. $f(x) = 0$ has more than 25 solutions in the interval $\left(\frac{1}{\pi^2}, \frac{1}{\pi}\right)$.

5. Let S be the set of all $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ such that

$$\lim_{x\to\infty} \frac{\sin(x^2)(\log_e x)^\alpha \sin\left(\frac{1}{x^2}\right)}{x^{\alpha\beta}(\log_e(1+x))^\beta} = 0.$$

Then which of the following is (are) correct?

A. $(-1, 3) \in S$
B. $(-1, 1) \in S$
C. $(1, -1) \in S$
D. $(1, -2) \in S$

6. A straight line drawn from the point $P(1, 3, 2)$, parallel to the line

$$\frac{x-2}{1} = \frac{y-4}{2} = \frac{z-6}{1},$$

intersects the plane $L_1 : x - y + 3z = 6$ at the point Q. Another straight line which passes through Q and is perpendicular to the plane L_1 intersects the plane $L_2 : 2x - y + z = -4$ at the point R. Then which of the following statements is (are) TRUE?

A. The length of the line segment PQ is $\sqrt{6}$
B. The coordinates of R are $(1, 6, 3)$
C. The centroid of the triangle PQR is $(()\frac{4}{3}, \frac{14}{3}, \frac{5}{3}())$
D. The perimeter of the triangle PQR is $\sqrt{2} + \sqrt{6} + \sqrt{11}$

7. Let A_1, B_1, C_1 be three points in the xy-plane. Suppose that the lines A_1C_1 and B_1C_1 are tangents to the curve $y^2 = 8x$ at A_1 and B_1, respectively. If $O = (0,0)$ and $C_1 = (-4, 0)$, then which of the following statements is (are) TRUE?

   A. The length of the line segment OA_1 is $4\sqrt{5}$
   B. The length of the line segment A_1B_1 is 16
   C. The orthocenter of the triangle $A_1B_1C_1$ is $(0,0)$
   D. The orthocenter of the triangle $A_1B_1C_1$ is $(1,0)$

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$, and $g : \mathbb{R} \to (0, \infty)$ be a function such that $g(x+y) = g(x)g(y)$ for all $x, y \in \mathbb{R}$. If

$$f\left(-\frac{3}{5}\right) = 12 \quad \text{and} \quad g\left(-\frac{1}{3}\right) = 2,$$

   then the value of

$$f\left(\frac{1}{4}\right) + g(-2) - 8g(0)$$

   is _____.

9. A bag contains N balls out of which 3 balls are white, 6 balls are green, and the remaining balls are blue. Assume that the balls are identical otherwise. Three balls are drawn randomly one after the other without replacement. For $i = 1, 2, 3$, let W_i, G_i, and B_i denote the events that the ball drawn in the i^{th} draw is a white ball, green ball, and blue ball, respectively. If the probability

$$P\left(W_1 \cap G_2 \cap B_3\right) = \frac{2}{5N}$$

   and the conditional probability

$$P\left(B_3 \mid W_1 \cap G_2\right) = \frac{2}{9},$$

   then N equals _____.

10. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \frac{\sin x}{e^{\pi x}} \cdot \frac{x^{2023} + 2024x + 2025}{x^2 - x + 3} + \frac{2}{e^{\pi x}} \cdot \frac{x^{2023} + 2024x + 2025}{x^2 - x + 3}.$$

   Then the number of solutions of $f(x) = 0$ in $\mathbb{R}$ is _____.

11. Let $\mathbf{p} = 2\hat{\imath} + \hat{\jmath} + 3\hat{k}$ and $\mathbf{q} = \hat{\imath} - \hat{\jmath} + \hat{k}$. If for some real numbers α, β, and γ, we have

$$15\hat{\imath} + 10\hat{\jmath} + 6\hat{k} = \alpha\,(2\mathbf{p} + \mathbf{q}) + \beta\,(\mathbf{p} - 2\mathbf{q}) + \gamma\,(\mathbf{p} \times \mathbf{q})\,,$$

then the value of γ is ____.

12. A normal with slope $\frac{1}{\sqrt{6}}$ is drawn from the point $(0, -\alpha)$ to the parabola $x^2 = -4\alpha y$, where $a > 0$. Let L be the line passing through $(0, -\alpha)$ and parallel to the directrix of the parabola. Suppose that L intersects the parabola at two points A and B. Let r denote the length of the latus rectum and s denote the square of the length of the line segment AB. If $r : s = 1 : 16$, then the value of $24a$ is ____.

13. Let the function $f : [1, \infty) \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} (-1)^{n+1} 2, & \text{if } t = 2n-1, n \in \mathbb{N}, \\ \frac{(2n+1)-t}{2} f(2n-1) + \frac{(t-(2n-1))}{2} f(2n+1), & \text{if } 2n-1 < t < 2n+1, n \in \mathbb{N}. \end{cases}$$

Define $g(x) = \int_1^x f(t)\,dt, x \in (1, \infty)$. Let α denote the number of solutions of the equation $g(x) = 0$ in the interval $(1, 8]$ and

$$\beta = \lim_{x \to 1^+} \frac{g(x)}{x - 1}.$$

Then the value of $\alpha + \beta$ is ____.

PARAGRAPH "I"

Let $S = \{1, 2, 3, 4, 5, 6\}$ and X be the set of all relations R from S to S that satisfy both the following properties:

A. R has exactly 6 elements.

B. For each $(a, b) \in R$, we have $|a - b| \geq 2$.

Let $Y = \{R \in X : \text{The range of } R \text{ has exactly one element}\}$ and $Z = \{R \in X : R \text{ is a function from } S \text{ to } S\}$.

Let $n(A)$ denote the number of elements in a set A.

(There are two questions based on PARAGRAPH "I", the question given below is one of them)

14. If $n(X) = \binom{m}{6}$, then the value of m is ______

15. If the value of $n(Y) + n(Z)$ is k^2, then $|k|$ is ______

PARAGRAPH "II"

Let $f : \left(0, \frac{\pi}{2}\right) \to [0, 1]$ be the function defined by

$$f(x) = \sin^2 x$$

and let $g : \left(0, \frac{\pi}{2}\right) \to [0, \infty)$ be the function defined by

$$g(x) = \sqrt{\frac{\pi x}{2} - x^2}.$$

(There are two questions based on PARAGRAPH "II", the question given below is one of them.)

16. The value of

$$2\int_0^{\frac{\pi}{2}} f(x)g(x)dx - \int_0^{\frac{\pi}{2}} g(x)dx$$

is ______

17. The value of

$$\frac{16}{\pi^3}\int_0^{\frac{\pi}{2}} f(x)g(x)dx$$

is ______